DPW potentials for compact symmetric CMC surfaces in the 3-sphere

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Zusammenfassung

Minimal- und CMC-Immersionen kompakter Riemannscher Flächen in der 3-Sphäre lassen sich anhand der Familie der ihnen zugeordneten flachen $\mathbf{SL}(2,\mathbb{C})$-Zusammenhänge auf einem Rang-2 holomorphen Vektorbündel $E \to M$ untersuchen. Allerdings ist die Beschreibung der Familie flacher Zusammenhänge für Riemannsche Flächen mit einem Genus $g \geq 2$ komplizierter. Es ist in diesem Fall einfacher, eine verwandte Familie meromorpher flacher Verbidungen zu betrachten und daraus die assoziierte Familie der Immersion zu rekonstruieren. Das Ziel dieser Arbeit ist es, die Möglichkeit der Definition meromorpher flacher Zusammenhänge auf einer Klasse von CMC-Flächen $f : M \to \mathbb{S}^3$ mit einer Gruppe von Symmetrien zu zeigen, welche endlich ist und deren Flächenquotient die Riemannsche Kugel $\mathbb{CP}^1$ ist. Gezeigt wird, dass die von Lawson 1970 sowie von Karcher, Pinkall und Sterling 1988 konstruierten Flächen zu der gleichen Klasse von Flächen gehören. Zunächst wird ein holomorphes Vektorbündel $\tilde{E}$ auf $\mathbb{CP}^1$ mit parabolischer Struktur definiert. Anschliessend betrachten wir eine Familie logarithmischer flacher Zusammenhänge $\tilde{\nabla}^\lambda$ auf $\tilde{E}$ und zeigen, dass $\tilde{\nabla}^\lambda$ eine festgelegte Asymptote für $\lambda = 0$ hat. Der Hauptsatz der Arbeit zeigt, dass $\tilde{\nabla}^\lambda$ dazu genutzt werden kann, ein DPW-Potential auf der CMC-Fläche zu definieren, welches die notwendigen Bedingungen erfüllt, um anhand einer Schleifen-Gruppenfaktorisation die Immersion $f : M \to \mathbb{S}^3$ zu rekonstruieren.

Introduction

A natural variational problem in the geometry of surfaces is the Isoperimetric problem which consists in finding the surface of minimum area among all the surfaces enclosing a fixed volume. The answer is, of course, the round sphere, and a proof of this has been known for a long time.

A more general question in Differential Geometry of surfaces in a space form is to determine the surfaces whose area is critical either under deformations which keep the volume unchanged or under deformations with no constrains on the volume. The differential equation characterizing the first class of surfaces is $H = constant$, where H is the mean curvature, and are called *constant mean curvature surfaces* (CMC). The second class is characterized by the equation $H = 0$ and these surfaces are known as *minimal surfaces*.

Although the theory of non compact minimal or CMC surfaces in $\mathbb{R}^3$ is very rich and has been a driving force for the development of many beautiful theories, the class of compact surfaces embedded in $\mathbb{R}^3$, minimally or CMC, contains only one example: the round sphere. Indeed, since the corresponding immersion of a minimal surface is given by a harmonic function, a maximum principle argument shows that there exist no compact minimal surfaces in $\mathbb{R}^3$ whether they are embedded or not.

As for CMC surfaces in $\mathbb{R}^3$, in 1956 Hopf [41] proved that the only compact CMC surface of genus 0 immersed in $\mathbb{R}^3$ is the round sphere. Later, in 1958, Alexandrov [1] proved that the only compact CMC surface embedded in $\mathbb{R}^3$ is the round sphere. The situation is different for surfaces embedded in $\mathbb{S}^3$. In fact, in 1970, Lawson [48] proved the existence of compact surfaces minimally embedded in $\mathbb{S}^3$ for every genus g. In 1988 Karcher, Pinkall and Sterling [44] provided other examples of compact surfaces minimally embedded in $\mathbb{S}^3$.

In 1998 Dorfmeister, Pedit and Wu [22] introduced a method (called the DPW method) which allows the construction of CMC immersions of simply connected Riemann surfaces in $\mathbb{S}^3$ given a holomorphic $\mathfrak{sl}(2,\mathbb{C})$-valued 1-form $\xi(z,\lambda)$ with series expansion of the form

$$\xi(z,\lambda) = \sum_{i=-1}^{\infty} \xi_i(z)\lambda^i. \tag{1}$$

In order to find the immersion $f : M \to \mathbb{S}^3$ it is necessary to solve the ODE

$$d\Psi(z,\lambda) = \Psi(z,\lambda)\xi(z,\lambda) \tag{2}$$

with respect to the variable z. A solution $\Psi : M \times \mathbb{C}^* \to \mathbf{SL}(2,\mathbb{C})$ can be decomposed via the Iwasawa decomposition [42] as

$$\Psi(z,\lambda) = F(z,\lambda)B(z,\lambda). \tag{3}$$

The element $F(z,\lambda)$ is called the unitary component of $\Psi(z,\lambda)$. A CMC immersion $f : M \to \mathbb{S}^3$ is obtained from the Sym-Bobenko formula [9]

$$f(z) = F(z,\eta\lambda)F^{-1}(z,\lambda) \tag{4}$$

for every $\eta, \lambda \in \mathbb{S}^1$, with $\eta \neq 1$. The mean curvature of f is given by

$$H = i\frac{1+\eta}{1-\eta}. \tag{5}$$

Although the DPW method was first meant to construct simply connected surfaces the method has been used for surfaces with more complicated topology. In this case, it is necessary to impose additional conditions to ensure that the immersed surface closes along non trivial loops on the surface ([56], [12], [46], [59]).

Hitchin and Bobenko, ([37], [9]) in the early '90s studied minimal and CMC immersions of compact surfaces of genus 1 in $\mathbb{S}^3$ using integrable system methods. They introduced the associated family of flat $\mathbf{SL}(2,\mathbb{C})$-connections of an immersion $f : M \to \mathbb{S}^3$ on a topologically trivial rank 2 vector bundle $E \to M$, given by

$$\nabla^\lambda = \nabla + \lambda^{-1}\Phi - \lambda\Phi^* \tag{6}$$

where $\Phi \in H^0(M, \mathrm{End}_0(E) \otimes K)$ is a nilpotent 1-form called the Higgs field.

Moreover, it was proven that, given a family of flat $\mathbf{SL}(2,\mathbb{C})$-connections ∇^λ, of the form (6), such that

- for $\lambda \in \mathbb{S}^1$ the connection ∇^λ is unitary;
- there exist $\lambda_1, \lambda_2 \in \mathbb{S}^1$ such that ∇^{λ_i} is the trivial connection;
- the residue of ∇^λ at $\lambda = 0$ is a nilpotent and nowhere vanishing Higgs field Φ,

it is possible to reconstruct the associated immersion as the gauge transformation between the trivial connections ∇^{λ_1} and ∇^{λ_2} (cf. Section 3.1).

Hitchin [37] studied in detail the case of M being a torus. He classified all the families of flat connections ∇^{λ} and parametrized the associated CMC immersions $f : M \to \mathbb{S}^3$.

For M being a torus the connections ∇^{λ} split into a direct sum of flat line bundle connections, for generic $\lambda \in \mathbb{C}^*$. Thus, the associated family of flat $\mathbf{SL}(2,\mathbb{C})$-connections can be described in terms of a corresponding family of flat line bundles on M.

Unfortunately, this is no longer applicable for higher genus surfaces. In fact, in this case flat $\mathbf{SL}(2,\mathbb{C})$-connections, generically, do not decompose as a direct sum of flat line bundle connections. In 2013 Heller [35] considered the Lawson surface $\Sigma_{2,1}$ of genus 2 and proved that its associated family of flat connections can be determined by a family of flat connections on the four punctured sphere.

Heller used the fact that the Lawson surface $\Sigma_{2,1}$ of genus 2 has several symmetries, including a $\mathbb{Z}_3$ symmetry, such that the quotient $\Sigma_{2,1}/\mathbb{Z}_3$ is the Riemann sphere $\mathbb{CP}^1$ and the covering map $\pi : \Sigma_{2,1} \to \mathbb{CP}^1$ has four branch points. Heller [35, Theorem 4.2] proved that the associated family of holomorphic flat connections ∇^{λ} of $\Sigma_{2,1}$ is gauge equivalent to a holomorphic family of meromorphic flat connections $\hat{\nabla}^{\lambda}$ with prescribed singularities, where the gauge transformation is singular at the branch points of the covering $\pi : \Sigma_{2,1} \to \mathbb{CP}^1$. Moreover, using the symmetries of $\Sigma_{2,1}$ Heller [35, Theorem 4.3] proved that the holomorphic family $\hat{\nabla}^{\lambda}$ is gauge equivalent to the holomorphic family of meromorphic connections $d + \eta(\lambda)$, where the meromorphic 1-form $\eta(\lambda)$ is given by the pullback of a meromorphic 1-form $\xi(\lambda)$ on $\mathbb{CP}^1$ under $\pi : \Sigma_{2,1} \to \mathbb{CP}^1$.

The 1-form $\xi(\lambda)$ is completely determined up to two unknown holomorphic functions in λ, called the *accessory parameters*. The connection 1-form $\eta(\lambda)$ gives a meromorphic DPW potential on $\Sigma_{2,1}$ from which it is possible to reconstruct the minimal immersion $f : \Sigma_{2,1} \to \mathbb{S}^3$.

Even though this approach can be used for other CMC or minimal surfaces in $\mathbb{S}^3$ of genus $g > 2$, it is more difficult to describe explicitly the corresponding family of flat connections on the four punctured sphere, and has not been carried out so far.

The aim of this thesis is to show that there exists a DPW potential for every CMC embedding $f : M \to \mathbb{S}^3$, of a compact Riemann surface M with genus $g \geq 2$, such that

(i) there exists a finite subgroup $G \subset \mathbf{SO}(4)$, with a presentation of the form

$$G = \langle g_1, g_2, g_3, g_4 \mid g_1 \cdots g_4 = 1 \rangle, \tag{7}$$

where 1 denotes the identity element of G, which acts faithfully on $f(M) \simeq M$ by orientation preserving automorphisms;

(ii) the quotient M/G is the Riemann sphere $\mathbb{CP}^1$;

(iii) the covering map $\pi : M \to M/G \simeq \mathbb{CP}^1$ is a $|G|$-fold covering, branched at four points $z_1, \dots, z_4 \in \mathbb{CP}^1$.

We will call a Riemann surface which satisfies $(i) - (iii)$ a *symmetric CMC surfaces* (cf. Definition 4.1). In Section 2.5 and Section 2.6 we will show that the Lawson's surfaces $\Sigma_{k,l}$ and the surfaces found by Karcher, Pinkall and Sterling are examples of symmetric CMC surfaces.

The first step, in order to construct a DPW potential for a symmetric CMC surface, consists in consider a lift of the action $G \times M \to M$ to an action $\Gamma \times M \to M$ where $\Gamma \subset \mathbf{SU}(2) \times \mathbf{SU}(2)$ is a finite group double covering G (cf. Section 4.1). The action of Γ on M can be lifted to an action on the holomorphic vector bundle $E \to M$ where the associated family of flat $\mathbf{SL}(2, \mathbb{C})$-connections of the immersions $f : M \to \mathbb{S}^3$ is defined (cf. Section 4.2)

In order to define the appropriate vector bundle on $\mathbb{CP}^1$ (cf. Section 4.3), which we will use to construct a DPW potential for M, we need a faithful action of Γ on M. Since this is not the case in our situation, we prove in Proposition 4.1 that there exists an abstract Riemann surface $\tilde{M}$, double covering M, on which Γ acts faithfully. Moreover, the surface $\tilde{M}$ is such that the quotient $\tilde{M}/\Gamma$ is the Riemann sphere $\mathbb{CP}^1$ and the covering map $\tilde{\pi} : \tilde{M} \to \tilde{M}/\Gamma$ is branched at the points $z_1, \dots, z_4$.

Then, we consider the pullback bundle $\tilde{E} = \tau^* E \to \tilde{M}$, where $\tau : \tilde{M} \to M$ is an appropriately chosen double covering, branched at $z_1, \dots, z_4 \in \mathbb{CP}^1$. The pullback of the associated family of flat connections ∇^λ under the map τ gives a family of Γ-equivariant connections on the bundle $\tilde{E} \to \tilde{M}$ (cf. Proposition 4.2)

We are in the right position to state the main result of the thesis.

Theorem. *4.1. Let M be a symmetric CMC surface with symmetry group $G \subset \boldsymbol{SO}(4)$ from the Table (4.3). Let ∇^λ be the associated family of flat $\boldsymbol{SL}(2, \mathbb{C})$-connections of the immersion $f : M \to \mathbb{S}^3$. Then, there exists a holomorphic family of logarithmic connections*

$$\tilde{\nabla}^\lambda = \lambda^{-1}\tilde{\Phi} + \tilde{\nabla} + \text{ higher order terms in } \lambda$$

on the four punctured sphere $\mathbb{CP}^1$, singular at the four branch points $z_1, \dots, z_4$ of $\pi : M \to M/G = \mathbb{CP}^1$, where $\tilde{\Phi}$ is a nilpotent $\mathfrak{sl}(2, \mathbb{C})$-valued complex linear 1-form, which satisfies the following:

(i) *there exists a flat connection* $\hat{\nabla}$ *on* M *with* $\mathbb{Z}_2$*-monodromy representation, such that the families of connections* ∇^λ *and* $\pi^*\tilde{\nabla}^\lambda \otimes \hat{\nabla}$ *are gauge equivalent via a family of gauge transformations* $g(\lambda)$ *which extends holomorphically at* $\lambda = 0$*;*

(ii) *there is an open neighborhood* U *of* $\lambda = 0$ *such that* $\tilde{\nabla}^\lambda$ *can be represented by a* λ*-family of Fuchsian systems for* $\lambda \in U$*. More specifically, for* $\lambda \in U$*, we have*

$$\tilde{\nabla}^\lambda = d + \eta(z, \lambda) = d + \sum_{j=-1}^{\infty} \eta_j(z)\lambda^j,$$

where, for every j*,* $\eta_j(z)$ *is a* $\mathfrak{sl}(2, \mathbb{C})$*-valued 1-form with simple poles at the branch points* $z_1, \ldots, z_4$ *and holomorphic on* $\mathbb{CP}^1 \setminus \{z_1, \ldots, z_4\}$*;*

(iii) *the map* $\lambda \mapsto \eta(z, \lambda)$ *extends meromorphically to* $\mathbb{C}^*$ *and the connection* $\tilde{\nabla}^\lambda = d + \eta(z, \lambda)$ *has unitarizable monodromy representation for every* $\lambda \in \mathbb{S}^1$ *such that* $\eta(z, \lambda)$ *does not have a pole;*

(iv) *the eigenvalues of the local residues of* $\tilde{\nabla}^\lambda$ *are given by the eigenvalues (of the first or second factor in* $\boldsymbol{SU}(2) \times \boldsymbol{SU}(2)$*) of the four generators* $\gamma_1, \ldots, \gamma_4$ *of the finite group* $\Gamma \subset \boldsymbol{SU}(2) \times \boldsymbol{SU}(2)$ *which double covers* G*.*

In particular, all of these CMC surfaces can be constructed from a meromorphic DPW potential on the four punctured sphere.

In order to prove Theorem 4.1 we will first use the correspondence between orbifold bundles and parabolic vector bundles introduced by Biswas [6] in 1997.

Given a vector bundle $\tilde{E} \to \tilde{M}$, equipped with a Γ-action, we consider the push-forward bundle $\tilde{\pi}_*\tilde{E} \to \mathbb{CP}^1$ (cf. Subsection 1.4.3). The appropriate vector bundle on $\mathbb{CP}^1$, necessary to define the DPW potential for M, is given by the Γ-invariant sub-bundle of $\tilde{\pi}_*\tilde{E}$, denoted with $(\tilde{\pi}_*\tilde{E})^\Gamma$ (cf. Section 4.3).

Biswas [6] showed that it is possible to define a parabolic structure on $(\tilde{\pi}_*\tilde{E})^\Gamma$ (cf. Section 4.3). Parabolic structures on vector bundles, first introduced by Mehta and Seshandri [51] in 1980, allow to understand the behaviour of a vector bundle defined over a space with singular points, for example Riemann surfaces with cusps. In our situation, the singular locus is given by the set of branch points of the covering $\tilde{\pi} : \tilde{M} \to \mathbb{CP}^1$.

Following the work of Biswas and Heu [8], we show that given a Γ-equivariant connection ∇ on $\tilde{E} \to \tilde{M}$ there exists a logarithmic connection $\tilde{\nabla}$ on $(\tilde{\pi}_*\tilde{E})^\Gamma$ whose pull-back under $\tilde{\pi}$ is gauge equivalent to ∇ (cf. Section 4.4). A *logarithmic connection* is a connection whose connection

1-form has logarithmic singularities at the points of a prescribed subspace D of the base space (cf. Definition 4.6).

It is possible to define a parabolic structure on $(\tilde{\pi}_*\tilde{E})^\Gamma$ using the logarithmic connection $\tilde{\nabla}$ (cf. Subsection 4.4.2), which turns out to be equivalent to the parabolic structure defined by the Biswas's approach (cf. Proposition 4.5). This equivalence allows to define a λ-family of parabolic structures on $(\tilde{\pi}_*\tilde{E})^\Gamma$ which extends to $\lambda = 0$ (cf. Subsection 4.4.3).

The next step is to show that the Higgs field Φ on E induces a Higgs field $\tilde{\Phi}$ on $(\tilde{\pi}_*\tilde{E})^\Gamma$ such that it is nilpotent, preserves the parabolic structure and makes $(\tilde{\pi}_*\tilde{E})^\Gamma$ a strictly stable parabolic Higgs bundle (cf. Section 4.5). The family of logarithmic connections $\tilde{\nabla}^\lambda$ can be written as

$$\tilde{\nabla}^\lambda = \lambda^{-1}\tilde{\Phi} + \tilde{\nabla} + \text{ higher order terms in } \lambda. \tag{8}$$

The pullback of $\tilde{\nabla}^\lambda$ under $\tilde{\pi}$ gives a meromorphic DPW potential on $\tilde{E}$ which is gauge equivalent to the λ-family of connections $\tau^*\nabla^\lambda$.

The thesis is organised as follows: Chapter 1 contains some background material about the theory of Riemann surfaces, group actions on Riemann surfaces and holomorphic vector bundle.

In Chapter 2 we introduce some of the most relevant results about CMC and minimal immersions in a three dimensional manifold. Section 2.5 and Section 2.6 contain the description of the Lawson's surfaces [48] and the surfaces constructed by Karcher, Pinkall and Sterling [44]. We also show that those surfaces are symmetric CMC surfaces in the above sense.

The gauge theoretical formalism for CMC surfaces in $\mathbb{S}^3$ is introduced in Chapter 3, where we describe the construction of the associated family of flat $\mathbf{SL}(2,\mathbb{C})$-connections. In Section 3.2 we explain how Heller [35] used the DPW method to describe the Lawson's surface of genus 2.

The last chapter contains the main results of this thesis. In Section 4.1 we show how to lift the $\mathbf{SO}(4)$-action on a symmetric CMC surface M to a $\mathbf{SU}(2)\times\mathbf{SU}(2)$-action and in Proposition 4.1 we prove the existence of the Riemann surface $\tilde{M}$ on which the $\mathbf{SU}(2) \times \mathbf{SU}(2)$-action is faithful. The Subsections 4.1.1, 4.1.2 and 4.1.3 contains the description of the action of the finite group $\Gamma \subset \mathbf{SU}(2) \times \mathbf{SU}(2)$ for the Lawson surfaces $\Sigma_{d-1,1}$, $\Sigma_{k-1,l-1}$ and some of the KPS surfaces.

Section 4.2 contains the description of the action of Γ on the holomorphic vector bundle $E \to M$ and the action of Γ on sections and connections.

The Γ-invariant vector bundle $(\tilde{\pi}_*\tilde{E})^\Gamma \to \mathbb{CP}^1$ is described in Section 4.3, together with the parabolic structure on $(\tilde{\pi}_*\tilde{E})^\Gamma$ defined by Biswas in [6].

In Section 4.4 we prove, following the work of Biswas and Heu in [8], the existence of a logarithmic connection $\tilde{\nabla}$ on $(\tilde{\pi}_*\tilde{E})^\Gamma \to \mathbb{CP}^1$. We also describe the parabolic structure on $(\tilde{\pi}_*\tilde{E})^\Gamma$ induced by $\tilde{\nabla}$ and Proposition 4.5 shows the equivalence of this parabolic structure with the parabolic structure defined by Biswas. In Subsection 4.4.3 we describe how it is possible to define a λ-family of parabolic structured on $(\tilde{\pi}_*\tilde{E})^\Gamma$ and Proposition 4.6 gives a description of the admissible holomorphic structure on $(\tilde{\pi}_*\tilde{E})^\Gamma$ for $\lambda \neq 0$.

Section 4.5 deals with the study of the residue in $\lambda = 0$ of the family of logarithmic connections $\tilde{\nabla}^\lambda$ defined by using the family of Γ-equivariant connections $\tau^*\nabla^\lambda$ on $\tilde{E} \to \tilde{M}$. We show how to define a Higgs field on $(\tilde{\pi}_*\tilde{E})^\Gamma$ and we prove in Proposition 4.8 that it is a nilpotent parabolic Higgs field, with non vanishing residues at the branch points , which makes $(\tilde{\pi}_*\tilde{E})^\Gamma$ a stable parabolic Higgs bundle. In Subsection 4.5.2 and 4.5.3, we study which holomorphic bundles on $\mathbb{CP}^1$ admit a nilpotent, nowhere vanishing, parabolic stable Higgs field and we conclude that the only possibility is that the bundle $(\tilde{\pi}_*\tilde{E})^\Gamma$, at $\lambda = 0$, is the bundle $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$. We conclude with some final remarks and the proof of the main Theorem 4.1.

Chapter 1

Preliminary Theory

1.1 Riemann surfaces

We begin this chapter with some preliminary definitions and results about Riemann surfaces. We will mainly refer to [21], [52], [31] and [27].

Definition 1.1. A *Riemann surface* is given by a Hausdorff topological space M together with an open covering $\{U_\alpha\}$ of M and a collection $\{\varphi_\alpha\}$ of homeomorphisms $\varphi_\alpha : U_\alpha \to \tilde{U}_\alpha$ to an open set $\tilde{U}_\alpha \subset \mathbb{C}$, such that the composition

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta) \tag{1.1}$$

is a holomorhic function, for every α, β with $U_\alpha \cap U_\beta \neq \emptyset$.

The maps $\{\varphi_\alpha\}$ are called *local charts* and sometimes we will denote them with $z_\alpha = x_\alpha + iy_\alpha$.

A function $f : M \to \mathbb{C}$ on a Riemann surface is called *holomorphic* (resp. *meromorphic*) around a point $p \in M$ if the function $f \circ \varphi^{-1}$ is a holomorphic (resp. meromorphic) function, where φ is a local chart around p. In a similar way, identifying $\mathbb{C} \simeq \mathbb{R}^2$, one says a function $f : M \to \mathbb{C}$ is *smooth* if $f \circ \varphi^{-1}$ is a smooth function (with respect to the coordinate (x, y) given by the real and imaginary parts of the complex coordinate z).

Definition 1.2. Let M and N be Riemann surfaces. A map $F : M \to N$ is called holomorphic if the composition

$$\varphi_\alpha \circ F \circ \psi_\beta^{-1} : \psi_\beta^{-1}(U_\alpha \cap F^{-1}(V_\beta)) \to \tilde{V}_\beta \tag{1.2}$$

is a holomorphic function for every chart $(U_\alpha, \varphi_\alpha)$ of M and (V_β, ψ_β) of N.

The Riemann surfaces M and N are said to be equivalent if there exists a holomorphic map $F : M \to N$ with holomorphic inverse.

Let $F : M \to N$ be a non constant holomorphic map between Riemann surfaces. For every point p in M there exists a local chart around p and a local chart around $F(p)$ such that F is locally represented by the map

$$z \mapsto z^k, \tag{1.3}$$

for an integer $k = k_p \geq 1$, called the *multiplicity of F at p* ([21, Proposition 5 p. 43]).

Definition 1.3. Let $F : M \to N$ be a non constant holomorphic map between Riemann surfaces and $R \subset M$ the set of points of M such that F has multiplicity $k > 1$ at them. The set $B := F(R) \subset N$ is called the set of *branch points of F*.

If F is proper, that is the preimage under F of a compact set of N is a compact set in M, the set B is discrete and the preimage $F^{-1}(q)$ is a finite set for every $q \in N$ ([21, Proposition 6 p. 44]).

The *degree of the map $F : M \to N$ at a point $q \in N$* is given by

$$d(q) := \sum_{p \in F^{-1}(q)} k_p. \tag{1.4}$$

If M and N are connected the degree of F does not depend on the point $q \in N$ ([21, Proposition 7 p. 44]).

Given a Riemann surface M and a point $p \in M$, let $f : U \to \mathbb{C}$ be a function defined on the domain of a local chart (U, φ) aroun p. If $\varphi = z = x + iy$ we can consider the function f as a function on the real variables (x, y) and it is possible to consider the derivatives of f with respect to x and y

$$\partial f / \partial x, \quad \partial f / \partial y. \tag{1.5}$$

The *tangent space of M at p* is the vector space of $\mathbb{R}$-linear derivations on the ring of smooth functions on a local chart (U, φ) around p, which can be represented by

$$T_p M := \mathbb{R}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}. \tag{1.6}$$

Analogously, the *complex tangent space $T_{\mathbb{C},p}M$ of M at p* is the vector space of $\mathbb{C}$-linear derivations on the ring of smooth complex valued functions on U:

$$T_{\mathbb{C},p} M := \mathbb{C}\left\{\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right\}. \tag{1.7}$$

The *complex cotangent space of M at p* is given by the dual $T^*_{\mathbb{C},p}M$ of $T_{\mathbb{C},p}M$. A basis for $T^*_{\mathbb{C},p}M$ is given by the elements $dz, d\bar{z}$ which are the dual elements of $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ respectively.

A *differential 1-form* ω on M is a map $\omega : M \to T^*_{\mathbb{C},p}M$, which can be written in local coordinates as

$$\omega = \omega_1 dz + \omega_2 d\bar{z}, \tag{1.8}$$

where $\omega_1, \omega_2 : M \to \mathbb{C}$ are smooth functions.

We will refer to [21, Chapter 5] and [52, Chapter 4] for more details about differential forms on a Riemann surface. Here we just recall that the space $\Omega^1(M)$ of 1-forms on M can be decomposed as

$$\Omega^1(M) = \Omega^{1,0}(M) \oplus \Omega^{0,1}(M), \tag{1.9}$$

where the elements of $\Omega^{1,0}(M)$ can be written locally as $\omega = \omega_1 dz$ and the elements of $\Omega^{0,1}(M)$ as $\eta = \eta_1 d\bar{z}$ for ω_1, η_1 complex valued functions.

1.2 Finite group actions on Riemann surfaces

Let G be a finite group and M a Riemann surface. We recall the notion of *group action* on a Riemann surface:

Definition 1.4. An *action* of G on M is a map $G \times M \to M$, which we will denote by the multiplicative notation, $(g, p) \mapsto g \cdot p$, such that

- $(gh) \cdot p = g \cdot (h \cdot p)$, for all $g, h \in G$ and all $p \in M$;
- $e \cdot p = p$ for any $p \in M$ and e the identity element of G.

The *orbit* of a point $p \in M$ is the set $G \cdot p = \{g \cdot p \mid g \in G\} \subset M$ (sometimes we will denote the orbit of p with $[p]$). The *stabilizer* of a point $p \in M$ is the subgroup $G_p = \{g \in G \mid g \cdot p = p\} \subset G$. Sometimes the stabilizer of a point $p \in M$ is called the *isotropy subgroup* of G at p.

The action is said to be continuous (resp. holomorphic) if for every $g \in G$ the bijection sending $p \to g \cdot p$ is a continuous (resp. holomorphic) map from M to itself. Moreover, if the kernel of the action $K = \{g \in G \mid g \cdot p = p, \text{ for all } p \in M\}$ is trivial we will say that the action is *effective* or *faithful*.

The quotient space M/G is the set of orbits and there is a natural quotient map $\pi : M \to M/G$ sending a point to its orbit. We consider M/G having the quotient topology. The following result shows under which hypothesis the quotient M/G is a Riemann surface:

Theorem 1.1 ([52, Theorem 3.4 p. 78])**.** *Let G be a finite group acting holomorphically and effectively on a Riemann surface M. Then it is possible to define a system of local charts on*

M/G which makes M/G into a Riemann surface. Moreover, the quotient map $\pi : M \to M/G$ is holomorphic of degree $|G|$, and $k_p(\pi) = |G_p|$ for any point $p \in M$.

Theorem 1.1 implies the following result, which describe how a finite group acts, locally, on a Riemann surface.

Corollary 1.1 ([52, Corollary 3.5 p. 79])**.** *Let G be a finite group acting holomorphically and effectively on a Riemann surface M. Fix a point $p \in M$ with nontrivial stabilizer of order m and let $g \in G_p$ generate the stabilizer subgroup. Then there is a local coordinate z on M centered at p such that $g(z) = \lambda z$, where λ is a primitive $m-$th root of unity. Moreover, by a suitable choice of a different generator g of G_p, it is possible to assume $\lambda = e^{\frac{2\pi i}{m}}$.*

1.3 Monodromy representation of holomorphic maps between Riemann surfaces

1.3.1 Covering spaces and the fundamental group

We will introduce the notion of covering map between topological spaces and we will briefly describe how this notion is related to the theory of Riemann surfaces. Since Riemann surfaces are locally homeomorphic to a disc in the complex plane, in what follows we will consider topological spaces which are Hausdorff, second countable and locally pathwise connected, even if the results hold for more general spaces.

Definition 1.5. Let $F : P \to Q$ be a map between topological spaces. F is a *covering map* if, around each point $q \in Q$ there is an open neighbourhood V such that $F^{-1}(V)$ is a disjoint union of open sets U_α in P and $F_{|U_\alpha}$ is a homeomorphism from U_α to V. Two coverings $F : P \to Q$ and $F' : P' \to Q$ are *equivalent* if there is a homeomorphism $g : P \to P'$ such that $F = F' \circ g$.

If P and Q are connected, for any two points $q_0, q_1 \in Q$ the number of points in $F^{-1}(q_0)$ and in $F^{-1}(q_1)$ is the same ([25, Theorem 4.16 p. 26]). This number is called the *degree* of the covering map $F : P \to Q$.

The relation between the notion of covering map and the theory of Riemann surfaces is given by the following result

Lemma 1.1 ([52, Lemma 4.7 p. 89])**.** *Let $F : M \to N$ be a covering map where N is a Riemann surface and M is a connected topological space. Then, there is a unique Riemann surface structure on M such that F is a holomorphic map.*

In general a non constant proper holomorphic map between connected Riemann surfaces has ramification points, thus it is not exactly a covering map in the sense of Definition 1.5. However, the number of points in the preimage $F^{-1}(q)$ counted with the multiplicities of F at these points is constant ([21, Proposition 7 p. 44]). Thus, the degree of F as a non constant, proper, holomorphic map is equal to the degree of F as a covering map.

Definition 1.6. A *path* in a Riemann surface M is a continuous map $\gamma : [0,1] \to M$. A path γ such that $\gamma(0) = \gamma(1)$ is called a *loop*.

Definition 1.7. Let M be a Riemann surface and $p_0, p_1 \in M$. Two paths $\gamma_1, \gamma_2 : [0,1] \to M$ with $\gamma_1(0) = \gamma_2(0) = p_0$ and $\gamma_1(1) = \gamma_2(1) = p_1$ are called *homotopic* if there exists a continuous map $A : [0,1] \times [0,1] \to M$ with the following properties:

(i) $A(t,0) = \gamma_1(t), \forall t \in [0,1]$;

(ii) $A(t,1) = \gamma_2(t), \forall t \in [0,1]$;

(iii) $A(0,t) = p_0$ and $A(1,t) = p_1, \forall t \in [0,1]$.

It is possible to prove that the notion of homotopy is an equivalence relation on the set of all paths in a Riemann surface M which connects a point p_0 to a point p_1, we refer to [25, Theorem 3.2 p. 14] for a proof in the general case of topological spaces.

In order to define a group structure on the set of homotopy classes of loops on a Riemann surface, based at a given point $p_0 \in M$ we first recall the definition of the product of two generic paths on M: Let p_0, p_1 and p_2 be three points in a Riemann surface M, γ_1 a path in M from p_0 to p_1 and γ_2 a path in M from p_1 to p_2. The *product path* $\gamma_1 \cdot \gamma_2$ is the path from p_0 to p_2 defined by

$$\gamma_1 \cdot \gamma_2(t) := \begin{cases} \gamma_1(2t), & \text{for } 0 \leq t \leq \frac{1}{2} \\ \gamma_2(2t-1), & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases} \tag{1.10}$$

The *inverse path* $\gamma^- : [0,1] \to M$ of a path γ on M is defined as

$$\gamma^-(t) := \gamma(1-t), \forall t \in [0,1]. \tag{1.11}$$

The following result defines the *fundamental group* of a Riemann surface:

Theorem 1.2 ([25, Theorem 3.8 p. 17])**.** *Let M be a Riemann surface and $p_0 \in M$. The set $\pi_1(M, p_0)$ of homotopy classes of loops based at p_0 forms a group under the operation induced by the above definitions of product and inverse of paths. This group is called the fundamental group of M based in p_0.*

We conclude this subsection describing the relation between a non constant, proper holomorphic maps $F : M \to N$ between Riemann surfaces and subgroups of the fundamental group of N.

Given a covering map $F : M \to N$ and a path γ in N a path $\tilde{\gamma}$ in M is a *lift* of γ if $\gamma = F \circ \tilde{\gamma}$. Given a path γ in N there always exists a lift $\tilde{\gamma}$ in M and if two lifts of γ coincide at one point then they are the same path ([25, Theorem 4.14 p. 25], [25, Theorem 4.8 p. 22]). Moreover, it is possible to also lift homotopy classes of paths in N with starting point $q_0 \in N$ to homotopy classes of paths in M starting at some $p_0 \in F^{-1}(q_0)$ ([25, Theorem 4.10 p. 23]).

Thus, the covering map $F : M \to N$ induces a map $F_* : \pi_1(M, p_0) \to \pi_1(N, q_0)$. The map F_* is injective and the image subgroup $F_*(\pi_1(P, p_0))$ consists of the homotopy classes of loops in N, based at q_0 whose lifts in M, starting at p_0, are loops (we refer to [31, Proposition 1.31 p. 61] for a proof). The degree of the covering map $F : M \to N$ is equal to the index of the subgroup $F_*(\pi_1(M, p_0))$ in $\pi_1(N, q_0)$ ([31, Proposition 1.32 p. 61]).

The following result shows that it is possible to go in the other direction, that is find a covering map of a given Riemann surface starting from a subgroup of its fundamental group:

Proposition 1.1 ([31, Proposition 1.36 p.66])**.** *Let N be a Riemann surface. Then for every subgroup $H \subset \pi_1(N, q_0)$ there is a connected Riemann surface M_H and a covering map $F : M_H \to N$ such that*

$$H = F_*(\pi_1(M_H, p_0)) \tag{1.12}$$

for a suitably chosen base point $p_0 \in M_H$.

The correspondence between connected covering spaces of finite degree of a Riemann surface M and subgroup $H \subset \pi_1(M, p_0)$ of index d is called *Galois correspondence* and it is summarized by the following:

Theorem 1.3 (Galois correspondence)**.** *Let N be a connected Riemann surface and $q_0 \in N$ a base point. Then there is a bijection between the set of isomorphism classes of covering maps $F : M \to N$ fixing the point q_0 and conjugacy classes of subgroups of $\pi_1(N, q_0)$, obtained by associating the subgroup $F_*(\pi_1(M, p_0))$ to the covering map $F : M \to N$.*

A proof of this Theorem for more general topological spaces is given in [31, Theorem 1.38 p. 67].

1.3.2 Monodromy representation

Let $F : P \to Q$ be a covering map between connected topological spaces of finite degree d. Let $q \in Q$ and $\{p_1, \ldots, p_d\}$ the set of points in $F^{-1}(q) \subset P$. As we recalled in Subsection 1.3.1, every loop γ in Q based at q can be lifted to d paths $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_d$, where $\tilde{\gamma}_j$ is the unique lift of γ starting at p_j. The endpoints $\tilde{\gamma}_1(1), \ldots, \tilde{\gamma}_d(1)$ lie over q, that is $F(\tilde{\gamma}_j(1)) = q$, and they form the entire set $\{p_1, \ldots, p_d\}$. Thus, there exists a permutation σ of $\{1, \ldots, d\}$ such that

$$\tilde{\gamma}_j(1) = p_{\sigma(j)}, \quad j = 1, \ldots, d. \tag{1.13}$$

The permutation σ depends only on the homotopy class of γ (cf. [52, Chapter 3 p. 86]) and the map

$$\begin{aligned} \rho : \pi_1(Q, q) &\to \mathcal{S}_d \\ [\gamma] &\mapsto \sigma \end{aligned} \tag{1.14}$$

where $\mathcal{S}_d$ is the space of permutations on d elements, is well defined. Given two elements $[\gamma_1], [\gamma_2] \in \pi_1(Q, q)$, the definition of product of paths (1.10) implies

$$\rho([\gamma_1][\gamma_2]) = \sigma_1\sigma_2, \tag{1.15}$$

where $\sigma_j = \rho([\gamma_j])$. This means that ρ is a group homomorphism and we can give the following:

Definition 1.8. Let $F : P \to Q$ be a covering map between connected topological spaces of finite degree d. The group homomorphism $\rho : \pi_1(Q, q) \to \mathcal{S}_d$ defined in (1.14) is called the *monodromy representation* of $F : P \to Q$.

The image H_ρ of ρ in $\mathcal{S}_d$ is a transitive subgroup of $\mathcal{S}_d$, that is for every pair of indices $j, l \in \{1, \ldots, d\}$ there exists a permutation $\sigma \in H_\rho$ such that $\sigma(j) = l$ ([52, Lemma 4.4 p.87]).

Conversely, given a connected topological space Q, a point $q \in Q$ and a group homomorphism $\rho : \pi_1(Q, q) \to \mathcal{S}_d$ with transitive image, it is possible to define a covering space $F : P \to Q$ such that its monodromy representation coincides with ρ. We briefly describe how this construction works: Fix an element in $\{1, \ldots, d\}$, for example 1, and let $H \subset \pi_1(Q, q)$ be the subgroup

$$H := \{[\gamma] \in \pi_1(Q, q) \mid \rho([\gamma])(1) = 1\}. \tag{1.16}$$

The index of H is d and, by the results in Subsection 1.3.1, it is possible to define a covering space $F : P \to Q$ associated to H. Moreover, the monodromy representation of this covering space coincide with the map ρ (we refer to [52, Chapter 3 pp. 88-89] and [31, Chapter 1 pp. 68-70] for more details).

We can now describe how it is possible to define a monodromy representation of a non constant holomorphic map between compact Riemann surfaces with finite degree d. Let $F : M \to N$ be such a map and $B \subset N$ the set of branch points of F (cf. Definition 1.3). The restriction of F to $M \smallsetminus F^{-1}(B) \to N \smallsetminus B$ is a holomorphic map without ramification points and branch points and it can be considered as a covering map in the sense of Definition 1.5. Therefore, there exists a monodromy representation $\rho : \pi_1(N \smallsetminus B, q) \to \mathcal{S}_d$ associated to $F_{|M \smallsetminus F^{-1}(B)}$, where $q \in N \smallsetminus B$ is a fixed base point. The map ρ is called the *monodromy representation of the holomorphic map* $F : M \to N$.

Given a compact Riemann surface N, a finite subset $B \subset N$, the next result shows that it is possible to construct a compact Riemann surface M and a non constant holomorphic map $F : M \to N$ of finite degree d, with branch points lying in B from a group homomorphism $\rho : \pi_1(N \smallsetminus B, q) \to \mathcal{S}_d$ with transitive image.

Theorem 1.4 (Riemann's existence Theorem, [21, Theorem 2 p. 49])**.** *Let N be a compact and connected Riemann surface and B a finite subset of N. Given $d \geq 1$ and a group homomorphism $\rho : \pi_1(N \smallsetminus B, q) \to \mathcal{S}_d$ with transitive image, there exists a compact Riemann surface M and a non constant holomorphic map $F : M \to N$ which realizes ρ as its monodromy representation. Moreover, F and M are unique up to equivalence.*

We conclude this section with an example which we will use in Chapter 4.

Example 1.1. Let M be a compact Riemann surface and G a finite group acting faithfully on M, generated by a finite number k of elements, which satisfy

$$\Pi_j g_j = 1. \tag{1.17}$$

Moreover, assume that the quotient M/G is the Riemann sphere $\mathbb{CP}^1$ and the quotient map $\pi : M \to \mathbb{CP}^1$ is a holomorphic map of degree d, branched over the points $z_1, \dots, z_k \in \mathbb{CP}^1$. Given a point $z_0 \in \mathbb{CP}^1 \smallsetminus \{z_1, \dots, z_k\}$, let γ_j be a simple loop based at z_0 around the points z_j. The fundamental group $\pi_1(\mathbb{CP}^1 \smallsetminus \{z_1, \dots, z_k\}, z_0)$ is generated by the loops γ_j with the relation

$$[\gamma_1] \cdots [\gamma_k] = 1, \tag{1.18}$$

where 1 is the identity element of $\pi_1(\mathbb{CP}^1 \smallsetminus \{z_1, \dots, z_k\}, z_0)$.

Let $\rho : \pi_1(\mathbb{CP}^1 \smallsetminus \{z_1, \dots, z_k\}, z_0) \to \mathcal{S}_d$ be the monodromy represenation of $\pi : M \to \mathbb{CP}^1$ and $\sigma_j := \rho([\gamma_j])$. Since ρ is a group homomorphism we have

$$\rho([\gamma_1] \cdots [\gamma_k]) = \rho([\gamma_1]) \cdots \rho([\gamma_k]) = \sigma_1 \cdots \sigma_k = \mathrm{Id}_{\mathcal{S}_d} \tag{1.19}$$

Thus, the monodromy representation ρ, in this case, can be determined by choosing d permutations $\sigma_1, \ldots, \sigma_d$ such that $\sigma_1 \cdots \sigma_k = 1$. The image of ρ is the subgroup of $\mathcal{S}_d$ generated by the σ_j's.

The preimage $\pi^{-1}(z_0)$ consists of d distinct points $p_1, \ldots, p_d$ of M. For every generator g_j of G, the set $g_j \cdot \pi^{-1}(z_0) := \{g_j \cdot p_1, \ldots, g_j \cdot p_d\}$ contains the same elements of $\pi^{-1}(z_0)$ by definition. Thus,

$$g_j \cdot p_l = p_{\sigma_j(l)}$$

for some permutation $\sigma_j \in \mathcal{S}_d$. From the relation $g_1 \cdots g_k = 1$ we have that $\sigma_1 \cdots \sigma_k = 1$. Therefore, we can consider the monodromy representation ρ as the group homomorphism

$$\begin{aligned} \rho : \pi_1(\mathbb{CP}^1 \setminus \{z_1, \ldots, z_k\}, z_0) &\to G \\ [\gamma_j] &\mapsto g_j. \end{aligned} \tag{1.20}$$

1.4 Complex and holomorphic vector bundles

As we will see in Chapter 3, it is possible to study CMC and minimal surfaces immersed in the 3-sphere (cf. Chapter 2) using holomorphic bundles over the surfaces and linear connections. In this section we will introduce these notions and we will recall some standard results.

1.4.1 Complex vector bundles over Riemann surfaces

Definition 1.9. Let M be a Riemann surface, a C^∞ *complex vector bundle* (or, simply, a *complex vector bundle*) on M consists of a family $\{E_p\}_{p\in M}$ of complex vector spaces parametrized by M, together with a C^∞ manifold structure on $E = \cup_{p\in M} E_p$ such that:

- The projection map $\pi : E \to M$ taking E_p to p is C^∞;
- For every $p_0 \in M$ there exists an open set U in M containing p_0 and a diffeomorphism

  $$\phi_U : \pi^{-1}(U) \to U \times \mathbb{C}^k$$

  taking the vector space E_p isomorphically onto $\{p\} \times \mathbb{C}^k$ for each $p \in U$. The map ϕ_U is called a *trivialization of E over U*. An open set $U \subset M$ is called a *trivializing set* if there exists a trivialization map defined on U.

The dimension of the fibers E_p of E is called the *rank* of E. A rank 1 vector bundle is called a *line bundle*.

Definition 1.10. Let $\pi_1 \to M$ and $\pi_2 \to N$ be complex vector bundles over the Riemann surfaces M and N. A *complex vector bundle map* (or *morphism*) is given by a pair of smooth maps $\tilde{f} : E_1 \to E_2$ and $f : M \to N$ such that:

- $f \circ \pi_1 = \pi_2 \circ \tilde{f}$;
- The restriction of $\tilde{f}$ on each fiber, $\tilde{f} : E_{1_{|p}} \to E_{2_{|f(p)}}$ is $\mathbb{C}$-linear for every $p \in M$.

Two complex vector bundles $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ are called *isomorphic* if there exists a vector bundle morphism $\tilde{f} : E_1 \to E_2$ such that it is a diffeomorphism between differentiable manifolds.

Given two trivializing sets U_α and U_β of a rank k complex vector bundle $\pi : E \to M$ with $U_\alpha \cap U_\beta \neq \emptyset$, the map

$$\phi_\alpha \circ \phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C}^k \to (U_\alpha \cap U_\beta) \times \mathbb{C}^k, \tag{1.21}$$

is linear on each fiber. Thus,

$$\phi_\alpha \circ \phi_\beta^{-1}(p, v) = (p, g_{\alpha\beta}(p)v), \tag{1.22}$$

where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbf{GL}(k, \mathbb{C})$ is called the *transition function* of the complex vector bundle $\pi : E \to M$ on $U_\alpha \cap U_\beta$.

The transition function $g_{\alpha\beta}$ describes how the trivialization maps ϕ_α and ϕ_β are related in the intersection $U_\alpha \cap U_\beta$. Moreover, for every trivializing sets U_α, U_β and U_γ of M with non empty intersection, the transition functions $g_{\alpha\beta}, g_{\beta\gamma}$ and $g_{\gamma\alpha}$ satisfy the *cocycle conditions*:

$$\begin{aligned} g_{\alpha\beta} \cdot g_{\beta\alpha} &= \mathrm{Id}, \qquad \text{on } U_\alpha \cap U_\beta; \\ g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} &= \mathrm{Id}, \qquad \text{on } U_\alpha \cap U_\beta \cap U_\gamma. \end{aligned} \tag{1.23}$$

The transition functions characterize completely the vector bundle E. More precisely, given an open cover $\{U_\alpha\}$ of M and smooth maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbf{GL}(k, \mathbb{C})$ satisfying the cocycle conditions for every α and β with $U_\alpha \cap U_\beta \neq \emptyset$, there exists a unique complex vector bundle $E \to M$ with transition functions $\{g_{\alpha\beta}\}$ (for more details we refer to [15, Chapter 1 p. 2]).

Definition 1.11. A *section* s of a vector bundle $\pi : E \to M$ over $U \subset M$ is a smooth map $s : U \to E$, such that $s(p) \in E_p$ for every $p \in U$. A *local frame* for E over U is a collection $s_1, \ldots, s_k$ of sections of M over U such that $\{s_1(p), \ldots, s_k(p)\}$ is a basis for E_p, for all $p \in U$. The space of sections over $U \subset M$ is denoted with $\Gamma(U, E)$.

Given a trivialization ϕ_U of a vector bundle $E \to M$ over an open set $U \subset M$, every section s of E over U can be represented uniquely as a smooth vector valued function $f = (f_1, \dots, f_k)$ by

$$s(p) = \sum_j f_j(p)\phi_U^{-1}(p, e_j) = \sum_j \phi_U^{-1}(p, f_j(p)e_j).$$

Let U_α, U_β be two trivializing sets of $E \to M$ with trivializing maps ϕ_α and ϕ_β, respectively. A section s of $E \to M$ defined on $U_\alpha \cap U_\beta$, is represented by the function $f = (f_1, \dots f_k)$ with respect to ϕ_α and by $f' = (f'_1, \dots f'_k)$ with respect to ϕ_β. Then, the following holds ([27, Chapter 0.5 p. 69]):

$$f(p) = g_{\alpha\beta}(p)f'(p), \quad \forall p \in U_\alpha \cap U_\beta. \tag{1.24}$$

Thus, given an open cover $\{U_\alpha\}$ of M of trivializing sets for a rank k complex vector bundle $E \to M$ with trivializing maps $\{\phi_\alpha\}$, the sections of $E \to M$ correspond to collections $\{f^\alpha = (f_1^\alpha, \dots, f_k^\alpha) : U_\alpha \to \mathbb{C}^k\}$ of vector valued functions such that

$$f^\alpha = g_{\alpha\beta} f^\beta, \quad \text{on } U_\alpha \cap U_\beta$$

for all α and β with $U_\alpha \cap U_\beta \neq \emptyset$, where $\{g_{\alpha\beta}\}$ is the collection of the transition functions of $E \to M$ relative to $\{\phi_\alpha\}$.

Definition 1.12. A complex structure on a complex vector bundle $E \to M$ is given by a bundle map $J : E \to E$, such that on each fiber E_p of E, the linear map $J_p : E_p \to E_p$ satisfies

$$J_p^2 = -\operatorname{Id}. \tag{1.25}$$

Definition 1.13. A *subbundle* $F \subset E$ of a bundle $\pi : E \to M$ is a collection $\{F_p \subset E_p\}_{p \in M}$ of subspaces of the fibers E_p such that $F = \bigcup_{p \in M} F_p \subset E$ is a submanifold and $\pi_{|F} : F \to M$ is still a vector bundle.

Example 1.2 (Trivial bundle)**.** Let M be a Riemann surface. The product $M \times \mathbb{C}^k$ together with the projection to the first factor, given by $\pi(p, v) = p$, is called the *trivial complex vector bundle of rank k*. Sometimes we will denote it with $\pi : \underline{\mathbb{C}}^k \to M$.

Example 1.3 (Tangent and cotangent bundle)**.** Let M be a Riemann surface with local charts $(U_\alpha, \varphi_\alpha)$, $T_{\mathbb{C}}M := \bigcup_{p \in M} T_{\mathbb{C},p}M$ (cf. Equation (1.7)) and $\pi : T_{\mathbb{C}}M \to M$ the projection to the first factor. The collection $\{g_{\alpha\beta}\}$:

$$g_{\alpha\beta} := \mathcal{J}_{\mathbb{R}}(\varphi_\alpha \circ \varphi_\beta^{-1}) : U_\alpha \cap U_\beta \to \mathbf{GL}(2, \mathbb{C}) \tag{1.26}$$

where $\mathcal{J}_{\mathbb{R}}$ is the Jacobian with respect to the real variables x_β, y_β, satisfies the cocycle conditions 1.23 and gives $\pi : T_{\mathbb{C}}M \to M$ the structure of complex vector bundle of rank 2 called the *complex tangent bundle of M*.

In analogous way it is possible to define the *complex cotangent bundle* $T^*_{\mathbb{C}}(M)$ of M, where $T^*_{\mathbb{C}}(M) := \bigcup_{p \in M} T^*_{\mathbb{C},p}M$ and the transition functions $\{h_\alpha \beta\}$ are given by

$$h_{\alpha\beta} = (\mathcal{J}_{\mathbb{R}}(\varphi_\alpha \circ \varphi_\beta^{-1})^{-1})^t. \tag{1.27}$$

The space of local sections on $T^*_{\mathbb{C}}(M)$ over a local chart U_α is the space of 1-forms $\Omega^1(U_\alpha)$.

We will briefly recall some operations on vector bundles. Let $E \to M$ and $F \to M$ be two complex vector bundles of rank k and l, respectively, with transition functions $\{g_{\alpha\beta}\}$ and $\{h_{\alpha\beta}\}$:

- Dual bundle([27, Chapter 0.5 p. 66]):

  The dual bundle $E^* \to M$ has fibers given by $E^*_p = (E_p)^*$ and transition functions

  $$j_{\alpha\beta}(p) = (g_{\alpha\beta}^{-1})^t.$$

- Direct sum bundle([27, Chapter 0.5 p. 67]):

  The rank $(k+l)$ complex vector bundle $E \oplus F$ is given by the transition functions

  $$j_{\alpha\beta}(p) = \begin{pmatrix} g_{\alpha\beta}(p) & 0 \\ 0 & h_{\alpha\beta}(p) \end{pmatrix} \in \mathbf{GL}(k+l, \mathbb{C}). \tag{1.28}$$

- Tensor product bundle([27, Chapter 0.5 p. 67]):

  The tensor product bundle $E \otimes F$ having rank kl is defined by the transition functions

  $$j_{\alpha\beta}(p) = g_{\alpha\beta}(p) \otimes h_{\alpha\beta}(p) \in \mathbf{GL}(kl, \mathbb{C}).$$

- Alternating product bundle([27, Chapter 0.5 p. 67]):

  The bundle $\bigwedge^r E$ is given by the transition functions

  $$j_{\alpha\beta}(p) = \bigwedge^r g_{\alpha\beta}(p).$$

  In particular if $r = k$, $\bigwedge^k E$ is a line bundle called the *determinant bundle* having transition functions

  $$j_{\alpha\beta}(p) = \det(g_{\alpha\beta}(p)).$$

Let $f : M \to N$ be a smooth map between manifolds and $\pi : E \to N$ a complex vector bundle. It is possible to define a complex vector bundle over M induced by f as follows ([27, Chapter 0.5 p. 68]): Let $f^*E := \{(p, v) \in M \times E \mid f(p) = \pi(v)\} \subset M \times E$ equipped with the subspace topology. The projection map $\pi' : f^*E \to M$ given by the projection to the first factor is a complex vector bundle of the same rank as $\pi : E \to N$ called the *pull-back bundle of E by f*. If (U, φ) is a local trivialization for $\pi : E \to N$ then, $(f^{-1}(U), \psi)$ is a local trivialization for $\pi' : f^*E \to M$ defined by

$$\psi(p, v) = (p, \pi_2(\varphi(v))), \tag{1.29}$$

where π_2 denotes the projection to the second factor.

It is also possible to consider differential k-forms on a Riemann surface M with values on a complex vector bundle $\pi : E \to M$:

Definition 1.14. A *E-valued differential k-form* ω on $U \subset M$ is a section of the complex vector bundle $E \otimes \bigwedge^k T^*_{\mathbb{C}}U$. We denote the space of E-valued differential k-forms with $\Omega^k(U, E)$ (or simply $\Omega^k(E)$ when the domain of definition is M).

Definition 1.15. Let $E \to M$ be a rank k complex vector bundle over a Riemann surface M. A *hermitian metric on E* is a hermitian inner product $\langle , \rangle$ on each fiber E_p varying smoothly with $p \in M$. More, precisely, if $s = (s_1, \dots, s_k)$ is a frame for $E \to M$ over an open set $U \subset M$, the function

$$h_{jl} = \langle s_j, s_l \rangle, \quad j, l = 1, \dots, k \tag{1.30}$$

is smooth in $p \in U$. A complex vector bundle $E \to M$ together with a hermitian metric is called *hermitian vector bundle*.

1.4.2 Holomorphic vector bundles

Definition 1.16. A rank k complex vector bundle $\pi : E \to M$ on a Riemann surface M is called *holomorphic* if the trivialization maps $\{\phi_\alpha\}$ are holomorphic. It follows that the transition functions $g_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$ of a holomorphic vector bundle are holomorphic.

A section $s \in \Gamma(U, E)$ of a rank k holomorphic vector bundle $E \to M$, represented (with respect to a local holomorphic frame) by k functions $(f_1, \dots, f_k)$, $f_j : U \to \mathbb{C}$, is a *holomorphic section* if every f_j is holomorphic. The space of holomorphic sections of $E \to M$ on an open set $U \subset M$ is denoted with $H^0(U, E)$.

Definition 1.17. A *meromorphic section* of a holomorphic vector bundle $E \to M$ is a section s, such that the the function $f = (f_1, \dots, f_k) : M \to \mathbb{C}^k$ representing s with respect to a given trivialization is a meromorphic function. The *order of s at a point* $p \in M$ is given by the order of $f = (f_1, \dots, f_k)$ at p.

Given a Riemann surface, it is always possible to define the following holomorphic line bundle on it:

Definition 1.18. Let M be a Riemann surface. The *canonical bundle* $K \to M$ is the line bundle, where

$$K := \bigcup_{p \in M} \{p\} \times T'^{*}_{\mathbb{C},p}(M), \tag{1.31}$$

and $T'^{*}_{\mathbb{C},p}(M)$ is the complex vector space spanned by the element dz, for z being a local coordinate around p. The space of sections $\Gamma(U, K)$ of $K \to M$ on an open set $U \subset M$ is given by the space $\Omega^{1,0}(U)$ of $(1,0)$-forms on U. Given two local charts (U_α, z_α) and (U_β, z_β) on M with $U_\alpha \cap U_\beta \neq \emptyset$, the transition function for the line bundle $K \to M$ on $U_\alpha \cap U_\beta$ is given by

$$g_{\alpha,\beta}(p) = \frac{dz_\beta}{dz_\alpha}(p), \tag{1.32}$$

which is holomorphic in its domain of definition.

Similarly, the *anti-canonical bundle* $\bar{K} \to M$ is the complex line bundle over M given by,

$$\bar{K} := \bigcup_{p \in M} \{p\} \times T''^{*}_{\mathbb{C},p}(M), \tag{1.33}$$

where $T''^{*}_{\mathbb{C},p}(M)$ is the complex vector space spanned by the element $d\bar{z}$. The space of sections $\Gamma(U, \bar{K})$ of $\bar{K} \to M$ on an open set $U \subset M$ is given by the space $\Omega^{0,1}(U)$ of $(0,1)$-forms on U.

Given two local charts (U_α, z_α) and (U_β, z_β) on M with $U_\alpha \cap U_\beta \neq \emptyset$, the transition function for the anti-canonical bundle $\bar{K} \to M$ on $U_\alpha \cap U_\beta$ is given by

$$h_{\alpha,\beta}(p) = \frac{d\bar{z}_\beta}{d\bar{z}_\alpha}(p), \tag{1.34}$$

Given a complex vector bundle $E \to M$, let $\Omega^1(U, E)$ be the space of C^∞ 1-forms on $U \subset M$ with values in E. It is possible to decompose the space $\Omega^1(U, E)$ as

$$\Omega^1(U, E) = \Omega^{1,0}(U, E) \oplus \Omega^{0,1}(U, E), \tag{1.35}$$

where $\Omega^{1,0}(U, E) = \Gamma(U, K \otimes E)$ and $\Omega^{0,1}(U, E) = \Gamma(U, \bar{K} \otimes E)$ ([27, Chapter 0.5 p.73]).

Given a rank k holomorphic vector bundle $E \to M$ it is possible to define an operator $\bar{\partial} : \Gamma(U, E) \to \Gamma(U, \bar{K} \otimes E)$ on a trivializing open set $U \subset M$ as follows: Let $s \in \Gamma(U, E)$ be a section represented by the functions $(f_1, \ldots, f_k)$ on U. The element $\bar{\partial} s \in \Gamma(U, \bar{K} \otimes E)$ is the section on U represented by the functions $(\frac{\partial f_1}{\partial \bar{z}}, \ldots, \frac{\partial f_k}{\partial \bar{z}})$ where z is a local coordinate for M on U. If $s \in H^0(U, E)$ it follows that

$$\bar{\partial} s = 0. \tag{1.36}$$

From the fact that the transition functions of a holomorphic vector bundle $E \to M$ are holomorphic it follows that the $\bar{\partial}$-operator defined above is well defined on the intersection of any two trivializing sets $U_\alpha, U_\beta \subset M$ of E with $U_\alpha \cap U_\beta \neq \emptyset$.

The operator $\bar{\partial}$ is called a *holomorphic structure* of the holomorphic vector bundle $E \to M$ (we refer to [15, Chapter 1] and [27, Chapter 0.5] for more details about holomorphic structures).

We briefly recall the relation between divisors on a compact Riemann surface M and holomorphic line bundles $L \to M$.

Definition 1.19. A *divisor* D on a compact Riemann surface M is a finite formal sum

$$D = \sum_{j=1}^{k} n_j p_j, \quad p_j \in M, n_j \in \mathbb{Z}. \tag{1.37}$$

The *degree of a divisor* D is defined as $\sum_j n_j$. A divisor D such that $n_j > 0$ for all j is called *effective*.

Given a divisor $D = \sum_j n_j p_j$ on a compact Riemann surface M it is possible to define a holomorphic line bundle $L = L(D) \to M$ ([53, Chapter 6 p. 29]). The space of holomorphic sections of the line bundle $L(D)$ is denoted with $\mathcal{O}(D)$. The holomorphic line bundle $L(D)$ admits a meromorphic section $s = s_D$ such that s_D has a zero (resp. pole) of order $|n_j|$ at the point p_j, if $n_j > 0$ (resp. $n_j < 0$).

Conversely, given a holomorphic line bundle $L \to M$, there exists a divisor D on M such that $L = L(D)$ ([53, Chapter 6], [27, Chapter 1 pp. 129-135]).

Definition 1.20. Given a holomorphic line bundle $L \to M$ on a compact Riemann surface the *degree of* L is the degree of the corresponding divisor D on M. Given a rank k holomorphic vector bundle $E \to M$, the *degree of* E is defined as the degree of its determinant bundle $\bigwedge^2 E \to M$.

Let M be a compact Riemann surface and $D = \sum_j n_j p_j$ an effective divisor on M. Given a holomorphic vector bundle $E \to M$, the space of holomorphic sections of the holomorphic vector bundle $E \otimes \mathcal{O}(D)$ is given by the space of meromorphic sections of E having a pole of order at most n_j at the point p_j. Analogously, the space of holomorphic sections of the holomorphic vector bundle $E \otimes \mathcal{O}(-D)$ is given by the space of sections of E having a zero of order at least n_j at the point p_j ([27, Chapter 1 p. 138]).

As examples that we will encounter in the next chapters we want to give an explicit description of the holomorphic vector bundles over the Riemann sphere $\mathbb{CP}^1$.

Example 1.4 (Holomorphic vector bundles over $\mathbb{CP}^1$)**.** Let $\mathbb{CP}^1$ denote the set of complex 1-dimensional subspace of $\mathbb{C}^2$. If (z, w) is a nonzero vector in $\mathbb{C}^2$, then the 1-dimensional subspace generated by (z, w) is a point of $\mathbb{CP}^1$ denoted with $[z : w]$. Every point of $\mathbb{CP}^1$ can be written as $[z : w]$ with z and w both not equal to zero. Moreover, we have

$$[z : w] = [\lambda z : \lambda w], \quad \forall \lambda \in \mathbb{C}^*.$$

Let (U_0, ψ_0) and (U_1, ψ_1) be the two local charts on $\mathbb{CP}^1$ where $U_0 = \{[z : w] \in \mathbb{CP}^1 \,|z \neq 0\}$, $U_1 = \{[z : w] \in \mathbb{CP}^1 \,|w \neq 0\}$ and

$$\begin{aligned} \psi_0 : U_0 &\to \mathbb{C} \\ [z : w] &\mapsto \frac{w}{z}, \end{aligned} \tag{1.38}$$

$$\begin{aligned} \psi_1 : U_1 &\to \mathbb{C} \\ [z : w] &\mapsto \frac{z}{w}. \end{aligned} \tag{1.39}$$

The *tautological line bundle* of $\mathbb{CP}^1$ is obtained by attaching to each point $p \in \mathbb{CP}^1$ the one-dimensional subspace of $\mathbb{C}^2$ associated to p as a fiber. This bundle is denoted with $\pi : \mathcal{O}(-1) \to \mathbb{CP}^1$. The local trivialization Φ_0 on U_0 is given by $\Phi_0([1 : z], (\lambda, \lambda z)) = ([1 : z], \lambda)$ and the local trivialization Φ_1 on U_1 is given by $\Phi_1([w : 1], (\lambda w, \lambda)) = ([w : 1], \lambda)$. Thus,

$$\Phi_0 \circ \Phi_1^{-1}([1 : w^{-1}], \lambda) = \Phi_0([1 : w^{-1}], (\lambda w, \lambda)) = ([1 : w^{-1}], \lambda w),$$

and the transition function $g_{01} : U_0 \cap U_1 \to \mathbb{C}$ is given by

$$g_{01}([1 : z]) = w = z^{-1},$$

which is holomorphic because z cannot be zero in $U_0 \cap U_1$.

The dual bundle of $\mathcal{O}(-1)$ is denoted by $\mathcal{O}(1)$ and its transition function is given by

$$g^*_{01}([1 : z]) = w^{-1} = z.$$

The holomorphic line bundle $\mathcal{O}(n) \to \mathbb{CP}^1$, for $n \in \mathbb{Z}$, is defined as the repeated tensor products of $\mathcal{O}(\pm 1)$.

An important result about holomorphic vector bundles on $\mathbb{CP}^1$ is given by the following:

Theorem 1.5 (Grothendieck splitting, [28, Theorem 2.1 p. 126])**.** *Let $E \to \mathbb{CP}^1$ be a rank r holomorphic vector bundle, then there exist r holomorphic line bundles $\mathcal{O}(n_i)$, with $n_i \in \mathbb{Z}$ such that*

$$E = \mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_r)$$

and deg$(E) = \sum_{i=1}^{r} n_i$.

1.4.3 Sheaves and Push-forward bundle

It is well known that there exists a relation between the theory of sheaves and the theory of vector bundles. We briefly recall this relation in order to introduce the *push-forward bundle* (we refer to [53, Chapter 5] or [62, Chapter 13.1] for more details about the theory of sheaves and the relation with the theory of vector bundles).

Definition 1.21. Let P be a topological space. A *pre-sheaf* $\mathcal{F}$ associates to every open set $U \subset P$ a set $\mathcal{F}(U)$. Moreover, for every pair of open subset $U, V \subset P$ with $U \subset V$ there exists a map, called *restriction map*, $r_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$ such that:

(i) The map $r_{U,U} = \mathrm{Id}_{\mathcal{F}(U)}$, for every open subset $U \subset P$;

(ii) Given three open subsets U, V, W of P with $U \subset V \subset W$ we have

$$r_{W,U} = r_{W,V} \circ r_{V,U}. \tag{1.40}$$

The elements of $\mathcal{F}(U)$ are called the *sections* of $\mathcal{F}$ over $U \subset P$.

Definition 1.22. A pre-sheaf $\mathcal{F}$ over a topological space P is a *sheaf* if it satisfies the two additional conditions:

(1) Let U be an open subset of P. If $\{U_\alpha\}$ is an open cover of U and $s_1, s_2 \in \mathcal{F}(U)$ are such that $r_{U,U_\alpha}(s_1) = r_{U,U_\alpha}(s_2)$ for every α, then $s_1 = s_2$.

(2) Let U be an open subset of P and $\{U_\alpha\}$ an open cover of U. Given $s_\alpha \in \mathcal{F}(U_\alpha)$ for every α such that

$$r_{U_\alpha, U_\alpha \cap U_\beta}(s_\alpha) = r_{U_\beta, U_\alpha \cap U_\beta}(s_\beta), \quad \forall \alpha, \beta \text{ with } U_\alpha \cap U_\beta \neq \emptyset$$

then, there exists a section $s \in \mathcal{F}(U)$ such that

$$r_{U,U_\alpha}(s) = s_\alpha, \quad \forall \alpha.$$

It is possible to consider a sheaf $\mathcal{F}$ such that the sets $\mathcal{F}(U)$ have an additional structure, for example group structure, ring structure or module structure, for every open subset U of P. In this case we refer to the sheaf $\mathcal{F}$ as a sheaf of groups, rings, modules etc.

Example 1.5. Let $\pi : E \to M$ be a holomorphic vector bundle on a Riemann surface M and $\mathcal{O}_U$ the space of holomorphic functions on an open set $U \subset M$. The map $\mathcal{F}$ which associate to every open subset $U \subset M$ the $\mathcal{O}_U$-module $H^0(U, E)$ of local holomorphic sections of E over U is a sheaf of $\mathcal{O}_U$-modules.

The previous example shows that we can associate to every holomorphic vector bundle $\pi : E \to M$ a sheaf of $\mathcal{O}_M$-modules. Conversely, given a locally free sheaf $\mathcal{F}$ of rank n on a manifold M, that is a sheaf such that for every $p \in M$ there exists a neighbourhood U of p with

$$\mathcal{F}(U) = \mathcal{O}_U^n = \mathcal{O}_U \oplus \cdots \oplus \mathcal{O}_U, \tag{1.41}$$

there exists a rank n holomorphic vector bundle $\pi : E \to M$ such that

$$\mathcal{F}(U) = H^0(U, E), \quad \forall \text{ open subsets } U \subset M. \tag{1.42}$$

Thus, there is a $1-1$ correspondence between locally free sheaves of rank n and holomorphic vector bundles of rank n ([30, Chapter 5 p. 128]).

We want to use this correspondence to define the push-forward bundle on a Riemann surface.

Definition 1.23. Let $f : M \to N$ be a holomorphic map between compact Riemann surfaces of degree d and $\pi : E \to M$ a holomorphic vector bundle of rank n. The *push-forward bundle of $E \to M$ under the map f* is the rank dn holomorphic vector bundle $f_*E \to N$ associated to the locally free sheaf ([11, Section 4 p.179]) $\mathcal{F}$ of $\mathcal{O}_N$-modules with

$$\mathcal{F}(U) = H^0(f^{-1}(U), E), \quad \forall \text{ open subsets } U \subset N \tag{1.43}$$

We will need the following properties of the push-forward bundle in the rest of this thesis:

- Let $f : M \to N$ be a holomorphic map between Riemann surfaces and $E \to M$ and $\tilde{E} \to N$ holomorphic vector bundles. Then

$$f_*(E \otimes f^*(\tilde{E})) = f_*E \otimes \tilde{E}, \tag{1.44}$$

  for a proof see [39, Proposition 4.2 p. 33];

- Let $f : M \to N$ be a holomorphic map between compact Riemann surfaces and $E_1 \to M$ and $E_2 \to M$ two holomorphic vector bundles on M. Given a holomorphic vector bundle map $\Phi : E_1 \to E_2$ it is possible to define a holomorphic vector bundle map $f_*\Phi : f_*E_1 \to f_*E_2$ as follows: The map Φ induces a map $\tilde{\Phi}$ between the sheaves of local sections of E_1 and E_2

$$\begin{aligned} \tilde{\Phi} : H^0(U, E_1) &\to H^0(U, E_2) \\ s &\mapsto \tilde{\Phi}(s) := \Phi \circ s \end{aligned} \tag{1.45}$$

for every open subset $U \subset M$ which trivializes both E_1 and E_2. Given an open subset $V \subset N$ we recall that $H^0(V, f_*E_1) = H^0(f^{-1}(V), E_1)$ and $H^0(V, f_*E_2) = H^0(f^{-1}(V), E_2)$ from Definition 1.23. The map $f_*\tilde{\Phi}$ is given by

$$\begin{aligned} f_*\tilde{\Phi} : H^0(V, f_*E_1) = H^0(f^{-1}(V), E_1) &\to H^0(f^{-1}(V), E_2) = H^0(V, f_*E_2) \\ s &\mapsto f_*\tilde{\Phi}(s) := \tilde{\Phi}(s). \end{aligned} \tag{1.46}$$

Moreover, for every holomorphic function $h : V \to \mathbb{C}$, $s \in H^0(V, f_*E_1)$, $q \in V$ and $p \in f^{-1}(q)$ we have

$$\begin{aligned} f_*\tilde{\Phi}(hs)(q) &= \tilde{\Phi}((h \circ f)(p)s)(p) \\ &= (h \circ f)(p)\tilde{\Phi}(s)(p) \\ &= h(q)f_*\tilde{\Phi}(s)(q). \end{aligned} \tag{1.47}$$

We provide an example to show more precisely how this construction works.

Example 1.6. Let M be a compact Riemann surface and $\varphi : M \times \mathbb{Z}_3 \to M$ an effective group action. Suppose that the quotient $M/\mathbb{Z}_3$ is the Riemann sphere $\mathbb{CP}^1$ and the map $f : M \to \mathbb{CP}^1$, given by the projection to the quotient, is a holomorphic map between Riemann surfaces of degree 3, branched at four points $z_1, \ldots, z_4 \in \mathbb{CP}^1$.

Consider a rank 2 holomorphic vector bundle $E \to M$. We want to describe the rank 6 push-forward bundle $f_*E \to \mathbb{CP}^1$ by defining its transition functions.

We need to consider three different cases.

Case 1) $U \subset \mathbb{CP}^1$ open set not containing any branch point of the map f.

Since U does not contain any branch point, its preimage $f^{-1}(U)$ is given by the disjoint union of three open sets of M:

$$f^{-1}(U) = U_1 \sqcup U_2 \sqcup U_3, \quad U_1, U_2, U_3 \subset M. \tag{1.48}$$

From the Definition 1.23, the space of holomorphic sections $H^0(U, f_*E)$ is given by the space of holomorphic sections of E on the open set $f^{-1}(U) \subset M$. Let s_j, t_j be a holomorphic local frame of E over the open set $U_j \subset M, j = 1, 2, 3$. Then, a local frame for f_*E over U is given by the sections

$$(s_1 \oplus 0 \oplus 0, t_1 \oplus 0 \oplus 0, 0 \oplus s_2 \oplus 0, 0 \oplus t_2 \oplus 0, 0 \oplus 0 \oplus s_3, 0 \oplus 0 \oplus t_3). \tag{1.49}$$

Let $\tilde{U} \subset \mathbb{CP}^1$ be an open set not containing branch point of f and such that $U \cap \tilde{U} \neq \emptyset$. The preimage of $\tilde{U}$under f is given by the disjoint union of three open set $\tilde{U}_1, \tilde{U}_2, \tilde{U}_3 \subset M$. Let $\tilde{s}_j, \tilde{t}_j$ be a holomorphic local frame for E over $\tilde{U}_j \subset M, j = 1, 2, 3$.

For every $j = 1, 2, 3$, the transition function of the bunde $E \to M$ over $U_j \cap \tilde{U}_j$ is given by

$$\begin{aligned} g^j : U_j \cap \tilde{U}_j &\to \mathbf{GL}(2, \mathbb{C}) \\ p &\mapsto \begin{pmatrix} g^j_{11} & g^j_{12} \\ g^j_{21} & g^j_{22} \end{pmatrix} \end{aligned} \tag{1.50}$$

and they are such that

$$(\tilde{s}_j, \tilde{t}_j) = g^j(s_j, t_j). \tag{1.51}$$

Thus, the transition function for the bundle $f_*E \to \mathbb{CP}^1$ over $U \cap \tilde{U}$ is given by the map $g : U \cap \tilde{U} \to \mathbf{GL}(6, \mathbb{C})$, where

$$g = \begin{pmatrix} g^1_{11} & g^1_{12} & 0 & 0 & 0 & 0 \\ g^1_{21} & g^1_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & g^2_{11} & g^2_{12} & 0 & 0 \\ 0 & 0 & g^2_{21} & g^2_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & g^3_{11} & g^3_{12} \\ 0 & 0 & 0 & 0 & g^3_{21} & g^3_{22} \end{pmatrix}. \tag{1.52}$$

Case 2) $U \subset \mathbb{CP}^1$ containing one branch point of f, which is assumed to be $z = 0 \in \mathbb{CP}^1$.

By the Local normal form Theorem ([21, Proposition 5 p. 43]) there exists a local coordinate w on $f^{-1}(U) \subset M$ and a local coordinate z on $U \subset \mathbb{CP}^1$ such that

$$w^3 = z. \tag{1.53}$$

As in the previous case, consider a holomorphic local frame s, t of E over the open set $f^{-1}(U) \subset M$. Then, the sections s, t, ws, wt, w^2s, w^2t considered as holomorphic sections of the pushforward bundle $f_*E \to \mathbb{CP}^1$ over U, are linearly independent and they define a holomorphic local frame for f_*E over U.

Let $\tilde{U} \subset \mathbb{CP}^1$ be an open set containing the branch point $z = 0$ (and no other branch points of the map f). Given a holomorphic local frame $\tilde{s}, \tilde{t}$ of E over $f^{-1}(\tilde{U})$, we can write

$$\begin{aligned} \tilde{s} &= a_1(z)s + a_2(z)ws + a_3(z)w^2 s, \\ \tilde{t} &= b_1(z)t + b_2(z)wt + b_3(z)w^2 t, \end{aligned} \tag{1.54}$$

where the complex valued functions $a_1, a_2, a_3, b_1, b_2, b_3$ on $U \cap \tilde{U} \subset \mathbb{CP}^1$ satisfy the following relations (obtained from the factorization in prime factors of the expression w^3):

$$\begin{aligned} a_1(z)a_2(z)a_3(z) &= 1, \\ a_1(z) + a_2(z) + a_3(z) &= 0, \\ a_1(z)a_2(z) + a_1(z)a_3(z) + a_2(z)a_3(z) &= 0, \\ b_1(z)b_2(z)b_3(z) &= 1, \\ b_1(z) + b_2(z) + b_3(z) &= 0, \\ b_1(z)b_2(z) + b_1(z)b_3(z) + b_2(z)b_3(z) &= 0. \end{aligned} \tag{1.55}$$

In this example we will consider

$$a_1(z) = b_1(z) = 1, \quad a_2(z) = b_2(z) = \alpha, \quad a_3(z) = b_3(z) = \alpha^2, \tag{1.56}$$

where $\alpha = e^{\frac{2\pi i}{3}}$.

The sections $w\tilde{s}, w\tilde{t}, w^2\tilde{s}, w^2\tilde{t}$ can be written as

$$\begin{aligned} (w\tilde{s}, w\tilde{t}) &= (\alpha^2 zs + ws + \alpha w^2 s, \alpha^2 zt + wt + \alpha w^2 t) \\ (w^2\tilde{s}, w^2\tilde{t}) &= (\alpha zs + \alpha^2 zws + w^2 s, \alpha zt + \alpha^2 zwt + w^2 t). \end{aligned} \tag{1.57}$$

The transition function $g : U \cap \tilde{U} \to \mathbf{GL}(6, \mathbb{C})$ for $f_*E \to \mathbb{CP}^1$ is given by

$$g = \begin{pmatrix} 1 & 0 & \alpha^2 z & 0 & \alpha z & 0 \\ 0 & 1 & 0 & \alpha^2 z & 0 & \alpha z \\ \alpha & 0 & 1 & 0 & \alpha^2 z & 0 \\ 0 & \alpha & 0 & 1 & 0 & \alpha^2 z \\ \alpha^2 & 0 & \alpha & 0 & 1 & 0 \\ 0 & \alpha^2 & 0 & \alpha & 0 & 1 \end{pmatrix} \tag{1.58}$$

whose determinant in $U \cap \tilde{U}$ is non zero.

Case 3) Let $U \subset \mathbb{CP}^1$ an open set containing a branch point of $f : M \to \mathbb{CP}^1$, which we assume to be $z = 0$.

Consider an open set $\tilde{U} \subset U$ such that $0 \notin \tilde{U}$. Thus, the set $\tilde{U}$ does not contain any branch point and

$$f^{-1}(\tilde{U}) = \tilde{U}_1 \sqcup \tilde{U}_2 \sqcup \tilde{U}_3, \tag{1.59}$$

where $\tilde{U}_j \subset M$ is an open set for $j = 1, 2, 3$.

Let s, t be a holomorphic local frame of E over $f^{-1}(U) \subset M$. A local frame for f_*E over U is given by $(s, t, ws, wt, w^2s, w^2t)$, as in Case 2.

Let $\tilde{w} : \tilde{U} \to \mathbb{C}$ be the function defined by

$$\tilde{w}([p]) = w(p), \quad p \in \tilde{U}, \tag{1.60}$$

where $[p]$ is the orbit of the points $p \in U$ under the action of $\mathbb{Z}_3$.

A holomorphic local frame for f_*E over the open set $\tilde{U}$ is given by the direct sum of the following holomorphic local frames defined, respectively, over $\tilde{U}_1$, $\tilde{U}_2$ and $\tilde{U}_3$

$$\begin{aligned}
(s_1, t_1) &= \frac{1}{3\tilde{w}^2}(\tilde{w}^2 s + \tilde{w}ws + w^2 s, \tilde{w}^2 t + \tilde{w}wt + w^2 t) \\
(s_2, t_2) &= \frac{1}{3\tilde{w}^2}(\tilde{w}^2 s + \alpha\tilde{w}ws + \alpha^2 w^2 s, \tilde{w}^2 t + \alpha\tilde{w}wt + \alpha^2 w^2 t) \\
(s_3, t_3) &= \frac{1}{3\tilde{w}^2}(\tilde{w}^2 s + \alpha^2\tilde{w}ws + \alpha w^2 s, \tilde{w}^2 t + \alpha^2\tilde{w}wt + \alpha w^2 t),
\end{aligned} \tag{1.61}$$

where $\alpha = e^{\frac{2\pi i}{3}}$.

It is possible to write the sections $(s, t, (ws, wt, w^2s, w^2, t)$ in terms of the sections $(s_1, t_1, s_2, t_2, s_3, t_3)$ as follows:

$$\begin{aligned}
(s, t) &= (s_1, t_1) + (s_2, t_2) + (s_3, t_3) \\
(ws, wt) &= \tilde{w}^2(s_1, t_1) + \alpha^2\tilde{w}^2(s_2, t_2) + \alpha\tilde{w}^2(s_3, t_3) \\
(w^2 s, w^2 t) &= \tilde{w}^2(s_1, t_1) + \alpha\tilde{w}^2(s_2, t_2) + \alpha^2\tilde{w}^2(s_3, t_3).
\end{aligned} \tag{1.62}$$

The transition function g for f_*E over the set $\tilde{U} = U \cap \tilde{U}$ is given by

$$g = \begin{pmatrix}
1 & 0 & \tilde{w}^2 & 0 & \tilde{w}^2 & 0 \\
0 & 1 & 0 & \tilde{w}^2 & 0 & \tilde{w}^2 \\
1 & 0 & \alpha^2\tilde{w}^2 & 0 & \alpha\tilde{w}^2 & 0 \\
0 & 1 & 0 & \alpha^2\tilde{w}^2 & 0 & \alpha\tilde{w}^2 \\
1 & 0 & \alpha\tilde{w}^2 & 0 & \alpha^2\tilde{w}^2 & 0 \\
0 & 1 & 0 & \alpha\tilde{w}^2 & 0 & \alpha^2\tilde{w}^2
\end{pmatrix}, \tag{1.63}$$

and the determinant of g is non zero in $\tilde{U}$.

The above description gives a system of transition functions for the push-forward bundle $f_*E \to \mathbb{CP}^1$, which determine the bundle uniquely up to vector bundle isomorphism (cf. Subsection 1.4.1).

1.4.4 Connections on complex vector bundles

A *connection* on a complex vector bundle $E \to M$ is an object used to define the derivative of sections. We briefly give the definition of a connection and recall some important results. For more details about the theory of connections on complex vector bundles we refer to [27, pp. 71-80] and [64, Chapter 3]

Definition 1.24. Let $E \to M$ be a rank k complex vector bundle over a Riemann surface M. A *connection* ∇ on E over an open set $U \subset M$ is a linear operator

$$\nabla : \Gamma(U, E) \to \Omega^1(U, E), \tag{1.64}$$

which satisfies the Leibniz rule

$$\nabla(fs) = f\nabla(s) + df s, \tag{1.65}$$

for every function $f : U \to \mathbb{C}$ and section $s \in \Gamma(U, E)$.

Let $E \to M$ be a complex vector bundle and $s = (s_1, \dots, s_k)$ a local frame over an open set $U \subset M$. A connection ∇ on E over the set U can be written, with respect to the local frame s as

$$\nabla s_j = \sum_l \theta_{jl} s_l \tag{1.66}$$

where $\omega = (\theta_{jl})$ is a matrix valued 1-form called the *connection 1-form of* ∇.

Example 1.7. Let $\underline{\mathbb{C}}^k = M \times \mathbb{C}^k$ be the rank k trivial bundle over M. The space $\Gamma(U, \underline{\mathbb{C}}^k)$ can be considered as the space of complex valued functions on $U \subset M$. The differential of these functions are elements of $\Omega^1(U)$ which correspond to the space $\Omega^1(U, \underline{\mathbb{C}}^k)$. Thus, we obtain a connection $d : \Gamma(U, \underline{\mathbb{C}}^k) \to \Omega^1(U, \underline{\mathbb{C}}^k)$ on $\underline{\mathbb{C}}^k$.

The connection d is called the *trivial connection* and it is possible to define d, locally, on every complex vector bundle $E \to M$ since every such bundle is locally isomorphic to the trivial bundle via the trivialization maps.

Given a connection ∇ on $E \to M$ over $U \subset M$, it is possible to define another operator $\hat{\nabla} : \Omega^1(U,E) \to \Omega^2(U,E)$ which satisfies

$$\hat{\nabla}(f\omega) = df \wedge \omega + f\hat{\nabla}\omega, \quad f : U \to \mathbb{C}, \omega \in \Omega^1(U,E). \tag{1.67}$$

The composition of ∇ with $\hat{\nabla}$ gives an operator $\nabla^2 : \Gamma(U,E) \to \Omega^2(U,E)$ ([27, Chapter 0 p. 74]). The connections ∇ is called a *flat connection* if $\nabla^2 \equiv 0$ (sometimes the curvature of a connection ∇ is denoted with F^{∇}). The operator $\hat{\nabla}$ is called the *exterior derivative* of the connection ∇.

Given a holomorphic vector bundle $E \to M$, from the decomposition of differential 1-forms on E over an open set $U \subset M$, given by (cf. (1.35))

$$\Omega^1(U,E) = \Omega^{1,0}(U,E) \oplus \Omega^{0,1}(U,E), \tag{1.68}$$

it follows that it is possible to decompose a connection ∇ on $E \to M$ as $\nabla = \nabla' + \nabla''$, where

$$\nabla' : \Gamma(U,E) \to \Omega^{1,0}(U,E), \quad \nabla'' : \Gamma(U,E) \to \Omega^{0,1}(U,E). \tag{1.69}$$

The component ∇'' gives a holomorphic structure on $E \to M$. Moreover, if $\bar{\partial}$ is the holomorphic structure of $E \to M$ on U, the connection ∇ is called *compatible with the holomorphic structure of* $E \to M$ if $\nabla'' = \bar{\partial}$.

The following result shows that given a hermitian holomorphic vector bundle $E \to M$ (cf. Definition 1.15) on a Riemann surface it is possible to find a canonical connection on it:

Lemma 1.2 ([27, Chapter 0.5 p. 73])**.** *Let $E \to M$ be a hermitian holomorphic vector bundle on a Riemann surface with holomorphic structure $\bar{\partial}$ and hermitian metric $\langle , \rangle$. There exists a unique connection ∇ on $E \to M$ such that*

- *∇ is compatible with the holomorphic structure $\bar{\partial}$;*
- *∇ is compatible with the hermitian product on E, that is*

$$\nabla\langle s,t\rangle = \langle \nabla s, t\rangle + \langle s, \nabla t\rangle. \tag{1.70}$$

for sections s and t of $E \to M$.

We conclude this subsection recalling the definition of *holonomy* of a connection ∇ on a vector bundle $E \to M$.

Definition 1.25. Let $E \to M$ be a complex vector bundle, ∇ be a connection on E and $\gamma : [0,1] \to M$ be a smooth curve. A section s of E along γ, is called *parallel* if

$$\nabla_{\dot{\gamma}(t)} s = 0, \quad t \in [0,1]. \tag{1.71}$$

Let $p = \gamma(0)$ and $v_0 \in E_p$, the *parallel transport of* v_0 *along* γ is the unique solution to the differential system

$$\begin{cases} \nabla_{\dot{\gamma}} s = 0 \\ s(p) = v_0. \end{cases} \tag{1.72}$$

More generally, it is possible to define a linear isomorphism between the fibers of E at points along a curve $\gamma : [0,1] \to M$ by

$$P_\gamma^\nabla : E_{\gamma(0)} \to E_{\gamma(1)}, \tag{1.73}$$

which is called the *parallel transport* associated to γ.

Let γ be a loop in M based at a point p, the parallel transport P_γ^∇ defines an element of the group of automorphisms of the fiber E_p, $\mathbf{GL}(E_p)$. The *holonomy group* of the connection ∇ based at p is defined as

$$\mathrm{Hol}_p(\nabla) := \{P_\gamma^\nabla \in \mathbf{GL}(E_p) \mid \gamma \text{ is a loop based at } p\}. \tag{1.74}$$

The holonomy group depends on the base point p only up to conjugation by $\mathbf{GL}(2,\mathbb{C})$ since M is connected ([47, Chapter 4]).

Chapter 2

CMC and minimal surfaces in $\mathbb{R}^3$ and $\mathbb{S}^3$

2.1 Immersed surfaces

In this chapter we will introduce the definition and some results about immersed surfaces in a three dimensional manifold. We will refer to [18, Chapter 1] for the first part.

Definition 2.1. Let M be a surface and N^3 a three dimensional manifold. A differentiable map $f : M \to N^3$ is an *immersion* if the differential $df(p) : T_pM \to T_{f(p)}N^3$ is injective for all $p \in M$. The map f is called an *embedding* if it is a homeomorphism to its image $f(M)$, with respect to the induced topology.

Definition 2.2. Let $f : M \to N^3$ be an immersion of a surface M in a three dimensional manifold N^3. Let $\langle , \rangle_N$ be a metric on N^3 and $\langle , \rangle_M$ a metric on M. The map f is called an *isometric immersion* if the following holds:

$$\langle X, Y \rangle_M = \langle df_p(X), df_p(Y) \rangle_N, \quad \forall X, Y \in T_pM, p \in M. \tag{2.1}$$

Given an immersion $f : M \to N^3$ into a Riemannian manifold $(N^3, \langle , \rangle_N)$, it is possible to define an induced metric on M by setting at each point $p \in M$

$$\langle X, Y \rangle_M := \langle df(X), df(Y) \rangle_N, \quad X, Y \in T_pM. \tag{2.2}$$

The immersion f, with respect to the metrics $\langle , \rangle_M$ on M and $\langle , \rangle_N$ on N^3 is an isometric immersion.

Let $f : M \to N^3$ be an isometric immersion, then for every $p \in M$ there exists a neighbourhood $U \subset M$ such that f is an embedding when restricted to U. Thus, it is possible to identify U and $f(U)$.

It follows that the tangent bundle of N^3 at a point $p \in M$ can be decomposed as

$$T_pN^3 = T_pM \oplus (T_pM)^\perp, \tag{2.3}$$

where $(T_pM)^\perp$ is the orthogonal complement of T_pM in T_pN^3 with respect to the metric on N^3.

Definition 2.3. The disjoint union $TM^\perp := \bigcup_{p\in M}\{p\} \times (T_pM)^\perp$, together with the projection to the first factor $\pi : TM^\perp \to M$ is called the *normal bundle of M in N^3*.

The Levi-Civita connection on N^3 is the unique connection $\tilde{\nabla}$ on TN^3 such that it is compatible with the metric (cf. 1.70) and satisfies

$$\tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y] = 0, \quad \forall X, Y \in \Gamma(TN^3) \tag{2.4}$$

(we refer to [47, Chapter 4.2 pp. 158-162] for more details about the Levi-Civita connection on a Riemannian manifold).

According to the decomposition (2.3) it is possible to decompose the connection $\tilde{\nabla}$ as

$$\tilde{\nabla}_X Y = (\tilde{\nabla}_X Y)^\top + (\tilde{\nabla}_X Y)^\perp. \tag{2.5}$$

From the uniqueness of the Levi-Civita connection on a Riemannian manifold ([47, Theorem 2.2 p. 158]) it follows that $(\tilde{\nabla})^\top$ is the Levi-Civita connection on M with respect to the metric (2.2) induced by f.

Definition 2.4. Given an isometric immersion $f : M \to N^3$ of a surface M, the *second fundamental form* of f is the bilinear and symmetric operator $\beta : \Gamma(TM)\times\Gamma(TM) \to \Gamma((TM)^\perp)$ given by

$$\beta(X,Y) := \tilde{\nabla}_X Y - (\tilde{\nabla}_X Y)^\top, \tag{2.6}$$

where $\tilde{\nabla}$ and $\tilde{\nabla}^\top$ are as above.

The *shape operator* $S : \Gamma(TM) \times \Gamma(TM^\perp) \to \Gamma(TM)$ of f is defined by

$$S(X,\xi) = S_\xi(X) := (\tilde{\nabla}_X \xi)^\top. \tag{2.7}$$

The shape operator of an isometric immersion $f : M \to N^3$ satisfies the following properties ([18, Chapter 1 p. 3]):

(i)

$$\langle S_\xi X, Y \rangle = \langle \beta(X,Y), \xi \rangle, \tag{2.8}$$

where $\langle , \rangle$ is the metric on N;

(ii) S is a bilinear and symmetric operator.

Remark 2.1. Since we are interested in the study of Riemann surfaces, which are always oriented ([63, Theorem 2.2.1 p. 23]), immersed in the three sphere $\mathbb{S}^3$, in what follows we will assume that the surface M and the Riemannian manifold N^3 are oriented.

Given an oriented surface M isometrically immersed in a three dimensional oriented manifold N^3, it is possible to define a unique vector field $\eta \in (TM)^\perp$ such that it has unit norm and it forms a positive oriented basis of T_pN^3 together with the positive basis of T_pM for every $p \in M$.

We will denote by $S = S_\eta$ the shape operator of an isometric immersion $f : M \to N^3$ with respect to this unique unit vector field $\eta \in \Gamma(TM^\perp)$.

Definition 2.5. Let $f : M \to N^3$ be an isometric immersion of a surface M and S the shape operator of f. The quantities

$$K := \det(S), \qquad H := \frac{1}{2}\operatorname{tr}(S) \tag{2.9}$$

are called, respectively, the *Gauss curvature* and the *mean curvature* of f at a given point $p \in M$.

The *mean curvature vector field* is the vector field $\vec{H} \in \Gamma(TM^\perp)$ given by

$$\vec{H}(p) := \frac{1}{2}\sum_{j=1}^{2} \beta(X_j, X_j), \tag{2.10}$$

where $X_1, X_2 \in T_pM$ form an orthonormal frame of T_pM for every $p \in M$.

From (2.8) it follows that the mean curvature H of Definition 2.5 coincides with

$$\langle \vec{H}, \eta \rangle, \tag{2.11}$$

where η is the unit normal vector field considered to define the shape operator S of the isometric immersion f.

Definition 2.6. An isometric immersion $f : M \to N^3$ of a surface M is called minimal if $H \equiv 0$ and CMC (constant mean curvature) if $H = const. \neq 0$, where H is the mean curvature of f.

Remark 2.2. Sometimes we will call a surface M minimal (resp. CMC) meaning that there exists a minimal (resp. CMC) immersion $f : M \to N^3$, from M to a three dimensional Riemannian manifold N^3.

We conclude this section with a brief description of the fundamental equations of an isometric immersion of a Riemann surface M into a three dimensional space form $N^3(c)$, that is $N^3(c) = \mathbb{R}^3, \mathbb{S}^3$ or $\mathbb{H}^3$.

In this situation it is possible to write the metric on M and the second fundamental form of an isometric immersion $f : M \to N^3(c)$, on an open set $U \subset M$ with local coordinate z, as (for more details in the case of $N^3(c) = \mathbb{R}^3$ we refer to [40, Chapter 6], which can be easily generalized to the other three dimensional space forms $\mathbb{S}^3$ and $\mathbb{H}^3$)

$$\begin{aligned} g &:= 4e^{2u} dz d\bar{z}, \\ \beta &:= Q dz^2 + 4e^{2u} H dz d\bar{z} + \bar{Q} d\bar{z}^2, \end{aligned} \tag{2.12}$$

where $u : U \to \mathbb{R}$ is a smooth function, H is the mean curvature of the immersion f and $Q : U \to \mathbb{C}$ is called the *coefficient of the Hopf differential* $Q dz^2$. An immersion $f : U \to N^3(c)$ such that the induced metric g can be written as in (2.12) is called a *conformal immersion.*

The metric g and the second fundamental form b satisfy the following equations ([26, Section 5 p. 95])

$$4u_{z\bar{z}} + 4e^{2u}(H + c) - Q\bar{Q}e^{-2u} = 0, \tag{2.13}$$

$$Q_{\bar{z}} = 2e^{2u} H_z, \tag{2.14}$$

where c is the sectional curvature of the ambient space $N^3(c)$ ($c = 0$ corresponds to $\mathbb{R}^3$, $c > 0$ to $\mathbb{S}^3$ and $c < 0$ to $\mathbb{H}^3$). Equation (2.13) is called the *Gauss equation* and equation 2.14 the *Codazzi equation.*

The fundamental theorem for immersed surfaces ([18, Theorem 1.1 p.7]) shows that on a surface M, given a metric g and a 2 form β on any simply connected domain $U \subset M$, which satisfy the Gauss and Codazzi equations, there exists an isometric immersion $f : U \to N^3(c)$ (unique up to isometries of the ambient space) such that g corresponds to the induced metric of f and β to the second fundamental form of f.

2.2 Harmonic maps

We will recall the notion of harmonic maps into a Riemannian manifold N and briefly describe the relation with minimal and CMC immersions of a surfaces. We will manly refer to [4, Chapter 3.3 pp. 71-81], [20, Chapter 2 pp. 53-58].

Let $f : M \to N$ be a smooth map between Riemannian manifold.

Definition 2.7. The *energy density* of the map $f : M \to N$ is defined by

$$|df_p|^2 := \sum_{j=1}^{2} \langle df_p(X_j), df_p(X_j)\rangle_N, \quad \forall p \in M, \tag{2.15}$$

where $X_1, X_2 \in T_pM$ forms an orthonormal basis and $\langle , \rangle_N$ is the Riemannian metric on N.

For any compact sect $U \subset M$, the *energy of f on U* is

$$E_U(f) := \frac{1}{2}\int_U |df|^2 dV, \tag{2.16}$$

where dV is the volume form of M.

Given $\varepsilon > 0$, a *smooth variation* of $f : M \to N$ is a smooth map

$$\begin{aligned} \varphi : (-\varepsilon, \varepsilon) \times M &\to N \\ (t, p) &\mapsto \varphi_t(p) \end{aligned} \tag{2.17}$$

such that the map $\varphi_t : M \to N$ is smooth for every $t \in (-\varepsilon, \varepsilon)$ and $\varphi_0 = f$.

The *variation vector field* of φ is defined by

$$\psi_\varphi(p) := \frac{\partial}{\partial t}\varphi_t(p)_{|t=0} \in T_{f(p)}N. \tag{2.18}$$

In [4, Chapter 3 p. 72] the authors show that, given a vector filed ψ on N along f, it is possible to define a smooth variation of f such that ψ is its variation vector field.

Definition 2.8. A smooth map $f : M \to N$ between Riemannian manifold is called *harmonic* if

$$\frac{d}{dt}E_U(\varphi_t)_{|t=0} = 0 \tag{2.19}$$

for all compact sets $U \subset M$ and smooth variations φ_t of f such that

$$\varphi_t = f, \quad \text{on } M \setminus \operatorname{Int}(U), \quad \forall t. \tag{2.20}$$

The *first variation formula for the energy* is given by ([4, Proposition 3.3.3 p. 72])

$$\frac{d}{dt}E_U(\varphi_t)_{|t=0} = -\int_U \langle \tau(f), \psi_\varphi\rangle dV, \tag{2.21}$$

where ψ_φ is the variation vector field of the smooth variation φ of f and

$$\tau(f) := \operatorname{tr} \nabla df \tag{2.22}$$

is called the *tension field* of f.

As an implication of the first variation formula for the energy there is the following:

Theorem 2.1 ([24, Section 2 p. 116]). *Let $f : M \to N$ be a smooth map. Then, f is harmonic if and only if $\tau(f) = 0$.*

In the situation where $f : M \to N^3$ is an isometric immersion of a surface M in a three dimensional Riemannian manifold N^3, the tension field and the mean curvature vector (2.10) of f are related. In fact, a direct computation shows that

$$\tau(f) = 2\vec{H}. \tag{2.23}$$

In particular, an isometric immersion$f$ is harmonic if and only if it is minimal.

There exists an analogous characterization for CMC isometric immersions of a surface in $\mathbb{R}^3$ or $\mathbb{S}^3$, which involves the *Gauss map* of the immersion (we refer to [18, Chapter 7 p .122] for the definition of the Gauss map)

In [57, Theorem pp. 571-572] the authors proved that, given an isometric immersion $f : M \to \mathbb{R}^3$ with Gauss map $g : M \to \mathbb{S}^2$, the following holds:

$$\tau(g) = \nabla\vec{H}, \tag{2.24}$$

where ∇ is the Levi-Civita connection on N. Thus, the isometric immersion f is CMC if and only if its Gauss map g is harmonic.

In the case of an isometric immersion $f : M \to \mathbb{S}^3$ it has been proven the following:

Theorem 2.2 ([49, Theorem 1 p. 85]). *Let $f : M \to \mathbb{S}^3$ be an isometric immersion with Gauss map $g : M \to \mathbb{S}^2$. Then, f is a CMC immersion if and only if g is a harmonic map.*

In order to prove Theorem 2.2, the author identified the three sphere $\mathbb{S}^3$ with the Lie group $\mathbf{SU}(2)$. The Gauss map of the isometric immersion $f : M \to \mathbb{S}^3$ can be defined as the map

$$\begin{aligned} g : M &\to \mathbb{S}^2 \\ p &\mapsto (dL_{f(p)^{-1}})(\eta), \end{aligned} \tag{2.25}$$

where L_q is the left translation by the element $q \in M \subset \mathbf{SU}(2)$ and η is the unique unit normal vector field to M (cf. Remark 2.1).

2.3 Minimal and CMC immersions into $\mathbb{R}^3$

In this section we will recall the variational description of minimal and CMC immersions of a surface M in the euclidean three space $\mathbb{R}^3$, together with some important results.

Let $f : M \to \mathbb{R}^3$ be an isometric immersion of a surface M and η the unique unit normal vector field to M (cf. Remark 2.1).

Let $\varphi : (-\varepsilon, \varepsilon) \times M \to \mathbb{R}^3$ be a smooth variation of f such that φ_t is an immersion for all $t \in (-\varepsilon, \varepsilon)$ and $\varphi_t \equiv 0$ on $M \smallsetminus U$ for a compact set $U \subset M$.

The variation vector field of φ is given by $h\eta \in TM^\perp$, where $h : M \to \mathbb{R}$ is a smooth function. This follows from the fact that η gives a global frame for $TM^\perp$.

Definition 2.9. The *area* and *volume functionals* are defined, respectively, by

$$A(t) := \int_U dM_t, \quad V(t) := \int_{[0,t]\times U} \varphi_t^*(dV_{\mathbb{R}^3}), \tag{2.26}$$

where dM_t is the area element of the metric on M induced by the immersion φ_t and $\varphi_t^*(dV_{\mathbb{R}^3})$ is the pullback under φ_t of the standard volume form of $\mathbb{R}^3$.

It is proved, in [20, Chapter 2 p. 56], that the first variation formula for the area and volume functionals are given by

$$A'(0) = -\int_U 2HhdM, \quad V'(0) = \int_U hdM, \tag{2.27}$$

where H is the mean curvature of the immersion $f : M \to \mathbb{R}^3$.

Using the above description, the following results gives a variational characterization of minimal and CMC immersions of surfaces in $\mathbb{R}^3$ in terms of the area functionals.

Theorem 2.3 ([20, Theorem 1 p. 56])**.** *Let $f : M \to \mathbb{R}^3$ be an isometric immersion of a surface M. Then the immersion f is minimal if and only if it is a critical point for the area functional $A(t)$.*

Moreover, considering volume preserving variations of f, that is $V(t) = const$, it is possible to obtain the analogous characterization for CMC surfaces:

Proposition 2.1 ([5, Proposition 2.3 p. 126])**.** *Let $f : M \to \mathbb{R}^3$ be an isometric immersion, then f has constant mean curvature if and only if f is a critical point of the area functional $A(t)$ for every volume preserving variation of f.*

In the last centuries mathematicians provided a huge variety of examples and results about minimal surfaces. On the other hand the study of CMC surfaces was not so prolific. In 1841 Delaunay [19] proved that the only CMC surfaces of revolution in $\mathbb{R}^3$ are the surfaces obtained by rotating the roulettes of the conics, which are called *unduloids* and *nodoids*.

In 1956 Hopf [41] proved that the only compact CMC surface of genus 0 immersed in $\mathbb{R}^3$ with constant mean curvature is the round sphere. In 1958 Alexandrov [1] proved that the only embedded compact CMC surface in $\mathbb{R}^3$ is the round sphere.

As a consequence of the latter result, it was conjectured that the only compact CMC surface immersed in $\mathbb{R}^3$ is the round sphere. It was in 1984 that Wente [65] provided the first example of a compact surface of genus 1, called the Wente torus, immersed in $\mathbb{R}^3$ with constant mean curvature.

During the last decades the use of new techniques allowed mathematicians to provide more examples. For example, Kapouleas, Mazzeo and Pollack ([43], [50]) proved that it is possible to glue together spheres and pieces of Delaunay surfaces to obtain new complete CMC surfaces immersed in $\mathbb{R}^3$.

In this thesis we are interested in studying CMC immersions of compact surfaces, and due to the results of Hopf [41] and Alexandrov [1] there are no CMC embeddings of compact surfaces in $\mathbb{R}^3$ except for the round sphere. In the next section we will show that for $\mathbb{S}^3$ as ambient space the situation is different and there is a huge variety of compact surfaces minimally or CMC embedded in $\mathbb{S}^3$.

2.4 Minimal and CMC surfaces into $\mathbb{S}^3$

Let $f : M \to \mathbb{S}^3$ be an isometric immersion of a surface M into the three dimensional sphere. It is possible to give a variational characterization of minimal and CMC immersions into $\mathbb{S}^3$, analogous to the characterization of immersions in $\mathbb{R}^3$ (cf. Section 2.3):

Proposition 2.2 ([5, Proposition 2.3 p. 126])**.** *Let $f : M \to \mathbb{S}^3$ be an isometric immersion of a surface M. Then f is minimal if and only if $A'(0) = 0$ for every variation of f, where $A(t)$ is the area functional. While, f is a CMC immersion if and only if $A'(0) = 0$ for every volume preserving variation of f.*

In 1970, Lawson [48] proved that there exist compact surfaces minimally immersed in $\mathbb{S}^3$ for every genus g (cf. Section 2.5). Karcher, Pinkall and Sterling [44] proved in 1988, the existence of other surfaces minimally embedded in $\mathbb{S}^3$ (cf. Section 2.6).

Lawson [48] showed also that there exists a local correspondence between immersions in $\mathbb{R}^3$ with mean curvature $H = 1$, in $\mathbb{S}^3$ with mean curvature $H = const$ and in $\mathbb{H}^3$ with mean curvature H such that $|H| > 1$. In fact, given a simply connected domain $U \subset M$ of a Riemann surface M and a conformal CMC immersion $f : U \to \mathbb{R}^3$ (resp. $\mathbb{S}^3$), the first fundamental form

g and the shape operator β of f satisfy the Gauss and Codazzi equations (cf. Equations (2.13) and (2.14)) in $\mathbb{R}^3$ (resp. $\mathbb{S}^3$).

Lawson proved that the two differential forms $\tilde{g} := g$ and $\tilde{\beta} := kg + \beta$ satisfy the Gauss and Codazzi equations in $\mathbb{H}^3$ (resp. $\mathbb{R}^3$), where k is a constant depending on the mean curvature H of f. Therefore, there exists a conformal immersion $\tilde{f} : U \to \mathbb{H}^3$ (resp. $\mathbb{R}^3$) with metric $\tilde{g}$ and shape operator $\tilde{\beta}$. Moreover, $\tilde{f}$ has constant mean curvature given by $H + k$ ([26, Section 5 p. 93])

In the case of genus 1, Lawson conjectured [48] that the Clifford torus is the only minimally embedded surface in $\mathbb{S}^3$. In 2012 Brendle [14] proved this conjecture using a maximum principle argument. Andrews and Li [2] extended this result to CMC embeddings, showing that the only embedded CMC tori in $\mathbb{S}^3$ are the unduloidal rotational Delaunay tori and the homogeneous tori.

Around 1990 Hitchin, Bobenko, Pinkall and Sterling ([37], [9], [55]) used integrable system methods to describe compact minimal and CMC tori immersed in $\mathbb{S}^3$, using the so called associated family of flat connections (cf. Section 3.1). In more recent years S. Heller, L. Heller and Schmitt ([35], [33], [34]) extended those arguments to the case of higher genus compact surfaces embedded minimally or with constant mean curvature in $\mathbb{S}^3$ (cf. Section 3.2).

2.5 Lawson's surfaces

We briefly recall the general method of constructing minimal surfaces in $\mathbb{S}^3$ due to Lawson ([48, Section 4 pp. 341-346]). Since this method relies on a solution of a Plateau problem, we first recall what a Pleateau problem is:

Let N be a Riemannian manifold and Γ a closed curve in N. The Plateau problem consists of finding a surface M immersed in N, such that

- Γ bounds the surface M, that is $\partial M = \Gamma$;
- the surface M minimizes the area of all surfaces having Γ as boundary.

Let $\Gamma \subset \mathbb{S}^3$ be a convex geodesic polygon with vertices $v_0, \ldots, v_n = v_0$ and edges $\gamma_0, \ldots, \gamma_n = \gamma_0$, such that the edge γ_j meets the edge γ_{j-1} at an angle $\frac{\pi}{k_j+1}$, where $k_j \in \mathbb{N}^+$ for $j = 1, \ldots, n$.

Let $\Psi : U \to \mathbb{S}^3$ be a solution of the Plateau problem for $\Gamma \subset \mathbb{S}^3$ and $\mathcal{M}_\Gamma = \Psi(U) \subset \mathbb{S}^3$. The surface $\mathcal{M}_\Gamma$ can be reflected about the edges $\gamma_1, \ldots, \gamma_n$ in order to obtain a complete, non singular surface M_Γ embedded in $\mathbb{S}^3$ ([48, Section 4 pp. 342-345]), under the assumptions $(A) - (D)$ on the curve Γ in [48, Section 4 p. 341].

If $r_{\gamma_j} : \mathbb{S}^3 \to \mathbb{S}^3$ denotes the reflection about the edge γ_j and G_Γ the subgroup of $\mathbf{O}(4)$ generated by these reflections, the surface M_Γ is given by

$$M_\Gamma = \bigcup_{g \in G_\Gamma} g(\mathcal{M}_\Gamma). \tag{2.28}$$

Moreove, the surface M_Γ is a compact surface if and only if the group G_Γ is finite ([48, Section 4 p. 345]).

2.5.1 The Lawson's Σ_{kl} surfaces

We describe one family of minimal surfaces discovered by Lawson in [48], with the construction method we have recalled in Section 2.5.

Let $\mathbb{S}^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$ and $C_1, C_2 \subset \mathbb{S}^3$ be the great circles given by

$$C_1 := \{(0, w) \mid |w|^2 = 1\}, \quad C_2 := \{(z, 0) \mid |z|^2 = 1\}. \tag{2.29}$$

Given two positive integers $k, l \in \mathbb{N}^+$, let $P_1, P_2 \in C_1$ and $Q_1, Q_2 \in C_2$ such that

$$\mathrm{dist}(P_1, P_2) = \frac{\pi}{l+1}, \quad \mathrm{dist}(Q_1, Q_2) = \frac{\pi}{k+1}, \tag{2.30}$$

and Γ_{kl} the closed geodesic convex polygon in $\mathbb{S}^3$ with vertices P_1, Q_1, P_2, Q_2.

Under the stererographic projection from $\mathbb{S}^3$ to $\mathbb{R}^3$, it is possible to obtain the following picture:

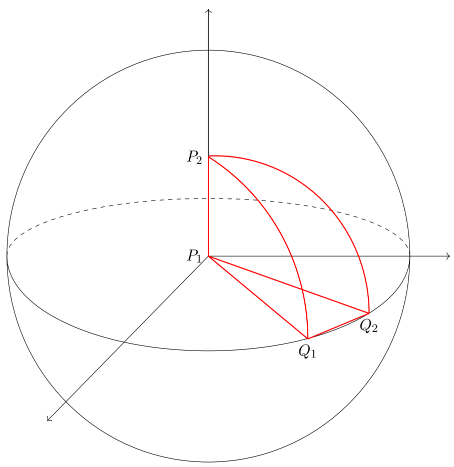

Figure 2.1: The stereographic projection of the geodesic polygon Γ_{kl} (in red), with angles $P_1\widehat{Q_1}P_2 = P_1\widehat{Q_2}P_2 = \frac{\pi}{l+1}$ and $Q_1\widehat{P_1}Q_2 = Q_1\widehat{P_2}Q_2 = \frac{\pi}{k+1}$

The geodesic polygon $\Gamma_{kl} \subset \mathbb{S}^3$ is mapped to the region of $\mathbb{R}^3$ bounded by the red curves in Figure 2.1.

The polygon Γ_{kl} satisfies the hypothesis of Theorem 1 in [48, Section 4 p. 345], therefore there exists a complete, non singular minimal surfaces determined by Γ_{kl} according to the Lawson construction (cf. Section 2.5), which we will denote by Σ_{kl}.

In order to show how the surface Σ_{kl} is obtained from the polygon Γ_{kl}, Lawson ([48, Section 6]) considered the surface $\mathcal{M}_{\Gamma_{kl}}$, solution to the Plateau problem for Γ_{kl} in $\mathbb{S}^3$ and the surface $\tilde{\mathcal{M}}_{\Gamma_{kl}}$ obtained from $\mathcal{M}_{\Gamma_{kl}}$ by reflection across the edge P_1Q_1 of Γ_{kl}.

The surface Σ_{kl} is obtained as the image of the action of the group $G := \mathbb{Z}_{k+1} \times \mathbb{Z}_{l+1}$ on $\tilde{\mathcal{M}}_{\Gamma_{kl}}$, generated by the rotations ([48, Section 6 p. 349])

$$\begin{aligned} R_k &: \mathbb{S}^3 \to \mathbb{S}^3 \\ (z,w) &\mapsto (e^{\frac{2\pi i}{k+1}} z, w), \end{aligned} \tag{2.31}$$

and

$$\begin{aligned} R_l &: \mathbb{S}^3 \to \mathbb{S}^3 \\ (z,w) &\mapsto (z, e^{\frac{2\pi i}{l+1}} w). \end{aligned} \tag{2.32}$$

The finiteness of G implies that the surface Σ_{kl} is compact ([48, Section 4 p. 345]). The genus g of Σ_{kl} can be computed as ([48, Proposition 4.4 p. 345]):

$$2 - 2g = 2(k+1)(l+1)\left(1 - \frac{k}{k+1} - \frac{l}{l+1}\right) = 2 - 2kl \Longrightarrow g = kl. \tag{2.33}$$

We conclude this subsection with the description of the map

$$\pi : \Sigma_{kl} \to \Sigma_{kl}/(\mathbb{Z}_{k+1} \times \mathbb{Z}_{l+1}). \tag{2.34}$$

Let $P_1, \dots, P_{2k+2} \in \Sigma_{kl}$ be the points on the great circle C_1 in (2.29), such that $\mathrm{dist}(P_j, P_{j+1}) = \frac{\pi}{l+1}$ and $Q_1, \dots, Q_{2l+2} \in \Sigma_{kl}$ the points on the great circle C_2 in (2.29) such that $\mathrm{dist}(Q_i, Q_{i+1}) = \frac{\pi}{k+1}$. The points $P_1, \dots, P_{2k+2}$ are the fixed points for the $\mathbb{Z}_{l+1}$-action on Σ_{kl} generated by the map R_l in (2.32).

The quotient $\Sigma_{kl}/\mathbb{Z}_{l+1}$ admits a unique Riemann surface structure such that the projection to the quotient

$$\pi_{l+1} : \Sigma_{kl} \to \Sigma_{kl}/\mathbb{Z}_{l+1} \tag{2.35}$$

is a holomorphic map between Riemann surfaces, branched at the points $p_j := \pi_{l+1}(P_j), j = 1, \dots, 2k+2$ (cf. Lemma 1.1). The branch order of each point p_j can be computed as follows ([52, Lemma 3.6 p. 80]):

$$b_{\pi_{l+1}}(p_j) = \frac{|\,\mathbb{Z}_{l+1}\,|}{|\,\mathrm{Stab}(p_j)|}(|\,\mathrm{Stab}(p_j)| - 1) = l, \quad j = 1, \dots, 2k+2, \tag{2.36}$$

where $\mathrm{Stab}(p_j)$ denotes the stabilizer group of p_j (cf. Definition 1.4), which in this case coincides with $\mathbb{Z}_{l+1}$. The Riemann-Hurwitz formula ([52, Corollary 3.7 p. 80]) implies that the quotient $\Sigma_{kl}/\mathbb{Z}_{l+1}$ is the Riemann sphere $\mathbb{CP}^1$ (cf. Example 1.4).

The set of points $\{Q_1, \ldots, Q_{2l+2}\}$, which are the fixed point for the $\mathbb{Z}_{k+1}$-action on Σ_{kl} (and on $\Sigma_{kl}/\mathbb{Z}_{l+1}$) generated by the map R_k in (2.31), is mapped under π_{l+1} to the two points $q_-, q_+ \in \mathbb{CP}^1$. There are exactly two points q_-, q_+ because the $\mathbb{Z}_{k+1}$-action and the $\mathbb{Z}_{l+1}$-action commute.

The map $\pi_{k+1} : \Sigma_{kl}/\mathbb{Z}_{l+1} \to (\Sigma_{kl}/\mathbb{Z}_{l+1})/\mathbb{Z}_{k+1}$ is a holomorphic map between Riemann surfaces (cf. Lemma 1.1) and the Riemann-Hurwitz formula implies that $(\Sigma_{kl}/\mathbb{Z}_{l+1})/\mathbb{Z}_{k+1}$ is the Riemann sphere $\mathbb{CP}^1$, since the branch points $\pi_{k+1}(q_\pm)$ have branch order k (from [52, Lemma 3.6 p. 80]). The points $p_1, \ldots, p_{2k+2}$ are mapped to two points p_-, p_+.

Therefore, we obtain that, the map

$$\pi : \Sigma_{kl} \xrightarrow{\pi_{l+1}} \Sigma_{kl}/\mathbb{Z}_{l+1} \xrightarrow{\pi_{k+1}} (\Sigma_{kl}/\mathbb{Z}_{l+1})/\mathbb{Z}_{k+1} \simeq \mathbb{CP}^1 \tag{2.37}$$

is a branched covering of degree $(k+1)(l+1)$ branched at the four points $p_-, p_+, q_-, q_+ \in \mathbb{CP}^1$. It is possible to show that the points $p_\pm$ have branch order $l(k+1)$ and the points $q_\pm$ have branch order $k(l+1)$ ([52, Lemma 3.6 p. 80]).

2.6 KPS surfaces

In 1998 Karcher, Pinkall and Sterling [44] constructed minimal embeddings of compact surfaces in $\mathbb{S}^3$, which we will call *KPS surfaces*. We briefly recall the construction of the KPS surfaces and we will give a description of their symmetry groups.

Let Δ be a tessellation of $\mathbb{S}^3$ such that each cell of Δ has the symmetries of a platonic solid in $\mathbb{R}^3$ (we refer to [17, Chapter 4] for more details about regular tessellations of $\mathbb{S}^3$). By subdividing one cell of Δ with respect to its planes of symmetry one obtains a tetrahedron, whose dihedral angles are given by ([44, Section 1 p. 169])

$$\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \eta, \beta_1, \beta_2. \tag{2.38}$$

The following table lists the data of the tessellations of $\mathbb{S}^3$ which the authors in [44] considered ([44, Table 1 p. 169])

η, β_1, β_2	Cell type	# cells in tasselation
$\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$	Tetrahedral (self-dual)	5
$\frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{3}$	Octahedral (self-dual)	24
$\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{4}$	Tetrahedral (Cubical)	16 (8)
$\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{5}$	Tetrahedral (Dodecahedral)	600 (120)
$\frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}$	Tetrahedral	2
$\frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{4}$	Cubical	2
$\frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{5}$	Dodecahedral	2
$\frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{3}$	Octahedral	2
$\frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{3}$	Icosahedral	2

(2.39)

Let Δ be a tessellation of $\mathbb{S}^3$ corresponding to the dihedral angles η, β_1, β_2 in Table 2.39 and T one of the tetrahedrons obtained by subdividing a cell of Δ with its planes of symmetry. The idea is to find a minimal surface M obtained as the solution of a Plateau problem in the tetrahedron T such that M intersects all faces of T perpendicularly and meets the edges of T corresponding to the dihedral angles $\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \eta$.

In order to find such a surface M, Karcher, Pinkall and Sterling in [44] considered a geodesic quadrilateral $\mathcal{Q} = ABCD$ in $\mathbb{S}^3$ with angles $\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \eta$ at A, B, C and D, respectively. The quadrilateral $\mathcal{Q}$ is completely determined by the lengths of the edges l_1, l_2 in A, since the other two edges are uniquely determined by l_1, l_2 and the angle η.

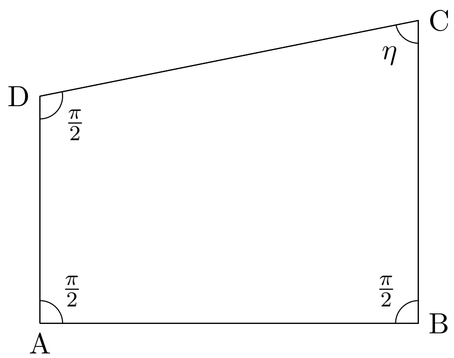

Figure 2.2: The geodesic quadrilateral bounding the surface $\tilde{M}(l_1, l_2, \eta)$.

Let $\tilde{M} = \tilde{M}(l_1, l_2, \eta)$ be the minimal surface obtained as the solution to the Plateau problem for the geodesic quadrilateral $\mathcal{Q}$ in $\mathbb{S}^3$. The conjugated minimal surface $M = M(l_1, l_2, \eta)$ of

$\tilde{M}(l_1, l_2, \eta)$ is the minimal surface obtained from the fundamental theorem of immersed surfaces [18, Theorem 1.1 p. 7] considering the same metric of $\tilde{M}(l_1, l_2, \eta)$ and the shape operator (cf. 2.7)

$$S := R_{\frac{\pi}{2}} \circ \tilde{S}, \tag{2.40}$$

where $\tilde{S}$ is the shape operator of $\tilde{M}(l_1, l_2, \eta)$ and $R_{\frac{\pi}{2}}$ is a rotation of $\pi/2$ in the tangent space $T_p\tilde{M}$ for all $p \in \tilde{M}(l_1, l_2, \eta)$.

The surface $M(l_1, l_2, \eta)$ determines a tetrahedron having dihedral angles $\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \eta, \delta_1, \delta_2$ ([44, Section 2 p. 173]). It is possible to choose the values l_1, l_2 such that the dihedral angles δ_1, δ_2 coincides with the dihedral angles β_1, β_2 corresponding to the angle η in Table 2.39.

In fact, let $F(l_1, l_2)$ be the function to $\mathbb{R}^2$ which gives the dihedral angles δ_1, δ_2 in terms of l_1 and l_2. Using a maximum principle argument, it is possible to prove that the function F is continuous ([44, Lemma 4 p. 175]). Moreover, the value $(\beta_1, \beta 2)$ is contained in the range of the function F as proved in [44, Theorem p. 178].

Therefore, there exists a choice of l_1, and l_2 such that the surface M is a minimal surface contained in the tetrahedron T obtained by subdividing a cell of the tessellation Δ of $\mathbb{S}^3$, for all the tessellation considered by Karcher, Pinkall and Sterling in [44].

In order to construct a compact, minimal surface in $\mathbb{S}^3$ from the simply connected surface M, the surface M is reflected across the faces of the tetrahedron T which are not contained in the faces of the corresponding cell of Δ. The resulting surface $\mathcal{B}$ is called a *bone* for the compact, non singular minimal surface Σ in $\mathbb{S}^3$ generated by reflecting $\mathcal{B}$ across the faces of all the cells in Δ [44, Section 1 p. 169 and Proposition 8 p. 183].

2.6.1 Symmetries of the KPS surfaces

In this subsection we consider the KPS surfaces corresponding to the tessellations of $\mathbb{S}^3$ whose cells have the symmetries of the platonic solid in $\mathbb{R}^3$. We will call these surfaces *platonic KPS surfaces* and they correspond to the last five rows in Table 2.39.

Proposition 2.3. *Let Σ be a platonic KPS surface. There exists a finite subgroup $G \subset \boldsymbol{SO}(4)$ acting on Σ, such that the quotient map $\pi : \Sigma \to \Sigma/G$ is a $|G|$-fold covering to the Riemann sphere $\mathbb{CP}^1$, branched at four points.*

Proof. Let Σ be a KPS surface and Δ the corresponding tessellation of $\mathbb{S}^3$ used to construct Σ. The tessellation Δ is obtained as the image of the action of a finite subgroup $\Gamma \subset \mathbf{SU}(2) \times \mathbf{SU}(2)$ of one cell of Δ ([23, Chapter 3 and 4]).

From the construction described in Section 2.6, the surface Σ is obtained as the image of the action of Γ on the bone $\mathcal{B}$ of Σ. The idea is to consider a larger fundamental piece for Σ such that Σ is obtained as the image of the action of a subgroup $G \subset \Gamma$ of such fundamental piece, which does not contain reflections.

Let $\tilde{M}$ be the simply connected, minimal surface in $\mathbb{S}^3$, bounded by the quadrilateral $\mathcal{Q}$ with angles $\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \eta$ (cf. Section 2.6). By reflecting $\tilde{M}$ across the edge AB (cf. Figure 2.2), we obtain a minimal surface $\tilde{N}$ bounded by a quadrilateral with angles $\frac{\pi}{2}, \frac{\pi}{2}, \eta, \eta$ (cf. Figure 2.3).

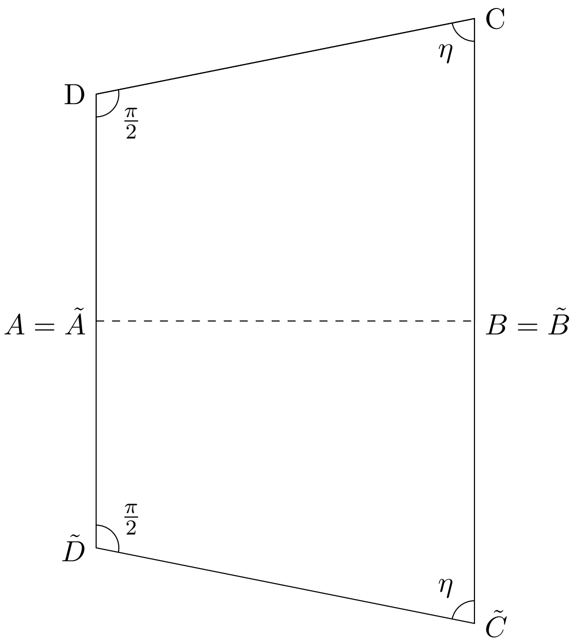

Figure 2.3: The geodesic quadrilateral bounding the surface $\tilde{N}$.

The conjugate minimal surface of $\tilde{N}$ is a minimal surface N in $\mathbb{S}^3$ bounded by a geodesic quadrilateral inside a tetrahedron with dihedral angles $\frac{\pi}{2}, \frac{\pi}{2}, \eta, \eta, \beta_1, \beta_2$, where β_1, β_2 are the dihedral angles corresponding to η in Table 2.39.

The surface Σ is then obtained as the image of the action of $G \subset \Gamma$ of N, where G is the subgroup of Γ consisting of all the rotations in Γ. Under the 2-fold covering $\mathbf{SU}(2) \times \mathbf{SU}(2) \to \mathbf{SO}(4)$ ([23, Chapter 3.17]), the group G can be identified with a finite subgroup of $\mathbf{SO}(4)$.

The quotient Σ/G corresponds to the surface N, which is a four punctured sphere with punctures at the points corresponding to the vertices of the quadrilateral bounding N. In fact, in the geodesic quadrilateral bounding the conjugate minimal surface $\tilde{N}$ of N, the vertex C is identified with the vertex $\tilde{C}$ and the vertex D with the vertex $\tilde{D}$. The other two punctures are given by the vertices $A = \tilde{A}$ and $B = \tilde{B}$ (cf. Figure 2.3).

Let $\pi : \Sigma \to \Sigma/G$ the projection to the quotient. There exists a unique Riemann surface structure on Σ/G such that the map π is a holomorphic map between Riemann surfaces (cf. Lemma 1.1). Let $z_1, \ldots, z_4 \in \Sigma/G$ be the points corresponding to the vertices of the quadrilateral bounding the surface N with angles $\frac{\pi}{2}, \eta, \frac{\pi}{2}, \eta$ respectively.

The branch orders of the points $z_1, \ldots, z_4$ are given by ([52, Lemma 3.6 p. 80]):

$$b_\pi(z_1) = b_\pi(z_3) = \frac{d}{2}, \tag{2.41}$$

$$b_\pi(z_2) = b_\pi(z_4) = d - \frac{d}{n}, \tag{2.42}$$

where $d = |G| = \frac{|\Gamma|}{2}$ and $n \in \mathbb{N}^+$ is such that $\eta = \frac{\pi}{n}$. Finally, the Riemann-Hurwitz formula ([52, Corollary 3.7 p. 80]) implies that the Riemann surface Σ/G is the Riemann sphere $\mathbb{CP}^1$, for all the platonic KPS surfaces. $\square$

Let A_n be the alternating group and S_n the symmetric group. The following table lists the platonic KPS surfaces together with the genus and the corresponding finite subgroup $G \subset \mathbf{SO}(4)$:

η, β_1, β_2	Cell type	genus	$\mathbf{SO}(4)$-subgroup
$\frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}$	Tetrahedral	3	A_4
$\frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{3}$	Octahedral	7	S_4
$\frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{4}$	Cubical	5	S_4
$\frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{5}$	Dodecahedral	11	A_5
$\frac{\pi}{5}, \frac{\pi}{2}, \frac{\pi}{3}$	Icosahedral	19	A_5

(2.43)

We expect that Proposition 2.3 holds also for the KPS surfaces corresponding to the first four rows of Table 2.39. However, the finite subgroup $\Gamma \subset \mathbf{SU}(2) \times \mathbf{SU}(2)$ acting on these surfaces is more complicated. We didn't have the time to carry out all the details and understand the corresponding finite subgroup of $\mathbf{SO}(4)$.

Chapter 3

Integrable system methods for minimal and CMC immersions into $\mathbb{S}^3$

In this chapter we will describe the gauge theoretic formalism for minimal and CMC immersions of compact Riemann surfaces into $\mathbb{S}^3$.

In 1989, Hitchin [37] described harmonic maps from a compact Riemann surfaces of genus 1 into $\mathbb{S}^3$ in terms of algebro geometric data. Since minimal immersions into $\mathbb{S}^3$ and the Gauss maps of CMC immersions into $\mathbb{S}^3$ are harmonic maps (cf. Section 2.2), the approach of Hitchin yields a description of minimal and CMC tori in $\mathbb{S}^3$.

Unfortunately, this approach cannot be generalized to surfaces of higher genus, due to the fact that it relies on the feature that the fundamental group of the surface is abelian.

In 2013 Heller [35] proved that it is possible to reconstruct a CMC immersion of a compact Riemann surface of genus 2 applying a version of the DPW method, introduced by Dorfmeister, Pedit and Wu in 1998 [22].

More recently, Heller, Heller and Schmitt in [33] showed a way to generalize Hitchin's approach for compact, minimal or CMC immersions into $\mathbb{S}^3$ of compact Riemann surfaces of genus $g \geq 2$.

3.1 Gauge theoretic formalism for minimal and CMC immersions in $\mathbb{S}^3$

We briefly describe a way of studying minimal and CMC immersions into $\mathbb{S}^3$ using flat connections defined on an appropriate complex vector bundle. We will mainly refer to [35], [37, Section 1] and [33].

We first recall the definition of gauge transformation

Definition 3.1. Let $E \to M$ be a complex vector bundle. A *gauge transformation* g is a section of the endomorphism bundle $\mathrm{End}(E)$ which is invertible everywhere. If $E \to M$ is a holomorphic vector bundle we will say that a gauge transformation g is holomorphic if it is an element of the space of holomorphic section of $\mathrm{End}(E)$, that is, $g \in H^0(M, \mathrm{End}(E))$.

The set of gauge transformations on a complex vector bundle $E \to M$ forms a group with respect to the composition, denoted with $\mathcal{G}_E$.

Given a connection ∇ on a complex vector bundle $E \to M$ it is possible to define the action of a gauge transformation $g \in \mathcal{G}_E$ on ∇ as follows:

$$\nabla \cdot g := g \circ \nabla \circ g^{-1}. \tag{3.1}$$

Consider the three sphere $\mathbb{S}^3$ identified with the compact Lie group $\mathbf{SU}(2)$ via the map

$$\begin{aligned} \mathbb{S}^3 &\to \mathbf{SU}(2) \\ (z, w) &\mapsto \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \qquad z, w \in \mathbb{C}, |z|^2 + |w|^2 = 1. \end{aligned} \tag{3.2}$$

The tangent bundle of $\mathbb{S}^3$ is trivial ([13, Chapter 10 p. 89]) and it is possible to write it, with respect to the trivializations given by left translations, as

$$T\,\mathbb{S}^3 = \mathbb{S}^3 \times \operatorname{Im}\mathbb{H}, \tag{3.3}$$

where $\operatorname{Im}\mathbb{H} \simeq \mathbb{R}^3$ is the space of pure quaternions.

With respect to the same trivializations of $T\,\mathbb{S}^3$, the Levi-Civita connection on $T\,\mathbb{S}^3$ (cf. Section 2.1) is given by

$$\tilde{\nabla} = d + \frac{1}{2}\omega, \tag{3.4}$$

where $\omega \in \Omega^1(\mathbb{S}^3, \operatorname{Im}\mathbb{H})$ is the Maurer-Cartan form which acts on $\operatorname{Im}\mathbb{H}$ via the adjoint representation [35, Section 2 p. 747].

It is possible to define a hermitian vector bundle on $\mathbb{S}^3$ by

$$\tilde{E} := \mathbb{S}^3 \times \mathbb{H} \to \mathbb{S}^3, \tag{3.5}$$

where $\mathbb{H} \simeq \mathbb{C}^2$ is the space of quaternions. The complex structure on $\tilde{E}$ (cf. Definition 1.12) is given by the right multiplication by $i \in \mathbb{H}$ and the hermitian metric $\langle , \rangle$ by the trivializations and the identification $\mathbb{H} \simeq \mathbb{C}^2$ ([35, Section 2 p. 747]). It is possible to consider the connection $\tilde{\nabla}$ on the hermitian vector bundle $\tilde{E}$, where the 1-form ω acts on $\mathbb{H}$ by left multiplication.

In this way, the tangent bundle $T\,\mathbb{S}^3$ is identified with the bundle $\mathrm{End}_0(\tilde{E})$ of skew-symmetric, trace free, complex linear endomorphisms of $\tilde{E}$ ([35, Section 2 p. 747]).

Let M be a compact Riemann surface and $f : M \to \mathbb{S}^3$ a conformal immersion (cf. Section 2.1). We first consider the case of f being a minimal immersion. Let $E := f^*\tilde{E}$ the pullback bundle of $\tilde{E}$ under f and

$$\nabla := f^*\tilde{\nabla} = d + \frac{1}{2}\phi \tag{3.6}$$

the pullback of the Levi-Civita connection defined on $\tilde{E}$. The 1-form $\phi \in \Omega^1(M, \mathrm{Im}\,\mathbb{H})$ acts on $\mathrm{Im}\,\mathbb{H}$ via the adjoint representation.

The connection 1-form ϕ satisfies the equation

$$d^{\nabla}\phi = 0, \tag{3.7}$$

where d^{∇} is the exterior derivative of ∇ (cf. Subsection 1.4.4). The minimality of f translates into the equation ([37, Equation 1.2 p. 632])

$$d^{\nabla} * \phi = 0, \tag{3.8}$$

where $* : \Omega^1(M) \to \Omega^1(M)$ is the Hodge star operator defined, with respect to a local coordinate $z : U \subset M \to \mathbb{C}$, by

$$* \, dz = i dz, \quad *d\bar{z} = -i d\bar{z}. \tag{3.9}$$

Using the identification of $T\,\mathbb{S}^3$ with the bundle $\mathrm{End}_0(\tilde{E})$ it is possible to consider the 1-form ϕ as an element of $\Omega^1(M, \mathrm{End}_0(E))$. Therefore, ϕ can be decomposed into its $(1,0)$ and $(0,1)$ components (cf. Equation 1.35):

$$\frac{1}{2}\phi = \Phi - \Phi^*, \tag{3.10}$$

where $\Phi \in \Gamma(M, K \otimes \mathrm{End}_0(E))$ is called the *Higgs field* of the immersion f and Φ^* is the adjoint operator of Φ, with respect to the metric on M induced by $\langle , \rangle$.

The equations (3.7) and (3.8) imply

$$\nabla''\Phi = 0, \tag{3.11}$$

where ∇'' is the $(0,1)$ part of the connection ∇ on $E \to M$, which defines a holomorphic structure on E (cf. Equation (1.69)). With respect to this holomorphic structure, the $(1,0)$-form Φ is an element of $H^0(M, K \otimes \mathrm{End}_0(E))$.

From the fact that the connection $\nabla - \frac{1}{2}\phi = d$ is trivial, it is possible to obtain another equation, which can be locally (on simply connected subsets of M) written as ([37, Equation 1.7 p. 633])

$$F^{\nabla} - [\Phi \wedge \Phi^*] = 0, \tag{3.12}$$

where F^{∇} is the curvature of the connection ∇ (cf. Subsection 1.4.4).

In [37, Proposition 1.8 p. 635] it is proven that the Higgs field Φ of a conformal, minimal immersion $f : M \to \mathbb{S}^3$ from a compact Riemann surface M, is a nilpontent, nowhere vanishing, with zero determinant, holomorphic $(1,0)$-form with values in $\mathrm{End}_0(E)$.

Thus, given a conformal minimal immersion $f : M \to \mathbb{S}^3$ is is possible to define a connection ∇ on the hermitian vector bundle $E \to M$ and a Higgs field Φ which satisfy the system of differential equations

$$\begin{cases} \nabla''\Phi = 0 \\ F^{\nabla} = [\Phi \wedge \Phi^*]. \end{cases} \tag{3.13}$$

Conversely, Hitchin proved ([37, Section 1 p. 641]) that, on simply connected subsets of M, it is possible to reconstruct the immersion $f : U \to M$ from a connection ∇ on $E \to M$ and a nilpotent, nowhere vanishing Higgs field Φ which satisfy the system (3.13). In fact, in this case the immersion f is given by the gauge transformation between the two trivial connections $\nabla - \phi$ and $\nabla + \phi$, where $\frac{1}{2}\phi := \Phi - \Phi^*$.

We can now introduce the *associated family of flat connections* of a conformal minimal immersion into $\mathbb{S}^3$.

Definition 3.2. Let $f : M \to \mathbb{S}^3$ be a conformal, minimal immersion of a compact Riemann surface M into $\mathbb{S}^3$. Let ∇ be the connection defined in (3.6) and Φ the Higgs field of the immersion f (cf. Equation (3.10)). The *associated family of* $\boldsymbol{SL}(2,\mathbb{C})$*-connections* of $f : M \to \mathbb{S}^3$, defined on the complex vector bundle $E \to M$, is given by

$$\nabla^{\lambda} := \nabla + \lambda^{-1}\Phi - \lambda\Phi^*, \quad \lambda \in \mathbb{C}^*. \tag{3.14}$$

From the equations (3.13) it follows that the connections ∇^{λ} are flat for every $\lambda \in \mathbb{C}^*$. Moreover, the connections $\nabla^1 = \nabla + \Phi - \Phi^* = \nabla + \frac{1}{2}\phi$ and $\nabla^{-1} = \nabla - \Phi + \Phi^* = \nabla - \frac{1}{2}\phi$ are trivial and the immersion f (on simply connected subset of M) is given by the gauge transformation between ∇^1 and ∇^{-1} ([35, Section 2 p. 748]).

Thus, the associated family of flat $\mathbf{SL}(2,\mathbb{C})$-connections of a conformal, minimal immersion $f : M \to \mathbb{S}^3$ contains all the information of the immersion.

Let $f : M \to \mathbb{S}^3$ be a conformal CMC immersion of a compact Riemann surface M into $\mathbb{S}^3$, with mean curvature $H \neq 0$. In this case the map f is no longer harmonic. However, the fact that its Gauss map is harmonic (cf. Theorem 2.2) allows the definition of an associated family of flat connections of the immersion f as follows.

Consider the 1-form $\phi := f^{-1}df$ on the hermitian vector bundle $E \to M$, as in the case of minimal immersion. It is possible to decompose ϕ into its $(1,0)$ and $(0,1)$ components as ([33, Section 1 p. 417])

$$\phi = \Psi - \Psi^*. \tag{3.15}$$

Let Φ and Φ^* be defined by

$$\Phi := \frac{\lambda_2}{1+\lambda_2}\Psi, \quad \Phi^* := \frac{1}{1+\lambda_2}\Psi^*, \tag{3.16}$$

where $\lambda_2 := \frac{-iH+1}{iH+1} \in \mathbb{S}^1 \subset \mathbb{C}^*$. It is possible to define a flat connection on the hermitian vector bundle $E \to M$ by

$$\nabla := d + \Phi - \Phi^*. \tag{3.17}$$

The associated family of $\mathbf{SL}(2,\mathbb{C})$-connections of the immersion $f : M \to \mathbb{S}^3$ is given by

$$\nabla^\lambda := \nabla + \lambda^{-1}\Phi - \lambda\Phi^*, \quad \lambda \in \mathbb{C}^*. \tag{3.18}$$

Analogously to the minimal case, the connections ∇^λ are flat for every $\lambda \in \mathbb{C}^*$, the 1-form $\Phi \in \Gamma(M, K \otimes \mathrm{End}_0(E))$ is nilpotent, nowhere vanishing and it is holomorphic with respect to the holomorphic structure on E induced by the connection ∇. Moreover, the connections $\nabla^{-1} = d$ and $\nabla^{\lambda_2} = d + \phi$ are trivial ([33, Section 1 p. 418]).

The following Theorem (due to Bobenko [9] and Hitchin [37]) shows that it is possible to reconstruct the CMC immersion $f : M \to \mathbb{S}^3$ from its associated family of flat $\mathbf{SL}(2,\mathbb{C})$-connections as in the minimal case:

Theorem 3.1 ([33, Theorem 1.1 p. 418]). *Let $f : M \to \mathbb{S}^3$ be a conformal CMC immersion of mean curvature $H \neq 0$. Then, its associated family of flat $\boldsymbol{SL}(2,\mathbb{C})$-connections ∇^λ given by (3.18), is unitary for $\lambda \in \mathbb{S}^1$ and trivial for $\lambda_1 \neq \lambda_2 \in \mathbb{S}^1$.*

Conversely, given such a family of flat $\boldsymbol{SL}(2,\mathbb{C})$-connections with nilpotent Φ, the immersion f, given by the gauge transformation between ∇^{λ_1} and ∇^{λ_2}, is conformal and of constant mean curvature

$$H = i\frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2}. \tag{3.19}$$

Moreover, the associated family of the immersion obtained in this way coincides with the family ∇^λ.

The points λ_1, λ_2 in Theorem 3.1, are called the *Sym points*. The existence of the Sym points is the *extrinsic closing condition* and the unitarity of the connections ∇^λ for $\lambda \in \mathbb{S}^1$ the *intrinsic closing condition* for the immersion f.

In [37] Hitchin considered the case of M being a torus. He classified all the families of flat connections ∇^λ and parametrized the associated minimal and CMC immersions $f : M \to \mathbb{S}^3$. He showed that it is possible to define an algebraic curve Σ associated to a solution (∇, Φ) of (3.13). Moreover, given such an algebraic curve Σ it is possible to find a solution of (3.13) associated to a conformal harmonic map in $\mathbb{S}^3$ or to a harmonic map to a totally geodesic 2-sphere in $\mathbb{S}^3$ depending on the properties the curve Σ satisfies ([37, Theorem 8.20]).

3.2 The DPW approach for higher genus surfaces

In 1998, Dorfmeister, Pedit and Wu [22] introduced a method (which we will call DPW method) that allows the construction (or reconstruction) of all harmonic maps $f : U \to G/K$ from a simply connected subset $U \subset M$ of a Riemann surface into a symmetric space G/K (we refer to [47, Chapter 11.2] for the definition and more details about symmetric spaces).

The initial data for the DPW method is given by a holomorphic (or, in some situations, meromorphic) $\mathfrak{sl}(2, \mathbb{C})$-valued 1-form $\xi(z, \lambda)$ which depends on a local coordinate z on $U \subset M$ and a parameter $\lambda \in \mathbb{C}^*$.

We have recalled in Section 2.2 that there exists a correspondence between harmonic maps from a Riemann surface M into $\mathbb{R}^3$ or $\mathbb{S}^3$ and minimal immersions $f : M \to \mathbb{R}^3, \mathbb{S}^3$. Therefore, the DPW method gives a way to reconstruct all such immersions, if one considers $G/K = \mathbb{R}^3$ or $\mathbb{S}^3$.

CMC immersions in $\mathbb{R}^3$ or $\mathbb{S}^3$ can also be obtained via the DPW method considering $G/K = \mathbb{S}^2$. In fact, the harmonic maps $g : U \subset M \to \mathbb{S}^2$ constructed via the DPW method can be considered as the Gauss maps of CMC immersions $f : U \subset M \to \mathbb{R}^3, \mathbb{S}^3$ (cf. Section 2.2).

If one wants to construct minimal or CMC immersions of a non simply connected surface M via the DPW method, one has to ensure that the immersions of two different simply connected domains in M patch together. In [45] it has been shown how to obtain CMC immersions of the thrice punctured sphere in $\mathbb{R}^3$ via DPW method. The surfaces obtained are called *trinoids with Delaunay ends*.

For compact surfaces of genus $g \geq 2$, the situation is more complicated. Heller, Heller and Schmitt in [33] proved that, in this case, it is more appropriate to consider holomorphic families of flat connections, defined on a hermitian vector bundle $E \to M$, up to gauge equivalence (cf. Definition 3.1).

Let M be a compact Riemann surface and $\mathcal{A}^2(M)$ the moduli space of flat connections (for more details about the moduli space of flat connections we refer to [29, Chapter 9]), defined on the hermitian vector bundle $E \to M$ described in Section 3.1. Heller, Heller and Schmitt proved the following theorem, which can be considered as a generalization of the DPW method for compact surfaces of genus $g \geq 2$, minimally or CMC immersed in $\mathbb{S}^3$.

Theorem 3.2 ([33, Theorem 1.2 p. 419]). *Let M be a compact Riemann surface of genus $g \geq 2$ and $\mathcal{D} : \mathbb{C}^* \to \mathcal{A}^2(M)$ a holomorphic map satisfying*

(1) *the unit circle $\mathbb{S}^1 \subset \mathbb{C}^*$ is mapped into the set consisting of gauge equivalence classes of unitary flat connections;*

(2) *around $\lambda = 0$, there exists a local lift $\tilde{\nabla}^\lambda$ of $\mathcal{D}$ with an expansion in λ*

$$\tilde{\nabla}^\lambda \sim \lambda^{-1}\Psi + \tilde{\nabla}^0 + \text{ higher order terms in } \lambda \tag{3.20}$$

for a nilpotent $\Psi \in \Gamma(M, K \otimes End_0(E))$;

(3) *there are two distinct points $\lambda_1, \lambda_2 \in \mathbb{S}^1 \subset \mathbb{C}^*$ such that $\mathcal{D}(\lambda_j)$ $j = 1, 2$ represents the trivial gauge equivalence class.*

Then, there exists a (possibly branched) CMC immersion $f : M \to \mathbb{S}^3$ inducing the map $\mathcal{D}$ as the family of gauge equivalence classes

$$\mathcal{D}(\lambda) = \nabla^\lambda, \quad \lambda \in \mathbb{C}^*, \tag{3.21}$$

where ∇^λ is the associated family of flat connections of f (cf. Equation (3.18)). The branch points of f are given by the zeros of Ψ. Conversely, every CMC immersion determines a holomorphic $\mathbb{C}^*$-curve into $\mathcal{A}^2(M)$ via (3.21).

The CMC immersions $f : M \to \mathbb{S}^3$ obtained by Theorem 3.2 are not uniquely determined by the map $\mathcal{D}$. The immersions corresponding to the same $\mathcal{D}$ are related by the so-called *dressing transformations* (we refer to [16] or [26, Section 4] for more details about dressing transformations).

The idea is to find a holomorphic family of flat connections on $E \to M$, of the form

$$d + \xi(z, \lambda) \tag{3.22}$$

which is gauge equivalent to the associated family of flat connections of f and satisfies the conditions of Theorem 3.2. Moreover, $\xi(z, \lambda)$ (which is called *DPW potential*) must be a holomorphic family of meromorphic 1-forms with values in $\mathfrak{sl}(2, \mathbb{C})$ with expansion series in λ given by

$$\xi(z, \lambda) = \sum_{j=-1}^{\infty} \xi_j(z) \lambda^j dz, \tag{3.23}$$

where $\xi_j(z)$ is a function with values in $\mathfrak{sl}(2, \mathbb{C})$ for every j and $\xi_{-1}(z)$ is nilpotent and upper triangular.

The immersion $f : M \to \mathbb{S}^3$ can be reconstructed via the DPW method, which can be summarized in the following steps ([26, Section 2]):

(i) solve the ODE for the variable $\Phi(z, \lambda)$

$$d\Phi(z, \lambda) = \Phi(z, \lambda)\xi(z, \lambda) \tag{3.24}$$

with respect to z, where $\Phi(z, \lambda) \in \mathbf{SL}(2, \mathbb{C})$;

(ii) Apply the Iwasawa decomposition [22, Theorem 2.2 p. 638] to Φ, to obtain a decomposition of the form

$$\Phi(z, \lambda) = F(z, \lambda)B(z, \lambda), \tag{3.25}$$

where $F(z, \lambda) \in \mathbf{SU}(2)$ for all $\lambda \in \mathbb{S}^1$ and the expansion series of $B(z, \lambda)$ in λ does not contains negative powers in λ. Moreover, $B(z, \lambda)$ is holomorphic in λ for λ in $D_1 := \{\lambda \in \mathbb{C}^* \mid |\lambda| < 1\}$;

(iii) The Sym-Bobenko formula [10, Chapter 3] for $\mathbb{S}^3$, given by

$$f(z, \lambda) = F(z, \eta\lambda)F^{-1}(z, \lambda), \tag{3.26}$$

gives an immersion $f : M \to \mathbb{S}^3$ for every $\eta \in \mathbb{S}^1, \eta \neq 1$ and $\lambda \in \mathbb{S}^1$. Moreover, the mean curvature of $f(z, \lambda)$ is constant and it is given by

$$H = i\frac{1 + \eta}{1 - \eta}. \tag{3.27}$$

The fact that the family of flat connections (3.22) satisfies the conditions of Theorem 3.2, guarantees that the immersion $f(z, \lambda)$ in (3.26) closes up around non trivial loops on the surface M.

For the Lawson surface $\Sigma_{2,1}$ (cf. Subsection 2.5.1), Heller [35] described explicitly a DPW potential, up to two unknown functions depending on the parameter $\lambda \in \mathbb{C}^*$. He used the fact that there is a $\mathbb{Z}_3$-action on $\Sigma_{2,1}$ such that the quotient $\Sigma_{2,1}/\mathbb{Z}_3 \simeq \mathbb{CP}^1$ and the projection $\pi : \Sigma_{2,1} \to \mathbb{CP}^1$ is a holomorphic map between Riemann surfaces, branched at four points $z_1, \ldots, z_4 \in \mathbb{CP}^1$ (cf. Subsection 2.5.1).

In [35, Theorem 4.2 p. 754] and [35, Theorem 4.3 p. 756] it is shown that the associate family of flat connections of the minimal immersion $f : \Sigma_{2,1} \to \mathbb{S}^3$ is gauge equivalent to the pull-back of the family of flat connections on $\mathbb{CP}^1$ $d + \xi(z,\lambda)$, where

$$\xi(z,\lambda) = \begin{pmatrix} -\frac{4}{3}\frac{z^3}{z^4-1} + \frac{A(\lambda)}{z} & \lambda^{-1} + B(\lambda)z^2 \\ \frac{G(\lambda)}{z^4-1} + \frac{\lambda H(\lambda)}{z^2(z^4-1)} & \frac{4}{3}\frac{z^3}{z^4-1} - \frac{A(\lambda)}{z} \end{pmatrix} dz, \tag{3.28}$$

and the branch points $z_1, \ldots, z_4$ of $\pi : \Sigma_{2,1} \to \mathbb{CP}^1$ are chosen to be $1, i, -1, -i$.

The functions A, B, G, H are holomorphic in $\lambda \in \mathbb{C}^*$ and they are called the *accessory parameters* of the family of flat connections $d + \xi(z,\lambda)$.

The symmetries on $\mathbb{CP}^1$, induced from the orientation preserving symmetries of $\Sigma_{2,1}$ imply the following relations on the accessory parameters ([35, Theorem 4.3 p. 756])

$$\begin{aligned} H &= A + A^2 \\ B &= -\frac{1}{G}\Big(-\frac{1}{3} + A + (\frac{1}{3} - A)^2 \Big). \end{aligned} \tag{3.29}$$

Thus, the family of flat connections $d+\xi(z,\lambda)$ is determined up to two holomorphic function $A(\lambda)$ and $G(\lambda)$ in $\lambda \in \mathbb{C}^*$.

The connection 1-form of the pull-back connection $d + \pi^*\xi(z,\lambda)$ on the surface $\Sigma_{2,1}$ is a meromorphic 1-form which is a DPW potential for $\Sigma_{2,1}$ ([35, Theorem 4.3 p. 756]). The minimal immersion $f : \Sigma_{2,1} \to \mathbb{S}^3$ can be reconstructed via the DPW method using the potential $\pi^*\xi(z,\lambda)$, after an appropriate choice of the accessory parameters that guarantee that the immersion f is well defined.

In the next chapter, we will show that there exists a DPW potential for other compact surfaces immersed in $\mathbb{S}^3$ of genus $g \geq 2$.

Chapter 4

DPW potentials for symmetric CMC surfaces in $\mathbb{S}^3$

The aim of this chapter is to show that there exists a DPW potential (cf. Section 3.2) for every compact CMC surface M in $\mathbb{S}^3$ which satisfies appropriate properties (cf. Definition 4.1 below). Therefore, it is possible to reconstruct the CMC immersion $f : M \to \mathbb{S}^3$ via the DPW method described in Section 3.2.

We first define the class of surfaces we will consider.

Definition 4.1. Let M be a compact Riemann surface. We say that M is a *symmetric CMC surface* if there exists a CMC embedding $f : M \to \mathbb{S}^3$ and the following conditions are satisfied:

(i) There exists a finite subgroup $G \subset \mathbf{SO}(4)$ with a presentation of the form

$$G = \langle g_1, \dots, g_4 \mid g_1 \cdots g_4 = 1 \rangle, \tag{4.1}$$

which acts faithfully (cf. Definition 1.4) on M, where 1 denotes the identity element of G;

(ii) The quotient M/G is the Riemann sphere $\mathbb{CP}^1$;

(iii) The projection to the quotient $\pi : M \to \mathbb{CP}^1$ is a holomorphic map between Riemann surfaces of degree $|G|$, branched at four points $z_1, \dots, z_4 \in \mathbb{CP}^1$.

Remark 4.1. The group G acts transitively on the set $\pi^{-1}(z)$ for each $z \in \mathbb{CP}^1$, that is, for every two points $p, \tilde{p} \in \pi^{-1}(z)$ there exists an element $g \in G$ such that

$$g \cdot p = \tilde{p}. \tag{4.2}$$

A covering map which satisfies this property is called a *Galois covering* and G the *Galois group* of the covering map (we refer to [61, Chapter 2] for more details on the theory of Galois coverings).

The following table lists the surfaces we will consider, together with their symmetry group $G \subset \mathbf{SO}(4)$:

Surface	Genus	$\mathbf{SO}(4)$-symmetry group
Lawson's surface $\xi_{(g-1,1)}$	$g-1$	$\mathbb{Z}_g \times \mathbb{Z}_2$
Lawson's surface $\xi_{(k-1,l-1)}$	$(k-1)(l-1)$	$\mathbb{Z}_k \times \mathbb{Z}_l$
KPS Tetrahedral	3	A_4
KPS Octahedral	7	S_4
KPS Cubical	5	S_4
KPS Icosahedral	19	A_5
KPS Dodecahedral	11	A_5
Octahedral join	11	S_4
Icosahedral join	29	A_5

(4.3)

We saw in Sections 2.5 and 2.6 that the Lawson's surfaces and the platonic KPS surfaces satisfy conditions $(i)-(iii)$ of Definition 4.1.

The Octahedral join surface can be constructed using the method of Karcher, Pinkall and Sterling (cf. Section 2.6) using a tessellation of $\mathbb{S}^3$ with symmetry group S_4 (the symmetric group of order 4) and dihedral angle η used to construct it (cf. Subsection 2.6) equal to $\frac{\pi}{12}$. Analogously, the Icosahedral join surface can be constructed using a tessellation of $\mathbb{S}^3$ with symmetry group A_5 (the alternating group of order 5) and dihedral angle η equal to $\frac{\pi}{30}$.

In the rest of this Chapter (unless otherwise stated) we will consider only the symmetric CMC surfaces of Table 4.3.

4.1 Lifting the SO(4)-action to a SU(2) × SU(2)-action

Let M be a symmetric CMC surface with symmetry group $G \subset \mathbf{SO}(4)$. In order to construct a DPW potential for M, we will need to lift the group action of G on a rank 2 holomorphic vector bundle $E \to M$ which induces an action on flat $\mathbf{SL}(2,\mathbb{C})$-connections on E. Therefore, it is better to consider the action of a subgroup $\Gamma \subset \mathbf{SU}(2) \times \mathbf{SU}(2)$ on the surface M which acts as the group G.

This can be realized as follows: Let $\varepsilon : \mathbf{SU}(2) \times \mathbf{SU}(2) \to \mathbf{SO}(4)$ the 2-fold spin convering of $\mathbf{SO}(4)$ ([23, Section 3.17]). An element $g \in G$ acts on M as

$$g \cdot p \mapsto apb, \tag{4.4}$$

for some $a, b \in \mathbf{SU}(2)$ and $p \in M \subset \mathbb{S}^3 \simeq \mathbf{SU}(2)$.

The finite subgroup $\Gamma := \varepsilon^{-1}(G) \subset \mathbf{SU}(2) \times \mathbf{SU}(2)$ acts on M in the same way as G. However, the Γ-action on M is not faithful (for example $-\mathrm{Id} \in \Gamma$ acts as the identity element of G). We will need a faithful action to construct a DPW potential for the immersion $f : M \to \mathbb{S}^3$.

We want to show that it is possible to define a Riemann surface $\tilde{M}$, related to M, on which Γ acts faithfully.

Let G be the Galois group of the covering map $\pi : M \to \mathbb{CP}^1$, with a presentation

$$G = \langle g_1, \ldots, g_4 \mid g_1 \cdots g_4 = 1 \rangle. \tag{4.5}$$

Consider $z_0 \in \mathbb{CP}^1 \smallsetminus \{z_1, \ldots, z_4\}$, where $z_1, \ldots, z_4$ are the branch points of π, and let

$$\rho : \pi_1(\mathbb{CP}^1 \smallsetminus \{z_1, \ldots, z_4\}, z_0) \to G \tag{4.6}$$

be its monodromy representation (cf. Example 1.1). A simple loop η_j around the point z_j is mapped to the generator g_j of G, for $j = 1, \ldots, 4$.

Geometrically, we can interpretate this as follows: If d_j is the order of the element $g_j \in G$, the loop $\eta_j^{d_j}$ in $\mathbb{CP}^1$ is lifted to a closed loop around the points in $\pi^{-1}(z_j)$. Thus, the surface M closes, locally around each of the point in $\pi^{-1}(z_j)$, after d_j rotations of angle $\frac{2\pi}{d_j}$.

Let $\gamma_j \in \varepsilon^{-1}(g_j)$, $j = 1, \ldots, 4$, we can define a group homomorphism

$$\tilde{\rho} : \pi_1(\mathbb{CP}^1 \smallsetminus \{z_1, \ldots, z_4\}, z_0) \to \Gamma \tag{4.7}$$

which maps the simple loop η_j around z_j to the element γ_j. Moreover, after an appropriate choice of the elements $\gamma_1, \ldots, \gamma_4$, we have

$$\gamma_1 \cdots \gamma_4 = 1, \tag{4.8}$$

where 1 is the identity element of Γ.

Proposition 4.1. *Let M be a symmetric CMC surface, $G \subset \boldsymbol{SO}(4)$ its symmetry group and $\tilde{\rho} : \pi_1(\mathbb{CP}^1 \smallsetminus \{z_1, \ldots, z_4\}, z_0) \to \Gamma$ the group homomorphism defined in (4.7). There exists a Riemann surface $\tilde{M}$ such that:*

(1) *The group* Γ *acts faithfully on* $\tilde{M}$ *and the quotient* $\tilde{M}/\Gamma$ *is the Riemann sphere* $\mathbb{CP}^1$*;*

(2) *The holomorphic map* $\tilde{\pi} : \tilde{M} \to \tilde{M}/\Gamma$ *is branched at the points* $z_1, \ldots, z_4 \in \mathbb{CP}^1$ *and its monodromy representation is given by* $\tilde{\rho}$*;*

(3) *There exists a holomorphic covering map* $\tau : \tilde{M} \to M$ *of degree 2, branched at the fixed points of the action of* G *on* M*.*

Proof. The existence of the Riemann surface $\tilde{M}$ and the conditions (1) and (2) come from the Riemann's existence Theorem 1.4 and its proof (see, for example, [52, Chapter 3 pp. 90-91]).

Condition (3) comes from the definition of the surface M via the Riemann's existence Theorem using the group homomorphism ρ in (4.6) and from the fact that the group $\Gamma \subset \mathbf{SU}(2) \times \mathbf{SU}(2)$ double covers the group $G \subset \mathbf{SO}(4)$.

□

In the next Subsections we will consider the symmetric surfaces of Table 4.3. We will describe the group $\Gamma \subset \mathbf{SU}(2) \times \mathbf{SU}(2)$ acting on each of them and the monodromy representation of the holomorphic map $\tilde{\pi} : \tilde{M} \to \mathbb{CP}^1$ defined in Proposition 4.1.

4.1.1 Lawson's $(d-1, 1)$ surfaces

Let M be the Lawson's surface $\Sigma_{d-1,1}$ of genus $g = d - 1$ (cf. Subsection 2.5.1). We consider the action of the group $\mathbb{Z}_d \subset \mathbf{SO}(4)$ on M. The group $\Gamma \subset \mathbf{SU}(2) \times \mathbf{SU}(2)$ double covering G is the cyclic group $\mathbb{Z}_{2d}$. We denote with $P_1, \ldots, P_4 \in M$ the fixed points of the action of $\mathbb{Z}_d$ on M.

Let $\rho : \pi_1(\mathbb{CP}^1 \smallsetminus \{z_1, \ldots, z_4\}, z_0) \to \mathbb{Z}_d$ be the monodromy representation of the covering map $\pi : M \to \mathbb{CP}^1$, branched at the points $z_1, \ldots, z_4 \in \mathbb{CP}^1$. We denote with $g_j \in \mathbb{Z}_d$ the image under ρ, of a simple loop η_j based in z_0 around the point z_j, for $j = 1, \ldots 4$.

From the construction of the surface M in Subsection 2.5.1, it follows that the elements $g_1, \ldots, g_4 \in \mathbb{Z}_d$ satisfy the following conditions

$$g_2 = g_1^{-1}, \quad g_3 = g_1, \quad g_4 = g_1^{-1}. \tag{4.9}$$

We first assume that d is an even number. Let $\tau : \tilde{M} \to M$ be the double covering map of Proposition 4.1. It is possible to find a local coordinate around the point $P_1 \in M$ and a local coordinate w around the point $\tilde{P}_1 := \tau^{-1}(P_1)$ such that $w^2 = z$ ([21, Proposition 5 p. 43]).

We can assume that g_1 acts, locally around P_1, as

$$g_1(z) = e^{\frac{2\pi i}{d}} z. \tag{4.10}$$

If $\tilde{\gamma}_1, \gamma_1$ are the preimages of g_1 in $\mathbb{Z}_{2d}$, they both have order $2d$ and one of them, for example $\tilde{\gamma}_1$, acts around $\tilde{P}_1$ as

$$\tilde{\gamma}_1(w) = e^{\frac{2\pi i}{2d}} w. \tag{4.11}$$

Around the point $P_2 \in M$, the element g_1 acts as a rotation of and angle $\frac{2\pi}{d}$ in the opposite direction of the rotation around P_1 (cf. Equation (4.9)), that is

$$g_1(\tilde{z}) = e^{\frac{2\pi i(d-1)}{d}} \tilde{z} \tag{4.12}$$

where $\tilde{z}$ is a local coordinate on M around P_2. Therefore, $\tilde{\gamma}_1$ act around $\tilde{P}_2 := \tau^{-1}(P_2)$ as

$$\tilde{\gamma}_1(\tilde{w}) = e^{\frac{2\pi i(d-1)}{2d}} \tilde{w}, \tag{4.13}$$

where $\tilde{w}$ is a local coordinate on $\tilde{M}$ around $\tilde{P}_2$ such that $\tilde{w}^2 = \tilde{z}$.

Up to rotations and sign, we can consider the representation of Γ in $\mathbf{SU}(2) \times \mathbf{SU}(2)$ such that (on the first factor)

$$\tilde{\gamma}_1 \mapsto \begin{pmatrix} e^{\frac{2\pi i}{2d}} & 0 \\ 0 & e^{-\frac{2\pi i}{2d}} \end{pmatrix}. \tag{4.14}$$

From the properties of the covering map $\varepsilon : \mathbf{SU}(2) \times \mathbf{SU}(2) \to \mathbf{SO}(4)$, the element $\gamma_1 \in \varepsilon^{-1}(g_1)$ is mapped, under the same representation, to the element whose first factor is given by

$$\gamma_1 \mapsto -\begin{pmatrix} e^{\frac{2\pi i}{2d}} & 0 \\ 0 & e^{-\frac{2\pi i}{2d}} \end{pmatrix} = \begin{pmatrix} e^{\frac{2\pi i(d+1)}{2d}} & 0 \\ 0 & e^{\frac{2\pi i(d-1)}{2d}} \end{pmatrix}. \tag{4.15}$$

Because $\mathbb{Z}_{2d}$ is a cyclic group generated by $\tilde{\gamma}_1$, Equation (4.15) implies that $\gamma_1 = \tilde{\gamma}^{d-1}$. Therefore, γ_1 acts around $\tilde{P}_1$ as

$$\gamma_1(w) = e^{\frac{2\pi i(d-1)}{2d}} w, \tag{4.16}$$

and around $\tilde{P}_2$ as

$$\gamma_1(\tilde{w}) = e^{\frac{2\pi i(d+1)}{2d}} \tilde{w}. \tag{4.17}$$

After analogous consideration on the action of $\mathbb{Z}_d$ around the points P_3 and P_4 and the action of $\mathbb{Z}_{2d}$ around the points $\tilde{P}_3 := \tau^{-1}(P_3)$ and $\tilde{P}_4 := \tau^{-1}(P_4)$, we define the monodromy representation $\tilde{\rho} : \pi_1(\mathbb{CP}^1 \smallsetminus \{z_1, \ldots, z_4\}, z_0) \to \mathbb{Z}_{2d}$ such that it maps simple loops $\eta_1, \ldots, \eta_4$ around the points $z_1, \ldots, z_4$ to the elements $\gamma_1, \gamma_1^{-1}, \gamma_1, \gamma_1^{-1}$, respectively,

Let now d be an odd number and g_1 one of the generator of $\mathbb{Z}_d$. The preimages $\tilde{\gamma}_1, \gamma_1$ of g_1 in $\mathbb{Z}_{2d}$ are such that one has order $2d$ and the other has order d. Without loss of generality, we assume that $\tilde{\gamma}_1$ has order $2d$ and γ_1 order d.

Thus, there is a 1:1 correspondence between $\mathbb{Z}_d \subset \mathbf{SO}(4)$ and a subgroup of $\mathbb{Z}_{2d} \subset \mathbf{SU}(2) \times \mathbf{SU}(2)$, obtained by mapping the generators g_j of $\mathbb{Z}_d$ to the elements $\gamma_j \in \mathbb{Z}_{2d}$ having the same order d.

Up to rotations and sign, we can consider the representation of $\mathbb{Z}_{2d}$ in $\mathbf{SU}(2) \times \mathbf{SU}(2)$ such that (on the first factor)

$$\tilde{\gamma}_1 \mapsto \begin{pmatrix} e^{\frac{2\pi i}{2d}} & 0 \\ 0 & e^{-\frac{2\pi i}{2d}} \end{pmatrix}, \quad \gamma_1 \mapsto \begin{pmatrix} e^{\frac{2\pi i(d+1)}{2d}} & 0 \\ 0 & e^{\frac{2\pi i(d-1)}{2d}} \end{pmatrix} \tag{4.18}$$

Therefore, γ_1 is given by $\tilde{\gamma}_1^{d-1}$ and the monodromy representation $\tilde{\rho} : \pi_1(\mathbb{CP}^1 \setminus \{z_1, \dots, z_4\}, z_0) \to \mathbb{Z}_{2d}$ is defined in the same way as for the case of d being even.

The local eigenvalues of the monodromy representation $\tilde{\rho}$ at the points $z_1, \dots, z_4$ are given by

$$\frac{d-1}{2d}, \quad \text{and} \quad \frac{d+1}{2d}. \tag{4.19}$$

4.1.2 Lawson's (k, l) surfaces

Let M be the Lawson's surface $\Sigma_{k-1,l-1}$ of genus $(k-1)(l-1)$ equipped with the action of the group $\mathbb{Z}_k \times \mathbb{Z}_l \subset \mathbf{SO}(4)$ (cf. Subsection 2.5.1). Let $P_1, \dots, P_{2l+2} \in M$ be the points with stabilizer group $\mathbb{Z}_k$ and $Q_1, \dots, Q_{2k+2} \in M$ the points with stabilizer group $\mathbb{Z}_l$.

From the construction of M in Subsection 2.5.1, it follows that the generator g_1 for the stabilizer group of one point P_{2r+1}, $r = 0, \dots, l$ acts locally as a rotation around them. The generator g_2 for the stabilizer group of the points P_{2r+2}, $r = 0, \dots, l$ acts locally as g_1^{-1}. The same holds for the generators g_3 and g_4 for the stabilizer groups of the points Q_{2s+1} and Q_{2s+2}, $s = 0, \dots, k$, respectively.

We consider the presentation of $\mathbb{Z}_k \times \mathbb{Z}_l$ given by

$$\mathbb{Z}_k \times \mathbb{Z}_l = \langle g_1, \dots, g_4 \mid g_1^k = g_2^k = g_3^l = g_4^l = 1, g_1 \cdots g_4 = 1 \rangle, \tag{4.20}$$

where $g_2 = g_1^{-1}$ and $g_4 = g_3^{-1}$.

Let $\tau : \tilde{M} \to M$ be the double covering defined in Proposition 4.1. The map τ is branched at the points P_r and Q_s for $r = 1, \dots, 2l+2$ and $s = 1, \dots, 2k+2$. The group $\Gamma \subset \mathbf{SU}(2) \times \mathbf{SU}(2)$ double covering $\mathbb{Z}_k \times \mathbb{Z}_l$ is given by the group $\mathbb{Z}_{2k} \times \mathbb{Z}_{2l}$.

Assume that k is an odd number and l is an even number. The preimages $\tilde{\gamma}_1, \gamma_1 \in \mathbb{Z}_{2k} \times \mathbb{Z}_{2l}$ of the generators g_1 for the stabilizer group of the point P_1 (the situation for the points P_{2r+1}, $r = 1, \dots, l$ is analogous) satisfy

$$\tilde{\gamma}_1^{2k} = \gamma_1^k = 1. \tag{4.21}$$

An argument analogous to the one used in Subsection 4.1.1, shows that there is a 1:1 correspondence between the stabilizer group of P_1 and a subgroup of the stabilizer group $\mathbb{Z}_{2k} \subset \mathbb{Z}_{2k} \times \mathbb{Z}_{2l}$ of the point $\tilde{P}_1 := \tau^{-1}(P_1) \in \tilde{M}$.

Let z be a local coordinate around P_1 and w a local coordinate around $\tilde{P}_1$ such that $w^2 = z$. The element g_1 acts around P_1 as

$$g_1(z) = e^{\frac{2\pi i}{k}} z \tag{4.22}$$

and the element $\tilde{\gamma}_1$ acts around $\tilde{P}_1$ as

$$\tilde{\gamma}_1(w) = e^{\frac{2\pi i}{2k}} w. \tag{4.23}$$

Up to rotations and sign, we can consider the representation of $\mathbb{Z}_{2k} \times \mathbb{Z}_{2l}$ in $\mathbf{SU}(2) \times \mathbf{SU}(2)$ such that (on the first factor)

$$\tilde{\gamma}_1 \mapsto \begin{pmatrix} e^{\frac{2\pi i}{2k}} & 0 \\ 0 & e^{-\frac{2\pi i}{2k}} \end{pmatrix}. \tag{4.24}$$

Therefore, the element γ_1 is mapped to the element whose first factor is given by

$$\gamma_1 \mapsto -\begin{pmatrix} e^{\frac{2\pi i}{2k}} & 0 \\ 0 & e^{-\frac{2\pi i}{2k}} \end{pmatrix} = \begin{pmatrix} e^{\frac{2\pi i(k+1)}{2k}} & 0 \\ 0 & e^{\frac{2\pi i(k-1)}{2k}} \end{pmatrix}. \tag{4.25}$$

Since $g_2 = g_1^{-1}$, the preimages $\tilde{\gamma}_2, \gamma_2 \in \mathbb{Z}_{2k} \times \mathbb{Z}_{2l}$ of g_2 are such that γ_2 is mapped, under the representation of $\mathbb{Z}_{2k} \times \mathbb{Z}_{2l}$ in $\mathbf{SU}(2) \times \mathbf{SU}(2)$, to the element whose first factor is given by

$$\gamma_2 \mapsto \begin{pmatrix} e^{\frac{2\pi i(k-1)}{2k}} & 0 \\ 0 & e^{\frac{2\pi i(k+1)}{2k}} \end{pmatrix}. \tag{4.26}$$

The local eigenvalues of the monodromy representation $\tilde{\rho} : \pi_1(\mathbb{CP}^1 \smallsetminus \{z_1, \ldots, z_4\}, z_0) \to \mathbb{Z}_{2k} \times \mathbb{Z}_{2l}$ mapping the simple loops η_1, η_2 around z_1, z_2 respectively to γ_1, γ_2, are given by

$$\frac{k+1}{2k}, \quad \text{and} \quad \frac{k-1}{2k}. \tag{4.27}$$

Consider now the generator g_3 of the stabilizer group of the point Q_1 (the situation for the points Q_{2s+1}, $s = 1, \ldots, k$ is analogous). Its preimages $\tilde{\gamma}_3, \gamma_3 \in \mathbb{Z}_{2k} \times \mathbb{Z}_{2l}$ satisfy

$$\tilde{\gamma}_3^{2l} = \gamma_3^{2l} = 1. \tag{4.28}$$

Let z be a local coordinate around Q_1 and w a local coordinate around $\tilde{Q}_1 := \tau^{-1}(Q_1) \in \tilde{M}$, such that $w^2 = z$. The element g_3 acts, locally around Q_1, as

$$g_3(z) = e^{\frac{2\pi i}{l}} z. \tag{4.29}$$

Then, the element $\tilde{\gamma}_3$ acts around $\tilde{Q}_1$ as

$$\tilde{\gamma}_3(w) = e^{\frac{2\pi i}{2l}} w. \tag{4.30}$$

Moreover, the element $\tilde{\gamma}_3$ acts as a rotation of the same angle, but in the opposite direction, around $\tilde{Q}_2 := \tau^{-1}(Q_2) \in \tilde{M}$ (and around the points $\tilde{Q}_{2s+2} := \tau^{-1}(Q_{2s+2})$ for $s = 1, \ldots, k$ in the same way):

$$\tilde{\gamma}_3(\tilde{w}) = e^{\frac{2\pi i(2l-1)}{2l}} \tilde{w}, \tag{4.31}$$

where $\tilde{w}$ is a local coordinate around $\tilde{Q}_2$.

Up to rotations and sign, we can consider the representation of $\mathbb{Z}_{2k} \times \mathbb{Z}_{2l}$ in $\mathbf{SU}(2) \times \mathbf{SU}(2)$ such that (on the first factor)

$$\tilde{\gamma}_3 \mapsto \begin{pmatrix} e^{\frac{2\pi i l}{2l}} & 0 \\ 0 & e^{-\frac{2\pi i l}{2l}} \end{pmatrix}. \tag{4.32}$$

Under the same representation, we obtain

$$\gamma_3 \mapsto \begin{pmatrix} e^{\frac{2\pi i(l+1)}{2l}} & 0 \\ 0 & e^{\frac{2\pi i(l-1)}{2l}} \end{pmatrix} \tag{4.33}$$

and the element $\gamma_4 \in \varepsilon^{-1}(g_4)$ is mapped to the same element as γ_3^{-1}.

The local eigenvalues of the monodromy representation $\tilde{\rho}$ mapping the simple loops η_3, η_4 around the points $z_3, z_4 \in \mathbb{CP}^1$ to the elements $\gamma_3, \gamma_4 \in \mathbb{Z}_{2k} \times \mathbb{Z}_{2l}$, respectively, are given by

$$\frac{l+1}{2l}, \quad \text{and} \quad \frac{l-1}{2l}. \tag{4.34}$$

The computations for the case of k and l being both odd (or both even) numbers can be carry out analogously and the local eigenvalues of the monodromy representation $\tilde{\rho}$ are given by (4.27) and (4.34).

4.1.3 Platonic KPS surfaces

We consider now the platonic KPS surfaces described in Section 2.6. We will carry out the computations for the platonic KPS surface M of genus 3 with symmetry group $A_4 \subset \mathbf{SO}(4)$ (the alternating group of order 4) of order 12. The computations for the other platonic KPS surfaces can be done in an analogous way.

We consider the presentation of A_4 given by

$$A_4 = \langle a, b | a^3 = b^3 = (ab)^2 = 1 \rangle. \tag{4.35}$$

Let $g_1 = a, g_2 = a^{-1}, g_3 = ab, g_4 = (ab)^{-1}$, which give a presentation of A_4 of the form

$$\langle g_1, \ldots, g_4 | g_1 \cdots g_4 = 1 \rangle. \tag{4.36}$$

The finite subgroup $\Gamma \subset \mathbf{SU}(2) \times \mathbf{SU}(2)$ which double covers A_4 is the *binary tetrahedral group* A_4^* of order 24, having a presentation

$$A_4^* = \langle x, y | x^3 = y^3 = (xy)^2 = -1 \rangle. \tag{4.37}$$

Let $\pi : M \to \mathbb{CP}^1$ be the $12:1$ covering map branched at the points $z_1, \ldots, z_4 \in \mathbb{CP}^1$. From the construction of the platonic KPS surfaces in Section 2.6 we have

$$\begin{aligned} \pi^{-1}(z_1) &= \{P_1, P_3, P_5, P_7\} \\ \pi^{-1}(z_2) &= \{P_2, P_4, P_6, P_8\} \\ \pi^{-1}(z_3) &= \{Q_1, Q_3, Q_5, Q_7, Q_9, Q_{11}\} \\ \pi^{-1}(z_4) &= \{Q_2, Q_4, Q_6, Q_8, Q_{10}, Q_{12}\}, \end{aligned} \tag{4.38}$$

where the points P_r have branch order 2, $r = 1, \ldots, 8$ and the points Q_s have branch order 1, $s = 1, \ldots, 12$.

The stabilizer group of the points P_r acts locally around P_r as a cyclic group of order 3. The stabilizer group of the points Q_s acts locally as a cyclic group of order 2.

Let $\tau : \tilde{M} \to M$ be the double covering map defined in Proposition 4.1. The map τ is branched at the points P_r, $r = 1, \ldots, 8$ and at the points Q_s, $s = 1, \ldots, 12$.

Consider the point $P_1 \in M$ (for the other points P_3, P_5, P_7 the situation is the same) and let z be a local coordinate around it and w a local coordinate around $\tilde{P}_1 := \tau^{-1}(P_1) \in \tilde{M}$ such that $w^2 = z$. The element $g_1 \in A_4$ acts around P_1 as

$$g_1(z) = e^{\frac{2\pi i}{3}} z. \tag{4.39}$$

The preimages $\tilde{\gamma}_1, \gamma_1 \in A_4^*$ of g_1, under the double covering $\varepsilon : \mathbf{SU}(2) \times \mathbf{SU}(2) \to \mathbf{SO}(4)$ satisfy

$$\tilde{\gamma}_1^6 = \gamma_1^3 = 1. \tag{4.40}$$

The element $\tilde{\gamma}_1$ acts around $\tilde{P}_1$ as

$$\tilde{\gamma}_1(w) = e^{\frac{2\pi i}{6}} w. \tag{4.41}$$

We can choose the element $\tilde{\gamma}_1$ to be the element x in the presentation (4.37) of A_4^*, and γ_1 to be the element $x^2 \in A_4^*$ of order 3.

Up to rotations and sign, we can consider the representation of the stabilizer group of $\tilde{P}_1$ in $\mathbf{SU}(2) \times \mathbf{SU}(2)$ such that (on the first factor)

$$\tilde{\gamma}_1 \mapsto \begin{pmatrix} e^{\frac{2\pi i}{6}} & 0 \\ 0 & e^{-\frac{2\pi i}{6}} \end{pmatrix}. \tag{4.42}$$

From the properties of the covering map ε, the element γ_1 is mapped to the element in $\mathbf{SU}(2) \times \mathbf{SU}(2)$ whose first factor is given by

$$\gamma_1 \mapsto - \begin{pmatrix} e^{\frac{2\pi i}{6}} & 0 \\ 0 & e^{-\frac{2\pi i}{6}} \end{pmatrix} = \begin{pmatrix} e^{\frac{4\pi i}{3}} & 0 \\ 0 & e^{\frac{2\pi i}{3}} \end{pmatrix}. \tag{4.43}$$

Since the element g_2 acts, locally around the point P_2 (the same holds for the points $P_4, P_6, P_8 \in M$), as g_1^{-1}, the preimage $\gamma_2 \in A_4^*$ of g_2 has order 3 and acts as γ_1^{-1}. Therefore, under the representation in $\mathbf{SU}(2) \times \mathbf{SU}(2)$, the element γ_2 is mapped to the element, whose first factor is given by

$$\gamma_2 \mapsto \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{\frac{4\pi i}{3}} \end{pmatrix}. \tag{4.44}$$

Consider the point Q_1 (for the other points Q_{2s+1}, $s = 1, \dots, 5$ the situation is analogous) and the action of the element g_3 around it. After an appropriate choice of a local coordinate z around Q_1, we can assume that g_3 acts locally as

$$g_3(z) = e^{\frac{2\pi i}{2}} z. \tag{4.45}$$

Let $\tilde{\gamma}_3, \gamma_3 \in A_4^*$ be the preimages of g_3 under ε, which satisfy

$$\tilde{\gamma}_3^4 = \gamma_3^4 = 1. \tag{4.46}$$

We can assume that either $\tilde{\gamma}_3$ or γ_3 is the element xy in the presentation (4.37) of A_4^*. Let γ_3 be such element. The element $\tilde{\gamma}_3$ acts around $\tilde{Q}_1 := \tau^{-1}(Q_1)$ as

$$\tilde{\gamma}_3(w) = e^{\frac{2\pi i}{4}} w. \tag{4.47}$$

Up to rotations and sign, we can consider the representation of the stabilizer group of $\tilde{Q}_1$ in $\mathbf{SU}(2) \times \mathbf{SU}(2)$ such that (on the first factor)

$$\tilde{\gamma}_3 \mapsto \begin{pmatrix} e^{\frac{2\pi i}{4}} & 0 \\ 0 & e^{-\frac{2\pi i}{4}} \end{pmatrix}. \tag{4.48}$$

The element γ_3 is then mapped to the element whose first factor is given by

$$\gamma_3 \mapsto - \begin{pmatrix} e^{\frac{2\pi i}{4}} & 0 \\ 0 & e^{-\frac{2\pi i}{4}} \end{pmatrix} = \begin{pmatrix} e^{\frac{6\pi i}{4}} & 0 \\ 0 & e^{\frac{2\pi i}{4}} \end{pmatrix}. \tag{4.49}$$

From the construction of the surface M (cf. Section 2.6), the element $g_4 \in A_4$ acts around the point Q_2 (the same holds for the points Q_{2s+2}, $s = 1, \dots, 5$) as the element g_3^{-1}. Thus, around the point $\tilde{Q}_2 := \tau^{-1}(Q_2) \in \tilde{M}$ the preimage $\gamma_4 \in A_4^*$ of g_4 acts as γ_3^{-1} and it is mapped to the element of $\mathbf{SU}(2) \times \mathbf{SU}(2)$ whose first factor is given by

$$\gamma_4 \mapsto \begin{pmatrix} e^{\frac{2\pi i}{4}} & 0 \\ 0 & e^{\frac{6\pi i}{4}} \end{pmatrix}. \tag{4.50}$$

We can conclude that the local eigenvalues of the monodromy representation

$$\tilde{\rho} : \pi_1(\mathbb{CP}^1 \setminus \{z_1, \dots, z_4\}, z_0) \to A_4^* \tag{4.51}$$

which maps a simple loop η_j around the point z_j to the element γ_j described above, for $j = 1, \dots, 4$, are given by

$$\begin{cases} \frac{1}{3}, \frac{2}{3} & \text{at } z_1, z_2 \\ \frac{1}{4}, \frac{3}{4} & \text{at } z_3, z_4. \end{cases} \tag{4.52}$$

It is possible to write the local eigenvalues of $\tilde{\rho}$ in the same form of the local eigenvalues of the monodromy representation of the Lawson's surfaces described in Subsections 4.1.1 and 4.1.2:

$$\frac{d_j - 1}{2d_j} \quad \text{and} \quad \frac{d_j + 1}{2d_j}, \tag{4.53}$$

where d_j is the order of the stabilizer group of the points in $\tilde{\pi}^{-1}(z_j)$, $j = 1, \dots, 4$.

Moreover, the elements $\gamma_1, \dots, \gamma_4 \in A_4^*$ generates the whole group and satisfy $\gamma_1 \cdots \gamma_4 = 1$. Therefore, we obtain a presentation of A_4^* of the form

$$\langle \gamma_1, \dots, \gamma_4 | \gamma_1 \cdots \gamma_4 = 1 \rangle. \tag{4.54}$$

The same arguments used above can be applied to the other platonic KPS surfaces and to the Octahedral and Icosahedral join surfaces. For the surfaces having symmetry group $S_4 \subset \mathbf{SO}(4)$, it is possible to consider the presentation

$$S_4 = \langle a, b | a^3 = b^4 = (ab)^2 = 1 \rangle. \tag{4.55}$$

The subgroup $\Gamma \subset \mathbf{SU}(2) \times \mathbf{SU}(2)$ double covering S_4 is the *binary octahedral group* S_4^* which has a presentation given by

$$S_4^* = \langle x, y | x^3 = y^4 = (xy)^2 = -1 \rangle. \tag{4.56}$$

For the surfaces having symmetry group $A_5 \subset \mathbf{SO}(4)$, it is possible to consider the presentation

$$A_5 = \langle a, b | a^3 = b^5 = (ab)^2 = 1 \rangle. \tag{4.57}$$

The subgroup $\Gamma \subset \mathbf{SU}(2) \times \mathbf{SU}(2)$ double covering A_5 is the *binary icosahedral group* A_5^* which has a presentation given by

$$A_5^* = \langle x, y | x^3 = y^5 = (xy)^2 = -1 \rangle. \tag{4.58}$$

4.2 The action of Γ on holomorphic vector bundles and connections

Let M be a symmetric CMC surface, $G \subset \mathbf{SO}(4)$ its symmetry group and $f : M \to \mathbb{S}^3$ the CMC immersion of M into $\mathbb{S}^3$. Consider the group action $\Gamma \times M \to M$, where $\Gamma = \varepsilon^{-1}(G) \subset \mathbf{SU}(2) \times \mathbf{SU}(2)$ (cf. Section 4.1) and the rank 2 complex vector bundle $E \to M$ where the associated family of flat $\mathbf{SL}(2, \mathbb{C})$-connections ∇^λ of the immersion f is defined (cf. Section 3.1).

For every element $\gamma \in \Gamma$ there is a biholomorphic map, which we denote with the same symbol, $\gamma : M \to M$, where

$$\gamma(p) := \gamma \cdot p, \quad p \in M. \tag{4.59}$$

It is possible to lift the action of Γ on M to an action of Γ on the complex vector bundle $E \to M$ as follows: for each $\gamma \in \Gamma$ we consider the pullback bundle $\gamma^* E \to M$ of E, which is isomorphic to E. Thus, there is a representation of the group Γ into the gauge group $\mathcal{G}$ (cf. Definition 3.1) of the complex vector bundle $E \to M$, given by

$$\begin{aligned} &\Gamma \to \mathcal{G} \\ &\gamma \mapsto g_\gamma, \end{aligned} \tag{4.60}$$

where $g_\gamma : E \to \gamma^* E \simeq E$.

The complex vector bundle $E \to M$, together with the action $\Gamma \times E \to E$, is called an *orbifold bundle* (we refer to [60] and [58] for more details about orbifold bundles).

The action of Γ on $E \to M$ induces an action of Γ on the space of sections of E. In fact, let s be a section of E, then

$$(\gamma \cdot s)(p) := g_{\gamma^{-1}}(p) s(p), \quad p \in M. \tag{4.61}$$

From the fact that (4.60) is a representation, the following hold:

$$\begin{aligned} g_1 &= \mathrm{Id} \\ g_{\gamma_1,\gamma_2} &= g_{\gamma_1} g_{\gamma_2}, \quad \forall \gamma_1,\gamma_2 \in \Gamma. \end{aligned} \tag{4.62}$$

Therefore, (4.61) is a well-defined Γ-action on the space of sections of E.

Definition 4.2. A section $s \in \Gamma(U \subset M, E)$ is Γ-*invariant* if

$$s(p) = (\gamma \cdot s)(p), \quad \forall p \in U, \gamma \in \Gamma. \tag{4.63}$$

We now consider the associated family of flat connections ∇^λ of the CMC immersion $f : M \to \mathbb{S}^3$. We first give the following

Definition 4.3. Let $E \to M$ a orbifold bundle over a compact Riemann surface M with symmetry group Γ. A flat connection ∇ on $E \to M$ is Γ-*equivariant* if, for every $\gamma \in \Gamma$ the gauge transformation $g_\gamma \in \mathcal{G}$ given by the image of γ under the representation (4.60) satisfies

$$\gamma^* \nabla = \nabla \cdot g_\gamma. \tag{4.64}$$

Let M be a symmetric CMC surface with symmetry group G. Consider the rank 2 complex vector bundle $E \to M$ described in Section 3.1 and the flat $\mathbf{SL}(2,\mathbb{C})$-connection $\nabla = d + \frac{1}{2} f^{-1} df$, where $f : M \to \mathbb{S}^3$ is the CMC immersion of M into $\mathbb{S}^3$.

Let $p \in M$ be a point with non trivial stabilizer group with respect to the action of $\Gamma = \varepsilon^{-1}(G) \subset \mathbf{SU}(2) \times \mathbf{SU}(2)$ on M. It is possible to normalize the immersion $f : M \to \mathbb{S}^3$ such that

$$f(p) = \mathrm{Id} \in \mathbb{S}^3 \simeq \mathbf{SU}(2). \tag{4.65}$$

Consider the action of $\gamma \in \Gamma$ on a neighbourhood of $p \in M$, we obtain (cf. (4.4))

$$\mathrm{Id} = f(p) = (\gamma \circ f)(p) = a f(p) b^{-1} = ab^{-1} \Rightarrow a = b, \quad a, b \in \mathbf{SU}(2). \tag{4.66}$$

Therefore, locally around a point p with non trivial stabilizer group, Γ acts on M by the conjugation of an element $b \in \mathbf{SU}(2)$.

Let $\omega = f^{-1} df$ be the connection 1-form of the connection ∇ on $E \to M$. Around the point $p \in M$, Γ acts on ω as

$$\begin{aligned} \gamma^* \omega &= (\gamma \circ f)^{-1} d(\gamma \circ f) \\ &= b f^{-1} b^{-1} b df b^{-1} \\ &= b f^{-1} df b^{-1} \\ &= b \omega b^{-1}. \end{aligned} \tag{4.67}$$

Thus, the local action of Γ on the connection ∇ is given by the constant gauge

$$\gamma \cdot \nabla = \nabla \cdot b = b \nabla b^{-1}. \tag{4.68}$$

Let Φ, Φ^* be the normalized $(1,0)$ and, respectively, $(0,1)$ components of the 1-form ω (cf. (3.16)):

$$\Phi = \frac{\lambda_2}{1+\lambda_2}(\omega - i * \omega), \quad \Phi^* = \frac{1}{1+\lambda_2}(\omega + i * \omega), \quad \lambda_2 \in \mathbb{S}^1 . \tag{4.69}$$

Using (4.67) it is possible to conclude that Γ acts, locally, on the associated family of flat $\mathbf{SL}(2,\mathbb{C})$-connections $\nabla^\lambda = \nabla + \lambda^{-1}\Phi - \lambda \Phi^*$ of the CMC immersion $f : M \to \mathbb{S}^3$ as

$$\gamma \cdot \nabla^\lambda = \nabla^\lambda \cdot b. \tag{4.70}$$

Therefore, we have proved the first part of the following

Proposition 4.2. *Let M be a symmetric CMC surface with symmetry group G, $\Gamma = \varepsilon^{-1}(G) \subset \boldsymbol{SU}(2) \times \boldsymbol{SU}(2)$. The associated family of flat $\boldsymbol{SL}(2,\mathbb{C})$-connections ∇^λ of the immersion $f : M \to \mathbb{S}^3$ is Γ-equivariant.*

Moreover, the pullback connections $\tau^*\nabla^\lambda$ is Γ-equivariant, where $\tau : \tilde{M} \to M$ is the holomorphic map of degree 2 defined in Proposition 4.1.

Proof. The only thing left to prove is the second part of the statement. From the construction of the holomorphic map τ and the Riemann surface $\tilde{M}$, we have the following commutative diagram

$$\begin{tikzcd} \Gamma \times \tilde{M} \arrow[r, "\gamma"] \arrow[d, "(\mathrm{Id},\tau)"] & \tilde{M} \arrow[d, "\tau"] \\ \Gamma \times M \arrow[r, "\gamma"] & M \end{tikzcd} \tag{4.71}$$

Given $\gamma \in \Gamma$, from the commutativity of the diagram (4.71), the equation (4.70) and the properties of the pullback we obtain

$$\begin{aligned} \gamma^*(\tau^*\nabla^\lambda) &= (\tau \circ \gamma)^*\nabla^\lambda = (\gamma \circ \tau)^*\nabla^\lambda \\ &= \tau^*(\gamma^*\nabla^\lambda) = \tau^*(\nabla^\lambda \cdot b) \\ &= \tau^*(b^{-1}\nabla^\lambda b) = b^{-1}\tau^*\nabla^\lambda b \\ &= (\tau^*\nabla^\lambda) \cdot b. \end{aligned} \tag{4.72}$$

□

4.3 A parabolic bundle on $\mathbb{CP}^1$ associated to $E \to M$

In this Section we will follow the work of Biswas in [6] and define a parabolic bundle over the Riemann sphere $\mathbb{CP}^1$ associated to the orbifold bundle $E \to M$ described in Section 4.2.

We will use this construction to define a λ-family of parabolic bundles on $\mathbb{CP}^1$, for $\lambda \in \mathbb{C}$, on which it is possible to define a λ-family of Fuchsian systems (we refer to [3] for more details on Fuchsian systems on Riemann surfaces). Moreover, we will show how this family is related to the associated family of flat $\mathbf{SL}(2,\mathbb{C})$-connections of the symmetric CMC surface M.

We first give the following ([6, Definition 2.1 p. 306]):

Definition 4.4. Let $E \to M$ be a holomorphic vector bundle over a compact Riemann surface M and D an effective divisor on M (cf. Definition 1.19). A *quasi-parabolic structure* on E, with respect to D, is a filtration of sub-bundles

$$E = F_1(E) \supset F_2(E) \supset \cdots \supset F_l(E) \supset F_{l+1}(E) = E(-D), \tag{4.73}$$

where $E(-D)$ is the bundle described in Subsection 1.4.2. The integer l is called the *length of the parabolic filtration* (4.73).

A *parabolic structure* on E is given by a quasi-parabolic structure together with a system of *parabolic weights* $\{\alpha_1, \ldots, \alpha_l\}$, such that

$$0 \le \alpha_1 < \cdots < \alpha_l < 1. \tag{4.74}$$

The weight α_j corresponds to the sub-bundle $F_j(E) \subset E$.

A holomorphic vector bundle E equipped with a parabolic structure is called a *parabolic bundle*.

Let $E \to M$ be a parabolic bundle. It is possible to define a *continuous* version of its parabolic filtration as follows ([6, Section 2.1 pp. 306-307]): For any $t \in \mathbb{R}$ consider the sub-bundle

$$E_t = F_j(E)(-\lfloor t \rfloor D), \tag{4.75}$$

where $\lfloor t \rfloor$ is the integral part of t and the index j is such that

$$\alpha_{j-1} < t - \lfloor t \rfloor \le \alpha_j, \quad j = 2, \ldots l, \quad \alpha_0 = \alpha_l - 1 \text{ and } \alpha_{l+1} = 1. \tag{4.76}$$

The set $\{E_t\}_{t \in \mathbb{R}}$ gives a decreasing filtration of sub-bundles of E. In fact

$$E_t \subset E_{t'}, \quad \text{for } t \ge t'. \tag{4.77}$$

The parabolic filtration given by $\{E_t\}$ is also left continuous, that is, there exists a $\epsilon > 0$ such that

$$E_{t-\epsilon} = E_t \quad \forall t \in \mathbb{R}. \tag{4.78}$$

Moreover, $E_{t+1} = E_t(-D)$ ([6, Section 2.a]).

Definition 4.5. The filtration $\{E_t\}$ is said to have a *jump* in $t \in \mathbb{R}$ if, for any $\epsilon > 0$

$$E_{t+\epsilon} \neq E_t. \tag{4.79}$$

From the construction, it follows that the parabolic filtration $\{E_t\}$ has a jump if and only if

$$t - \lfloor t \rfloor = \alpha_j, \qquad \text{for some } j = 1, \dots, l. \tag{4.80}$$

Let M be a symmetric CMC surface with symmetry group $G \subset \mathbf{SO}(4)$ and $\tau : \tilde{M} \to M$ the holomorphic map between Riemann surfaces defined in Proposition 4.1. Consider the rank 2 holomorphic vector bundle $\tilde{E} := \tau^* E \to \tilde{M}$, where $E \to M$ is the holomorphic vector bundle where the associated family of flat $\mathbf{SL}(2, \mathbb{C})$-connections of M is defined (cf. Section 3.1).

From Proposition 4.1, there exists a holomorphic map $\tilde{\pi} : \tilde{M} \to \tilde{M}/\Gamma = \mathbb{CP}^1$ of degree $2d := |\Gamma|$, where $\Gamma = \varepsilon^{-1}(G) \subset \mathbf{SU}(2) \times \mathbf{SU}(2)$.

The push-forward bundle of $\tilde{E}$ under the map $\tilde{\pi}$ is a holomorphic vector bundle $\tilde{\pi}_* \tilde{E} \to \mathbb{CP}^1$ of rank $4d$ (cf. Subsection 1.4.3).

The action of Γ on $\tilde{E}$ (cf. Section 4.2) induces an action of Γ on $\tilde{\pi}_* \tilde{E}$ as follows: Let $U \subset \mathbb{CP}^1$ be an open set not containing any branch point of the map $\tilde{\pi}$ and $U_1, \dots, U_{2d}$ the disjoint open subsets of $\tilde{M}$ covering U. A holomorphic local frame for $\tilde{\pi}_* \tilde{E}$ over U is given by (with abuse of notation, cf. (1.49))

$$(s_1, t_1) \oplus \cdots \oplus (s_{2d}, t_{2d}), \tag{4.81}$$

where (s_j, t_j) is a local frame for $\tilde{E} \to \tilde{M}$ over the set U_j, $j = 1, \dots, 2d$.

The action of $\gamma \in \Gamma$ on the local frame (4.81) is given by

$$\gamma \cdot ((s_1, t_1) \oplus \cdots \oplus (s_{2d}, t_{2d})) = (g_{\gamma^{-1}} s_1, g_{\gamma^{-1}} t_1) \oplus \cdots \oplus (g_{\gamma^{-1}} s_{2d}, g_{\gamma^{-1}} t_{2d}), \tag{4.82}$$

where $g_\gamma \in \mathcal{G}$ is the image of the element γ under the representation 4.60.

Let now $U \subset \mathbb{CP}^1$ be an open set containing a branch point of $\tilde{\pi}$, which we denote with z_1. There exists a local coordinate z on $\mathbb{CP}^1$ around z_1 and a local coordinate w on $\tilde{M}$ around $p_1 \in \tilde{\pi}^{-1}(z_1)$ such that ([21, Proposition 5 p. 43])

$$w^{2d_1} = z, \tag{4.83}$$

where $2d_1$ is the order of the stabilizer group Γ_{p_1} of $p_1 \in \tilde{M}$ (cf. Section 4.1).

Let $(s,t) \in H^0(\tilde{\pi}^{-1}(U), \tilde{E})$ be a holomorphic local frame. Then, a holomorphic local frame for $\tilde{\pi}_*\tilde{E}$ over U is given by (cf. Example 1.6)

$$(s, t, ws, wt, \dots, w^{2d_1-1}s, w^{2d_1-1}t). \tag{4.84}$$

The stabilizer group Γ_{p_1} acts on $U \subset \tilde{M}$ as a cyclic group of order $2d_1$. It is possible to write the action of a generator γ of Γ_{p_1} on the sections s and t as (see the proof of [8, Proposition 2.2])

$$\begin{aligned} \gamma \cdot s(p) &= \alpha s(\gamma^{-1} \cdot p) \\ \gamma \cdot t(p) &= \alpha^{-1} s(\gamma^{-1} \cdot p), \end{aligned} \tag{4.85}$$

for every $p \in U$, where $\alpha = e^{\frac{2\pi i(d_1-1)}{2d_1}}$.

Therefore, the action of γ on the holomorphic local frame (4.84) is given by

$$\begin{aligned} &\gamma \cdot (s,t, ws, wt, \dots, w^{2d_1-1}s, w^{2d_1-1}t)(q) := \\ &\quad = (\alpha s(\gamma^{-1} \cdot p), \alpha^{-1}t(\gamma^{-1} \cdot p), \dots, \alpha^{2d_1-1}w^{2d_1-1}\alpha s(\gamma^{-1} \cdot p), \alpha^{2d_1-1}w^{2d_1-1}\alpha^{-1}t(\gamma^{-1} \cdot p)), \end{aligned} \tag{4.86}$$

where $q = \tilde{\pi}(p)$.

Following [6], we want to define a rank 2 sub-bundle of $\tilde{\pi}_*\tilde{E}$ on which it is possible to define a parabolic structure.

Consider the vector bundle map $\Omega : \tilde{\pi}_*\tilde{E} \to \tilde{\pi}_*\tilde{E}$ given, on each fiber, by

$$\begin{aligned} \Omega_p : (\tilde{\pi}_*\tilde{E})_p &\to (\tilde{\pi}_*\tilde{E})_p \\ v &\mapsto \sum_{\gamma \in \Gamma} \gamma \cdot v. \end{aligned} \tag{4.87}$$

From the definition, it follows that Ω satisfies

$$\Omega^2 = |\Gamma|\Omega, \tag{4.88}$$

which implies

$$\operatorname{Im}\Omega \cap \operatorname{Ker}\Omega = \{0\}. \tag{4.89}$$

In fact, if $s \in \Gamma(U, \tilde{\pi}_*\tilde{E})$ is a local section with $s \in \operatorname{Im}\Omega \cap \operatorname{Ker}\Omega$, there exists another section t such that $s = \Omega(t)$. The following computation

$$\begin{aligned} 0 = \Omega(s) &= \Omega(\Omega(t)) = \Omega^2(t) \\ &= |\Gamma|\Omega(t) = |\Gamma|s, \end{aligned} \tag{4.90}$$

implies $s = 0$.

It is also possible to prove that the bundle $\tilde{\pi}_*\tilde{E}$ is given by the direct sum

$$\tilde{\pi}_*\tilde{E} = \operatorname{Im}\Omega \oplus \operatorname{Ker}\Omega, \tag{4.91}$$

by writing a section s of $\tilde{\pi}_*\tilde{E}$ as

$$s = \frac{\Omega(s)}{|\Gamma|} + \left(s - \frac{\Omega(s)}{|\Gamma|}\right) \tag{4.92}$$

where $\frac{\Omega(s)}{|\Gamma|} \in \operatorname{Im}\Omega$ and $s - \frac{\Omega(s)}{|\Gamma|} \in \operatorname{Ker}\Omega$.

The image of Ω is generated by the Γ-invariant sections of $\tilde{\pi}_*\tilde{E}$ (cf. Definition 4.2). The sheaf of Γ-invariant sections of $\tilde{\pi}_*\tilde{E}$ is locally free of rank 2 ([6, Section 2.c]). Therefore, $\operatorname{Im}\Omega$ is a Γ-invariant rank 2 sub-bundle of $\tilde{\pi}_*\tilde{E}$, which we will denote with $(\tilde{\pi}_*\tilde{E})^\Gamma$.

Biswas in [6], considered the rank 2 vector bundle $(\tilde{\pi}_*\tilde{E})^\Gamma \to \mathbb{CP}^1$ and defined the parabolic structure on it, given by the parabolic filtration

$$\tilde{E}_t := \left(\tilde{\pi}_*\left(\tilde{E} \otimes L\Big(\sum_{j=1}^{4} \lfloor -2td_j \rfloor \tilde{\pi}^{-1}(z_j)\Big)\right)\right)^\Gamma, \quad t \in \mathbb{R}, \tag{4.93}$$

where $z_1, \ldots, z_4 \in \mathbb{CP}^1$ are the branch points of the map $\tilde{\pi} : \tilde{M} \to \mathbb{CP}^1$. The number $2d_j$ is the order of the stabilizer group $\Gamma_{p_j^k}$ of the points $p_j^k \in \tilde{\pi}^{-1}(z_j) \subset \tilde{M}$.

The action of Γ on the second factor of (4.93) can be described as follows: Let $D = \sum_{j=1}^n n_j z_j$ be an effective divisor on $\mathbb{CP}^1$ and s_{-D} the meromorphic frame of the line bundle $L(-D)$ (cf. Subsection 1.4.2). Locally, around the point z_j, Γ acts on s_{-D} as

$$\gamma \cdot s_{-D} = \beta s_{-D}(\gamma^{-1} \cdot z), \tag{4.94}$$

where $\beta = e^{\frac{2\pi i}{k_j}}$, γ is a generator of the stabilizer group Γ_{z_j} of the point z_j and $k_j = |\Gamma_{z_j}|$.

The parabolic weights $\alpha_0, \ldots, \alpha_l$ of the parabolic structure on $(\tilde{\pi}_*\tilde{E})^\Gamma$ given by (4.93) are given by the values of $t \in \mathbb{R}$ such that $\{\tilde{E}_t\}$ has a jump (cf. Definition 4.5).

In the next Section we will show that this parabolic structure on $(\tilde{\pi}_*\tilde{E})^\Gamma$ is equivalent to another parabolic structure defined using a singular connection $\tilde{\nabla}$ on $(\tilde{\pi}_*\tilde{E})^\Gamma$.

4.4 Logarithmic connections and parabolic structures on $(\tilde{\pi}_*\tilde{E})^\Gamma \to \mathbb{CP}^1$

Following the work of Biswas and Heu [8], we want to define a logarithmic connection on the holomorphic vector bundle $(\tilde{\pi}_*\tilde{E})^\Gamma \to \mathbb{CP}^1$ defined in Section 4.3, starting from a Γ-equivariant connection on $\tilde{E}$ (cf. Definition 4.3).

We first recall the following ([8, Section 2])

Definition 4.6. Let $E \to M$ be a holomorphic vector bundle on a compact Riemann surface M, $U \subset M$ a trivializing open set for $E \to M$ and K_M the canonical bundle of M . A *logarithmic connection* ∇, singular over an effective divisor D of M, is given by a linear map

$$\nabla : H^0(U, E) \to H^0(U, E \otimes K_M \otimes L(D)), \tag{4.95}$$

which satisfies the Leibniz rule

$$\nabla(fs) = f\nabla s + sdf, \tag{4.96}$$

for every holomorphic function $f : U \to \mathbb{C}$ and section $s \in H^0(U, E)$.

Consider a symmetric CMC surface M and the holomorphic map between Riemann surfaces $\tilde{\pi} : \tilde{M} \to \mathbb{CP}^1$ defined in Proposition 4.1. Let $\tau^*\nabla$ be the Γ-equivariant connection on the holomorphic vector bundle $\tilde{E} \to \tilde{M}$ described in Section 4.3.

We consider $\tau^*\nabla$ as a linear map, which satisfies the Leibniz rule

$$\tau^*\nabla : H^0(U, \tilde{E}) \to H^0(U, \tilde{E} \otimes K_{\tilde{M}}), \tag{4.97}$$

where $U \subset \tilde{M}$ is an open set.

There exists an inclusion map ([8, Lemma 2.1])

$$H^0(U, K_{\tilde{M}}) \hookrightarrow H^0(U, \tilde{\pi}^*(K_{\mathbb{CP}^1} \otimes L(D))), \tag{4.98}$$

where $D = z_1 + \cdots + z_4$ is the branch divisor of the map $\tilde{\pi} : \tilde{M} \to \mathbb{CP}^1$.

In fact, let $\{P_1^j, \ldots, P_{k_j}^j\} = \tilde{\pi}^{-1}(z_j), j = 1, \ldots, 4$, be the ramification points of $\tilde{\pi}$ and d_j the branch order of the points $P_l^j, l = 1, \ldots, k_j$.

The differential $d\tilde{\pi}$ can be considered as a section of the holomorphic vector bundle $\mathrm{Hom}(\tilde{\pi}^*K_{\mathbb{CP}^1}, K_{\tilde{M}})$, whose divisor is given by

$$\tilde{D} = d_1 \sum_{j=1}^{k_1} P_j^1 + d_2 \sum_{j=1}^{k_2} P_j^2 + d_3 \sum_{j=1}^{k_3} P_j^3 + d_4 \sum_{j=1}^{k_4} P_j^4. \tag{4.99}$$

The holomorphic line bundles $\tilde{\pi}^*K_{\mathbb{CP}^1} \otimes L(\tilde{D})$ and $K_{\tilde{M}}$ are isomorphic, since they correspond to the same divisor on $\tilde{M}$ (Subsection 1.4.2). In fact, the divisor of $\tilde{\pi}^*K_{\mathbb{CP}^1}$ is equal to $-2d$ (where d is the degree of the holomorphic map $\tilde{\pi} : \tilde{M} \to \mathbb{CP}^1$) and the Riemann Hurwitz formula ([52, Corollary 3.7 p. 80]) implies that the divisor of $K_{\tilde{M}}$ is given by

$$2\tilde{g} - 2 = b - 2d, \tag{4.100}$$

where $\tilde{g}$ is the genus of the Riemann surface $\tilde{M}$ and b is the total branch order of the map $\tilde{\pi}$.

Moreover, the pullback bundle $\tilde{\pi}^*(L(D))$ is given by the holomorphic line bundle $L(\hat{D})$, where

$$\hat{D} = (d_1+1)\sum_{j=1}^{k_1} P_j^1 + (d_2+1)\sum_{j=1}^{k_2} P_j^2 + (d_3+1)\sum_{j=1}^{k_3} P_j^3 + (d_4+1)\sum_{j=1}^{k_4} P_j^4. \tag{4.101}$$

Therefore, we obtain

$$\begin{aligned} K_{\tilde{M}} &\simeq \tilde{\pi}^* K_{\mathbb{CP}^1} \otimes L(\tilde{D}), \\ \tilde{\pi}^*(K_{\mathbb{CP}^1} \otimes L(D)) &= \tilde{\pi}^* K_{\mathbb{CP}^1} \otimes \tilde{\pi}^* L(D) \simeq \tilde{\pi}^* K_{\mathbb{CP}^1} \otimes L(\hat{D}), \end{aligned} \tag{4.102}$$

and the inclusion map

$$H^0(U, K_{\tilde{M}}) \hookrightarrow H^0(U, \tilde{\pi}^*(K_{\mathbb{CP}^1} \otimes L(D)))) \tag{4.103}$$

is the map whose divisor is given by

$$\sum_{j=1}^{k_1} P_j^1 + \sum_{j=1}^{k_2} P_j^2 + \sum_{j=1}^{k_3} P_j^3 + \sum_{j=1}^{k_4} P_j^4. \tag{4.104}$$

We can now prove the following (for a proof in sheaf theoretic terms see [8, Lemma 2.1]).

Proposition 4.3. *Let M be a symmetric CMC surface and $\tilde{\pi} : \tilde{M} \to \tilde{M}/\Gamma = \mathbb{CP}^1$ the holomorphic map defined in Proposition 4.1. There exists a logarithmic connection $\tilde{\nabla}$ on the vector bundle $(\tilde{\pi}_*\tilde{E})^\Gamma \to \mathbb{CP}^1$ described in Section 4.3.*

Proof. Let $U \subset \tilde{M}$ be an open set and $\tau^*\nabla : H^0(U, \tilde{E}) \to H^0(U, \tilde{E} \otimes K_{\tilde{M}})$ the Γ-equivariant connection on the holomorphic vector bundle $\tilde{E} \to \tilde{M}$ described in Section 4.3.

The composition of $\tau^*\nabla$ with the inclusion map (4.103) gives a map

$$h : H^0(U, \tilde{E}) \to H^0(U, \tilde{E} \otimes \tilde{\pi}^*(K_{\mathbb{CP}^1} \otimes L(D))), \tag{4.105}$$

where $D = z_1 + \cdots + z_4$ is the branch divisor of the map $\tilde{\pi}$.

From (1.46), it follows that the map h induces a map

$$\tilde{\pi}_*(\tau^*\nabla) : H^0(\tilde{U}, \tilde{\pi}_*\tilde{E}) \to H^0(\tilde{U}, \tilde{\pi}_*(\tilde{E} \otimes \tilde{\pi}^*(K_{\mathbb{CP}^1} \otimes L(D)))), \tag{4.106}$$

where $\tilde{U} \subset \mathbb{CP}^1$ is an open set such that $\tilde{\pi}(U) = \tilde{U}$.

Using (1.44), we obtain

$$H^0(\tilde{U}, \tilde{\pi}_*(\tilde{E} \otimes \tilde{\pi}^*(K_{\mathbb{CP}^1} \otimes L(D)))) = H^0(\tilde{U}, \tilde{\pi}_*\tilde{E} \otimes K_{\mathbb{CP}^1} \otimes L(D)). \tag{4.107}$$

We can conclude that the map

$$\tilde{\pi}_*(\tau^*\nabla) : H^0(\tilde{U}, \tilde{\pi}_*\tilde{E}) \to H^0(\tilde{U}, \tilde{\pi}_*\tilde{E} \otimes K_{\mathbb{CP}^1} \otimes L(D)) \tag{4.108}$$

satisfies the Leibniz rule, since $\tau^*\nabla$ satisfies it and gives a logarithmic connection on the holomorphic vector bundle $\tilde{\pi}_*\tilde{E}$.

Finally, the logarithmic connection $\tilde{\pi}_*(\tau^*\nabla)$ induces a logarithmic connection on the Γ-invariant bundle $(\tilde{\pi}_*\tilde{E})^\Gamma$. In fact, let $s \in H^0(\tilde{U}, \tilde{\pi}_*\tilde{E})$ be a Γ-invariant section and $\gamma \in \Gamma$, then

$$\begin{aligned} \tilde{\pi}_*(\tau^*\nabla)(s) &= \tilde{\pi}_*(\tau^*\nabla)(\gamma \cdot s) = \tau^*\nabla(\gamma \cdot s) \\ &= \gamma^*(\tau^*\nabla)(s) = \gamma^*(\tilde{\pi}_*(\tau^*\nabla))(s), \end{aligned} \tag{4.109}$$

where we have used the fact that s can be considered as a section of $\tilde{E}$ and that $\tau^*\nabla$ is Γ-equivariant.

We denote with $\tilde{\nabla}$ the logarithmic connection on $(\tilde{\pi}_*\tilde{E})^\Gamma$ induced by the connection $\tilde{\pi}_*(\tau^*\nabla)$.

□

4.4.1 The local residues of the connection $\tilde{\nabla}$

Let M be a symmetric CMC surface and $\tilde{\pi} : \tilde{M} \to \tilde{M}/\Gamma = \mathbb{CP}^1$ the holomorphic map between Riemann surfaces defined in Proposition 4.1, where $\Gamma \subset \mathbf{SU}(2) \times \mathbf{SU}(2)$ is the finite group acting on M and $\tilde{M}$.

We want to determine the local residues of the logarithmic connection $\tilde{\nabla}$ on the Γ-invariant vector bundle $(\tilde{\pi}_*\tilde{E})^\Gamma \to \mathbb{CP}^1$ (cf. Proposition 4.3) at the branch points $z_1, \ldots, z_4$ of the map $\tilde{\pi}$.

Let $p_1 \in \tilde{\pi}^{-1}(z_1) \subset \tilde{M}$ be a point with branch order $2d_1 - 1$ (cf. Section 4.1). There exists a local coordinate w around p_1 and a local coordinate around z_1 such that

$$w^{2d_1} = z. \tag{4.110}$$

Up to shrinking the domain of the coordinate w on $\tilde{M}$, we can consider w defined on a trivializing set $U \subset \tilde{M}$ for the holomorphic vector bundle $\tilde{E} \to \tilde{M}$. Therefore, over U we can consider $\tilde{E}$ to be the trivial bundle $\underline{\mathbb{C}}^2 \to U$, together with the Γ-equivariant trivial connection d (cf. Example 1.7).

Let (s, t) be a holomorphic local frame for $\underline{\mathbb{C}}^2 \to U$. The stabilizer group Γ_{p_1} of the point $p_1 \in \tilde{M}$ acts on U as the cyclic group $\mathbb{Z}_{2d_1}$ (cf. Subsections 4.1.1, 4.1.2, 4.1.3). The action of a generator γ of Γ_{p_1} on (s, t) is given by

$$\gamma \cdot (s(p), t(p)) = (\alpha s(\gamma^{-1} \cdot p), \alpha^{-1} t(\gamma^{-1} \cdot p)), \tag{4.111}$$

where $\alpha = e^{\frac{2\pi i(d_1-1)}{2d_1}}$ (cf. (4.85)).

A local frame for the vector bundle $(\tilde{\pi}_*\underline{\mathbb{C}})^{\mathbb{Z}_{2d_1}}$ around the point $z_1 \in \mathbb{CP}^1$ is given by (cf. (4.84))

$$(s, t, ws, wt, \dots, w^{2d_1-1}s, w^{2d_1-1}t). \tag{4.112}$$

The following computation shows that the sections $w^{d_1-1}s$ and $w^{d_1+1}t$ are both Γ_{p_1}-invariant

$$\begin{aligned}\gamma \cdot (w^{d_1-1}s(p), w^{d_1+1}t(p)) &= (\alpha^{d_1-1}\alpha w^{d_1-1}s(\gamma^{-1}\cdot p), \alpha^{d_1+1}\alpha^{-1}w^{d_1+1}t(\gamma^{-1}\cdot p)) \\ &= (\alpha^{d_1}w^{d_1-1}s(\gamma^{-1}\cdot p), \alpha^{d_1}w^{d_1+1}t(\gamma^{-1}\cdot p)) \\ &= (w^{d_1-1}s(\gamma^{-1}\cdot p), w^{d_1+1}t(\gamma^{-1}\cdot p)).\end{aligned} \tag{4.113}$$

Therefore, $(w^{d_1-1}s, w^{d_1+1}t)$ gives a local frame around $z_1 \in \mathbb{CP}^1$ for the Γ-invariant vector bundle $(\tilde{\pi}_*\tilde{E})^\Gamma \to \mathbb{CP}^1$.

The logarithmic connection $\tilde{\nabla}$ on $(\tilde{\pi}_*\tilde{E})^\Gamma$, induced by the trivial connection d over the open set $U \subset \tilde{M}$, is such that

$$\begin{aligned}\tilde{\nabla}(w^{d_1-1}s, w^{d_1+1}t) &= d(w^{d_1-1}s, w^{d_1+1}t) \\ &= ((d_1-1)w^{d_1-2}s, (d_1+1)w^{d_1}t)dw \\ &= ((d_1-1)w^{d_1-1}s, (d_1+1)w^{d_1+1}t)\frac{dw}{w}.\end{aligned} \tag{4.114}$$

Thus, with respect to the local frame $(w^{d_1-1}s, w^{d_1+1}t)$, the connection $\tilde{\nabla}$ is given by

$$\tilde{\nabla} = d + \begin{pmatrix} (d_1-1)\frac{dw}{w} & 0 \\ 0 & (d_1+1)\frac{dw}{w} \end{pmatrix}. \tag{4.115}$$

The 1-form $\frac{dw}{w}$ has residue $\frac{1}{2d_1}$ at the point $z_1 \in \mathbb{CP}^1$. Therefore, the local residue of the connection $\tilde{\nabla}$ at z_1 is given, with respect to the local frame $(w^{d_1-1}s, w^{d_1+1}t)$, by

$$\operatorname{Res}_{z_1}\tilde{\nabla} = \begin{pmatrix} \frac{(d_1-1)}{2d_1} & 0 \\ 0 & \frac{(d_1+1)}{2d_1} \end{pmatrix}. \tag{4.116}$$

Analogous computations shows that the residues of $\tilde{\nabla}$ at the other branch points $z_2, z_3, z_4 \in \mathbb{CP}^1$ of the map $\tilde{\pi}$ are given by

$$\operatorname{Res}_{z_j}\tilde{\nabla} = \begin{pmatrix} \frac{(d_j-1)}{2d_j} & 0 \\ 0 & \frac{(d_j+1)}{2d_j} \end{pmatrix}, \quad j = 2,3,4, \tag{4.117}$$

where $2d_j - 1$ is the branch oder of the points in $\tilde{\pi}^{-1}(z_j)$.

Let $U \subset \tilde{M}$ be the set given by

$$U := \bigcup_{j=1}^{4} \tilde{\pi}^{-1}(z_j). \tag{4.118}$$

Consider the regular connections $\tau^*\nabla$ and $\tilde{\pi}^*\tilde{\nabla}$ on $\tilde{M} \setminus U$ and a point $p_0 \in \tilde{M} \setminus U$. It is possible to obtain two representations of the fundamental group $\pi_1(\tilde{M} \setminus U, p_0)$ into $\mathbf{GL}(2, \mathbb{C})$ via the parallel transport of $\tau^*\nabla$ and $\tilde{\pi}^*\tilde{\nabla}$ along loops in $\pi_1(\tilde{M} \setminus U, p_0)$ (cf. (1.73)). Let ρ_1 (resp. ρ_2) be the representation corresponding to $\tau^*\nabla$ (resp. $\tilde{\pi}^*\tilde{\nabla}$).

From the construction of the connection $\tilde{\nabla}$ (cf. Proposition 4.3), it follows that the representations ρ_1 and ρ_2 are equivalent up to conjugation. Therefore, the Riemann-Hilbert correspondence ([36, Theorem 3.6]) implies that there exists a gauge transformation $\tilde{g}$ such that

$$\tau^*\nabla = \tilde{\pi}^*\tilde{\nabla} \cdot \tilde{g}, \tag{4.119}$$

where $\tilde{g}$ is defined on $\tilde{M} \setminus U$ ([36, Theorem 3.11]).

Moreover, [36, Corollary 3.13] implies that it is possible to extend the connection $\tilde{\pi}^*\tilde{\nabla}$ on $\tilde{M}$ such that it is still gauge equivalent to $\tau^*\nabla$.

We can summarize the above argument with the following

Proposition 4.4. *The connection $\tau^*\nabla$ on the holomorphic vector bundle $\tilde{E} \to \tilde{M}$, described in Section 4.3, is gauge equivalent to the pullback connection $\tilde{\pi}^*\tilde{\nabla}$ of the logarithmic connection defined on the Γ-invariant vector bundle $(\tilde{\pi}_*\tilde{E})^\Gamma \to \mathbb{CP}^1$, under the holomorphic map $\tilde{\pi} : \tilde{M} \to \mathbb{CP}^1$.*

4.4.2 The parabolic structure on $(\tilde{\pi}_*\tilde{E})^\Gamma \to \mathbb{CP}^1$ induced by the connection $\tilde{\nabla}$

It is possible to define a parabolic structure on the vector bundle $(\tilde{\pi}_*\tilde{E})^\Gamma \to \mathbb{CP}^1$ using the logarithmic connection $\tilde{\nabla}$ given by Proposition 4.3 ([32, Section 2]).

Let $\operatorname{Res}_{z_j} \tilde{\nabla} \in \operatorname{End}((\tilde{\pi}_*\tilde{E})^\Gamma)_{z_j}$ be the local residue of $\tilde{\nabla}$ at the branch point $z_j \in \mathbb{CP}^1$, $j = 1, \dots, 4$, of the holomorphic map $\tilde{\pi} : \tilde{M} \to \mathbb{CP}^1$.

The eigenvalues of $\operatorname{Res}_{z_j} \tilde{\nabla}$ are given, with respect to an appropriate local frame (cf. (4.116)), by

$$\mu_1^j = \frac{d_j - 1}{2d_j}, \quad \mu_2^j = \frac{d_j + 1}{2d_j}. \tag{4.120}$$

The eigenline L_j of $\operatorname{Res}_{z_j} \tilde{\nabla}$, corresponding to the higher eigenvalue μ_2^j, is given by

$$L_j := \operatorname{Ker}(\operatorname{Res}_{z_j} \tilde{\nabla} - \mu_2^j \operatorname{Id}). \tag{4.121}$$

The line L_j is contained in the fiber of the vector bundle $(\tilde{\pi}_*\tilde{E})^\Gamma \to \mathbb{CP}^1$ at the point z_j and we obtain a filtration

$$0 \subset L_j \subset ((\tilde{\pi}_*\tilde{E})^\Gamma)_{z_j}. \tag{4.122}$$

This filtration, together with the system of weights $\{\mu_1^j, \mu_2^j\}$ gives a parabolic structure on $(\tilde{\pi}_*\tilde{E})^\Gamma$ ([32, Section 2.1]).

We want to show that the parabolic structure on $(\tilde{\pi}_*\tilde{E})^\Gamma$ defined via (4.122) is equivalent to the parabolic structure defined by Biswas in [6], using the parabolic filtration (4.93).

We consider the Lawson's surface $M = \Sigma_{k-1,l-1}$ (cf. Subsection 2.5.1). The computations for the other symmetric CMC surfaces in Table 4.3 can be done in an analogous way.

Let $\tilde{M}$ be the Riemann surface double covering M and $\Gamma = \mathbb{Z}_{2k} \times \mathbb{Z}_{2l} \subset \mathbf{SU}(2) \times \mathbf{SU}(2)$ the group acting faithfully on $\tilde{M}$.

Consider the branch points $z_1, \dots, z_4 \in \mathbb{CP}^1$ of the holomorphic map $\tilde{\pi} : \tilde{M} \to \mathbb{CP}^1$ of degree $4kl$ (cf. Proposition 4.1). We recall that

$$\begin{aligned}
\tilde{\pi}^{-1}(z_1) &= \{P_1, P_3, \dots, P_{2l+1}\}\\
\tilde{\pi}^{-1}(z_2) &= \{P_2, P_4, \dots, P_{2l+2}\}\\
\tilde{\pi}^{-1}(z_3) &= \{Q_1, Q_3, \dots, Q_{2k+1}\}\\
\tilde{\pi}^{-1}(z_4) &= \{Q_2, Q_4, \dots, Q_{2k+2}\},
\end{aligned} \tag{4.123}$$

where the points P_r, $r = 1, \dots, 2l+2$, have stabilizer group $\mathbb{Z}_{2k}$ and the points Q_s, $s = 1, \dots, 2k+2$, have stabilizer group $\mathbb{Z}_{2l}$.

The parabolic filtration for the vector bundle $(\tilde{\pi}_*\tilde{E})^\Gamma \to \mathbb{CP}^1$ defined by Biswas is given by (cf. (4.93))

$$(\tilde{\pi}_*\tilde{E})^\Gamma_t = \left(\tilde{\pi}_*\left(\tilde{E}\otimes L\Big(\sum_{j=0}^{l}\lfloor -2kt\rfloor P_{2j+1}+\sum_{j=0}^{l}\lfloor -2kt\rfloor P_{2j+2}+\sum_{j=0}^{k}\lfloor -2lt\rfloor Q_{2j+1}+\sum_{j=0}^{k}\lfloor -2lt\rfloor Q_{2j+2}\Big)\right)\right)^\Gamma, \tag{4.124}$$

for $t \in \mathbb{R}$.

We study in detail the situation around the points P_{2r+1}. For the points P_{2r+2} the computations are the same and for the points Q_s it is only sufficient to use the integer l instead of k.

There are three cases, according to the value of $t \in \mathbb{R}$, to consider

- $-\frac{k-1}{2k} < t \leq \frac{k-1}{2k}$

$$(\tilde{\pi}_*\tilde{E})^\Gamma_{t_1} = \left(\tilde{\pi}_*\left(\tilde{E} \otimes L\Big(\sum_{r=0}^{l} -(k-1)P_{2r+1}\Big)\right)\right)^\Gamma. \tag{4.125}$$

The bundle $(\tilde{\pi}_*\tilde{E})^\Gamma_{t_1}$ is generated by the sections of $\tilde{\pi}_*\tilde{E}$ which are $\Gamma_{P_{2r+1}}$-invariant and have at least a $(k-1)$-order zero at the points P_{2r+1} when considered as sections of the holomorphic vector bundle $\tilde{E}$.

We can consider the local frame, around the point $z_1 \in \mathbb{CP}^1$, of $(\tilde{\pi}_*\tilde{E})^{\Gamma}_{t_1}$ given by

$$(w^{k-1}s, w^{k+1}t), \tag{4.126}$$

where (s,t) is a holomorphic local frame of $\tilde{E} \to \tilde{M}$ around P_{2r+1}.

A local frame for $(\tilde{\pi}_*\tilde{E})^{\Gamma}$ around $z_1 \in \mathbb{CP}^1$ is given by $(w^{k-1}s, w^{k+1}t)$ (cf. Subsection 4.4.1). Therefore, the bundles $(\tilde{\pi}_*\tilde{E})^{\Gamma}$ and $(\tilde{\pi}_*\tilde{E})^{\Gamma}_{t_1}$ coincides.

- $\frac{k-1}{2k} < t \leq \frac{k+1}{2k}$

$$(\tilde{\pi}_*\tilde{E})^{\Gamma}_{t_2} = \left(\tilde{\pi}_*\left(\tilde{E} \otimes L\Big(\sum_{r=0}^{l} -(k+1)P_{2r+1}\Big)\right)\right)^{\Gamma}. \tag{4.127}$$

Similarly to the previous case, the bundle $(\tilde{\pi}_*\tilde{E})^{\Gamma}_{t_2}$ is generated by the $\Gamma_{P_{2r+1}}$-invariant sections of $\tilde{\pi}_*\tilde{E}$ having at least a $(k+1)$-order zero at P_{2r+1} when considered as sections of the bundle $\tilde{E}$.

We can consider the local frame for $(\tilde{\pi}_*\tilde{E})^{\Gamma}_{t_2}$ given by

$$(w^{2k}s, w^{k+1}t), \tag{4.128}$$

where (s,t) is a holomorphic local frame for $\tilde{E}$ around P_{2r+1}.

Comparing the local frame (4.126) for $(\tilde{\pi}_*\tilde{E})^{\Gamma}_{t_1}$ with the local frame (4.128) for $(\tilde{\pi}_*\tilde{E})^{\Gamma}_{t_2}$, we observe that we have an inclusion of vector bundles

$$(\tilde{\pi}_*\tilde{E})^{\Gamma}_{t_2} \subset (\tilde{\pi}_*\tilde{E})^{\Gamma}_{t_1}. \tag{4.129}$$

Moreover, the parabolic line (the fiber of $(\tilde{\pi}_*\tilde{E})^{\Gamma}_{t_2}$) at the point $z_1 \in \mathbb{CP}^1$ is given, with respect to these local frames, by

$$[0 : \tilde{t}(z_1)], \tag{4.130}$$

where $\tilde{t} := w^{k+1}t$.

We recall that (cf. Subsection 4.4.1) the local residue of the logarithmic connection $\tilde{\nabla}$ on $(\tilde{\pi}_*\tilde{E})^{\Gamma}$ at the point z_1 can be written as

$$\mathrm{Res}_{z_1}\tilde{\nabla} = \begin{pmatrix} \frac{(k-1)}{2k} & 0 \\ 0 & \frac{(k+1)}{2k} \end{pmatrix} \tag{4.131}$$

with respect to the local frame $(\tilde{s},\tilde{t}) = (w^{k-1}s, w^{k+1}t)$ of $(\tilde{\pi}_*\tilde{E})^{\Gamma}$ around $z_1 \in \mathbb{CP}^1$.

Therefore, the parabolic line L_1 given by (4.121), is the line $[0 : \tilde{t}(z_1)]$. We can conclude that the parabolic line at z_1 for the parabolic structures on $(\tilde{\pi}_*\tilde{E})^\Gamma$ defined via (4.93) and via the logarithmic connection $\tilde{\nabla}$ are the same.

- $\frac{k+1}{2k} < t \leq 1 + \frac{k-1}{2k}$

$$(\tilde{\pi}_*\tilde{E})^\Gamma_{t_3} = \left(\tilde{\pi}_*\left(\tilde{E} \otimes L\left(\sum_{r=0}^{l} -2\tilde{k}P_{2r+1}\right)\right)\right)^\Gamma, \quad \tilde{k} \geq k. \tag{4.132}$$

In this case we have to consider $\Gamma_{P_{2r+1}}$-invariant sections of $(\tilde{\pi}_*\tilde{E})$ which have at least a $2k$-order zero at P_{2r+1} when considered as sections of $\tilde{E}$.

A local frame for $(\tilde{\pi}_*\tilde{E})^\Gamma_{t_3}$ is given by

$$(w^{2k}s, w^{2k}t), \tag{4.133}$$

where (s, t) are as above.

We recall that the local coordinates w and z satisfy $w^{2k} = z$. Therefore, the bundle $(\tilde{\pi}_*\tilde{E})^\Gamma_{t_3}$ is locally generated by the sections (zs, zt), which give a holomorphic local frame for the bundle (cf. Subsection 1.4.2)

$$(\tilde{\pi}_*\tilde{E})^\Gamma \otimes L(-z_1) = (\tilde{\pi}_*\tilde{E})^\Gamma(-z_1). \tag{4.134}$$

We can conclude that the parabolic filtration, given by $\{(\tilde{\pi}_*\tilde{E})^\Gamma_{t_i}\}$, $1 = 1, \ldots, 3$, at the branch points $z_1, \ldots, z_4 \in \mathbb{CP}^1$ of the holomorphic map $\tilde{\pi} : \tilde{M} \to \mathbb{CP}^1$, is given by

$$((\tilde{\pi}_*\tilde{E})^\Gamma)_{z_j} \subset L_j \subset ((\tilde{\pi}_*\tilde{E})^\Gamma(-z_j))_{z_j}, \quad j = 1, \ldots, 4, \tag{4.135}$$

where L_j is the parabolic line defined in (4.121).

In order to obtain a global parabolic structure on $(\tilde{\pi}_*\tilde{E})^\Gamma$, it is sufficient to relate the local frames for the bundles $(\tilde{\pi}_*\tilde{E})^\Gamma_{t_j}$, $j = 1, 2, 3$, around the points $z_1, \ldots, z_4 \in \mathbb{CP}^1$. This can be done considering the restriction of the transition functions of the holomorphic vector bundle $\tilde{\pi}_*\tilde{E}$ (cf. Example 1.6), on the Γ-invariant sub-bundle $(\tilde{\pi}_*\tilde{E})^\Gamma \subset \tilde{\pi}_*\tilde{E}$.

We summarize the above construction with the following

Proposition 4.5. *Let M be a symmetric CMC surface and $\tilde{\pi} : \tilde{M} \to \tilde{M}/\Gamma = \mathbb{CP}^1$ the holomorphic map between Riemann surfaces defined in Proposition 4.1.*

The logarithmic connection $\tilde{\nabla}$ on the Γ-invariant vector bundle $(\tilde{\pi}_*\tilde{E})^\Gamma \to \mathbb{CP}^1$ given by Proposition 4.3, induces a parabolic structure on $(\tilde{\pi}_*\tilde{E})^\Gamma$ via the parabolic filtration (4.122), *with parabolic weights given by the eigenvalues of the local residues of $\tilde{\nabla}$ at the branch points $z_1, \ldots, z_4$ of the map $\tilde{\pi}$.*

Moreover, this parabolic structure is equivalent to the parabolic structure defined by Biswas [6] using (4.93), with jumps at the values of t equal to the eigenvalues of the local residues of the connection $\tilde{\nabla}$ at the points $z_1, \ldots, z_4$.

4.4.3 The λ-family of parabolic bundles on $\mathbb{CP}^1$ induced by the associated family of flat connections

Let M a symmetric CMC surface and ∇^λ the associated family of flat $\mathbf{SL}(2,\mathbb{C})$-connections of the CMC immersion $f : M \to \mathbb{S}^3$.

If $\tau : \tilde{M} \to M$ is the holomorphic map of degree 2, defined in Proposition 4.1, it is possible to consider the λ-family of Γ-equivariant flat connections $\tau^*\nabla^\lambda$ on the vector bundle $\tilde{E} \to \tilde{M}$ (cf. Section 4.3), for $\lambda \in \mathbb{C}^*$.

From Proposition 4.3, it is possible to define a λ-family of logarithmic connection $\tilde{\nabla}^\lambda$, $\lambda \in \mathbb{C}^*$, on the Γ-invariant vector bundle $(\tilde{\pi}_*\tilde{E})^\Gamma \to \mathbb{CP}^1$. Moreover, the family $\tilde{\nabla}^\lambda$ induces a λ-family of parabolic structures on $(\tilde{\pi}_*\tilde{E})^\Gamma$ (cf. Subsection 4.4.2).

We denote with $(\tilde{\pi}_*\tilde{E})^{\Gamma,\lambda}$ the parabolic bundle given by $(\tilde{\pi}_*\tilde{E})^\Gamma$ together with the holomorphic structure (cf. Subsection 1.4.4) and the parabolic structure induced by the connection $\tilde{\nabla}^\lambda$.

Proposition 4.5 ensures that it is possible to extends the λ-family of parabolic bundles $(\tilde{\pi}_*\tilde{E})^{\Gamma,\lambda}$ at $\lambda = 0$. In fact, the construction due to Biswas of the parabolic structure given by (4.93) does not depend on λ.

We will study the situation at $\lambda = 0$ more in details in the next Section.

We conclude this Subsection showing that, for $\lambda \neq 0$, there are only two possible holomorphic structures on $(\tilde{\pi}_*\tilde{E})^{\Gamma,\lambda}$.

Proposition 4.6. *Given a symmetric CMC surface M, let $\tilde{M}$ be the Riemann surface double covering M given by Proposition 4.1 and ∇^λ the associated family of flat $\boldsymbol{SL}(2,\mathbb{C})$-connections of the immersion $f : M \to \mathbb{S}^3$.*

The holomorphic vector bundle $(\tilde{\pi}_*\tilde{E})^{\Gamma,\lambda} \to \mathbb{CP}^1$, for $\lambda \neq 0$, with holomorphic structure given by $\bar{\partial}^\lambda := (\tilde{\nabla}^\lambda)^{0,1}$ (cf. Subsection 1.4.4), can be only one of the following two bundles

$$(\tilde{\pi}_*\tilde{E})^{\Gamma,\lambda} \begin{cases} \mathcal{O}(-2) \oplus \mathcal{O}(-2) \\ \mathcal{O}(-1) \oplus \mathcal{O}(-3). \end{cases} \tag{4.136}$$

Proof. Let $z_1, \ldots, z_4$ be the branch points of the holomorphic map $\tilde{\pi} : \tilde{M} \to \mathbb{CP}^1$ (cf. Proposition 4.1) and $2d_j$, $j = 1, \ldots, 4$, the order of the stabilizer group of the points $p \in \tilde{\pi}^{-1}(z_j)$ under the action of $\Gamma \subset \mathbf{SU}(2) \times \mathbf{SU}(2)$ on $\tilde{M}$.

In Subsection 4.4.1, we showed that the eigenvalues of the local residues of the connection $\tilde{\nabla}^\lambda$ at the points $z_1, \ldots, z_4$ do not depends on λ and are given by

$$\mu_1^j = \frac{d_j - 1}{2d_j}, \quad \mu_2^j = \frac{d_j + 1}{2d_j}, \quad j = 1, \ldots, 4. \tag{4.137}$$

The degree of the holomorphic vector bundle $(\tilde{\pi}_* \tilde{E})^{\Gamma, \lambda} \to \mathbb{CP}^1$ can be computed using the formula ([7, Proposition 1.2])

$$\deg((\tilde{\pi}_* \tilde{E})^{\Gamma, \lambda}) = -\sum_{j=1}^{4} (\mu_1^j + \mu_2^j) = -4. \tag{4.138}$$

Therefore, the Grothendieck Splitting Theorem 1.5 implies

$$(\tilde{\pi}_* \tilde{E})^{\Gamma, \lambda} = \mathcal{O}(n) \oplus \mathcal{O}(m), \quad \lambda \in \mathbb{C}^*, \tag{4.139}$$

with $m + n = -4$.

Suppose that $n \in \mathbb{N}$ and $m \leq -4$ (or viceversa). The logarithmic connection $\tilde{\nabla}^\lambda$ induces a logarithmic connection on the line sub-bundle $\mathcal{O}(m) \to \mathbb{CP}^1$.

Let $\alpha_1, \ldots, \alpha_4$ be the local residues of such logarithmic connection on $\mathcal{O}(m)$, which must satisfies

$$-4 \geq m = -\sum_{j=1}^{4} \alpha_j. \tag{4.140}$$

However, the values $\alpha_1, \ldots, \alpha_4$ are induced by the local residues of the connection $\tilde{\nabla}^\lambda$ and must be contained in the interval $(0, 1)$ since the values μ_1^j, μ_2^j are contained in the same interval.

Hence, we obtain

$$-\sum_{j=1}^{4} \alpha_j > -4 \tag{4.141}$$

which gives a contraddiction.

We conclude that the only two possibilities for the values of m and n are given by

$$\begin{cases} n = m = -2 \\ n = -1, m = -3. \end{cases} \tag{4.142}$$

$\square$

4.5 The parabolic Higgs field on $(\pi_*\tilde{E})^\Gamma$

In this section we will study the residue at $\lambda = 0$ of the λ-family of logarithmic connections $\tilde{\nabla}^\lambda$ defined on the parabolic vector bundle $(\tilde{\pi}_*\tilde{E})^\Gamma \to \mathbb{CP}^1$ in Subection 4.4.3.

We first recall the definitions of parabolic sub-bundle and parabolic degree.

Definition 4.7. Let E and V be parabolic bundles over a compact Riemann surface M and $\{E_t\}$, $\{V_t\}$ the corresponding parabolic filtrations (cf. (4.75)). The parabolic bundle V is a *parabolic sub-bundle* of E if the following conditions hold:

(1) V is a vector sub-bundle of E;

(2) $V_t \subseteq E_t, \quad \forall t \in \mathbb{R}$;

(3) for $s, t \in \mathbb{R}$ with $t > s$, if $V_s \subseteq E_t$ then $V_s = V_t$.

Definition 4.8. Let $E \to M$ be a parabolic bundle over a compact Riemann surface M. The *parabolic degree* of E is given by

$$\text{par-deg}(E) := \int_{-1}^{0} \deg(E_t)dt, \tag{4.143}$$

where $\{E_t\}$ is the parabolic filtration of E.

Let $E \to M$ be a rank 2 parabolic bundle over a compact Riemann surface M. If there exists a logarithmic connection $\tilde{\nabla}$ on E, it is possible to define the parabolic degree of E using $\tilde{\nabla}$ ([32, Section 2.1]).

Let μ_1^j, μ_2^j with $\mu_2^j \geq \mu_1^j$, be the eigenvalues of the local residues of $\tilde{\nabla}$ at the singular point $p_j \in M$, $j = 1, \ldots, n$. Then, the parabolic degree of E is given by

$$\text{par-deg}(E) = \deg(E) + \sum_{j=1}^{n} \mu_1^j + \mu_2^j. \tag{4.144}$$

Moreover, for every parabolic line sub-bundle V of E, the parabolic degree of V is given by

$$\text{par-deg}(V) = \deg(V) + \sum_{j=1}^{n} \alpha_j, \tag{4.145}$$

where

$$\alpha_j = \begin{cases} \mu_2^j & \text{if } V_{p_j} = L_j \\ \mu_1^j & \text{otherwise} \end{cases} \tag{4.146}$$

and L_j is the parabolic line of $E \to M$ at the point $p_j \in M$ (cf. (4.121)).

Definition 4.9. A parabolic bundle $E \to M$ on a compact Riemann surface M is called *parabolic semi-stable* (resp. *parabolic stable*) if for every parabolic sub-bundle V of E with $0 < \mathrm{rank}(V) < \mathrm{rank}(E)$

$$\frac{\text{par-deg}(V)}{\mathrm{rank}(V)} \leq \frac{\text{par-deg}(E)}{\mathrm{rank}(E)} \quad \left(resp. \quad \frac{\text{par-deg}(V)}{\mathrm{rank}(V)} < \frac{\text{par-deg}(E)}{\mathrm{rank}(E)} \right). \tag{4.147}$$

The formula for the parabolic degree (4.144) implies the following

Lemma 4.1. *Let M be a symmetric CMC surface, $(\tilde{\pi}_* \tilde{E})^{\Gamma,\lambda} \to \mathbb{CP}^1$ the parabolic bundle defined in Section 4.4.3 together with the family of logarithmic connections $\tilde{\nabla}^\lambda$. The parabolic degree of $(\tilde{\pi}_* \tilde{E})^{\Gamma,\lambda}$ is equal to zero.*

Proof. The eigenvalues of the local residues of $\tilde{\nabla}^\lambda$ at the branch points $z_1, \ldots, z_4$ of the map $\tilde{\pi} : \tilde{M} \to \mathbb{CP}^1$ are of the form (cf. (4.116))

$$\mu_1^j = \frac{d_j - 1}{2d_j}, \quad \mu_2^j = \frac{d_j + 1}{2d_j}. \tag{4.148}$$

Moreover, from (4.138), the degree of $(\tilde{\pi}_* \tilde{E})^{\Gamma,\lambda}$ is equal to -4.

Therefore, equation (4.144) implies

$$\text{par-deg}((\tilde{\pi}_* \tilde{E})^{\Gamma,\lambda}) = -4 + \sum_{j=1}^{4} \mu_1^j + \mu_2^j = -4 + 4 = 0. \tag{4.149}$$

□

4.5.1 The induced Higgs field on $(\tilde{\pi}_* \tilde{E})^{\Gamma,\lambda}$ from the Higgs field Φ of M

Let M be a symmetric CMC surface, ∇^λ the associated family of flat $\mathbf{SL}(2, \mathbb{C})$-connections of the immersion $f : M \to \mathbb{S}^3$ and $\Phi \in H^0(M, \mathrm{End}_0(E) \otimes K_M)$ the Higgs field of M (cf. Section 3.1).

The pullback of Φ under the map $\tau : \tilde{M} \to M$, defined in Proposition 4.1, can be considered as a map

$$\tau^* \Phi : H^0(U, \tilde{E}) \to H^0(U, \tilde{E} \otimes K_{\tilde{M}}), \tag{4.150}$$

where $U \subset \tilde{M}$ is an open set and $\tilde{E} \to \tilde{M}$ is the holomorphic vector bundle defined in Section 4.3.

In Section 4.4, we described the inclusion map

$$H^0(U, K_{\tilde{M}}) \hookrightarrow H^0(U, \tilde{\pi}^*(K_{\mathbb{CP}^1} \otimes L(D))), \tag{4.151}$$

where $D = z_1 + \cdots + z_4$ is the branch divisor of the map $\tilde{\pi} : \tilde{M} \to \mathbb{CP}^1$.

The composition of $\tau^*\Phi$ with the inclusion map (4.151) gives a map

$$h : H^0(U, \tilde{E}) \to H^0(U, \tilde{E} \otimes \tilde{\pi}^*(K_{\mathbb{CP}^1} \otimes L(D))). \tag{4.152}$$

Similarly to the construction of the logarithmic connection $\tilde{\nabla}$ on $(\tilde{\pi}_*\tilde{E})^\Gamma$ (cf. Proposition 4.3), from the formulas (1.44) and (1.46) we obtain a map

$$\tilde{\pi}_*(\tau^*\Phi) : H^0(\tilde{U}, \tilde{\pi}_*\tilde{E}) \to H^0(\tilde{U}, \tilde{\pi}_*\tilde{E} \otimes K_{\mathbb{CP}^1} \otimes L(D)), \tag{4.153}$$

where $\tilde{U} \subset \mathbb{CP}^1$ is an open set such that $\tilde{\pi}(U) = \tilde{U}$.

We can now prove the following

Proposition 4.7. *Given a symmetric CMC surface M with associated family of flat $\boldsymbol{SL}(2,\mathbb{C})$-connections $\nabla^\lambda = \nabla + \lambda^{-1}\Phi - \lambda\Phi^*$, the map $\tilde{\pi}_*(\tau^*\Phi)$, given by (4.153) described above, induces a Higgs field $\tilde{\Phi} \in H^0(\tilde{U}, End_0((\tilde{\pi}_*\tilde{E})^\Gamma) \otimes K_{\mathbb{CP}^1} \otimes L(D))$ on the Γ-invariant vector bundle $(\tilde{\pi}_*\tilde{E})^\Gamma \to \mathbb{CP}^1$.*

Proof. Let $s \in H^0(\tilde{U}, \tilde{\pi}_*\tilde{E})$ be a Γ-invariant section, where $\Gamma \subset \mathbf{SU}(2) \times \mathbf{SU}(2)$ is the finite group acting faithfully on the surface $\tilde{M}$ (cf. Section 4.1).

Let $\gamma \in \Gamma$, then

$$\begin{aligned} \tilde{\pi}_*(\tau^*\Phi)(s) &= \tilde{\pi}_*(\tau^*\Phi)(\gamma \cdot s) = \tau^*\Phi(\gamma \cdot s) \\ &= \gamma^*(\tau^*\Phi)(s) = \gamma^*(\tilde{\pi}_*(\tau^*\Phi))(s), \end{aligned} \tag{4.154}$$

where we have used the fact that $\tau^*\Phi$ is Γ-equivariant (cf. Proposition 4.2) and that s can be considered as a section of $\tilde{E} \to \tilde{M}$. Therefore, we obtain a well-defined Higgs field $\tilde{\Phi}$ on $(\tilde{\pi}_*\tilde{E})^\Gamma \to \mathbb{CP}^1$ and the λ-family of logarithmic connections $\tilde{\nabla}^\lambda$ on $(\tilde{\pi}_*\tilde{E})^{\Gamma,\lambda} \to \mathbb{CP}^1$ (cf. Subsection 4.4.3) has the form

$$\tilde{\nabla}^\lambda = \lambda^{-1}\tilde{\Phi} + \tilde{\nabla} + \text{ higher order terms in } \lambda. \tag{4.155}$$

$\square$

The next Proposition shows some properties of the Higgs field $\tilde{\Phi}$ defined in Proposition 4.7.

Proposition 4.8. *The Higgs field $\tilde{\Phi} \in H^0(\mathbb{CP}^1, End_0((\tilde{\pi}_*\tilde{E})^\Gamma) \otimes K_{\mathbb{CP}^1} \otimes L(D))$ defined in Proposition 4.7 satisfies the following:*

(i) $\tilde{\Phi}$ *is nilpotent, that is,* $\tilde{\Phi}^2 = 0$*;*

(ii) $\tilde{\phi}_j := Res_{z_j}(\tilde{\Phi}) \neq 0$, *for* $j = 1, \ldots, 4$*;*

(iii) *$\tilde{\Phi}$ is a parabolic Higgs field, that is*

$$L_j \in Ker(\tilde{\phi}_j), \quad j = 1, \ldots, 4, \tag{4.156}$$

where L_j is the parabolic line of $(\tilde{\pi}_* \tilde{E})^\Gamma$ at z_j given by (4.121);

(iv) *for every holomorphic line sub-bundle $L \subset (\tilde{\pi}_* \tilde{E})^\Gamma$ with $\tilde{\Phi}(L) \subset L \otimes K$ the following holds*

$$\textit{par-deg}(L) < \textit{par-deg}((\tilde{\pi}_* \tilde{E})^\Gamma) = 0. \tag{4.157}$$

If this condition is satisfied we say that $((\tilde{\pi}_* \tilde{E})^\Gamma, \tilde{\Phi})$ is a parabolic Higgs stable bundle.

Proof. (i) Let $\tilde{U} \subset \mathbb{CP}^1$ be an open set and $s \in H^0(\tilde{U}, (\tilde{\pi}_* \tilde{E})^\Gamma)$. From the construction of $\tilde{\Phi}$, we have

$$\tilde{\Phi}(s) = \tau^* \Phi(s), \tag{4.158}$$

where, in the right hand side, we consider s as a local section of the holomorphic vector bundle $\tilde{E}$ on an open set $U \subset \tilde{M}$ such that $\tilde{\pi}(U) = \tilde{U}$. Since $\tau^*\Phi$ is nilpotent, it follows that $\tilde{\Phi}$ is nilpotent.

(ii) Consider the local frame $(w^{d_j-1}s, w^{d_j+1}t)$ of $(\tilde{\pi}_* \tilde{E})^\Gamma$ around one branch point $z_j \in \mathbb{CP}^1$ of the map $\tilde{\pi} : \tilde{M} \to \mathbb{CP}^1$, where (s,t) is a local frame for $\tilde{E}$ on an open set $U \subset \tilde{M}$ (cf. Subsection 4.4.1).

It is possible to write the Higgs field $\tau^*\Phi$, locally on U, as $\tau^*\Phi = A(w)dw$, where

$$A(w) = \begin{pmatrix} a(w) & b(w) \\ c(w) & -a(w) \end{pmatrix}, \tag{4.159}$$

for some functions $a, b, c : U \to \mathbb{C}$. Therefore,

$$\begin{aligned} \tau^*\Phi(s) &= (a(w)s + c(w)t)dw \\ \tau^*\Phi(t) &= (b(w)s - a(w)t)dw. \end{aligned} \tag{4.160}$$

Writing $\tau^*\Phi$ with respect to the local frame $(w^{d_j-1}s, w^{d_j+1}t)$, we obtain

$$\begin{aligned} \tau^*\Phi(w^{d_j-1}s) &= (a(w)w^{d_j-1}s + c(w)w^{d_j-1}t)dw = (a(w)w^{d_j-1}s + \frac{c(w)}{w^2}w^{d_j+1}t)dw \\ \tau^*\Phi(w^{d_j+1}t) &= (b(w)w^{d_j+1}s - a(w)w^{d_j+1}t)dw = (w^2 b(w)w^{d_j-1}s - a(w)w^{d_j+1}t)dw. \end{aligned} \tag{4.161}$$

Hence, the Higgs field $\tilde{\Phi}$, with respect to the local frame $(w^{d_j-1}s, w^{d_j+1}t)$ on the open set $\tilde{U} = \tilde{\pi}(U)$, is given by $\tilde{\Phi} = \tilde{A}(w)dw$, where

$$\tilde{A}(w) = \begin{pmatrix} a(w) & w^2 b(w) \\ \frac{c(w)}{w^2} & -a(w) \end{pmatrix}. \tag{4.162}$$

We can conclude that the function $c(w)$ must have a simple zero at $w = 0$ and there are no conditions on the functions $a(w)$ and $b(w)$. Therefore, the residue of $\tilde{\Phi}$ at the point z_j is given by

$$\begin{pmatrix} 0 & 0 \\ c_0 & 0 \end{pmatrix}, \quad c_0 \in \mathbb{C}^*, \tag{4.163}$$

which is non vanishing.

(iii) Let μ_1^1, μ_2^1 be the eigenvalues of the local residues of the λ-family of logarithmic connections $\tilde{\nabla}^\lambda$ on $(\tilde{\pi}_* \tilde{E})^\Gamma \to \mathbb{CP}^1$ (cf. (4.116)).

The parabolic line L_1 at the branch point $z_1 \in \mathbb{CP}^1$ of the map $\tilde{\pi} : \tilde{M} \to \mathbb{CP}^1$, is given by (cf. (4.121))

$$L_1 = \mathrm{Ker}(\mathrm{Res}_{z_1} \tilde{\nabla}^\lambda + \mu_2^1 \,\mathrm{Id}). \tag{4.164}$$

From the discussion in Subsection 4.4.2, we have that, with respect to the local frame $(w^{d_1-1}s, w^{d_1+1}t)$ of $(\tilde{\pi}_* \tilde{E})^\Gamma$ around z_1, the parabolic line L_1 is given by

$$L_1 = [0 : w^{d_1+1} t(z_1)]. \tag{4.165}$$

From (ii), the residue of $\tilde{\Phi}$ at z_1 is given, with respect to the same local frame, by

$$\tilde{\phi}_1 = \begin{pmatrix} 0 & 0 \\ c_0 & 0 \end{pmatrix}, \quad c_0 \in \mathbb{C}^*. \tag{4.166}$$

Therefore, we obtain

$$\begin{pmatrix} 0 & 0 \\ c_0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ w^{d_1+1} t(z_1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{4.167}$$

which implies $L_1 \in \mathrm{Ker}(\tilde{\phi}_1)$. Analogous computations can be made for the parabolic lines at the other branch points $z_2, z_3, z_4 \in \mathbb{CP}^1$.

(iv) Since the holomorphic vector bundle $\tilde{E} \to \tilde{M}$ is stable ([38]), this follows from [54, Theor. 3.1].

□

The existence of a parabolic Higgs field $\tilde{\Phi}$ allows the description of the admissible holomorphic structures on $(\tilde{\pi}_*\tilde{E})^{\Gamma,\lambda}$ at $\lambda = 0$.

Proposition 4.9. *Let M be a symmetric CMC surface and $(\tilde{\pi}_*\tilde{E})^{\Gamma,\lambda} \to \mathbb{CP}^1$ the parabolic bundle defined in Subsection 4.4.3. The holomorphic structure of $(\tilde{\pi}_*\tilde{E})^{\Gamma,\lambda}$, induced by the λ-family of logarithmic connections $\tilde{\nabla}^\lambda$ (cf. Subsection 4.4.3), at $\lambda = 0$ can be either $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$ or $\mathcal{O}(-1) \oplus \mathcal{O}(-3)$.*

Proof. Analogously to the proof of Proposition 4.6, we have that $(\tilde{\pi}_*\tilde{E})^{\Gamma,\lambda}$ at $\lambda = 0$ is given by $\mathcal{O}(n) \oplus \mathcal{O}(m)$ with $n + m = -4$. Suppose that $n \geq 0$ and $m \leq -4$.

Let (s, t) be a frame for $\mathcal{O}(n) \oplus \mathcal{O}(m)$, where s has divisor $n \cdot 0$ and t divisor $m \cdot 0$, for $0 \in \mathbb{CP}^1$. If we write the parabolic Higgs field locally as

$$\tilde{\Phi}(w) = \begin{pmatrix} a(w) & b(w) \\ c(w) & -a(w) \end{pmatrix} dw, \tag{4.168}$$

for some complex-valued functions a, b, c, we have

$$\tilde{\Phi}(s) = (as + ct)dw. \tag{4.169}$$

Equation (4.169) and the assumption on m and n imply that the function c must be zero and $\tilde{\Phi}$ is upper triangular. From (i) of Proposition 4.8, $\tilde{\Phi}$ is nilpotent. Therefore, the function a must be also zero and $\tilde{\Phi}$ is given locally by

$$\tilde{\Phi}(w) = \begin{pmatrix} 0 & b(w) \\ 0 & 0 \end{pmatrix} dw, \tag{4.170}$$

From the fact that the Higgs field $\tau^*\Phi$ on the holomorphic vector bundle $\tilde{E} \to \tilde{M}$ is nowhere vanishing (because the Higgs field Φ on M is nowhere vanishing, cf. Section 3.1), it follows that $\tilde{\Phi}$ must be nowhere vanishing. But the Higgs field $\tilde{\Phi}$ of the form (4.170) admits points where it vanishes and we obtain a contradiction.

We conclude that the only possibilities for the values of m and n are

$$\begin{cases} n = m = -2 \\ n = -1, m = -3 \end{cases}. \tag{4.171}$$

$\square$

In the next subsections we will investigate which parabolic structures on $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$ and on $\mathcal{O}(-1) \oplus \mathcal{O}(-3)$ admit a nilpotent parabolic Higgs field $\tilde{\Phi}$ with non vanishing residues at the points $z_1, \ldots, z_4$ such that the pair $((\tilde{\pi}_*\tilde{E})^\Gamma, \tilde{\Phi})$ is parabolic Higgs stable.

4.5.2 The case of $\mathcal{O}(-2)\oplus\mathcal{O}(-2)$

Up to Möbius transformation on $\mathbb{CP}^1$, it is possible to assume that the branch points $z_1,\dots,z_4$ of the holomorphic map $\tilde{\pi}:\tilde{M}\to\mathbb{CP}^1$ are given by

$$z_1=0, z_2=1, z_3=-1, z_4=m, \quad m\in\mathbb{C}\smallsetminus\{0,1,-1\}. \tag{4.172}$$

Let $U_0=\mathbb{C}$, $U_1=\mathbb{CP}^1\smallsetminus\{0\}$ be two open sets of $\mathbb{CP}^1$ which gives a trivializing cover for $\mathcal{O}(-2)\oplus\mathcal{O}(-2)\to\mathbb{CP}^1$ (cf. Example 1.4). Let $s=(s_1,s_2)$ be a local frame on U_0 and $t=(t_1,t_2)$ a local frame on U_1. The transition function between s and t is given by

$$g=\begin{pmatrix} z^{-2} & 0\\ 0 & z^{-2}\end{pmatrix} \tag{4.173}$$

and $t=gs$.

We check under which conditions the Higgs field $\tilde{\Phi}$ on $(\tilde{\pi}_*\tilde{E})^\Gamma\to\mathbb{CP}^1$, defined in Proposition 4.7, can be extended holomorphically to infinity. With respect to the local frame s, $\tilde{\Phi}$ can be written as

$$\tilde{\Phi}(z)=\begin{pmatrix} a(z) & b(z)\\ c(z) & -a(z)\end{pmatrix}dz, \tag{4.174}$$

where $a,b,c:U_0\to\mathbb{C}$ are meromorphic functions with simple poles at the branch points $z_1,\dots,z_4$ (that is, they are rational functions in z).

Around infinity, it is possible to write $\tilde{\Phi}$, with respect to the local frame t, as

$$g\tilde{\Phi}(z)g^{-1}=\begin{pmatrix} a(z) & b(z)\\ c(z) & -a(z)\end{pmatrix}dz. \tag{4.175}$$

Since it is not possible to use the local coordinate z around infinity, we consider the coordinate $w=\frac{1}{z}$ and we obtain

$$g\tilde{\Phi}(1/w)g^{-1}=\begin{pmatrix} a(1/w) & b(1/w)\\ c(1/w) & -a(1/w)\end{pmatrix}\frac{dw}{w^2}. \tag{4.176}$$

Therefore, the functions a,b and c must vanish to order two at $w=0$.

With respect to the frame s, it is possible to write $\tilde{\Phi}$ as

$$\tilde{\Phi}(z)=\sum_{j=1}^{4}\tilde{\phi}_j\frac{dz}{z-z_j}+A(z)dz, \tag{4.177}$$

where

$$\tilde{\phi}_j=\operatorname{Res}_{z_j}\tilde{\Phi}=\begin{pmatrix} a_j & b_j\\ c_j & -a_j\end{pmatrix},\quad A(z)=\begin{pmatrix} p(z) & q(z)\\ r(z) & -p(z)\end{pmatrix}, \tag{4.178}$$

for some complex numbers a_j, b_j, c_j and polynomials $p(z), q(z)$ and $r(z)$.

Consider the upper left entry of $\tilde{\Phi}$, given by

$$a(z) = \frac{a_1}{z} + \frac{a_2}{z-1} + \frac{a_3}{z+1} + \frac{a_4}{z-m} + p(z). \tag{4.179}$$

For z going to infinity, the first four summands in (4.179) vanish. Since $a(z)$ must vanish at infinity (cf. (4.176)), it follows that $p(z)$ must vanish at infinity as well. Thus, $p(z)$ is a polynomial with a zero at infinity, which implies $p(z) \equiv 0$.

Looking at the series expansion of $a(z)$ at $z = \infty$, using the coordinate $w = \frac{1}{z}$, we observe that the condition that $a(z)$ must vanish to second order at $w = 0$ translates to

$$\sum_{j=1}^{4} a_j = 0. \tag{4.180}$$

Analogous arguments can be applied to the other entries of the Higgs field $\tilde{\Phi}$. We conclude that for $(\tilde{\pi}_* \tilde{E})^\Gamma = \mathcal{O}(-2) \oplus \mathcal{O}(-2)$, the matrix $A(z)$ is identically zero and the residues $\tilde{\phi}_j$ of $\tilde{\Phi}$ at the branch points $z_1, \ldots, z_4$ satisfy

$$\sum_{j=1}^{4} \tilde{\phi}_j = 0. \tag{4.181}$$

We now check which parabolic structures on $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$ admit a nilpotent, parabolic Higgs field $\tilde{\Phi}$ of the form (4.177), with non vanishing residues satisfying condition (4.181).

Let the parabolic lines be (with respect to a local frame)

$$L_j := \mathbb{C} \begin{pmatrix} v_j \\ w_j \end{pmatrix}, \quad v_j, w_j \in \mathbb{CP}^1 . \tag{4.182}$$

An automorphism of $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$ is given by a $\mathbf{SL}(2, \mathbb{C})$-matrix

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1. \tag{4.183}$$

We want to see in which ways it is possible to normalize the parabolic lines L_j in $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$. For example, we can ask if it is possible to have one parabolic line contained in the first $\mathcal{O}(-2)$ summand and none in the second $\mathcal{O}(-2)$ summand or if three parabolic lines can be contained in one summand and none in the other one, and so on.

Here we consider the case where the parabolic lines can be normalized to correspond to four different points of $\mathbb{CP}^1$. Therefore, after the composition with an automorphism of the form (4.183), we can write

$$L_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad L_4 = \begin{pmatrix} u \\ 1 \end{pmatrix}, \tag{4.184}$$

where $u \in \mathbb{CP}^1$.

The strictly parabolicity of the Higgs field $\tilde{\Phi}$ (cf. (iii) Proposition 4.8) implies

$$\tilde{\phi}_j L_j = 0, \quad j = 1, \ldots, 4. \tag{4.185}$$

Thus, the residues $\tilde{\phi}_j$ at the points $z_1, \ldots, z_4$ are of the form

$$\tilde{\phi}_1 = \begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\phi}_2 = \begin{pmatrix} 0 & 0 \\ c_2 & 0 \end{pmatrix}, \quad \tilde{\phi}_3 = \begin{pmatrix} a_3 & -a_3 \\ a_3 & -a_3 \end{pmatrix}, \quad \tilde{\phi}_4 = \begin{pmatrix} a_4 & -a_4 u \\ \frac{a_4}{u} & -a_4 \end{pmatrix}. \tag{4.186}$$

From the condition (4.181) on the residues $\tilde{\phi}_1, \ldots, \tilde{\phi}_4$, we obtain the linear system

$$\begin{cases} a_3 + a_4 = 0 \\ b_1 - a_3 - a_4 u = 0 \\ c_2 + a_3 + \frac{a_4}{u} = 0. \end{cases} \tag{4.187}$$

Therefore, the Higgs field $\tilde{\Phi}$ can be written as

$$\begin{aligned} \tilde{\Phi}(z) = &\begin{pmatrix} 0 & a_4(u-1) \\ 0 & 0 \end{pmatrix} \frac{dz}{z} + \begin{pmatrix} 0 & 0 \\ \frac{a_4(u-1)}{u} & 0 \end{pmatrix} \frac{dz}{z-1} + \\ &+ \begin{pmatrix} -a_4 & a_4 \\ -a_4 & a_4 \end{pmatrix} \frac{dz}{z+1} + \begin{pmatrix} a_4 & -a_4 u \\ \frac{a_4}{u} & -a_4 \end{pmatrix} \frac{dz}{z-m}, \quad a_4 \in \mathbb{C}^*. \end{aligned} \tag{4.188}$$

The nilpotency of $\tilde{\Phi}$ gives

$$\tilde{\Phi}^2 = \begin{pmatrix} \frac{a_4^2(u-1)(1+m(2u-1))}{u(m-z)z(z^2-1)} & 0 \\ 0 & \frac{a_4^2(u-1)(1+m(2u-1))}{u(m-z)z(z^2-1)} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.189}$$

which implies that u can only be either 1 or $\frac{m-1}{2m}$.

It remains to check the cases for $u = 0$ and $u = \infty$. For $u = 0$, the parabolic line L_4 corresponds to the point $[0:1] \in \mathbb{CP}^1$ and the residues $\tilde{\phi}_4$ is given by

$$\tilde{\phi}_4 = \begin{pmatrix} 0 & 0 \\ c_4 & 0 \end{pmatrix}. \tag{4.190}$$

The condition (4.181) implies $a_3 = 0$, thus $\tilde{\phi}_3 = 0$ which is not admissible since the residues of $\tilde{\Phi}$ must be non vanishing (cf. (ii) Proposition 4.8). Analogous computations shows that the case $u = \infty$ is not admissible as well.

Finally, for $u = 1$ the residues $\tilde{\phi}_1$ and $\tilde{\phi}_2$ vanish (cf. (4.188)) and we conclude that the only admissible possibility is given by $u = \frac{m-1}{2m}$ and the Higgs field $\tilde{\Phi}$ has the expression

$$\begin{aligned}\tilde{\Phi}(z) = \begin{pmatrix} 0 & -a_4\frac{1+m}{2m} \\ 0 & 0 \end{pmatrix}\frac{dz}{z} + \begin{pmatrix} 0 & 0 \\ a_4\frac{1+m}{1-m} & 0 \end{pmatrix}\frac{dz}{z-1} + \\ + \begin{pmatrix} -a_4 & a_4 \\ -a_4 & a_4 \end{pmatrix}\frac{dz}{z+1} + \begin{pmatrix} a_4 & -a_4\frac{m-1}{2m} \\ \frac{2a_4m}{m-1} & -a_4 \end{pmatrix}\frac{dz}{z-m}, \quad a_4 \in \mathbb{C}^*.\end{aligned} \tag{4.191}$$

The last thing to check is if the Higgs field $\tilde{\Phi}$ of the form (4.191) makes the bundle $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$ parabolic Higgs stable (cf. *(iv)* Proposition 4.8). The only $\tilde{\Phi}$-invariant sub-bundles of $(\tilde{\pi}_*\tilde{E})^\Gamma$ are $\mathrm{Ker}(\tilde{\Phi})$ and $\mathrm{Im}(\tilde{\Phi})$. Since the residues $\tilde{\phi}_j$ of $\tilde{\Phi}$ are non vanishing and satisfy $\tilde{\phi}_j L_j = 0$, all the parabolic lines are contained in $\mathrm{Ker}(\tilde{\Phi})$.

Moreover, from the nilpotency of $\tilde{\Phi}$, it follows

$$\mathrm{Im}(\tilde{\Phi}) \subset \mathrm{Ker}(\tilde{\Phi}), \tag{4.192}$$

thus, it is enough to compute the parabolic degree of $\mathrm{Ker}(\tilde{\Phi})$ and check if it is less than the parabolic degree of $(\tilde{\pi}_*\tilde{E})^\Gamma$, which is equal to zero (cf. Lemma 4.1).

The degree of $\mathrm{Ker}(\tilde{\Phi})$ can be computed counting the order of the zeros of the composition

$$\mathrm{Ker}(\tilde{\Phi}) \hookrightarrow \mathcal{O}(-2) \oplus \mathcal{O}(-2) \xrightarrow{\pi^2} \mathcal{O}(-2), \tag{4.193}$$

where π^2 is the projection to the second summand.

We have

$$\tilde{\Phi}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} \frac{a_4(1+m)(d(z-1)-2cz)}{2z(m-z)(1+z)} \\ \frac{a_4(1+m)(d(z-1)-2cz}{(m-z)(z^2-1)} \end{pmatrix}. \tag{4.194}$$

A simple computation shows that $(c, d) \in \mathrm{Ker}(\tilde{\Phi})$ only if d has a simple zero. Therefore, $\mathrm{Ker}(\tilde{\Phi})$ has degree equal to -3.

Using the formula (4.145), we can compute the parabolic degree of $\mathrm{Ker}(\tilde{\Phi})$ for the symmetric CMC surfaces in Table 4.3.

- Lawson's surfaces $\Sigma_{k-1,1}$

  The eigenvalues of the local residues of the λ-family of logarithmic connections $\tilde{\nabla}^\lambda$ on $(\tilde{\pi}_*\tilde{E})^\Gamma$ at the points $z_1, \ldots, z_4$ are given by $\frac{k-1}{2k}$, $\frac{k+1}{2k}$ (cf. Subsection 4.1.1). Thus,

$$\begin{aligned}\text{par-deg}\,\mathrm{Ker}\,\tilde{\Phi} &= -3 + \frac{k+1}{2k} + \frac{k+1}{2k} + \frac{k+1}{2k} + \frac{k+1}{2k} = \\ &= -3 + \frac{2k+2}{k} = \frac{-3k+2k+2}{k} = \frac{2-k}{k}\end{aligned} \tag{4.195}$$

which is always negative since $k > 2$.

- Lawson's surfaces $\Sigma_{k-1,l-1}$

  The eigenvalues of the local residues of the λ-family of logarithmic connections $\tilde{\nabla}^\lambda$ on $(\tilde{\pi}_*\tilde{E})^\Gamma$ are given by $\frac{k-1}{2k}$, $\frac{k+1}{2k}$ at the points z_1, z_2 and $\frac{l-1}{2l}$, $\frac{l+1}{2l}$ at the points z_3, z_4 (cf. Subsection 4.1.2). Thus,

$$\begin{aligned}\text{par-deg}\,\mathrm{Ker}\,\tilde{\Phi} &= -3 + \frac{k+1}{2k} + \frac{k+1}{2k} + \frac{l+1}{2l} + \frac{l+1}{2l} = \\ &= \frac{-3kl + kl + l + kl + k}{kl} = \frac{-kl + l + k}{kl}\end{aligned} \tag{4.196}$$

  which is always negative since k and l are greater than 2.

- Platonic KPS surfaces

  The eigenvalues of the local residues of the λ-family of logarithmic connections $\tilde{\nabla}^\lambda$ on $(\tilde{\pi}_*\tilde{E})^\Gamma$ are of the form $\frac{k-1}{2k}$, $\frac{k+1}{2k}$ at the points z_1, z_2 and $\frac{1}{4}$, $\frac{3}{4}$ at the points z_3, z_4 (cf. Subsection 4.1.3). Thus,

$$\begin{aligned}\text{par-deg}\,\mathrm{Ker}\,\tilde{\Phi} &= -3 + \frac{d+1}{2d} + \frac{d+1}{2d} + \frac{3}{4} + \frac{3}{4} = \\ &= \frac{2-d}{2d}\end{aligned} \tag{4.197}$$

  which is always negative since $d > 2$.

Appendix A.1 contains the computations for the other parabolic structures on $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$ and it is shown that there are no other parabolic structures which admits a nilpotent, parabolic Higgs field $\tilde{\Phi}$ with non zero residues such that $(\tilde{\pi}_*\tilde{E})^\Gamma$ is parabolic Higgs stable.

4.5.3 The case of $\mathcal{O}(-1) \oplus \mathcal{O}(-3)$

As in Subsection 4.5.2, we consider the branch points $z_1, \dots, z_4 \in \mathbb{CP}^1$ of the map $\tilde{\pi} : \tilde{M} \to \mathbb{CP}^1$, to be

$$z_1 = 0, z_2 = 1, z_3 = -1, z_4 = m, \quad m \in \mathbb{C} \setminus \{0, 1, -1\}. \tag{4.198}$$

Let $U_0 = \mathbb{C}$, $U_1 = \mathbb{CP}^1 \setminus \{0\}$ be two open sets of $\mathbb{CP}^1$ which give a trivializing cover for $\mathcal{O}(-1) \oplus \mathcal{O}(-3) \to \mathbb{CP}^1$ and s (resp. t) a local frame on U_0 (resp. U_1). The transition function between s and t is given by

$$g = \begin{pmatrix} z^{-1} & 0 \\ 0 & z^{-3} \end{pmatrix}, \tag{4.199}$$

and $t = gs$.

We check under which conditions the Higgs field $\tilde{\Phi}(z)$ can be extended holomorphically to infinity. We write $\tilde{\Phi}$, with respect to the local frame s, as

$$\tilde{\Phi}(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & -a(z) \end{pmatrix} dz, \tag{4.200}$$

for some meromorphic functions a, b, c on U_0, with simple poles at the points $z_1, \ldots, z_4 \in \mathbb{CP}^1$.

Using the transition function g, it is possible to write $\tilde{\Phi}$ with respect to the local frame t as

$$g\tilde{\Phi}(z)g^{-1} = \begin{pmatrix} a(z) & \frac{b(z)}{z^2} \\ z^2 c(z) & -a(z) \end{pmatrix} dz. \tag{4.201}$$

In order to study the behaviour of $\tilde{\Phi}$ around infinity, we use the coordinate $w = \frac{1}{z}$, and we obtain

$$g\tilde{\Phi}(1/w)g^{-1} = \begin{pmatrix} a(1/w) & w^2 b(1/w) \\ \frac{c(1/w)}{w^2} & -a(1/w) \end{pmatrix} \frac{dw}{w^2}. \tag{4.202}$$

Therefore, the function $a(1/w)$ must vanish to second order at $w = 0$, $b(1/w)$ can take any values in $\mathbb{C}^*$ at $w = 0$ and $c(1/w)$ must vanish to fourth order at $w = 0$.

It is possible to write $\tilde{\Phi}$, with respect to the frame s, as

$$\tilde{\Phi}(z) = \sum_{j=1}^{4} \tilde{\phi}_j \frac{dz}{z - z_j} + A(z)dz, \tag{4.203}$$

where

$$\tilde{\phi}_j = \operatorname{Res}_{z_j} \tilde{\Phi} = \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix}, \quad A(z) = \begin{pmatrix} p(z) & q(z) \\ r(z) & -p(z) \end{pmatrix}, \tag{4.204}$$

for some complex numbers a_j, b_j, c_j and polynomials $p(z)$, $q(z)$ and $r(z)$.

In the following, we describe the conditions that each entry of $\tilde{\Phi}(z)$ of the form (4.203) must satisfy:

- Upper left and lower right entries

$$a(z) = \frac{a_1}{z} + \frac{a_2}{z-1} + \frac{a_3}{z+1} + \frac{a_4}{z-m} + p(z). \tag{4.205}$$

For z going to infinity, the first four summands in (4.205) vanish. Since $a(z)$ must vanish at infinity (cf. (4.202)), it follows that $p(z)$ must vanish at infinity as well, thus $p(z) \equiv 0$.

As in Subsection 4.5.2, the values $a_1, \ldots, a_4$ must satisfy the condition

$$\sum_{j=1}^{4} a_j = 0. \tag{4.206}$$

- Upper right entry

$$b(z) = \frac{b_1}{z} + \frac{b_2}{z-1} + \frac{b_3}{z+1} + \frac{b_4}{z-m} + q(z). \tag{4.207}$$

For z going to infinity the function b does not vanish and we have $b(z) \equiv q(z)$. The polynomial q does not have a pole at infinity, thus it is a nonzero constant which we will denote with b (with abuse of notation).

- Lower left entry

$$c(z) = \frac{c_1}{z} + \frac{c_2}{z-1} + \frac{c_3}{z+1} + \frac{c_4}{z-m} + r(z). \tag{4.208}$$

An argument similar to the one used for the upper left entry implies that $r(z) \equiv 0$. Looking at the series expansion of the function $c(z)$ at $z = \infty$, using the coordinate $w = \frac{1}{z}$, we observe that the condition that $c(z)$ vanishes to fourth order at $w = 0$ translates into the linear system

$$\begin{cases} c_2 = ic_1 \\ c_3 = -c_1 \\ c_4 = -ic_1 \end{cases}, \quad c_1 \in \mathbb{C}^*. \tag{4.209}$$

We conclude that $\tilde{\Phi}$ can be written as

$$\tilde{\Phi}(z) = \sum_{j=1}^{4} \tilde{\phi}_j \frac{dz}{z - z_j} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} dz, \tag{4.210}$$

where the residues $\tilde{\phi}_1, \ldots, \tilde{\phi}_4$ satisfy the conditions described above.

We now check which parabolic structures on $\mathcal{O}(-1) \oplus \mathcal{O}(-3)$ admit a nilpotent parabolic Higgs field of the form (4.210), with non vanishing residues.

Assume that the parabolic lines $L_1, \ldots, L_4$ are contained in the $\mathcal{O}(-1)$ summand of $\mathcal{O}(-1) \oplus \mathcal{O}(-3)$. The strictly parabolicity of $\tilde{\Phi}$ (cf. (iii) Proposition 4.8) implies

$$\tilde{\phi}_j L_j = 0, \quad j = 1, \ldots, 4. \tag{4.211}$$

Therefore, the residues of $\tilde{\Phi}$ at the points $z_1, \ldots, z_4$ are given by

$$\tilde{\phi}_1 = \begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\phi}_2 = \begin{pmatrix} 0 & b_2 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\phi}_3 = \begin{pmatrix} 0 & b_3 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\phi}_4 = \begin{pmatrix} 0 & b_4 \\ 0 & 0 \end{pmatrix}. \tag{4.212}$$

It follows that there are no additional conditions on the residues of $\tilde{\Phi}$ and we can write the parabolic Higgs field as

$$\tilde{\Phi}(z) = \begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix} \frac{dz}{z} + \begin{pmatrix} 0 & b_2 \\ 0 & 0 \end{pmatrix} \frac{dz}{z-1} + \begin{pmatrix} 0 & b_3 \\ 0 & 0 \end{pmatrix} \frac{dz}{z+1} + \begin{pmatrix} 0 & b_4 \\ 0 & 0 \end{pmatrix} \frac{dz}{z-m} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} dz, \tag{4.213}$$

which is nilpotent for every values of b_1, b_2, b_3, b_4 and $b \in \mathbb{C}^*$.

The last thing to check is if the Higgs field $\tilde{\Phi}(z)$ of the form (4.213) makes the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-3)$ parabolic Higgs stable (cf. (iv) Proposition 4.8).

As in the case of $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$, we need to consider only the bundle $\mathrm{Ker}(\tilde{\Phi})$, whose degree can be computed counting the zeros of the composition

$$\mathrm{Ker}\, \tilde{\Phi} \hookrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-3) \xrightarrow{\pi^1} \mathcal{O}(-1), \tag{4.214}$$

, where π^1 denotes the projection to the first summand.

We have

$$\tilde{\Phi}(z) \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} d(b + \frac{b_1}{z} + \frac{b_2}{z-1} + \frac{b_3}{z+1} + \frac{b_4}{z-m} \\ 0 \end{pmatrix}, \tag{4.215}$$

and it is immediate to conclude that $\mathrm{Ker}(\tilde{\Phi}) = \mathcal{O}(-1)$ and has degree -1.

Using the formula (4.145), we compute the parabolic degree of $\mathrm{Ker}(\tilde{\Phi})$ for the symmetric CMC surfaces in Table 4.3.

- Lawson's surfaces $\Sigma_{k-1,1}$

  The eigenvalues of the local residues of the λ-family of logarithmic connections $\tilde{\nabla}^\lambda$ on $(\tilde{\pi}_* \tilde{E})^\Gamma$ at the points $z_1, \ldots, z_4$ are given by $\frac{k-1}{2k}$, $\frac{k+1}{2k}$ (cf. Subsection 4.1.1). Thus,

$$\begin{aligned} \text{par-deg}\, \mathrm{Ker}\, \tilde{\Phi} &= -1 + \frac{k+1}{2k} + \frac{k+1}{2k} + \frac{k+1}{2k} + \frac{k+1}{2k} = \\ &= -1 + \frac{2k+2}{k} = \frac{k+2}{k} \end{aligned} \tag{4.216}$$

  which is always positive since $k > 2$.

- Lawson's surfaces $\Sigma_{k-1,l-1}$

  The eigenvalues of the local residues of the λ-family of logarithmic connections $\tilde{\nabla}^\lambda$ on $(\tilde{\pi}_* \tilde{E})^\Gamma$ are given by $\frac{k-1}{2k}$, $\frac{k+1}{2k}$ at the points z_1, z_2 and $\frac{l-1}{2l}, \frac{l+1}{2l}$ at the points z_3, z_4 (cf. Subsection 4.1.2). Thus,

$$\begin{aligned} \text{par-deg}\, \mathrm{Ker}\, \tilde{\Phi} &= -1 + \frac{k+1}{2k} + \frac{k+1}{2k} + \frac{l+1}{2l} + \frac{l+1}{2l} = \\ &= -1 + \frac{k+1}{k} + \frac{l+1}{l} = \frac{l+kl+k}{kl} \end{aligned} \tag{4.217}$$

which is always positive.

- Platonic KPS surfaces

  The eigenvalues of the local residues of the λ-family of logarithmic connections $\tilde{\nabla}^\lambda$ on $(\tilde{\pi}_*\tilde{E})^\Gamma$ are of the form $\frac{k-1}{2k}$, $\frac{k+1}{2k}$ at the points z_1, z_2 and $\frac{1}{4}, \frac{3}{4}$ at the points z_3, z_4 (cf. Subsection 4.1.3). Thus,

$$\begin{aligned}\text{par-deg}\,\mathrm{Ker}\,\tilde{\Phi} &= -1 + \frac{d+1}{2d} + \frac{d+1}{2d} + \frac{3}{4} + \frac{3}{4} = \\ &= \frac{3d+2}{2d}\end{aligned} \tag{4.218}$$

  which is always positive.

Therefore, the parabolic structure we considered does not admit a nilpotent parabolic Higgs field which makes the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-3)$ parabolic Higgs stable. Analogous computations shows that there is no parabolic structure on $\mathcal{O}(-1)\oplus\mathcal{O}(-3)$ which admits a nilpotent parabolic Higgs field, with nonzero residues, such that the pair $((\tilde{\pi}_*\tilde{E})^\Gamma, \tilde{\Phi})$ is parabolic Higgs stable. (see Appendix A.2).

The above computations in the case of $(\tilde{\pi}_*\tilde{E})^\Gamma = \mathcal{O}(-2)\oplus\mathcal{O}(-2)$ or $\mathcal{O}(-1)\oplus\mathcal{O}(-3)$, together with the Appendices A.1 and A.2, prove the following

Proposition 4.10. *Let M be a symmetric CMC surface and $(\tilde{\pi}_*\tilde{E})^{\Gamma,\lambda} \to \mathbb{CP}^1$, the parabolic vector bundle defined in Subsection 4.4.3. The only possible holomorphic structure on $(\tilde{\pi}_*\tilde{E})^{\Gamma,\lambda}$ at $\lambda = 0$ is $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$.*

4.6 Main result

Let M be a symmetric CMC surface and ∇^λ the associated family of flat $\mathbf{SL}(2,\mathbb{C})$-connections of the immersion $f : M \to \mathbb{S}^3$ (cf. Section 3.1). In this section we want to prove that the λ-family of logarithmic connections $\tilde{\nabla}^\lambda$ defined on the Γ-invariant parabolic bundle $(\tilde{\pi}_*\tilde{E})^{\Gamma,\lambda} \to \mathbb{CP}^1$ (cf. Subsection 4.4.3), gives a DPW potential on the four punctured Riemann sphere, from which it is possible to reconstruct the immersion $f : M \to \mathbb{S}^3$ (cf. Section 3.2).

The following diagram shows the various objects we introduced in this Chapter, together with the maps between them:

$$
\begin{tikzcd}
(\tilde{E}=\tau^*E,\tau^*\nabla^\lambda) \arrow[d] & (E,\nabla^\lambda) \arrow[l, dashed, "\tau^*"'] \arrow[d] \arrow[r, bend left, "\tilde{\pi}_*"] & ((\tilde{\pi}_*\tilde{E})^{\Gamma,\lambda},\tilde{\nabla}^\lambda) \arrow[l, dashed, "\pi^*"'] \arrow[d] \\
\tilde{M} \arrow[r, "\tau"] \arrow[rr, bend right, "\tilde{\pi}"'] & M \arrow[r, "\pi"] & \mathbb{CP}^1
\end{tikzcd}
$$

The next result shows that it is possible to find a DPW potential for the symmetric CMC surfaces in Table (4.3).

Theorem 4.1. *Let M be a symmetric CMC surface with symmetry group $G \subset \boldsymbol{SO}(4)$ from the Table (4.3). Let ∇^λ be the associated family of flat $\boldsymbol{SL}(2,\mathbb{C})$-connections of the immersion $f : M \to \mathbb{S}^3$. Then, there exists a holomorphic family of logarithmic connections*

$$\tilde{\nabla}^\lambda = \lambda^{-1}\tilde{\Phi} + \tilde{\nabla} + \text{ higher order terms in } \lambda$$

on the four punctured sphere $\mathbb{CP}^1$, singular at the four branch points $z_1, \dots, z_4$ of $\pi : M \to M/G = \mathbb{CP}^1$, where $\tilde{\Phi}$ is a nilpotent $\mathfrak{sl}(2,\mathbb{C})$-valued complex linear 1-form, which satisfies the following:

(i) there exists a flat connection $\hat{\nabla}$ on M with $\mathbb{Z}_2$-monodromy representation, such that the families of connections ∇^λ and $\pi^*\tilde{\nabla}^\lambda \otimes \hat{\nabla}$ are gauge equivalent via a family of gauge transformations $g(\lambda)$ which extends holomorphically at $\lambda = 0$;

(ii) there is an open neighborhood U of $\lambda = 0$ such that $\tilde{\nabla}^\lambda$ can be represented by a λ-family of Fuchsian systems for $\lambda \in U$. More specifically, for $\lambda \in U$, we have

$$\tilde{\nabla}^\lambda = d + \eta(z,\lambda) = d + \sum_{j=-1}^{\infty} \eta_j(z)\lambda^j, \tag{4.219}$$

where, for every j, $\eta_j(z)$ is a $\mathfrak{sl}(2,\mathbb{C})$-valued 1-form with simple poles at the branch points $z_1, \dots, z_4$ and holomorphic on $\mathbb{CP}^1 \setminus \{z_1, \dots, z_4\}$;

(iii) the map $\lambda \mapsto \eta(z,\lambda)$ extends meromorphically to $\mathbb{C}^*$ and the connection $\tilde{\nabla}^\lambda = d + \eta(z,\lambda)$ has unitarizable monodromy representation for every $\lambda \in \mathbb{S}^1$ such that $\eta(z,\lambda)$ does not have a pole;

(iv) the eigenvalues of the local residues of $\tilde{\nabla}^\lambda$ are given by the eigenvalues (of the first or second factor in $\boldsymbol{SU}(2) \times \boldsymbol{SU}(2)$) of the four generators $\gamma_1, \dots, \gamma_4$ of the finite group $\Gamma \subset \boldsymbol{SU}(2) \times \boldsymbol{SU}(2)$ which double covers G.

In particular, all of these CMC surfaces can be constructed from a meromorphic DPW potential on the four punctured sphere.

Proof. In Subsection 4.4.3 we have defined a holomorphic family of logarithmic connections $\tilde{\nabla}^\lambda$ on the parabolic vector bundle $(\tilde{\pi}_*\tilde{E})^\Gamma$. We want to prove that $\tilde{\nabla}^\lambda$ satisfies conditions $(i)-(iv)$.

(i) Let $\tau : \tilde{M} \to M$ be the holomorphic map between Riemann surfaces defined in Proposition 4.1, branched at the fixed points of the action of $G \subset \mathbf{SO}(4)$ on M.

From the Riemann's existence Theorem 1.4, τ is uniquely determined by a monodromy representation

$$\rho : \pi_1(M \smallsetminus \{P_1, \ldots, P_k\}, P_0) \to \mathbb{Z}_2, \tag{4.220}$$

where $P_1, \ldots, P_k$ are the fixed points of the action $G \times M \to M$ and $P_0 \in M \smallsetminus \{P_1, \ldots, P_k\}$.

The representation ρ determines a flat connections $\hat{\nabla}$ on M via the Riemann-Hilbert correspondence ([36, Theorem 3.5]). Moreover, the constructions of τ and $\tilde{M}$ imply that the pullback connection $\tau^*\hat{\nabla}$ on $\tilde{M}$ is trivial.

From Proposition 4.4 and Subsection 4.4.3, it follows that the λ-family of flat connections $\tau^*\nabla^\lambda$ and $\tilde{\pi}^*\tilde{\nabla}^\lambda$ are gauge equivalent. Therefore, if we consider the associated family of flat connections ∇^λ on M and the pullback under the map $\pi : M \to \mathbb{CP}^1$ of the λ-family of logarithmic connections $\tilde{\nabla}^\lambda$ on $\mathbb{CP}^1$, an argument similar to the one used in the discussion prior to Proposition 4.4, shows that there only two possible cases:

(1) ∇^λ is gauge equivalent to $\tilde{\pi}^*\tilde{\nabla}^\lambda$ under a holomorphic family of meromorphic gauge transformations $g(\lambda)$, which extends to $\lambda = 0$;

(2) ∇^λ is gauge equivalent to $\tilde{\pi}^*\tilde{\nabla}^\lambda \otimes \hat{\nabla}$ under a holomorphic family of meromorphic gauge transformations $g(\lambda)$, which extends to $\lambda = 0$.

Case (1) cannot occur due to the choice for the eigenvalues of the local residues of the family of logarithmic connections $\tilde{\nabla}^\lambda$ at the points $z_1, \ldots, z_4 \in \mathbb{CP}^1$ we have made (cf. Subsection 4.4.1). Therefore, it remains only the situation given in case (2).

(ii) Proposition 4.10 shows that the underlying holomorphic vector bundle of the parabolic bundle $(\tilde{\pi}_*\tilde{E})^{\Gamma,\lambda} \to \mathbb{CP}^1$ at $\lambda = 0$ is given by $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$.

Since the generic holomorphic rank 2 vector bundle of degree -4 on $\mathbb{CP}^1$ is $\mathcal{O}(-2)\oplus\mathcal{O}(-2)$, it is possible to find an open neighbourhood $U \subset \mathbb{C}^*$ of $\lambda = 0$ such that

$$(\tilde{\pi}_*\tilde{E})^{\Gamma,\lambda} = \mathcal{O}(-2) \oplus \mathcal{O}(-2), \qquad \text{for every } \lambda \in U. \tag{4.221}$$

The expression (4.219) for the λ-family of logarithmic connections $\tilde{\nabla}^\lambda$, for $\lambda \in U$, can be obtained by writing the connection 1-form of $\tilde{\nabla}^\lambda$ with respect to the frame

$$\left(\frac{1}{(z-z_1)(z-z_3)}e_1, \frac{1}{(z-z_1)(z-z_3)}e_2\right), \tag{4.222}$$

where (e_1, e_2) is the meromorphic frame for $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$ such that e_1, e_2 have simple poles at the branch points $z_2, z_4 \in \mathbb{CP}^1$.

(*iii*) Since for a generic $\lambda \in \mathbb{C}^*$ the underlying holomorphic vector bundle of the parabolic bundle $(\tilde{\pi}_*\tilde{E})^{\Gamma,\lambda} \to \mathbb{CP}^1$ is $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$, the values of $\lambda \in \mathbb{C}$ such that this not happens form a discrete subset $\tilde{U} \subset \mathbb{C}^*$.

By parametrizing the λ-family of logarithmic connections $\tilde{\nabla}^\lambda$ of the form (4.219) similarly to [32, Section 2.3], it follows that the entries of the λ-family of 1-forms $\eta(z, \lambda)$ cannot have essential singularities. Therefore, it is possible to extends holomorphically to $\mathbb{C}^*$ the map

$$\lambda \mapsto \eta(z, \lambda). \tag{4.223}$$

The unitarizability of $\tilde{\nabla}^\lambda = d + \eta(z, \lambda)$ for $\lambda \in \mathbb{S}^1$ comes from the construction of $\tilde{\nabla}^\lambda$ and from the fact that the connection ∇^λ is unitary for $\lambda \in \mathbb{S}^1$ (cf. Theorem 3.1).

(*iv*) This follows from the computations in Subsections 4.1.1, 4.1.2 and 4.1.3 and the description of the local residues of $\tilde{\nabla}^\lambda$ in Subsection 4.4.1 (cf. (4.116)).

The CMC immersion $f : M \to \mathbb{S}^3$ can be constructed via the DPW method using the family of 1-forms $\eta(z, \lambda)$ in (4.219). There are two cases to consider:

(*a*) for all $\lambda \in D_1 = \{\lambda \in \mathbb{C}^* \mid |\lambda| \leq 1\}$ the underlying holomorphic vector bundle of $(\tilde{\pi}_*\tilde{E})^{\Gamma,\lambda} \to \mathbb{CP}^1$ is $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$.

In order to obtain a closed immersion along non trivial loops in M, the monodromy matrices of $\eta(z, \lambda)$ must be simultaneously unitarizable (see for example [59, Section 7]). It is possible to find a unitarizer for the monodromy matrices of $\eta(z, \lambda)$ for all $\lambda \in D_1$ ([59, Theorem 4]).

Therefore, the immersion $f : M \to \mathbb{S}^3$ constructed using the steps of the DPW method described in Section 3.2, is well-defined.

(*b*) There exists a discrete subset $\tilde{U} \subset D_1$ such that the underlying holomorphic vector bundle of $(\tilde{\pi}_*\tilde{E})^{\Gamma,\lambda} \to \mathbb{CP}^1$ is $\mathcal{O}(-1) \oplus \mathcal{O}(-3)$.

Similarly to case (a), [59, Theorem 4] gives a unitarizer for the monodromy matrices of $\eta(z, \lambda)$, which is singular at the values of $\lambda \in \tilde{U}$.

In order to obtain a well-defined immersion $f : M \to \mathbb{S}^3$, in this case it is necessary to apply the Iwasawa factorization (step (ii) of the DPW method in Section 3.2) on a disc D_r of radius $r < 1$ such that ([59, Section 3.1])

$$D_r \cap \tilde{U} = \emptyset. \tag{4.224}$$

Since $\tilde{U}$ is a discrete subset, there exists a $r < 1$ such that (4.224) holds. Therefore, the immersion $f : M \to \mathbb{S}^3$ constructed via the DPW method is well defined.

$\square$

Appendix A

A.1 Other parabolic structures for $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$

The following computations shows that there are no parabolic structure on $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$ which admits a nilpotent parabolic Higgs field $\tilde{\Phi}$ with non vanishing residues at the points $z_1, \dots, z_4$ except the one provided in Subsection 4.5.2. We denote with S_1 the first $\mathcal{O}(-2)$ summand and with S_2 the second summand in $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$. Let $L_1, \dots, L_4$ be the parabolic lines at the branch points $z_1, \dots, z_4$ of the map $\tilde{\pi} \to \tilde{M} \to \mathbb{CP}^1$

- $L_1 \in S_1, L_2, L_3 \in S_2$, L_4 not contained in S_1 nor in S_2

  After normalization we can consider

$$L_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad L_4 = \begin{pmatrix} u \\ 1 \end{pmatrix}. \tag{A.1}$$

  Using the condition $\tilde{\phi}_j L_j = 0$, $j = 1, \dots, 4$ we obtain that the residues of the Higgs field $\tilde{\Phi}$ have the form:

$$\tilde{\phi}_1 = \begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\phi}_2 = \begin{pmatrix} 0 & 0 \\ c_2 & 0 \end{pmatrix}, \quad \tilde{\phi}_3 = \begin{pmatrix} 0 & 0 \\ c_3 & 0 \end{pmatrix}, \quad \tilde{\phi}_4 = \begin{pmatrix} a_4 & -a_4 u \\ \frac{a_4}{u} & -a_4 \end{pmatrix}. \tag{A.2}$$

  From the fact that the sum of the residues is zero we get the linear system

$$\begin{cases} a_4 = 0 \\ b_1 - a_4 u = 0 \\ c_2 + c_3 + \frac{a_4}{u} = 0 \end{cases} \tag{A.3}$$

  which implies that $a_4 = b_1 = 0$ and $c_2 = -c_3$. Even if $\tilde{\Phi}(z)$ in this case is nilpotent, we will exclude this parabolic structure since a Higgs field with vanishing residues is not admissible in our situation.

- $L_1 \in S_2, L_2, L_3, L_4$ not contained in S_1 nor in S_2

  After normalization we can consider

$$L_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad L_4 = \begin{pmatrix} u \\ 1 \end{pmatrix}. \tag{A.4}$$

  Using the condition $\tilde{\phi}_j L_j = 0$, $j = 1, \dots, 4$ we obtain that the residues of the Higgs field $\tilde{\Phi}$ have the form:

$$\tilde{\phi}_1 = \begin{pmatrix} 0 & 0 \\ c_1 & 0 \end{pmatrix}, \quad \tilde{\phi}_2 = \begin{pmatrix} a_2 & -a_2 \\ a_2 & -a_2 \end{pmatrix}, \quad \tilde{\phi}_3 = \begin{pmatrix} a_3 & -a_3 \\ a_3 & -a_3 \end{pmatrix}, \quad \tilde{\phi}_4 = \begin{pmatrix} a_4 & -a_4 u \\ \frac{a_4}{u} & -a_4 \end{pmatrix}. \tag{A.5}$$

  From the fact that the sum of the residues is zero we get the linear system

$$\begin{cases} a_2 + a_3 + a_4 = 0 \\ -a_2 - a_3 - a_4 u = 0 \\ c_1 + a_2 + a_3 + \frac{a_4}{u} = 0 \end{cases} \tag{A.6}$$

  which implies $c_1 = 0, a_2 = -a_3, a_4 = 0$. Even if $\tilde{\Phi}(z)$ in this case is nilpotent, we will exclude this parabolic structure since a Higgs field with vanishing residues is not admissible in our situation.

- $L_1, L_2 \in S_1, L_3, L_4$ not contained in S_1 nor in S_2

  After normalization we can consider

$$L_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad L_4 = \begin{pmatrix} u \\ 1 \end{pmatrix}. \tag{A.7}$$

  Using the condition $\tilde{\phi}_j L_j = 0$, $j = 1, \dots, 4$ we obtain that the residues of the Higgs field $\tilde{\Phi}$ have the form:

$$\tilde{\phi}_1 = \begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\phi}_2 = \begin{pmatrix} 0 & b_2 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\phi}_3 = \begin{pmatrix} a_3 & -a_3 \\ a_3 & -a_3 \end{pmatrix}, \quad \tilde{\phi}_4 = \begin{pmatrix} a_4 & -a_4 u \\ \frac{a_4}{u} & -a_4 \end{pmatrix}. \tag{A.8}$$

  From the fact that the sum of the residues is zero we get the linear system

$$\begin{cases} a_3 + a_4 = 0 \\ b_1 + b_2 - a_3 - a_4 u = 0 \\ a_3 + \frac{a_4}{u} = 0 \end{cases} \tag{A.9}$$

and from the first and the third equations we get $a_3 = a_4 = 0$. Even if $\tilde{\Phi}(z)$ in this case is nilpotent, we will exclude this parabolic structure since a Higgs field with vanishing residues is not admissible in our situation.

- $L_1, L_2 \in S_1, L_3 \in S_2$ and L_4 not contained in S_1 nor in S_2

  After normalization we can consider

$$L_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad L_4 = \begin{pmatrix} u \\ 1 \end{pmatrix}. \tag{A.10}$$

  Using the condition $\tilde{\phi}_j L_j = 0$, $j = 1, \ldots, 4$ we obtain that the residues of the Higgs field $\tilde{\Phi}$ have the form:

$$\tilde{\phi}_1 = \begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\phi}_2 = \begin{pmatrix} 0 & b_2 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\phi}_3 = \begin{pmatrix} 0 & 0 \\ c_3 & 0 \end{pmatrix}, \quad \tilde{\phi}_4 = \begin{pmatrix} a_4 & -a_4 u \\ \frac{a_4}{u} & -a_4 \end{pmatrix}. \tag{A.11}$$

  From the fact that the sum of the residues is zero we get the linear system

$$\begin{cases} a_4 = 0 \\ b_1 + b_2 - a_4 u = 0 \\ c_3 + \frac{a_4}{u} = 0 \end{cases} \tag{A.12}$$

  which implies $a_4 = 0, c_3 = 0, b_1 = -b_2$. Even if $\tilde{\Phi}(z)$ in this case is nilpotent, we will exclude this parabolic structure since a Higgs field with vanishing residues is not admissible in our situation.

- $L_1, L_2, L_3 \in S_1$, L_4 not contained in S_1 nor in S_2

  After normalization we can consider

$$L_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad L_4 = \begin{pmatrix} u \\ 1 \end{pmatrix}. \tag{A.13}$$

  Using the condition $\tilde{\phi}_j L_j = 0$, $j = 1, \ldots, 4$ we obtain that the residues of the Higgs field $\tilde{\Phi}$ have the form:

$$\tilde{\phi}_1 = \begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\phi}_2 = \begin{pmatrix} 0 & b_2 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\phi}_3 = \begin{pmatrix} 0 & b_3 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\phi}_4 = \begin{pmatrix} a_4 & -a_4 u \\ \frac{a_4}{u} & -a_4 \end{pmatrix}. \tag{A.14}$$

From the fact that the sum of the residues is zero we get the linear system

$$\begin{cases} a_4 = 0 \\ b_1 + b_2 + b_3 - a_4 u = 0 \\ \frac{a_4}{u} = 0 \end{cases} \tag{A.15}$$

which implies $a_4 = 0, b_1 = -b_2 - b_3$. Even if $\tilde{\Phi}(z)$ in this case is nilpotent, we will exclude this parabolic structure since a Higgs field with vanishing residues is not admissible in our situation.

- $L_1, L_2 \in S_2$, L_3, L_4 not contained in S_1 nor in S_2

  After normalization we can consider

$$L_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad L_4 = \begin{pmatrix} u \\ 1 \end{pmatrix}. \tag{A.16}$$

  Using the condition $\tilde{\phi}_j L_j = 0$, $j = 1, \ldots, 4$ we obtain that the residues of the Higgs field $\tilde{\Phi}$ have the form:

$$\tilde{\phi}_1 = \begin{pmatrix} 0 & 0 \\ c_1 & 0 \end{pmatrix}, \quad \tilde{\phi}_2 = \begin{pmatrix} 0 & 0 \\ c_2 & 0 \end{pmatrix}, \quad \tilde{\phi}_3 = \begin{pmatrix} a_3 & -a_3 \\ a_3 & -a_3 \end{pmatrix}, \quad \tilde{\phi}_4 = \begin{pmatrix} a_4 & -a_4 u \\ \frac{a_4}{u} & -a_4 \end{pmatrix}. \tag{A.17}$$

  From the fact that the sum of the residues is zero we get the linear system

$$\begin{cases} a_3 + a_4 = 0 \\ -a_3 - a_4 u = 0 \\ c_1 + c_2 + a_3 + \frac{a_4}{u} = 0 \end{cases} \tag{A.18}$$

  and from the first and from the second equations we have that $a_3 = a_4 = 0$ and so $c_1 = -c_2$. Even if $\tilde{\Phi}(z)$ in this case is nilpotent, we will exclude this parabolic structure since a Higgs field with vanishing residues is not admissible in our situation.

- $L_1, L_2, L_3 \in S_2$, L_4 not contained in S_1 nor in S_2

  After normalization we can consider

$$L_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad L_4 = \begin{pmatrix} u \\ 1 \end{pmatrix}. \tag{A.19}$$

Using the condition $\tilde{\phi}_j L_j = 0,\ j = 1, \ldots, 4$ we obtain that the residues of the Higgs field $\tilde{\Phi}$ have the form:

$$\tilde{\phi}_1 = \begin{pmatrix} 0 & 0 \\ c_1 & 0 \end{pmatrix}, \quad \tilde{\phi}_2 = \begin{pmatrix} 0 & 0 \\ c_2 & 0 \end{pmatrix}, \quad \tilde{\phi}_3 = \begin{pmatrix} 0 & 0 \\ c_3 & 0 \end{pmatrix}, \quad \tilde{\phi}_4 = \begin{pmatrix} a_4 & -a_4 u \\ \frac{a_4}{u} & -a_4 \end{pmatrix}. \tag{A.20}$$

From the fact that the sum of the residues is zero we get the linear system

$$\begin{cases} a_4 = 0 \\ c_1 + c_2 + c_3 + \frac{a_4}{u} = 0 \\ -a_4 u = 0 \end{cases} \tag{A.21}$$

which implies $a_4 = 0, c_1 = -c_2 - c_3$. Even if $\tilde{\Phi}(z)$ in this case is nilpotent, we will exclude this parabolic structure since a Higgs field with vanishing residues is not admissible in our situation.

A.2 Other parabolic structures for $\mathcal{O}(-1) \oplus \mathcal{O}(-3)$

The following computation shows that there are no parabolic structure on $\mathcal{O}(-1) \oplus \mathcal{O}(-3)$ which admits a nilpotent parabolic Higgs field $\tilde{\Phi}$ with non vanishing residues at the points $z_1, \ldots, z_4$ like the one considered in Subsection 4.5.3. Let $L_1, \ldots L_4$ be the parabolic lines at the branch points $z_1, \ldots, z_4$ of the map $\tilde{\pi} : \tilde{M} \to \mathbb{CP}^1$.

- $L_1, L_2, L_3 \in \mathcal{O}(-3)$ and L_4 not contained in $\mathcal{O}(-3)$ nor in $\mathcal{O}(-1)$

  Using the condition $\tilde{\phi}_j L_j = 0,\ j = 1, \ldots, 4$ we obtain that the residues of the Higgs field $\tilde{\Phi}$ have the form:

$$\tilde{\phi}_1 = \begin{pmatrix} 0 & 0 \\ c_1 & 0 \end{pmatrix}, \quad \tilde{\phi}_2 = \begin{pmatrix} 0 & 0 \\ c_2 & 0 \end{pmatrix}, \quad \tilde{\phi}_3 = \begin{pmatrix} 0 & 0 \\ c_3 & 0 \end{pmatrix}, \quad \tilde{\phi}_4 = \begin{pmatrix} a_4 & -a_4 u \\ \frac{a_4}{u} & -a_4 \end{pmatrix}. \tag{A.22}$$

  Since the residues of $\tilde{\Phi}$ must satisfies the condition described in Subsection 4.5.3, we obtain the linear system:

$$\begin{cases} a_4 = 0 \\ c_2 = ic_1 \\ c_3 = -c_1 \\ \frac{a_4}{u} = -ic_1 \end{cases} \tag{A.23}$$

Thus, even if $\tilde{\Phi}(z)$ in this case is nilpotent, we will exclude this parabolic structure since the all the residues are vanishing and this is not admissible.

- $L_1, L_2, L_3, L_4 \in \mathcal{O}(-3)$

  Using the condition $\tilde{\phi}_j L_i = 0$, $j = 1, \dots, 4$ we obtain that the residues of the Higgs field $\tilde{\Phi}$ have the form:

$$\tilde{\phi}_1 = \begin{pmatrix} 0 & 0 \\ c_1 & 0 \end{pmatrix}, \quad \tilde{\phi}_2 = \begin{pmatrix} 0 & 0 \\ c_2 & 0 \end{pmatrix}, \quad \tilde{\phi}_3 = \begin{pmatrix} 0 & 0 \\ c_3 & 0 \end{pmatrix}, \quad \tilde{\phi}_4 = \begin{pmatrix} 0 & 0 \\ c_4 & 0 \end{pmatrix}. \tag{A.24}$$

  Since the residues of $\tilde{\Phi}$ must satisfies the condition described in Subsection 4.5.3, we obtain the linear system:

$$\begin{cases} c_2 = ic_1 \\ c_3 = -c_1 \\ c_4 = -ic_1 \end{cases} \tag{A.25}$$

  Thus the parabolic Higgs field $\tilde{\Phi}$ is given by

$$\tilde{\Phi}(z) = \begin{pmatrix} 0 & 0 \\ c_1 & 0 \end{pmatrix} \frac{dz}{z-1} + \begin{pmatrix} 0 & 0 \\ ic_1 & 0 \end{pmatrix} \frac{dz}{z-i} + \begin{pmatrix} 0 & 0 \\ -c_1 & 0 \end{pmatrix} \frac{dz}{z+1} + \begin{pmatrix} 0 & 0 \\ -ic_1 & 0 \end{pmatrix} \frac{dz}{z+i} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} dz. \tag{A.26}$$

  We can now check for which values of b and c_1 the parabolic Higgs field $\tilde{\Phi}$ is nilpotent.

$$\tilde{\Phi}^2(z) = \begin{pmatrix} 4bc_1 & 0 \\ 0 & 4bc_1 \end{pmatrix} \frac{dz}{z^4 - 1}. \tag{A.27}$$

  Thus, the only possibilities are $b = 0$ or $c_1 = 0$, which contradicts the fact that b and c_1 are non zero constant. We conclude that this parabolic structure on $\mathcal{O}(-1) \oplus \mathcal{O}(-3)$ is not admissible.

- $L_1, L_2, L_3 \in \mathcal{O}(-3)$ and $L_4 \in \mathcal{O}(-1)$

  Using the condition $\tilde{\phi}_j L_j = 0$, $j = 1, \dots, 4$ we obtain that the residues of the Higgs field $\tilde{\Phi}$ have the form:

$$\tilde{\phi}_1 = \begin{pmatrix} 0 & 0 \\ c_1 & 0 \end{pmatrix}, \quad \tilde{\phi}_2 = \begin{pmatrix} 0 & 0 \\ c_2 & 0 \end{pmatrix}, \quad \tilde{\phi}_3 = \begin{pmatrix} 0 & 0 \\ c_3 & 0 \end{pmatrix}, \quad \tilde{\phi}_4 = \begin{pmatrix} 0 & b_4 \\ 0 & 0 \end{pmatrix}. \tag{A.28}$$

Since the residues of $\tilde{\Phi}$ must satisfies the condition described in Subsection 4.5.3, we obtain the linear system:

$$\begin{cases} c_2 = ic_1 \\ c3 = -c_1 \\ 0 = -ic_1 \end{cases} \tag{A.29}$$

Thus, $c_1 = 0$ and we obtain that the residues $\tilde{\phi}_1, \tilde{\phi}_2$ and $\tilde{\phi}_3$ are vanishing which makes the parabolic structure non admissible.

- $L_1, L_2 \in \mathcal{O}(-3)$ and $L_3, L_4 \in \mathcal{O}(-1)$

  Using the condition $\tilde{\phi}_j L_j = 0$, $j = 1, \ldots, 4$ we obtain that the residues of the Higgs field $\tilde{\Phi}$ have the form:

$$\tilde{\phi}_1 = \begin{pmatrix} 0 & 0 \\ c_1 & 0 \end{pmatrix}, \quad \tilde{\phi}_2 = \begin{pmatrix} 0 & 0 \\ c_2 & 0 \end{pmatrix}, \quad \tilde{\phi}_3 = \begin{pmatrix} 0 & b_3 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\phi}_4 = \begin{pmatrix} 0 & b_4 \\ 0 & 0 \end{pmatrix}. \tag{A.30}$$

  Since the residues of $\tilde{\Phi}$ must satisfies the condition described in Subsection 4.5.3, we obtain the linear system:

$$\begin{cases} c_2 = ic_1 \\ 0 = -c_1 \\ 0 = -ic_1 \end{cases} \tag{A.31}$$

  Thus, $c_1 = c_2 = 0$ and we obtain that the residues $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are vanishing which makes the parabolic structure non admissible.

- $L_1, L_2, L_3 \in \mathcal{O}(-1)$ and $L_4 \in \mathcal{O}(-3)$

  Using the condition $\tilde{\phi}_j L_j = 0$, $j = 1, \ldots, 4$ we obtain that the residues of the Higgs field $\tilde{\Phi}$ have the form:

$$\tilde{\phi}_1 = \begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\phi}_2 = \begin{pmatrix} 0 & b_2 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\phi}_3 = \begin{pmatrix} 0 & b_3 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\phi}_4 = \begin{pmatrix} 0 & 0 \\ c_4 & 0 \end{pmatrix}. \tag{A.32}$$

  Since the residues of $\tilde{\Phi}$ must satisfies the condition described in Subsection 4.5.3, we obtain the linear system:

$$\begin{cases} c_2 = ic_1 \\ c_3 = -c_1 \\ c_4 = -ic_1 \end{cases} \tag{A.33}$$

Thus, $c_4 = 0$ and we obtain that the residues $\tilde{\phi}_4$ is vanishing which makes the parabolic structure non admissible.

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