# Stringy Invariants of Algebraic Varieties and Lattice Polytopes

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# Chapter 1

## Introduction

Mirror symmetry is a pillar of modern mathematical and physical research driven by an active and interdisciplinary scientific community. Following proceedings at the International Congress of Mathematicians (ICM), Victor Batyrev stated that

'*mirror symmetry is a remarkable discovery by physicists who suggest that partition functions of two physical theories obtained from two different Calabi-Yau manifolds $V$ and $V^*$ can be identified* [Mor95]. *So far mathematicians could not find any appropriate language for a rigorous formulation of this identification [...]. Without knowing a mathematical reason for mirror symmetry it simply remains to believe in its existence. This belief is supported by many computational experiments followed by attempts to find rigorous mathematical explanations of their results. [...] Loosely speaking, toric geometry provides some kind of 'platonic' approach to mirror symmetry, because it replaces the highly non-trivial duality between some mathematical objects, which we still do not completely know, by an elementary duality of certain polyhedra. Of course, such a simplification cannot reflect the whole nature of mirror symmetry, but it helps to form our intuition and find reasonable mathematical tests for this duality.*'[1]

To reach a better understanding of this broad topic, we refer the reader to the book *Mirror Symmetry* [HKK$^+$03], which develops mirror symmetry from both physical and mathematical perspectives in order to facilitate further interaction between the two fields.

Physicists have predicted the following basic and extended versions of *topological mirror symmetry tests* involving Hodge numbers and Euler numbers of two $d$-dimensional *smooth* Calabi-Yau mirror symmetry candidates $V$ and $V^*$:

$$e(V) = (-1)^d e(V^*) \qquad \text{and} \qquad h^{p,q}(V) = h^{d-p,q}(V^*) \ \ (0 \leq p, q \leq d)$$

[Wit92, Mor99]. The basic topological mirror symmetry test is passed if the Euler numbers of both tested partners are equal up to sign. To pass the extended topological mirror symmetry test, the Hodge numbers of the tested pair have to fulfil symmetry conditions. To this day, no universally accepted mathematical definition exists to answer the

[1] [Bat98a, page 1, line 1-17]

question when two varieties are considered to be mirror dual. But if we have reason to believe that $(V, V^*)$ are a mirror pair, passing the above mirror symmetry tests provide further evidence to support this assumption. In the future, we would, of course, prefer an independent definition comprising these two mirror symmetry tests as implications.

The discovery of mirror symmetry by the physics community is based on a lot of examples of mirror pairs $(V, V^*)$, where both mirror partners are smooth [GP90, CLS90, CdlOGP91]. Nevertheless, they also found a lot of examples of mirror pairs $(X, X^*)$ consisting of at least one non-smooth Calabi-Yau variety [GP90, Bat94, BvS95, BB97, Bor93]. From a mathematical perspective, these circumstances lead to substantial difficulties because the presence of at least one non-smooth Calabi-Yau mirror symmetry partner results in negative topological mirror symmetry tests. The reason for these difficulties is due to the fact that both applied mirror symmetry tests include usual Hodge numbers and usual Euler numbers of both partners [Bat98b].

In this thesis, we see traditional mirror symmetry as a starting point for new developments from the mathematical perspective, with a small contribution from physics. To establish the possibility for slightly modified topological mirror symmetry tests as

$$e_{\mathrm{str}}(X) = (-1)^d e_{\mathrm{str}}(X^*),$$

where $e_{\mathrm{str}}(\cdot)$ denotes the stringy Euler number of certain non-smooth varieties, we investigate various stringy invariants. More specifically, this thesis aims to understand stringy invariants on toric varieties from a combinatorial perspective. Our interest is motivated by the well-known construction of many examples of Calabi-Yau varieties and their mirrors as hypersurfaces and complete intersections in Gorenstein toric Fano varieties [Bat94, BB97, BD96]. Here, the mirror symmetry property agrees with a combinatorial duality between certain polytopes, in conformance with the 'platonic' approach mentioned in the proceedings quoted at the beginning. This is attributable to the fact that toric varieties $X$ arise together with a variety-fan dictionary, *i.e.*, they are completely determined by combinatorial objects, called *fans*, that are certain finite collections of rational polyhedral cones in the real vector space $N_{\mathbb{R}} \cong \mathbb{R}^d$. The potential of this correspondence is apparent: we can easily read off a multitude of properties about $X$ through a combinatorial exploration of the associated fan, including statements about the variety's smoothness and orbits. It has been a long-held belief in the mathematical community that many more invariants are combinatorially encoded in this tight relationship [CLS11]. This potential in the interaction of toric geometry and combinatorial methods is exploited in principled and elegant ways by applications to lattice polytopes [Oda85, Ful93, CLS11]. It connects hard algebra-geometric objects, such as projective toric varieties, and intuitive convex-geometric objects, such as lattice polytopes. This replaces the above-mentioned dictionary dealing with fan explorations. Studying this interplay in its full breadth and uncovering the relationship between these two worlds are among the most central aspects of this thesis.

In particular, this thesis on *Stringy Invariants of Algebraic Varieties and Lattice Polytopes* investigates topological invariants in a more general, non-smooth setting and provides answers to the following questions: (1) Are there stringy, *i.e.*, non-smooth, versions of topological invariants, such as Hodge numbers, Euler numbers, and Chern classes?; (2) What are the implications if we drop the smoothness assumption of the toric variety?; (3) Can we discover formulas to compute these stringy invariants in combinatorial ways that are as elegant as the ones for smooth toric varieties?; (4) Do applications to toric varieties corresponding to lattice polytopes end in plain combinatorial identities?

## 1.1 Results and Overview

Before we begin to answer these questions in detail, Chapter 2 fixes our terminology and provides the necessary prerequisites for the subsequent chapters. The results included in Chapter 3, 4, 5, and 6 arose as collaborative work authored with Victor Batyrev [BS17, BS18]. The contents of Chapter 7 are intermediate results of an ongoing research project with Victor Batyrev and Alexander Kasprzyk [BKS18]. A brief overview of all results is provided in the remainder of this Introduction. All statements in form of 'theorems' or 'propositions' have been obtained through these joint collaborative efforts.

### 1.1.1 Algebraic Varieties

Let $X$ be a $d$-dimensional normal projective $\mathbb{Q}$-Gorenstein variety with at worst log-terminal singularities, *i.e.*, the canonical divisor $K_X$ on $X$ is a $\mathbb{Q}$-Cartier divisor and for some desingularization $\rho : Y \to X$ of $X$, whose exceptional locus is a union of smooth irreducible divisors $D_1, \ldots, D_s$ with only simple normal crossings [CLS11, page 525], one has

$$K_Y = \rho^* K_X + \sum_{i=1}^{s} a_i D_i \tag{1.1}$$

for some rational numbers $a_i > -1$ ($1 \leq i \leq s$). The above desingularization $\rho$ is called *log-desingularization* of $X$. For any non-empty subset $J \subseteq I \coloneqq \{1, \ldots, s\}$, we define $D_J$ to be the subvariety $\cap_{j \in J} D_j$ together with its closed embedding $e_J : D_J \hookrightarrow Y$ and set $D_\emptyset \coloneqq Y$. We note that $D_J \subseteq Y$ is either empty or a smooth projective subvariety of $Y$ of codimension $|J|$.

Chapter 3 centers on the definition of stringy Chern classes of the above-mentioned singular varieties $X$ and formulas for stringy Chern classes of complete intersections of generic semi-ample Cartier divisors on these varieties. This allows us to compute intersection numbers of stringy Chern classes and $\mathbb{Q}$-Cartier divisors. But first, let us briefly make some already known remarks:

If $V$ is an arbitrary smooth projective variety of dimension $d$, the *E-polynomial* of $V$ is defined as

$$E(V; u, v) \coloneqq \sum_{0 \leq p,q \leq d} (-1)^{p+q} h^{p,q}(V)\, u^p v^q, \tag{1.2}$$

where $h^{p,q}(V)$ denotes the Hodge numbers of $V$. Furthermore, the *Euler number* $e(V) = c_d(V)$ of $V$ equals $E(V;1,1)$. The *stringy $E$-function* of the singular variety $X$ given above is a rational algebraic function in two variables $u, v$ defined by the formula

$$E_{\mathrm{str}}(X;u,v) \coloneqq \sum_{\emptyset \subseteq J \subseteq I} E(D_J;u,v) \prod_{j \in J} \left( \frac{uv-1}{(uv)^{a_j+1}-1} - 1 \right). \tag{1.3}$$

One can prove that the stringy $E$-function $E_{\mathrm{str}}(X;u,v)$ does not depend on the choice of the log-desingularization $\rho$ [Bat98b, Theorem 3.4]. As a special case, this formula implies $E_{\mathrm{str}}(X;u,v) = E(X;u,v)$ if $X$ is smooth and $E_{\mathrm{str}}(X;u,v) = E(Y;u,v)$ if $\rho$ is a crepant desingularization of $X$ (*i.e.*, $a_i = 0$ for all $i \in I$). The *top stringy Chern class* $c_d^{\mathrm{str}}(X)$ (or the *stringy Euler number* $e_{\mathrm{str}}(X)$) of $X$ is defined as the limit of the stringy $E$-function, *i.e.*,

$$c_d^{\mathrm{str}}(X) = e_{\mathrm{str}}(X) \coloneqq \lim_{u,v \to 1} E_{\mathrm{str}}(X;u,v) = \sum_{\emptyset \subseteq J \subseteq I} c_{d-|J|}(D_J) \prod_{j \in J} \left( \frac{-a_j}{a_j+1} \right), \tag{1.4}$$

where $c_{d-|J|}(D_J) = e(D_J)$ denotes the usual Euler number of the smooth subvariety $D_J \subseteq Y$ with $\dim(D_J) = d - |J|$ [Bat98b, Definition 3.3]. We view the stringy top Chern class $c_d^{\mathrm{str}}(X)$ as a special case of the *$k$-th stringy Chern classes* defined in [Alu05, dFLNU07] for any $k$ ($0 \le k \le d$). As is the case with stringy $E$-functions, the definition of stringy Chern classes is also independent of the log-desingularization $\rho$ [dFLNU07].

In Chapter 3, Section 3.1, we prove and apply the following formula for the computation of $k$-th stringy Chern classes using the usual Chern classes of smooth projective subvarieties:

**Proposition 3.1.1.** *Let $X$ be a $d$-dimensional normal projective $\mathbb{Q}$-Gorenstein variety with at worst log-terminal singularities. Then*

$$c_k^{\mathrm{str}}(X) = \rho_* \left( \sum_{\emptyset \subseteq J \subseteq I} e_{J*} c_{k-|J|}(D_J) \prod_{j \in J} \left( \frac{-a_j}{a_j+1} \right) \right) \in A_{d-k}(X)_{\mathbb{Q}},$$

*where $A_{d-k}(X) = A^k(X)$ denotes the Chow group of $(d-k)$-dimensional cycles on $X$ modulo rational equivalence and $A_\bullet(X)_{\mathbb{Q}} = \bigoplus_{k=0}^d A_{d-k}(X)_{\mathbb{Q}}$ with $A_{d-k}(X)_{\mathbb{Q}} = A_{d-k}(X) \otimes \mathbb{Q}$ the rational Chow ring of $X$. Moreover, $\rho_* : A_{d-k}(Y) \to A_{d-k}(X)$ and $e_{J*} : A_{d-k}(D_J) \to A_{d-k}(Y)$ are push-forward homomorphisms corresponding to the proper birational morphism $\rho : Y \to X$ and the closed embedding $e_J : D_J \hookrightarrow Y$, respectively.*

As a special case, this formula implies

$$c_k^{\mathrm{str}}(X) = c_k(X) \tag{1.5}$$

if $X$ is smooth and $c_k^{\mathrm{str}}(X) = c_k(Y)$ if $\rho$ is a crepant desingularization of $X$.

In addition, we prove that the well-known formula expressing total Chern classes $c_\bullet(\cdot)$ of smooth complete intersections via the total Chern classes of the ambient smooth varieties $V$ can be generalized to total stringy Chern classes $c_\bullet^{\mathrm{str}}(\cdot)$ of generic hypersurfaces and complete intersections in the singular ambient varieties $X$:

**Theorem 3.1.5.** *Let $X$ be a normal projective $\mathbb{Q}$-Gorenstein variety with at worst log-terminal singularities and $Z_1, \ldots, Z_r$ generic semi-ample Cartier divisors on $X$ such that $[Z] \coloneqq [Z_1] = \ldots = [Z_r]$. Then*

$$i_* c_\bullet^{\mathrm{str}}(Z_1 \cap \ldots \cap Z_r) = c_\bullet^{\mathrm{str}}(X). \prod_{j=1}^{r} [Z_j] (1 + [Z_j])^{-1},$$

*where $i : Z_1 \cap \ldots \cap Z_r \hookrightarrow X$ denotes the corresponding closed embedding.*

In particular, Corollary 3.1.4 shows that the top stringy Chern class (or stringy Euler number) of generic semi-ample Cartier divisors $Z$ on $X$ can be computed via stringy Chern classes of the ambient variety $X$ as

$$c_{d-1}^{\mathrm{str}}(Z) = e_{\mathrm{str}}(Z) = \sum_{k=1}^{d} (-1)^{k-1} [Z]^k . c_{d-k}^{\mathrm{str}}(X). \tag{1.6}$$

In Corollary 3.1.6, we give a similar formula for the top stringy Chern class

$$c_{d-r}^{\mathrm{str}}(Z_1 \cap \ldots \cap Z_r)$$

of complete intersections $Z_1 \cap \ldots \cap Z_r$, where $Z_1, \ldots, Z_r$ are generic semi-ample Cartier divisors on the singular variety $X$.

In Section 3.2, we study intersection numbers of stringy Chern classes with $\mathbb{Q}$-Cartier divisors. A particular case of such an intersection number appears in the stringy version of the Libgober-Wood identity

$$\frac{d^2}{du^2} E_{\mathrm{str}}(X; u, 1)\Big|_{u=1} = \frac{3d^2 - 5d}{12} c_d^{\mathrm{str}}(X) + \frac{1}{6} c_1(X).c_{d-1}^{\mathrm{str}}(X) \tag{1.7}$$

[Bat00, Theorem 3.8], where the intersection number $c_1(X).c_{d-1}^{\mathrm{str}}(X)$ is defined as

$$c_1(X).c_{d-1}^{\mathrm{str}}(X) \coloneqq \sum_{\emptyset \subseteq J \subseteq I} \rho^* c_1(X).e_{J*} c_{d-|J|-1}(D_J) \prod_{j \in J} \left( \frac{-a_j}{a_j + 1} \right) \tag{1.8}$$

and its independence on the choice of the log-desingularization $\rho$ is shown in [Bat00, Corollary 3.9]. In this thesis, we consider more general intersection numbers $[Z_1]. \ldots .[Z_k]. c_{d-k}^{\mathrm{str}}(X)$, which can be defined by a similar formula:

**Theorem 3.2.1.** *Let $X$ be a $d$-dimensional normal projective $\mathbb{Q}$-Gorenstein variety with at worst log-terminal singularities. Then*

$$[Z_1]. \ldots .[Z_k].c_{d-k}^{\mathrm{str}}(X) \coloneqq \sum_{\emptyset \subseteq J \subseteq I} \rho^*[Z_1]. \ldots .\rho^*[Z_k].e_{J*} c_{d-|J|-k}(D_J) \prod_{j \in J} \left( \frac{-a_j}{a_j + 1} \right),$$

*where $Z_1, \ldots, Z_k \subseteq X$ are arbitrary $\mathbb{Q}$-Cartier divisors on $X$.*

We present proof for its independence on the choice of the log-desingularization $\rho$ without using the definition of stringy Chern classes.

### 1.1.2 Toric Varieties

In the chapters following our treatment of algebraic varieties, we leave this general setting and turn to the subclass of toric varieties, also referred to as the 'platonic' approach mentioned in the quote at the very beginning of the Introduction.

A *toric variety* $X$ is a normal variety over the field of complex numbers $\mathbb{C}$ containing a torus $\mathbb{T}^d \cong (\mathbb{C}^*)^d$ as a Zariski open set such that the action of $(\mathbb{C}^*)^d$ on itself extends to an action on $X$ [Ful93, CLS11]. For additional information, we refer the reader to Chapter 2.

Chapter 4 applies our contributions to stringy Chern classes of toric varieties in order to reveal formulas that are completely determined by combinatorial objects. In Section 4.1, we focus on stringy Chern classes of projective $\mathbb{Q}$-Gorenstein toric varieties $X$ associated with fans $\Sigma$ of rational polyhedral cones in $N_{\mathbb{R}}$. $\Sigma(k)$ denotes the set of all $k$-dimensional cones in a fan $\Sigma$, $v(\sigma)$ the normalized volume of a cone $\sigma \in \Sigma$ (*i.e.*, the normalized volume of a lattice polytope obtained as the convex hull of the origin and all primitive ray generators of the cone $\sigma$), and $[X_\sigma]$ the class of the closed torus orbit $X_\sigma$ corresponding to a given cone $\sigma \in \Sigma$. In this setting, the basis for further research is our formula presented below:

**Theorem 4.1.2.** *Let $X$ be a projective $\mathbb{Q}$-Gorenstein toric variety of dimension $d$ associated with a fan $\Sigma$. Then*

$$c_k^{\mathrm{str}}(X) = \sum_{\sigma \in \Sigma(k)} v(\sigma) \cdot [X_\sigma]$$

*for $0 \leq k \leq d$.*

Here, the stringy Chern classes are computed in a purely combinatorial way and our formula simultaneously illustrates their independence of the chosen desingularization. If $X$ is smooth, we already know from Equation (1.5) that $c_k(X) = c_k^{\mathrm{str}}(X)$. A combination of these two facts yields the well-known formula $c_k(X) = \sum_{\sigma \in \Sigma(k)} [X_\sigma]$ because each cone of a smooth variety $X$ is smooth and therefore has normalized volume 1.

In addition, we compute in Theorem 4.1.10 intersection numbers

$$[D_1]. \ldots .[D_k].c_{d-k}^{\mathrm{str}}(X),$$

where $D_i = \sum_{\rho \in \Sigma(1)} a_\rho D_{i_\rho}$ $(1 \leq i \leq k)$ are semi-ample torus-invariant $\mathbb{Q}$-Cartier divisors on the toric variety $X$. This results in a simple combinatorial formula for the computation of the stringy Euler number

$$c_{d-r}^{\mathrm{str}}\,(Z_1 \cap \ldots \cap Z_r)$$

of complete intersections of $r$ generic semi-ample Cartier divisors $Z_i$ $(1 \leq i \leq r)$ on $X$ with $[Z_1] = \ldots = [Z_r] = [D]$ (Corollary 4.1.9).

In Section 4.2, we establish an equivalent combinatorial interpretation of the stringy Libgober-Wood identity (1.7) for projective $\mathbb{Q}$-Gorenstein toric varieties $X$ of Gorenstein index $q_X$ associated with a fan $\Sigma$ in $N_{\mathbb{R}}$ and a semi-ample anticanonical torus-invariant $\mathbb{Q}$-Cartier divisor $-K_X = \sum_{\rho\in\Sigma(1)} D_\rho$ on $X$. To this end, we show that the stringy $E$-function $E_{\text{str}}(X; u, v)$ of $X$ can be written as a finite sum:

**Proposition 4.2.1.** *Let $X$ be a $d$-dimensional projective $\mathbb{Q}$-Gorenstein toric variety of Gorenstein index $q_X$ associated with a fan $\Sigma$ in $N_{\mathbb{R}}$. Then*

$$E_{\text{str}}(X; u, v) = \sum_{\alpha\in[0,d]\cap\frac{1}{q_X}\mathbb{Z}} \psi_\alpha(\Sigma)(uv)^\alpha,$$

*where the coefficients $\psi_\alpha(\Sigma)$ are non-negative integers satisfying the conditions $\psi_0(\Sigma) = \psi_d(\Sigma) = 1$ and $\psi_\alpha(\Sigma) = \psi_{d-\alpha}(\Sigma)$ for all $\alpha \in [0, d] \cap \frac{1}{q_X}\mathbb{Z}$.*

A comparison with the usual $E$-polynomial, which has Hodge numbers as coefficients, and its properties leads us to call the coefficients $\psi_\alpha(\Sigma)$ of the stringy $E$-function *generalized stringy Hodge numbers*. With this notion of the stringy $E$-function in mind, the combinatorial version of the stringy Libgober-Wood identity (1.7) looks as follows:

**Theorem 4.2.2.** *Let $X$ be a $d$-dimensional projective $\mathbb{Q}$-Gorenstein toric variety of Gorenstein index $q_X$ associated with a fan $\Sigma$ in $N_{\mathbb{R}}$ and $-K_X = \sum_{\rho\in\Sigma(1)} D_\rho$ a semi-ample anticanonical torus-invariant $\mathbb{Q}$-Cartier divisor on $X$. Then*

$$\sum_{\alpha\in[0,d]\cap\frac{1}{q_X}\mathbb{Z}} \psi_\alpha(\Sigma)\left(\alpha - \frac{d}{2}\right)^2 = \frac{d}{12}v(\Sigma) + \frac{1}{6}\sum_{\sigma\in\Sigma(d-1)} v(\sigma)\cdot v\left(\Delta^\sigma_{-K_X}\right),$$

*where $v(\Sigma) = \sum_{\sigma\in\Sigma(d)} v(\sigma)$ denotes the normalized volume of the fan $\Sigma$, $\Delta^\sigma_{-K_X} = \{y \in \Delta_{-K_X} \mid \langle y, u_\rho\rangle = -1\ \forall\rho \in \Sigma(1) \text{ with } \rho \subseteq \sigma\} \preceq \Delta_{-K_X}$ a face of the rational polytope $\Delta_{-K_X} = \{y \in M_{\mathbb{R}} \mid \langle y, u_\rho\rangle \geq -1\ \forall\rho \in \Sigma(1)\} \subseteq M_{\mathbb{R}}$ corresponding to a cone $\sigma \in \Sigma$, and $u_\rho$ the primitive ray generator of a $1$-dimensional cone $\rho \in \Sigma(1)$.*

### 1.1.3 Lattice Polytopes

Chapter 5 and 6 are dedicated to the direct correspondence between projective toric varieties and lattice polytopes. We show how this relationship gives rise to mostly combinatorial proofs, thereby reflecting the simple elegance of the combinatorial formulas and identities themselves. For more information on polytope-fan and fan-variety constructions, we refer the reader to Chapter 2.

Chapter 5 examines the combinatorial formulas computing the stringy $E$-functions $E_{\text{str}}(X_i;\, u, v)$ $(1 \leq i \leq 3)$ of Gorenstein toric Fano varieties $X_1$, log del Pezzo surfaces $X_2$, and canonical toric Fano threefolds $X_3$. The common principle underlying the three formulas in the following theorem is their usage of intrinsic information provided by the underlying polytopes, which are reflexive polytopes $\Delta_1$, $LDP$-polygons $\Delta_2$, and canonical Fano 3-polytopes $\Delta_3$, respectively:

**Theorem 5.1.3, 5.2.2, 5.3.1.** *Let $X_i$ and $\Delta_i$ $(1 \leq i \leq 3)$ be given as above. Then the stringy $E$-function of a*

◇ *Gorenstein toric Fano variety* $X_1$ *is given by*

$$E_{\mathrm{str}}(X_1; u, v) = \psi_d(\Delta_1)(uv)^d + \ldots + \psi_1(\Delta_1)(uv) + \psi_0(\Delta_1),$$

*where the generalized stringy Hodge numbers* $\psi_\alpha(\Sigma_1)$ *are the non-negative integral coefficients* $\psi_\alpha(\Delta_1)$ $(\alpha \in [0, d] \cap \mathbb{Z})$ *in the numerator of the Ehrhart power series in* Definition 5.1.1.

◇ *log del Pezzo surface* $X_2$ *is given by*

$$E_{\mathrm{str}}(X_2; u, v) = ((uv)^2 + 1) + r \cdot (uv) + \sum_{n \in \Delta_2^\circ \setminus \{0\}} \left((uv)^{2+\kappa_{\Delta_2}(n)} + (uv)^{-\kappa_{\Delta_2}(n)}\right)$$

*with* $r := |\partial\Delta_2 \cap N| - 2$*, where* $\kappa_{\Delta_2}$ *denotes the* $\Sigma_{\Delta_2}$*-piecewise linear function corresponding to the anticanonical divisor* $-K_{X_2}$ *on* $X_2$ (Equation (2.3))*,* $\Delta_2^\circ$ *the interior of the polytope* $\Delta_2$*, and* $\partial\Delta_2$ *the boundary of* $\Delta_2$*.*

◇ *canonical toric Fano threefold* $X_3$ *is given by*

$$E_{\mathrm{str}}(X_3; u, v) = \left((uv)^3 + 1\right) + r \cdot \left((uv)^2 + (uv)\right) + \sum_{\substack{\theta \preceq \Delta_3 \\ \dim(\theta)=2, n_\theta > 1}} v(\theta) \cdot \left(\sum_{k=1}^{n_\theta - 1} (uv)^{\frac{k}{n_\theta}+1}\right)$$

*with* $r := |\Delta_3 \cap N| - 4$*, where* $n_\theta$ *denotes the lattice distance from a face* $\theta \preceq \Delta_3$ *of* $\Delta_3$ *to the origin* (Definition 2.3.5) *and* $v(\theta)$ *the normalized volume of a polytope* $\theta$ (Definition 2.3.6).

Applying these results, we can once more use the toric approach to obtain in Chapter 6 five combinatorial identities relating reflexive polytopes of arbitrary dimension (Theorem 6.1.1), $LDP$-polygons (Theorem 6.2.1), almost reflexive 3-polytopes (Theorem 6.3.1), canonical Fano 3-polytopes (Theorem 6.4.1), and certain Gorenstein polytopes (Proposition 6.5.4 and 6.5.2) to the number 24. Each of these identities is a generalization of at least one of the following well-known identities for reflexive polytopes $\Delta$ in dimension $d = 2$,

$$12 = v(\Delta) \cdot v(\Delta^*) \tag{1.9}$$

(Corollary 6.1.2 and Theorem 6.2.1) and $d = 3$,

$$24 = \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=1}} v(\theta) \cdot v(\theta^*) \tag{1.10}$$

(Corollary 6.1.4 and 6.3.5, [BCF+05, Theorem 4.3]), where a *reflexive polytope* is a lattice polytope with the origin in its interior such that its dual polytope $\Delta^*$ is also a lattice polytope. The dual polytope $\Delta^*$ of a lattice polytope $\Delta$ containing the origin in its interior is again a convex polytope, in general with rational vertices (Definition 2.3.3).

Moreover, $\theta^* \preceq \Delta^*$ denotes a face of $\Delta^*$ dual to the face $\theta \preceq \Delta$ of $\Delta$ (Definition 2.3.4). A possible proof of the identity (1.10) can be obtained from the fact that the Euler number of a smooth $K3$-surface equals 24. Now, one considers a generic affine hypersurface $Z_\Delta$ defined by a Laurent polynomial $g_\Delta \in \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]$ with the reflexive Newton polytope $\Delta$ in the 3-dimensional torus $(\mathbb{C}^*)^3 \subseteq X_{\Sigma_{\Delta^*}}$, where $X_{\Sigma_{\Delta^*}}$ denotes the toric variety corresponding to the normal fan $X_{\Sigma_{\Delta^*}}$ of $\Delta^*$ (Definition 2.3.8). In addition, one applies the following two statements: The projective closure $\overline{Z_\Delta} \subseteq X_{\Sigma_{\Delta^*}}$ of $Z_\Delta$ is a $K3$-surface with at worst canonical singularities [Bat94, Theorem 4.1.9] and the stringy Euler number of $\overline{Z_\Delta}$ equals $\sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=1}} v(\theta) \cdot v(\theta^*)$ [BD96, Corollary 7.10].

The mentioned identity for reflexive polytopes of arbitrary dimension looks as follows:

**Theorem 6.1.1.** *Let $\Delta \subseteq N_\mathbb{R}$ be a $d$-dimensional reflexive polytope. Then*

$$\sum_{\alpha \in [0,d] \cap \mathbb{Z}} \psi_\alpha(\Delta) \left(\alpha - \frac{d}{2}\right)^2 = \frac{d}{12} v(\Delta) + \frac{1}{6} \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=d-2}} v(\theta) \cdot v(\theta^*),$$

*where the generalized stringy Hodge numbers $\psi_\alpha(\Delta)$ are the coefficients in the numerator of the Ehrhart power series* (Theorem 5.1.3).

Choosing $d = 2$ or $d = 3$, this identity leads to the well-known and (with respect to duality) symmetric identities $v(\Delta) + v(\Delta^*) = 12$ (Equation (1.9)) and $\sum_{\theta \preceq \Delta, \dim(\theta)=1} v(\theta) \cdot v(\theta^*) = 24$ (Equation (1.10)), respectively. Theorem 6.1.1 thus provides a positive answer to the natural question as to whether, there is a generalization to reflexive polytopes of arbitrary dimension for these well-known formulas relating reflexive polytopes in dimension 2 and 3 to the number 12 and 24, respectively.

Similar results reveal the following relationship between *LDP-polygons* (*i.e.*, lattice polygons containing the origin in its interior such that its vertices are primitive lattice points) and the number 12:

**Theorem 6.2.1.** *Let $\Delta \subseteq N_\mathbb{R}$ be a LDP-polygon. Then*

$$v(\Delta) + v(\Delta^*) = 12 \sum_{n \in \Delta \cap N} (\kappa_\Delta(n) + 1)^2,$$

*where $X_{\Sigma_\Delta}$ denotes the toric variety corresponding to the spanning fan $\Sigma_\Delta$ of $\Delta$* (Definition 2.3.10) *and $\kappa_\Delta$ the $\Sigma_\Delta$-piecewise linear function corresponding to the anticanonical divisor $-K_{X_{\Sigma_\Delta}}$ on $X_{\Sigma_\Delta}$* (Equation (2.3)).

We continue with two further generalizations of the well-known identity for reflexive polytopes in dimension 3 (Equation (1.10)) to arbitrary 3-dimensional *canonical Fano polytopes* (*i.e.*, lattice polytopes containing only the origin as an interior lattice point). First, we consider a 3-dimensional lattice polytope $\Delta \subseteq N_\mathbb{R}$ such that a generic affine hypersurface $Z_\Delta \subseteq (\mathbb{C}^*)^3$ defined by a Laurent polynomial $f_\Delta \in \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]$ with Newton polytope $\Delta$ is birational to a smooth $K3$-surface. Such a lattice polytope $\Delta$ has to be a 3-dimensional canonical Fano polytope because the number of interior lattice points in $\Delta$ equals the geometric genus of the affine surface $Z_\Delta \subseteq (\mathbb{C}^*)^3$ [Kho78] and it equals 1 for a smooth $K3$-surface. Moreover, one can show that the condition on $Z_\Delta$ to be birational to a smooth $K3$-surface is satisfied if and only if $\Delta$ is contained in some reflexive polytope $\Delta'$. Those 3-dimensional canonical Fano polytopes are called *almost reflexive*. Many equivalent characterizations exist of 3-dimensional almost reflexive polytopes $\Delta$ [Bat17], e.g., $\Delta \subseteq N_\mathbb{R}$ is almost reflexive if and only if the convex hull $[\Delta^*] := \mathrm{conv}(\Delta^* \cap M)$ of all lattice points in the dual polytope $\Delta^* \subseteq M_\mathbb{R}$ is a reflexive polytope. By classifying all toric Fano threefolds with at worst canonical singularities, Kasprzyk found 674,688 isomorphism classes of 3-dimensional canonical Fano polytopes [Kas10]. Using this list of polytopes, Kasprzyk further showed that among all these polytopes there exist exactly 9,089 lattice polytopes $\Delta$ that are not almost reflexive. In other words, there exist exactly 665,599 almost reflexive polytopes in dimension 3. Any almost reflexive polytope $\Delta$ allows us to construct a family of smooth $K3$-surfaces using the toric variety $X_{\Sigma^\Delta}$ associated with the normal fan $\Sigma^\Delta$ of $\Delta$ (Definition 2.3.8).

The first generalization for arbitrary 3-dimensional almost reflexive polytopes is as follows:

**Theorem 6.3.1.** *Let $\Delta \subseteq N_\mathbb{R}$ be a 3-dimensional almost reflexive polytope. Then*

$$24 = v(\Delta) - \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=2}} \frac{1}{n_\theta} \cdot v(\theta) + \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=1}} v(\theta) \cdot v(\theta^*).$$

The proof of this combinatorial formula is built around the more general combinatorial formula

$$e_{\mathrm{str}}(Y) = \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)\geq 1}} (-1)^{\dim(\theta)-1}\, v(\theta) \cdot v(\sigma^\theta \cap \Delta^*) \tag{1.11}$$

[Bat17, Theorem 4.11] computing the stringy Euler number $e_{\mathrm{str}}(Y)$ of a canonical Calabi-Yau hypersurface $Y$ with at worst canonical singularities which is birational to an affine hypersurface $Z_\Delta \subseteq (\mathbb{C}^*)^d$ defined by a generic Laurent polynomial $f_\Delta \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ with Newton polytope $\Delta$. If $d = 3$, as already mentioned, such a Calabi-Yau hypersurface $Y$ exists if and only if $\Delta$ is a 3-dimensional almost reflexive polytope.

It is remarkable that the same identity holds for all 9,089 3-dimensional canonical Fano polytopes that are not almost reflexive, *i.e.*, it holds for all 674,688 3-dimensional canonical Fano polytopes:

**Theorem 6.4.1.** *Let $\Delta \subseteq N_\mathbb{R}$ be a canonical Fano 3-polytope. Then*

$$24 = v(\Delta) - \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=2}} \frac{1}{n_\theta} \cdot v(\theta) + \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=1}} v(\theta) \cdot v(\theta^*).$$

Our proof does not rely on electronic verification, *i.e.*, at this point we have not checked the identity for all 9,089 polytopes step by step.

The key principle underlying these four generalizations is our combinatorial interpretation of the stringy Libgober-Wood identity in terms of generalized stringy Hodge numbers and intersection products of stringy Chern classes (Theorem 4.2.2) as well as our combinatorial version of the stringy $E$-function (Theorem 5.1.3, 5.2.2, and 5.3.1).

In addition to that, we obtain two identities connecting *Gorenstein polytopes of index* $r$ (*i.e.*, lattice polytopes such that their $r$-th dilates are reflexive polytopes) with an index that differs by 1 or 2 from dimension to the number 12 and 24, respectively:

**Proposition 6.5.4, 6.5.2.** *Let $\Delta \subseteq M_{\mathbb{R}}$ be a $d$-dimensional Gorenstein polytope of index $r$. Then*

$$12 = \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta) = r-1}} v(\theta) \cdot v\left(\theta^*\right) + \frac{r(1-r)+2}{2} v\left(\Delta\right)$$

*if $r = d-1$ and*

$$24 = \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta) = r}} v(\theta) \cdot v\left(\theta^*\right) + \frac{r(1-r)}{2} v(\Delta)$$

*if $r = d-2$.*

Chapter 7 takes the purely combinatorial setting from Chapter 6 back to the geometric world. In particular, we classify all surfaces given by generic Laurent polynomials corresponding to 3-dimensional canonical Fano polytopes as Newton polytopes. We consider this project as the starting point for advancing our understanding of stringy invariants and their use in mirror symmetry. In addition, we provide new examples of non-degenerate affine surfaces of geometric genus 1.

The mentioned classification of all toric Fano threefolds with at worst canonical singularities, *i.e.*, of all 674,688 3-dimensional canonical Fano polytopes, opens up an avenue for the targeted surface classification: We apply the combinatorial notion of Fine interior $\Delta^{FI}$ of a lattice polytope $\Delta$, which is an *a priori* infinite intersection of certain affine halfspaces (Definition 2.3.24). By checking whether the Fine interior of such a polytope $\Delta$ consists of a single lattice point, we obtain the result that, among all 674,688 3-dimensional canonical Fano polytopes, only 9,089 correspond to surfaces that are not birational to $K3$-surfaces (Subsection 7.1.1 and 7.2.1). These are exactly the 9,089 non-almost reflexive polytopes. According to a preliminary assessment, we conjecture that, among all 9,089 canonical Fano polytopes, only 49 define surfaces of general type and are of particular interest (Subsection 7.1.4 and 7.2.4). Furthermore, we are claiming that the remaining 9,020 + 20 polytopes define surfaces that are birational to elliptic surfaces of Kodaira dimension 1 (Subsection 7.1.2, 7.1.3, and 7.2.2, 7.2.3). The entire structure of the elliptic fibration over $\mathbb{P}^1$ seems to be completely combinatorially encoded by the unique primitive lattice point $v_\Delta$ of the line containing the 1-dimensional Fine interior $\Delta^{FI}$, where $\Delta$ is one of those 9,020 + 20 polytopes. Such elliptic surfaces have been obtained by Corti and Golyshev [CG11] from $9 = 104 - 95$ examples stemming from 3-dimensional weighted projective spaces with at worst canonical singularities. Their work is a prime example of a paper in which a known result inspired a proof of a more general statement while, at the same time, showing the importance of hidden combinatorial information. Using a generalization of Theorem 6.3.3 that is not limited to $K3$-surfaces, our parallel step is to compute the stringy Euler numbers of the considered surfaces in a purely combinatorial way (Conjecture 7.2.2).

Chapter 8 is a short digression relating our interests in computing the stringy Euler numbers of certain hypersurfaces to physics and, thereby, again branches off into the world of mirror symmetry, exemplified in the following way: In Theorem 8.2.5, we describe the relation that the Witten index of a quasi-smooth Calabi-Yau hypersurface in a weighted projective space computed with the well-known formula of Vafa [Vaf89] in Equation (8.2) equals, up to sign, the stringy Euler number of a Calabi-Yau variety that is birational to a hypersurface in the algebraic torus (Theorem 6.3.2).

Appendix A collects various combinatorial information supporting the surface classification of Chapter 7. By investigating the Fine interior, we sort all 9,089 non-almost reflexive polytopes in dimension 3 into three classes with 9,020, 20, and 49 non-almost reflexive polytopes. In addition, we visualize all three polytope classes, namely: 9 out of 9,020, all 20, and all 49 polytopes that we are claiming to correspond to elliptic surfaces of Kodaira dimension 1, elliptic surfaces of Kodaira dimension 1, and surfaces of general type, respectively.

# Chapter 2

## Background

This background chapter fixes our notation while recalling basic notions and provides a short summary of fundamental concepts used throughout this thesis. All of its content is well-known, does not contain results by the author, and the main references are [CLS11, Ful93, Ful98].

Throughout the thesis, a variety is meant to be an algebraic variety over the complex numbers $\mathbb{C}$. The structure of this chapter takes up the Introduction structure.

## 2.1 Algebraic Varieties

Looking at the wide range of algebraic varieties, we focus, above all, on the ones introduced below:

**Definition 2.1.1.** A normal variety $X$ is called *Gorenstein variety* if the canonical divisor $K_X$ on $X$ is a Cartier divisor and $\mathbb{Q}$-*Gorenstein variety* if the canonical divisor $K_X$ on $X$ is a $\mathbb{Q}$-Cartier divisor.

**Definition 2.1.2.** A normal projective variety $X$ is called *Fano variety* if the anticanonical divisor $-K_X$ on $X$ is an ample $\mathbb{Q}$-Cartier divisor. Moreover, a normal projective surface $X$ with an ample anticanonical $\mathbb{Q}$-Cartier divisor $-K_X$ is called *log del Pezzo surface*.

**Definition 2.1.3.** A complete normal variety $X$ is called *Gorenstein Fano variety* if the anticanonical divisor $-K_X$ on $X$ is an ample Cartier divisor. Thus Gorenstein Fano varieties are projective.

**Definition 2.1.4.** A normal projective variety $Y$ is called *Calabi-Yau variety* if it has at worst canonical Gorenstein singularities and the canonical divisor $K_Y$ on $Y$ is trivial. Moreover, we call $Y$ a $K3$-*surface* if $Y$ is 2-dimensional.

In this thesis, we waive the requirements that the cohomologies of the structure sheaf $\mathcal{O}_Y$ of a Calabi-Yau variety $Y$ has to satisfy the vanishing conditions $H^i(Y, \mathcal{O}_Y) = 0$ for $0 < i < \dim(Y)$.

**Definition 2.1.5.** Let $X$ be a normal variety and $\omega_X$ the canonical sheaf of $X$. Then there exists a Weil divisor $D$ on $X$ such that $\omega_X \cong \mathcal{O}_X(D)$. The divisor $D$ is called *canonical divisor* on $X$ and denoted by $K_X$.

### 2.1.1 Singularities

The present thesis focuses on singular varieties. Therefore, it is important to distinguish several occurring singularities:

**Definition 2.1.6.** Let $X$ be a normal projective variety. A birational morphism $\rho\colon Y \to X$ is called a *log-desingularization* of $X$ if $Y$ is smooth and the exceptional locus of $\rho$ is a union of smooth irreducible divisors $D_1, \ldots, D_s$ with only simple normal crossings [CLS11, page 525].

Assume that $X$ is $\mathbb{Q}$-Gorenstein. Then the canonical divisors of $X$ and $Y$ are related by the ramification formula

$$K_Y = \rho^* K_X + \sum_{i=1}^{s} a_i D_i,$$

where $a_1, \ldots, a_s$ are rational numbers called *discrepancies* . The singularities of $X$ are at worst *log-terminal/canonical/terminal* if $a_i > -1/a_i \geq 0/a_i > 0$ for $1 \leq i \leq s$.

Furthermore, we call $\rho$ *crepant* if $K_Y = \rho^* K_X$, *i.e.*, if all $a_i$ are zero ($1 \leq i \leq s$).

**Definition 2.1.7.** Let $X$ be a normal projective $\mathbb{Q}$-Gorenstein variety. Therefore, $K_X$ is a $\mathbb{Q}$-Cartier divisor, *i.e.*, there exists a positive integer $q_X$ such that $q_X K_X$ is a Cartier divisor on $X$. The *Gorenstein index* of $X$ is the smallest such integer $q_X$.

### 2.1.2 Minimal Model Program

The minimal model program is an important part of algebraic geometry. It deals with the birational classification of algebraic varieties and its basic idea is to simplify this classification by finding in each birational equivalence class a representative variety that roughly speaking is as simple as possible. Considered from the historical point of view, it has its origin in the Enriques-Kodaira classification of surfaces, which categorizes birational equivalence classes of surfaces. A deeper insight can be gained by consulting survey references [Kol89, Rei87, Mor87, KMM87].

In this thesis, the main focus is not on the minimal model program, therefore we need only the below-defined objects:

**Definition 2.1.8.** Let $X$ be a normal projective variety. A normal projective variety $Y$ is called *canonical model* of $X$ if $Y$ is birationally equivalent to $X$ and has at worst canonical $\mathbb{Q}$-Gorenstein singularities.

**Definition 2.1.9.** Let $X$ be a normal projective variety. A normal projective variety $Y$ is called *minimal model* of $X$ if $Y$ is birationally equivalent to $X$, has at worst terminal $\mathbb{Q}$-Gorenstein singularities, and the canonical divisor $K_Y$ on $Y$ is nef.

**Remark 2.1.10.** Let $Y$ be a normal variety and $D$ a Cartier divisor on $Y$. Then $D$ is *nef* (numerically effective) if $D \cdot C \geq 0$ for any irreducible complete curve $C \subseteq Y$.

A normal projective variety $Y$ is called $\mathbb{Q}$-*factorial* if every Weil divisor on $Y$ is a $\mathbb{Q}$-Cartier divisor.

The minimal model program conjectures that a projective variety with positive Kodaira dimension (Definition 7.2.9) has a minimal model with certain characteristics while a projective variety with Kodaira dimension $-\infty$ has a birationally equivalent model with at worst terminal singularities fulfilling certain properties. The conjecture is classically known to hold in the surface case. In the case of 3-dimensional varieties it is proved by Mori and Kawamata in [Mor88, Kaw92].

Especially in Chapter 7, our applications are limited to surfaces making use of the facts below:

**Theorem 2.1.11.** *Let $Y$ be a normal projective variety with* $\dim(Y) = 2$. *Then $Y$ is birationally equivalent to some normal projective variety $Y'$ that is $\mathbb{Q}$-factorial, has at worst terminal singularities, and is a minimal model. In particular, $Y'$ is smooth because terminal singularities in dimension* 2 *are smooth.*

Let $Y$ be a normal projective surface with at worst canonical singularities. Then $Y$ has only Du Val singularities because a 2-dimensional terminal singularity is smooth and canonical singularities are exactly the Du Val singularities [Kol89, Proposition 2.6]. Therefore, there exists a crepant desingularization $\rho : Y' \to Y$ (*i.e.*, $K_{Y'} = \rho^* K_Y$), where $Y'$ is a minimal model of $Y$ [Rei04, Theorem 2.1].

### 2.1.3 Chow Rings and Chern Classes

In this thesis, intersection theory is considered as a part of algebraic geometry, introduced under this aspect, and later applied like this. We start with an introduction in Chow rings that provide the basic objects of intersection theory. Furthermore, we present an algebra-geometric generalization of the divisor concept. A comprehensive standard reference is [Ful98].

We begin by sketching the construction of Chow rings. Therefore, we need to define cycles, rational equivalence, and especially Chow groups. These objects can be understood as a kind of generalizations of divisors, linear equivalence, and divisor class groups.

**Definition 2.1.12.** A *cycle $\alpha$ of dimension $k$* (or a *$k$-cycle*) on a normal variety $X$ is a finite formal linear sum of the form

$$\alpha = \sum_i n_i [Y_i],$$

where $Y_i$ are $k$-dimensional irreducible subvarieties of $X$ and $n_i \in \mathbb{Z}$. The set of all $k$-cycles on $X$ forms a additive group, is generated by $k$-dimensional subvarieties $Y_i \subseteq X$, and denoted by $Z_k(X)$.

Moreover, we call a $k$-cycle $\alpha \in Z_k(X)$ on $X$ *rational equivalent to zero* (written $\alpha \sim 0$) if there are finitely many $(k+1)$-dimensional irreducible subvarieties $W_i \subseteq X$ and non-zero rational functions $f_i \in \mathbb{K}(W_i)^*$ such that $\alpha = \sum_i [\mathrm{div}_{W_i}(f_i)]$, where $\mathrm{div}_{W_i}(f_i)$ is the divisor of the rational function $f_i$ on $W_i$. It should be noted that the notation $\mathrm{div}_{W_i}$ makes clear that the formal sum $\mathrm{div}_{W_i}(f_i) = \sum_j \nu_{V_j}(f_i) \cdot V_j$ running over prime divisors $V_j \subseteq W_i$ is a principal divisor on $W_i$. The set $\{\alpha \in Z_k(X) |\, \alpha \sim 0\}$ of all $k$-cycles that are rational equivalent to zero is denoted by $R_k(X)$. Moreover, $R_k(X)$ is a subgroup of $Z_k(X)$, which leads to the following factor group definition

$$A_k(X) \coloneqq Z_k(X)/R_k(X).$$

Analogously, it is possible to define this group using cycles of codimension $k$, where the difference is made explicit through the exponent representation $A^k(X) = A_{d-k}(X)$. If the codimension equals 0 and $r > d$, we get $A^0(X) = A_d(X) \cong \mathbb{Z}$ and $A^r(X) = 0$. $A^\bullet(X)$ is called *Chow group* of $X$. It is a graduated group defined as $A^\bullet(X) = \bigoplus_{k=0}^d A^k(X)$. For reasons of simplicity a $k$-cycle $\alpha$ could mean $\alpha \in Z_k(X)$ or $\alpha \in A_k(X)$.

We continue with an introduction of two maps that are in combination used to compute the intersection number of two correctly chosen cycles $\alpha \in A^l(X)$ and $\beta \in A^m(X)$, *i.e.*, $l + m = d$, where $X$ is a $d$-dimensional normal variety.

**Proposition 2.1.13 [Har77, page 426].** *Let $X$ be a $d$-dimensional normal complete variety. Then there exists a canonical group homomorphism*

$$\deg : A^d(X) = A_0(X) \to \mathbb{Z},\ \sum_i n_i P_i \mapsto \sum_i n_i.$$

In practice, this map is often used without writing its name at length.

**Definition 2.1.14.** Let $X$ be a smooth irreducible projective variety of dimension $d$. Then there exists a *intersection product* (or *intersection pairing*)

$$\cdot.\cdot : A^l(X) \times A^m(X) \to A^{l+m}(X),\ (\alpha, \beta) \mapsto \alpha.\beta$$

such that $A^\bullet(X) = \bigoplus_{k=0}^d A^k(X)$ is an associative commutative graded ring called *Chow ring* of $X$.

**Remark 2.1.15.** If $X$ is a smooth irreducible projective variety of dimension $d$, then the *rational Chow group* $A^\bullet(X)_\mathbb{Q} \coloneqq A^\bullet(X) \otimes_\mathbb{Z} \mathbb{Q} = \bigoplus_{k=0}^d A^k(X) \otimes_\mathbb{Z} \mathbb{Q} = \bigoplus_{k=0}^d A^k(X)_\mathbb{Q}$ of $X$ is a graded $\mathbb{Q}$-algebra, thus a $\mathbb{Q}$-module. In combination with the intersection product $\cdot.\cdot : A^l(X)_\mathbb{Q} \times A^m(X)_\mathbb{Q} \to A^{l+m}(X)_\mathbb{Q}$ is a graded ring called *rational Chow ring* of $X$. Similarly, there exists a map $\deg : A^d(X)_\mathbb{Q} = A_0(X)_\mathbb{Q} \to \mathbb{Q}$ on the rational Chow group of $X$.

Let $i : Y \hookrightarrow X$ a closed embedding of a smooth normal projective subvariety $Y \subseteq X$ with $\dim(Y) = k$ in a smooth normal projective variety $X$. Moreover, let $\phi : X' \to X$ be a desingularization of a non-smooth normal projective variety $X$.

**Definition 2.1.16.** The *push-forward* $i_* : A_l(Y) \to A_l(X)$ and $\phi_* : A_l(X') \to A_l(X)$ are dimension preserving homomorphisms of Chow groups. In particular, $i_*(Z) = Z$ for each $Z \subseteq Y$ with $\dim(Z) = l$.

Let $\alpha \in A_0(Y)$ and $\alpha' \in A_0(X')$ be two cycles. Then $\deg(\alpha) = \deg(i_*(\alpha))$ and $\deg(\alpha') = \deg(\phi_*(\alpha'))$ using Proposition 2.1.13 (cf. [Har77, page 426], [Ful98, 1.4]).

**Definition 2.1.17.** The *pullback* $i^* : A^l(X) \to A^l(Y)$ and $\phi^* : A^l(X) \to A^l(X')$ are codimension preserving homomorphism of Chow rings.

**Proposition 2.1.18 – Projection Formula [Ful98, 2.5].** *Let $X$ be a smooth normal projective variety, $Y \subseteq X$ a smooth normal projective subvariety of $X$ with* $\dim(Y) = k$*, $D$ a Cartier divisor on $X$* (i.e., $D \in A^1(X)$)*, $\beta \in A_l(Y)$, and $l \in \mathbb{N}$ with $1 \leq l \leq k$. Then*

$$i_*(i^*(D^l).\beta) = D^l.i_*(\beta),$$

*where $i_*$ denotes the push-forward and $i^*$ the pullback of the closed embedding $i : Y \hookrightarrow X$. An analogue projektion formula exists also for desingularizations $\phi : X' \to X$ with $D \in A^1(X)$, $\beta \in A_l(X')$, and $l \in \mathbb{N}$ with $1 \leq l \leq d = \dim(X')$.*

The following part generalizes the well-known relationship between Cartier divisors and invertible sheaves [CLS11, Theorem 6.0.20]. In technical terms, this is achieved with the theory of Chern classes because it assigns to each locally free sheaf of rank $r$ exactly $r$ objects, called Chern classes.

Let $X$ be a smooth irreducible projective variety of dimension $d$ and $\mathcal{F}$ a locally free sheaf of rank $r$ on $X$. Furthermore, let $\mathbb{P}(\mathcal{F})$ be the sheaf associated projective bundle of $\mathcal{F}$ with associated morphism $\pi : \mathbb{P}(\mathcal{F}) \to X$ [Har77, page 162]. $\mathcal{L}$ denotes the associated invertible sheaf on $\mathbb{P}(\mathcal{F})$ and $\xi \in A^1(\mathbb{P}(\mathcal{F}))$ the corresponding divisor class.

**Definition 2.1.19.** The *$k$-th Chern class* $c_k(\mathcal{F}) \in A^k(X)$ of $\mathcal{F}$ $(0 \leq k \leq r)$ is defined through the following two conditions

$$c_0(\mathcal{F}) = 1 \text{ and } \sum_{0 \leq k \leq r} (-1)^k \pi^* (c_k(\mathcal{F})) .\xi^{r-k} = 0,$$

where the second condition is a class in $A^r(\mathbb{P}(\mathcal{F}))$. In particular, the $k$-th Chern class is determined by $c_k(\mathcal{F}) = 0$ if $k > r$ [Ful98, 3.2].

The Chern classes, we are especially interested in, are the ones associated with the following locally free sheaf:

**Definition 2.1.20.** Let $X$ be a smooth irreducible variety of dimension $d$. Then there exists a sheaf $\mathcal{T}_X$ with a geometric origin, called *tangent sheaf*, that is a locally free sheaf of rank $d$ [Bro89, page 405]. Another possibility to introduce this sheaf is via the dualisation of the *differential sheaf* $\Omega_X$ because $\mathcal{T}_X \cong \Omega_X^\vee$ [Har77, page 180, II.8.15], [Bro89, page 409-416].

To simplify our notation, we call a Chern class $c_k(\mathcal{T}_X)$ of $\mathcal{T}_X$ Chern class $c_k(X)$ of the given variety $X$ $(0 \leq k \leq d)$.

**Definition 2.1.21.** Let $X$ be a smooth irreducible variety of dimension $d$. Then the top Chern class $c_d(X)$ of $X$ is called *Euler number* of $X$.

## 2.2 Cones, Fans, and Toric Varieties

Toric geometry unites two strands of mathematics: algebraic geometry and combinatorics. The potential of this correspondence is apparent: we can easily read off a multitude of properties about a toric variety through a combinatorial exploration of the associated fan, including statements about the variety's smoothness and orbits. In this section, we state the central aspects of the mentioned relationship.

**Definition 2.2.1.** A *toric variety* of dimension $d$ is a normal variety $X$ over the field $\mathbb{C}$ of complex numbers containing a torus $\mathbb{T}^d \cong (\mathbb{C}^*)^d$ as an open dense subset such that the action of $\mathbb{T}^d$ on itself extends to an algebraic action $\mathbb{T}^d \times X \to X$ of $\mathbb{T}^d$ on $X$.

Let $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R} \cong \mathbb{R}^d$ be a real vector space obtained by an extension of a $d$-dimensional lattice (*i.e.*, a free abelian group of rank $d$) $N \cong \mathbb{Z}^d$. Furthermore, $M := \operatorname{Hom}(M, \mathbb{Z})$ denotes the dual lattice to $N$ and $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$ the natural pairing which extends to a pairing $\langle \cdot, \cdot \rangle : M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}$, where $M_\mathbb{R} := M \otimes_\mathbb{Z} \mathbb{R} \cong \mathbb{R}^d$ is the corresponding real vector space to $M$.

A subset $C \subseteq N_\mathbb{R}$ is called *convex* if $c_1, c_2 \in C$ implies $\lambda \cdot c_1 + (1 - \lambda) \cdot c_2 \in C$ for all $\lambda \in \mathbb{R}$ with $0 \leq \lambda \leq 1$.

**Definition 2.2.2.** A *cone* $\sigma$ (or more precisely, a *finitely generated convex rational polyhedral cone*) in $N_\mathbb{R}$ is a set of the form

$$\sigma = \left\{ \sum_{u \in S} \lambda_u u \;\middle|\; \lambda_u \in \mathbb{R}_{\geq 0} \right\} \subseteq N_\mathbb{R},$$

where $S \subseteq N$ is finite.

Let $\sigma \subseteq N_\mathbb{R}$ be a cone. Then the *relative interior of the cone* $\sigma$ is denoted with $\sigma^\circ$ [CLS11, page 24]. The *dimension* $\dim(\sigma)$ *of the cone* $\sigma \subseteq N_\mathbb{R}$ is the dimension of the smallest subspace of $N_\mathbb{R}$ containing $\sigma$. A *primitive ray generator* $u_\rho \in \rho \cap N$ is the unique element generating the semigroup $\rho \cap N$ for any ray $\rho$ of the given cone $\sigma$.

**Definition 2.2.3.** Let $\sigma \subseteq N_\mathbb{R}$ be a cone. Then

(i) $\sigma$ is *smooth* if its primitive ray generators form a part of a $\mathbb{Z}$-basis of N.

(ii) $\sigma$ is *simplicial* if its primitive ray generators are linearly independent over $\mathbb{R}$.

**Definition 2.2.4.** The *dual cone* $\sigma^\vee \subseteq M_\mathbb{R}$ of a cone $\sigma \subseteq N_\mathbb{R}$ is defined as

$$\sigma^\vee := \{ m \in M_\mathbb{R} \mid \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma \} \subseteq M_\mathbb{R}.$$

The introduced duality has the property that the dual of a convex rational polyhedral cone is a convex rational polyhedral cone and $(\sigma^\vee)^\vee = \sigma$ [CLS11, Proposition 1.2.4].

**Definition 2.2.5.** A *face* $\tau$ *of a cone* $\sigma \subseteq N_\mathbb{R}$ (written $\tau \preceq \sigma$) is a set $\tau = H_m \cap \sigma \subseteq N_\mathbb{R}$ for some $m \in \sigma^\vee$, where $H_m := \{ u \in N_\mathbb{R} \mid \langle m, u \rangle = 0 \} \subseteq N_\mathbb{R}$ is a hyperplane.

Given a cone $\sigma \subseteq N_{\mathbb{R}}$, the set of lattice points $S_\sigma := \sigma^\vee \cap M \subseteq M$ is finitely generated and hence an affine semigroup by Gordan's Lemma [CLS11, Proposition 1.2.17].

**Theorem 2.2.6 [CLS11, Theorem 1.2.18].** *Let $\sigma \subseteq N_{\mathbb{R}}$ be a cone. Then $X_\sigma := \operatorname{Spec}(\mathbb{C}[S_\sigma]) = \operatorname{Spec}(\mathbb{C}[\sigma^\vee \cap M])$ is an affine toric variety.*

**Remark 2.2.7.** The semigroup $S_\sigma = \sigma^\vee \cap M$ is finitely generated by Gordan's Lemma and the lattice points in it can be regarded as monomials in the coordinate ring of the affine toric variety $X_\sigma$. The semigroup algebra $\mathbb{C}[S_\sigma]$ is a vector space over $\mathbb{C}$ with basis $S_\sigma$, *i.e.*, an arbitrary element is given by a finite sum $\sum_{m \in S_\sigma} c_m \chi^m$ with $c_m \in \mathbb{C}$ and $\chi^m$ the element in the $\mathbb{C}$-algebra corresponding to the semigroup element $m \in S_\sigma$, where the multiplication is induced by $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$.

The combinatorial object to glue a collection of affine toric varieties together to yield a toric variety is given as follows:

**Definition 2.2.8.** A *fan* $\Sigma$ in $N_{\mathbb{R}}$ is a finite collection of cones $\sigma \subseteq N_{\mathbb{R}}$ such that

(i) every $\sigma \in \Sigma$ is a *strongly convex rational polyhedral cone* (*i.e.*, $\sigma \cap (-\sigma) = \{0\}$).

(ii) for all $\sigma \in \Sigma$, each face $\tau \preceq \sigma$ of $\sigma$ is also an element in $\Sigma$.

(iii) the intersection $\sigma_1 \cap \sigma_2$ of any two cones $\sigma_1, \sigma_2 \in \Sigma$ in $\Sigma$ is a face of each (*i.e.*, $\sigma_1 \cap \sigma_2 \preceq \sigma_1, \sigma_2$) and hence also an element in $\Sigma$.

Furthermore, if $\Sigma$ is a fan, then $\Sigma(k)$ denotes the set of all $k$-dimensional cones of $\Sigma$.

Considering the collection of affine toric varieties $X_\sigma$, where $\sigma$ runs over all cones in a given fan $\Sigma$. All compatibility conditions for gluing the affine varieties $X_\sigma$ along the relevant subvarieties are satisfied [CLS11, Section 3] and hence we obtain a normal toric variety $X_\Sigma$ [CLS11, Theorem 3.1.5].

**Theorem 2.2.9 [CLS11, Theorem 3.1.19, Theorem 3.4.6].** *Let $X_\Sigma$ be a toric variety defined by a fan $\Sigma$ in $N_{\mathbb{R}}$. Then $X_\Sigma$ is smooth* [CLS11, Definition 3.0.13] *if and only if every cone in $\Sigma$ is smooth. Furthermore, $X_\Sigma$ is complete* [CLS11, Definition 3.4.3] *if and only if the fan $\Sigma$ is* complete, i.e., *$\cup_{\sigma\in\Sigma}\sigma = N_{\mathbb{R}}$.*

**Proposition 2.2.10 [Rei83].** *Let $X$ be a $\mathbb{Q}$-Gorenstein toric variety. Then $X$ has at worst log-terminal singularities.*

**Remark 2.2.11.** Let $X_\Sigma$ be a toric variety defined by a fan $\Sigma$ in $N_{\mathbb{R}}$. Then the canonical divisor $K_{X_\Sigma}$ equals $-\sum_{\rho\in\Sigma(1)} D_\rho$, where $D_\rho$ is the closure of a codimension 1 orbit corresponding to a 1-dimensional cone $\rho \in \Sigma(1)$ and a torus-invariant prime divisor.

**Proposition 2.2.12 [CLS11, Theorem 4.2.12],[Bat98b, Section 4].** *Let $X$ be a normal $\mathbb{Q}$-Gorenstein toric variety associated with a fan $\Sigma$ in $N_{\mathbb{R}}$. Then the property of $X$ to be $\mathbb{Q}$-Gorenstein is equivalent to the existence of a $\Sigma$-piecewise linear function*

$$\kappa : N_{\mathbb{R}} \to \mathbb{R}$$

*corresponding to the anticanonical divisor $-K_X$ on $X$ that is linear on each cone $\sigma$ of $\Sigma$ and has value $-1$ on every primitive ray generator of a 1-dimensional cone of $\Sigma$.*

**Definition 2.2.13.** The *normalized volume $v(\sigma)$ of a $k$-dimensional cone $\sigma \subseteq N_{\mathbb{R}}$* is defined to be the normalized volume of the lattice polytope $\theta_\sigma$ (Definition 2.3.6) obtained as the convex hull of the origin and all primitive ray generators of the given cone $\sigma$, *i.e.*,

$$v(\sigma) = k! \cdot \mathrm{vol}_k\left(\theta_\sigma\right),$$

where $\mathrm{vol}_k\left(\theta_\sigma\right)$ denotes the $k$-dimensional volume of the lattice polytope $\theta_\sigma$ with respect to the sublattice $\langle\sigma\rangle_{\mathbb{R}} \cap N$.

We denote by $v(\Sigma)$ the *normalized volume of a fan* $\Sigma$ defined as the sum running over all $d$-dimensional (*i.e.*, maximal-dimensional) cones of the normalized volumes of these cones, *i.e.*,

$$v(\Sigma) \coloneqq \sum_{\sigma \in \Sigma(d)} v(\sigma). \tag{2.1}$$

**Example 2.2.14.** Let $\Sigma_m \coloneqq \{\{0\}, \rho_1, \rho_2, \rho_3, \sigma_1, \sigma_2, \sigma_3\}$ be a fan in $N_{\mathbb{R}}$, where the cones are defined as

$$\rho_1 \coloneqq \mathbb{R}_{\geq 0} e_2, \quad \rho_2 \coloneqq \mathbb{R}_{\geq 0} e_1, \quad \rho_3 \coloneqq \mathbb{R}_{\geq 0}(-e_1 - m e_2)$$

and

$$\begin{aligned} \sigma_1 &\coloneqq \{\lambda_1 e_2 + \lambda_2 e_1 \mid \lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}\}, \\ \sigma_2 &\coloneqq \{\lambda_1 e_1 + \lambda_3(-e_1 - m e_2) \mid \lambda_1, \lambda_3 \in \mathbb{R}_{\geq 0}\}, \\ \sigma_3 &\coloneqq \{\lambda_3(-e_1 - m e_2) + \lambda_2 e_2 \mid \lambda_3, \lambda_2 \in \mathbb{R}_{\geq 0}\} \end{aligned}$$

(Figure 2.1). To compute the normalized volume of a cone in $\Sigma_m$ (Definition 2.2.13), we have to compute the normalized volume of the lattice polytope obtained as the convex hull of the origin and all primitive ray generators of the given cone. Figure 2.1 pictures all seven cones together with the needed lattice polytopes to compute the following normalized volumes: $v(\{0\}) = v(\theta_{\{0\}}) = 1$, $v(\rho_1) = v(\theta_{\rho_1}) = v(\mathrm{conv}(0, e_2)) = 1! \cdot 1 = 1$, $v(\rho_2) = v(\theta_{\rho_2}) = v(\mathrm{conv}(0, e_1)) = 1$, $v(\rho_3) = v(\theta_{\rho_3}) = v(\mathrm{conv}(0, -e_1 - m e_2)) = 1$, and

$$v(\sigma_1) = v(\theta_{\sigma_1}) = v(\mathrm{conv}(0, e_2, e_1)) = 2! \cdot \left(\frac{1}{2} \cdot 1 \cdot 1\right) = 1,$$

$$v(\sigma_2) = v(\theta_{\sigma_2}) = v(\mathrm{conv}(0, e_1, -e_1 - m e_2)) = 2! \cdot \left(\frac{1}{2} \cdot 1 \cdot m\right) = m,$$

$$v(\sigma_3) = v(\theta_{\sigma_3}) = v(\mathrm{conv}(0, -e_1 - m e_2, e_2)) = 2! \cdot \left(\frac{1}{2} \cdot 1 \cdot 1\right) = 1.$$

In particular, the normalized volume $v(\Sigma_m)$ of the fan $\Sigma_m$ equals $v(\sigma_1) + v(\sigma_2) + v(\sigma_3) = 1 + m + 1 = 2 + m$. There is a close link between this example and Example 6.2.2 including Figure 6.4 because $\Sigma_m$ is the (spanning) fan of the toric log del Pezzo surface appearing there.

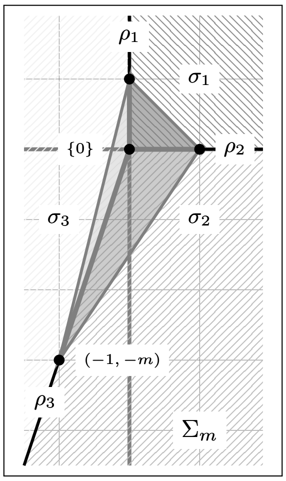



**Figure 2.1: Normalized Cone Volumes.** We consider the fan $\Sigma_m = \{\{0\}, \rho_1, \rho_2, \rho_3, \sigma_1, \sigma_2, \sigma_3\}$ (here: $m = 3$), where the 2-dimensional cones $\sigma_1$, $\sigma_2$, and $\sigma_3$ are hatched dark grey, grey, and light grey, respectively, and the black dots are the primitive ray generators of the 1-dimensional cones $\rho_1$, $\rho_2$, and $\rho_3$. Additionally, the polytopes to compute the normalized volumes of all seven cones are included.

### 2.2.1 Chow Rings and Chern Classes of Toric Varieties

We revisit Chow rings and Chern classes (Subsection 2.1.3) in the particular case of toric varieties. Here, the Chow group is generated by torus orbit closures:

**Proposition 2.2.15 [CLS11, Theorem 3.2.6, Lemma 12.5.1, Section 12.5].** *Let $X$ be a complete simplicial toric variety of dimension $d$ associated with a fan $\Sigma$ in $N_{\mathbb{R}}$. For each $\sigma \in \Sigma$, the torus orbit closure $X_\sigma$ is a subvariety of codimension* $\dim(\sigma)$ *and its rational equivalence class $[X_\sigma]$ lives in $A^{\dim(\sigma)}(X)$. Moreover, these torus orbit closure classes generate $A^\bullet(X)$ as an abelian group. In addition, $A^\bullet(X)$ is also a ring.*

The usual Chern classes of a toric variety are given in a purely combinatorial way using the fact that the Chow ring is generated by torus orbit closure classes and intrinsic information provided by the underlying fan:

**Proposition 2.2.16 [CLS11, Proposition 13.1.2].** *Let $X$ be a $d$-dimensional smooth complete toric variety associated with a fan $\Sigma$ in $N_{\mathbb{R}}$. Then*

(i) $c_\bullet(X) = \sum_{\sigma\in\Sigma}[X_\sigma]$, *where* $c_\bullet(X) \coloneqq c_d(X) + \ldots + c_1(X) + 1$ *is called* total Chern class *of $X$.*

(ii) $c_1(X) = [-K_X]$, *where* $-K_X = \sum_{\rho\in\Sigma(1)} D_\rho$ *is the anticanonical divisor on $X$.*

(iii) $c_d(X) = |\Sigma(d)| \cdot [\text{point}]$, *where* $[\text{point}] \in A^d(X)$.

## 2.3 Polytopes, Fans, and Toric Varieties

A specialization to the interaction of toric geometry and combinatorial methods applied to lattice polytopes exploits its potential (mentioned at the beginning of Section 2.2) in principled and elegant ways [Oda85, Ful93, CLS11]. It connects hard algebra-geometric objects, such as projective toric varieties, and intuitive convex-geometric objects, such as lattice polytopes. This replaces the dictionary from the previous section dealing with fan explorations.

**Definition 2.3.1.** A $d$-dimensional convex polytope $\Delta \subseteq N_{\mathbb{R}}$ is called *lattice polytope* (or *$d$-polytope*) if $\Delta = \operatorname{conv}(\Delta \cap N)$, *i.e.*, if all vertices of $\Delta$ belong to the lattice $N$.

**Definition 2.3.2.** A *face* $\theta \preceq \Delta$ of a given polytope $\Delta \subseteq N_{\mathbb{R}}$ is an intersection of $\Delta$ with an affine hyperspace, *i.e.*, there exists $m \in M_{\mathbb{R}} \setminus \{0\}$ and $b \in \mathbb{R}$ such that $\theta = \Delta \cap H_{mb}$ with $H_{mb} \coloneqq \{x \in N_{\mathbb{R}} \mid \langle m, x\rangle = b\}$.

A *vertex* is a 0-dimensional face of a $d$-dimensional polytope $\Delta$, an *edge* is a 1-dimensional face of $\Delta$, and a *facet* is a $(d-1)$-dimensional face of $\Delta$. The set of all vertices of $\Delta$ is denoted by $\operatorname{vert}(\Delta)$.

Let $\Delta \subseteq N_{\mathbb{R}}$ be a $d$-dimensional lattice polytope and $\theta \preceq \Delta$ a face of $\Delta$. Then the *boundary of the polytope* $\Delta$ is denoted by $\partial\Delta$ and the *relative interior of the polytope* $\theta$ by $\theta^{\circ}$.

**Definition 2.3.3.** Let $\Delta \subseteq N_{\mathbb{R}}$ be a $d$-dimensional convex polytope that contains the origin $0 \in N$ in its interior. Then one defines the *dual polytope* $\Delta^* \subseteq M_{\mathbb{R}}$ as

$$\Delta^* \coloneqq \{y \in M_{\mathbb{R}} \mid \langle y, x\rangle \geq -1 \;\forall x \in \Delta\} \subseteq M_{\mathbb{R}}.$$

**Definition 2.3.4.** Let $\Delta \subseteq N_{\mathbb{R}}$ be a $d$-dimensional convex lattice polytope that contains the origin in its interior and $\theta \preceq \Delta$ a face of $\Delta$. Then

$$\theta^* \coloneqq \{y \in \Delta^* \mid \langle y, x\rangle = -1 \;\forall x \in \theta\} \preceq \Delta^*$$

is a face of $\Delta^*$ and called *dual face of* $\theta$.

The duality between $\Delta$ and $\Delta^*$ implies a one-to-one order-reversing duality between $k$-dimensional faces $\theta \preceq \Delta$ of $\Delta$ and $(d-k-1)$-dimensional dual faces $\theta^* \preceq \Delta^*$ of $\Delta^*$ such that $\dim(\theta) + \dim(\theta^*) = d - 1$ [Ful93, Section 1.5].

**Definition 2.3.5.** The non-negative integer $|\langle m, n\rangle - b|$ is called *lattice distance* between $n \in N$ and the hyperplane $H_{mb}$, where $m \in M$ is a primitive lattice vector and $b \in \mathbb{Z}$ with $H_{mb} = \{x \in N_{\mathbb{R}} \mid \langle m, x\rangle = b\}$.

Let $\Delta \subseteq N_{\mathbb{R}}$ be a $d$-dimensional lattice polytope that contains the origin in its interior and $\theta \preceq \Delta$ a face of $\Delta$. Then we denote by $n_\theta$ the lattice distance from $\theta$ (or the affine hyperplane spanned by $\theta$) to the origin.

**Definition 2.3.6.** Let $\Delta \subseteq N_{\mathbb{R}}$ be a $d$-dimensional lattice polytope. Then we define $v(\Delta)$ to be the *normalized volume* of $\Delta$, *i.e.*, the positive integer

$$v(\Delta) := d! \cdot \mathrm{vol}_d(\Delta),$$

where $\mathrm{vol}_d(\Delta)$ denotes the $d$-dimensional volume of $\Delta$ with respect to the lattice $N$. Similarly, we define the positive integer $v(\theta) := k! \cdot \mathrm{vol}_k(\theta)$ for a $k$-dimensional face $\theta \preceq \Delta$ of $\Delta$, where $\mathrm{vol}_k(\theta)$ denotes the $k$-dimensional volume of $\theta$ with respect to the sublattice $\mathrm{span}(\theta) \cap N$. If $\Delta \subseteq N_{\mathbb{R}}$ is a $d$-dimensional polytope having vertices in $N_{\mathbb{Q}} := N \otimes \mathbb{Q}$, *i.e.*, $\Delta$ is a *rational polytope*, then we can similarly define the positive rational number $v(\theta)$ for any $k$-dimensional face $\theta \preceq \Delta$. For this purpose, we consider an integer $l$ such that $l\Delta$ is a lattice polytope and define for a $k$-dimensional face $\theta \preceq \Delta$ its normalized volume as $v(\theta) := \frac{1}{l^k} v(l\theta)$.

**Example 2.3.7.** Example 2.2.14 instances as an example because while computing the normalized volumes of certain cones in a given fan, we computed normalized volumes of lattice polytopes in three different dimensions.

### 2.3.1 Fans

In the previous section, fans have been introduced to define toric varieties. Now, this subsection presents two different ways to obtain a fan from a given lattice polytope:

**Definition 2.3.8.** Let $\Delta \subseteq M_{\mathbb{R}}$ be a $d$-dimensional lattice polytope. We define $\Sigma^{\Delta}$ to be the *normal fan* of $\Delta$ in $N_{\mathbb{R}}$, *i.e.*, $\Sigma^{\Delta} := \left\{\sigma^{\theta} \,\middle|\, \theta \preceq \Delta\right\}$, where $\sigma^{\theta}$ is the cone generated by all inward-pointing facet normals of facets containing the face $\theta \preceq \Delta$ of $\Delta$ with $\dim(\sigma^{\theta}) = d - \dim(\theta)$. In particular, the normal fan is a fan [CLS11, Theorem 2.3.2].

Let $\Delta \subseteq M_{\mathbb{R}}$ be a $d$-dimensional lattice polytope and $\Sigma^{\Delta}$ the associated normal fan in $N_{\mathbb{R}}$. Then the function

$$\kappa^{\Delta} : N_{\mathbb{R}} \to \mathbb{R},\ x \mapsto \min_{y \in \Delta} \{\langle y, x\rangle\} = \min_{y \in \mathrm{vert}(\Delta)} \{\langle y, x\rangle\} \tag{2.2}$$

is a support function for $\Sigma^{\Delta}$ (*i.e.*, it is linear on each cone of $\Sigma^{\Delta}$) and integral with respect to $N$ (*i.e.*, $\kappa^{\Delta}(N) \subseteq \mathbb{Z}$) [CLS11, Proposition 4.2.14]. In other words, the cones in the normal fan $\Sigma^{\Delta}$ are given by

$$\sigma^{\theta} = \left\{x \in N_{\mathbb{R}} \,\middle|\, \kappa^{\Delta}(x) = \langle y, x\rangle \ \forall y \in \theta\right\} \subseteq N_{\mathbb{R}}.$$

Moreover,

$$\sigma^{\theta} \cap \Delta^* = \left\{x \in \sigma^{\theta} \,\middle|\, \langle y, x\rangle \geq -1 \ \forall y \in \theta\right\} \subseteq N_{\mathbb{R}}$$

is the associated rational polytope, where $\Delta^*$ is the dual polytope of $\Delta$, that has rational not necessarily integer vertices with $\Delta^* = \cup_{\theta \preceq \Delta} \sigma^{\theta} \cap \Delta^*$.

**Theorem 2.3.9 [CLS11, Theorem 3.1.5, Theorem 3.1.6].** *Let $\Delta \subseteq M_{\mathbb{R}}$ be a $d$-dimensional lattice polytope. Then $X_{\Sigma^{\Delta}}$ is a normal projective toric variety, where $\Sigma^{\Delta}$ is the normal fan of $\Delta$.*

There is a second possibility to define a fan associated with a polytope:

**Definition 2.3.10.** Let $\Delta \subseteq N_{\mathbb{R}}$ be a $d$-dimensional lattice polytope with $0 \in \Delta^\circ \cap N$. We define $\Sigma_\Delta$ to be the *spanning fan* of $\Delta$ in $N_{\mathbb{R}}$, *i.e.*, $\Sigma_\Delta := \{\sigma_\theta \,|\, \theta \preceq \Delta\}$, where $\sigma_\theta$ is the cone $\mathbb{R}_{\geq 0}\theta$ spanned by the face $\theta \preceq \Delta$ of $\Delta$ with $\dim(\sigma_\theta) = \dim(\theta)+1$. In particular, the spanning fan is a fan.

**Theorem 2.3.11 [Kas06, Proposition 2.3.15], [CLS11, Theorem 3.1.5].** *Let $\Delta \subseteq N_{\mathbb{R}}$ be a $d$-dimensional lattice polytope with $0 \in \Delta^\circ \cap N$. Then $X_{\Sigma_\Delta}$ is a normal projective toric variety, where $\Sigma_\Delta$ is the spanning fan of $\Delta$.*

Let $\Delta \subseteq N_{\mathbb{R}}$ be a $d$-dimensional lattice polytope with $0 \in \Delta^\circ \cap N$. Then there exists a $\Sigma_\Delta$-piecewise linear function

$$\kappa_\Delta : N_{\mathbb{R}} \to \mathbb{R} \tag{2.3}$$

corresponding to the anticanonical divisor $-K_{X_{\Sigma_\Delta}}$ on $X_{\Sigma_\Delta}$ that is linear on each cone $\sigma$ of $\Sigma_\Delta$ and has value $-1$ on every primitive ray generator of a 1-dimensional cone of $\Sigma_\Delta$. In this case, the given polytope $\Delta$ is of the form

$$\Delta = \{x \in N_{\mathbb{R}} \,|\, \kappa_\Delta(x) \geq -1\} \subseteq N_{\mathbb{R}}$$

and there are lattice points $n \in N$ with $\kappa_\Delta(n) \in \mathbb{Q} \setminus \mathbb{Z}$.

**Example 2.3.12.** Let $\Delta$ be a 2-dimensional polytope containing the origin in its interior and defined as $\mathrm{conv}(e_1, e_2, -e_1 - e_2)$ (Figure 2.2(a)). The associated normal fan $\Sigma^\Delta$ and the associated spanning fan $\Sigma_\Delta$ consist of one 0-dimensional cone, three 1-dimensional cones and three 2-dimensional cones (Figure 2.2(b) and (c)). The corresponding toric varieties are $X_{\Sigma^\Delta} = \mathbb{P}^2/\mathbb{Z}_2$ and $X_{\Sigma_\Delta} = \mathbb{P}^2$.

In the remaining part of this section, we look at special lattice polytopes (namely, reflexive polytopes, $LDP$-polygons, almost pseudoreflexive polytopes, canonical Fano polytopes, and Gorenstein polytopes of index $r$) and apply in each case the dictionary knowledge from this section.

### 2.3.2 Reflexive Polytopes

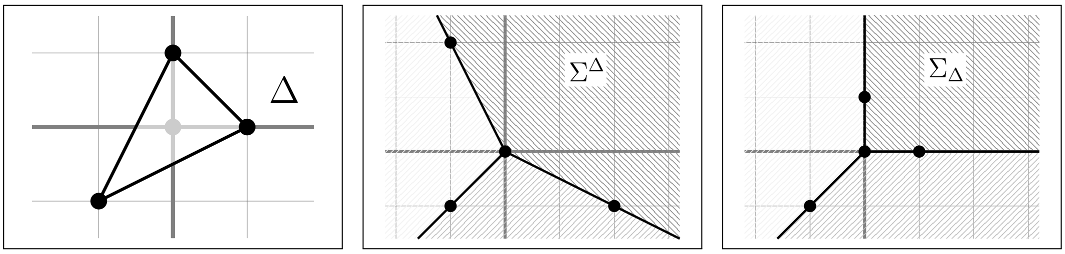
A detailed introduction to reflexive polytopes can be found in [Bat94].

**Definition 2.3.13.** A $d$-dimensional lattice polytope $\Delta \subseteq N_{\mathbb{R}}$ containing the origin $0 \in N$ in its interior is called *reflexive* if all vertices of the dual polytope

$$\Delta^* = \{y \in M_{\mathbb{R}} | \langle y, x\rangle \geq -1 \; \forall x \in \Delta\} \subseteq M_{\mathbb{R}}$$

belong to the dual lattice $M$, *i.e.*, if $\Delta^*$ is also a lattice polytope.

By Definition 2.1.3, a complete normal toric variety $X$ is a Gorenstein toric Fano variety if the anticanonical divisor $-K_X$ is an ample Cartier divisor.




**(a)** Lattice polytope $\Delta$. **(b)** Normal fan $\Sigma^{\Delta}$. **(c)** Spanning fan $\Sigma_{\Delta}$.

**Figure 2.2: Lattice Polytope $\boldsymbol{\Delta}$ with Normal fan $\boldsymbol{\Sigma^{\Delta}}$ and Spanning fan $\boldsymbol{\Sigma_{\Delta}}$.** ***(a)*** The lattice polytope $\Delta$ is given as the convex hull of $e_1$, $e_2$, and $-e_1-e_2$. ***(b)*** The normal fan $\Sigma^{\Delta}$ of $\Delta$ consists of a 0-dimensional cone $\{0\}$, three 1-dimensional cones pictured as black lines with the primitive ray generators as black dots, and three 2-dimensional cones hatched dark grey, grey, and light grey, respectively. ***(c)*** The spanning fan $\Sigma_{\Delta}$ of $\Delta$ consists of a 0-dimensional cone $\{0\}$, three 1-dimensional cones pictured as black lines with the primitive ray generators as black dots, and three 2-dimensional cones hatched dark grey, grey, and light grey, respectively.

**Theorem 2.3.14 [CLS11, Theorem 8.3.4], [Bat98b, Corollary 4.2].** *Let $\Delta \subseteq M_{\mathbb{R}}$ be a reflexive polytope. Then the associated variety $X_{\Sigma^{\Delta}}$ defined via the normal fan $\Sigma^{\Delta}$ of $\Delta$ is a projective Gorenstein toric Fano variety with at worst log-terminal singularities. In particular, the Gorenstein index $q_{X_{\Sigma^{\Delta}}}$ equals* 1.

**Remark 2.3.15.** The statement of Theorem 2.3.14 holds also true if the associated variety $X_{\Sigma_{\Delta}}$ is defined via the spanning fan $\Sigma_{\Delta}$ of a reflexive polytope $\Delta \subseteq N_{\mathbb{R}}$ because $\Delta$ and $\Delta^*$ are reflexive polytopes and therefore $\Sigma_{\Delta} = \Sigma^{\Delta^*}$.

Moreover, if $\Delta \subseteq N_{\mathbb{R}}$ is reflexive, then $\Delta^*$ is also reflexive and one has $(\Delta^*)^* = \Delta$. Any facet $\theta \preceq \Delta$ of a reflexive polytope $\Delta$ is defined by an equation $\langle m, x\rangle = -1$ for some lattice vertex $m \in M$ of the dual reflexive polytope $\Delta^*$. This means that each facet $\theta \preceq \Delta$ of a reflexive polytope $\Delta$ has lattice distance 1 to the origin $0 \in N$ and the origin is the only interior lattice point of $\Delta$. In particular, every reflexive polytope is a lattice polytope with exactly one interior lattice point. The converse is not true. If $\Delta \subseteq N_{\mathbb{R}}$ is a $d$-dimensional lattice polytope containing $0 \in N$ in its interior, then the lattice distance $n_\theta$ from a $(d-1)$-dimensional face $\theta \preceq \Delta$ to the origin can be larger than 1, *i.e.*, the facet $\theta$ is defined by an equation $-\langle m, x\rangle = n_\theta > 1$ for some primitive lattice point $m \in M$. One can show that any 2-dimensional lattice polytope containing the origin in its interior is always reflexive, but the latter is not true if $d \geq 3$. There exist exactly 16 isomorphism classes of 2-dimensional reflexive polytopes [CLS11, Theorem 8.3.7], exactly 4,319 isomorphism classes of 3-dimensional reflexive polytopes, and exactly 473,800,776 isomorphism classes of 4-dimensional reflexive polytopes classified by Kreuzer and Skarke [KS98b, KS00].

**Remark 2.3.16.** Let $\Delta$ and $\Delta' \subseteq N_{\mathbb{R}}$ be two $d$-polytopes. Then we call $\Delta$ and $\Delta'$ *isomorphic* if there exists a lattice isomorphism (isomorphism of abelian groups) $N \to N$

mapping the vertices $\mathrm{vert}(\Delta)$ of $\Delta$ onto the vertices $\mathrm{vert}(\Delta')$ of $\Delta'$. In addition, a linear map $l : N_{\mathbb{R}} \to N_{\mathbb{R}}$ induces a lattice automorphism on $N$ if and only if the matrix corresponding to $l$ is an element of the general linear group $\mathrm{GL}(d, \mathbb{Z})$ of degree $d$.

**Example 2.3.17.** A complete list of all 16 isomorphism classes of 2-dimensional reflexive polytopes can be found in Figure 6.1. Furthermore, there are examples of 3-dimensional reflexive polytopes in Figure 6.2 and of 4-dimensional reflexive polytopes in Figure 6.3.

### 2.3.3 $LDP$-polygons

**Definition 2.3.18.** A 2-dimensional lattice polytope $\Delta \subseteq N_{\mathbb{R}}$ containing the origin $0 \in N$ in its interior such that all vertices of $\Delta$ are primitive lattice points in $N$ is called *$LDP$-polygon*, where $LDP$ is an abbreviation for 'log del Pezzo'.

In general, the vertices of the dual polygon $\Delta^* \subseteq M_{\mathbb{R}}$ to a $LDP$-polygon $\Delta$ are not lattice points in $M$, *i.e.*, $\Delta^*$ is in general a rational polytope.

**Remark 2.3.19.** Let $\Delta \subseteq N_{\mathbb{R}}$ be a reflexive polygon. Using Subsection 2.3.2, the origin $0 \in N$ is the only interior lattice point of $\Delta$. Hence, all vertices of $\Delta$ are primitive lattice points in $N$, *i.e.*, $LDP$-polygons build a superclass of reflexive polygons.

By Definition 2.1.2 and Proposition 2.2.10, a normal projective toric surface $X$ with at worst log-terminal singularities and an ample anticanonical $\mathbb{Q}$-Cartier divisor $-K_X$ is a toric log del Pezzo surface.

**Theorem 2.3.20 [KKN10, Introduction].** *Let $\Delta \subseteq N_{\mathbb{R}}$ be a $LDP$-polygon. Then the associated variety $X_{\Sigma_\Delta}$ defined via the spanning fan $\Sigma_\Delta$ of $\Delta$ consisting of cones over faces of the given $LDP$-polygon $\Delta$ is a toric log del Pezzo surface.*

Moreover, there exists a one-to-one correspondence between toric log del Pezzo surfaces and $LDP$-polygons. In particular, any $LDP$-polygon is the convex hull of all primitive ray generators of 1-dimensional cones in the fan $\Sigma$ associated with a toric log del Pezzo surface.

**Example 2.3.21.** Following Remark 2.3.19, Example 2.3.17 presents 16 reflexive polygons, which are in particular $LDP$-polygons. In addition, Figure 5.1 and 6.4(a) present some $LDP$-polygons that are not reflexive.

### 2.3.4 Almost Pseudoreflexive Polytopes

Let $\Delta \subseteq M_{\mathbb{R}}$ be a $d$-dimensional lattice polytope containing the origin $0 \in M$ in its interior. We denote by $[\Delta^*]$ the convex hull $\mathrm{conv}(\Delta^* \cap N)$ of all lattice points in the dual polytope $\Delta^*$.

**Definition 2.3.22.** Let $\Delta \subseteq M_{\mathbb{R}}$ be a $d$-dimensional lattice polytope containing the origin $0 \in M$ in its interior. The lattice polytope $\Delta$ is called *almost pseudoreflexive* if the convex hull $[\Delta^*] = \mathrm{conv}(\Delta^* \cap N)$ of all lattice points in the dual polytope $\Delta^*$

also contains the origin $0 \in N$ in its interior and both polytopes are $d$-dimensional. An almost pseudoreflexive polytope $\Delta$ is called *pseudoreflexive* if $\Delta = [[\Delta^*]^*]$. If $\Delta$ is an almost pseudoreflexive polytope, then $[\Delta^*]$ is always a pseudoreflexive polytope. We call an almost pseudoreflexive polytope $\Delta$ *almost reflexive* if $[\Delta^*]$ is a reflexive polytope.

**Remark 2.3.23.** Let $\Delta \subseteq M_{\mathbb{R}}$ be a reflexive $d$-polytope. Using Subsection 2.3.2, the origin $0 \in M$ is contained in the interior of $\Delta$ and $\Delta^*$ is also a reflexive $d$-polytope. Hence, the origin $0 \in N$ is also contained in the interior of $[\Delta^*] = \Delta^*$, *i.e.*, almost pseudoreflexive $d$-polytopes build a superclass of reflexive $d$-polytopes.

To check whether a lattice polytope is almost pseudoreflexive or not, we can use the equivalent criterions in Proposition 2.3.25 for which we need the following notion:

**Definition 2.3.24.** Let $\Delta \subseteq M_{\mathbb{R}}$ be a $d$-dimensional lattice polytope containing the origin $0 \in M$ in its interior. Then the *Fine interior* $\Delta^{\mathrm{FI}}$ of $\Delta$ is defined as

$$\Delta^{\mathrm{FI}} := \bigcap_{0 \neq n \in N} \Gamma_1^{\Delta}(n)$$

with halfspaces $\Gamma_1^{\Delta}(n) := \left\{ y \in M_{\mathbb{R}} \,\middle|\, \langle y, n \rangle \geq \kappa^{\Delta}(n) + 1 \right\}$ for any non-zero lattice point $n \in N$, where $\kappa^{\Delta} : N_{\mathbb{R}} \to \mathbb{R}$, $x \mapsto \min_{y \in \mathrm{vert}(\Delta)} \{\langle y, x \rangle\}$ (Equation (2.2)).

In particular, the Fine interior $\Delta^{\mathrm{FI}}$ is a convex subset in the interior of $\Delta$ and unlike the interior of $\Delta$ maybe empty. *A priori* the Fine interior $\Delta^{\mathrm{FI}}$ is an infinite intersection of affine halfspaces, but it can be obtained as a finite intersection of halfspaces $\Gamma_1^{\Delta}(n)$, where $n$ appears as a minimal generator of the semigroup $\sigma^{\theta} \cap N$ for a face $\theta \preceq \Delta$ [Bat17, Remark 2.6].

**Proposition 2.3.25 [Bat17, Proposition 3.4].** *Let $\Delta \subseteq M_{\mathbb{R}}$ a $d$-dimensional lattice polytope. Then the following conditions are equivalent:*

(i) *$\Delta$ is almost pseudoreflexive.*

(ii) $\Delta^{\mathrm{FI}} = \{0\}$.

(iii) *$\Delta$ contains the origin $0 \in M$ in its interior and $\Delta$ is contained in a pseudoreflexive polytope $\Delta'$.*

**Remark 2.3.26.** Using a result of Skarke [Ska96], one obtains that every pseudoreflexive polytope $\Delta$ of dimension $d \leq 4$ is reflexive. In particular, every almost pseudoreflexive polytope of dimension $d \leq 4$ is almost reflexive. There exists an equivalent description of almost pseudoreflexive reflexive polytopes $\Delta$ of dimension $d \leq 4$: $\Delta \subseteq M_{\mathbb{R}}$ is almost pseudoreflexive if and only if $\Delta$ contains $0 \in M$ in its interior and is contained in some reflexive polytope $\Delta'$ [Bat17, Proposition 3.4 and Theorem 3.12].

In this thesis, we consider the following three running examples of 3-polytopes $\Delta_1$, $\Delta_2$, and $\Delta_3$:

**Example 2.3.27.** Set $\Delta_1 := \mathrm{conv}(e_1, e_2, e_3, -e_1 - e_2 - e_3)$, $\Delta_2 := \mathrm{conv}(e_1, e_2, e_3, -e_1 - e_2 - 2e_3)$, and $\Delta_3 := \mathrm{conv}(e_1, e_2, e_3, -5e_1 - 6e_2 - 8e_3)$ (Figure 2.3). All these polytopes contain the origin 0 as an unique interior lattice point (*i.e.*, $\Delta_i^\circ \cap M = \{0\}$ $(1 \le i \le 3)$), but only $\Delta_1$ is a reflexive polytope. Checking if $[\Delta_i^*] = \mathrm{conv}(\Delta_i^* \cap N)$ $(1 \le i \le 3)$ contains the origin in its interior, we get that the simplices $\Delta_1$, $\Delta_2$ are almost reflexive and $\Delta_3$ is not almost reflexive (Figure 6.6(b), 6.5(b), and 6.7(b)). Using Proposition 2.3.25, the corresponding Fine interiors are $\Delta_1^{\mathrm{FI}} = \{0\}$, $\Delta_2^{\mathrm{FI}} = \{0\}$, and $\Delta_3^{\mathrm{FI}} \neq \{0\}$. To be precise, a concrete computation yields $\Delta_3^{\mathrm{FI}} = \mathrm{conv}((0,0,0), (-1, -1/2, -1/2))$.

Now, we are interested in algebraic varieties that are birational to canonical models of a generic affine hypersurface $Z_\Delta \subseteq (\mathbb{C}^*)^d$ defined by a generic Laurent polynomial $f_\Delta$ whose Newton polytope is a given $d$-dimensional lattice polytope $\Delta \subseteq M_\mathbb{R}$. The generic open condition on the coefficients of the Laurent polynomial $f_\Delta$ can be formulated in a more precise form:

**Definition 2.3.28.** Let $\Delta \subseteq M_\mathbb{R}$ be a $d$-dimensional lattice polytope and $Z_\Delta \subseteq \mathbb{T}^d \cong (\mathbb{C}^*)^d$ an affine hypersurface in the $d$-dimensional algebraic torus $\mathbb{T}^d$, which is the zero set of a Laurent polynomial

$$f_\Delta(x) = \sum_{m \in \Delta \cap M} a_m x^m$$

with the Newton polytope $\Delta$ and some sufficiently general coefficients $a_m \in \mathbb{C}$. $\Delta$ is called *Newton polytope* of the Laurent polynomial $f_\Delta$ because it is the convex hull of all lattice points $m \in M$ such that $a_m \neq 0$.

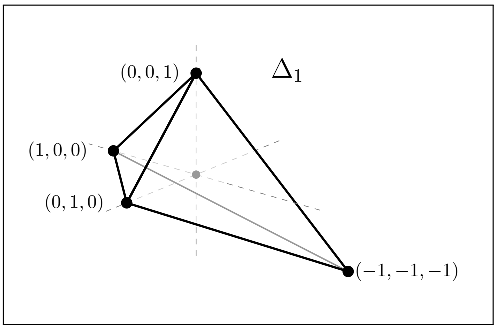
The Laurent polynomial $f_\Delta$ and the affine hypersurface $Z_\Delta$ are called $\Delta$-*non-degenerate* if for every face $\theta \preceq \Delta$ of $\Delta$ the zero locus $Z_\theta := \{x \in \mathbb{T}^d \mid f_\theta(x) = 0\}$ of the $\theta$-part of $f_\Delta$ is empty or a smooth affine hypersurface in the $d$-dimensional algebraic torus $\mathbb{T}^d$.

Almost pseudoreflexive $d$-polytopes are in the following sense closely related to Calabi-Yau varieties (Definition 2.1.4):

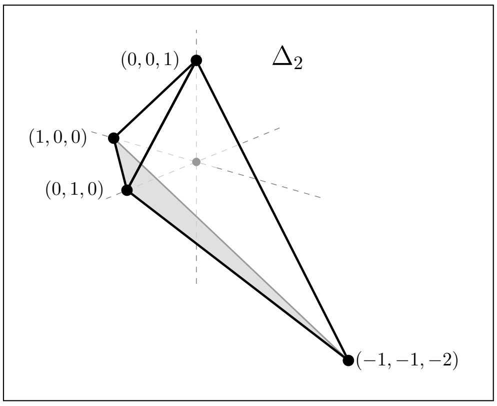
**Proposition 2.3.29 [Bat17, Theorem 2.23].** *Let $\Delta \subseteq M_\mathbb{R}$ be a $d$-dimensional lattice polytope containing the origin $0 \in M$ in its interior. Then a canonical model* (Definition 2.1.8) *of a $\Delta$-non-degenerate affine hypersurface $Z_\Delta \subseteq \mathbb{T}^d$ is birational to a $(d-1)$-dimensional Calabi-Yau variety $Y$ with at worst Gorenstein canonical singularities if and only if the polytope $\Delta$ is almost pseudoreflexive. In particular, in the case $d = 3$ a $\Delta$-non-degenerate affine surface $Z_\Delta \subseteq \mathbb{T}^3$ is birational to a $K3$-surface if and only if $\Delta$ is an almost reflexive polytope.*

**Example 2.3.30.** Following Remark 2.3.23, a complete list of all 16 isomorphism classes of 2-dimensional reflexive polytopes can be found in Figure 6.1. Furthermore, there are examples of 3-dimensional reflexive polytopes in Figure 6.2 and of 4-dimensional reflexive polytopes in Figure 6.3. In particular, all these polytopes are almost pseudoreflexive. In addition, Figure 2.3(b) and 6.5(a) present an almost pseudoreflexive polytope that is not reflexive.

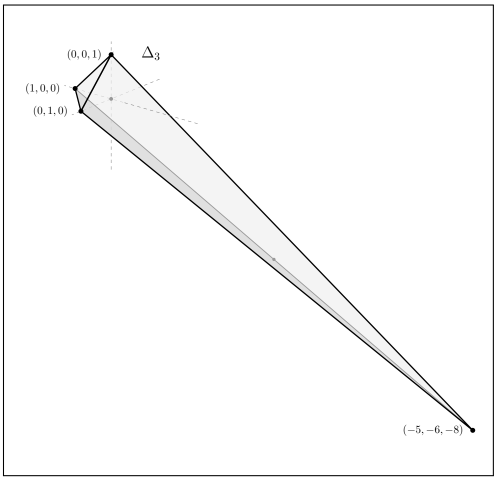
To recapitulate the content of this subsection, we refer the reader to [Bat17].




**(a)** Reflexive polytope $\Delta_1$.




**(b)** Almost reflexive polytope $\Delta_2$.




**(c)** Non-almost reflexive polytope $\Delta_3$.

**Figure 2.3: Three Types of Canonical Fano 3-polytopes.** Shaded faces are occluded. ***(a)*** Reflexive polytope $\Delta_1$. All facets of $\Delta_1$ have lattice distance 1 to the origin. ***(b)*** Almost reflexive polytope $\Delta_2$. The grey coloured facet of $\Delta_2$ has lattice distance 2 and all other facets have lattice distance 1 to the origin. ***(c)*** Non-almost reflexive polytope $\Delta_3$. The grey coloured facet of $\Delta_3$ has lattice distance 2, the light grey coloured facet has lattice distance 3, and all other facets have lattice distance 1 to the origin.

### 2.3.5 Canonical Fano Polytopes

In addition to this subsection that focuses on the essentials regarding the needs of this thesis, we refer to [Kas10, Nil05].

**Definition 2.3.31.** A $d$-dimensional lattice polytope $\Delta \subseteq N_{\mathbb{R}}$ containing only the origin as an interior lattice point (*i.e.*, $\Delta^{\circ} \cap N = \{0\}$) is called *canonical Fano polytope* .

**Remark 2.3.32.** Let $\Delta \subseteq N_{\mathbb{R}}$ be an almost pseudoreflexive $d$-polytope. By Proposition 2.3.25, $\Delta^{\mathrm{FI}} = \{0\}$ and $\mathrm{conv}(\Delta^{\circ} \cap N) \subseteq \Delta^{\mathrm{FI}}$ [Bat17, Remark 2.7], *i.e.*, canonical Fano $d$-polytopes build a superclass of almost pseudoreflexive $d$-polytopes. In particular, combining this fact with Remark 2.3.23 provides that canonical Fano $d$-polytopes build a superclass of reflexive $d$-polytopes.

Canonical Fano 3-polytopes are classified up to isomorphism (Remark 2.3.16) and there exist exactly 674,688 isomorphism classes of 3-dimensional canonical Fano polytopes [Kas10]. Furthermore, 2-dimensional canonical Fano varieties are fully classified. There exist exactly 16 isomorphism classes corresponding to the 16 reflexive polygons in Figure 6.1. Full classifications of $d$-dimensional canonical Fano varieties with $d > 3$ are still pending.

A *toric Fano variety* $X$ is a normal projective toric variety with at worst log-terminal singularities such that $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor.

**Theorem 2.3.33 [Dan78, Rei83].** *Let $\Delta \subseteq N_{\mathbb{R}}$ be a canonical Fano $d$-polytope. Then the associated normal projective toric variety $X_{\Sigma_\Delta}$ defined via the spanning fan $\Sigma_\Delta$ of $\Delta$ consisting of cones over faces of the given polytope $\Delta$ is a toric Fano variety with at worst canonical singularities, called* canonical toric Fano variety*.*

Moreover, there exists a one-to-one correspondence between $d$-dimensional canonical toric Fano varieties and canonical Fano $d$-polytopes [Nil05, Proposition 2.3.7].

The properties $X$ occupies can be combinatorially characterized as follows: $X$ is a toric Fano variety if and only if the primitive ray generators of all 1-dimensional cones in the associated fan $\Sigma$ with $X$ correspond to vertices of a lattice polytope in $N$ [Dan78, Ewa96].

**Proposition 2.3.34 [Rei83].** *A toric variety $X$ has at worst canonical singularities if and only if the primitive ray generators of 1-dimensional cones of each cone $\sigma \in \Sigma$ are contained in an affine hyperplane $H$ and there are no other elements of the lattice $N$ in the part of $\sigma$ under $H$.*

**Example 2.3.35.** Following Remark 2.3.32, 2-dimensional reflexive polytopes can be found in Figure 6.1, 3-dimensional reflexive polytopes in Figure 6.2 and 2.3(a) and 4-dimensional reflexive polytopes in Figure 6.3. Moreover, Figure 2.3(b) and 6.5(a) present an almost pseudoreflexive polytope. In addition, Figure 2.3(c) and 6.7(a) present a canonical Fano polytope that is not almost pseudoreflexive. A whole string of examples that are not almost pseudoreflexive is listed in the Appendix A, Figure A.1, A.2, and A.3.

The existence of canonical and minimal models (Definition 2.1.8 and 2.1.9) of non-degenerate affine hypersurfaces (Definition 2.3.28) defined by canonical Fano d-polytopes is guaranteed:

**Remark 2.3.36.** Let $\Delta \subseteq N_{\mathbb{R}}$ be a canonical Fano $d$-polytope. Then $\Delta^{\mathrm{FI}} \neq \emptyset$ because $\operatorname{conv}(\Delta^\circ \cap N) \subseteq \Delta^{\mathrm{FI}}$ and $\operatorname{conv}(\Delta^\circ \cap N) = \{0\}$ [Bat17, Remark 2.7]. Therefore, the $\Delta$-non-degenerate affine hypersurface $Z_\Delta$ has a canonical and a minimal model [Bat17, Theorem 2.8].

### 2.3.6 Gorenstein Polytopes

We recommend consulting, in addition, other articles, such as [BN08] and [BJ10].

**Definition 2.3.37.** Let $r$ be a positive integer. A $d$-dimensional lattice polytope $\Delta \subseteq M_{\mathbb{R}}$ is called *Gorenstein polytope of index* $r$ if $r\Delta - m$ is a reflexive polytope for some lattice point $m \in M$.

To simplify our notation, we skip $m$ and write $r\Delta$ instead of $r\Delta - m$ while considering the associated reflexive polytope.

**Example 2.3.38.** Reflexive polytopes are Gorenstein polytopes of index $r = 1$.

Let $\Delta \subseteq M_{\mathbb{R}}$ be a $d$-dimensional Gorenstein polytope of index $r$. Then the normal fan $\Sigma^\Delta$ in $N_{\mathbb{R}}$ of the polytope $\Delta$ (or, equivalently, of $r\Delta$) defines a Gorenstein toric Fano variety $X := X_{\Sigma^\Delta}$ such that its anticanonical class $c_1(X)$ is divisible by $r$ in $\operatorname{Pic}(X)$ with at worst log-terminal singularities (Proposition 2.2.10).

There exists a duality for Gorenstein polytopes that generalizes the duality for reflexive polytopes: For this purpose, we associate with a Gorenstein polytope $\Delta \subseteq M_{\mathbb{R}}$ of index $r$ the $(d+1)$-dimensional cone

$$C_\Delta := \{(y, \lambda) \in M_{\mathbb{R}} \oplus \mathbb{R} \mid y \in \lambda\Delta\} \subseteq M_{\mathbb{R}} \oplus \mathbb{R}.$$

The dual cone $C_\Delta^\vee \subseteq N_{\mathbb{R}} \oplus \mathbb{R}$ is defined as

$$C_\Delta^\vee := \{(x, \mu) \in N_{\mathbb{R}} \oplus \mathbb{R} \mid \langle y, x\rangle + \lambda\mu \geq 0 \ \forall (y, \lambda) \in C_\Delta\} \tag{2.4}$$

and the $l$-th slice $C_\Delta^\vee(l)$ of $C_\Delta^\vee$ is defined as the lattice polytope

$$C_\Delta^\vee(l) := C_\Delta^\vee \cap \{(x, \mu) \in N_{\mathbb{R}} \oplus \mathbb{R} \mid \langle m, x\rangle + r\mu = l\} \subseteq N_{\mathbb{R}} \oplus \mathbb{R}.$$

**Definition 2.3.39.** Let $\Delta \subseteq M_{\mathbb{R}}$ be a Gorenstein polytope of index $r$. Then the lattice polytope $\Delta^* := C_\Delta^\vee(1)$ is again a Gorenstein polytope of index $r$ and called *dual Gorenstein polytope* to $\Delta$.

The duality between two $(d+1)$-dimensional cones $C_\Delta$ and $C_\Delta^\vee$ establishes a one-to-one order-reversing correspondence between faces of $C_\Delta$ and $C_\Delta^\vee$ that induces a duality between faces of the Gorenstein polytopes $\Delta$ and $\Delta^* = C_\Delta^\vee(1)$. It is important to note

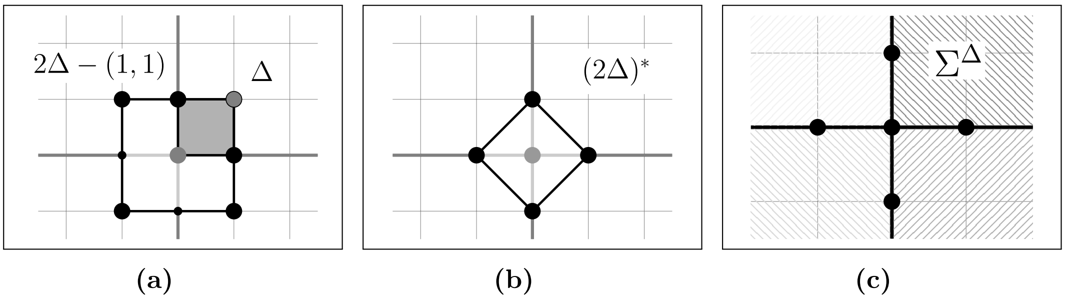



**Figure 2.4: Gorenstein Polytope of Index $r$.** ***(a)*** Gorenstein polytope $\Delta$ (grey coloured) of index $r$ (here: $r = 2$) and reflexive polytope $r\Delta - m$ (here: $m = (1,1)$). ***(b)*** Dual reflexive polytope $(r\Delta)^*$. ***(c)*** Normal fan $\Sigma^\Delta$ associated with $\Delta$ and $r\Delta$, respectively.

that the reflexive polytope $(r\Delta)^*$ and the Gorenstein polytope $\Delta^*$ are not only naturally combinatorially isomorphic, but this isomorphism also induces isomorphisms between proper faces of $(r\Delta)^*$ and $\Delta^*$ considered as lattice polytopes [BN08].

The normal fan $\Sigma^\Delta$ in $N_\mathbb{R}$ can be constructed via the projection

$$N_\mathbb{R} \oplus \mathbb{R} \to (N_\mathbb{R} \oplus \mathbb{R})/\mathbb{R}(n, \underline{\mathrm{r}}) \cong N_\mathbb{R} \tag{2.5}$$

of all proper faces of the dual cone $C_\Delta^\vee$ along the 1-dimensional subspace generated by the unique interior lattice point $(n, \underline{\mathrm{r}})$ in the reflexive polytope $C_\Delta^\vee(r)$.

**Example 2.3.40.** Let $\Delta \subseteq M_\mathbb{R}$ be a lattice polytope given by

$$\operatorname{conv}((0,0),(1,0),(1,1),(0,1))$$

(grey coloured in Figure 2.4(a)). Then $\Delta$ is a Gorenstein polytope of index $r = 2$ because $2\Delta - (1,1)$ with $m = (1,1) \in M$ is the reflexive polytope in Figure 6.1(n). To construct the normal fan $\Sigma^\Delta$ of $\Delta$, we can look at the dual polytope $(r\Delta)^*$ of $r\Delta$ pictured in Figure 2.4(b), which is the reflexive polytope in Figure 6.1(d). The corresponding normal fan $\Sigma^\Delta$ consists of the following nine cones: $\{0\}$, $\operatorname{conv}(\pm e_1)$, $\operatorname{conv}(\pm e_2)$, and $\operatorname{conv}(\pm e_1, \pm e_2)$ (Figure 2.4(c)) and defines the Gorenstein toric Fano variety $X_{\Sigma^\Delta} = \mathbb{P}^1 \times \mathbb{P}^1$ with at worst log-terminal singularities.

To see the duality for this Gorenstein polytope, we look at the (2+1)-dimensional cone $C_\Delta \subseteq M_\mathbb{R} \oplus \mathbb{R}$ (Figure 2.4(d)) and its dual cone $C_\Delta^\vee \subseteq N_\mathbb{R} \oplus \mathbb{R}$ (Figure 2.4(e)) defined in Equation (2.4). In Figure 2.4(d) the grey coloured polytope at height 1 in the cone $C_\Delta$ corresponds to $\Delta$. Finally, the grey coloured polytope $C_\Delta^\vee(1)$ at height 1 in the dual cone $C_\Delta^\vee$ in Figure 2.4(e) is again a Gorenstein polytope of index $r = 2$ that is defined to be the dual Gorenstein polytope $\Delta^*$ of $\Delta$.

Moreover, we construct the normal fan $\Sigma^\Delta$ via the projection in Equation (2.5) of all proper faces of the cone $C_\Delta^\vee$ along the 1-dimensional subspace generated by the unique interior lattice point $(n, \underline{\mathrm{r}}) = (0,0,1)$ in the reflexive polytope $C_\Delta^\vee(2)$, that is the 2-th slice of $C_\Delta^\vee$ (Figure 2.4(e)).

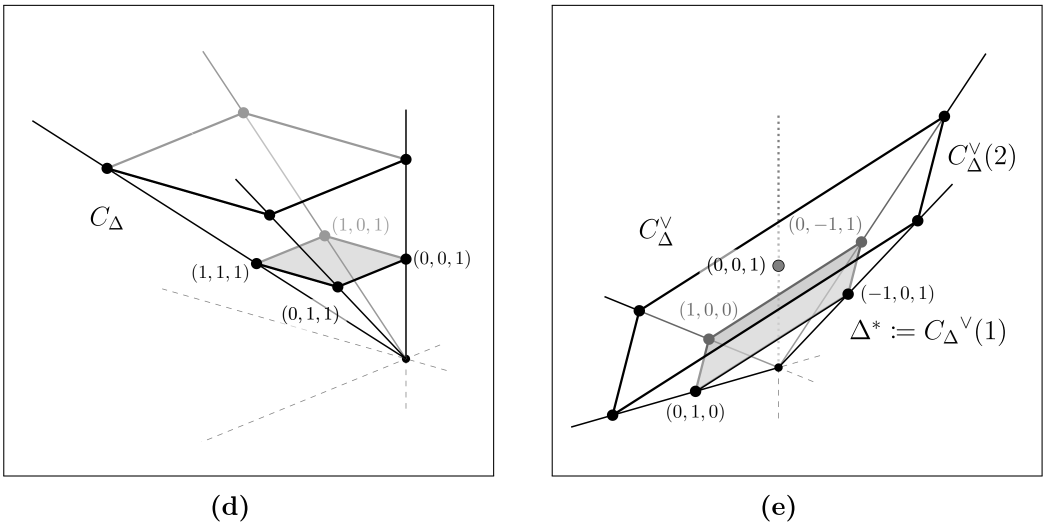



**Figure 2.4: Gorenstein Polytope of Index *r*.** Shaded faces are occluded. ***(d)*** Cone $C_\Delta \subseteq M_\mathbb{R} \oplus \mathbb{R}$ associated with the Gorenstein polytope $\Delta$ of index $r$. The grey coloured polytope at height 1 corresponds to $\Delta$. ***(e)*** Dual cone $C^\vee_\Delta \subseteq N_\mathbb{R} \oplus \mathbb{R}$. The grey coloured polytope $C^\vee_\Delta(1)$ at height 1 is the dual Gorenstein polytope $\Delta^*$ of index $r$. The unique interior lattice point (n, $\underline{\text{r}}$)=(1,0,0) of the reflexive polytope $C^\vee_\Delta(2)$ is needed to construct the normal fan $\Sigma^\Delta$ via a projection.

Note that almost all combinatorial definitions in this section rely only on the fact that $\Delta$ is a lattice polytope and so it does not matter in which lattice the polytope is given, *i.e.*, we can define all objects in an analogous way if the given lattice polytope $\Delta$ lies in the dual space.

# Chapter 3

## Stringy Chern Classes and Their Intersection Numbers

This chapter starts in Section 3.1 with a simplification of the general definition for $k$-th stringy Chern classes due to de Fernex, Luperico, Nevins, and Uribe [dFLNU07]. Applying this definition, we prove that the well-known formula expressing the total Chern class of smooth complete intersections via the total Chern class of the ambient smooth variety remains valid also for the total stringy Chern class of generic hypersurfaces and complete intersections in an ambient singular variety. In particular, we show that the top stringy Chern class (or stringy Euler number) of generic semi-ample Cartier divisors can be computed via the stringy Chern classes of the ambient singular variety. In addition, we give a similar formula for the top stringy Chern class of complete intersections $Z_1 \cap \ldots \cap Z_r$, where $Z_1, \ldots, Z_r$ are generic semi-ample Cartier divisors on the ambient singular variety.

In Section 3.2, we look at the intersection numbers of stringy Chern classes with $\mathbb{Q}$-Cartier divisors. A particular case of such an intersection number appears in the stringy version of the Libgober-Wood identity (1.7) [Bat00, Theorem 3.8]. For arbitrary $\mathbb{Q}$-Cartier divisors $Z_1, \ldots, Z_k$, we consider more general intersection numbers $[Z_1]. \ldots .[Z_k].c^{\mathrm{str}}_{d-k}(X)$ which can be defined by a similar formula. In addition, we prove the formula's independence on the chosen desingularization without using the definition of stringy Chern classes.

The results of this chapter have been published in our paper *Stringy Chern Classes of Singular Toric Varieties and Their Applications* [BS17] that is joint work of the author of this thesis and Victor Batyrev.

## 3.1 Stringy Chern Classes of Complete Intersections

Let $X$ be a $d$-dimensional $\mathbb{Q}$-Gorenstein variety (Definition 2.1.1) with at worst log-terminal singularities (Definition 2.1.6), *i.e.*, $X$ is in particular normal. First, we note that the general definition of the total stringy Chern class $c^{\mathrm{str}}_{\bullet}(X) \in A_{\bullet}(X)_{\mathbb{Q}}$ of $X$ [dFLNU07] can be simplified as follows, where the *total stringy Chern class* of $X$ is defined to be $c^{\mathrm{str}}_d(X) + \ldots + c^{\mathrm{str}}_2(X) + c_1(X) + 1$:

**Proposition 3.1.1.** *Let $X$ be a projective $\mathbb{Q}$-Gorenstein variety of dimension $d$ with at worst log-terminal singularities and $\rho : Y \to X$ a log-desingularization of $X$* (Definition 2.1.6). *Then the total stringy Chern class $c_\bullet^{\mathrm{str}}(X)$ of $X$ can be computed through total Chern classes $c_\bullet(D_J)$ of smooth projective subvarieties $D_J \subseteq Y$ by*

$$c_\bullet^{\mathrm{str}}(X) := \rho_*\left(\sum_{\emptyset \subseteq J \subseteq I} e_{J*} c_\bullet(D_J) \prod_{j \in J}\left(\frac{-a_j}{a_j+1}\right)\right) \in A_\bullet(X)_{\mathbb{Q}},$$

*where $D_J$ is the subvariety $\cap_{j\in J} D_j$ of codimension $|J|$ for any non-empty subset $J \subseteq I = \{1, \ldots, s\}$, $e_J : D_J \hookrightarrow Y$ its closed embedding, and $\rho_* : A_{d-k}(Y) \to A_{d-k}(X)$ and $e_{J*} : A_{d-k}(D_J) \to A_{d-k}(Y)$ push-forward homomorphisms corresponding to the proper birational morphism $\rho$ and the closed embedding $e_J$, respectively. In particular, one obtains*

$$c_k^{\mathrm{str}}(X) := \rho_*\left(\sum_{\emptyset \subseteq J \subseteq I} e_{J*} c_{k-|J|}(D_J) \prod_{j \in J}\left(\frac{-a_j}{a_j+1}\right)\right) \in A_{d-k}(X)_{\mathbb{Q}}$$

*for the $k$-th* stringy Chern class *of $X$ and*

$$c_k^{\mathrm{str}}(X) = \rho_* c_k(Y) \in A_{d-k}(X)_{\mathbb{Q}}$$

*if $\rho : Y \to X$ is a crepant log-desingularization* $(0 \leq k \leq d)$.

*Proof.* The definition of total stringy Chern classes given by de Fernex, Luperico, Nevins, and Uribe in [dFLNU07] uses the group homomorphism of MacPherson $c : F(Y) \to A_\bullet(Y)$ from the group $F(Y)$ of constructible functions on $Y$ to the Chow group $A_\bullet(Y)$ of $Y$. If $\mathbb{1}_Y$ is the characteristic function of the smooth variety $Y$, then $c(\mathbb{1}_Y) = c_\bullet(Y)$. Using the stratification of $Y$ by locally closed subsets $D_J^\circ := D_J \setminus \left(\cup_{i \in I\setminus J} D_i\right)$ $(\emptyset \subseteq J \subseteq I)$ their definition looks as follows:

$$c_\bullet^{\mathrm{str}}(X) := \rho_*\left(\sum_{\emptyset \subseteq J \subseteq I} c\left(\mathbb{1}_{D_J^\circ}\right) \prod_{j \in J}\left(\frac{1}{a_j+1}\right)\right). \tag{3.1}$$

Using the stratification $\mathbb{1}_{D_J} = \sum_{J' \supseteq J} \mathbb{1}_{D_{J'}^\circ}$ and $c\left(\mathbb{1}_{D_J}\right) = \sum_{J' \supseteq J} c\left(\mathbb{1}_{D_{J'}^\circ}\right)$ for all $\emptyset \subseteq J \subseteq I$, we conclude

$$\begin{aligned}\sum_{\emptyset \subseteq J \subseteq I} c\left(\mathbb{1}_{D_J^\circ}\right) \prod_{j \in J}\left(\frac{1}{a_j+1}\right) &= \sum_{\emptyset \subseteq J \subseteq I}\left(\sum_{J' \supseteq J} c\left(\mathbb{1}_{D_{J'}^\circ}\right)\right) \prod_{j \in J}\left(\frac{1}{a_j+1} - 1\right) \\ &= \sum_{\emptyset \subseteq J \subseteq I} c\left(\mathbb{1}_{D_J}\right) \prod_{j \in J}\left(\frac{-a_j}{a_j+1}\right).\end{aligned} \tag{3.2}$$

It remains to apply $\rho_*$ to Equation (3.2) and the property $c\left(\mathbb{1}_{D_J}\right) = e_{J*} c_\bullet(D_J)$, which follows from the commutative diagram

$$\begin{tikzcd} F(D_J) \arrow[r] \arrow[d, "c"'] & F(Y) \arrow[d, "c"] \\ A_\bullet(D_J) \arrow[r, "e_{J*}"'] & A_\bullet(Y), \end{tikzcd}$$

where $e_{J*} : A_\bullet(D_J) \to A_\bullet(Y)$ is the push-forward homomorphism corresponding to the closed embedding $e_J : D_J \hookrightarrow Y$.

If $\rho : Y \to X$ is a crepant log-desingularization ($i.e.$, $a_i = 0$ for all $i \in I$), then we obtain

$$\begin{aligned} c_k^{\mathrm{str}}(X) &= \rho_*\left(\sum_{\emptyset\subseteq J\subseteq I} e_{J*}c_{k-|J|}(D_J)\prod_{j\in J}\left(\frac{-a_j}{a_j+1}\right)\right) = \rho_*\left(\sum_{\emptyset\subseteq J\subseteq I} e_{J*}c_{k-|J|}(D_J)\prod_{j\in J}0\right) \\ &= \rho_*\Big(\sum_{\emptyset\subseteq J\subseteq I} e_{\emptyset*}c_{k-|\emptyset|}(D_\emptyset)\prod_{j\in\emptyset}0+0\ldots+0\Big) = \rho_*c_k(Y) \in A_{d-k}(X)_{\mathbb{Q}} \end{aligned}$$

because $D_\emptyset = Y$, $e_\emptyset$ is the identity on $Y$, and $\prod_{j\in\emptyset} 0 = 1$. □

**Remark 3.1.2.** The central aspect in the definition of total stringy Chern classes $c_\bullet^{\mathrm{str}}(X)$ and $k$-th stringy Chern classes $c_k^{\mathrm{str}}(X)$ is their independence on the choice of the log-desingularization $\rho$. This property is proved in [dFLNU07, Proposition 3.2].

Let $V$ be a smooth projective variety of dimension $d$ and $Z$ a smooth divisor on $V$ together with the closed embedding $i : Z \hookrightarrow V$. Using the exact sequence of vector bundles $0 \to \mathcal{T}_Z \to i^*\mathcal{T}_V \to \mathcal{O}_Z(Z) \to 0$, one obtains a formula that computes the total Chern class of $Z$ in terms of the total Chern class of the ambient variety $V$:

$$i_*c_\bullet(Z) = c_\bullet(V).[Z](1+[Z])^{-1} = c_\bullet(V).\left(\sum_{k=1}^{\infty}(-1)^{k-1}[Z]^k\right) \tag{3.3}$$

[Ful98, Example 3.2.12], where $\mathcal{T}_.$ denotes the tangent bundle, $[Z]$ the divisor class of $Z$ in $A_{d-1}(V)$, and $c_\bullet(V) = \sum_k c_k(V)$ the total Chern class of $V$. In particular, the Euler number of $Z$ can be computed as

$$e(Z) = c_{d-1}(Z) = \sum_{k=1}^{d}(-1)^{k-1}[Z]^k.c_{d-k}(V). \tag{3.4}$$

We show that the same formulas hold for the total stringy Chern class and the stringy Euler number of generic hypersurfaces in singular varieties:

**Theorem 3.1.3.** *Let $X$ be a projective $\mathbb{Q}$-Gorenstein variety with at worst log-terminal singularities and $Z$ a generic semi-ample Cartier divisor on $X$. Then the total stringy Chern class of $Z$ is*

$$i_*c_\bullet^{\mathrm{str}}(Z) = c_\bullet^{\mathrm{str}}(X).[Z]\,(1+[Z])^{-1},$$

*where $i : Z \hookrightarrow X$ is the closed embedding of $Z$ in $X$.*

*Proof.* Let $\rho : Y \to X$ be a log-desingularization of $X$ with $K_Y = \rho^* K_X + \sum_{i=1}^{s} a_i D_i$ [CLS11, Theorem 11.2.2]. By the Theorem of Bertini [Har77, Chapter III, Section 10], we can assume that $Z' := \rho^{-1}(Z)$ is a smooth divisor on $Y$. Using the adjunction formula $K_{Z'} = \left(K_Y + Z'\right)|_{Z'}$, we obtain $K_{Z'} = \rho^* K_Z + \sum_{i=1}^{s} a_i D_i'$, where $D_i' := D_i \cap Z'$. Define $D_J' := \cap_{j \in J} D_j'$ and note that $D_J' = D_J \cap Z' \subseteq D_J$ is a smooth divisor on $D_J$. Let $e'_{J*} : A_\bullet\left(D_J'\right) \to A_\bullet\left(Z'\right)$ be the push-forward homomorphism corresponding to the closed embedding $e_J' : D_J' \hookrightarrow Z'$. Consider the commutative diagram

$$\begin{tikzcd}
D_J \arrow[r, hook, "e_J"] & Y \arrow[r, "\rho"] & X \\
D_J' \arrow[u, hook, "i_J"] \arrow[r, hook, "e_J'"'] & Z' \arrow[r, "\rho_Z"'] & Z, \arrow[u, hook, "i"']
\end{tikzcd}$$

where $\rho_Z : Z' \to Z$ and $i_J : D_J' \to D_J$ are restrictions of $\rho$ and $i$, respectively. We apply Equation (3.3) to the smooth divisor $D_J' \subseteq D_J$ and obtain

$$i_* \rho_{Z*} e'_{J*} c_\bullet\left(D_J'\right) = \rho_* e_{J*} i_{J*} c_\bullet\left(D_J'\right) = \rho_* e_{J*}\left(c_\bullet\left(D_J\right) . \left[D_J'\right]\left(1 + \left[D_J'\right]\right)^{-1}\right).$$

Using the projection formula (Proposition 2.1.18) twice provides

$$e_{J*}\left(c_\bullet\left(D_J\right) . \left[D_J'\right]\left(1 + \left[D_J'\right]\right)^{-1}\right) = e_{J*} c_\bullet\left(D_J\right) . \left[Z'\right]\left(1 + \left[Z'\right]\right)^{-1}$$

and

$$\rho_*\left(e_{J*} c_\bullet\left(D_J\right) . \left[Z'\right]\left(1 + \left[Z'\right]\right)^{-1}\right) = \rho_* e_{J*} c_\bullet\left(D_J\right) . [Z]\left(1 + [Z]\right)^{-1}$$

because ${e_J}^* Z' = D_J'$, $\rho^* Z = Z'$, and the pullbacks ${e_J}^*$ and $\rho^*$ are homomorphisms. Therefore, we get

$$i_* \rho_* e'_{J*} c_\bullet\left(D_J'\right) = \rho_* e_{J*} c_\bullet\left(D_J\right) . [Z]\left(1 + [Z]\right)^{-1}.$$

An application of Proposition 3.1.1 to $Z$ and $X$ provides

$$\begin{aligned}
i_* c_\bullet^{\mathrm{str}}(Z) &= i_* \rho_* \left( \sum_{\emptyset \subseteq J \subseteq I} e'_{J*} \left( c_\bullet\left(D_J'\right) \prod_{j \in J} \left( \frac{-a_j}{a_j + 1} \right) \right) \right) \\
&= \rho_* \left( \sum_{\emptyset \subseteq J \subseteq I} e_{J*} c_\bullet\left(D_J\right) \prod_{j \in J} \left( \frac{-a_j}{a_j + 1} \right) \right) . [Z]\left(1 + [Z]\right)^{-1} \\
&= c_\bullet^{\mathrm{str}}(X) . [Z]\left(1 + [Z]\right)^{-1}.
\end{aligned}$$

□

**Corollary 3.1.4.** *Let $X$ be a projective $\mathbb{Q}$-Gorenstein variety of dimension $d$ with at worst log-terminal singularities and $Z$ a generic semi-ample Cartier divisor on $X$. Then the stringy Euler number of $Z$ is*

$$e_{\mathrm{str}}(Z) = c_{d-1}^{\mathrm{str}}(Z) = \sum_{k=1}^{d} (-1)^{k-1} [Z]^k . c_{d-k}^{\mathrm{str}}(X).$$

*If $[Z] = c_1(X)$, then $Z$ has trivial anticanonical class and the formula simplifies to*

$$e_{\mathrm{str}}(Z) = c_{d-1}^{\mathrm{str}}(Z) = \sum_{k=1}^{d-2} (-1)^{k-1} [Z]^k . c_{d-k}^{\mathrm{str}}(X).$$

The formulas in Theorem 3.1.3 and Corollary 3.1.4 for total stringy Chern classes and top stringy Chern classes of generic hypersurfaces can be generalized to generic complete intersections in singular varieties:

**Theorem 3.1.5.** *Let $X$ be a projective $\mathbb{Q}$-Gorenstein variety with at worst log-terminal singularities, $Z_1, \ldots, Z_r$ generic semi-ample Cartier divisors on $X$, and $i : Z_1 \cap \ldots \cap Z_r \hookrightarrow X$ the closed embedding of $Z_1 \cap \ldots \cap Z_r$ in $X$. Then the total stringy Chern class of the complete intersection $Z_1 \cap \ldots \cap Z_r$ is*

$$i_* c_{\bullet}^{\mathrm{str}} (Z_1 \cap \ldots \cap Z_r) = c_{\bullet}^{\mathrm{str}}(X). \prod_{j=1}^{r} [Z_j] (1 + [Z_j])^{-1} .$$

*Proof.* We apply induction on $r$ and use Theorem 3.1.3 $r$-times, since for any $2 \le r' \le r$ the complete intersection $Z_1 \cap \ldots \cap Z_{r'}$ is a generic hypersurface in $Z_1 \cap \ldots \cap Z_{r'-1}$. $\square$

**Corollary 3.1.6.** *Let $X$ be a projective $\mathbb{Q}$-Gorenstein variety of dimension $d$ with at worst log-terminal singularities and $Z_1, \ldots, Z_r$ generic semi-ample Cartier divisors on $X$. Then the stringy Euler number $c_{d-r}^{\mathrm{str}}(Z_1 \cap \ldots \cap Z_r)$ of the complete intersection $Z_1 \cap \ldots \cap Z_r$ is*

$$\sum_{k=0}^{d-r} (-1)^k [Z_1] . \ \ldots \ . [Z_r] . \left( \sum_{\substack{j_0, \ldots, j_k = 1 \\ j_0 \le \cdots \le j_k}}^{r} [Z_{j_0}] . \ \ldots \ . [Z_{j_k}] \right) . c_{d-r-k}^{\mathrm{str}} (X) .$$

**Corollary 3.1.7.** *Let $X$ be a projective $\mathbb{Q}$-Gorenstein variety of dimension $d$ with at worst log-terminal singularities and $Z_1, \ldots, Z_r$ generic semi-ample Cartier divisors on $X$ such that $[Z] \coloneqq [Z_1] = \ldots = [Z_r]$. Then the stringy Euler number of the complete intersection $Z_1 \cap \ldots \cap Z_r$ is*

$$c_{d-r}^{\mathrm{str}} (Z_1 \cap \ldots \cap Z_r) = \sum_{k=0}^{d-r} (-1)^k \binom{k+r-1}{r-1} [Z]^{r+k} . c_{d-r-k}^{\mathrm{str}} (X) .$$

## 3.2 Intersection Numbers with Stringy Chern Classes

Let $X$ be a projective $\mathbb{Q}$-Gorenstein variety (Definition 2.1.1) of dimension $d$ with at worst log-terminal singularities (Definition 2.1.6, Proposition 2.2.10), $\rho : Y \to X$ a log-desingularization of $X$ (Definition 2.1.6), and $Z_1, \dots, Z_k$ arbitrary $\mathbb{Q}$-Cartier divisors on $X$. Intersecting the divisor classes $[Z_1], \dots, [Z_k] \in \operatorname{Pic}(X)_{\mathbb{Q}}$ with the stringy Chern class $c^{\text{str}}_{d-k}(X) \in A_k(X)_{\mathbb{Q}}$ $(0 \leq k \leq d)$, one obtains the rational intersection number

$$[Z_1]. \ \dots \ .[Z_k].c^{\text{str}}_{d-k}(X),$$

which can be considered as a generalization of $c_1(X).c^{\text{str}}_{d-1}(X)$ in Equation (1.8). Using the second formula in Proposition 3.1.1 for the definition of stringy Chern classes $c^{\text{str}}_{d-k}(X) \in A_k(X)_{\mathbb{Q}}$ and the projection formula of Proposition 2.1.18 for the proper morphism $\rho$, one receives a formula to compute the above-mentioned rational intersection number:

**Theorem 3.2.1.** *Let $X$ be a projective $\mathbb{Q}$-Gorenstein variety of dimension $d$ with at worst log-terminal singularities, $\rho : Y \to X$ a log-desingularization of $X$, and $Z_1, \dots, Z_k \subseteq X$ $\mathbb{Q}$-Cartier divisors on $X$. Then*

$$[Z_1]. \ \dots \ .[Z_k].c^{\text{str}}_{d-k}(X) = \sum_{\emptyset \subseteq J \subseteq I} \rho^*[Z_1]. \ \dots \ .\rho^*[Z_k].e_{J*}c_{d-|J|-k}(D_J) \prod_{j \in J} \left( \frac{-a_j}{a_j+1} \right),$$

*where the homomorphism $\rho^* : \operatorname{Pic}(X)_{\mathbb{Q}} \to \operatorname{Pic}(Y)_{\mathbb{Q}}$ is determined by the pullback of line bundles and the intersection product*

$$\rho^*[Z_1]. \ \dots \ .\rho^*[Z_k].e_{J*}c_{d-|J|-k}(D_J)$$

*is the value of the multilinear map $\operatorname{Pic}(Y)^k_{\mathbb{Q}} \times A_k(Y) \to \mathbb{Q}$ defined by the intersection of $k$ divisor classes $\rho^*[Z_1], \dots, \rho^*[Z_k]$ of $\mathbb{Q}$-Cartier divisors on $Y$ with the $k$-dimensional cycle $e_{J*}c_{d-|J|-k}(D_J) \in A_k(Y)$ composed with the natural map $A_0(Y)_{\mathbb{Q}} \to \mathbb{Q}$.*

*Proof.* The desired formula holds because

$$\begin{aligned} [Z_1]. \ \dots \ .[Z_k].c^{\text{str}}_{d-k}(X) &= [Z_1]. \ \dots \ .[Z_k].\rho_*\left( \sum_{\emptyset \subseteq J \subseteq I} e_{J*}c_{d-|J|-k}(D_J) \prod_{j \in J} \left( \frac{-a_j}{a_j+1} \right) \right) \\ &= \rho_*\left( \rho^*\left([Z_1]. \ \dots \ .[Z_k]\right). \sum_{\emptyset \subseteq J \subseteq I} e_{J*}c_{d-|J|-k}(D_J) \prod_{j \in J} \left( \frac{-a_j}{a_j+1} \right) \right) \\ &= \sum_{\emptyset \subseteq J \subseteq I} \rho^*[Z_1]. \ \dots \ .\rho^*[Z_k].e_{J*}c_{d-|J|-k}(D_J) \prod_{j \in J} \left( \frac{-a_j}{a_j+1} \right). \end{aligned}$$

$\square$

In particular, we obtain with the same prerequisites as in the last theorem:

**Theorem 3.2.2.** *Let $X$, $\rho : Y \to X$, and $Z_1, \ldots, Z_k \subseteq X$ be given as in* Theorem 3.2.1. *Then the rational intersection number*

$$\sum_{\emptyset \subseteq J \subseteq I} \rho^*[Z_1].\ \ldots\ .\rho^*[Z_k].e_{J*}c_{d-|J|-k}(D_J) \prod_{j \in J} \left( \frac{-a_j}{a_j+1} \right)$$

*does not depend on the choice of the log-desingularization $\rho : Y \to X$.*

In the special case of $k = 1$, i.e., $c_1(X).c^{\mathrm{str}}_{d-1}(X)$ (Equation (1.8)), Theorem 3.2.2 is proved in [Bat00, Corollary 3.9].

**Corollary 3.2.3.** *Let $X$ be a projective $\mathbb{Q}$-Gorenstein variety of dimension $d$ with at worst log-terminal singularities, $\rho : Y \to X$ a log-desingularization of $X$, and $\rho^*c_1(X) = [-\rho^*K_X]$ the pullback of the anticanonical class of $X$. Then the intersection number*

$$\sum_{\emptyset \subseteq J \subseteq I} \rho^*c_1(X)^k.e_{J*}c_{d-|J|-k}(D_J) \prod_{j \in J} \left( \frac{-a_j}{a_j+1} \right)$$

*is independent of the log-desingularization $\rho$.*

We observe that the intersection number $[Z_1].\ \ldots\ .[Z_k].c^{\mathrm{str}}_{d-k}(X)$ can be computed by a formula that does not involve stringy Chern classes of the ambient singular variety $X$, but only usual Chern classes of smooth subvarieties $D_J \subseteq Y$. We give below a proof of Theorem 3.2.2, which does not use the general definition of stringy Chern classes, but only the definition of the stringy Euler number $c^{\mathrm{str}}_d(X)$ and its independence of log-desingularization.

*Proof of Theorem 3.2.2.* Let us denote the number

$$\sum_{\emptyset \subseteq J \subseteq I} \rho^*[Z_1].\ \ldots\ .\rho^*[Z_k].e_{J*}c_{d-|J|-k}(D_J) \prod_{j \in J} \left( \frac{-a_j}{a_j+1} \right)$$

by $\imath_\rho(Z_1, \ldots, Z_k)$. It is clear that the map

$$\imath_\rho : \mathrm{Pic}(X)^k_{\mathbb{Q}} \to \mathbb{Q},\ ([Z_1], \ldots, [Z_k]) \mapsto \imath_\rho(Z_1, \ldots, Z_k)$$

is symmetric and multilinear. Since the group $\mathrm{Pic}(X)$ is generated by classes of very ample Cartier divisors, it suffices to show the statement of Theorem 3.2.2 in the case of very ample Cartier divisors $Z_1, \ldots, Z_k$. For any sequence of positive integers $n_1, \ldots, n_k$, the linear combination $n_1[Z_1] + \ldots + n_k[Z_k]$ represents a class of a very ample Cartier divisor $Z$ on $X$. It follows from the symmetry and multilinearity of $\imath_\rho$ that $\imath_\rho(Z, Z, \ldots, Z)$ is a homogeneous polynomial of degree $k$ in $n_1, \ldots, n_k$ whose coefficients are the rational numbers $\imath_\rho(Z_{i_1}, Z_{i_2}, \ldots, Z_{i_k})$, where $1 \leq i_1, \ldots, i_k \leq k$. Therefore, it is sufficient to prove the statement of Theorem 3.2.2 only for the rational numbers $\imath_\rho(Z, Z, \ldots, Z)$, where $Z$ is a generic very ample Cartier divisor on $X$. By the Theorem of Bertini [Har77, Chapter III, Section 10], we can assume that $Z' := \rho^{-1}(Z)$ is a smooth divisor on $Y$ and the restriction of $\rho$ to $Z'$ defines a log-desingularization of $Z$ with the exceptional divisors $D'_i := D_i \cap Z'$ such that $K_{Z'} = \rho^* K_Z + a_1 D'_1 + \ldots + a_s D'_s$. One can compute the stringy Euler number of $Z$ (Equation (1.4)) by

$$e_{\text{str}}(Z) = \sum_{\emptyset \subseteq J \subseteq I} e(D'_J) \prod_{j \in J} \left( \frac{-a_j}{a_j + 1} \right),$$

where $e(D'_J)$ denotes the usual Euler number of the smooth variety $D'_J = D_J \cap Z'$. The next step is an application of Equation (3.4) to each smooth divisor $D'_J \subseteq D_J$ to obtain

$$e(D'_J) = \sum_{k=1}^{d-|J|} (-1)^{k-1} \left[ D'_J \right]^k .c_{d-|J|-k}(D_J) = \sum_{k=1}^{d-|J|} (-1)^{k-1} \rho^* \left[ Z \right]^k .e_{J*} c_{d-|J|-k}(D_J)$$

because the projection formula (Proposition 2.1.18) for the proper morphism $e_J : D_J \hookrightarrow Y$ implies

$$\begin{aligned} \rho^* \left[ Z \right]^k .e_{J*} c_{d-|J|-k}(D_J) &= e_{J*}(e_J{}^* \rho^* \left[ Z \right]^k .c_{d-|J|-k}(D_J)) \\ &= e_J{}^* \rho^* \left[ Z \right]^k .c_{d-|J|-k}(D_J) \\ &= e_J{}^* \left[ Z' \right]^k .c_{d-|J|-k}(D_J) = \left[ D'_J \right]^k .c_{d-|J|-k}(D_J). \end{aligned}$$

Therefore, the stringy Euler number $e_{\text{str}}(Z)$ of $Z$ has the form

$$\begin{aligned} e_{\text{str}}(Z) &= \sum_{\emptyset \subseteq J \subseteq I} \left( \sum_{k=1}^{d-|J|} (-1)^{k-1} \rho^* \left[ Z \right]^k .e_{J*} c_{d-|J|-k}(D_J) \right) \prod_{j \in J} \left( \frac{-a_j}{a_j + 1} \right) \\ &= \sum_{k \geq 1} (-1)^{k-1} \rho^* [Z]^k \left( \sum_{\emptyset \subseteq J \subseteq I} e_{J*} c_{d-|J|-k}(D_J) \prod_{j \in J} \left( \frac{-a_j}{a_j + 1} \right) \right). \end{aligned}$$

For any positive integer $n$, the class $n[Z]$ can be again represented by a generic very ample Cartier divisor $Z^{(n)}$ such that $\rho^{-1}(Z^{(n)})$ is smooth. So we can repeat the same arguments for $Z^{(n)}$ to obtain that the stringy Euler number

$$e_{\text{str}} \left( Z^{(n)} \right) = \sum_{k \geq 1} (-1)^{k-1} n^k \rho^* [Z]^k \left( \sum_{\emptyset \subseteq J \subseteq I} e_{J*} c_{d-|J|-k}(D_J) \prod_{j \in J} \left( \frac{-a_j}{a_j + 1} \right) \right)$$

is a polynomial $p$ in $n$ whose $k$-th coefficient

$$\rho^* [Z]^k \left( \sum_{\emptyset \subseteq J \subseteq I} e_{J*} c_{d-|J|-k}(D_J) \prod_{j \in J} \left( \frac{-a_j}{a_j + 1} \right) \right)$$

is equal to $\imath_\rho(Z, \ldots, Z)$. Since the stringy Euler number of $Z^{(n)}$ does not depend on the choice of the log-desingularization $\rho$ [Bat98b, Theorem 3.4], the same is true for the polynomial $p$ and hence for its $k$-th coefficient $\imath_\rho(Z, \ldots, Z)$. □

# Chapter 4

## Stringy Invariants and Stringy Identities on Toric Varieties

In Chapter 4, we apply our results of the previous chapter to hypersurfaces and complete intersections in projective $\mathbb{Q}$-Gorenstein toric varieties. For this purpose, we prove in Section 4.1 how to compute $k$-th stringy Chern classes of a projective $\mathbb{Q}$-Gorenstein toric variety $X$ associated with a fan $\Sigma$ of rational polyhedral cones in $N_{\mathbb{R}}$ as a linear combination of classes of closed torus orbits $X_\sigma$ corresponding to $k$-dimensional cones $\sigma \in \Sigma(k)$. We apply this formula to compute intersection numbers $[D_1]. \dots .[D_k].c^{\mathrm{str}}_{d-k}(X)$ via mixed volumes of faces of divisor-associated lattice polytopes, where $D_1, \dots, D_k$ are semi-ample torus-invariant $\mathbb{Q}$-Cartier divisors on $X$.

Section 4.2 is devoted to combinatorial interpretations of the stringy Libgober-Wood identity for projective $\mathbb{Q}$-Gorenstein toric varieties of dimension $d$. To achieve these interpretations, we have to investigate the form of stringy $E$-functions of projective $\mathbb{Q}$-Gorenstein toric varieties and prove that they are polynomials with non-negative integral coefficients in non-negative rational powers. We should emphasize the importance of these two tools, the combinatorial stringy Libgober-Wood identity and the precise form of the stringy $E$-function, for the remaining Chapter 5 and 6, where we will use them intensively.

The results of both sections in this chapter have been published with minor modifications in our paper *Stringy Chern Classes of Singular Toric Varieties and Their Applications* [BS17] that is joint work of the author of this thesis and Victor Batyrev.

## 4.1 Stringy Chern Classes and Their Intersection Numbers

It is well-known that singularities of an arbitrary $\mathbb{Q}$-Gorenstein toric variety $X$ are at worst log-terminal (Proposition 2.2.10). Moreover, the stringy Euler number $e_{\mathrm{str}}(X)$ of the $d$-dimensional toric variety $X$ can be computed combinatorially using cones of maximal dimension in the associated fan $\Sigma$:

$$e_{\mathrm{str}}(X) = v(\Sigma) = \sum_{\sigma \in \Sigma(d)} v(\sigma)$$

[Bat98b, Proposition 4.10], where $v(\sigma)$ denotes the normalized volume of a cone $\sigma$ (Definition 2.2.13) and $v(\Sigma)$ the normalized volume of the fan $\Sigma$. In this section, we give a combinatorial computation formula for arbitrary stringy Chern classes $c_k^{\mathrm{str}}(X)$ $(0 \leq k \leq d)$ of $\mathbb{Q}$-Gorenstein toric varieties using intrinsic information provided by the associated fans.

We start with a well-known fact about usual Chern classes of smooth toric varieties that we already mentioned in Proposition 2.2.16:

**Theorem 4.1.1.** *Let $V$ be a smooth toric variety associated with a fan $\Sigma$ in $N_{\mathbb{R}}$. Then the total Chern class of $V$ is*

$$c_{\bullet}(V) = \sum_{\sigma \in \Sigma} [V_\sigma] \,,$$

*where $[V_\sigma]$ denotes the class of the closed torus orbit $V_\sigma$ corresponding to a cone $\sigma \in \Sigma$.*

**Theorem 4.1.2.** *Let $X$ be a $\mathbb{Q}$-Gorenstein toric variety associated with a fan $\Sigma$ in $N_{\mathbb{R}}$. Then the total stringy Chern class of $X$ is*

$$c_{\bullet}^{\mathrm{str}}(X) = \sum_{\sigma \in \Sigma} v(\sigma) \cdot [X_\sigma] \,,$$

*where the total stringy Chern class $c_{\bullet}^{\mathrm{str}}(X)$ of $X$ is defined on* page 35, *$v(\sigma)$ denotes the normalized volume of a cone $\sigma \in \Sigma$, and $[X_\sigma]$ the class of the closed torus orbit $X_\sigma$ corresponding to a cone $\sigma \in \Sigma$.*

*Proof.* Consider a log-desingularization $\rho : Y \to X$ of $X$ (Definition 2.1.6) obtained by a refinement $\Sigma'$ of the fan $\Sigma$ [CLS11, Theorem 11.2.2]. We start proofing the statement in the case of a crepant resolution $\rho$, *i.e.*, $Y$ is smooth and $K_Y = \rho^* K_X$ (Definition 2.1.6):

$$\begin{aligned} c_{\bullet}^{\mathrm{str}}(X) = \rho_* c_{\bullet}\,(Y) = \rho_* \Bigg( \sum_{\sigma' \in \Sigma'} [Y_{\sigma'}] \Bigg) &= \sum_{\sigma \in \Sigma} \sum_{\substack{\sigma' \in \Sigma' \\ \sigma' \preceq \sigma}} \rho_* \,[Y_{\sigma'}] \\ &= \sum_{\sigma \in \Sigma} \Bigg( \sum_{\substack{\sigma' \in \Sigma' \\ \sigma' \preceq \sigma}} 1 \Bigg) \cdot [X_\sigma] = \sum_{\sigma \in \Sigma} \Bigg( \sum_{\substack{\sigma' \in \Sigma' \\ \sigma' \preceq \sigma}} v\,(\sigma') \Bigg) \cdot [X_\sigma] = \sum_{\sigma \in \Sigma} v\,(\sigma) \cdot [X_\sigma] \,. \end{aligned}$$

Now, consider $\rho : Y \to X$ to be an arbitrary log-desingularization of $X$ obtained by a refinement $\Sigma'$ of the fan $\Sigma$. There is a natural bijection between the set of exceptional divisors $\{D_1, \ldots, D_s\}$ on $Y$ and the set of 1-dimensional cones in $\Sigma'(1) \setminus \Sigma(1)$. We denote by $\{D_{s+1}, \ldots, D_r\}$ the set of all remaining torus-invariant divisors on $Y$ whose elements one-to-one correspond to 1-dimensional cones in $\Sigma(1)$ and set $I := \{1, \ldots, s\}$, $I' := I \cup \{s+1, \ldots, r\}$, and $a_j := 0$ for all $j \in I' \setminus I$.

Let $\{u_1, \ldots, u_r\} = \{u_i \,|\, i \in I'\}$ be the set of all primitive ray generators of 1-dimensional cones of $\Sigma'(1)$ corresponding to all torus-invariant divisors $D_1, \ldots, D_r$. For any subset $J' \subseteq I'$, the subset $D_{J'} = \cap_{j \in J'} D_j$ is either empty or a smooth toric subvariety $Y_{\sigma'}$ of $Y$. The latter holds if and only if the set $\{u_j | j \in J'\}$ generates a cone $\sigma' \in \Sigma'$

of dimension $|J'|$. Then the locally closed subset $D^\circ_{J'} := D_{J'} \setminus \big(\cup_{i\in I'\setminus J'} D_i\big)$ is the dense open torus orbit $T_{\sigma'}$ in $Y_{\sigma'}$. By Theorem 4.1.1, we have $c_\bullet(T_{\sigma'}) = [T_{\sigma'}]$. This implies $c\big(\mathbb{1}_{D^\circ_{J'}}\big) = [Y_{\sigma'}]$. Using Equation (3.1), we get

$$c_\bullet^{\mathrm{str}}(X) = \rho_*\left(\sum_{\emptyset\subseteq J'\subseteq I'} c\left(\mathbb{1}_{D^\circ_{J'}}\right)\prod_{j\in J'}\left(\frac{1}{a_j+1}\right)\right) = \sum_{\sigma'\in\Sigma'}\rho_*\left[Y_{\sigma'}\right]\prod_{u_j\in\sigma'}\left(\frac{1}{a_j+1}\right).$$

Let us compute $\rho_*[Y_{\sigma'}]$: If $\sigma\in\Sigma$ is the minimal cone of $\Sigma$ containing $\sigma'\in\Sigma'$, then $\rho(Y_{\sigma'}) = X_\sigma$. In order to compute the corresponding cycle map $\rho_* : A_\bullet(Y)\to A_\bullet(X)$, we need to compare the dimensions of $Y_{\sigma'}$ and $X_\sigma$. If $\dim(Y_{\sigma'}) > \dim(X_\sigma)$, then $\rho_*[Y_{\sigma'}] = 0$. Otherwise, we have $\rho_*[Y_{\sigma'}] = [X_\sigma]$. Therefore, we get

$$c_\bullet^{\mathrm{str}}(X) = \sum_{\sigma\in\Sigma}\left(\sum_{\substack{\sigma'\in\Sigma',\sigma'\preceq\sigma\\ \dim(\sigma')=\dim(\sigma)}}\prod_{u_j\in\sigma'}\left(\frac{1}{a_j+1}\right)\right)\cdot[X_\sigma].$$

Furthermore, $\prod_{u_j\in\sigma'}\left(\frac{1}{a_j+1}\right) = v(\sigma')$ for every cone $\sigma'\in\Sigma'$ and this implies

$$c_\bullet^{\mathrm{str}}(X) = \sum_{\sigma\in\Sigma}\Bigg(\sum_{\substack{\sigma'\in\Sigma',\sigma'\preceq\sigma\\ \dim(\sigma')=\dim(\sigma)}} v(\sigma')\Bigg)\cdot[X_\sigma] = \sum_{\sigma\in\Sigma} v(\sigma)\cdot[X_\sigma].$$

□

**Corollary 4.1.3.** *Let $X$ be a $d$-dimensional $\mathbb{Q}$-Gorenstein toric variety associated with a fan $\Sigma$. Then the $k$-th stringy Chern class of $X$ $(0\le k\le d)$ is*

$$c_k^{\mathrm{str}}(X) = \sum_{\sigma\in\Sigma(k)} v(\sigma)\cdot[X_\sigma].$$

**Example 4.1.4.** Let $X$ be the 2-dimensional $\mathbb{Q}$-Gorenstein toric variety associated with the fan $\Sigma_m$ of Example 2.2.14. Then

$$\begin{aligned} c_2^{\mathrm{str}}(X) &= \sum_{\sigma\in\Sigma(2)} v(\sigma) = v(\sigma_1)+v(\sigma_2)+v(\sigma_3)\\ &= v(\mathrm{conv}(0,e_2,e_1)) + v(\mathrm{conv}(0,e_1,-e_1-me_2)) + v(\mathrm{conv}(0,-e_1-me_2,e_2))\\ &= 1+m+1 = 2+m \end{aligned}$$

because $\sigma\in\Sigma(2) = \{\sigma_1,\sigma_2,\sigma_3\}$ with $\sigma_1 = \{\lambda_1 e_2+\lambda_2 e_1 \mid \lambda_1,\lambda_2\in\mathbb{R}_{\ge 0}\}$, $\sigma_2 = \{\lambda_1 e_1 + \lambda_3(-e_1-me_2)\mid \lambda_1,\lambda_3\in\mathbb{R}_{\ge0}\}$, and $\sigma_3 = \{\lambda_3(-e_1-me_2)+\lambda_2 e_2 \mid \lambda_3,\lambda_2\in\mathbb{R}_{\ge0}\}$ (Figure 2.1).

In a second step, we apply this formula to compute intersection numbers

$$[D_1]\,.\,\ldots\,.\,[D_k]\,.c_{d-k}^{\mathrm{str}}(X)$$

via mixed volumes of certain polytopes, where $D_1, \ldots, D_k$ are semi-ample torus-invariant $\mathbb{Q}$-Cartier divisors on $X$. First, we compute in a combinatorial way the intersection number of the stringy Chern class $c_{d-1}^{\mathrm{str}}(X)$ of a $d$-dimensional projective $\mathbb{Q}$-Gorenstein toric variety $X$ associated with a fan $\Sigma$ in $N_{\mathbb{R}}$ with an arbitrary torus-invariant $\mathbb{Q}$-Cartier divisor $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ on $X$.

Therefore, we define for any cone $\sigma \in \Sigma(d-1)$ the rational number $l_D(\sigma)$:

**Definition 4.1.5.** Let $\sigma'$, $\sigma'' \in \Sigma(d)$ be two $d$-dimensional cones such that $\sigma = \sigma' \cap \sigma''$. Denote by $m_{\sigma'}$ and $m_{\sigma''}$ elements in $M_{\mathbb{Q}}$ that are defined by the conditions $\langle m_{\sigma'}, u_\rho \rangle = -a_\rho \ \forall \rho \subseteq \sigma'$ and $\langle m_{\sigma''}, u_\rho \rangle = -a_\rho \ \forall \rho \subseteq \sigma''$, respectively, where $\rho \in \Sigma(1)$ and $u_\rho \in N$ denotes its primitive ray generator. Now, choose the primitive ray generator $u$ of the 1-dimensional sublattice $M(\sigma) := \{m \in M | \langle m, u' \rangle = 0 \ \forall u' \in \sigma\}$ such that $u|_{\sigma'} \leq 0$ and $u|_{\sigma''} \geq 0$. Since $m_{\sigma'} - m_{\sigma''}$ vanishes on $\sigma$, there exists a unique rational number $l_D(\sigma) \in \mathbb{Q}$ such that $m_{\sigma'} - m_{\sigma''} = l_D(\sigma) \cdot u$.

**Proposition 4.1.6.** *Let $X$ be a $d$-dimensional projective $\mathbb{Q}$-Gorenstein toric variety associated with a fan $\Sigma$ and $D$ a torus-invariant $\mathbb{Q}$-Cartier divisor on $X$. Then*

$$[D] \,.c_{d-1}^{\mathrm{str}}(X) = \sum_{\sigma \in \Sigma(d-1)} v(\sigma) \cdot l_D(\sigma),$$

*where the rational number $l_D(\sigma) \in \mathbb{Q}$ is defined in* Definition 4.1.5.

*Proof.* Using Corollary 4.1.3, we obtain

$$[D] \,.c_{d-1}^{\mathrm{str}}(X) = \sum_{\sigma \in \Sigma(d-1)} v(\sigma) \cdot [D] \,.\, [X_\sigma] \,.$$

It remains to apply the equality $[D] \,.\, [X_\sigma] = l_D(\sigma)$ for every cone $\sigma \in \Sigma(d-1)$ [CLS11, Proposition 6.3.8]. $\square$

Secondly, we compute intersection numbers $[D]^k.c_{d-k}^{\mathrm{str}}(X)$, where $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ is a semi-ample torus-invariant $\mathbb{Q}$-Cartier divisor on a toric variety $X$ associated with a fan $\Sigma$ in $N_{\mathbb{R}}$.

Therefore, we define the rational polytope $\Delta_D \subseteq M_{\mathbb{R}}$ of dimension $\leq d$ corresponding to the divisor $D$ on $X$ as

$$\Delta_D := \{y \in M_{\mathbb{R}} | \langle y, u_\rho \rangle \geq -a_\rho \ \forall \rho \in \Sigma(1)\} \subseteq M_{\mathbb{R}} \tag{4.1}$$

[CLS11, page 189], where $u_\rho \in N$ denotes the primitiv ray generator of a 1-dimensional cone $\rho \in \Sigma(1)$. In general, *i.e.*, for arbitrary torus-invariant $\mathbb{Q}$-Cartier divisors $D' = \sum_{\rho \in \Sigma(1)} a_\rho D'_\rho$ the polyhedron $\Delta_{D'}$ need not be a polytope. Let $\sigma \in \Sigma(d-k)$ be a $(d-k)$-dimensional cone of the fan $\Sigma$. Then $\Delta_D^\sigma \preceq \Delta_D$ denotes a face of $\Delta_D$ of dimension $\leq k$ defined as

$$\Delta_D^\sigma := \{y \in \Delta_D | \langle y, u_\rho \rangle = -a_\rho \ \forall \rho \in \Sigma(1) \text{ with } \rho \subseteq \sigma\} \subseteq M_{\mathbb{R}}. \tag{4.2}$$

The normalized volume of the polytope $\Delta_D^\sigma$ is given by

$$v\left(\Delta_D^\sigma\right) = k! \cdot \mathrm{vol}_k\left(\Delta_D^\sigma\right) \in \mathbb{Q},$$

where $\mathrm{vol}_k\left(\Delta_D^\sigma\right)$ denotes the volume of $\Delta_D^\sigma$ with respect to the $k$-dimensional sublattice $M(\sigma) = \{m \in M | \langle m, u' \rangle = 0 \ \forall u' \in \sigma\}$ of $M$ (Definition 2.3.6). In particular, one has $v\left(\Delta_D^\sigma\right) = 0$ if $\dim\left(\Delta_D^\sigma\right) < k$.

**Theorem 4.1.7.** *Let $X$ be a $d$-dimensional projective $\mathbb{Q}$-Gorenstein toric variety associated with a fan $\Sigma$ and $D$ a semi-ample torus-invariant $\mathbb{Q}$-Cartier divisor on $X$. Then*

$$[D]^k . c_{d-k}^{\mathrm{str}}(X) = \sum_{\sigma \in \Sigma(d-k)} v(\sigma) \cdot v\left(\Delta_D^\sigma\right)$$

*($0 \le k \le d$), where the face $\Delta_D^\sigma \preceq \Delta_D$ of $\Delta_D$ for a cone $\sigma \in \Sigma(d-k)$ is defined in* Equation (4.1) *and* (4.2).

*Proof.* By Corollary 4.1.3, we have

$$[D]^k . c_{d-k}^{\mathrm{str}}(X) = \sum_{\sigma \in \Sigma(d-k)} v(\sigma) \cdot [D]^k . [X_\sigma] .$$

Let $D^\sigma$ be the restriction of the semi-ample torus-invariant $\mathbb{Q}$-Cartier divisor $D$ to the $k$-dimensional toric subvariety $X_\sigma$ of $X$. Then $[D]^k . [X_\sigma]$ equals the intersection number $[D^\sigma]^k$ of the semi-ample torus-invariant $\mathbb{Q}$-Cartier divisor $D^\sigma$ on the $k$-dimensional variety $X_\sigma$. It remains to note that the number $[D^\sigma]^k$ equals $v\left(\Delta_D^\sigma\right)$ [CLS11, Section 13.4]. $\square$

Using Theorem 4.1.7 and Corollary 3.1.4 and 3.1.7, respectively, we derive combinatorial formulas to compute stringy Euler numbers of generic hypersurfaces and complete intersections in toric varieties:

**Corollary 4.1.8.** *Let $X$ be a $d$-dimensional projective $\mathbb{Q}$-Gorenstein toric variety associated with a fan $\Sigma$ and $D$ a semi-ample torus-invariant Cartier divisor on $X$ together with the corresponding lattice polytope $\Delta_D$. Denote by $Z \subseteq X$ a generic semi-ample Cartier divisor on $X$ such that $[Z] = [D]$. Then the stringy Euler number of $Z$ is*

$$e_{\mathrm{str}}(Z) = c_{d-1}^{\mathrm{str}}(Z) = \sum_{k=0}^{d-1} (-1)^k \sum_{\sigma \in \Sigma(d-1-k)} v(\sigma) \cdot v\left(\Delta_D^\sigma\right),$$

*where $\Delta_D^\sigma \preceq \Delta_D$ is a face of $\Delta_D$ corresponding to a cone $\sigma \in \Sigma$. If $[Z] = c_1(X)$, the formula simplifies to*

$$e_{\mathrm{str}}(Z) = c_{d-1}^{\mathrm{str}}(Z) = \sum_{k=0}^{d-3} (-1)^k \sum_{\sigma \in \Sigma(d-1-k)} v(\sigma) \cdot v\left(\Delta_D^\sigma\right).$$

**Corollary 4.1.9.** *Let $X$ be a $d$-dimensional projective $\mathbb{Q}$-Gorenstein toric variety associated with a fan $\Sigma$ and $D$ a semi-ample torus-invariant Cartier divisor on $X$ together with the corresponding lattice polytope $\Delta_D$. Denote by $Z_1, \ldots, Z_r \subseteq X$ generic semi-ample Cartier divisors on $X$ such that $[Z_1] = \ldots = [Z_r] = [D]$. Then the stringy Euler number of the complete intersection $Z_1 \cap \ldots \cap Z_r$ is*

$$c_{d-r}^{\text{str}}\left(Z_1 \cap \ldots \cap Z_r\right) = \sum_{k=0}^{d-r} (-1)^k \binom{k+r-1}{r-1} \sum_{\sigma \in \Sigma(d-r-k)} v(\sigma) \cdot v\left(\Delta_D^\sigma\right),$$

*where $\Delta_D^\sigma \preceq \Delta_D$ is a face of $\Delta_D$ corresponding to a cone $\sigma \in \Sigma$.*

One can generalize Theorem 4.1.7 and combinatorially compute intersection numbers $[D_1] . \ \ldots \ . [D_k] . c_{d-k}^{\text{str}}(X)$, where $D_1, \ldots, D_k$ are different semi-ample torus-invariant $\mathbb{Q}$-Cartier divisors on $X$. For this purpose, we use mixed volumes of faces of some convex rational polytopes and refer the reader for their definition to [Ful93, Chapter 5.4]:

**Theorem 4.1.10.** *Let $X$ be a $d$-dimensional projective $\mathbb{Q}$-Gorenstein toric variety associated with a fan $\Sigma$ and $D_1, \ldots, D_k$ semi-ample torus-invariant $\mathbb{Q}$-Cartier divisors on $X$. Then*

$$[D_1] . \ \ldots \ . [D_k] . c_{d-k}^{\text{str}}(X) = \sum_{\sigma \in \Sigma(d-k)} v(\sigma) \cdot v\left(\Delta_{D_1}^\sigma, \ldots, \Delta_{D_k}^\sigma\right),$$

*where $\Delta_{D_i}^\sigma \preceq \Delta_{D_i}$ is a face of $\Delta_{D_i}$ $(1 \leq i \leq k)$ corresponding to a cone $\sigma \in \Sigma$ and $v(\Delta_{D_1}^\sigma, \ldots, \Delta_{D_k}^\sigma)$ denotes the mixed volume of the polytopes $\Delta_{D_1}^\sigma, \ldots, \Delta_{D_k}^\sigma$ with respect to the sublattice $M(\sigma) \subseteq M$.*

*Proof.* Let $\sigma \in \Sigma(d-k)$ be a $(d-k)$-dimensional cone of $\Sigma$. Then we restrict the semi-ample torus-invariant $\mathbb{Q}$-Cartier divisors $D_1, \ldots, D_k$ to the corresponding projective $k$-dimensional toric subvariety $X_\sigma$ of $X$ and obtain $k$ semi-ample torus-invariant $\mathbb{Q}$-Cartier divisors $D_1^\sigma, \ldots, D_k^\sigma$ on $X_\sigma$. It remains to apply Corollary 4.1.3 and the formula in [Ful93, Section 5.4] that says that the intersection number

$$[D_1] . \ \ldots \ . [D_k] . [X_\sigma] = [D_1^\sigma] . \ \ldots \ . [D_k^\sigma]$$

can be computed as the mixed volume $v(\Delta_{D_1}^\sigma, \ldots, \Delta_{D_k}^\sigma)$ of the polytopes $\Delta_{D_i}^\sigma$ $(1 \leq i \leq k)$. □

## 4.2 Stringy $E$-functions and Stringy Libgober-Wood Identities

The original *Libgober-Wood identity*

$$\frac{d^2}{du^2} E\left(V; u, 1\right)\Big|_{u=1} = \frac{3d^2 - 5d}{12} c_d(V) + \frac{1}{6} c_1(V) . c_{d-1}(V)$$

for arbitrary smooth projective varieties $V$ of dimension $d$ has been proved by Libgober and Wood [LW90], where $E\left(V; u, v\right)$ denotes the $E$-polynomial of $V$ (Equation (1.2)). The stated identity is equivalent to

$$\sum_{0 \leq p,q \leq d} (-1)^{p+q} h^{p,q}(V) \left(p - \frac{d}{2}\right)^2 = \frac{d}{12} c_d(V) + \frac{1}{6} c_1(V).c_{d-1}(V)$$

and so the intersection number $c_1(V).c_{d-1}(V)$ can be expressed via the Hodge numbers $h^{p,q}(V)$ of $V$ [Bor97].

There exists a *stringy version of the Libgober-Wood identity*

$$\frac{d^2}{du^2} E_{\text{str}}\left(X; u, 1\right)\Big|_{u=1} = \frac{3d^2 - 5d}{12} c_d^{\text{str}}(X) + \frac{1}{6} c_1(X).c_{d-1}^{\text{str}}(X), \tag{4.3}$$

which holds for any $d$-dimensional projective variety $X$ with at worst log-terminal singularities [Bat00, Theorem 3.8], where $E_{\text{str}}\left(X; u, v\right)$ denotes the stringy $E$-function of $X$ (Equation (1.3)). Moreover, if the singularities of $X$ are at worst canonical Gorenstein and the stringy $E$-function $E_{\text{str}}(X; u, v)$ is a polynomial $\sum_{p,q} \psi_{p,q} u^p v^q$, then one can define the *stringy Hodge numbers* $h_{\text{str}}^{p,q}(X)$ of $X$ [Bat98b, Bat00] as

$$h_{\text{str}}^{p,q}(X) \coloneqq (-1)^{p+q} \psi_{p,q}.$$

In this case, the stringy Libgober-Wood identity (4.3) can be equivalently reformulated [Bat00, Corollary 3.10] as

$$\sum_{0 \leq p,q \leq d} (-1)^{p+q} h_{\text{str}}^{p,q}\left(X\right) \left(p - \frac{d}{2}\right)^2 = \frac{d}{12} c_d^{\text{str}}(X) + \frac{1}{6} c_1(X).c_{d-1}^{\text{str}}(X). \tag{4.4}$$

In this section, we are interested in combinatorial interpretations of the stringy Libgober-Wood identities (4.3) and (4.4) for arbitrary $d$-dimensional projective $\mathbb{Q}$-Gorenstein toric varieties $X$ associated with a fan $\Sigma$ in $N_{\mathbb{R}}$. Such varieties have at worst log-terminal singularities (Proposition 2.2.10). Note that the stringy $E$-function $E_{\text{str}}\left(X; u, v\right)$ of such a toric variety $X$ can be computed combinatorially as

$$E_{\text{str}}\left(X; u, v\right) = (uv - 1)^d \sum_{\sigma \in \Sigma} \sum_{n \in \sigma^\circ \cap N} (uv)^{\kappa(n)} \tag{4.5}$$

[Bat98b, Theorem 4.3], where $\kappa$ denotes the $\Sigma$-piecewise linear function corresponding to the anticanonical divisor $-K_X$ on $X$ (Proposition 2.2.12) and $\sigma^\circ$ the relative interior of a cone $\sigma \in \Sigma$. We note that $\kappa$ has value $-1$ on every primitive ray generator of a 1-dimensional cone $\rho \in \Sigma(1)$ and that the value $\kappa(n)$ belongs to $\frac{1}{q_X}\mathbb{Z}$ $(n \in N)$.

Moreover, for any cone $\sigma \in \Sigma$, we denote by $\square_\sigma$ the *parallelepiped* spanned by the primitive ray generators of the cone $\sigma$, *i.e.*,

$$\square_\sigma \coloneqq \left\{ \sum_{\rho} \lambda_\rho u_\rho \,\middle|\, 0 \leq \lambda_\rho \leq 1 \ \forall \rho \in \Sigma'(1) \text{ with } \rho \subseteq \sigma \right\}, \tag{4.6}$$

where $u_\rho$ denotes the primitive ray generator of a 1-dimensional cone $\rho \in \Sigma(1)$. For any cone $\sigma \in \Sigma$, we denote by $\square_\sigma^\circ$ the relative interior of the parallelepiped $\square_\sigma$.

First, we show that the stringy $E$-function $E_{\text{str}}(X; u, v)$ of $X$ is a polynomial with non-negative integral coefficients $\psi_\alpha(\Sigma)$ in non-negative rational powers $\alpha \in [0, d] \cap \frac{1}{q_X}\mathbb{Z}$ of $uv$, where $q_X$ denotes the Gorenstein index of $X$ (Definition 2.1.7):

**Proposition 4.2.1.** *Let $X$ be a $d$-dimensional projective $\mathbb{Q}$-Gorenstein toric variety of Gorenstein index $q_X$ associated with a fan $\Sigma$ in $N_\mathbb{R}$ and $\Sigma'$ a simplicial subdivision of the fan $\Sigma$ such that $\Sigma'(1) = \Sigma(1)$* [CLS11, Proposition 11.1.7]. *Then the stringy $E$-function of $X$ can be computed as a finite sum*

$$E_{\text{str}}(X; u, v) = \sum_{\sigma \in \Sigma'} (uv - 1)^{d - \dim(\sigma)} \sum_{n' \in \square_\sigma^\circ \cap N} (uv)^{\dim(\sigma) + \kappa(n')}.$$

*Moreover, the stringy $E$-function can be written as a finite sum*

$$E_{\text{str}}(X; u, v) = \sum_{\alpha \in [0,d] \cap \frac{1}{q_X}\mathbb{Z}} \psi_\alpha(\Sigma)(uv)^\alpha,$$

*where the coefficients $\psi_\alpha(\Sigma)$ are non-negative integers satisfying the conditions $\psi_0(\Sigma) = \psi_d(\Sigma) = 1$ and $\psi_\alpha(\Sigma) = \psi_{d-\alpha}(\Sigma)$ for all $\alpha \in [0, d] \cap \frac{1}{q_X}\mathbb{Z}$.*

*Proof.* Any $s$-dimensional simplicial cone $\sigma$ of $\Sigma'(s)$ (Definition 2.2.3) is generated by $s$ linearly independent primitive lattice vectors $u_1, \ldots, u_s$. Therefore, any lattice point $n \in \sigma^\circ \cap N$ has a unique representation as a sum $n = n' + n''$, where $n' = \sum_{i=1}^s \lambda_i u_i \in \square_\sigma^\circ \cap N$ $(0 \leq \lambda_i \leq 1)$ and $n''$ is a linear combination $n'' = \sum_{i=1}^s k_i u_i$ with non-negative integral coefficients $k_i$. Therefore, one has

$$\begin{aligned}
(uv - 1)^s \sum_{n \in \sigma^\circ \cap N} (uv)^{\kappa(n)} &= (uv - 1)^s \sum_{n' \in \square_\sigma^\circ \cap N} (uv)^{\kappa(n')} \cdot \left( \sum_{k_1, \ldots, k_s \in \mathbb{Z}_{\geq 0}} (uv)^{-k_1 - \ldots - k_s} \right) \\
&= (uv - 1)^s \sum_{n' \in \square_\sigma^\circ \cap N} (uv)^{\kappa(n')} \prod_{i=1}^s \left( \sum_{k_i \in \mathbb{Z}_{\geq 0}} (uv)^{-k_i} \right) \\
&= (uv - 1)^s \sum_{n' \in \square_\sigma^\circ \cap N} (uv)^{\kappa(n')} \cdot \left( \frac{1}{1 - (uv)^{-1}} \right)^s \\
&= \sum_{n' \in \square_\sigma^\circ \cap N} (uv)^{s + \kappa(n')}
\end{aligned}$$

and the first statement of this proposition follows from Equation (4.5). Since $\kappa$ has value $-1$ on every primitive ray generator $u_i$ and $q_X \cdot \kappa(n) \in \mathbb{Z}$ for all $n \in N$, we obtain that $s + \kappa(n') = s - \sum_{i=1}^s \lambda_i$ is a non-negative rational number in $\frac{1}{q_X}\mathbb{Z}_{\geq 0}$. Therefore, $E_{\text{str}}(X; u, v)$ can be written as a finite sum $E_{\text{str}}(X; u, v) = \sum_\alpha \psi_\alpha(\Sigma)(uv)^\alpha$ for some integral coefficients $\psi_\alpha(\Sigma)$ and some non-negative rational numbers $\alpha$ in $\frac{1}{q_X}\mathbb{Z}_{\geq 0}$. The Poincaré duality [Bat98b, Theorem 3.7] for the stringy $E$-function

$$E_{\mathrm{str}}\left(X; u, v\right) = (uv)^d E_{\mathrm{str}}\left(X; u^{-1}, v^{-1}\right)$$

provides the equalities $\psi_\alpha(\Sigma) = \psi_{d-\alpha}(\Sigma)$ for all $\alpha \in [0, d] \cap \frac{1}{q_X}\mathbb{Z}$. This implies $\alpha \leq d$ as soon as $\psi_\alpha(\Sigma) \neq 0$. Therefore, we obtain

$$E_{\mathrm{str}}\left(X; u, v\right) = \sum_{\alpha \in [0,d] \cap \frac{1}{q_X}\mathbb{Z}} \psi_\alpha(\Sigma)(uv)^\alpha.$$

The non-negativity of the coefficients $\psi_\alpha(\Sigma)$ can be shown using an interpretation of the coefficients $\psi_\alpha(\Sigma)$ as dimensions of graded homogenous components of a graded artinian ring $R$ obtained as a quotient of a graded Cohen-Macaulay ring $S$ by a regular sequence of homogeneous elements [Bat93, Theorem 2.11]. □

The equality $E_{\mathrm{str}}\left(X; u, v\right) = \sum_\alpha \psi_\alpha\left(\Sigma\right)(uv)^\alpha$ in Proposition 4.2.1 suggests to interpret the non-negative integral coefficients $\psi_\alpha\left(\Sigma\right)$ as *generalized stringy Hodge numbers* $h^{\alpha,\alpha}_{\mathrm{str}}(X)$ of the toric variety $X$ for some rational numbers $\alpha \in [0, d] \cap \frac{1}{q_X}\mathbb{Z}$.

The following theorem presents a combinatorial interpretation of the second stringy Libgober-Wood identity version (4.4) using generalized stringy Hodge numbers of toric varieties:

**Theorem 4.2.2.** *Let $X$ be a $d$-dimensional projective $\mathbb{Q}$-Gorenstein toric variety of Gorenstein index $q_X$ associated with a fan $\Sigma$ and $-K_X = \sum_{\rho \in \Sigma(1)} D_\rho$ the anticanonical torus-invariant $\mathbb{Q}$-Cartier divisor on $X$. Then the stringy Libgober-Wood identity is equivalent to*

$$\sum_{\alpha \in [0,d] \cap \frac{1}{q_X}\mathbb{Z}} \psi_\alpha\left(\Sigma\right)\left(\alpha - \frac{d}{2}\right)^2 = \frac{d}{12}v(\Sigma) + \frac{1}{6}\sum_{\sigma \in \Sigma(d-1)} v(\sigma) \cdot l_{-K_X}(\sigma),$$

*where $\psi_\alpha\left(\Sigma\right)$ are the generalized stringy Hodge numbers from* Proposition 4.2.1*, $v\left(\Sigma\right) = \sum_{\sigma \in \Sigma(d)} v(\sigma)$ the normalized volume of the fan $\Sigma$, and $l_{-K_X}(\sigma) \in \mathbb{Q}$ the intersection number $[-K_X].[X_\sigma]$* (Definition 4.1.5)*. If $-K_X$ is semi-ample, then*

$$\sum_{\alpha \in [0,d] \cap \frac{1}{q_X}\mathbb{Z}} \psi_\alpha\left(\Sigma\right)\left(\alpha - \frac{d}{2}\right)^2 = \frac{d}{12}v(\Sigma) + \frac{1}{6}\sum_{\sigma \in \Sigma(d-1)} v(\sigma) \cdot v\left(\Delta^\sigma_{-K_X}\right),$$

*where $\Delta^\sigma_{-K_X} \preceq \Delta_{-K_X}$ is a face of the polytope $\Delta_{-K_X}$ corresponding to a cone $\sigma \in \Sigma$* (Equation (4.1) and (4.2))*.*

*Proof.* Using the equality $E_{\mathrm{str}}\left(X; u, v\right) = \sum_\alpha \psi_\alpha\left(\Sigma\right)(uv)^\alpha$ from Proposition 4.2.1, we obtain

$$\frac{d^2}{du^2}E_{\mathrm{str}}\left(X; u, 1\right)|_{u=1} = \sum_\alpha \alpha \cdot (\alpha - 1)\psi_\alpha\left(\Sigma\right),$$

*i.e.*, the stringy Libgober-Wood identity (4.3) is given by

$$\sum_{\alpha}\left(\alpha^2-\alpha\right)\psi_\alpha\left(\Sigma\right)=\frac{3d^2-5d}{12}c_d^{\mathrm{str}}(X)+\frac{1}{6}c_1(X).c_{d-1}^{\mathrm{str}}(X).$$

Applying $\sum_\alpha \alpha\psi_\alpha\left(\Sigma\right)=\frac{d}{du}E_{\mathrm{str}}\left(X;u,1\right)|_{u=1}=\frac{d}{2}c_d^{\mathrm{str}}(X)$ [Bat00, Proposition 3.4] a short calculation yields

$$\begin{aligned}\sum_{\alpha}\alpha^2\psi_\alpha\left(\Sigma\right)&=\left(\frac{d}{12}-\frac{d}{2}+\frac{d^2}{4}\right)c_d^{\mathrm{str}}(X)+\frac{1}{6}c_1(X).c_{d-1}^{\mathrm{str}}(X)+\sum_{\alpha}\alpha\psi_\alpha\left(\Sigma\right)\\&=\left(\frac{d}{12}-\frac{d}{2}+\frac{d^2}{4}\right)c_d^{\mathrm{str}}(X)+\frac{1}{6}c_1(X).c_{d-1}^{\mathrm{str}}(X)+\frac{d}{2}c_d^{\mathrm{str}}(X)\\&=\left(\frac{d}{12}+\frac{d^2}{4}\right)c_d^{\mathrm{str}}(X)+\frac{1}{6}c_1(X).c_{d-1}^{\mathrm{str}}(X)\end{aligned}$$

and implies

$$\begin{aligned}\sum_{\alpha}\psi_\alpha\left(\Sigma\right)\left(\alpha-\frac{d}{2}\right)^2&=\sum_{\alpha}\left(\frac{d^2}{4}-\alpha d+\alpha^2\right)\psi_\alpha\left(\Sigma\right)\\&=\frac{d^2}{4}\sum_{\alpha}\psi_\alpha\left(\Sigma\right)-d\sum_{\alpha}\alpha\psi_\alpha\left(\Sigma\right)+\sum_{\alpha}\alpha^2\psi_\alpha\left(\Sigma\right)\\&=\frac{d^2}{4}c_d^{\mathrm{str}}(X)-\frac{d^2}{2}c_d^{\mathrm{str}}(X)+\sum_{\alpha}\alpha^2\psi_\alpha\left(\Sigma\right)\\&=-\frac{d^2}{4}c_d^{\mathrm{str}}(X)+\left(\frac{d}{12}+\frac{d^2}{4}\right)c_d^{\mathrm{str}}(X)+\frac{1}{6}c_1(X).c_{d-1}^{\mathrm{str}}(X)\\&=\frac{d}{12}c_d^{\mathrm{str}}(X)+\frac{1}{6}c_1(X).c_{d-1}^{\mathrm{str}}(X)\end{aligned}$$

because $\sum_\alpha \psi_\alpha\left(\Sigma\right)=E_{\mathrm{str}}\left(X;u,1\right)|_{u=1}=c_d^{\mathrm{str}}(X)$ [Bat00, Definition 2.1]. To finish, it remains to note that $c_d^{\mathrm{str}}(X)=v(\Sigma)$ by Corollary 4.1.3,

$$c_1(X).c_{d-1}^{\mathrm{str}}(X)=\left[-K_X\right].c_{d-1}^{\mathrm{str}}(X)=\sum_{\sigma\in\Sigma(d-1)}v(\sigma)\cdot l_{-K_X}(\sigma)$$

by Proposition 4.1.6, and

$$c_1(X).c_{d-1}^{\mathrm{str}}(X)=\left[-K_X\right].c_{d-1}^{\mathrm{str}}(X)=\sum_{\sigma\in\Sigma(d-1)}v(\sigma)\cdot v\left(\Delta^\sigma_{-K_X}\right)$$

by Theorem 4.1.7 if $-K_X$ is semi-ample. □

We formulate one more combinatorial version of the stringy Libgober-Wood identity containing only intrinsic information coming from the associated fan of a toric variety. To achieve this, we describe the left side of the stringy Libgober-Wood identity (4.3) in purely combinatorial terms using Proposition 4.2.1:

**Proposition 4.2.3.** *Let $X$ be a $d$-dimensional projective $\mathbb{Q}$-Gorenstein toric variety associated with a fan $\Sigma$ in $N_{\mathbb{R}}$ and $-K_X = \sum_{\rho \in \Sigma(1)} D_\rho$ the anticanonical torus-invariant $\mathbb{Q}$-Cartier divisor on $X$. Then the stringy Libgober-Wood identity is equivalent to*

$$2 \cdot \sum_{\sigma \in \Sigma'(d-2)} |\square_\sigma^\circ \cap N| + 2 \cdot \sum_{\sigma \in \Sigma'(d-1)} \sum_{n' \in \square_\sigma^\circ \cap N} \big(d + \kappa(n') - 1\big) + \sum_{\sigma \in \Sigma'(d)} \sum_{n' \in \square_\sigma^\circ \cap N} \big(d + \kappa(n')\big)\big(d + \kappa(n') - 1\big) = \frac{(3d^2 - 5d)}{12} v(\Sigma) + \frac{1}{6} c_1(X).c_{d-1}^{\mathrm{str}}(X),$$

*where $\Sigma'$ is a simplicial subdivision of the fan $\Sigma$ such that $\Sigma'(1) = \Sigma(1)$* [CLS11, Proposition 11.1.7]*, $\square_\sigma^\circ$ the relative interior of the parallelepiped $\square_\sigma$ spanned by the primitive ray generators of a cone $\sigma \in \Sigma$* (Equation (4.6))*, $\kappa$ the $\Sigma$-piecewise linear function corresponding to the anticanonical divisor $-K_X$ on $X$* (Proposition 2.2.12)*, and $v(\Sigma) = \sum_{\sigma \in \Sigma(d)} v(\sigma)$ the normalized volume of the fan $\Sigma$. The rational number $c_1(X).c_{d-1}^{\mathrm{str}}(X) \in \mathbb{Q}$ is computable as*

$$c_1(X).c_{d-1}^{\mathrm{str}}(X) = \sum_{\sigma \in \Sigma(d-1)} v(\sigma) \cdot l_{-K_X}(\sigma),$$

*where $l_{-K_X}$ is the intersection number $[-K_X].[X_\sigma]$* (Definition 4.1.5)*. If $-K_X$ is semi-ample, then*

$$c_1(X).c_{d-1}^{\mathrm{str}}(X) = \sum_{\sigma \in \Sigma(d-1)} v(\sigma) \cdot v\left(\Delta_{-K_X}^\sigma\right),$$

*where $\Delta_{-K_X}^\sigma \preceq \Delta_{-K_X}$ is a face of the polytope $\Delta_{-K_X}$ corresponding to a cone $\sigma \in \Sigma$* (Equation (4.1) and (4.2))*.*

*Proof.* We insert $v = 1$ in the stringy $E$-function

$$E_{\mathrm{str}}(X; u, v) = \sum_{\sigma \in \Sigma'} (uv - 1)^{d - \dim(\sigma)} \sum_{n' \in \square_\sigma^\circ \cap N} (uv)^{\dim(\sigma) + \kappa(n')}$$

given in the form of Proposition 4.2.1 and derivate one summand twice to get

$$\frac{d}{du}(u-1)^{d-s} \sum_{n' \in \square_\sigma^\circ \cap N} u^{s + \kappa(n')} = (d-s)(u-1)^{d-s-1} \sum_{n' \in \square_\sigma^\circ \cap N} u^{s+\kappa(n')} + (u-1)^{d-s} \sum_{n' \in \square_\sigma^\circ \cap N} (s + \kappa(n')) u^{s + \kappa(n') - 1}$$

and

$$\frac{d^2}{du^2}(u-1)^{d-s} \sum_{n' \in \square_\sigma^\circ \cap N} u^{s+\kappa(n')} = (d-s)(d-s-1)(u-1)^{d-s-2} \sum_{n' \in \square_\sigma^\circ \cap N} u^{s+\kappa(n')}$$

$$+ 2 \cdot (d-s)(u-1)^{d-s-1} \sum_{n' \in \square_\sigma^\circ \cap N} (s+\kappa(n'))u^{s+\kappa(n')-1}$$
$$+ (u-1)^{d-s} \sum_{n' \in \square_\sigma^\circ \cap N} (s+\kappa(n'))(s+\kappa(n')-1)u^{s+\kappa(n')-2},$$

where $\sigma$ is a $s$-dimensional cone of $\Sigma'(s)$. Inserting $u=1$, the relevant cones of $\Sigma'$ are these of dimension $d$, $d-1$, and $d-2$, *i.e.*,

$$\begin{aligned}
\frac{d^2}{du^2} E_{\mathrm{str}}\left(X;u,1\right)\Big|_{u=1} &= \sum_{\sigma\in\Sigma'(d-2)} (d-(d-2))(d-(d-2)-1) \sum_{n'\in\square_\sigma^\circ\cap N} 1^{(d-2)+\kappa(n')} + 0 + 0 \\
&\quad + 0 + 2\cdot \sum_{\sigma\in\Sigma'(d-1)} (d-(d-1)) \sum_{n'\in\square_\sigma^\circ\cap N} ((d-1)+\kappa(n'))1^{(d-1)+\kappa(n')-1} + 0 \\
&\quad + 0 + 0 + \sum_{\sigma\in\Sigma'(d)} \sum_{n'\in\square_\sigma^\circ\cap N} (d+\kappa(n'))(d-1+\kappa(n'))1^{d+\kappa(n')-2} \\
&= \sum_{\sigma\in\Sigma'(d-2)} 2 \sum_{n'\in\square_\sigma^\circ\cap N} 1 + 2\cdot \sum_{\sigma\in\Sigma'(d-1)} \sum_{n'\in\square_\sigma^\circ\cap N} (d+\kappa(n')-1) \\
&\quad + \sum_{\sigma\in\Sigma'(d)} \sum_{n'\in\square_\sigma^\circ\cap N} (d+\kappa(n'))(d+\kappa(n')-1) \\
&= 2\cdot \sum_{\sigma\in\Sigma'(d-2)} |\square_\sigma^\circ\cap N| + 2\cdot \sum_{\sigma\in\Sigma'(d-1)} \sum_{n'\in\square_\sigma^\circ\cap N} \big(d+\kappa(n')-1\big) \\
&\quad + \sum_{\sigma\in\Sigma'(d)} \sum_{n'\in\square_\sigma^\circ\cap N} \big(d+\kappa(n')\big)\big(d+\kappa(n')-1\big).
\end{aligned}$$

By Equation (4.3), we obtain the equality

$$\begin{aligned}
&2\cdot \sum_{\sigma\in\Sigma'(d-2)} |\square_\sigma^\circ\cap N| + 2\cdot \sum_{\sigma\in\Sigma'(d-1)} \sum_{n'\in\square_\sigma^\circ\cap N} \big(d+\kappa(n')-1\big) \\
&\quad + \sum_{\sigma\in\Sigma'(d)} \sum_{n'\in\square_\sigma^\circ\cap N} \big(d+\kappa(n')\big)\big(d+\kappa(n')-1\big) = \frac{(3d^2-5d)}{12} v\left(\Sigma\right) + \frac{1}{6} c_1(X).c_{d-1}^{\mathrm{str}}(X)
\end{aligned}$$

because $c_d^{\mathrm{str}}(X) = v(\Sigma)$ by Corollary 4.1.3. Furthermore,

$$c_1(X).c_{d-1}^{\mathrm{str}}(X) = \left[-K_X\right].c_{d-1}^{\mathrm{str}}(X) = \sum_{\sigma\in\Sigma(d-1)} v(\sigma)\cdot l_{-K_X}(\sigma)$$

by Proposition 4.1.6 and

$$c_1(X).c_{d-1}^{\mathrm{str}}(X) = \left[-K_X\right].c_{d-1}^{\mathrm{str}}(X) = \sum_{\sigma\in\Sigma(d-1)} v(\sigma)\cdot v\left(\Delta_{-K_X}^{\sigma}\right)$$

by Theorem 4.1.7 if $-K_X$ is semi-ample. $\square$

# Chapter 5

## Stringy $E$-functions of Toric Varieties

This chapter is dedicated to stringy $E$-functions of projective $\mathbb{Q}$-Gorenstein toric varieties and their combinatorial computations using underlying intrinsic information provided by associated fans and associated polytopes of these varieties. In detail, this means that we obtain three combinatorial formulas to compute stringy $E$-functions for the following classes of toric varieties: Gorenstein toric Fano varieties (Section 5.1), toric log del Pezzo surfaces (Section 5.2), and canonical Fano threefolds (Section 5.3). We will revisit this results in Chapter 6.

The results of Section 5.1 and Section 5.2 have been published as joint work of the author of this thesis and Victor Batyrev in our paper *Stringy Chern Classes of Singular Toric Varieties and Their Applications* [BS17]. The results of Section 5.3 are based on our joint preprint *Stringy E-functions of Canonical Toric Fano Threefolds and Their Applications* [BS18]. Compared to [BS17] and [BS18], the present chapter contains minor modifications and additional examples.

We begin by repeating a central result of Chapter 4, which is that the stringy $E$-function of a projective $\mathbb{Q}$-Gorenstein toric variety is a polynomial with non-negative integral coefficients in non-negative rational powers:

**Proposition 4.2.1.** *Let $X$ be a $d$-dimensional projective $\mathbb{Q}$-Gorenstein toric variety of Gorenstein index $q_X$ associated with a fan $\Sigma$ in $N_{\mathbb{R}}$ and $\Sigma'$ a simplicial subdivision of the fan $\Sigma$ such that $\Sigma'(1) = \Sigma(1)$. Then the stringy $E$-function can be computed as a finite sum*

$$E_{\text{str}}(X; u, v) = \sum_{\sigma \in \Sigma'} (uv-1)^{d-\dim(\sigma)} \sum_{n \in \square_\sigma^\circ \cap N} (uv)^{\dim(\sigma)+\kappa(n)},$$

*where $\kappa$ denotes the $\Sigma$-piecewise linear function corresponding to the anticanonical divisor $-K_X$ on $X$* (Proposition 2.2.12)*. Moreover, the stringy $E$-function can be written as a finite sum*

$$E_{\text{str}}(X; u, v) = \sum_{\alpha \in [0,d] \cap \frac{1}{q_X}\mathbb{Z}} \psi_\alpha(\Sigma)(uv)^\alpha,$$

*where the coefficients* $\psi_\alpha(\Sigma)$ *are non-negative integers satisfying the conditions* $\psi_0(\Sigma) = \psi_d(\Sigma) = 1$ *and* $\psi_\alpha(\Sigma) = \psi_{d-\alpha}(\Sigma)$ *for all* $\alpha \in [0,d] \cap \frac{1}{q_X}\mathbb{Z}$.

## 5.1 Gorenstein Toric Fano Varieties

Let $\Delta \subseteq N_\mathbb{R}$ be a $d$-dimensional reflexive polytope (*i.e.*, a lattice polytope containing the origin in its interior such that the dual polytope $\Delta^*$ is also a lattice polytope (Definition 2.3.13)). $\Sigma_\Delta$ denotes the associated spanning fan in $N_\mathbb{R}$ consisting of cones over faces of the given polytope $\Delta$ that defines a normal projective toric variety $X_\Delta := X_{\Sigma_\Delta}$ (Theorem 2.3.11). In this case, $X_\Delta$ is a projective Gorenstein toric Fano variety (Theorem 2.3.14) with at worst log-terminal singularities (Proposition 2.2.10). In particular, the Gorenstein index $q_{X_\Delta}$ equals 1.

We are interested in a combinatorial formula for the stringy $E$-function of Gorenstein toric Fano varieties $X_\Delta$ corresponding to reflexive polytopes $\Delta$. For this purpose, we investigate the generalized stringy Hodge numbers $\psi_\alpha(\Sigma_\Delta)$ appearing as coefficients in the stringy $E$-function

$$E_{\mathrm{str}}(X_\Delta; u, v) = \sum_{\alpha \in [0,d] \cap \mathbb{Z}} \psi_\alpha(\Sigma_\Delta)(uv)^\alpha.$$

**Definition 5.1.1.** The *Ehrhart power series* of a $d$-dimensional lattice polytope $\Delta \subseteq N_\mathbb{R}$ is defined as

$$P_\Delta(t) := \sum_{k \geq 0} |k\Delta \cap N| \, t^k,$$

where $|k\Delta \cap N|$ denotes the number of lattice points in $k\Delta$.

**Remark 5.1.2.** The Ehrhart power series can be equivalently rewritten as the rational function

$$P_\Delta(t) = \frac{\psi_d(\Delta)\, t^d + \ldots + \psi_1(\Delta)\, t + \psi_0(\Delta)}{(1-t)^{d+1}},$$

where $\psi_\alpha(\Delta)$ are non-negative integers for all $0 \leq \alpha \leq d$ [BR06, Theorem 3.12] with $\psi_0(\Delta) = 1$ [BR06, Lemma 3.13] and $\psi_1(\Delta) = |\Delta \cap N| - d - 1$ [BR06, Corollary 3.16]. If $\Delta \subseteq N_\mathbb{R}$ is a reflexive polytope, then $\psi_\alpha(\Delta) = \psi_{d-\alpha}(\Delta)$ for all $\alpha \in [0,d] \cap \mathbb{Z}$ [BR06, Theorem 4.6].

This in mind, we observe that in case of a reflexive polytope $\Delta$, the generalized stringy Hodge numbers $\psi_\alpha(\Sigma_\Delta)$ are equal to the non-negative integral coefficients $\psi_\alpha(\Delta)$ in the numerator of the Ehrhart power series $P_\Delta(t)$:

**Theorem 5.1.3.** *Let* $\Delta \subseteq N_\mathbb{R}$ *be a reflexive* $d$*-polytope and* $X_\Delta$ *the associated Gorenstein toric Fano variety. Then*

$$E_{\mathrm{str}}(X_\Delta; u, v) = \psi_d(\Delta)(uv)^d + \ldots + \psi_1(\Delta)(uv) + \psi_0(\Delta),$$

i.e., $\psi_\alpha(\Sigma_\Delta) = \psi_\alpha(\Delta)$ *for all* $\alpha \in [0,d] \cap \mathbb{Z}$*, where* $\psi_\alpha(\Sigma_\Delta)$ *and* $\psi_\alpha(\Delta)$ *are given as above.*

*Proof.* By Equation (4.5), we have

$$E_{\text{str}}(X_\Delta; u, v) = (uv-1)^d \sum_{\sigma \in \Sigma_\Delta} \sum_{n \in \sigma^\circ \cap N} (uv)^{\kappa_\Delta(n)},$$

where $\kappa_\Delta : N_\mathbb{R} \to \mathbb{R}$ is the $\Sigma_\Delta$-piecewise linear function corresponding to the anti-canonical divisor $-K_{X_{\Sigma_\Delta}}$ on $X_\Delta$ and has value $-1$ on every primitive ray generator of a 1-dimensional cone of $\Sigma_\Delta(1)$ (Equation (2.3)). We obtain

$$E_{\text{str}}(X_\Delta; u, v) = (uv-1)^d \sum_{n \in N} (uv)^{\kappa_\Delta(n)} = (uv-1)^d \sum_{k \geq 0} \sum_{\substack{n \in N \\ \kappa_\Delta(n) = -k}} (uv)^{-k},$$

since the spanning fan $\Sigma_\Delta$ defining $X_\Delta$ is complete (Theorem 2.2.9). We note that the number of lattice points $n \in N$ such that $\kappa_\Delta(n) = -k$ equals $|k\Delta \cap N| - |(k-1)\Delta \cap N|$. Therefore, we get

$$E_{\text{str}}(X_\Delta; u, v) = (uv-1)^d \left(1 - (uv)^{-1}\right) \sum_{k \geq 0} |k\Delta \cap N| \, (uv)^{-k}.$$

Using the definition of $P_\Delta\left((uv)^{-1}\right)$ and Remark 5.1.2, this implies

$$\begin{aligned}
E_{\text{str}}(X_\Delta; u, v) &= (uv)^d \left(1 - (uv)^{-1}\right)^{d+1} \cdot P_\Delta\left((uv)^{-1}\right) \\
&= (uv)^d \left( \psi_d(\Delta) \left(\frac{1}{uv}\right)^d + \ldots + \psi_1(\Delta) \left(\frac{1}{uv}\right) + \psi_0(\Delta) \right) \\
&= \psi_d(\Delta) + \ldots + \psi_1(\Delta)(uv)^{d-1} + \psi_0(\Delta)(uv)^d \\
&= \psi_d(\Delta)(uv)^d + \ldots + \psi_1(\Delta)(uv) + \psi_0(\Delta)
\end{aligned}$$

because $\psi_\alpha(\Delta) = \psi_{d-\alpha}(\Delta)$ for all $0 \leq \alpha \leq d$. $\square$

**Example 5.1.4.** Let $\Delta \subseteq N_\mathbb{R}$ be a reflexive $d$-polytope and $X_\Delta$ the associated Gorenstein toric Fano variety. Applying Theorem 5.1.3 and using the properties $\psi_d(\Delta) = \psi_0(\Delta) = 1$, $\psi_\alpha(\Delta) = \psi_{d-\alpha}(\Delta)$, and $\psi_1(\Delta) = |\Delta \cap N| - d - 1$ of the Ehrhart power series (Remark 5.1.2), the stringy $E$-functions of all 16 reflexive 2-polytopes in Figure 6.1 are listed in Table 5.1.

Looking at a small selection of all 4,319 reflexive 3-polytopes (Figure 6.2), the associated stringy $E$-functions can be found in Table 5.2. To be precise, the polytope in Figure 6.2(a) is given by

$$\begin{aligned}
\operatorname{conv}(&(1,1,1),(1,1,-1),(-1,1,-1),(-1,1,1),\\
&(-1,-1,-1),(-1,-1,1),(1,-1,1),(1,-1,-1)),
\end{aligned}$$

**Table 5.1: Stringy $E$-functions of Gorenstein toric Fano Varieties Associated with Reflexive 2-polytopes.** Table contains: links to pictures of reflexive 2-polytopes $\Delta$ and the stringy $E$-functions (computed with Theorem 5.1.3) of the corresponding Gorenstein toric Fano varieties $X_\Delta$ defined via the spanning fans $\Sigma_\Delta$ of $\Delta$.

| $\Delta$ | $E_{\text{str}}(X_\Delta; u, v)$ | $\Delta$ | $E_{\text{str}}(X_\Delta; u, v)$ |
|---|---|---|---|
| Figure 6.1(a) | $(uv)^2 + (uv) + 1$ | Figure 6.1(i) | $(uv)^2 + 4(uv) + 1$ |
| Figure 6.1(b) | $(uv)^2 + 2(uv) + 1$ | Figure 6.1(j) | $(uv)^2 + 4(uv) + 1$ |
| Figure 6.1(c) | $(uv)^2 + 2(uv) + 1$ | Figure 6.1(k) | $(uv)^2 + 7(uv) + 1$ |
| Figure 6.1(d) | $(uv)^2 + 2(uv) + 1$ | Figure 6.1(l) | $(uv)^2 + 6(uv) + 1$ |
| Figure 6.1(e) | $(uv)^2 + 3(uv) + 1$ | Figure 6.1(m) | $(uv)^2 + 6(uv) + 1$ |
| Figure 6.1(f) | $(uv)^2 + 3(uv) + 1$ | Figure 6.1(n) | $(uv)^2 + 6(uv) + 1$ |
| Figure 6.1(g) | $(uv)^2 + 4(uv) + 1$ | Figure 6.1(o) | $(uv)^2 + 5(uv) + 1$ |
| Figure 6.1(h) | $(uv)^2 + 4(uv) + 1$ | Figure 6.1(p) | $(uv)^2 + 5(uv) + 1$ |

the polytope in Figure 6.2(b) by

$$\operatorname{conv}((1,1,0),(-1,1,0),(-1,-1,0),(1,-1,0),(0,0,1),(0,0,-1)),$$

the polytope in Figure 6.2(c) by

$$\operatorname{conv}((1,0,0),(0,1,0),(0,0,1),(-1,-1,-1)),$$

and the polytope in Figure 6.2(d) by

$$\operatorname{conv}((-1,3,-1),(-1,-1,-1),(3,-1,-1),(-1,-1,3)).$$

Moreover, $\psi_2(\Delta) = \psi_1(\Delta) = |\Delta \cap N| - 4$ and the polytopes contain in the listed order 27, 7, 5, and 35 lattice points.

**Table 5.2: Stringy $E$-functions of Gorenstein toric Fano Varieties Associated with Reflexive 3-polytopes.** Table contains: links to pictures of reflexive 3-polytopes $\Delta$ and the stringy $E$-functions (computed with Theorem 5.1.3) of the corresponding Gorenstein toric Fano varieties $X_\Delta$ defined via the spanning fans $\Sigma_\Delta$ of $\Delta$.

| $\Delta$ | $E_{\text{str}}(X_\Delta; u, v)$ |
|---|---|
| Figure 6.2(a) | $(uv)^3 + 23(uv)^2 + 23(uv) + 1$ |
| Figure 6.2(b) | $(uv)^3 + 3(uv)^2 + 3(uv) + 1$ |
| Figure 6.2(c) | $(uv)^3 + 1(uv)^2 + 1(uv) + 1$ |
| Figure 6.2(d) | $(uv)^3 + 31(uv)^2 + 31(uv) + 1$ |

The stringy $E$-functions of the Gorenstein toric Fano varieties corresponding to the following small selection of the 473,800,776 reflexive 4-polytopes $\Delta$ (Figure 6.3) are listed in Table 5.3:

$\operatorname{conv}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$ (Figure 6.3(a)), $\operatorname{conv}(\pm e_1, \pm e_2, \pm e_3, \pm e_4)$ (Figure 6.3(b))

**Table 5.3: Stringy $E$-functions of Gorenstein toric Fano Varieties Associated with Reflexive 4-polytopes.** Table contains: links to pictures of reflexive 4-polytopes $\Delta$ and the stringy $E$-functions (computed with Theorem 5.1.3) of the corresponding Gorenstein toric Fano varieties $X_\Delta$ defined via the spanning fans $\Sigma_\Delta$ of $\Delta$.

| $\Delta$ | $E_{\text{str}}(X_\Delta; u, v)$ |
|---|---|
| Figure 6.3(a) | $(uv)^4 + 76(uv)^3 + 230(uv)^2 + 76(uv) + 1$ |
| Figure 6.3(b) | $(uv)^4 + 4(uv)^3 + 6(uv)^2 + 4(uv) + 1$ |
| Figure 6.3(c) | $(uv)^4 + (uv)^3 + (uv)^2 + (uv) + 1$ |
| Figure 6.3(d) | $(uv)^4 + 121(uv)^3 + 381(uv)^2 + 121(uv) + 1$ |

and

$$\operatorname{conv}(e_1, e_2, e_3, e_4, -(e_1 + e_2 + e_3 + e_4)) \text{ (Figure 6.3(c))},$$

$$5 \cdot \operatorname{conv}(e_1, e_2, e_3, e_4, 0) - (e_1 + e_2 + e_3 + e_4) \text{ (Figure 6.3(d))}.$$

Moreover, $\psi_3(\Delta) = \psi_1(\Delta) = |\Delta \cap N| - 5$ and the polytopes contain in the listed order 81, 9, 6, and 126 lattice points. The remaining generalized stringy Hodge numbers $\psi_2(\Delta)$ can be computed using $\psi_2(\Delta) = e_{\text{str}}(X_\Delta) - 2\psi_0(\Delta) - 2\psi_1(\Delta)$ and $e_{\text{str}}(X_\Delta) = v(\Delta)$ (Theorem 6.1.1 (proof), Corollary 4.1.3, and Equation (1.4)). The polytopes have in the listed order normalized volume 384, 16, 5, and 625. A combination of these facts provides the received stringy $E$-functions in Table 5.3.

## 5.2 Toric log del Pezzo Surfaces

First, we consider a more general case and look at 2-dimensional projective $\mathbb{Q}$-Gorenstein toric varieties that build a superclass of toric log del Pezzo surfaces:

**Proposition 5.2.1.** *Let $X$ be a 2-dimensional projective $\mathbb{Q}$-Gorenstein toric variety associated with a fan $\Sigma$ in $N_\mathbb{R}$. Then*

$$E_{\text{str}}(X; u, v) = (uv-1)^2 + \sum_{\substack{n \in N \\ \kappa(n) = -1}} uv + \sum_{\substack{n \in N \\ -1 < \kappa(n) < 0}} \left( (uv)^{2+\kappa(n)} + (uv)^{-\kappa(n)} \right),$$

*where $\kappa$ denotes the $\Sigma$-piecewise linear function corresponding to the anticanonical divisor $-K_X$ on $X$* (Proposition 2.2.12).

*Proof.* The fan $\Sigma$ already is simplicial because every cone $\sigma \in \Sigma$ is simplicial, *i.e.*, we do not need a simplicial subdivision $\Sigma'$ of $\Sigma$ to continue. Therefore, we set $\Sigma' = \Sigma$. By Proposition 4.2.1, we obtain

$$\begin{aligned} E_{\text{str}}(X; u, v) &= \sum_{\sigma \in \Sigma} (uv-1)^{2-\dim(\sigma)} \sum_{n \in \square_\sigma^\circ \cap N} (uv)^{\dim(\sigma)+\kappa(n)} \\ &= (uv-1)^2 + \sum_{\sigma \in \Sigma(1)} (uv-1) + \sum_{\sigma \in \Sigma(2)} \sum_{n \in \square_\sigma^\circ \cap N} (uv)^{2+\kappa(n)}, \end{aligned}$$

where $\square_\sigma$ denotes the parallelepiped in Equation (4.6) spanned by the primitive ray generators of a cone $\sigma$.

For any 2-dimensional cone $\sigma \in \Sigma(2)$, the set $\{x \in N_{\mathbb{R}} \ : \ \kappa(x) = -1\}$ divides the parallelogram $\square_\sigma$ into two isomorphic lattice triangles $\triangle^\sigma_{\leq -1}$ and $\triangle^\sigma_{\geq -1}$. Let $u_1, u_2 \in N$ be the primitive ray generators of $\sigma$. We can write every lattice point $n \in \square_\sigma$ as a linear combination $n = \lambda_1 u_1 + \lambda_2 u_2$ with rational coefficients $\lambda_1, \lambda_2 \in [0,1]$. A lattice point $n \in \square_\sigma$ belongs to the triangle $\triangle^\sigma_{\geq -1}$ if and only if the lattice point $n^* := u_1 + u_2 - n$ belongs to the triangle $\triangle^\sigma_{\leq -1}$. Since the boundary of the lattice parallelogram $\square_\sigma$ has no lattice points except vertices, we can use the bijection $n \leftrightarrow n^*$ together with the equation $\kappa(n) + \kappa(n^*) = -2$ to obtain

$$\sum_{n \in \square^\circ_\sigma \cap N} (uv)^{2+\kappa(n)} = 1 + \sum_{\substack{n \in \sigma^\circ \cap N \\ \kappa(n) = -1}} uv + \sum_{\substack{n \in \sigma \cap N \\ -1 < \kappa(n) < 0}} \left( (uv)^{2+\kappa(n)} + (uv)^{-\kappa(n)} \right).$$

It remains to apply the equalities $|\Sigma(1)| = |\Sigma(2)|$ and

$$\sum_{\sigma \in \Sigma(1)} uv + \sum_{\sigma \in \Sigma(2)} \sum_{\substack{n \in \sigma^\circ \cap N \\ \kappa(n) = -1}} uv = \sum_{\substack{n \in N \\ \kappa(n) = -1}} uv.$$

$\square$

Now, let $X$ be a toric log del Pezzo surface (*i.e.*, a normal projective toric surface with at worst log-terminal singularities and an ample anticanonical $\mathbb{Q}$-Cartier divisor $-K_X$ on $X$ (page 26)). Such a toric surface is uniquely defined by a $LDP$-polygon $\Delta \subseteq N_{\mathbb{R}}$ (*i.e.*, a 2-dimensional lattice polytope containing the origin in its interior such that all vertices of $\Delta$ are primitive lattice points in $N$ (Definition 2.3.18)). The fan $\Sigma_\Delta$ denotes the spanning fan in the 2-dimensional space $N_{\mathbb{R}}$ consisting of cones over faces of the given $LDP$-polygon $\Delta$, *i.e.*, $X_\Delta := X_{\Sigma_\Delta} = X$. In this case, we obtain the much more precise computation of the stringy $E$-function $E_{\text{str}}(X_\Delta; u, v)$ stated below:

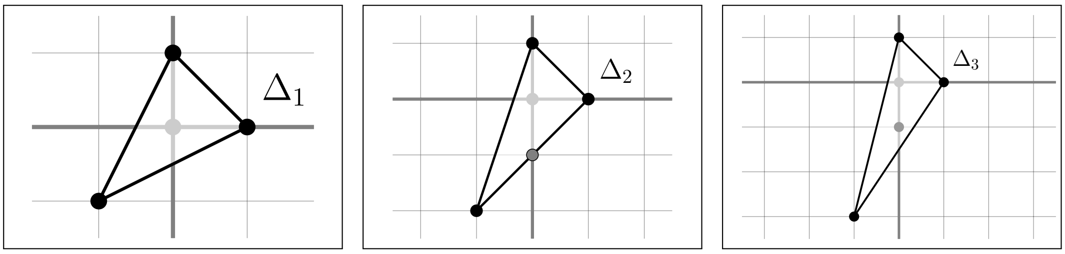
**Theorem 5.2.2.** *Let $X_\Delta$ be a toric log del Pezzo surface defined by a $LDP$-polygon $\Delta \subseteq N_{\mathbb{R}}$. Then the stringy $E$-function of $X_\Delta$ can be computed as*

$$E_{\text{str}}(X_\Delta; u, v) = ((uv)^2 + 1) + r \cdot (uv) + \sum_{n \in \Delta^\circ \setminus \{0\}} \left( (uv)^{2+\kappa_\Delta(n)} + (uv)^{-\kappa_\Delta(n)} \right)$$

*with $r := |\partial\Delta \cap N| - 2$, where $\kappa_\Delta$ is the $\Sigma_\Delta$-piecewise linear function corresponding to the anticanonical divisor $-K_{X_\Delta}$ on $X_\Delta$* (Equation (2.3)) *with value $-1$ on the boundary $\partial\Delta$ of $\Delta$.*

*Proof.* Using Proposition 5.2.1, one gets

$$E_{\text{str}}(X_\Delta; u, v) = (uv - 1)^2 + \sum_{\substack{n \in N \\ \kappa_\Delta(n) = -1}} uv + \sum_{\substack{n \in N \\ -1 < \kappa_\Delta(n) < 0}} \left( (uv)^{2+\kappa_\Delta(n)} + (uv)^{-\kappa_\Delta(n)} \right)$$




**(a)** Reflexive polygon $\Delta_1$. **(b)** Reflexive polygon $\Delta_2$. **(c)** *LDP*-polygon $\Delta_3$.

**Figure 5.1: Reflexive Polygons and *LDP*-polygons.** ***(a)*** The reflexive polygon $\Delta_1$ is given as the convex hull of $e_1$, $e_2$, and $-e_1-e_2$. In particular, it is a *LDP*-polygon. The lattice points in $\Delta_1$ are the origin (light grey dot) and the three vertices (black dots). ***(b)*** The reflexive polygon $\Delta_2$ is given as the convex hull of $e_1$, $e_2$, and $-e_1-2e_2$. In particular, it is a *LDP*-polygon. The lattice points in $\Delta_2$ are the origin (light grey dot), the three vertices (black dots), and one boundary lattice point (grey dot). ***(c)*** The *LDP*-polygon $\Delta_3$ is given as the convex hull of $e_1$, $e_2$, and $-e_1-3e_2$. The lattice points in $\Delta_3$ are the origin (light grey dot), the three vertices (black dots), and one interior lattice point (grey dot). $\Delta_3$ is not reflexive because it has more than one interior lattice point (page 25).

$$
\begin{aligned}
&= (uv)^2 - 2(uv) + 1 + |\partial\Delta \cap N| \cdot (uv) + \sum_{n \in \Delta^\circ \setminus \{0\}} \left( (uv)^{2+\kappa_\Delta(n)} + (uv)^{-\kappa_\Delta(n)} \right) \\
&= ((uv)^2 + 1) + (|\partial\Delta \cap N| - 2) \cdot (uv) + \sum_{n \in \Delta^\circ \setminus \{0\}} \left( (uv)^{2+\kappa_\Delta(n)} + (uv)^{-\kappa_\Delta(n)} \right).
\end{aligned}
$$

□

**Example 5.2.3.** Set $\Delta_m := \operatorname{conv}(e_1, e_2, -e_1 - me_2)$ for an integer $m \in \mathbb{N}$, *i.e.*, $\Delta_1$, $\Delta_2$, and $\Delta_3$ are *LDP*-polygons (Figure 5.1(a), (b), and (c)). In particular, $\Delta_1$ and $\Delta_2$ are reflexive polygons. Therefore, $X_{\Delta_1} = \mathbb{P}^2$, $X_{\Delta_2} = \mathbb{P}(1,1,2)$, and $X_{\Delta_3} = \mathbb{P}(1,1,3)$ are toric log del Pezzo surfaces with

$$E_{\mathrm{str}}\left(X_{\Delta_1}; u, v\right) = ((uv)^2 + 1) + (uv) = (uv)^2 + (uv) + 1,$$

$$E_{\mathrm{str}}\left(X_{\Delta_2}; u, v\right) = ((uv)^2 + 1) + 2(uv) = (uv)^2 + 2(uv) + 1,$$

$$
\begin{aligned}
E_{\mathrm{str}}\left(X_{\Delta_3}; u, v\right) &= ((uv)^2 + 1) + (uv) + \sum_{0 < n \leq \frac{3}{2}} \left( (uv)^{2+\kappa_\Delta(0,n)} + (uv)^{-\kappa_\Delta(0,n)} \right) \\
&= ((uv)^2 + 1) + (uv) + \sum_{0 < n \leq \frac{3}{2}} \left( (uv)^{2-\frac{2}{3}n} + (uv)^{\frac{2}{3}n} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= ((uv)^2+1)+(uv)+\sum_{0<n\leq\frac{3}{2}}\left((uv)^{\frac{2(3-n)}{3}}+(uv)^{\frac{2n}{3}}\right)\\
&= ((uv)^2+1)+(uv)+\left((uv)^{\frac{4}{3}}+(uv)^{\frac{2}{3}}\right)\\
&= (uv)^2+(uv)^{\frac{4}{3}}+(uv)+(uv)^{\frac{2}{3}}+1,
\end{aligned}
$$

and in general

$$
\begin{aligned}
E_{\mathrm{str}}\left(X_{\Delta_m};u,v\right) &= ((uv)^2+1)+(uv)+\sum_{0<n\leq\frac{m}{2}}\left((uv)^{2+\kappa_\Delta(0,n)}+(uv)^{-\kappa_\Delta(0,n)}\right)\\
&= ((uv)^2+1)+(uv)+\sum_{0<n\leq\frac{m}{2}}\left((uv)^{2-\frac{2}{m}n}+(uv)^{\frac{2}{m}n}\right)\\
&= ((uv)^2+1)+(uv)+\sum_{0<n\leq\frac{m}{2}}\left((uv)^{\frac{2(m-n)}{m}}+(uv)^{\frac{2n}{m}}\right)
\end{aligned}
$$

by using Theorem 5.2.2 because $\kappa_\Delta(0,-m/2)=-1$.

## 5.3 Canonical Toric Fano Threefolds

Let $\Delta\subseteq N_{\mathbb{R}}$ be a canonical Fano 3-polytope, $\Sigma_\Delta$ the associated spanning fan of $\Delta$, and $X_\Delta:=X_{\Sigma_\Delta}$ the corresponding toric Fano threefold with at worst canonical singularities (Theorem 2.3.33). We are interested in a combinatorial formula to compute the stringy $E$-function $E_{\mathrm{str}}\left(X_\Delta;u,v\right)$ of such a toric Fano threefold $X_\Delta$ by making use of intrinsic information provided by the lattice polytope $\Delta$ that yields the appearing coefficients $\psi_\alpha(\Sigma_\Delta)$ and rational powers of $uv$ in the stringy $E$-function $E_{\mathrm{str}}\left(X_\Delta;u,v\right)$ in a direct way (Proposition 4.2.1). To be precise, the following formula primarily depends on normalized volumes $v(\theta)$ (Definition 2.3.6) and lattice distances $n_\theta$ to the origin (Definition 2.3.5) of 2-dimensional faces $\theta\preceq\Delta$ of $\Delta$:

**Theorem 5.3.1.** *Let $\Delta\subseteq N_{\mathbb{R}}$ be a canonical Fano 3-polytope. Then the stringy $E$-function of the corresponding canonical toric Fano threefold $X_\Delta$ can be computed as*

$$
E_{\mathrm{str}}\left(X_\Delta;u,v\right)=\left((uv)^3+1\right)+r\cdot\left((uv)^2+(uv)\right)+\sum_{\substack{\theta\preceq\Delta\\ \dim(\theta)=2,n_\theta>1}} v(\theta)\cdot\left(\sum_{k=1}^{n_\theta-1}(uv)^{\frac{k}{n_\theta}+1}\right)
$$

*with $r:=|\Delta\cap N|-4$.*

*Proof.* Applying Proposition 4.2.1, the stringy $E$-function of $X_\Delta$ is given by

$$
E_{\mathrm{str}}\left(X_\Delta;u,v\right)=\sum_{\sigma\in\Sigma'_\Delta}(uv-1)^{3-\dim(\sigma)}\sum_{n\in\square_\sigma^\circ\cap N}(uv)^{\dim(\sigma)+\kappa_\Delta(n)},
$$

where $\kappa_\Delta$ denotes the $\Sigma_\Delta$-piecewise linear function corresponding to the anticanonical divisor $-K_{X_{\Sigma_\Delta}}$ on $X_{\Sigma_\Delta}$ (Equation (2.3)) and $\Sigma'_\Delta$ a simplicial subdivision of the fan $\Sigma_\Delta$

such that $\Sigma_\Delta(1) = \Sigma'_\Delta(1)$ with $|\Sigma'_\Delta(1)| = |\Delta \cap N| - 1$. Moreover, the fan $\Sigma'_\Delta$ is obtained by a triangulation of all 2-dimensional faces $\theta \preceq \Delta$ of $\Delta$ and $\square^\circ_\sigma = \sigma^\circ \cap \square_\sigma$ for a cone $\sigma \in \Sigma'_\Delta$ (Equation (4.6)).

In each 1-dimensional cone $\rho \in \Sigma'_\Delta(1)$, there exists exactly one lattice point $n \in \square^\circ_\rho \cap N$ with $\kappa_\Delta(n) = -1$ because $n$ is a boundary lattice point of $\Delta$. Each 2-dimensional cone $\tau \in \Sigma'_\Delta(2)$ contains also exactly one lattice point $n \in \square^\circ_\tau \cap N$ with $\kappa_\Delta(n) = -2$ because the minimal lattice generators $u_{\rho'}$ and $u_{\rho''} \in N$ of $\tau$ are linearly independent over $\mathbb{R}$ with $n = u_{\rho'} + u_{\rho''}$. These two facts yield

$$\begin{aligned}
E_{\mathrm{str}}\left(X_\Delta; u, v\right) &= (uv-1)^3 + \sum_{\rho \in \Sigma'_\Delta(1)} (uv-1)^2 \sum_{\substack{n \in \square^\circ_\rho \cap N \\ \text{with } |n \in \square^\circ_\rho \cap N| = 1}} \underbrace{(uv)^{1+\kappa_\Delta(n)}}_{=(uv)^{1-1}=1} \\
&\quad + \sum_{\tau \in \Sigma'_\Delta(2)} (uv-1) \sum_{\substack{n \in \square^\circ_\tau \cap N \\ \text{with } |n \in \square^\circ_\tau \cap N| = 1}} \underbrace{(uv)^{2+\kappa_\Delta(n)}}_{=(uv)^{2-2}=1} + \sum_{\sigma \in \Sigma'_\Delta(3)} \sum_{n \in \square^\circ_\sigma \cap N} (uv)^{3+\kappa_\Delta(n)} \\
&= (uv-1)^3 + \sum_{\rho \in \Sigma'_\Delta(1)} (uv-1)^2 + \sum_{\tau \in \Sigma'_\Delta(2)} (uv-1) + \sum_{\sigma \in \Sigma'_\Delta(3)} \sum_{n \in \square^\circ_\sigma \cap N} (uv)^{3+\kappa_\Delta(n)} \\
&= (uv-1)^3 + \sum_{\sigma \in \Sigma'_\Delta(1)} (uv-1)^2 + \sum_{\sigma \in \Sigma'_\Delta(2)} (uv-1) \qquad (5.1) \\
&\quad + \sum_{\sigma \in \Sigma'_\Delta(3)} \Big( \underbrace{(uv)^{3+\kappa_\Delta(n)}}_{=(uv)^{3-3}=1} + \sum_{\substack{n \in \square^\circ_\sigma \cap N \\ \kappa_\Delta(n) \in \mathbb{Q} \setminus \mathbb{Z}}} (uv)^{3+\kappa_\Delta(n)} \Big) \\
&= (uv-1)^3 + \sum_{\rho \in \Sigma'_\Delta(1)} (uv-1)^2 + \sum_{\tau \in \Sigma'_\Delta(2)} (uv-1) \\
&\quad + \sum_{\sigma \in \Sigma'_\Delta(3)} \Big( 1 + \sum_{\substack{n \in \square^\circ_\sigma \cap N \\ \kappa_\Delta(n) \in \mathbb{Q} \setminus \mathbb{Z}}} (uv)^{3+\kappa_\Delta(n)} \Big)
\end{aligned}$$

because each 3-dimensional cone $\sigma \in \Sigma'_\Delta(3)$ has also exactly one lattice point $n \in \square^\circ_\sigma \cap N$ with $\kappa_\Delta(n) = -3$. The intersection of all cones $\sigma \in \Sigma'_\Delta$ with the 2-dimensional sphere $S^2$ defines a triangulation of $S^2$. Using Euler's polyhedral formula

$$2 = \left|\Sigma'_\Delta(3)\right| - \left|\Sigma'_\Delta(2)\right| + \left|\Sigma'_\Delta(1)\right|$$

and $3 \cdot |\Sigma'_\Delta(3)| = 2 \cdot |\Sigma'_\Delta(2)|$, we obtain $3 \cdot |\Sigma'_\Delta(3)| = 2 \cdot (|\Sigma'_\Delta(3)| + |\Sigma'_\Delta(1)| - 2)$ because $|\Sigma'_\Delta(2)| = |\Sigma'_\Delta(3)| + |\Sigma'_\Delta(1)| - 2$. Therefore, we get $|\Sigma'_\Delta(3)| = 2 \cdot |\Sigma'_\Delta(1)| - 4$ and $|\Sigma'_\Delta(2)| = 3 \cdot |\Sigma'_\Delta(1)| - 6$. Using $|\Sigma'_\Delta(1)| = |\Delta \cap N| - 1$, we receive

$$\begin{aligned}
E_{\mathrm{str}}\left(X_\Delta; u, v\right) &= (uv-1)^3 + \left(|\Delta \cap N| - 1\right)(uv-1)^2 \\
&\quad + (3 \cdot \left|\Sigma'_\Delta(1)\right| - 6)(uv-1) + (2 \cdot \left|\Sigma'_\Delta(1)\right| - 4) + T \\
&= ((uv)^3 - 3(uv)^2 + 3(uv) - 1) + \left(|\Delta \cap N| - 1\right)\left((uv)^2 - 2(uv) + 1\right) \\
&\quad + (3 \cdot (|\Delta \cap N| - 1) - 6)(uv-1) + (2 \cdot (|\Delta \cap N| - 1) - 4) + T \qquad (5.2) \\
&= (uv)^3 - 3(uv)^2 + 3(uv) - 1 + (|\Delta \cap N| - 1)(uv)^2 - 2(|\Delta \cap N| - 1)(uv)
\end{aligned}$$

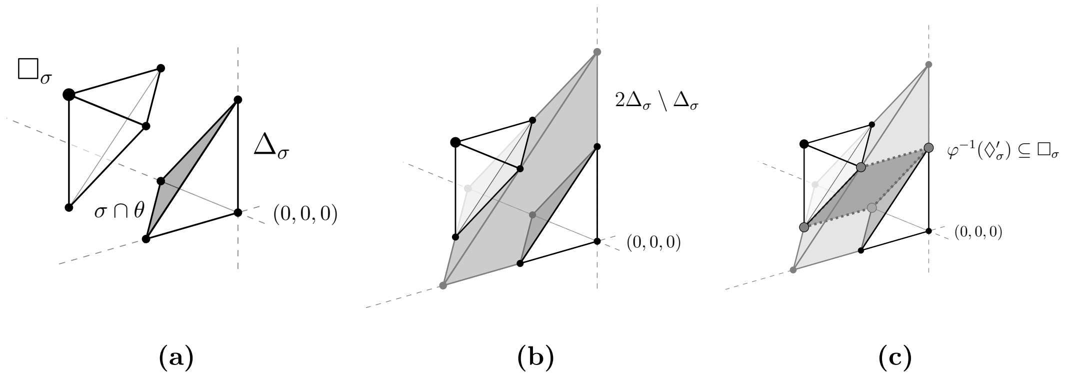



**Figure 5.2: Illustration to detect $n \in \Box^\circ_\sigma \cap N$ with $\kappa_\Delta(n) \in \mathbb{Q} \setminus \mathbb{Z}$.**

$$\begin{aligned} &+ (|\Delta \cap N| - 1) + 3(|\Delta \cap N| - 1)(uv) - 6(uv) - 3(|\Delta \cap N| - 1) \\ &+ 6 + 2(|\Delta \cap N| - 1) - 4 + T \\ = &\big((uv)^3 + 1\big) + (|\Delta \cap N| - 4)\big((uv)^2 + (uv)\big) + T, \end{aligned}$$

where

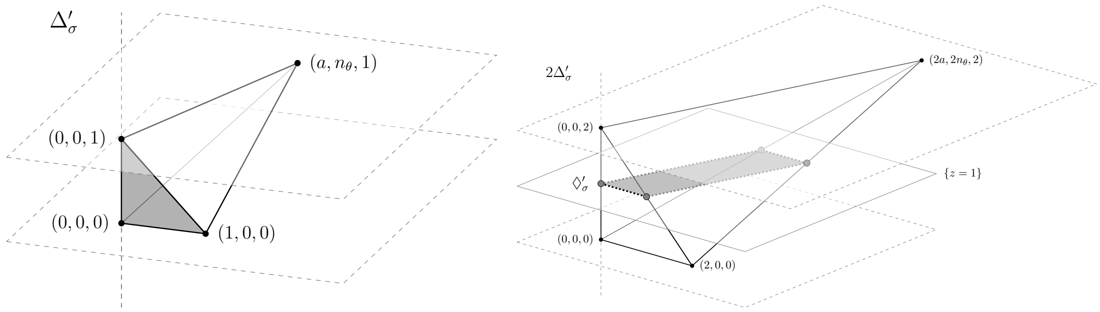
$$T := \sum_{\sigma \in \Sigma'_\Delta(3)} T_\sigma \quad \text{and} \quad T_\sigma := \sum_{\substack{n \in \Box^\circ_\sigma \cap N \\ \kappa_\Delta(n) \in \mathbb{Q}\setminus\mathbb{Z}}} (uv)^{3+\kappa_\Delta(n)} \quad \text{for } \sigma \in \Sigma'_\Delta(3).$$

Let $\sigma \in \Sigma'_\Delta(3)$ be a 3-dimensional cone. Then $\sigma \cap \theta$ is a 2-dimensional simplex with normalized volume $v(\sigma \cap \theta) = 1$ contained in a 2-dimensional face $\theta \preceq \Delta$ of $\Delta$. The only lattice points of the 3-dimensional lattice simplex $\Delta_\sigma := \sigma \cap \Delta$ are the origin $0 \in N$ and the three vertices $\nu_1$, $\nu_2$, and $\nu_3$ of the triangle $\sigma \cap \theta$ (Figure 5.2(a)) because $\Delta$ is a canonical Fano polytope. Therefore, we get $v(\Delta_\sigma) = v(\sigma \cap \theta) \cdot n_\theta = n_\theta$.

If $n_\theta = 1$, then $v(\Delta_\sigma) = 1$ and $\sigma$ is generated by a basis of $N$, *i.e.*, $T_\sigma = 0$. For the following considerations, let $n_\theta$ be greater than 1. A lattice point $n \in \Box^\circ_\sigma \cap N$ can not be contained in $\Delta_\sigma$ and $-\Delta_\sigma + (\nu_1 + \nu_2 + \nu_3)$ (Figure 5.2(a)) as well as not in one of the three simplices $\Delta_\sigma + \nu_1$, $\Delta_\sigma + \nu_2$, and $\Delta_\sigma + \nu_3$ (Figure 5.2(b)). Therefore, $n$ belongs to $(2\Delta_\sigma)^\circ \cap N$, *i.e.*,

$$T_\sigma = \sum_{\substack{n \in \Box^\circ_\sigma \cap N \\ \kappa_\Delta(n) \in \mathbb{Q}\setminus\mathbb{Z}}} (uv)^{3+\kappa_\Delta(n)} = \sum_{n \in (2\Delta_\sigma)^\circ \cap N} (uv)^{3+\kappa_\Delta(n)}. \tag{5.3}$$

To compute $T_\sigma$, we apply the Theorem of White [Whi64], which says that $\Delta_\sigma$ is isomorphic to a lattice simplex $\Delta'_\sigma := \operatorname{conv}(\nu'_1, \nu'_2, \nu'_3, \nu'_4)$ (Remark 2.3.16) such that $\nu'_1 := (0,0,0)$, $\nu'_2 := (1,0,0)$, $\nu'_3 := (0,0,1)$, and $\nu'_4 := (a, n_\theta, 1)$ for some integer $a$ with $\gcd(a, n_\theta) = 1$ (Figure 5.3(a)). In particular, $\Delta'_\sigma$ is contained between the two planes $\{z = 0\}$ and $\{z = 1\}$. The isomorphism $\varphi : \Delta_\sigma \to \Delta'_\sigma$ of White maps the vertex $0 \in \Delta_\sigma$ to the vertex $\nu'_4 \in \Delta'_\sigma$ and transforms the linear function $\kappa_\Delta$ on $\sigma$ into the affine linear function $-1 + \frac{y}{n_\theta}$ on a cone with origin $\nu'_4$ and generators $\nu'_i - \nu'_4$ $(1 \le i \le 3)$.




**(a)** **(b)**

**Figure 5.3: Theorem of White (here: $a = 1$ and $n_\theta = 3$).**

The double simplex $2\Delta'_\sigma$ is contained between the two planes $\{z = 0\}$ and $\{z = 2\}$. Therefore, an interior lattice point $n \in (2\Delta'_\sigma)^\circ$ belongs to an interior lattice point of $\lozenge'_\sigma := 2\Delta'_\sigma \cap \{z = 1\}$, which is a parallelogram with vertices $(0,0,1)$, $(1,0,1)$, $(a+1, n_\theta, 1)$, and $(a, n_\theta, 1)$ (Figure 5.3(b)). We note that the isomorphism of White maps $\lozenge'_\sigma$ into a subset of $\square_\sigma$ (Figure 5.2(c)).

The normalized volume of the parallelogram $\lozenge'_\sigma$ equals $n_\theta$ and it has exactly 4 boundary lattice points. By Pick's Theorem, there exist exactly $n_\theta - 1$ lattice points in the interior of $\lozenge'_\sigma$. We claim that the affine linear function $-1 + \frac{y}{n_\theta}$ takes $n_\theta - 1$ different values on these lattice points (Figure 5.4). Assume that there exist two lattice points $P = (P_x, P_y)$ and $P' = (P'_x, P'_y)$ with $P'_x \geq P_x$ in the interior of $\lozenge'_\sigma$ with the same affine linear function value, *i.e.*, $P_y = P'_y$ and $v(\mathrm{conv}(P, P')) \geq 1$. Set $E := \mathrm{conv}((0,0),(1,0))$, then $P' - P \in E^\circ \cap N = \emptyset$. Contradiction.

Therefore, we get the set equality

$$\{\kappa_\Delta(n) \in \mathbb{Q} \setminus \mathbb{Z} \,|\, n \in \square^\circ_\sigma \cap N\} = \big\{ -1 - \frac{l}{n_\theta} \,\big|\, l \in \{1, \ldots, n_\theta - 1\}\big\}$$

and Equation (5.3) can be rewritten in the form

$$T_\sigma = \sum_{\substack{n \in \square^\circ_\sigma \cap N \\ \kappa_\Delta(n) \in \mathbb{Q}\setminus\mathbb{Z}}} (uv)^{3+\kappa_\Delta(n)} = \sum_{l=1}^{n_\theta - 1} (uv)^{3-1-\frac{l}{n_\theta}} = \sum_{l=1}^{n_\theta - 1} (uv)^{2-\frac{l}{n_\theta}} = \sum_{k=1}^{n_\theta - 1} (uv)^{1+\frac{k}{n_\theta}}.$$

Using this, the remaining last term in Equation (5.1) can be computed as

$$\begin{aligned} T = \sum_{\sigma \in \Sigma'_\Delta(3)} \sum_{\substack{n' \in \square^\circ_\sigma \cap N \\ \kappa_\Delta(n') \in \mathbb{Q}\setminus\mathbb{Z}}} (uv)^{3+\kappa_\Delta(n')} &= \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=2, n_\theta > 1}} \sum_{\sigma \cap \theta \subseteq \theta} v(\sigma \cap \theta) \cdot \left( \sum_{k=1}^{n_\theta - 1} (uv)^{1+\frac{k}{n_\theta}} \right) \\ &+ \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=2, n_\theta = 1}} \sum_{\sigma \cap \theta \subseteq \theta} 0 = \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=2, n_\theta > 1}} v(\theta) \cdot \left( \sum_{k=1}^{n_\theta - 1} (uv)^{\frac{k}{n_\theta}+1} \right) \end{aligned} \tag{5.4}$$

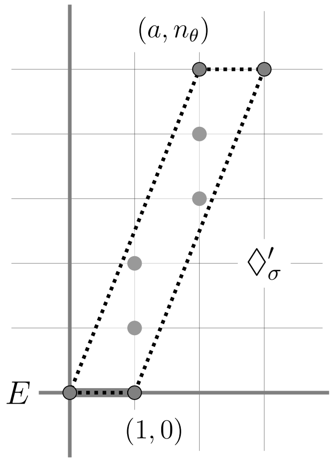



**Figure 5.4: Illustration of $\lozenge'_\sigma$ (here: $a = 2$ and $n_\theta = 5$).**

because $\sum_{\sigma\cap\theta\subseteq\theta} v(\sigma \cap \theta) = \sum_{\sigma\cap\theta\subseteq\theta} 1 = v(\theta)$.
A combination of Equation (5.2) and (5.4) yields the desired result

$$E_{\text{str}}\left(X_\Delta; u, v\right) = \big((uv)^3 + 1\big) + r \cdot \big((uv)^2 + (uv)\big) + \sum_{\substack{\theta\preceq\Delta \\ \dim(\theta)=2, n_\theta>1}} v(\theta) \cdot \left(\sum_{k=1}^{n_\theta-1} (uv)^{\frac{k}{n_\theta}+1}\right)$$

with $r = |\Delta \cap N| - 4$. $\square$

**Example 5.3.2.** Consider the lattice polytopes $\Delta_1 = \operatorname{conv}(e_1, e_2, e_3, -e_1 - e_2 - e_3)$, $\Delta_2 = \operatorname{conv}(e_1, e_2, e_3, -e_1 - e_2 - 2e_3)$, and $\Delta_3 = \operatorname{conv}(e_1, e_2, e_3, -5e_1 - 6e_2 - 8e_3)$ (Figure 2.3) with $\Delta_i^\circ \cap N = \{0\}$ $(1 \leq i \leq 3)$. Therefore, $\Delta_i$ are canonical Fano polytopes and $X_{\Delta_i} := X_{\Sigma_{\Delta_i}}$ are toric Fano threefolds with at worst canonical singularities $(1 \leq i \leq 3)$. In particular, $X_{\Delta_1} = \mathbb{P}(1,1,1,1) = \mathbb{P}^3$, $X_{\Delta_2} = \mathbb{P}(1,1,1,2)$, and $X_{\Delta_3} = \mathbb{P}(1,5,6,8)$. Using Theorem 5.3.1, the associated stringy $E$-functions are given by

$$E_{\text{str}}\left(X_{\Delta_1}; u, v\right) = (uv)^3 + (uv)^2 + (uv) + 1,$$

$$\begin{aligned} E_{\text{str}}\left(X_{\Delta_2}; u, v\right) &= \big((uv)^3 + 1\big) + 1 \cdot \big((uv)^2 + (uv)\big) + 1 \cdot \sum_{k=1}^{2-1} (uv)^{\frac{k}{2}+1} \\ &= \big((uv)^3 + 1\big) + \big((uv)^2 + (uv)\big) + (uv)^{\frac{3}{2}} \\ &= (uv)^3 + (uv)^2 + (uv)^{\frac{3}{2}} + (uv) + 1, \end{aligned}$$

and

$$E_{\text{str}}\left(X_{\Delta_3}; u, v\right) = \big((uv)^3 + 1\big) + 5 \cdot \big((uv)^2 + (uv)\big) + 4 \cdot \sum_{k=1}^{2-1} (uv)^{\frac{k}{2}+1} + 2 \cdot \sum_{k=1}^{3-1} (uv)^{\frac{k}{3}+1}$$

$$= \big((uv)^3 + 1\big) + 5 \cdot \big((uv)^2 + (uv)\big) + 4 \cdot (uv)^{\frac{3}{2}} + 2 \cdot \big((uv)^{\frac{4}{3}} + (uv)^{\frac{5}{3}}\big)$$
$$= (uv)^3 + 5 \cdot (uv)^2 + 2 \cdot (uv)^{\frac{5}{3}} + 4 \cdot (uv)^{\frac{3}{2}} + 2 \cdot (uv)^{\frac{4}{3}} + 5 \cdot (uv) + 1$$

because all facets $\theta_1$ of $\Delta_1$ have lattice distance $n_{\theta_1} = 1$ to the origin, all facets $\theta_2$ of $\Delta_2$ have lattice distance $n_{\theta_2} = 1$ to the origin except of one with $n_{\theta_2} = 2$ and $v(\theta_2) = 1$, and $\Delta_3$ has two facets with $n_{\theta_3} = 1$, one facet with $n_{\theta_3} = 2$ and $v(\theta_2) = 4$, and one facet with $n_{\theta_3} = 3$ and $v(\theta_2) = 2$.

# Chapter 6

## Applications

Our aim is to obtain five combinatorial identities relating the following classes of lattice polytopes to the number 24: reflexive polytopes of arbitrary dimension (Section 6.1), $LDP$-polygons (Section 6.2), almost reflexive 3-polytopes (Section 6.3), canonical Fano 3-polytopes (Section 6.4), and certain Gorenstein polytopes (Section 6.5).

The results of this chapter arose as joint work with Victor Batyrev. To be precise, Section 6.1, Section 6.2, and Section 6.5 have been published in the paper *Stringy Chern Classes of Singular Toric Varieties and Their Applications* [BS17]. The results of Section 6.3 and Section 6.4 are based on the preprint *Stringy E-functions of Canonical Toric Fano Threefolds and Their Applications* [BS18]. Compared to [BS17] and [BS18], this chapter contains minor modifications and additional examples.

The central elements to reach the mentioned identities are the combinatorial version of the stringy Libgober-Wood identity (Theorem 4.2.2, Equation (4.3)) and the combinatorial version of the stringy $E$-function for each polytope class (Chapter 5). The latter version specifies the generalized stringy Hodge numbers because they appear as coefficients in the stringy $E$-function (Proposition 4.2.1). Therefore, we briefly repeat the statements of these two elements: Let $X_\Sigma$ be a $d$-dimensional projective $\mathbb{Q}$-Gorenstein toric variety of Gorenstein index $q_{X_\Sigma}$ associated with a fan $\Sigma$ of rational polyhedral cones in $N_\mathbb{R}$ and $-K_{X_\Sigma} = \sum_{\rho\in\Sigma(1)} D_\rho$ an ample anticanonical torus-invariant $\mathbb{Q}$-Cartier divisor on $X_\Sigma$. The combinatorial version of the stringy Libgober-Wood identity is proved in Chapter 4 and revisited in the following paragraph:

**Theorem 4.2.2.** *Let $X_\Sigma$ and $-K_{X_\Sigma}$ be given as above. Then the stringy Libgober-Wood identity is equivalent to*

$$\sum_{\alpha\in[0,d]\cap\frac{1}{q_{X_\Sigma}}\mathbb{Z}} \psi_\alpha(\Sigma)\left(\alpha-\frac{d}{2}\right)^2 = \frac{d}{12}v(\Sigma)+\frac{1}{6}\sum_{\sigma\in\Sigma(d-1)} v(\sigma)\cdot v\left(\Delta^\sigma_{-K_{X_\Sigma}}\right),$$

*where* $\psi_\alpha(\Sigma)$ *denotes the generalized stringy Hodge numbers* (page 51), $v(\Sigma)$ *the normalized volume of the fan* $\Sigma$ (Equation (2.1)), $v(\sigma)$ *the normalized volume of a cone* $\sigma \in \Sigma$ (Definition 2.2.13), $\Delta^\sigma_{-K_{X_\Sigma}} \preceq \Delta_{-K_{X_\Sigma}}$ *a face of the polytope* $\Delta_{-K_{X_\Sigma}}$ *corresponding to a cone* $\sigma \in \Sigma$ *given as below, and* $v(\Delta^\sigma_{-K_{X_\Sigma}})$ *the normalized volume of the polytope* $\Delta^\sigma_{-K_{X_\Sigma}}$ (Definition 2.3.6).

Following Equation (4.1) and (4.2), we denote by $\Delta_{-K_{X_\Sigma}} \subseteq M_\mathbb{R}$ the $d$-dimensional polytope corresponding to the ample anticanonical divisor $-K_{X_\Sigma}$ on $X_\Sigma$ given by

$$\Delta_{-K_{X_\Sigma}} = \{y \in M_\mathbb{R} \mid \langle y, u_\rho\rangle \geq -1 \ \forall \rho \in \Sigma(1)\} \subseteq M_\mathbb{R}$$

and by $\Delta^\sigma_{-K_{X_\Sigma}} \preceq \Delta_{-K_{X_\Sigma}}$ a $k$-dimensional face of $\Delta_{-K_{X_\Sigma}}$ corresponding to a $(d-k)$-dimensional cone $\sigma \in \Sigma(d-k)$ given by

$$\Delta^\sigma_{-K_{X_\Sigma}} = \{y \in \Delta_{-K_X} \mid \langle y, u_\rho\rangle = -1 \ \forall \rho \in \Sigma(1) \text{ with } \rho \subseteq \sigma\},$$

where $u_\rho \in N$ denotes the primitive ray generator of a 1-dimensional cone $\rho \in \Sigma(1)$.

To apply the combinatorial version of the stringy Libgober-Wood identity stated in Theorem 4.2.2, we need to know the generalized stringy Hodge numbers $\psi_\alpha(\Sigma)$ concretely, *i.e.*, we have to look at the stringy $E$-function of $X_\Sigma$ written as a finite sum

$$E_{\text{str}}(X_\Sigma; u, v) = \sum_{\alpha \in [0,d] \cap \frac{1}{q_{X_\Sigma}}\mathbb{Z}} \psi_\alpha(\Sigma)(uv)^\alpha$$

(Proposition 4.2.1), where the coefficients $\psi_\alpha(\Sigma)$ are non-negative integers (the generalized stringy Hodge numbers) satisfying the conditions $\psi_0(\Sigma) = \psi_d(\Sigma) = 1$ and $\psi_\alpha(\Sigma) = \psi_{d-\alpha}(\Sigma)$ for all $\alpha \in [0,d] \cap \frac{1}{q_{X_\Sigma}}\mathbb{Z}$.

## 6.1 Reflexive Polytopes

In this section, the focus is on reflexive polytopes $\Delta \subseteq N_\mathbb{R}$ of arbitrary dimension $d$, where a reflexive polytope is a lattice polytope containing the origin in its interior such that the dual polytope $\Delta^*$ is also a lattice polytope (Definition 2.3.13 and 2.3.3).

**Theorem 6.1.1.** *Let* $\Delta \subseteq N_\mathbb{R}$ *be a reflexive d-polytope. Then the stringy Libgober-Wood identity for the Gorenstein toric Fano variety* $X_\Delta := X_{\Sigma_\Delta}$ *defined by the spanning fan* $\Sigma_\Delta$ *in* $N_\mathbb{R}$ *associated with* $\Delta$ (Theorem 2.3.14, Remark 2.3.15) *is equivalent to*

$$\sum_{\alpha \in [0,d] \cap \mathbb{Z}} \psi_\alpha(\Delta) \left(\alpha - \frac{d}{2}\right)^2 = \frac{d}{12} v(\Delta) + \frac{1}{6} \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta) = d-2}} v(\theta) \cdot v(\theta^*),$$

*where* $\psi_\alpha(\Delta)$ *are the coefficients in the numerator of the Ehrhart power series* $P_\Delta(t)$ (Definition 5.1.1) *and* $\theta^* \preceq \Delta^*$ *is a face of the dual polytope* $\Delta^*$ *and the dual face of the face* $\theta \preceq \Delta$ *of* $\Delta$ (Definition 2.3.4).

*Proof.* Using Theorem 5.1.3 and Theorem 4.2.2, we obtain

$$\sum_{\alpha \in [0,d] \cap \mathbb{Z}} \psi_\alpha(\Delta)\left(\alpha - \frac{d}{2}\right)^2 = \frac{d}{12} v(\Sigma_\Delta) + \frac{1}{6} \sum_{\sigma \in \Sigma(d-1)} v(\sigma) \cdot v\left(\Delta^{\sigma}_{-K_{X_\Delta}}\right),$$

since the anticanonical divisor $-K_{X_\Delta}$ on $X_\Delta$ is ample by Definition 2.1.3. In this case, the normalized volume of the spanning fan $\Sigma_\Delta$ (Equation (2.1), Definition 2.3.10) is given by

$$v(\Sigma_\Delta) = \sum_{\sigma \in \Sigma_\Delta(d)} v(\sigma) = v(\Delta)$$

because the primitive ray generators of a given $d$-dimensional cone $\sigma \in \Sigma_\Delta(d)$ are vertices of the reflexive polytope $\Delta$. It remains to use

$$\begin{aligned} \Delta_{-K_{X_\Delta}} &= \{y \in M_{\mathbb{R}} \mid \langle y, u_\rho \rangle \geq -1 \;\forall \rho \in \Sigma_\Delta(1)\} \\ &= \{y \in M_{\mathbb{R}} \mid \langle y, x \rangle \geq -1 \;\forall x \in \Delta\} = \Delta^* \end{aligned}$$

(Definition 2.3.4), where $u_\rho \in N$ denotes the primitive ray generator of a cone $\rho \in \Sigma(1)$. In combination with the facts that each $(d-1)$-dimensional cone $\sigma \in \Sigma_\Delta(d-1)$ is a cone over a face $\theta \preceq \Delta$ of $\Delta$ and

$$\begin{aligned} \Delta^{\sigma}_{-K_{X_\Delta}} &= \{y \in \Delta_{-K_X} \mid \langle y, u_\rho \rangle = -1 \;\forall \rho \in \Sigma(1) \text{ with } \rho \subseteq \sigma\} \\ &= \{y \in \Delta^* \mid \langle y, x \rangle = -1 \;\forall x \in \theta\} = \theta^* \end{aligned}$$

(Definition 2.3.4), we obtain

$$\sum_{\sigma \in \Sigma_\Delta(d-1)} v(\sigma) \cdot v\left(\Delta^{\sigma}_{-K_{X_\Delta}}\right) = \sum_{\substack{\theta \preceq \Delta, \\ \dim(\theta) = d-2}} v(\theta) \cdot v(\theta^*),$$

since every facet of the reflexive polytope $\Delta$ has lattice distance 1 to the origin (page 25). □

The well-known identities for reflexive polytopes in dimension 2 and 3 follow from the statement in Theorem 6.1.1:

**Corollary 6.1.2.** *Let $\Delta \subseteq N_{\mathbb{R}}$ be a reflexive 2-polytope. Then the stringy Libgober-Wood identity is equivalent to*

$$v(\Delta) + v(\Delta^*) = 12.$$

*Proof.* Using Theorem 6.1.1, we get

$$\sum_{\alpha \in [0,2] \cap \mathbb{Z}} \psi_\alpha(\Delta)(\alpha - 1)^2 = \frac{1}{6} v(\Delta) + \frac{1}{6} \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta) = 0}} v(\theta) \cdot v(\theta^*).$$

**Table 6.1: Computational Proof of Corollary 6.1.2.** Table contains: links to pictures of all reflexive 2-polytopes $\Delta$, links to pictures of dual reflexive 2-polytopes $\Delta^*$, and their normalized volumes $v(\Delta)$ and $v(\Delta^*)$, respectively. The last column confirms the statement of the corollary.

| $\Delta$ | $\Delta^*$ | $v(\Delta)$ | $v(\Delta^*)$ | $v(\Delta)+v(\Delta^*)$ |
|---|---|---|---|---|
| Figure 6.1(a) | Figure 6.1(k) | 3 | 9 | 12 |
| Figure 6.1(b) | Figure 6.1(l) | 4 | 8 | 12 |
| Figure 6.1(c) | Figure 6.1(m) | 4 | 8 | 12 |
| Figure 6.1(d) | Figure 6.1(n) | 4 | 8 | 12 |
| Figure 6.1(e) | Figure 6.1(o) | 5 | 7 | 12 |
| Figure 6.1(f) | Figure 6.1(p) | 5 | 7 | 12 |
| Figure 6.1(g) | Figure 6.1(g) | 6 | 6 | 12 |
| Figure 6.1(h) | Figure 6.1(h) | 6 | 6 | 12 |
| Figure 6.1(i) | Figure 6.1(i) | 6 | 6 | 12 |
| Figure 6.1(j) | Figure 6.1(j) | 6 | 6 | 12 |
| Figure 6.1(k) | Figure 6.1(a) | 9 | 3 | 12 |
| Figure 6.1(l) | Figure 6.1(b) | 8 | 4 | 12 |
| Figure 6.1(m) | Figure 6.1(c) | 8 | 4 | 12 |
| Figure 6.1(n) | Figure 6.1(d) | 8 | 4 | 12 |
| Figure 6.1(o) | Figure 6.1(e) | 7 | 5 | 12 |
| Figure 6.1(p) | Figure 6.1(f) | 7 | 5 | 12 |

Moreover,

$$\sum_{\alpha\in[0,2]\cap\mathbb{Z}} \psi_\alpha(\Delta)(\alpha-1)^2 = \psi_0(\Delta)(0-1)^2+\psi_1(\Delta)(1-1)^2+\psi_2(\Delta)(2-1)^2$$
$$= \psi_0(\Delta)+\psi_2(\Delta) = 2\psi_0(\Delta) = 2$$

because $\psi_0(\Delta)=\psi_2(\Delta)=1$ (Remark 5.1.2). It remains to apply the equalities

$$\sum_{\substack{\theta\preceq\Delta\\ \dim(\theta)=0}} v(\theta)\cdot v(\theta^*) = \sum_{\substack{\theta\preceq\Delta\\ \dim(\theta)=0}} v(\theta^*) = v(\Delta^*)$$

that hold because $\Delta^*$ is reflexive (page 25) and $v(\theta)=1$ if $\dim(\theta)=0$. $\square$

**Example 6.1.3.** Using a remark on page 25, a lattice polygon $\Delta \subseteq N_{\mathbb{R}}$ containing the origin in its interior is reflexive if and only if $\Delta^\circ \cap N = \{0\}$. Because there exist only 16 isomorphism classes of reflexive 2-polytopes (Figure 6.1), we prove Corollary 6.1.2 by checking if the identity holds for each reflexive polygon. Doing this, we get the computation results listed in Table 6.1.

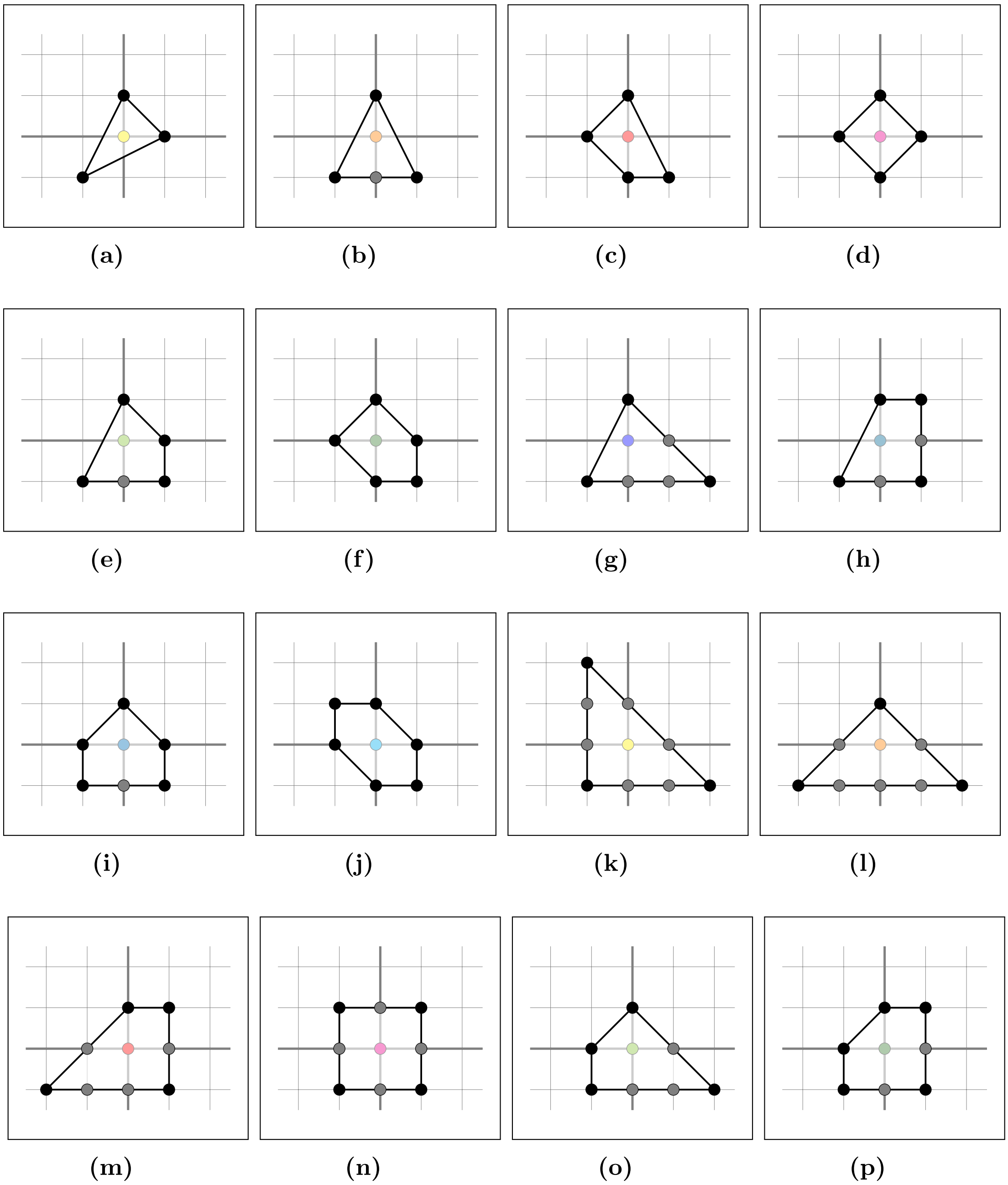



**Figure 6.1: 16 Reflexive Polygons.** Dual polygons have the same origin color. Vertices are coloured black and boundary points that are not vertices are coloured grey.

**Corollary 6.1.4.** *Let $\Delta \subseteq N_{\mathbb{R}}$ be a reflexive 3-polytope. Then the stringy Libgober-Wood identity is equivalent to*

$$\sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=1}} v(\theta) \cdot v(\theta^*) = 24.$$

*Proof.* Theorem 6.1.1 implies

$$\sum_{\alpha \in [0,3] \cap \mathbb{Z}} \psi_\alpha(\Delta) \left(\alpha - \frac{3}{2}\right)^2 = \frac{1}{4} v(\Delta) + \frac{1}{6} \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=1}} v(\theta) \cdot v(\theta^*).$$

The coefficients $\psi_\alpha(\Delta)$ in the numerator of the Ehrhart power series $P_\Delta(t)$ are $\psi_0(\Delta) = \psi_3(\Delta) = 1$ and $\psi_1(\Delta) = \psi_2(\Delta)$ (Remark 5.1.2), *i.e.*,

$$\begin{aligned}
\sum_{\alpha \in [0,3] \cap \mathbb{Z}} \psi_\alpha(\Delta) \left(\alpha - \frac{3}{2}\right)^2 &= \psi_0(\Delta) \left(0 - \frac{3}{2}\right)^2 + \psi_1(\Delta) \left(1 - \frac{3}{2}\right)^2 \\
&\quad + \psi_2(\Delta) \left(2 - \frac{3}{2}\right)^2 + \psi_3(\Delta) \left(3 - \frac{3}{2}\right)^2 \\
&= \frac{9}{4}\psi_0(\Delta) + \frac{1}{4}\psi_1(\Delta) + \frac{1}{4}\psi_2(\Delta) + \frac{9}{4}\psi_3(\Delta) \\
&= \frac{9}{2}\psi_0(\Delta) + \frac{1}{2}\psi_1(\Delta) = \frac{9}{2} + \frac{1}{2}\psi_1(\Delta).
\end{aligned}$$

This provides

$$\sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=1}} v(\theta) \cdot v(\theta^*) = 27 + 3\Big(\psi_1(\Delta) - \frac{1}{2} v(\Delta)\Big) = 24,$$

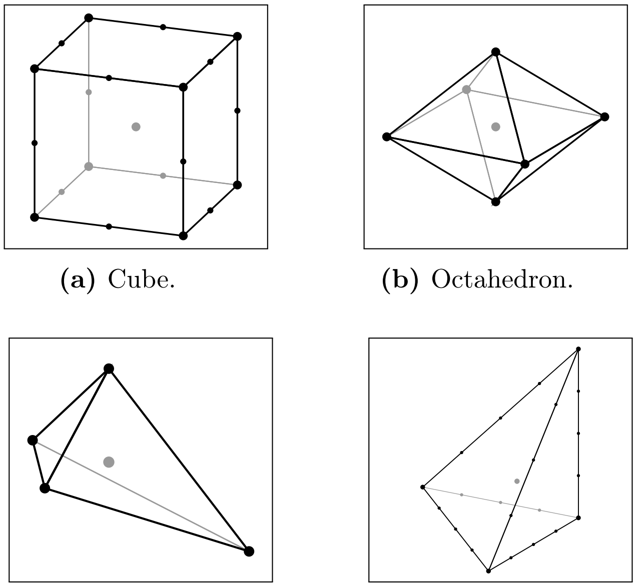
where the last equality holds because

$$v(\Delta) = v(\Sigma) = e_{\mathrm{str}}(X) = \sum_{\alpha \in [0,3] \cap \mathbb{Z}} \psi_\alpha(\Delta)$$

(Theorem 6.1.1 (proof), Corollary 4.1.3, and Equation (1.4)) and therefore

$$\begin{aligned}
\psi_1(\Delta) - \frac{1}{2} v(\Delta) &= \psi_1(\Delta) - \frac{1}{2} \sum_{\alpha \in [0,3] \cap \mathbb{Z}} \psi_\alpha(\Delta) \\
&= \psi_1(\Delta) - \frac{1}{2}(\psi_0(\Delta) + \psi_1(\Delta) + \psi_2(\Delta) + \psi_3(\Delta)) \\
&= \psi_1(\Delta) - \frac{1}{2}(2 + 2\psi_1(\Delta)) \\
&= \psi_1(\Delta) - 1 - \psi_1(\Delta) = -1.
\end{aligned}$$

□



**(a)** Cube. **(b)** Octahedron.

**(c)** 3-simplex. **(d)** Tetrahedron.

**Figure 6.2: Reflexive 3-polytopes.** Shaded faces are occluded. ***(a)*** The cube is a reflexive 3-polytope given as the convex hull of $(1,1,1)$, $(1,1,-1)$, $(-1,1,-1)$, $(-1,1,1)$, $(-1,-1,-1)$, $(-1,-1,1)$, $(1,-1,1)$, $(1,-1,-1)$ and dual to $(b)$. ***(b)*** The octahedron is a reflexive 3-polytope given as the convex hull of $(1,1,0)$, $(-1,1,0)$, $(-1,-1,0)$, $(1,-1,0)$, $(0,0,1)$, $(0,0,-1)$ and dual to $(a)$. ***(c)*** The 3-simplex is a reflexive 3-polytope given as the convex hull of $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(-1,-1,-1)$ and dual to $(d)$. ***(d)*** The tetrahedron is a reflexive 3-polytope given as the convex hull of $(-1,3,-1)$, $(-1,-1,-1)$, $(3,-1,-1)$, $(-1,-1,3)$ and dual to $(c)$.

**Table 6.2: Computational Examples for Corollary 6.1.4.** Table contains: links to pictures of reflexive 3-polytopes $\Delta$, links to pictures of dual reflexive 3-polytopes $\Delta^*$, and tuples $(v(\theta))$ and $(v(\theta^*))$ containing the normalized volumes of all 1-dimensional faces $\theta \preceq \Delta$ of $\Delta$ and $\theta^* \preceq \Delta^*$ of $\Delta^*$, respectively. The last column confirms the statement of the corollary.

| $\Delta$ | $\Delta^*$ | $(v(\theta))$ | $(v(\theta^*))$ | $\sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=1}} v(\theta) \cdot v(\theta^*)$ |
|---|---|---|---|---|
| Figure 6.2(a) | Figure 6.2(b) | (2,. . .,2) | (1,. . .,1) | 24 |
| Figure 6.2(b) | Figure 6.2(a) | (1,. . .,1) | (2,. . .,2) | 24 |
| Figure 6.2(c) | Figure 6.2(d) | (1,1,1,1,1,1) | (4,4,4,4,4,4) | 24 |
| Figure 6.2(d) | Figure 6.2(c) | (4,4,4,4,4,4) | (1,1,1,1,1,1) | 24 |

**Example 6.1.5.** We check the statement of Corollary 6.1.4 for a small selection of all 4,319 reflexive 3-polytopes $\Delta$ listed in Figure 6.2, where the detailed polytope data can be found in Example 5.1.4. The obtained results are listed in Table 6.2.

**Remark 6.1.6.** Gorenstein toric Fano varieties $X$ of dimension $d$ associated with reflexive polytopes $\Delta$ are used in mirror symmetry as ambient spaces for Calabi-Yau hypersurfaces [Bat94, BB97, BD96]. Using Corollary 4.1.8 and the proof of Theorem 6.1.1, the stringy Euler number of a generic ample Calabi-Yau hypersurface $Z$ in $X$ is combinatorially computable via

$$e_{\mathrm{str}}(Z) = c_{d-1}^{\mathrm{str}}(Z) = \sum_{k=0}^{d-3}(-1)^k \sum_{\substack{\theta\preceq\Delta\\ \dim(\theta)=k+1}} v(\theta)\cdot v(\theta^*)$$

[BD96, Corollary 7.10]. If $\Delta \subseteq N_{\mathbb{R}}$ is a reflexive 3-polytope, then the stringy Euler number of a generic ample $K3$-surface $Z$ in the associated Gorenstein toric variety $X$ is 24 (Theorem 6.3.1 (proof)) and one obtains the identity of Corollary 6.1.4

$$24 = e_{\mathrm{str}}(Z) = \sum_{k=0}^{0}(-1)^k \sum_{\substack{\theta\preceq\Delta\\ \dim(\theta)=k+1}} v(\theta)\cdot v(\theta^*) = \sum_{\substack{\theta\preceq\Delta\\ \dim(\theta)=1}} v(\theta)\cdot v(\theta^*)\,.$$

If the dimension $d$ of a reflexive polytope is greater than 3, the identity in Theorem 6.1.1 is not anymore a symmetric equation with respect to the duality between $\Delta$ and $\Delta^*$. The received identities for reflexive polytopes $\Delta$ and $\Delta^*$ of dimension $\geq 4$ are not equivalent to each other. The latter is easy to see in dimension $d = 4$:

**Corollary 6.1.7.** *Let $\Delta \subseteq N_{\mathbb{R}}$ be a reflexive 4-polytope. Then the stringy Libgober-Wood identity is equivalent to*

$$12\cdot|\partial\Delta\cap N| = 2\cdot v(\Delta) + \sum_{\substack{\theta\preceq\Delta\\ \dim(\theta)=2}} v(\theta)\cdot v(\theta^*)\,,$$

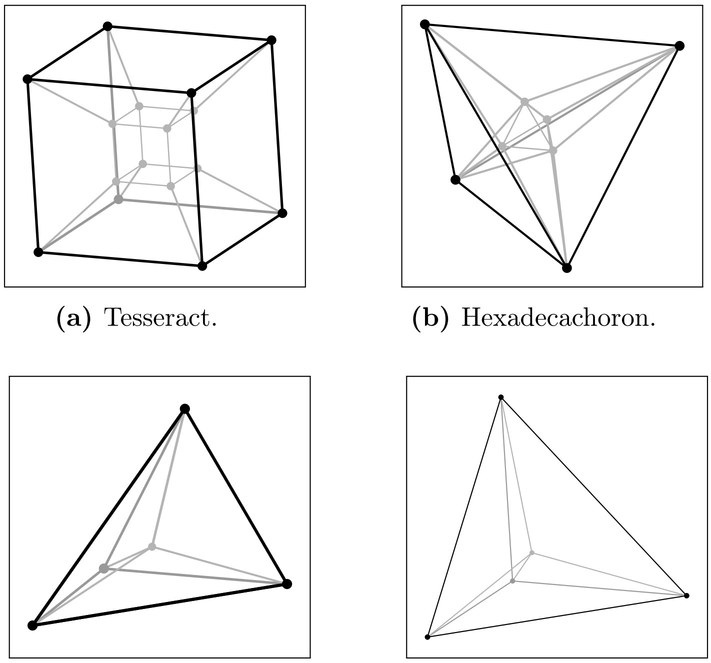
*where $\partial\Delta$ denotes the boundary of $\Delta$ and $|\partial\Delta\cap N|$ the number of lattice points on $\partial\Delta$.*

*Proof.* By Theorem 6.1.1, we have

$$\sum_{\alpha\in[0,4]\cap\mathbb{Z}} \psi_\alpha(\Delta)(\alpha-2)^2 = \frac{1}{3}v(\Delta) + \frac{1}{6}\sum_{\substack{\theta\preceq\Delta\\ \dim(\theta)=2}} v(\theta)\cdot v(\theta^*)\,.$$

Furthermore, $\psi_0(\Delta) = \psi_4(\Delta) = 1$ and $\psi_1(\Delta) = \psi_3(\Delta) = |\Delta\cap N| - 5$ (Remark 5.1.2), *i.e.*, we obtain

$$\sum_{\alpha\in[0,4]\cap\mathbb{Z}} \psi_\alpha(\Delta)(\alpha-2)^2 + \psi_2(\Delta)(2-2)^2 = \psi_0(\Delta)(0-2)^2 + \psi_1(\Delta)(1-2)^2 + \psi_3(\Delta)(3-2)^2 + \psi_4(\Delta)(4-2)^2$$



**(a)** Tesseract. **(b)** Hexadecachoron.

**(c)** 4-simplex. **(d)** Pentachoron.

**Figure 6.3: Reflexive 4-polytopes.** ***(a)*** The Tesseract is a reflexive 4-polytope given as the convex hull of $\pm e_1 \pm e_2 \pm e_3 \pm e_4$ and dual to $(b)$. ***(b)*** The hexadecachoron is a reflexive 4-polytope given as the convex hull of $\pm e_1$, $\pm e_2$, $\pm e_3$, $\pm e_4$ and dual to $(a)$. ***(c)*** The 4-simplex is a reflexive 4-polytope given as the convex hull of $e_1$, $e_2$, $e_3$ ,$e_4$, $-(e_1+e_2+e_3+e_4)$ and dual to $(d)$. ***(d)*** The pentachoron is a reflexive 4-polytope given as $5 \cdot \operatorname{conv}(e_1, e_2, e_3, e_4, 0) - (e_1+e_2+e_3+e_4)$ and dual to $(c)$.

$$
\begin{aligned}
&= 4\psi_0(\Delta) + \psi_1(\Delta) + \psi_3(\Delta) + 4\psi_4(\Delta) \\
&= 8\psi_0(\Delta) + 2\psi_1(\Delta) = 8 + 2 \cdot (|\Delta \cap N| - 5)\,.
\end{aligned}
$$

It remains to apply $|\partial\Delta \cap N| = |\Delta \cap N| - 1$ because a reflexive polytope $\Delta$ has exactly one interior lattice point (page 25) and therefore

$$
\begin{aligned}
8 + 2 \cdot (|\Delta \cap N| - 5) &= 8 + 2 \cdot (|\partial\Delta \cap N| + 1 - 5) \\
&= 8 + 2 \cdot (|\partial\Delta \cap N| - 4) = 2 \cdot |\partial\Delta \cap N|\,.
\end{aligned}
$$

A multiplication by 6 yields

$$
12 \cdot |\partial\Delta \cap N| = 6 \cdot \frac{1}{3} v(\Delta) + 6 \cdot \frac{1}{6} \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=2}} v(\theta) \cdot v(\theta^*)\,.
$$

$\square$

**Table 6.3: Computational Examples for Corollary 6.1.7.** Table contains: links to pictures of reflexive 4-polytopes $\Delta$, links to pictures of dual reflexive 4-polytopes $\Delta^*$, and the last column confirms the statement of the corollary.

| $\Delta$ | $\Delta^*$ | $12 \cdot \lvert\partial\Delta \cap N\rvert$ |
|---|---|---|
| Figure 6.3(a) | Figure 6.3(b) | $12 \cdot (16 + 32 + 24 + 8) = 960$ |
| Figure 6.3(b) | Figure 6.3(a) | $12 \cdot 8 = 96$ |
| | | $2 \cdot v\,(\Delta) + \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=2}} v(\theta) \cdot v\,(\theta^*)$ |
| Figure 6.3(a) | Figure 6.3(b) | $2 \cdot (4! \cdot 2^4) + 24 \cdot (2! \cdot 2^2)(1! \cdot 1) = 960$ |
| Figure 6.3(b) | Figure 6.3(a) | $2 \cdot (4! \cdot 2^4 \cdot \frac{1}{4!}) + 32 \cdot (2! \cdot \frac{1}{2})(1! \cdot 2) = 96$ |

| $\Delta$ | $\Delta^*$ | $12 \cdot \lvert\partial\Delta \cap N\rvert$ |
|---|---|---|
| Figure 6.3(c) | Figure 6.3(d) | $12 \cdot 5 = 60$ |
| Figure 6.3(d) | Figure 6.3(c) | $12 \cdot 125 = 1500$ |
| | | $2 \cdot v\,(\Delta) + \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=2}} v(\theta) \cdot v\,(\theta^*)$ |
| Figure 6.3(c) | Figure 6.3(d) | $2 \cdot (5! \cdot \frac{1}{4!}) + 10 \cdot (2! \cdot \frac{1}{2!})(1! \cdot 5) = 60$ |
| Figure 6.3(d) | Figure 6.3(c) | $2 \cdot (5^4 \cdot 1) + 10 \cdot (5^2 \cdot 1)(1! \cdot 1) = 1500$ |

**Example 6.1.8.** Let us choose a small selection of all 473,800,776 reflexive 4-polytopes $\Delta$ presented in Figure 6.3 to check that the identity of Corollary 6.1.7 holds, where the detailed polytope data can be found in Example 5.1.4. The computation results are listed in Table 6.3.

One may produce a more 'mirror symmetric' identity for arbitrary reflexive 4-polytopes by summing the equations from Corollary 6.1.7 for $\Delta$ and $\Delta^*$.

**Corollary 6.1.9.** *Let $\Delta \subseteq N_{\mathbb{R}}$ be a reflexive 4-polytope. Then*

$$12 \cdot (\lvert\partial\Delta \cap N\rvert + \lvert\partial\Delta^* \cap M\rvert) = 2 \cdot (v\,(\Delta) + v\,(\Delta^*)) + \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=1,2}} v(\theta) \cdot v\,(\theta^*)\,.$$

*Proof.* Corollary 6.1.7 yields

$$12 \cdot \lvert\partial\Delta \cap N\rvert = 2 \cdot v\,(\Delta) + \sum_{\theta \preceq \Delta,\, \dim(\theta)=2} v(\theta) \cdot v\,(\theta^*)\,.$$

A role reversal of $\Delta$ and $\Delta^*$, *i.e.*, applying Corollary 6.1.7 to $\Delta^*$ adds up to

$$12 \cdot \lvert\partial\Delta^* \cap M\rvert = 2 \cdot v\,(\Delta^*) + \sum_{\theta^* \preceq \Delta^*,\, \dim(\theta^*)=2} v\,(\theta^*) \cdot v\,(\theta)$$

because $\Delta^*$ is also a reflexive 4-polytope. Connecting these two equations provides

$$12 \cdot (|\partial\Delta \cap N| + |\partial\Delta^* \cap M|) = 2 \cdot (v(\Delta^*) + v(\Delta)) + \left( \sum_{\theta \preceq \Delta,\, \dim(\theta)=2} v(\theta) \cdot v(\theta^*) + \sum_{\theta \preceq \Delta,\, \dim(\theta)=1} v(\theta^*) \cdot v(\theta) \right).$$

□

## 6.2 $LDP$-polygons

This section treats $LDP$-polygons $\Delta \subseteq N_{\mathbb{R}}$, where a $LDP$-polygon is a lattice 2-polytope containing the origin in its interior such that all vertices are primitive lattice points (Definition 2.3.18).

We briefly recap that a normal projective surface is a log del Pezzo surface if it has at worst log-terminal singularities and if its anticanonical divisor is an ample $\mathbb{Q}$-Cartier divisor (Definition 2.1.2). Moreover, toric log del Pezzo surfaces one-to-one correspond to $LDP$-polygons (page 26). The fan $\Sigma$ defining a toric log del Pezzo surface $X$ is the spanning fan $\Sigma_\Delta$ of the corresponding $LDP$-polygon $\Delta$ and consists of cones over faces of $\Delta$ (Definition 2.3.10). In particular, any $LDP$-polygon $\Delta$ is the convex hull of all primitive ray generators of 1-dimensional cones of $\Sigma(1)$. We note that in general the dual polygon $\Delta^* \subseteq M_{\mathbb{R}}$ is a rational and not a lattice polygon (Definition 2.3.3).

We present a combinatorial identity that is equivalent to the stringy Libgober-Wood identity (4.3) and relates $LDP$-polygons $\Delta$ to the number 12:

**Theorem 6.2.1.** *Let $\Delta \subseteq N_{\mathbb{R}}$ be a $LDP$-polygon. Then the stringy Libgober-Wood identity for the toric log del Pezzo surface $X_\Delta := X_{\Sigma_\Delta}$ defined by the spanning fan $\Sigma_\Delta$ in $N_{\mathbb{R}}$ associated with $\Delta$ is equivalent to*

$$v(\Delta) + v(\Delta^*) = 12 \sum_{n \in \Delta \cap N} (\kappa_\Delta(n) + 1)^2,$$

*where $\kappa_\Delta$ the $\Sigma_\Delta$-piecewise linear function corresponding to the anticanonical divisor $-K_{X_\Delta}$ on $X_\Delta$* (Equation (2.3))*. In particular, one always has $v(\Delta) + v(\Delta^*) \geq 12$ and equality holds if and only if $\Delta$ is a reflexive polygon.*

*Proof.* We use the formula for the stringy $E$-function from Proposition 5.2.1 and obtain

$$E_{\text{str}}(X_\Delta; u, 1) = (u-1)^2 + \sum_{\substack{n \in N \\ \kappa_\Delta(n) = -1}} u + \sum_{\substack{n \in N \\ -1 < \kappa_\Delta(n) < 0}} \left( u^{2+\kappa_\Delta(n)} + u^{-\kappa_\Delta(n)} \right).$$

Therefore,

$$\frac{d^2}{du^2} E_{\text{str}}(X_\Delta; u, 1) \Big|_{u=1} = 2 + \sum_{0 \neq n \in \Delta^\circ \cap N} ((2 + \kappa_\Delta(n))(1 + \kappa_\Delta(n)))$$

$$
\begin{aligned}
&+ (-\kappa_\Delta(n))(-\kappa_\Delta(n)-1)) \\
&= 2 \sum_{n\in\Delta^\circ\cap N} (\kappa_\Delta(n)+1)^2 = 2 \sum_{n\in\Delta\cap N} (\kappa_\Delta(n)+1)^2,
\end{aligned}
$$

where $\Delta^\circ$ denotes the interior of the polygon $\Delta$. By Equation (4.3), we get the equality

$$
2 \sum_{n\in\Delta\cap N} (\kappa_\Delta(n)+1)^2 = \frac{1}{6} c_2^{\mathrm{str}}(X_\Delta) + \frac{1}{6} c_1(X_\Delta)^2 = \frac{1}{6}(v(\Delta)+v(\Delta^*))
$$

because $c_2^{\mathrm{str}}(X_\Delta) = v(\Sigma_\Delta) = v(\Delta)$ and $c_1(X_\Delta)^2 = v(\Delta^*)$. Moreover, one has

$$
\sum_{n\in\Delta\cap N} (\kappa_\Delta(n)+1)^2 \geq 1
$$

because the origin is contained in $\Delta$ with $\kappa_\Delta(0) = 0$ and all boundary lattice points $n \in \Delta \cap N$ play no role because $\kappa_\Delta(n) = -1$. Furthermore, all remaining lattice points in $\Delta \cap N$ contribute with positive values because of the square. Equality holds, if and only if the origin is the unique interior lattice point in $\Delta$. In dimension 2, this is equivalent to the condition that $\Delta$ is a reflexive polygon (page 25). □

**Example 6.2.2.** Let us consider a family $\Delta_m \subseteq N_\mathbb{R}$ of *LDP*-polygons depending on $m \in \mathbb{N}$ (Figure 6.4(a)) and defined as

$$
\Delta_m := \mathrm{conv}(e_1, e_2, -e_1 - m e_2).
$$

The dual polytope $\Delta_m^*$ (Figure 6.4(c)) is given by

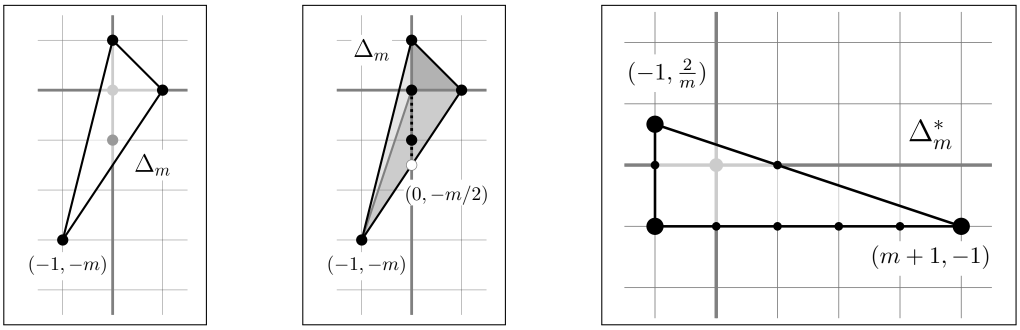
$$
\Delta_m^* = \mathrm{conv}((m+1,-1),(-1,2/m),(-1,-1)).
$$

To check that the identity of Theorem 6.2.1 holds, we subdivide for the left side $\Delta_m$ in 3 triangles (highlighted in three different shapes of grey in Figure 6.4(b)) and compute their normalized volumes to get the normalized volume of $\Delta_m$. In addition, we compute the normalized volume of $\Delta_m^*$ using again the formula 1/2 times base times height:

$$
\begin{aligned}
v(\Delta_m) + v\left(\Delta_m^*\right) &= 2! \cdot \left(\frac{1}{2}\cdot 1 \cdot 1 + \frac{1}{2}\cdot 1\cdot 1 + \frac{1}{2}\cdot 1 \cdot m\right) + 2! \cdot \left(\frac{1}{2}\left(1+\frac{2}{m}\right)(m+2)\right) \\
&= 2\cdot(1+m) + 2\cdot\left(\frac{1}{2}\left(m+4+\frac{4}{m}\right)\right) = 2m+6+\frac{4}{m}.
\end{aligned}
$$

To compute the right side, we have to consider the $\kappa_{\Delta_m}$-values of all lattice points $n$ in $\Delta_m$ (Figure 6.4(b)). Starting with the origin, we get $\kappa_{\Delta_m}(0) = 0$. All lattice points on the boundary have $\kappa_{\Delta_m}$-value $-1$. The remaining lattice points in $\Delta_m$ will appear on a line through the origin and $(0,-m/2)$ (dotted in Figure 6.4(b)). Because $(0,-m/2)$ lies on the boundary of $\Delta_m$, we know that $\kappa_{\Delta_m}$ as value $-1$ on it. Using this, we get the still pending $\kappa_{\Delta_m}$-values by a kind of back computation:

$$
12 \sum_{n\in\Delta\cap N} (\kappa_{\Delta_m}(n)+1)^2 = 12 \sum_{0\leq n\leq \frac{m}{2}} (\kappa_{\Delta_m}(0,n)+1)^2
$$




**(a)** **(b)** **(c)**

**Figure 6.4: *LDP*-polygon and its dual.** We illustrate the identity of Theorem 6.2.1 for a family of $LDP$-polygons: ***(a)*** The $LDP$-polygon $\Delta_m$ ($m \in \mathbb{N}$, here: $m = 3$) is given as the convex hull of $e_1$, $e_2$, and $-e_1 - me_2$. The lattice points in $\Delta_m$ are the origin (light grey dot), the three vertices (black dots), and one interior lattice point (grey dot). ***(b)*** To compute the normalized volume of $\Delta_m$, we subdivide the polytope in three triangles coloured in dark grey, grey, and light grey. $\kappa_{\Delta_m}$ has value 0 at the origin and $-1$ on the boundary, *i.e.*, in particular at $(0, -m/2)$. ***(c)*** The dual $LDP$-polygon $\Delta_m^*$ is given as the convex hull of $-e_1 - e_2$, $(m+1)e_1 - e_2$, and $-e_1 + 2/me_2$. The normalized volume of $\Delta_m^*$ is directly computable.

$$= 12 \sum_{0 \leq n \leq \frac{m}{2}} \left(-\frac{2}{m}n + 1\right)^2 = 2m + 6 + \frac{4}{m}.$$

By comparing both sides, the identity from Theorem 6.2.1 is confirmed for the whole family of $LDP$-polygons $\Delta_m$.

There is a close link between this example and Example 2.2.14 including Figure 2.1 because the normalized volume $v(\Delta_m)$ of $\Delta_m$ equals the normalized volume $v(\Sigma_{\Delta_m})$ of the spanning fan $\Sigma_{\Delta_m} = \Sigma_m$ (cf. Theorem 6.1.1 (proof)).

## 6.3 Almost Reflexive 3-polytopes

The objects of consideration are almost pseudoreflexive 3-polytopes $\Delta \subseteq M_{\mathbb{R}}$, *i.e.*, lattice 3-polytopes containing the origin in its interior such that the convex hull $[\Delta^*] = \mathrm{conv}(\Delta^* \cap N)$ also contains the origin in its interior and both polytopes are 3-dimensional (Definition 2.3.22). In particular, in dimension 3, being almost pseudoreflexive is the same than being almost reflexive (Remark 2.3.26).

Our aim is to introduce a combinatorial identity that is equivalent to the stringy Libgober-Wood identity (4.3) and relates almost reflexive 3-polytopes to the number 24:

**Theorem 6.3.1.** *Let $\Delta \subseteq M_{\mathbb{R}}$ be an almost reflexive 3-polytope. Then*

$$24 = v(\Delta) - \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=2}} \frac{1}{n_\theta} \cdot v(\theta) + \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=1}} v(\theta) \cdot v(\theta^*),$$

*where $n_\theta$ denotes the lattice distance from a face $\theta \preceq \Delta$ of $\Delta$ to the origin* (page 22).

To be able to prove this identity, we have to do preliminary work: The stringy Euler number $e_{\text{str}}(Y)$ of certain Calabi-Yau hypersurfaces $Y$ can be computed using the combinatorial formula in Theorem 6.3.2 that has been derived by Batyrev in [Bat17]. To be precise, $Y$ is a canonical Calabi-Yau model (Definition 2.1.8) of an affine $\Delta$-non-degenerate hypersurface $Z_\Delta \subseteq \mathbb{T}^d$ (Definition 2.3.28) corresponding to a $d$-dimensional almost pseudoreflexive polytope $\Delta \subseteq M_{\mathbb{R}}$, where $Z_\Delta$ is the zero set of a Laurent polynomial

$$f_\Delta(x) = \sum_{m \in \Delta \cap M} a_m x^m$$

with sufficiently general coefficients $a_m \in \mathbb{C}$ in $\mathbb{T}^d$ (Definition 2.3.28).

**Theorem 6.3.2 [Bat17, Theorem 4.11].** *Let $\Delta \subseteq M_{\mathbb{R}}$ be an almost pseudoreflexive $d$-polytope. Denote by $Y$ a canonical Calabi-Yau model of a $\Delta$-non-degenerate affine hypersurface $Z_\Delta$ in the $d$-dimensional algebraic torus $\mathbb{T}^d$. Then the stringy Euler number of $Y$ can be computed with the combinatorial formula*

$$e_{\text{str}}(Y) = \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta) \geq 1}} (-1)^{\dim(\theta)-1} v(\theta) \cdot v(\sigma^\theta \cap \Delta^*),$$

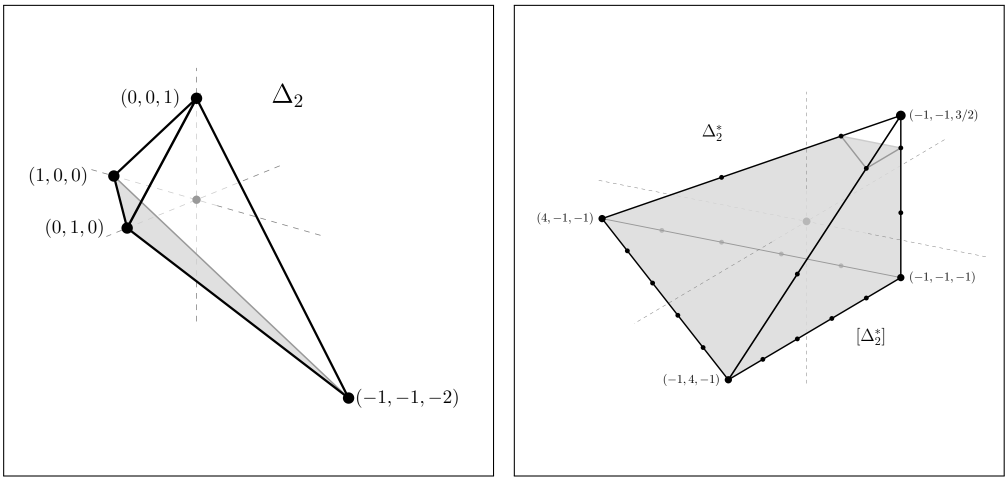
*where $\sigma^\theta = \mathbb{R}_{\geq 0}\theta^*$ is the cone in the normal fan $\Sigma^\Delta$ of $\Delta$ corresponding to the dual face $\theta^* \preceq \Delta^*$ of the dual polytope $\Delta^* \subseteq N_{\mathbb{R}}$* (Definition 2.3.8 and 2.3.3) *for a face $\theta \preceq \Delta$ of $\Delta$.*

The combinatorial formula for the stringy Euler number in Theorem 6.3.2 can be rewritten in the following equivalent form:

**Theorem 6.3.3.** *Let $\Delta \subseteq M_{\mathbb{R}}$ be an almost pseudoreflexive $d$-polytope. Denote by $Y$ a canonical Calabi-Yau model of a $\Delta$-non-degenerate affine hypersurface $Z_\Delta \subseteq \mathbb{T}^d$. Then the stringy Euler number of $Y$ can be computed with the combinatorial formula*

$$e_{\text{str}}(Y) = v(\Delta) - \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=d-1}} \frac{1}{n_\theta} \cdot v(\theta) + \sum_{\substack{\theta \preceq \Delta \\ 1 \leq \dim(\theta) \leq d-2}} (-1)^{\dim(\theta)-1} v(\theta) \cdot v(\theta^*).$$

*Proof.* Let $\theta \preceq \Delta$ be a face of $\Delta$. If $\theta = \Delta$, then $\sigma^\theta \cap \Delta^* = \{0\}$ and $v(\sigma^\theta \cap \Delta^*) = 1$. If $\theta \preceq \Delta$ is a $(d-1)$-dimensional face of $\Delta$ and $v_\theta$ denotes the primitive inward-pointing facet normal of $\theta$, then $\sigma^\theta \cap \Delta^* = \text{conv}(0, 1/n_\theta \cdot v_\theta)$ and $v(\sigma^\theta \cap \Delta^*) = 1/n_\theta$. If $\theta \preceq \Delta$ is face of $\Delta$ of dimension $k$ $(1 \leq k \leq d-2)$, then $\sigma^\theta \cap \Delta^*$ is a $(d-k)$-dimensional pyramid with vertex $0 \in N$ over the $(d-k-1)$-dimensional dual face $\theta^* \preceq \Delta^*$. Using the definition of the dual polytope $\Delta^*$, the lattice distance between 0 and $\theta^*$ equals 1 and implies $v(\sigma^\theta \cap \Delta^*) = v(\theta^*)$. Now, it remains to apply the formula of Theorem 6.3.2. □




**(a)** **(b)**

**Figure 6.5: Almost Reflexive Polytope $\boldsymbol{\Delta_2}$ and its dual.** Shaded faces are occluded. ***(a)*** The almost reflexive polytope $\Delta_2$ is given as the convex hull of $e_1$, $e_2$, $e_3$, and $-e_1 - e_2 - 2e_3$. The grey coloured facet of $\Delta_2$ has lattice distance 2 and all other facets have lattice distance 1 to the origin. ***(b)*** The whole polytope is the dual polytope $\Delta_2^*$ given as the convex hull of $(4, -1, -1)$, $(-1, 4, -1)$, $(-1, -1, 3/2)$, and $(-1, -1, -1)$. The grey coloured polytope is $[\Delta_2^*]$ given as the convex hull of $(4, -1, -1)$, $(-1, 4, -1)$, $(-1, -1, -1)$, $(-1, -1, 1)$, $(-1, 0, 1)$, and $(0, -1, 1)$.

We apply the last theorem in the case $d = 3$ to prove the combinatorial identity for the Euler number 24 stated at the beginning of this section:

*Proof of Theorem 6.3.1.* A canonical Calabi-Yau model $Y$ of a $\Delta$-non-degenerate affine hypersurface $Z_\Delta \subseteq \mathbb{T}^3$ is a $K3$-surface with at worst canonical singularities (Proposition 2.3.29). Its minimal crepant desingularization $\tilde{Y}$ is a smooth $K3$-surface (Theorem 2.1.11 and Equation (1.5)) and by Noether's Theorem [CLS11, Theorem 10.5.3]

$$2 = 1 + 0 + 1 = \chi(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) = \frac{1}{12}(c_1(\tilde{Y})^2 + c_2(\tilde{Y})) = \frac{1}{12} c_2(\tilde{Y})$$

the stringy Euler number $e_{\mathrm{str}}(Y)$ equals $c_2(\tilde{Y}) = 24$. Using the formula for $e_{\mathrm{str}}(Y)$ from Theorem 6.3.3, we get the desired identity. □

**Example 6.3.4.** Consider the almost reflexive polytope $\Delta_2 = \mathrm{conv}(e_1, e_2, e_3, -e_1 - e_2 - 2e_3)$ (Figure 6.5(a)). Its dual polytope $\Delta_2^*$ is rational:

$$\Delta_2^* = \mathrm{conv}((4, -1, -1), (-1, 4, -1), (-1, -1, 3/2), (-1, -1, -1))$$

(Figure 6.5(b)), *i.e.*, $\Delta_2$ is not reflexive. There exists exactly one 2-dimensional face $\theta \preceq \Delta_2$ of $\Delta_2$ (highlighted in grey in Figure 6.5(a)) having lattice distance $n_\theta = 2 > 1$

to the origin. The combinatorial identity from Theorem 6.3.1 holds because

$$
\begin{aligned}
&v(\Delta_2) - \sum_{\substack{\theta \preceq \Delta_2 \\ \dim(\theta)=2}} \frac{1}{n_\theta} \cdot v(\theta) + \sum_{\substack{\theta \preceq \Delta_2 \\ \dim(\theta)=1}} v(\theta) \cdot v(\theta^*) \\
&= 5 - \left(\frac{1}{2} \cdot (1! \cdot 1) + 1 \cdot (1! \cdot 1) + 1 \cdot (1! \cdot 1) + 1 \cdot (1! \cdot 1)\right) + \Big((1! \cdot 1) \cdot (1! \cdot \frac{5}{2}) + (1! \cdot 1) \\
&\qquad \cdot (1! \cdot \frac{5}{2}) + (1! \cdot 1) \cdot (1! \cdot \frac{5}{2}) + (1! \cdot 1) \cdot (1! \cdot 5) + (1! \cdot 1) \cdot (1! \cdot 5) + (1! \cdot 1) \cdot (1! \cdot 5)\Big) \\
&= 5 - 3.5 + 22.5 = 24
\end{aligned}
$$

(Definition 2.3.3, Definition 2.3.4, page 22).

The statement of Theorem 6.3.1 specializes to the well-known formula for reflexive polytopes in dimension 3 [BCF+05, Theorem 4.3]:

**Corollary 6.3.5.** *Let $\Delta \subseteq M_{\mathbb{R}}$ be a reflexive 3-polytope. Then*

$$24 = \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=1}} v(\theta) \cdot v\left(\theta^*\right).$$

*Proof.* We apply Theorem 6.3.1 and obtain

$$24 = v(\Delta) - \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=2}} \frac{1}{n_\theta} \cdot v(\theta) + \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=1}} v(\theta) \cdot v(\theta^*).$$

Since $\Delta$ is reflexive, we have $n_\theta = 1$ for all facets $\theta \preceq \Delta$ (page 25) and

$$v(\Delta) - \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=2}} \frac{1}{n_\theta} \cdot v(\theta) = v(\Delta) - \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=2}} v(\theta) = 0$$

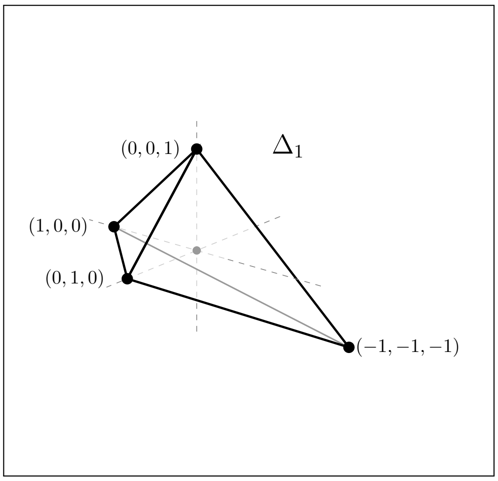
because each normalized volume of the $\Delta$-part below a facet $\theta \preceq \Delta$ of $\Delta$ equals the normalized volume of $\theta$. To be precise, by denoting the set of vertices of $\theta$ with $\mathrm{vert}(\theta) = \{P_{\theta_1}, \ldots, P_{\theta_t}\}$ and using the formula 1/3 times base times height, we get

$$v(\mathrm{conv}(0, P_{\theta_1}, \ldots, P_{\theta_t})) = 3! \cdot \left(\frac{1}{3} \cdot v(\theta) \cdot 1\right) = v(\theta)$$

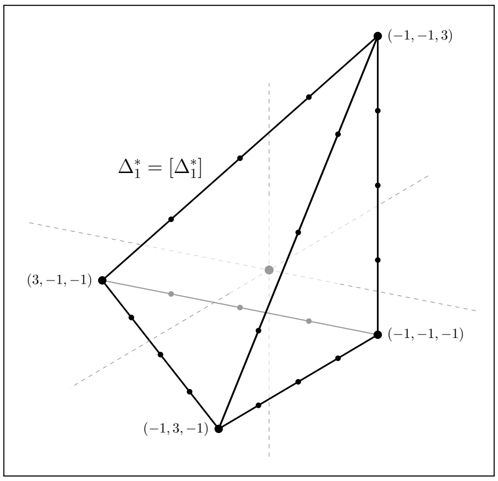
and therefore the normalized volumes of all $\Delta$-parts add up to $\sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)}} v(\theta) = v(\Delta)$. The equation simplifies to

$$24 = \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=1}} v(\theta) \cdot v(\theta^*).$$

$\square$




**(a)** 3-simplex $\Delta_1$.




**(b)** Tetrahedron $\Delta_1^*$.

**Figure 6.6: Reflexive Polytope $\Delta_1$ and its dual.** Shaded faces are occluded. ***(a)*** The reflexive polytope $\Delta_1$ is given as the convex hull of $e_1$, $e_2$, $e_3$, and $-e_1 - e_2 - e_3$. All facets of $\Delta_1$ have lattice distance 1 to the origin. ***(b)*** The polytope is the dual polytope $\Delta_1^*$ given as the convex hull of $(3, -1, -1)$, $(-1, 3, -1)$, $(-1, -1, 3)$, and $(-1, -1, -1)$. $\Delta_1^*$ is a reflexive polytope with $[\Delta_1^*] = \Delta_1^*$.

**Example 6.3.6.** For the reflexive polytope $\Delta_1 = \operatorname{conv}(e_1, e_2, e_3, -e_1 - e_2 - e_3)$ (Figure 6.6(a)), we get the identity of Corollary 6.3.5

$$\sum_{\substack{\theta \preceq \Delta_1 \\ \dim(\theta)=1}} v(\theta) \cdot v(\theta^*) = 6 \cdot (1! \cdot 1) \cdot (1! \cdot 4) = 24$$

because the normalized volume $v(\theta)$ of each 1-dimensional face $\theta \preceq \Delta_1$ of $\Delta_1$ equals $1! \cdot 1 = 1$. Moreover, the dual polytope is given by

$$\Delta_1^* = \operatorname{conv}((3, -1, -1), (-1, 3, -1), (-1, -1, 3), (-1, -1, -1))$$

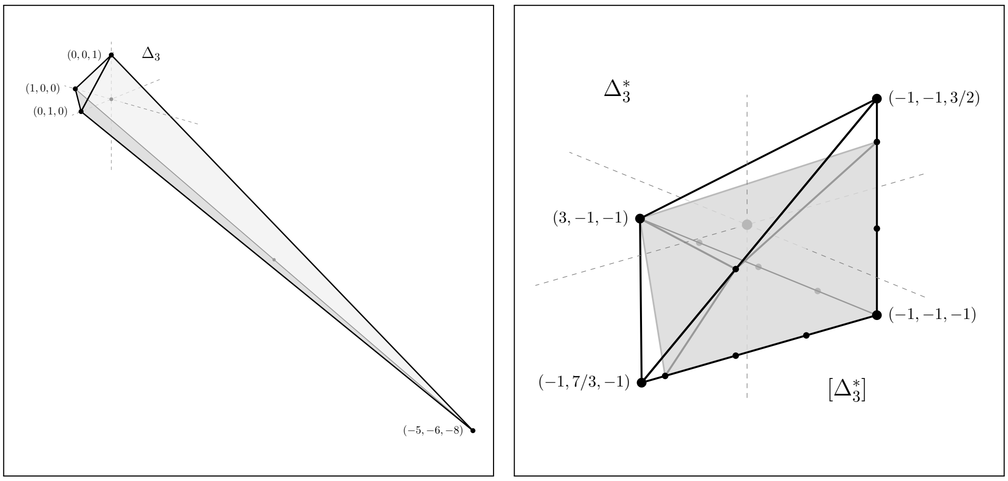
(Figure 6.6(b)) and $v(\theta^*) = 1! \cdot 4 = 4$ for each 1-dimensional face $\theta^* \preceq \Delta_1^*$ of $\Delta_1^*$ (Definition 2.3.3, Definition 2.3.4, page 22).

## 6.4 Canonical Fano 3-polytopes

The second last section is devoted to canonical Fano 3-polytopes $\Delta \subseteq N_{\mathbb{R}}$, *i.e.*, to 3-dimensional lattice polytopes containing only the origin as an interior lattice point (Definition 2.3.31).

The 3-polytope

$$\Delta_3 = \operatorname{conv}(e_1, e_2, e_3, -5e_1 - 6e_2 - 8e_3)$$




**(a)** **(b)**

**Figure 6.7: Non-almost Reflexive Polytope $\boldsymbol{\Delta_3}$ and its dual.** Shaded faces are occluded. ***(a)*** The non-almost reflexive polytope $\Delta_3$ is given as the convex hull of $e_1$, $e_2$, $e_3$, and $-5e_1 - 6e_2 - 8e_3$. The grey coloured facet of $\Delta_3$ has lattice distance 2, the light grey coloured facet has lattice distance 3, and all other facets have lattice distance 1 to the origin. ***(b)*** The whole polytope is the dual polytope $\Delta_3^*$ given as the convex hull of $(3,-1,-1)$, $(-1,7/3,-1)$, $(-1,-1,3/2)$, and $(-1,-1,-1)$. The grey coloured polytope is $[\Delta_3^*]$ given as the convex hull of $(3,-1,-1)$, $(-1,-1,-1)$, $(-1,-1,1)$, $(-1,1,0)$, and $(-1,2,-1)$.

(Figure 6.7(a)) is one of the simplest examples of a canonical Fano 3-polytope that is not almost reflexive. The canonical Fano polytope $\Delta_3$ corresponds to the canonical toric Fano threefold $X_{\Delta_3} := X_{\Sigma_{\Delta_3}}$ (Theorem 2.3.33), which is the weighted projective space $\mathbb{P}(1,5,6,8)$. It was observed by Corti and Golyshev [CG11] that an affine hypersurface $Z_{\Delta_3} \subseteq (\mathbb{C}^*)^3$ defined by a general Laurent polynomial $f_{\Delta_3} \in \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]$ with the Newton polytope $\Delta_3$ is birational to an elliptic surface of Kodaira dimension 1. In particular, $Z_{\Delta_3}$ is not birational to a $K3$-surface. The dual polytope $\Delta_3^*$ (Figure 6.7(b)) is the rational polytope

$$\operatorname{conv}((3,-1,-1),(-1,7/3,-1),(-1,-1,3/2),(-1,-1,-1)).$$

The two rational vertices of $\Delta_3^*$ are dual to two 2-dimensional faces of $\Delta_3$ (highlighted in grey and light grey in Figure 6.7(a)) having lattice distance 2 and 3 to the origin as well as normalized volume 4 and 2, respectively. The other two 2-dimensional faces of $\Delta_3$ have both lattice distance 1 to the origin with normalized volume 1 and 5, respectively. Using this knowledge, we get

$$\sum_{\substack{\theta \preceq \Delta_3 \\ \dim(\theta)=2}} \frac{1}{n_\theta} \cdot v(\theta) = \frac{1}{2} \cdot 4 + \frac{1}{3} \cdot 2 + 1 \cdot 1 + 1 \cdot 5 = \frac{26}{3}$$

and

$$\sum_{\substack{\theta\preceq\Delta_3\\ \dim(\theta)=1}} v(\theta)\cdot v(\theta^*) = 2\cdot\frac{5}{6}+1\cdot\frac{5}{2}+1\cdot\frac{1}{2}+1\cdot\frac{10}{3}+1\cdot\frac{2}{3}+1\cdot 4=\frac{38}{3}$$

because all edges of $\Delta_3$ have normalized volume 1 except of the edge $\operatorname{conv}(e_1, -5e_1 - 6e_2 - 8e_3)$ with normalized volume 2 (interior lattice point of the edge is coloured in grey in Figure 6.7(a)) and the corresponding normalized volumes of the 1-dimensional dual faces are noted in the second computation and can be read out in Figure 6.7(b). Therefore, we obtain the same identity as stated in Theorem 6.3.1

$$v(\Delta_3) - \sum_{\substack{\theta\preceq\Delta_3\\ \dim(\theta)=2}} \frac{1}{n_\theta}\cdot v(\theta) + \sum_{\substack{\theta\preceq\Delta_3\\ \dim(\theta)=1}} v(\theta)\cdot v(\theta^*) = 24,$$

because $v(\Delta_3) = 20$.

As we have mentioned on page 10, there exist exactly 9,089 canonical Fano polytopes in dimension 3 that are not almost reflexive. Our purpose is now to show that these polytopes also satisfy the identity of Theorem 6.3.1, so that one gets the further generalization:

**Theorem 6.4.1.** *Let $\Delta \subseteq N_{\mathbb{R}}$ be a canonical Fano 3-polytope. Then*

$$24 = v(\Delta) - \sum_{\substack{\theta\preceq\Delta\\ \dim(\theta)=2}} \frac{1}{n_\theta}\cdot v(\theta) + \sum_{\substack{\theta\preceq\Delta\\ \dim(\theta)=1}} v(\theta)\cdot v(\theta^*).$$

To prove this statement, we can not use the ideas mentioned in the Introduction on page 9 because there exist 9,089 canonical Fano polytopes $\Delta$ of dimension 3 that are not related to $K3$-surfaces. In contrast, we use a completely different idea based on the stringy Libgober-Wood identity and its combinatorial interpretation for toric varieties containing generalized stringy Hodge numbers and intersection products of stringy Chern classes:

*Proof of Theorem 6.4.1.* We apply Theorem 4.2.2 to the canonical toric Fano threefold $X_\Delta := X_{\Sigma_\Delta}$ corresponding to the canonical Fano 3-polytope $\Delta$ (Theorem 2.3.33). In this case, the associated fan $\Sigma_\Delta$ is the spanning fan of $\Delta$, *i.e.*, it consists of cones $\sigma_\theta = \mathbb{R}_{\geq 0}\theta$, where $\theta \preceq \Delta$ runs over all faces $\theta$ of $\Delta$ (Definition 2.3.10).

Due to the construction of the fan $\Sigma_\Delta$, we can show that $v(\Sigma_\Delta) = v(\Delta)$ (Theorem 6.1.1 (proof)). Moreover,

$$v(\sigma_\theta) = v(\theta)$$

for any 1-dimensional face $\theta \preceq \Delta$ because $v(\sigma_\theta) = v(\operatorname{conv}(0,\theta)) = v(\theta)\cdot v(h_\theta)$, where $h_\theta$ denotes the height of the lattice triangle $\operatorname{conv}(0,\theta)$ with base $\theta$. The normalized volume of $h_\theta$ equals 1 because all non-zero lattice points of the lattice triangle $\operatorname{conv}(0,\theta)$ are contained in its side $\theta$.

Using Definition 2.3.3 of the dual polytope $\Delta^*$, every 1-dimensional face $\theta^* \preceq \Delta^*$ of $\Delta^*$ has lattice distance $n_{\theta^*} = 1$ to the origin. This implies $\Delta_{-K_{X_\Delta}} = \Delta^*$ and $\Delta^{\sigma_\theta}_{-K_{X_\Delta}} = \theta^*$ (Theorem 6.1.1 (proof)) yielding

$$v(\Delta^{\sigma_\theta}_{-K_{X_\Delta}}) = v(\theta^*).$$

Therefore, the right hand side of the stringy Libgober-Wood identity in Theorem 4.2.2 equals

$$\frac{3}{12} v(\Delta) + \frac{1}{6} \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=1}} v(\theta) \cdot v\left(\theta^*\right).$$

Moreover, the normalized volume $v(\Delta)$ of $\Delta$ equals the stringy Euler number $e_{\text{str}}(X_\Delta)$ of $X_\Delta$ (Theorem 6.1.1 (proof), Corollary 4.1.3) and

$$e_{\text{str}}(X_\Delta) = E_{\text{str}}(X_\Delta; 1, 1) = 2 \cdot \psi_0(\Sigma_\Delta) + 2 \cdot \psi_1(\Sigma_\Delta) + \sum_{\alpha \in ([0,3] \cap \frac{1}{q_{X_\Delta}} \mathbb{Z}) \setminus \mathbb{Z}} \psi_\alpha(\Sigma_\Delta),$$

where the equality signs hold due to Equation (1.4), Proposition 4.2.1, and $\psi_0(\Sigma_\Delta) = \psi_3(\Sigma_\Delta)$ as well as $\psi_1(\Sigma_\Delta) = \psi_2(\Sigma_\Delta)$. A combination of these facts yields

$$\frac{1}{2} \cdot \psi_0(\Sigma_\Delta) + \frac{1}{2} \cdot \psi_1(\Sigma_\Delta) + \frac{3}{12} \sum_{\alpha \in ([0,3] \cap \frac{1}{q_{X_\Delta}} \mathbb{Z}) \setminus \mathbb{Z}} \psi_\alpha\left(\Sigma_\Delta\right) + \frac{1}{6} \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=1}} v(\theta) \cdot v\left(\theta^*\right)$$

for the right hand side of the stringy Libgober-Wood identity. The left hand side of the stringy Libgober-Wood identity in Theorem 4.2.2 can be computed with Theorem 5.3.1 as

$$2 \cdot \psi_0(\Sigma_\Delta) \left(0 - \frac{3}{2}\right)^2 + 2 \cdot \psi_1(\Sigma_\Delta) \left(1 - \frac{3}{2}\right)^2 + \sum_{\alpha \in ([0,3] \cap \frac{1}{q_{X_\Delta}} \mathbb{Z}) \setminus \mathbb{Z}} \psi_\alpha\left(\Sigma_\Delta\right) \left(\alpha - \frac{3}{2}\right)^2.$$

Comparing right and left hand sides, a short calculation provides

$$\begin{aligned}
24 + 6 \cdot \sum_{\alpha \in ([0,3] \cap \frac{1}{q_{X_\Delta}} \mathbb{Z}) \setminus \mathbb{Z}} \psi_\alpha\left(\Sigma_\Delta\right) \left(\alpha - \frac{3}{2}\right)^2 &= \frac{3}{2} \sum_{\alpha \in ([0,3] \cap \frac{1}{q_{X_\Delta}} \mathbb{Z}) \setminus \mathbb{Z}} \psi_\alpha\left(\Sigma_\Delta\right) + w \\
\iff 24 = 6 \cdot \sum_{\alpha \in ([0,3] \cap \frac{1}{q_{X_\Delta}} \mathbb{Z}) \setminus \mathbb{Z}} \psi_\alpha\left(\Sigma_\Delta\right) \left(\frac{1}{4} - \left(\alpha - \frac{3}{2}\right)^2\right) + w \\
\iff 24 = 6 \cdot \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=2, n_\theta > 1}} v(\theta) \cdot \sum_{k=1}^{n_\theta - 1} \left(\frac{1}{4} - \left(\frac{k}{n_\theta} - \frac{1}{2}\right)^2\right) + w
\end{aligned}$$

with $w := \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=1}} v(\theta) \cdot v(\theta^*)$, where the last equation holds due to Theorem 5.3.1. Using Lemma 6.4.3, this adds up to

$$\begin{aligned} 24 &= \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=2, n_\theta > 1}} v(\theta) \left( n_\theta - \frac{1}{n_\theta} \right) + \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=1}} v(\theta) \cdot v(\theta^*) \\ \Longleftrightarrow \quad 24 &= v(\Delta) - \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=2}} \frac{1}{n_\theta} \cdot v(\theta) + \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=1}} v(\theta) \cdot v(\theta^*) \end{aligned}$$

because $v(\Delta) = \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=2}} v(\theta) \cdot n_\theta$. □

**Remark 6.4.2.** The statement of Theorem 6.4.1 specializes to the well-known formula

$$24 = \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=1}} v(\theta) \cdot v(\theta^*)$$

for reflexive 3-polytopes $\Delta$ because this was already true for the statement of Theorem 6.3.1 concerning almost reflexive 3-polytopes $\Delta$ (Corollary 6.3.5).

**Lemma 6.4.3.** *Let $n$ be a positive integer. Then*

$$6 \cdot \sum_{k=1}^{n-1} \left( \frac{1}{4} - \left( \frac{k}{n} - \frac{1}{2} \right)^2 \right) = n - \frac{1}{n}.$$

*Proof.* Applying the well-known formulas for $\sum_{k=1}^{n-1} k$ and $\sum_{k=1}^{n-1} k^2$, we get

$$\begin{aligned} 6 \cdot \sum_{k=1}^{n-1} \left( \frac{1}{4} - \left( \frac{k}{n} - \frac{1}{2} \right)^2 \right) &= 6 \cdot \sum_{k=1}^{n-1} \left( \frac{1}{4} - \frac{k^2}{n^2} + \frac{k}{n} - \frac{1}{4} \right) \\ &= -\frac{6}{n^2} \cdot \sum_{k=1}^{n-1} k^2 + \frac{6}{n} \sum_{k=1}^{n-1} k \\ &= -\frac{6}{n^2} \frac{(n-1)((n-1)+1)(2(n-1)+1)}{6} \\ &\quad + \frac{6}{n} \frac{(n-1)((n-1)+1)}{2} \\ &= -\frac{(n-1)n(2n-1)}{n^2} + \frac{3(n-1)n}{n} \\ &= -\frac{2n^3 - 3n^2 + n}{n^2} + \frac{3(n^2 - n)}{n} \\ &= -\frac{2n^3 - 3n^2 + n - 3n^3 + 3n^2}{n^2} \\ &= \frac{n^3 - n}{n^2} = \frac{n^2 - 1}{n} = n - \frac{1}{n}. \end{aligned}$$

□

## 6.5 Gorenstein Polytopes

The chapter concludes with the treatment of Gorenstein $d$-polytopes $\Delta \subseteq M_{\mathbb{R}}$ of index $r$, *i.e.*, with $d$-dimensional lattice polytopes $\Delta$ such that $r\Delta - m$ is a reflexive polytope for some lattice point $m \in M$ (Definition 2.3.37).

The goal of this section is similar to the previous sections, we aim to achieve combinatorial identities relating Gorenstein $d$-polytopes of index $r = d-2$ and $r = d-1$ to the number 24 and 12, respectively. The difference is in the way we prove them and for this reason, we need a central statement (Proposition 6.5.1) about the combinatorial computation of the stringy Euler number of generic Calabi-Yau complete intersections.

It is well-known (Remark 6.1.6, [BD96, Corollary 7.10]) that the stringy Euler number of a generic Calabi-Yau hypersurface $Z$ in a Gorenstein toric Fano variety $X_{\Sigma_\Delta}$ (Theorem 2.3.14, Remark 2.3.15) can be computed via normalized volumes of faces $\theta \preceq \Delta$ and $\theta^* \preceq \Delta^*$ of reflexive $d$-polytopes $\Delta$ and $\Delta^*$, respectively, as

$$e_{\mathrm{str}}(Z) = c^{\mathrm{str}}_{d-1}(Z) = \sum_{k=0}^{d-3} (-1)^k \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=k+1}} v(\theta) \cdot v(\theta^*).$$

Using Corollary 4.1.9 and the duality between faces $\theta \preceq \Delta$ and $\theta^* \preceq \Delta^*$ of Gorenstein polytopes $\Delta$ and $\Delta^*$, respectively (page 31), we generalize this combinatorial formula to the case of generic Calabi-Yau complete intersections in Gorenstein toric Fano varieties:

**Proposition 6.5.1.** *Let $X$ be a Gorenstein toric Fano variety associated with a Gorenstein $d$-polytope $\Delta \subseteq M_{\mathbb{R}}$ of index $r$ and $D$ an ample torus-invariant Cartier divisor on $X$ such that $[D] = \frac{1}{r}c_1(X)$. Denote by $Z_1, \ldots, Z_r \subseteq X$ generic semi-ample Cartier divisors such that $[Z_1] = \ldots = [Z_r] = [D] = \frac{1}{r}c_1(X)$. Then the stringy Euler number of the Calabi-Yau complete intersection $S \coloneqq Z_1 \cap \ldots \cap Z_r$ is*

$$c^{\mathrm{str}}_{d-r}(S) = \sum_{k=0}^{d-r-1} (-1)^k \binom{k+r-1}{r-1} \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=k+r}} v(\theta) \cdot v(\theta^*) \quad + (-1)^{d-r}\binom{d-1}{r-1} v(\Delta).$$

*Proof.* Corollary 4.1.9 provides

$$c^{\mathrm{str}}_{d-r}(S) = \sum_{k=0}^{d-r} (-1)^k \binom{k+r-1}{r-1} \sum_{\sigma \in \Sigma^\Delta(d-r-k)} v(\sigma) \cdot v(\Delta_D^\sigma),$$

where $\Sigma^\Delta$ denotes the associated normal fan of $X$ (page 32) and $\Delta_D^\sigma \preceq \Delta_D$ a face of the lattice polytope $\Delta_D$ (Equation (4.1) and (4.2)). Let $\sigma \in \Sigma(d-r-k)$ be a $(d-r-k)$-dimensional cone of $\Sigma$ $(0 \le k \le d-r-1)$. Then the cone $\sigma$ can be considered as a cone over a $(d-r-k-1)$-dimensional proper face of the reflexive polytope $(r\Delta)^*$, which we naturally identify with the corresponding proper face $\theta^* \preceq \Delta^*$ of the dual Gorenstein polytope $\Delta^*$ [BN08, Proposition 1.16]. Therefore, we obtain $v(\sigma) = v(\theta^*)$. On the other hand, the lattice polytope $\Delta_D$ is exactly the Gorenstein polytope $\Delta$ and $\theta^* \preceq \Delta^*$ is the dual face to a $(k+r)$-dimensional face $\Delta_D^\sigma = \theta \preceq \Delta$ of $\Delta$. This implies

$$\sum_{\sigma \in \Sigma(d-r-k)} v(\sigma) \cdot v\left(\Delta_D^\sigma\right) = \sum_{\substack{\theta^* \preceq \Delta^* \\ \dim(\theta^*)=d-k-r-1}} v\left(\theta^*\right) \cdot v(\theta) = \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=k+r}} v\left(\theta\right) \cdot v\left(\theta^*\right)$$

for all $0 \leq k \leq d-r-1$. It remains to note that in the case $k = d-r$ (*i.e.*, $\dim(\sigma)=0$), one has $\Delta_D^\sigma = \Delta$, $v\left(\Delta_D^\sigma\right) = v(\Delta)$, and $v(\sigma)=1$. ☐

The well-known combinatorial formula from Corollary 6.1.4

$$24 = \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=1}} v(\theta) \cdot v\left(\theta^*\right)$$

for reflexive 3-polytopes $\Delta$ can be generalized to Gorenstein $d$-polytopes $\Delta$ ($d \geq 3$) of index $r = d-2$:

**Proposition 6.5.2.** *Let $\Delta \subseteq M_{\mathbb{R}}$ be a Gorenstein $d$-polytope of index $r = d-2$. Then*

$$24 = \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=r}} v(\theta) \cdot v\left(\theta^*\right) + \frac{r(1-r)}{2} v(\Delta).$$

*Proof.* Let $S \coloneqq Z_1 \cap \ldots \cap Z_r$ be a generic Calabi-Yau complete intersection in the Gorenstein toric Fano variety $X$ associated with $\Delta$, *i.e.*, $Z_1, \ldots, Z_r$ are $r$ generic ample Cartier divisors on $X$ such that $[Z_1] = \ldots = [Z_r] = \frac{1}{r} c_1(X)$. Then $\dim(S) = 2$, *i.e.*, $S$ is a (possibly singular) $K3$-surface. The stringy Euler number $c_2^{\mathrm{str}}(S)$ of $S$ equals the usual Euler number $c_2(\widetilde{S})$ of the minimal crepant desingularization $\widetilde{S}$ of $S$ (Theorem 2.1.11 and Equation (1.5)). Since $\widetilde{S}$ is a smooth $K3$-surface, we have $c_2^{\mathrm{str}}(S) = c_2(\widetilde{S}) = 24$ (Theorem 6.3.1 (proof)). Using Proposition 6.5.1, we obtain

$$24 = c_2^{\mathrm{str}}(S) = \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=r}} v(\theta) \cdot v\left(\theta^*\right) - r \cdot \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=d-1}} v(\theta) \cdot v\left(\theta^*\right) + \frac{(r+1)r}{2} v(\Delta).$$

Since $r\Delta$ is a reflexive polytope, one has

$$r^d v(\Delta) = v(r\Delta) = \sum_{\substack{r\theta \preceq r\Delta \\ \dim(r\theta)=d-1}} v(r\theta) = r^{d-1} \cdot \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=d-1}} v(\theta).$$

Moreover, $v(\theta^*) = 1$ if $\dim(\theta^*) = 0$. It remains to apply the equalities

$$\sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=d-1}} v(\theta) \cdot v\left(\theta^*\right) = \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=d-1}} v(\theta) = r v(\Delta).$$

☐

The well-known identity for reflexive polytopes in dimension 3 (Corollary 6.1.4) follows immediately from the statement in Proposition 6.5.2:

**Corollary 6.5.3.** *Let $\Delta \subseteq M_{\mathbb{R}}$ be a reflexive 3-polytope. Then*

$$24 = \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=1}} v(\theta) \cdot v(\theta^*) .$$

*Proof.* Using Definition 2.3.37, $\Delta$ is a Gorenstein polytope of index 1 because it is a reflexive polytope. In particular, the index differs by 2 from dimension $d = 3$ and an application of Proposition 6.5.2 yields

$$\begin{aligned} 24 &= \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=1}} v(\theta) \cdot v(\theta^*) + \frac{1(1-1)}{2} v(\Delta) \\ &= \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=1}} v(\theta) \cdot v(\theta^*) + 0 \cdot v(\Delta) = \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=1}} v(\theta) \cdot v(\theta^*) . \end{aligned}$$

□

It is proved in [BJ10, Proposition 3.4] that the combinatorial identity

$$12 = \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=1}} v(\theta) \cdot v(\theta^*) \tag{6.1}$$

holds for any Gorenstein 3-polytope $\Delta$ of index $r = 2$. We show that this identity can be generalized to arbitrary Gorenstein $d$-polytopes $\Delta$ ($d \geq 3$) of index $r = d - 1$:

**Proposition 6.5.4.** *Let $\Delta \subseteq M_{\mathbb{R}}$ be a Gorenstein $d$-polytope of index $r = d - 1$. Then*

$$12 = \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=r-1}} v(\theta) \cdot v(\theta^*) + \frac{r(1-r)+2}{2} v(\Delta) .$$

*Proof.* Let $S := Z_1 \cap \ldots \cap Z_{r-1}$ be a generic complete intersection in the Gorenstein toric Fano variety $X$ associated with $\Delta$, *i.e.*, $Z_1, \ldots, Z_{r-1}$ are $r-1$ generic ample Cartier divisors on $X$ such that $[D] := [Z_1] = \ldots = [Z_{r-1}] = \frac{1}{r} c_1(X)$. Then $\dim(S) = 2$, *i.e.*, $S$ is a (possibly singular) del Pezzo surface. The stringy Euler number $c_2^{\text{str}}(S)$ of $S$ equals the usual Euler number $c_2(\widetilde{S})$ of the minimal crepant desingularization $\widetilde{S}$ of $S$ (Theorem 2.1.11 and Equation (1.5)). Then $\widetilde{S}$ is a smooth rational surface and we have $c_2^{\text{str}}(S) = c_2(\widetilde{S})$ and $c_1(S)^2 = c_1(\widetilde{S})^2$, respectively. Using Noether's Theorem [CLS11, Theorem 10.5.3] for $\widetilde{S}$, we obtain

$$12 = 12 \cdot (1 + 0 + 0) = c_2(\widetilde{S}) + c_1(\widetilde{S})^2 = c_2^{\text{str}}(S) + c_1(S)^2 .$$

It remains to derive combinatorial formulas for $c_1(S)^2$ and $c_2^{\mathrm{str}}(S)$. Using the adjunction formula, the anticanonical class of $S$ is the restriction of $c_1(X) - (r-1)[D] = [D]$ to $S$. Therefore, $c_1(S)^2 = [D]^2.[D]^{r-1} = [D]^d = v(\Delta)$. By Corollary 4.1.9, we get

$$c_2^{\mathrm{str}}(S) = \sum_{\sigma \in \Sigma(2)} v(\sigma) \cdot v\left(\Delta_D^\sigma\right) - (r-1) \cdot \sum_{\sigma \in \Sigma(1)} v(\sigma) \cdot v\left(\Delta_D^\sigma\right) \quad + \frac{r(r-1)}{2} v(\Delta_D).$$

Using the same arguments as in the proof of Proposition 6.5.1, we can rewrite the last equation as

$$c_2^{\mathrm{str}}(S) = \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta) = d-2}} v(\theta) \cdot v\left(\theta^*\right) - (r-1) \cdot \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta) = d-1}} v(\theta) \cdot v\left(\theta^*\right) \quad + \frac{r(r-1)}{2} v(\Delta).$$

In addition, the last equation

$$\sum_{\substack{\theta \preceq \Delta \\ \dim(\theta) = d-1}} v(\theta) \cdot v\left(\theta^*\right) = rv(\Delta)$$

in the proof of Proposition 6.5.2 yields

$$c_2^{\mathrm{str}}(S) = \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta) = r-1}} v(\theta) \cdot v\left(\theta^*\right) + \frac{r(1-r)}{2} v\left(\Delta\right).$$

□

The known identity for Gorenstein 3-polytope of index $r = 2$ in Equation (6.1) follows immediately from the statement in Proposition 6.5.4:

**Corollary 6.5.5.** *Let $\Delta \subseteq M_{\mathbb{R}}$ be a Gorenstein 3-polytope of index $r = 2$. Then*

$$12 = \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta) = 1}} v(\theta) \cdot v\left(\theta^*\right).$$

*Proof.* Because the index $r = 2$ of the Gorenstein polytope $\Delta$ differs by 1 from dimension $d = 3$, we can apply Proposition 6.5.4 to get

$$\begin{aligned} 12 &= \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta) = 2-1}} v(\theta) \cdot v\left(\theta^*\right) + \frac{2(1-2)+2}{2} v\left(\Delta\right) \\ &= \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta) = 1}} v(\theta) \cdot v\left(\theta^*\right) + \frac{0}{2} v\left(\Delta\right) = \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta) = 1}} v(\theta) \cdot v\left(\theta^*\right). \end{aligned}$$

□

Chapter 7

# Non-degenerate Surfaces of Geometric Genus 1

This chapter connects the primarily combinatorial setting from Chapter 6 with geometry. Our aim is to classify all $\Delta$-non-degenerate affine surfaces $Z_\Delta$ of geometric genus 1 given by the zero sets of Laurent polynomials in the algebraic torus $\mathbb{T}^3$, where canonical Fano 3-polytopes $\Delta \subseteq M_\mathbb{R}$ are the Newton polytopes corresponding to these Laurent polynomials. The mentioned classification is meant up to birational equivalence, *i.e.*, up to an existing smooth projective minimal model (Definition 2.1.9). The existence of such a minimal model is guaranteed by Remark 2.3.36 and the fact that a crepant desingularization always exists in dimension 2 (page 15).

The *geometric genus* $p_g(Y)$ of a smooth projective variety $Y$ is defined as

$$p_g(Y) := \dim\left(H^0\left(Y, \Omega_Y^{\dim(Y)}\right)\right), \tag{7.1}$$

where $\Omega_Y^{\dim(Y)}$ denotes the canonical sheaf of $Y$. In our case, the geometric genus $p_g(Y)$ equals $|\Delta^\circ \cap M|$ by [Kho78, Remark page 42] and by definition this number equals 1 for canonical Fano polytopes $\Delta$.

This project advances our understanding of stringy invariants and their use in mirror symmetry. In addition, we provide new examples of non-degenerate affine surfaces of geometric genus 1. Classifying all canonical toric Fano threefolds, Kasprzyk showed that 674,688 isomorphism classes of canonical Fano 3-polytopes exist [Kas10]. This opens up an avenue for the targeted surface classification.

## 7.1 Classification of all Canonical Fano 3-polytopes

The above-mentioned classification of surfaces is preceded by a combinatorial classification of all canonical Fano 3-polytopes in four subclasses including closer investigations of their combinatorial properties.

All classification computations were done using the Graded Ring Database[1], including the data of all 674,688 canonical Fano 3-polytopes, MAGMA[2], and Maple[3]. Therefore, all appearing statements have been checked and validated by computer calculations. The canonical Fano 3-polytopes used as examples in this chapter appear with an ID that is the example's ID in the Graded Ring Database[4].

Let $\Delta \subseteq M_{\mathbb{R}}$ be one of the 674,688 canonical Fano 3-polytopes.[5] To sort these polytopes into four disjoint subclasses, we apply the combinatorial notion of the Fine interior

$$\Delta^{\mathrm{FI}} = \bigcap_{0 \neq n \in N} \Gamma_1^{\Delta}(n)$$

with $\Gamma_1^{\Delta}(n) = \left\{ y \in M_{\mathbb{R}} \,\middle|\, \langle y, n \rangle \geq \kappa^{\Delta}(n) + 1 \right\}$ for any non-zero lattice point $n \in N$ (Definition 2.3.24), where $\kappa^{\Delta} : N_{\mathbb{R}} \to \mathbb{R},\ x \mapsto \min_{y \in \mathrm{vert}(\Delta)} \{\langle y, x \rangle\}$ (Equation (2.2)).

### 7.1.1 Canonical Fano 3-polytopes $\Delta$ with $\Delta^{\mathrm{FI}} = \{0\}$

By an application of Proposition 2.3.25 and Definition 2.3.22 (*i.e.*, by checking if $[\Delta^*] = \mathrm{conv}(\Delta^* \cap N)$ contains the origin $0 \in N$ in its interior and is 3-dimensional), we obtain that exactly 665,599 canonical Fano 3-polytopes have Fine interior $\Delta^{\mathrm{FI}} = \{0\}$. In particular, we call these polytopes almost reflexive because we are in dimension 3 (Remark 2.3.26).

**Example 7.1.1.** Following Remark 2.3.23, all 4,319 reflexive 3-polytopes $\Delta$ are canonical Fano 3-polytopes with $\Delta^{\mathrm{FI}} = \{0\}$. A small selection of these is visualized in Figure 6.2, Figure 2.3(a), and Example 6.3.6 with Figure 6.6(a). Moreover, Figure 2.3(b) and Example 6.3.4 with Figure 6.5(a) present an almost reflexive polytope that is not reflexive.

For the remaining 9,089 non-almost reflexive canonical Fano 3-polytopes $\Delta \subseteq M_{\mathbb{R}}$, we have to compute the Fine interior $\Delta^{\mathrm{FI}}$ in detail: Using Gordan's Lemma (Remark 2.2.7) and the remark on page 27, the Fine interior $\Delta^{\mathrm{FI}}$ can be computed as a finite intersection of affine halfspaces $\Gamma_1^{\Delta}(n)$, where $n$ appears as a minimal generator of the semigroup $S_{\sigma^\theta}$ for a face $\theta \preceq \Delta$ of $\Delta$. Hence, we obtain the following finite computation formula

$$\Delta^{\mathrm{FI}} = \bigcap_{\substack{\theta \preceq \Delta \\ n \in \mathcal{H}(S_{\sigma^\theta})}} \Gamma_1^{\Delta}(n) = \bigcap_{\substack{\sigma \in \Sigma^{\Delta} \\ n \in \mathcal{H}(S_{\sigma})}} \Gamma_1^{\Delta}(n), \tag{7.2}$$

where $\sigma^\theta$ is a cone in the normal fan $\Sigma^{\Delta}$ corresponding to $\Delta$ (Definition 2.3.8), $\mathcal{H}(S_{\sigma^\theta}) \subseteq S_{\sigma^\theta}$ is called *Hilbert basis of* $S_{\sigma^\theta}$ and its elements are the minimal generators of $S_{\sigma^\theta}$ [CLS11, Proposition 1.2.23].

[1] `http://www.grdb.co.uk`

[2] `http://magma.maths.usyd.edu.au/magma`

[3] `https://www.maplesoft.com` using the convex package (`http://www.math.uwo.ca/~mfranz/convex`) and the MDSpackage (`http://www.math.uni-tuebingen.de/user/keicher/MDS`)

[4] `http://www.grdb.co.uk/forms/toricf3c`

[5] In order to avoid confusion with respect to the definitions in Chapter 2, we want to mention the lattice change, *i.e.*, from now on the given canonical Fano 3-polytopes $\Delta \subseteq M_{\mathbb{R}}$ are living in the real vector space $M_{\mathbb{R}}$.

**Table 7.1: Distribution with Respect to Projection Results.** Table contains: Distribution of all 9,020 canonical Fano 3-polytopes $\Delta$ with $\dim(\Delta^{\mathrm{FI}}) = 1$ and $0 \in \mathrm{vert}(\Delta^{\mathrm{FI}})$ with respect to their projection results $\overline{\Delta}$ that are reflexive polygons and links to pictures of the projection results. The third column displays the toric variety $X_{\Sigma_{\overline{\Delta}^*}}$ associated with the dual reflexive polygon $\overline{\Delta}^*$ constructed by corresponding spanning fan $\Sigma_{\overline{\Delta}^*}$.

| reflexive polygon $\overline{\Delta}$ | $\#\{\Delta \mid \dim(\Delta^{\mathrm{FI}}) = 1, 0 \in \mathrm{vert}(\Delta^{\mathrm{FI}})\}$ | $X_{\Sigma_{\overline{\Delta}^*}}$ |
|---|---|---|
| Figure 7.1(a) | 3038 | $\mathbb{P}^2$ |
| Figure 7.1(b) | 4663 | $\mathbb{P}(1,1,2)$ |
| Figure 7.1(c) | 1319 | $\mathbb{P}^1 \times \mathbb{P}^1$ |

### 7.1.2 Canonical Fano 3-polytopes $\Delta$ with $\dim(\Delta^{\mathrm{FI}}) = 1$ and $0 \in \mathrm{vert}(\Delta^{\mathrm{FI}})$

We apply Proposition 2.3.25 and Definition 2.3.22 to see that exactly 9,020 polytopes contain the origin $0 \in N$ in the interior of a facet $\Theta \preceq [\Delta^*] \subseteq N_{\mathbb{R}}$ of $[\Delta^*]$ and not in the interior of $[\Delta^*]$, *i.e.*, they are not almost reflexive (*i.e.*, $\Delta^{\mathrm{FI}} \neq \{0\}$). Simultaneously, an application of Equation (7.2) reveals that these canonical Fano 3-polytopes $\Delta \subseteq M_{\mathbb{R}}$ have 1-dimensional rational Fine interior $\Delta^{\mathrm{FI}}$ and the origin $0 \in M$ is a vertex of $\Delta^{\mathrm{FI}}$.

Subsequently, we investigate some combinatorial properties of this subclass more detailed: Let $\Delta \subseteq M_{\mathbb{R}}$ be a canonical Fano 3-polytope with $\dim(\Delta^{\mathrm{FI}}) = 1$ and $0 \in \mathrm{vert}(\Delta^{\mathrm{FI}})$. We denote by $v_\Delta \in M$ the unique primitive lattice point of the line containing the 1-dimensional rational polytope $\Delta^{\mathrm{FI}}$ with normalized volume smaller than 1. Another way to detect $v_\Delta$ is the following: Let $\Theta \preceq [\Delta^*] \subseteq N_{\mathbb{R}}$ be the facet of $[\Delta^*]$ from above, which contains the origin $0 \in N$ in its interior. Then $v_\Delta$ is the inward-pointing facet normal of $\Theta$.

Using this data and $b_1, b_2 \in M$ such that $\{v_\Delta, b_1, b_2\}$ is a lattice basis of $M$, we showed that the projection of $\Delta$ along $v_\Delta$ on the plane $\langle b_1, b_2\rangle$ with respect to the lattice basis provides in each case a reflexive polygon $\overline{\Delta}$. To be precise, we got the distribution of 3 different types out of 16 existing reflexive polygons (Figure 7.1) that is listed in Table 7.1.

**Example 7.1.2.** Let $\Delta \subseteq M_{\mathbb{R}}$ be a canonical Fano 3-polytope given by the convex hull

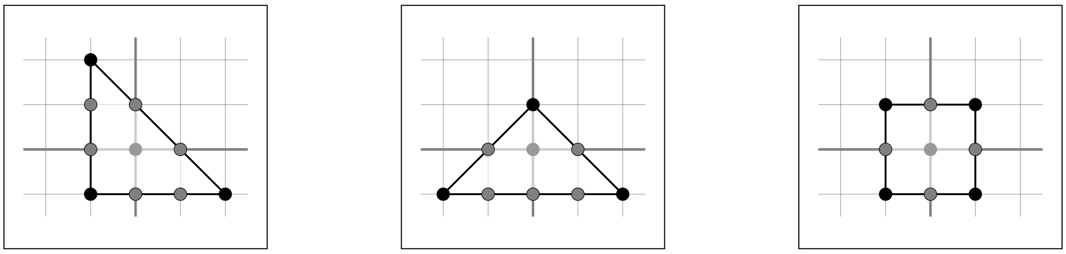
$$\mathrm{conv}((2,3,8),(1,0,0),(0,1,0),(-1,-1,-1))$$

(ID 547324, Figure A.1(a), Table A.1). Then the Fine interior $\Delta^{\mathrm{FI}}$ of $\Delta$ is 1-dimensional, equals

$$\mathrm{conv}((0,0,0),(1/2,1/2,1)),$$

and therefore $v_\Delta = (1,1,2)$ (Table A.2). To project $\Delta$ along $v_\Delta$ on a plane with respect to a lattice basis, we have to choose two lattice points $b_1, b_2 \in M$ such that $\{v_\Delta, b_1, b_2\}$ builds a lattice basis of $M$. In this case, we can take $b_1 = (1,0,0)$ and $b_2 = (0,0,1)$ to obtain

$$(2,3,8) = 3 \cdot v_\Delta + (-1) \cdot b_1 + 2 \cdot b_2 \qquad \longmapsto \quad (-1,2),$$



(a) (b) (c)

**Figure 7.1: Reflexive Projection Results.** Three types of reflexive polygons received as projection results $\overline{\Delta}$ along $v_\Delta$ of 9,020+20 canonical Fano 3-polytopes $\Delta$ in Subsection 7.1.2 and 7.1.3. Vertices are coloured black, boundary points that are not vertices grey, and the origin light grey. ***(a)*** Reflexive polygon $\overline{\Delta} = \operatorname{conv}((-1,2),(-1,-1),(2,-1))$ with $X_{\Sigma_{\overline{\Delta}^*}} = \mathbb{P}^2$. ***(b)*** Reflexive polygon $\overline{\Delta} = \operatorname{conv}((-2,-1),(0,1),(2,-1))$ with $X_{\Sigma_{\overline{\Delta}^*}} = \mathbb{P}(1,1,2)$. ***(c)*** Reflexive polygon $\overline{\Delta} = \operatorname{conv}((\pm 1, \pm 1))$ with $X_{\Sigma_{\overline{\Delta}^*}} = \mathbb{P}^1 \times \mathbb{P}^1$.

$$\begin{aligned} (1,0,0) &= 0 \cdot v_\Delta + 1 \cdot b_1 + 0 \cdot b_2 && \longmapsto (1,0), \\ (0,1,0) &= 1 \cdot v_\Delta + (-1) \cdot b_1 + (-2) \cdot b_2 && \longmapsto (-1,-2), \\ (-1,-1,-1) &= (-1) \cdot v_\Delta + 0 \cdot b_1 + 1 \cdot b_2 && \longmapsto (0,1). \end{aligned}$$

Therefore, we get the projection result (Table A.2)

$$\overline{\Delta} = \operatorname{conv}((-1,2),(1,0),(-1,-2),(0,1)) = \operatorname{conv}((-1,2),(1,0),(-1,-2))$$

that is a reflexive polygon isomorphic to the one in Figure 7.1(b) (Remark 2.3.16).

In order to achieve this projection result, we do not have to project the whole polytope $\Delta$ along $v_\Delta$. It is sufficient to project only one facet $\theta_{\text{proj}} \preceq \Delta$ of $\Delta$ that we detect using the following algorithm: Let $\{v_i \,|\, 1 \le i \le f\}$ be the set of all inward-pointing facet normals of the polytope $\Delta$, where $f$ denotes the number of facet of $\Delta$. Pairing computations with $v_\Delta$ showed that the $f$-tuple

$$(\langle v_\Delta, v_1 \rangle, \ldots, \langle v_\Delta, v_f \rangle)$$

contains exactly one negative integer. The requested facet $\theta_{\text{proj}}$ is the facet corresponding to the inward-pointing facet normal $v_i$ that produces the mentioned negative integer. But an easier way to detect this facet $\theta_{\text{proj}} \preceq \Delta$ of $\Delta$, is to check which facet of $\Delta$ contains the primitive lattice point $v_\Delta$ as an interior lattice point.

**Example 7.1.3.** Continuing Example 7.1.2 (ID 547324), the inward-pointing facet normals of $\Delta$ are $v_1 = (-1,-1,3)$, $v_2 = (7,-3,-1)$, $v_3 = (-1,3,-1)$, and $v_4 = (-2,-2,1)$ and therefore $(\langle v_\Delta, v_1 \rangle, \ldots, \langle v_\Delta, v_4 \rangle) = (4,2,0,-2)$ (Table A.3). The corresponding facet to $v_4$ is $\theta_{\text{proj}} = \operatorname{conv}((2,3,8),(1,0,0),(0,1,0))$ (Figure A.1) and a projection along $v_\Delta$ yields the same reflexive polygon $\overline{\Delta} = \operatorname{conv}((-1,2),(1,0),(-1,-2))$ as we received in Example 7.1.2 by projecting the polytope $\Delta$ (Table A.3).

**Remark 7.1.4.** The complete data and the corresponding pictures to a small selection of the 9,020 canonical Fano 3-polytopes with $\dim(\Delta^{\mathrm{FI}}) = 1$ and $0 \in \mathrm{vert}(\Delta^{\mathrm{FI}})$ are listed in the Appendix in Table A.1, A.2, and A.3 and can be viewed in Figure A.1, respectively. The polytope pictures include the Fine interior $\Delta^{\mathrm{FI}}$ (coloured red and the origin grey with a red margin) and the facet $\theta_{\mathrm{proj}} \preceq \Delta$ (grey dotted) that is used to project. The surfaces corresponding to the reflexive polygons $\overline{\Delta}$ (projection results) are boxed top right. These polytopes have been obtained by Corti and Golyshev [CG11] from $9 = 104 - 95$ examples stemming from 3-dimensional weighted projective spaces with at worst canonical singularities.

### 7.1.3 Canonical Fano 3-polytopes $\Delta$ with $\dim(\Delta^{\mathrm{FI}}) = 1$ and $0 \in (\Delta^{\mathrm{FI}})^{\circ}$

Using Equation (7.2), there exist exactly 20 canonical Fano 3-polytopes $\Delta \subseteq M_{\mathbb{R}}$ with 1-dimensional rational Fine interior $\Delta^{\mathrm{FI}}$ and the origin $0 \in M$ as the center interior lattice point of $\Delta^{\mathrm{FI}}$. Simultaneously, Proposition 2.3.25 and Definition 2.3.22 provides that these polytopes are not almost reflexive because the origin $0 \in N$ is contained in the interior of $[\Delta^*]$, but $[\Delta^*]$ is 2-dimensional (*i.e.*, not 3-dimensional as $\Delta$ is).

We continue investigating relevant combinatorial properties of all polytopes living in this small subclass in an analogue way to Subsection 7.1.2: Let $\Delta \subseteq M_{\mathbb{R}}$ be a canonical Fano 3-polytopes with $\dim(\Delta^{\mathrm{FI}}) = 1$ and $0 \in (\Delta^{\mathrm{FI}})^{\circ}$. We denote by $v_\Delta$ and $v'_\Delta = -v_\Delta \in M$ the unique primitive lattice points of the line containing the 1-dimensional rational Fine interior $\Delta^{\mathrm{FI}}$ that contains the origin in its interior.

In this subclass, a projection of $\Delta$ along $v_\Delta$ (or $v'_\Delta$) on the plane $\langle b_1, b_2\rangle$ with respect to a lattice basis $\{v_\Delta, b_1, b_2\}$ (or $\{v'_\Delta, b_1, b_2\}$) provides in each case a reflexive polygon $\overline{\Delta}$. To be precise, we got the distribution of 3 different types of reflexive polygons (Figure 7.1) listed in Table 7.2. In particular, these three types of reflexive polygons coincide with the three types we got in Subsection 7.1.2 (Table 7.1).

**Example 7.1.5.** Let $\Delta \subseteq M_{\mathbb{R}}$ be a canonical Fano 3-polytope given by the convex hull

$$\mathrm{conv}((0,1,0),(2,1,1),(-2,-3,-5),(2,1,9))$$

(ID 547393, Figure A.2(a), Table A.4). Then the Fine interior $\Delta^{\mathrm{FI}}$ of $\Delta$ is 1-dimensional with

$$\Delta^{\mathrm{FI}} = \mathrm{conv}((0,0,-1/2),(0,0,1/2))$$

and therefore $v_\Delta = (0,0,1), v'_\Delta = (0,0,-1)$ (Table A.5). To project $\Delta$ along $v_\Delta$ (or $v'_\Delta$) on a plane with respect to a lattice basis, we have to choose two lattice points $b_1, b_2 \in M$ such that $\{v_\Delta, b_1, b_2\}$ (or $\{v'_\Delta, b_1, b_2\}$) builds a basis of $M$. In this case, we can take $b_1 = (1,0,0)$ and $b_2 = (0,1,0)$ to obtain

$$\begin{aligned}
(0,1,0) &= 0\cdot v_\Delta + 0\cdot b_1 + 1\cdot b_2 && \longmapsto (0,1),\\
(2,1,1) &= 1\cdot v_\Delta + 2\cdot b_1 + 1\cdot b_2 && \longmapsto (2,1),\\
(-2,-3,-5) &= (-5)\cdot v_\Delta + (-2)\cdot b_1 + (-3)\cdot b_2 && \longmapsto (-2,-3),
\end{aligned}$$

**Table 7.2: Distribution with Respect to Projection Results.** Table contains: Distribution of all 20 canonical Fano 3-polytopes $\Delta$ with $\dim(\Delta^{\mathrm{FI}}) = 1$ and $0 \in (\Delta^{\mathrm{FI}})^{\circ}$ with respect to their projection results $\overline{\Delta}$ that are reflexive polygons and links to pictures of the projection results. The third column displays the toric variety $X_{\Sigma_{\overline{\Delta}^*}}$ associated with the dual reflexive polygon $\overline{\Delta}^*$ constructed by corresponding spanning fan $\Sigma_{\overline{\Delta}^*}$.

| reflexive polygon $\overline{\Delta}$ | $\#\{\Delta \mid \dim(\Delta^{\mathrm{FI}}) = 1, 0 \in (\Delta^{\mathrm{FI}})^{\circ}\}$ | $X_{\Sigma_{\overline{\Delta}^*}}$ |
|---|---|---|
| Figure 7.1(a) | 7 | $\mathbb{P}^2$ |
| Figure 7.1(b) | 9 | $\mathbb{P}(1,1,2)$ |
| Figure 7.1(c) | 4 | $\mathbb{P}^1 \times \mathbb{P}^1$ |

$$(2,1,9) = 9 \cdot v_{\Delta} + 2 \cdot b_1 + 1 \cdot b_2 \qquad\qquad \longmapsto \ (2,1).$$

Therefore, we get the projection result (Table A.6)

$$\overline{\Delta} = \mathrm{conv}((0,1),(2,1),(-2,-3),(2,1)) = \mathrm{conv}((0,1),(2,1),(-2,-3))$$

that is a reflexive polygon isomorphic to the one in Figure 7.1(b) (Remark 2.3.16).

Similar to Subsection 7.1.2, we do not have to project the whole polytope $\Delta$ along $v_{\Delta}$ (or $v'_{\Delta}$). It is sufficient to project only one of two possible facets $\theta_{\mathrm{proj}}, \theta'_{\mathrm{proj}} \preceq \Delta$ of $\Delta$ that we find using a slightly modified version of the algorithm applied in the previous subsection: Let $\{v_i \mid 1 \leq i \leq f\}$ be the set of all inward-pointing facet normals of the given polytope $\Delta$, where $f$ denotes the number of facet of $\Delta$. Pairing computations with $v_{\Delta}$ showed that the $f$-tuple

$$(\langle v_{\Delta}, v_1 \rangle, \ldots, \langle v_{\Delta}, v_f \rangle)$$

contains exactly one negative integer. The first requested facet $\theta_{\mathrm{proj}}$ is the facet corresponding to the inward-pointing facet normal which produces this negative integer. The second facet $\theta'_{\mathrm{proj}}$ is obtained by a repetition of the algorithm with $v'_{\Delta}$. An easier way to detect the facets $\theta_{\mathrm{proj}}$ and $\theta'_{\mathrm{proj}} \preceq \Delta$, is to check which facets of $\Delta$ contain the primitive lattice points $v_{\Delta}$ and $v'_{\Delta}$, respectively, as interior lattice points.

**Example 7.1.6.** Continuing Example 7.1.5 (ID 547393), the inward-pointing facet normals of $\Delta$ are $v_1 = (-1,-2,2)$, $v_2 = (-1,1,0)$, $v_3 = (0,-1,0)$, and $v_4 = (9,-2,-2)$. An double application of the algorithm from above yields $(\langle v_{\Delta}, v_1 \rangle, \ldots, \langle v_{\Delta}, v_4 \rangle) = (2,0,0,-2)$ and $(\langle v'_{\Delta}, v_1 \rangle, \ldots, \langle v'_{\Delta}, v_4 \rangle) = (-2,0,0,2)$. Therefore, the corresponding facets to $v_1$ and $v_4$ are

$$\theta_{\mathrm{proj}} = \mathrm{conv}((0,1,0),(2,1,1),(-2,-3,-5))$$

and

$$\theta'_{\mathrm{proj}} = \mathrm{conv}((0,1,0),(-2,-3,-5),(2,1,9)),$$

respectively (Figure A.2(a), Table A.6). Projections of both facets along $v_{\Delta}$ (or $v'_{\Delta}$) yield the same reflexive polygon $\overline{\Delta} = \mathrm{conv}((0,1),(2,1),(-2,-3))$ as we received in Example 7.1.5 by projecting the polytope $\Delta$.

**Remark 7.1.7.** The complete data of all 20 canonical Fano 3-polytopes with $\dim(\Delta^{\mathrm{FI}}) = 1$ and $0 \in (\Delta^{\mathrm{FI}})^{\circ}$ is listed in the Appendix in Table A.4, A.5, A.6, and A.7. The corresponding polytope pictures include the Fine interior $\Delta^{\mathrm{FI}}$ (coloured red and the origin grey with a red margin) and the facets $\theta_{\mathrm{proj}}, \theta'_{\mathrm{proj}} \preceq \Delta$ (grey dotted) that are used to project can be viewed in the Appendix in Figure A.2. The surfaces corresponding to the reflexive polygons $\overline{\Delta}$ (projection results) are boxed top right.

### 7.1.4 Canonical Fano 3-polytopes $\Delta$ with $\dim(\Delta^{\mathrm{FI}}) = 3$ and $0 \in \mathrm{vert}(\Delta^{\mathrm{FI}})$

We again used the Hilbert basis and Equation (7.2) to obtain that there exist exactly 49 canonical Fano 3-polytopes $\Delta$ with $\dim(\Delta^{\mathrm{FI}}) = 3$ and $0 \in M$ a vertex of $\Delta^{\mathrm{FI}}$. Simultaneously, the criterion in Proposition 2.3.25 and Definition 2.3.22 fails because the origin $0 \in N$ is a vertex of $[\Delta^*]$, *i.e.*, these polytopes are not almost reflexive.

**Remark 7.1.8.** The complete data and the corresponding pictures to all 49 canonical Fano 3-polytopes with $\dim(\Delta^{\mathrm{FI}}) = 3$ are listed in the Appendix in Table A.8 and A.9 and can be viewed in Figure A.3.

## 7.2 Classification of Non-degenerate Surfaces of Geometric Genus 1

Section 7.1 provides the classification of all canonical Fano 3-polytopes $\Delta$ into four subclasses depending on their Fine interiors, along with a large amount of combinatorial data associated with these polytopes. Now, we aim at classifying all $\Delta$-non-degenerate affine surfaces $Z_\Delta$ of geometric genus 1 up to birational equivalence.

Let $\Delta \subseteq M_{\mathbb{R}}$ a canonical Fano 3-polytope and $Z_\Delta \subseteq \mathbb{T}^3$ the $\Delta$-non-degenerate affine surfaces defined by a generic Laurent polynomial $f_\Delta$ whose Newton polytope is the given polytope $\Delta$ (Definition 2.3.28). Following Remark 2.3.36, the $\Delta$-non-degenerate surface $Z_\Delta$ has a canonical model $Y$ and a minimal model $Y'$ because the Fine interior $\Delta^{\mathrm{FI}}$ of each canonical Fano 3-polytope $\Delta$ is not empty. The fastest way to see this, is to apply again Remark 2.3.36. A more explicit way would be to use our concrete classification results from Section 7.1. Furthermore, a minimal model $Y'$ of $Z_\Delta$ is, in particular, a smooth surface (Theorem 2.1.11).

Using a generalization of Theorem 6.3.3 that is not limited to $K3$-surfaces, a parallel step to achieving our classification aim is to compute the stringy Euler numbers of a canonical model of the considered surfaces in a purely combinatorial way. Since our main interest is on the surface classification and not directly on the minimal model program, we compute the stringy Euler number $e_{\mathrm{str}}(Y)$ of a canonical model $Y$, whereby $e_{\mathrm{str}}(Y) = e(Y')$ for a minimal model $Y'$ (page 15). Before we follow up with the mentioned generalization, we need to introduce some general combinatorial objects:

**Definition 7.2.1.** Let $\Delta \subseteq M_{\mathbb{R}}$ be a lattice polytope with non-empty Fine interior $\Delta^{\mathrm{FI}}$ (Definition 2.3.24). Then the *support of the Fine interior* $\Delta^{\mathrm{FI}}$ of $\Delta$ is defined as

$$\mathrm{supp}(\Delta^{\mathrm{FI}}) := \{n \in N \mid \langle y, n\rangle = \kappa^{\Delta}(n) + 1 \quad \text{for some } y \in \Delta^{\mathrm{FI}}\} \subseteq N.$$

The convex rational polytope

$$\Delta^{\mathrm{can}} := \bigcap_{s \in \mathrm{supp}(\Delta^{\mathrm{FI}})} \Gamma_0^{\Delta}(s) \subseteq M_{\mathbb{R}}$$

containing $\Delta$ is called *canonical hull* of $\Delta$, where

$$\Gamma_0^{\Delta}(n) := \{y \in M_{\mathbb{R}} \mid \langle y, n\rangle \geq \kappa^{\Delta}(n)\}$$

is a halfspace for any non-zero lattice point $n \in N$ and

$$\kappa^{\Delta} : N_{\mathbb{R}} \to \mathbb{R},\ x \mapsto \min_{y \in \mathrm{vert}(\Delta)} \{\langle y, x\rangle\}$$

the support function for the normal fan $\Sigma^{\Delta}$ (Equation (2.2)).

The support $\mathrm{supp}(\Delta^{\mathrm{FI}})$ of the Fine interior $\Delta^{\mathrm{FI}}$ is a finite subset in $N$ consisting of finitely many non-zero primitive lattice points $v_1, \ldots, v_l \in N$ such that $\sum_{i=1}^{l} \mathbb{R}_{\geq 0} v_i = N_{\mathbb{R}}$ [Bat17, Remark 2.12].

**Conjecture 7.2.2.** *Let $\Delta \subseteq M_{\mathbb{R}}$ be a canonical Fano 3-polytope. Denote by $Y$ a canonical model of a $\Delta$-non-degenerate affine surface $Z_{\Delta} \subseteq \mathbb{T}^3$. Then the stringy Euler number of $Y$ can be computed by the combinatorial formula*

$$e_{\mathrm{str}}(Y) = v(\Delta) - \sum_{\substack{\theta \preceq \Delta^{\mathrm{can}} \\ \dim(\theta)=2}} \theta_2 + \sum_{\substack{\theta \preceq \Delta^{\mathrm{can}} \\ \dim(\theta)=1}} \theta_1 + \omega,$$

*where*

$$\theta_2 := \begin{cases} 0, & \dim([\theta \cap \Delta]) < 2 \\ v([\theta \cap \Delta]), & \dim([\theta \cap \Delta]) = 2 \end{cases}$$

*for any 2-dimensional face $\theta \preceq \Delta^{\mathrm{can}}$,*

$$\theta_1 := \begin{cases} 0, & \dim([\theta \cap \Delta]) < 1 \\ v([\theta \cap \Delta]) \cdot v(\theta^{\mathrm{supp}}), & \dim([\theta \cap \Delta]) = 1 \end{cases}$$

*with*

$$\theta^{\mathrm{supp}} := \mathrm{conv}(s \in \mathrm{supp}(\Delta^{\mathrm{FI}}) \mid s \text{ reaches its minimum } \kappa^{\Delta}(s) \text{ on } \mathrm{vert}([\theta \cap \Delta])) \subseteq N_{\mathbb{R}}$$

*for any 1-dimensional face $\theta \preceq \Delta^{\mathrm{can}}$, and*

$$\omega := |\{v \in \mathrm{vert}(\Delta^{\mathrm{can}}) \mid v \notin \mathrm{vert}(\Delta)\}|$$

*with $[\theta \cap \Delta] = \mathrm{conv}((\theta \cap \Delta) \cap M)$.*

**Remark 7.2.3.** Let $\Delta \subseteq M_{\mathbb{R}}$ be a canonical Fano 3-polytope. Denote by $Y'$ a smooth minimal model of a $\Delta$-non-degenerate affine surface $Z_\Delta \subseteq \mathbb{T}^3$ (Theorem 2.1.11). Then we obtain by Noether's Theorem [CLS11, Theorem 10.5.3]

$$12 \cdot \chi(Y', \mathcal{O}_{Y'}) = c_1(Y')^2 + c_2(Y').$$

Combining this equation with $\dim(H^0(Y', \mathcal{O}_{Y'})) = 1$, $\dim(H^1(Y', \mathcal{O}_{Y'})) = 0$, and

$$\dim(H^2(Y', \mathcal{O}_{Y'})) = p_g(Y') = |\Delta^\circ \cap M| = 1$$

(Equation (7.1)), we get

$$24 = 12 \cdot (1 + 0 + 1) = c_1(Y')^2 + c_2(Y').$$

### 7.2.1 $K3$-surfaces

Let $\Delta \subseteq M_{\mathbb{R}}$ be one of the 665,599 almost reflexive canonical Fano 3-polytopes (Subsection 7.1.1). Then Proposition 2.3.29 provides that a canonical model $Y$ of a $\Delta$-non-degenerate affine surface $Z_\Delta \subseteq \mathbb{T}^3$ is birational to a $K3$-surface. Computed with the formula in Theorem 6.3.3, the stringy Euler number of $Y$ is given by

$$e_{\mathrm{str}}(Y) = v(\Delta) - \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=2}} \frac{1}{n_\theta} \cdot v(\theta) + \sum_{\substack{\theta \preceq \Delta \\ \dim(\theta)=1}} v(\theta) \cdot v(\theta^*) = 24,$$

where $n_\theta$ denotes the lattice distance from a facet $\theta \preceq \Delta$ to the origin (Definition 2.3.5). The same result arises already from the fact that we are dealing with $K3$-surfaces (Theorem 6.3.1 (proof)).

**Remark 7.2.4.** The support $\mathrm{supp}(\Delta^{\mathrm{FI}})$ of $\Delta^{\mathrm{FI}}$ and the canonical hull $\Delta^{\mathrm{can}}$ of an almost reflexive canonical Fano 3-polytope $\Delta \subseteq M_{\mathbb{R}}$ are given by

$$\mathrm{supp}(\Delta^{\mathrm{FI}}) = [\Delta^*] \cap N$$

and

$$\Delta^{\mathrm{can}} = [\Delta^*]^*$$

that is reflexive because $\Delta$ is almost reflexive, *i.e.*, $[\Delta^*]$ is reflexive by Definition 2.3.22.

We close the $K3$-surface classification with two detailed examples computing the stringy Euler number of our first two running examples in Figure 2.3 with the predicted formula in Conjecture 7.2.2:

**Example 7.2.5.** Let us consider the (almost) reflexive canonical Fano 3-polytope $\Delta_1 = \mathrm{conv}(P_1, P_2, P_3, P_4) \subseteq M_{\mathbb{R}}$ (Figure 2.3(a)) with vertices

$$P_1 := (1, 0, 0),\ P_2 := (0, 1, 0),\ P_3 := (0, 0, 1), \text{ and } P_4 := (-1, -1, -1)$$

and ${\Delta_1}^{\mathrm{FI}} = \{0\}$. The normalized volume $v(\Delta_1)$ of $\Delta_1$ equals 4. To compute the stringy Euler number $e_{\mathrm{str}}(Y)$ of a canonical model $Y$ corresponding to $\Delta_1$ with the formula in Conjecture 7.2.2, we have to collect the missing relevant data: Using Remark 7.2.4, we get

$$\mathrm{supp}({\Delta_1}^{\mathrm{FI}}) = [{\Delta_1}^*] \cap N = \{S_i \,|\, 1 \leq i \leq 34\}$$

with $S_1 \coloneqq (-1,-1,-1)$, $S_2 \coloneqq (-1,-1,0)$, $S_3 \coloneqq (-1,-1,1)$, $S_4 \coloneqq (-1,-1,2)$, $S_5 \coloneqq (-1,-1,3)$, $S_6 \coloneqq (-1,0,-1)$, $S_7 \coloneqq (-1,0,0)$, $S_8 \coloneqq (-1,0,1)$, $S_9 \coloneqq (-1,0,2)$, $S_{10} \coloneqq (-1,1,-1)$, ..., $S_{34} \coloneqq (3,-1,-1)$ and

$${\Delta_1}^{\mathrm{can}} = [{\Delta_1}^*]^* = ({\Delta_1}^*)^* = \Delta_1$$

that is reflexive because $\Delta_1$ is reflexive (Figure 6.6(a)).

Let us denote the 2-dimensional faces of $\Delta_1$ with $\theta_{P_{ijk}} \preceq \Delta_1$ defined as $\theta_{P_{ijk}} \coloneqq \mathrm{conv}(P_i, P_j, P_k)$ $(1 \leq i < j < k \leq 4)$ and the 1-dimensional faces of $\Delta_1$ with $\theta_{P_{ij}} \preceq \Delta_1$ defined as $\theta_{P_{ij}} \coloneqq \mathrm{conv}(P_i, P_j)$ $(1 \leq i < j \leq 4)$. Of course, all polytopes $[\theta_{P_{ijk}} \cap \Delta_1]$ have dimension 2 because $\theta_{P_{ijk}} \preceq \Delta_1$, *i.e.*, $[\theta_{P_{ijk}} \cap \Delta_1] = \theta_{P_{ijk}} \cap \Delta_1 = \theta_{P_{ijk}}$ and $\dim(\theta_{P_{ijk}}) = 2$. Moreover, the normalized volume $v([\theta_{P_{ijk}} \cap \Delta_1]) = v(\theta_{P_{ijk}})$ equals 1. Applying the same arguments, all polytopes $[\theta_{P_{ij}} \cap \Delta_1]$ have dimension 1 with normalized volume $v([\theta_{P_{ij}} \cap \Delta_1]) = 1$.

In our last step, we determine the normalized volumes of all line segments $\theta^{\mathrm{supp}} = \mathrm{conv}(s \in \mathrm{supp}({\Delta_1}^{\mathrm{FI}}) \,|\, s$ reaches its minimum $\kappa^{\Delta_1}(s)$ on $\mathrm{vert}([\theta_{P_{ij}} \cap \Delta_1]))$ corresponding to 1-dimensional faces $\theta_{P_{ij}} \preceq \Delta_1$ of $\Delta_1$. Therefore, we compute $\kappa^{\Delta_1}(s)$ for all $s \in \mathrm{supp}({\Delta_1}^{\mathrm{FI}})$ (Equation (2.2)):

$$\begin{aligned}
\kappa^{\Delta_1}(S_1) &= \min\nolimits_{P \in \mathrm{vert}(\Delta_1)} \{\langle P, S_1 \rangle\} &&= \min\{-1,-1,-1,3\} = -1,\\
\kappa^{\Delta_1}(S_2) &= \min\nolimits_{P \in \mathrm{vert}(\Delta_1)} \{\langle P, S_2 \rangle\} &&= \min\{-1,-1,0,2\} = -1,\\
\kappa^{\Delta_1}(S_3) &= \min\nolimits_{P \in \mathrm{vert}(\Delta_1)} \{\langle P, S_3 \rangle\} &&= \min\{-1,-1,1,1\} = -1,\\
\kappa^{\Delta_1}(S_4) &= \min\nolimits_{P \in \mathrm{vert}(\Delta_1)} \{\langle P, S_4 \rangle\} &&= \min\{-1,-1,2,0\} = -1,\\
\kappa^{\Delta_1}(S_5) &= \min\nolimits_{P \in \mathrm{vert}(\Delta_1)} \{\langle P, S_5 \rangle\} &&= \min\{-1,-1,3,-1\} = -1,\\
\kappa^{\Delta_1}(S_6) &= \min\nolimits_{P \in \mathrm{vert}(\Delta_1)} \{\langle P, S_6 \rangle\} &&= \min\{-1,0,-1,2\} = -1,\\
\kappa^{\Delta_1}(S_7) &= \min\nolimits_{P \in \mathrm{vert}(\Delta_1)} \{\langle P, S_7 \rangle\} &&= \min\{-1,0,0,1\} = -1,\\
\kappa^{\Delta_1}(S_8) &= \min\nolimits_{P \in \mathrm{vert}(\Delta_1)} \{\langle P, S_8 \rangle\} &&= \min\{-1,0,1,0\} = -1,\\
\kappa^{\Delta_1}(S_9) &= \min\nolimits_{P \in \mathrm{vert}(\Delta_1)} \{\langle P, S_9 \rangle\} &&= \min\{-1,0,2,-1\} = -1,\\
\kappa^{\Delta_1}(S_{10}) &= \min\nolimits_{P \in \mathrm{vert}(\Delta_1)} \{\langle P, S_{10} \rangle\} &&= \min\{-1,1,-1,1\} = -1,\\
&\dots\\
\kappa^{\Delta_1}(S_{34}) &= \min\nolimits_{P \in \mathrm{vert}(\Delta_1)} \{\langle P, S_{34} \rangle\} &&= \min\{3,-1,-1,-1\} = -1
\end{aligned}$$

with $\mathrm{vert}(\Delta_1) = \{P_1, P_2, P_3, P_4\}$. Using this information, we get

$$\theta^{\mathrm{supp}}_{P_{12}} = \mathrm{conv}(S_1, S_2, S_3, S_4, S_5),\ \theta^{\mathrm{supp}}_{P_{13}} = \mathrm{conv}(S_1, S_6, S_{10}, S_{13}, S_{15}),$$

$$\theta_{P_{14}}^{\text{supp}} = \operatorname{conv}(S_5, S_9, S_{12}, S_{14}, S_{15}),\ \theta_{P_{23}}^{\text{supp}} = \operatorname{conv}(S_1, S_{16}, S_{25}, S_{31}, S_{34}),$$
$$\theta_{P_{24}}^{\text{supp}} = \operatorname{conv}(S_5, S_{19}, S_{27}, S_{32}, S_{34}),\ \theta_{P_{34}}^{\text{supp}} = \operatorname{conv}(S_{15}, S_{24}, S_{30}, S_{33}, S_{34})$$

with

$$v(\theta_{P_{12}}^{\text{supp}}) = v(\theta_{P_{13}}^{\text{supp}}) = v(\theta_{P_{14}}^{\text{supp}}) = v(\theta_{P_{23}}^{\text{supp}}) = v(\theta_{P_{24}}^{\text{supp}}) = v(\theta_{P_{34}}^{\text{supp}}) = 4.$$

A combination of all data yields

$$\begin{aligned} e_{\text{str}}(Y) &= v(\Delta_1) - \sum_{\substack{\theta \preceq {\Delta_1}^{\text{can}} \\ \dim(\theta)=2}} \theta_2 + \sum_{\substack{\theta \preceq {\Delta_1}^{\text{can}} \\ \dim(\theta)=1}} \theta_1 + \omega \\ &= 4 - (1+1+1+1) + (1\cdot 4 + 1\cdot 4 + 1\cdot 4 + 1\cdot 4 + 1\cdot 4 + 1\cdot 4) + 0 = 24, \end{aligned}$$

where $\omega = |\{v \in \operatorname{vert}({\Delta_1}^{\text{can}}) \,|\, v \notin \operatorname{vert}(\Delta_1)\}| = 0$ because ${\Delta_1}^{\text{can}} = \Delta_1$, *i.e.*, $c_2^{\text{str}}(Y) = 24$ and $c_1(Y)^2 = 0$ (Remark 7.2.3).

**Example 7.2.6.** Let us consider the almost reflexive canonical Fano 3-polytope $\Delta_2 = \operatorname{conv}(P_1, P_2, P_3, P_4) \subseteq M_{\mathbb{R}}$ (Figure 2.3(b) and 6.5(a)) with vertices

$$P_1 \coloneqq (1,0,0),\ P_2 \coloneqq (0,1,0),\ P_3 \coloneqq (0,0,1), \text{ and } P_4 \coloneqq (-1,-1,-2)$$

and ${\Delta_2}^{\text{FI}} = \{0\}$. The normalized volume $v(\Delta_2)$ of $\Delta_2$ equals 5. To compute the stringy Euler number $e_{\text{str}}(Y)$ of a canonical model $Y$ corresponding to $\Delta_2$ with the formula in Conjecture 7.2.2, we have to collect the missing relevant data: Using Remark 7.2.4, we get

$$\operatorname{supp}({\Delta_2}^{\text{FI}}) = [{\Delta_2}^*] \cap N = \{S_i \,|\, 1 \leq i \leq 33\}$$

with $S_1 \coloneqq (-1,-1,-1)$, $S_2 \coloneqq (-1,-1,0)$, $S_3 \coloneqq (-1,-1,1)$, $S_4 \coloneqq (-1,0,-1)$, $S_5 \coloneqq (-1,0,0)$, $S_6 \coloneqq (-1,0,1)$, $S_7 \coloneqq (-1,1,-1)$, $S_8 \coloneqq (-1,1,0)$, $S_9 \coloneqq (-1,2,-1)$, $S_{10} \coloneqq (-1,2,0)$, …, $S_{33} \coloneqq (4,-1,-1)$ and

$${\Delta_2}^{\text{can}} = [{\Delta_2}^*]^* = \operatorname{conv}(P_1, P_2, P_3, P_4, P_5)$$

with $P_5 \coloneqq (0,0,-1)$ that is reflexive because $\Delta_2$ is almost reflexive.

Let us denote the 2-dimensional faces of ${\Delta_2}^{\text{can}}$ with $\theta_{P_{ijk}} \preceq {\Delta_2}^{\text{can}}$ defined as $\theta_{P_{ijk}} \coloneqq \operatorname{conv}(P_i, P_j, P_k)$ ($1 \leq i < j < k \leq 5$) and the 1-dimensional faces of ${\Delta_2}^{\text{can}}$ with $\theta_{P_{ij}} \preceq {\Delta_2}^{\text{can}}$ defined as $\theta_{P_{ij}} \coloneqq \operatorname{conv}(P_i, P_j)$ ($1 \leq i < j \leq 5$). Then all polytopes $[\theta_{P_{ijk}} \cap \Delta_2]$ have dimension 2 and the same normalized volume $v([\theta_{P_{ijk}} \cap \Delta_2]) = 1$. All polytopes $[\theta_{P_{ij}} \cap \Delta_2]$ have dimension 1 and also the same normalized volume $v([\theta_{P_{ij}} \cap \Delta_2]) = 1$.

It remains to determine the normalized volumes of all line segments $\theta^{\text{supp}} = \operatorname{conv}(s \in \operatorname{supp}({\Delta_2}^{\text{FI}}) \,|\, s$ reaches its minimum $\kappa^{\Delta_2}(s)$ on $\operatorname{vert}([\theta_{P_{ij}} \cap \Delta_2]))$ corresponding to 1-dimensional faces $\theta_{P_{ij}} \preceq {\Delta_2}^{\text{can}}$ with $\dim([\theta_{P_{ij}} \cap \Delta_2]) = 1$. Therefore, we compute $\kappa^{\Delta_2}(s)$ for all $s \in \operatorname{supp}({\Delta_2}^{\text{FI}})$ (Equation (2.2)):

$$
\begin{aligned}
\kappa^{\Delta_2}(S_1) &= \min_{P\in\mathrm{vert}(\Delta_2)}\{\langle P,S_1\rangle\} &&= \min\{-1,-1,-1,4\} = -1,\\
\kappa^{\Delta_2}(S_2) &= \min_{P\in\mathrm{vert}(\Delta_2)}\{\langle P,S_2\rangle\} &&= \min\{-1,-1,0,2\} = -1,\\
\kappa^{\Delta_2}(S_3) &= \min_{P\in\mathrm{vert}(\Delta_2)}\{\langle P,S_3\rangle\} &&= \min\{-1,-1,1,0\} = -1,\\
\kappa^{\Delta_2}(S_4) &= \min_{P\in\mathrm{vert}(\Delta_2)}\{\langle P,S_4\rangle\} &&= \min\{-1,0,-1,3\} = -1,\\
\kappa^{\Delta_2}(S_5) &= \min_{P\in\mathrm{vert}(\Delta_2)}\{\langle P,S_5\rangle\} &&= \min\{-1,0,0,1\} = -1,\\
\kappa^{\Delta_2}(S_6) &= \min_{P\in\mathrm{vert}(\Delta_2)}\{\langle P,S_6\rangle\} &&= \min\{-1,0,1,-1\} = -1,\\
\kappa^{\Delta_2}(S_7) &= \min_{P\in\mathrm{vert}(\Delta_2)}\{\langle P,S_7\rangle\} &&= \min\{-1,1,-1,2\} = -1,\\
\kappa^{\Delta_2}(S_8) &= \min_{P\in\mathrm{vert}(\Delta_2)}\{\langle P,S_8\rangle\} &&= \min\{-1,1,0,0\} = -1,\\
\kappa^{\Delta_2}(S_9) &= \min_{P\in\mathrm{vert}(\Delta_2)}\{\langle P,S_9\rangle\} &&= \min\{-1,2,-1,1\} = -1,\\
\kappa^{\Delta_2}(S_{10}) &= \min_{P\in\mathrm{vert}(\Delta_2)}\{\langle P,S_{10}\rangle\} &&= \min\{-1,2,0,-1\} = -1,\\
&\dots\\
\kappa^{\Delta_2}(S_{33}) &= \min_{P\in\mathrm{vert}(\Delta_2)}\{\langle P,S_{33}\rangle\} &&= \min\{4,-1,-1,-1\} = -1
\end{aligned}
$$

with $\mathrm{vert}(\Delta_2) = \{P_1, P_2, P_3, P_4\}$. Using this information, we get

$$
\begin{aligned}
&\theta^{\mathrm{supp}}_{P_{24}} = \mathrm{conv}(S_{15}, S_{28}, S_{33}),\ \theta^{\mathrm{supp}}_{P_{13}} = \mathrm{conv}(S_1, S_4, S_7, S_9, S_{11}, S_{12}),\\
&\theta^{\mathrm{supp}}_{P_{14}} = \mathrm{conv}(S_6, S_{10}, S_{12}),\ \theta^{\mathrm{supp}}_{P_{34}} = \mathrm{conv}(S_{12}, S_{20}, S_{26}, S_{30}, S_{32}, S_{33}),\\
&\theta^{\mathrm{supp}}_{P_{12}} = \mathrm{conv}(S_1, S_2, S_3),\ \theta^{\mathrm{supp}}_{P_{23}} = \mathrm{conv}(S_1, S_{13}, S_{21}, S_{27}, S_{31}, S_{33})
\end{aligned}
$$

with

$$
v(\theta^{\mathrm{supp}}_{P_{24}}) = v(\theta^{\mathrm{supp}}_{P_{14}}) = v(\theta^{\mathrm{supp}}_{P_{12}}) = 2 \text{ and } v(\theta^{\mathrm{supp}}_{P_{13}}) = v(\theta^{\mathrm{supp}}_{P_{34}}) = v(\theta^{\mathrm{supp}}_{P_{23}}) = 5.
$$

A combination of all data yields

$$
\begin{aligned}
e_{\mathrm{str}}(Y) &= v(\Delta_2) - \sum_{\substack{\theta\preceq{\Delta_2}^{\mathrm{can}}\\ \dim(\theta)=2}} \theta_2 + \sum_{\substack{\theta\preceq{\Delta_2}^{\mathrm{can}}\\ \dim(\theta)=1}} \theta_1 + \omega\\
&= 5 - (1+1+1) + (1\cdot 2 + 1\cdot 5 + 1\cdot 2 + 1\cdot 5 + 1\cdot 2 + 1\cdot 5) + 1 = 24,
\end{aligned}
$$

where $\omega = |\{v \in \mathrm{vert}({\Delta_2}^{\mathrm{can}}) \mid v \notin \mathrm{vert}(\Delta_2)\}| = 1$, *i.e.*, $c_2^{\mathrm{str}}(Y) = 24$ and $c_1(Y)^2 = 0$ (Remark 7.2.3).

The applied Proposition 2.3.29 contains an if and only if condition, *i.e.*, all remaining 9,089 canonical models corresponding to canonical Fano 3-polytopes $\Delta$ with $\Delta^{\mathrm{FI}} \neq \{0\}$ are not birational to $K3$-surfaces.

### 7.2.2 Elliptic Surfaces of Kodaira Dimension 1 – Part I

Let $\Delta \subseteq M_{\mathbb{R}}$ be one of the 9,020 canonical Fano 3-polytopes with 1-dimensional Fine interior $\Delta^{\mathrm{FI}}$ having the origin as a vertex (Subsection 7.1.2). We are claiming that canonical models of surfaces defined by these 9,020 polytopes are birational to elliptic surfaces of Kodaira dimension 1:

**Conjecture 7.2.7.** *Let $\Delta \subseteq M_{\mathbb{R}}$ be a canonical Fano 3-polytope with $\dim(\Delta^{\mathrm{FI}}) = 1$ and $0 \in \mathrm{vert}(\Delta^{\mathrm{FI}})$. Then a canonical model of a $\Delta$-non-degenerate affine surface $Z_\Delta \subseteq \mathbb{T}^3$ is birational to an elliptic surfaces over $\mathbb{P}^1$ of Kodaira dimension 1.*

**Definition 7.2.8.** Let $C$ a smooth projective curve. An *elliptic surface* $Y$ over $C$ is a smooth projective surface $Y$ with an elliptic fibration over $C$, *i.e.*, a surjective morphism $m : Y \to C$ such that almost all fibres are smooth curves of genus 1 and no fibre contains an exceptional curve that is a smooth rational curve of self-intersection $-1$ [SS10].

**Definition 7.2.9.** Let $Y$ be a variety over $\mathbb{C}$ and $Y'$ a smooth complete variety birational to $Y$. Moreover, the canonical ring $R(Y)$ of $Y$ is defined as $\bigoplus_{m\geq 0} H^0(Y', \mathcal{O}_{Y'}(mK_{Y'}))$. Then the *Kodaira dimension* $\kappa(Y)$ of $Y$ is defined as $\mathrm{trans.deg}_{\mathbb{C}} R(Y) - 1$ if $R(Y) \neq \mathbb{C}$ and $-\infty$ otherwise.

**Remark 7.2.10.** Such varieties $Y'$ from the above definition exist by the resolution Theorem of Hironaka [Hir64]. Furthermore, the Kodaira dimension of an arbitrary variety is well-defined and a birational invariant because for any two birational smooth complete varieties $Y'$ and $Y''$, the canonical rings $R(Y')$ and $R(Y'')$ are isomorphic [KMM87].

The Enriques-Kodaira classification of smooth projective surfaces [BHPdV04, Chapter VI] tells us that if $Y$ is an elliptic surface, then the Kodaira dimension $\kappa(Y)$ of $Y$ is smaller or equal to 1.

Conjecture 7.2.7 has been proved for the $9 = 104 - 95$ examples stemming from 3-dimensional weighted projective spaces with at worst canonical singularities in [CG11] (Table A.2). Corti's and Golyshev's work is a example of a paper in which a known result inspired a conjecture of a more general statement while, at the same time, showing the importance of the hidden combinatorial information: The structure of the elliptic fibration over $\mathbb{P}^1$ seems to be combinatorially encoded by the unique primitive lattice point $v_\Delta$ of the line containing the 1-dimensional Fine interior $\Delta^{\mathrm{FI}}$ (page 97), where $\Delta$ is one of these 9,020 polytopes.

A strong indication that we are indeed dealing with elliptic surfaces with an explicit map to $\mathbb{P}^1$ is our result that the projection of all 9,020 canonical Fano 3-polytopes $\Delta$ with $\dim(\Delta^{\mathrm{FI}}) = 1$ and $0 \in \mathrm{vert}(\Delta^{\mathrm{FI}})$ along $v_\Delta$ onto a plane leads to reflexive polygons $\overline{\Delta}$ (page 97 and Table 7.1).

Let $\Delta \subseteq M_{\mathbb{R}}$ be a canonical Fano 3-polytope with $\dim(\Delta^{\mathrm{FI}}) = 1$ and $0 \in \mathrm{vert}(\Delta^{\mathrm{FI}})$. Furthermore, let $\{v_i \,|\, 1 \leq i \leq f\}$ be the set of all inward-pointing facet normals of $\Delta$ and $f$ the number of facet of $\Delta$. By Subsection 7.1.2, page 98, we know that

$$(\langle v_\Delta, v_1\rangle, \ldots, \langle v_\Delta, v_f\rangle)$$

contains exactly one negative integer.

**Conjecture 7.2.11.** *Let $\Delta \subseteq M_{\mathbb{R}}$ be a canonical Fano 3-polytope with $\dim(\Delta^{\mathrm{FI}}) = 1$ and $0 \in \mathrm{vert}(\Delta^{\mathrm{FI}})$ and $Y$ an elliptic surfaces over $\mathbb{P}^1$ of Kodaira dimension 1 that is birational to a canonical model of a $\Delta$-non-degenerate affine surface $Z_\Delta \subseteq \mathbb{T}^3$. Then the* *elliptic fibration* $m : Y \to \mathbb{P}^1$ *expressing* $Y$ *as an elliptic surface of Kodaira dimension* 1 *is given by*

$$m = z_0^{-\langle v_\Delta, v_1\rangle} \cdot \ldots \cdot z_{f-1}^{-\langle v_\Delta, v_f\rangle},$$

*where* $z_0, \ldots, z_{f-1}$ *denote the homogenous coordinates of the toric variety* $X_{\Sigma^\Delta}$ [Cox95].

We proved this conjecture for the 9 examples of Corti and Golyshev by comparing their results in [CG11, Table 2] with our results that are listed in Table A.3.

Parallel to the conjectures, we claim that the stringy Euler numbers $e_{\text{str}}(Y)$ of all canonical models corresponding to one of these 9,020 polytopes with $\dim(\Delta^{\text{FI}}) = 1$ and $0 \in \text{vert}(\Delta^{\text{FI}})$ equal 24 and are computable via the formula in Conjecture 7.2.2. We checked and confirmed this for the 9 examples of Corti and Golyshev (Table A.3).

In this sense, $Y$ meets the invariant properties $c_2^{\text{str}}(Y) \geq 0$ and $c_1(Y)^2 = 0$ of (minimal) surfaces of Kodaira dimension 1 as well as the invariant properties $c_2^{\text{str}}(Y) = 24$ and $c_1(Y)^2 = 0$ of $K3$-surfaces [BHPdV04, Chapter VI, Table 10]. Applying the if and only if condition in Proposition 2.3.29, we can exclude $Y$ to be a $K3$-surface. Moreover, knowing that $Y$ is a surface of Kodaira dimension 1 implies that $Y$ is an elliptic surface [BHPdV04, Chapter VI, Table 10]. The opposite direction is not true.

The following example computes the stringy Euler number of our third running example in Figure 2.3 that is at the same time one of Corti's and Golyshev's examples with the predicted formula in Conjecture 7.2.2:

**Example 7.2.12.** Let us consider the canonical Fano 3-polytope from Example 7.1.2 and 7.1.3 with ID 547324 (Figure A.1(a), Table A.1, A.2, and A.3), *i.e.*, $\Delta = \text{conv}(P_1, P_2, P_3, P_4) \subseteq M_{\mathbb{R}}$ with $\dim(\Delta^{\text{FI}}) = 1$, $0 \in \text{vert}(\Delta^{\text{FI}})$, and vertices

$$P_1 := (2, 3, 8),\ P_2 := (1, 0, 0),\ P_3 := (0, 1, 0), \text{ and } P_4 := (-1, -1, -1).$$

The polytope $\Delta$ pictured in Figure A.1(a) and the polytope $\Delta_3$ pictured in Figure 2.3(c) and 6.7(a) are isomorphic (Remark 2.3.16) with normalized volume 20.

To compute the stringy Euler number $e_{\text{str}}(Y)$ of a canonical model $Y$ corresponding to $\Delta$ with the formula in Conjecture 7.2.2, we obtain $\text{supp}(\Delta^{\text{FI}}) = \{S_i \,|\, 1 \leq i \leq 18\}$ with $S_1 := (-1, -1, 1)$, $S_2 := (-1, -1, 2)$, $S_3 := (-1, -1, 3)$, $S_4 := (-1, 0, 1)$, $S_5 := (-1, 0, 2)$, $S_6 := (-1, 1, 0)$, $S_7 := (-1, 1, 1)$, $S_8 := (-1, 2, 0)$, $S_9 := (-1, 3, -1)$, $S_{10} := (0, -1, 1)$, ..., $S_{18} := (-2, -2, 1)$, which leads to

$$\Delta^{\text{can}} = \text{conv}(P_1, P_2, P_3, P_4, P_5)$$

with $P_5 := (0, 1, 4)$.

Let us denote the 2-dimensional faces of $\Delta^{\text{can}}$ with $\theta_{P_{ijkl}} \preceq \Delta^{\text{can}}$ defined as $\theta_{P_{ijkl}} := \text{conv}(P_i, P_j, P_k, P_l)$ $(1 \leq i < j < k \leq l \leq 5)$ and the 1-dimensional faces of $\Delta^{\text{can}}$ with $\theta_{P_{ij}} \preceq \Delta^{\text{can}}$ defined as $\theta_{P_{ij}} := \text{conv}(P_i, P_j)$ $(1 \leq i < j \leq 5)$. The following two tables list all facets $\theta_{P_{ijkl}}$ with $\dim([\theta_{P_{ijkl}} \cap \Delta]) = 2$ and the normalized volumes of $[\theta_{P_{ijkl}} \cap \Delta]$ as well as all edges $\theta_{P_{ij}}$ with $\dim([\theta_{P_{ij}} \cap \Delta]) = 1$ and the normalized volumes of $[\theta_{P_{ij}} \cap \Delta]$ :

| facet $\theta_{P_{ijkl}} \preceq \Delta^{\text{can}}$ with $\dim([\theta_{P_{ijkl}} \cap \Delta]) = 2$ | $v([\theta_{P_{ijkl}} \cap \Delta])$ |
|---|---|
| $\theta_{P_{1245}}$ | 5 |
| $\theta_{P_{2344}}$ | 1 |
| $\theta_{P_{1233}}$ | 4 |

| edge $\theta_{P_{ij}} \preceq \Delta^{\text{can}}$ with $\dim([\theta_{P_{ij}} \cap \Delta]) = 1$ | $v([\theta_{P_{ij}} \cap \Delta])$ |
|---|---|
| $\theta_{P_{24}}$ | 1 |
| $\theta_{P_{12}}$ | 1 |
| $\theta_{P_{13}}$ | 2 |
| $\theta_{P_{34}}$ | 1 |
| $\theta_{P_{23}}$ | 1 |

In our last step, we determine the normalized volumes of all line segments $\theta^{\text{supp}} = \operatorname{conv}(s \in \operatorname{supp}(\Delta^{\text{FI}}) \mid s \text{ reaches its minimum } \kappa^{\Delta}(s) \text{ on } \operatorname{vert}([\theta_{P_{ij}} \cap \Delta]))$ corresponding to 1-dimensional faces $\theta_{P_{ij}} \preceq \Delta^{\text{can}}$ with $\dim([\theta_{P_{ij}} \cap \Delta]) = 1$. Therefore, we compute $\kappa^{\Delta}(s)$ for all $s \in \operatorname{supp}(\Delta^{\text{FI}})$ (Equation (2.2)):

$$\begin{aligned}
\kappa^{\Delta}(S_1) &= \min\nolimits_{P\in\operatorname{vert}(\Delta)} \{\langle P, S_1\rangle\} &&= \min\{11,-1,-1,0\} = -1,\\
\kappa^{\Delta}(S_2) &= \min\nolimits_{P\in\operatorname{vert}(\Delta)} \{\langle P, S_2\rangle\} &&= \min\{-1,-1,0,2\} = -1,\\
\kappa^{\Delta}(S_3) &= \min\nolimits_{P\in\operatorname{vert}(\Delta)} \{\langle P, S_3\rangle\} &&= \min\{19,-1,-1,-1\} = -1,\\
\kappa^{\Delta}(S_4) &= \min\nolimits_{P\in\operatorname{vert}(\Delta)} \{\langle P, S_4\rangle\} &&= \min\{6,-1,0,0\} = -1,\\
\kappa^{\Delta}(S_5) &= \min\nolimits_{P\in\operatorname{vert}(\Delta)} \{\langle P, S_5\rangle\} &&= \min\{14,-1,0,-1\} = -1,\\
\kappa^{\Delta}(S_6) &= \min\nolimits_{P\in\operatorname{vert}(\Delta)} \{\langle P, S_6\rangle\} &&= \min\{1,-1,1,0\} = -1,\\
\kappa^{\Delta}(S_7) &= \min\nolimits_{P\in\operatorname{vert}(\Delta)} \{\langle P, S_7\rangle\} &&= \min\{9,-1,1,-1\} = -1,\\
\kappa^{\Delta}(S_8) &= \min\nolimits_{P\in\operatorname{vert}(\Delta)} \{\langle P, S_8\rangle\} &&= \min\{-4,-1,2,-1\} = -1,\\
\kappa^{\Delta}(S_9) &= \min\nolimits_{P\in\operatorname{vert}(\Delta)} \{\langle P, S_9\rangle\} &&= \min\{-1,-1,3,-1\} = -1,\\
\kappa^{\Delta}(S_{10}) &= \min\nolimits_{P\in\operatorname{vert}(\Delta)} \{\langle P, S_{10}\rangle\} &&= \min\{5,0,-1,0\} = -1,\\
&\dots\\
\kappa^{\Delta}(S_{18}) &= \min\nolimits_{P\in\operatorname{vert}(\Delta)} \{\langle P, S_{33}\rangle\} &&= \min\{-2,-2,-2,3\} = -2
\end{aligned}$$

with $\operatorname{vert}(\Delta) = \{P_1, P_2, P_3, P_4\}$. Using this information, we get

$$\theta^{\text{supp}}_{P_{24}} = \operatorname{conv}(S_3, S_5, S_7, S_8, S_9),\ \theta^{\text{supp}}_{P_{12}} = \operatorname{conv}(S_9, S_{18}),\ \theta^{\text{supp}}_{P_{13}} = \operatorname{conv}(S_{14}, S_{18}),$$
$$\theta^{\text{supp}}_{P_{34}} = \operatorname{conv}(S_3, S_{11}, S_{15}, S_{17}),\ \theta^{\text{supp}}_{P_{23}} = \operatorname{conv}(S_1, S_2, S_3, S_{18})$$

with

$$v(\theta^{\text{supp}}_{P_{24}}) = 4,\ \ v(\theta^{\text{supp}}_{P_{12}}) = 1,\ \ v(\theta^{\text{supp}}_{P_{13}}) = 1,\ \ v(\theta^{\text{supp}}_{P_{34}}) = 3,\ \text{and } v(\theta^{\text{supp}}_{P_{23}}) = 3.$$

A combination of all data yields

$$\begin{aligned}
e_{\text{str}}(Y) &= v(\Delta) - \sum_{\substack{\theta\preceq\Delta^{\text{can}}\\ \dim(\theta)=2}} \theta_2 + \sum_{\substack{\theta\preceq\Delta^{\text{can}}\\ \dim(\theta)=1}} \theta_1 + \omega\\
&= 20 - (5+1+4) + (1\cdot 4 + 1\cdot 1 + 2\cdot 1 + 1\cdot 3 + 1\cdot 3) + 1 = 24,
\end{aligned}$$

where $\omega = |\{v \in \operatorname{vert}(\Delta^{\text{can}}) \mid v \notin \operatorname{vert}(\Delta)\}| = 1$, *i.e.*, $c_2^{\text{str}}(Y) = 24 \geq 0$ and $c_1(Y)^2 = 0$ (Remark 7.2.3).

**Remark 7.2.13.** The detailed information about a small selection of the 9,020 canonical Fano 3-polytopes with $\dim(\Delta^{\text{FI}}) = 1$ and $0 \in \text{vert}(\Delta^{\text{FI}})$ can be found in the Appendix. To be precise, it is listed in Table A.1, A.2, and A.3 and can be viewed in Figure A.1, respectively (cf. Remark 7.1.4).

### 7.2.3 Elliptic Surfaces of Kodaira Dimension 1 – Part II

Let $\Delta \subseteq M_{\mathbb{R}}$ be one of the 20 canonical Fano 3-polytopes with 1-dimensional Fine interior $\Delta^{\text{FI}}$ having the origin as the center interior lattice point (Subsection 7.1.3). We are claiming that canonical models of surfaces defined by these 20 polytopes are birational to elliptic surfaces of Kodaira dimension 1:

**Conjecture 7.2.14.** *Let $\Delta \subseteq M_{\mathbb{R}}$ be a canonical Fano 3-polytope with $\dim(\Delta^{\text{FI}}) = 1$ and $0 \in (\Delta^{\text{FI}})^{\circ}$. Then a canonical model of a $\Delta$-non-degenerate affine surface $Z_\Delta \subseteq \mathbb{T}^3$ is birational to an elliptic surfaces over $\mathbb{P}^1$ of Kodaira dimension 1.*

Analogous to Subsection 7.2.2, a strong indication that we are in fact dealing with elliptic surfaces with an explicit map to $\mathbb{P}^1$ is the fact that the projection of all these 20 canonical Fano 3-polytopes $\Delta$ along $v_\Delta$ onto a plane leads to a reflexive polygon $\overline{\Delta}$ (page 99 and Table 7.2).

Let $\Delta \subseteq M_{\mathbb{R}}$ be a canonical Fano 3-polytope with $\dim(\Delta^{\text{FI}}) = 1$ and $0 \in (\Delta^{\text{FI}})^{\circ}$. Furthermore, let $\{v_i \mid 1 \leq i \leq f\}$ be the set of all inward-pointing facet normals of $\Delta$ and $f$ the number of facet of $\Delta$. By Subsection 7.1.3, page 100, we know that

$$(\langle v_\Delta, v_1\rangle, \ldots, \langle v_\Delta, v_f\rangle)$$

contains exactly one negative integer.

**Conjecture 7.2.15.** *Let $\Delta \subseteq M_{\mathbb{R}}$ be a canonical Fano 3-polytope with $\dim(\Delta^{\text{FI}}) = 1$ and $0 \in (\Delta^{\text{FI}})^{\circ}$ and $Y$ an elliptic surfaces over $\mathbb{P}^1$ of Kodaira dimension 1 that is birational to a canonical model of a $\Delta$-non-degenerate affine surface $Z_\Delta \subseteq \mathbb{T}^3$. Then the elliptic fibration $m : Y \to \mathbb{P}^1$ expressing $Y$ as an elliptic surface of Kodaira dimension 1 is given by*

$$m = z_0^{-\langle v_\Delta, v_1\rangle} \cdot \ldots \cdot z_{f-1}^{-\langle v_\Delta, v_f\rangle},$$

*where $z_0, \ldots, z_{f-1}$ denote the homogenous coordinates of the toric variety $X_{\Sigma^\Delta}$ [Cox95].*

Parallel and analogous to Subsection 7.2.2, we claim that the stringy Euler numbers $e_{\text{str}}(Y)$ of all canonical models corresponding to one of the 20 canonical Fano 3-polytopes with $\dim(\Delta^{\text{FI}}) = 1$ and $0 \in (\Delta^{\text{FI}})^{\circ}$ equals 24 and is computable via the formula in Conjecture 7.2.2. The latter claim is validated by us for all 20 canonical Fano 3-polytopes (Table A.7).

**Example 7.2.16.** Let us consider the canonical Fano 3-polytope from Example 7.1.5 and 7.1.6 with ID 547393 (Figure A.2(a), Table A.4, A.5, A.6, and A.7), *i.e.*, $\Delta = \text{conv}(P_1, P_2, P_3, P_4) \subseteq M_{\mathbb{R}}$ with $\dim(\Delta^{\text{FI}}) = 1$, $0 \in (\Delta^{\text{FI}})^{\circ}$, and vertices

$$P_1 := (0,1,0),\ P_2 := (2,1,1),\ P_3 := (-2,-3,-5),\ \text{and}\ P_4 := (2,1,9).$$

The normalized volume $v(\Delta)$ of $\Delta$ equals 64.

To compute the stringy Euler number $e_{\mathrm{str}}(Y)$ of a canonical model $Y$ corresponding to $\Delta$ with the formula in Conjecture 7.2.2, we obtain $\mathrm{supp}(\Delta^{\mathrm{FI}}) = \{S_i \,|\, 1 \le i \le 6\}$ with $S_1 \coloneqq (-1,-2,2)$, $S_2 \coloneqq (-1,1,0)$, $S_3 \coloneqq (0,-1,0)$, $S_4 \coloneqq (1,-1,0)$, $S_5 \coloneqq (2,-1,0)$, and $S_6 \coloneqq (9,-2,-2)$, which leads to

$$\Delta^{\mathrm{can}} = \mathrm{conv}(P_1, P_2, P_3, P_4) = \Delta.$$

Let us denote the 2-dimensional faces of $\Delta$ with $\theta_{P_{ijk}} \preceq \Delta$ defined as $\theta_{P_{ijk}} \coloneqq \mathrm{conv}(P_i, P_j, P_k)$ $(1 \le i < j < k \le 4)$ and the 1-dimensional faces of $\Delta$ with $\theta_{P_{ij}} \preceq \Delta$ defined as $\theta_{P_{ij}} \coloneqq \mathrm{conv}(P_i, P_j)$ $(1 \le i < j \le 4)$. Of course, all polytopes $[\theta_{P_{ijk}} \cap \Delta]$ have dimension 2 because $\theta_{P_{ijk}} \preceq \Delta$, *i.e.*, $[\theta_{P_{ijk}} \cap \Delta] = \theta_{P_{ijk}} \cap \Delta = \theta_{P_{ijk}}$ and $\dim(\theta_{P_{ijk}}) = 2$. Moreover, $v([\theta_{P_{123}} \cap \Delta]) = v(\theta_{P_{123}}) = 4$, $v(\theta_{P_{124}}) = 16$, $v(\theta_{P_{134}}) = 4$, and $v(\theta_{P_{234}}) = 32$. Applying the same arguments, all polytopes $[\theta_{P_{ij}} \cap \Delta]$ have dimension 1 with normalized volume $v([\theta_{P_{12}} \cap \Delta]) = v(\theta_{P_{12}}) = v(\theta_{P_{13}}) = v(\theta_{P_{14}}) = 1$, $v(\theta_{P_{23}}) = v(\theta_{P_{34}}) = 2$, and $v(\theta_{P_{24}}) = 8$.

In our last step, we determine the normalized volumes of all line segments $\theta^{\mathrm{supp}} = \mathrm{conv}(s \in \mathrm{supp}(\Delta^{\mathrm{FI}}) \,|\, s$ reaches its minimum $\kappa^{\Delta}(s)$ on $\mathrm{vert}([\theta_{P_{ij}} \cap \Delta]))$ corresponding to 1-dimensional faces $\theta_{P_{ij}} \preceq \Delta$ of $\Delta$. Therefore, we compute $\kappa^{\Delta}(s)$ for all $s \in \mathrm{supp}(\Delta^{\mathrm{FI}})$ (Equation (2.2)):

$$\begin{aligned}
\kappa^{\Delta}(S_1) &= \min\nolimits_{P \in \mathrm{vert}(\Delta)} \{\langle P, S_1\rangle\} = \min\{-2,-2,-2,14\} = -2,\\
\kappa^{\Delta}(S_2) &= \min\nolimits_{P \in \mathrm{vert}(\Delta)} \{\langle P, S_2\rangle\} = \min\{1,-1,-1,-1\} = -1,\\
\kappa^{\Delta}(S_3) &= \min\nolimits_{P \in \mathrm{vert}(\Delta)} \{\langle P, S_3\rangle\} = \min\{-1,-1,3,-1\} = -1,\\
\kappa^{\Delta}(S_4) &= \min\nolimits_{P \in \mathrm{vert}(\Delta)} \{\langle P, S_4\rangle\} = \min\{-1,1,1,1\} = -1,\\
\kappa^{\Delta}(S_5) &= \min\nolimits_{P \in \mathrm{vert}(\Delta)} \{\langle P, S_5\rangle\} = \min\{-1,3,-1,3\} = -1,\\
\kappa^{\Delta}(S_6) &= \min\nolimits_{P \in \mathrm{vert}(\Delta)} \{\langle P, S_6\rangle\} = \min\{-2,14,-2,-2\} = -2
\end{aligned}$$

with $\mathrm{vert}(\Delta) = \{P_1, P_2, P_3, P_4\}$. Using this information, we get

$$\begin{aligned}
&\theta^{\mathrm{supp}}_{P_{12}} = \mathrm{conv}(S_1, S_2),\ \theta^{\mathrm{supp}}_{P_{13}} = \mathrm{conv}(S_1, S_3),\ \theta^{\mathrm{supp}}_{P_{14}} = \mathrm{conv}(S_1, S_5, S_6),\\
&\theta^{\mathrm{supp}}_{P_{23}} = \mathrm{conv}(S_2, S_3),\ \theta^{\mathrm{supp}}_{P_{24}} = \mathrm{conv}(S_2, S_6),\ \theta^{\mathrm{supp}}_{P_{34}} = \mathrm{conv}(S_3, S_6)
\end{aligned}$$

with

$$v(\theta^{\mathrm{supp}}_{P_{12}}) = v(\theta^{\mathrm{supp}}_{P_{13}}) = 1,\ \ v(\theta^{\mathrm{supp}}_{P_{14}}) = 2,\ \text{and}\ v(\theta^{\mathrm{supp}}_{P_{23}}) = v(\theta^{\mathrm{supp}}_{P_{24}}) = v(\theta^{\mathrm{supp}}_{P_{34}}) = 1.$$

A combination of all data yields

$$\begin{aligned}
e_{\mathrm{str}}(Y) = v(\Delta) - &\sum_{\substack{\theta \preceq \Delta^{\mathrm{can}} \\ \dim(\theta)=2}} \theta_2 + \sum_{\substack{\theta \preceq \Delta^{\mathrm{can}} \\ \dim(\theta)=1}} \theta_1 + \omega\\
&= 64 - (4+16+4+32) + (1\cdot 1 + 1\cdot 1 + 1\cdot 2 + 2\cdot 1 + 8\cdot 1 + 2\cdot 1) + 0 = 24,
\end{aligned}$$

where $\omega = |\{v \in \mathrm{vert}(\Delta^{\mathrm{can}}) \,|\, v \notin \mathrm{vert}(\Delta)\}| = 0$ because $\Delta^{\mathrm{can}} = \Delta$, *i.e.*, $c_2^{\mathrm{str}}(Y) = 24$ and $c_1(Y)^2 = 0$ (Remark 7.2.3).

**Remark 7.2.17.** The detailed information about all 20 canonical Fano 3-polytopes with $\dim(\Delta^{\mathrm{FI}}) = 1$ and $0 \in (\Delta^{\mathrm{FI}})^{\circ}$ can be found in the Appendix. To be precise, it is listed in Table A.4, A.5, A.6, and A.7 and can be viewed in Figure A.2, respectively (cf. Remark 7.1.7).

### 7.2.4 Surfaces of General Type

Let $\Delta \subseteq M_{\mathbb{R}}$ be one of the 49 canonical Fano 3-polytopes with 3-dimensional Fine interior $\Delta^{\mathrm{FI}}$ having the origin as a vertex (Subsection 7.1.4). It is very likely that, among all 9,089 canonical 3-polytopes, only these 49 define surfaces that are birational to surfaces of general type and are of particular interest:

**Conjecture 7.2.18.** *Let $\Delta \subseteq M_{\mathbb{R}}$ be a canonical Fano 3-polytope with $\dim(\Delta^{\mathrm{FI}}) = 3$. Then a canonical model of a $\Delta$-non-degenerate affine surface $Z_\Delta \subseteq \mathbb{T}^3$ is birational to a surface of general type.*

**Definition 7.2.19.** A surface $Y$ is called *of general type* if it is a surface of maximal Kodaira dimension, *i.e.*, of Kodaira dimension $\dim(Y) = 2$.

A strong indication that we are indeed dealing with surfaces of general type is the fact that the stringy Euler numbers $e_{\mathrm{str}}(Y)$ (computed with the formula in Conjecture 7.2.2) of all canonical models $Y$ corresponding to one of these 49 polytopes are strictly smaller than 24. To be precise, in most cases the stringy Euler number equals 23 and with a few exceptions 22, *i.e.*, $c_2^{\mathrm{str}}(Y) > 0$ and $c_1(Y)^2 > 0$ (Table A.10). This are exactly the invariants of (minimal) surfaces of general type [BHPdV04, Chapter VI, Table 10]. A further indication is the fact that $c_2^{\mathrm{str}}(Y) \in \{23, 22\}$ and $c_1(Y)^2 \in \{1, 2\}$ fulfill the following properties that (minimal) surfaces of general type would fulfill [BHPdV04, Chapter VII, Theorem 1.1]:

- $c_1(Y)^2 > 0$ and $c_2^{\mathrm{str}}(Y) > 0$,
- $c_1(Y)^2 + c_2^{\mathrm{str}}(Y) \equiv 0 \mod 12$,
- $c_1(Y)^2 \leq 3c_2^{\mathrm{str}}(Y)$,
- $5c_1(Y)^2 - c_2^{\mathrm{str}}(Y) + 36 \geq 0$ ($c_1(Y)^2$ even),
  $5c_1(Y)^2 - c_2^{\mathrm{str}}(Y) + 30 \geq 0$ ($c_1(Y)^2$ odd),

because $1 > 0, 23 > 0$, $1 + 23 = 24 \equiv 0 \mod 12$, $1 \leq 69$, and $5 - 23 + 30 = 48 > 0$ and $2 > 0, 22 > 0$, $2 + 22 = 24 \equiv 0 \mod 12$, $2 \leq 66$, and $10 - 22 + 36 = 24 > 0$.

We close this subsection with detailed examples computing the stringy Euler number of the two examples in Figure A.3(b) and (c) with the predicted formula in Conjecture 7.2.2:

**Example 7.2.20.** Let us consider the canonical Fano 3-polytope with ID 547465 (Figure A.3(b), Table A.8, A.9, and A.10), *i.e.*, $\Delta = \mathrm{conv}(P_1, P_2, P_3, P_4) \subseteq M_{\mathbb{R}}$ with $\dim(\Delta^{\mathrm{FI}}) = 3$, $0 \in \mathrm{vert}(\Delta^{\mathrm{FI}})$, and vertices

$$P_1 := (-3, -2, -2),\ P_2 := (1, 0, 0),\ P_3 := (1, 3, 1),\ \text{and}\ P_4 := (1, 1, 3).$$

The normalized volume $v(\Delta)$ of $\Delta$ equals 32. The Fine interior $\Delta^{\mathrm{FI}}$ of $\Delta$ is given by the convex hull $\mathrm{conv}(F_1, F_2, F_3, F_4)$ with vertices

$$F_1 \coloneqq (0,0,0),\ F_2 \coloneqq (-1,-1/2,-1/2),\ F_3 \coloneqq (0,3/4,1/4), \text{ and } F_4 \coloneqq (0,1/4,3/4).$$

The inward-pointing facet normals of $\Delta$ are

$$v_1 \coloneqq (2,-1,-1),\ v_2 \coloneqq (-1,3,-1),\ v_3 \coloneqq (-1,-1,3), \text{ and } v_4 \coloneqq (-1,0,0).$$

Looking at the inward-pointing facet normals of $\Delta^{\mathrm{FI}}$, we recognize that they are the same, *i.e.*, $\Delta^{\mathrm{FI}}$ is a scaling down of $\Delta$.

To compute the stringy Euler number $e_{\mathrm{str}}(Y)$ of a canonical model $Y$ corresponding to $\Delta$ with the formula in Conjecture 7.2.2, we have to collect the missing relevant data: In this special case, we get $\mathrm{supp}(\Delta^{\mathrm{FI}}) = \{v_1, v_2, v_3, v_4\}$ and therefore $\Delta^{\mathrm{can}} = \Delta$.

Let us denote the 2-dimensional faces of $\Delta$ with $\theta_{P_{ijk}} \preceq \Delta$ defined as $\theta_{P_{ijk}} \coloneqq \mathrm{conv}(P_i, P_j, P_k)$ $(1 \le i < j < k \le 4)$ and the 1-dimensional faces of $\Delta$ with $\theta_{P_{ij}} \preceq \Delta$ defined as $\theta_{P_{ij}} \coloneqq \mathrm{conv}(P_i, P_j)$ $(1 \le i < j \le 4)$. The following two tables list all facets along with their inward-pointing facet normals and normalized volumes as well as all edges along with their normalized volumes:

| facet | inward-pointing facet normal | normalized volume |
|---|---|---|
| $\theta_{P_{123}}$ | $v_4$ | 8 |
| $\theta_{P_{124}}$ | $v_3$ | 4 |
| $\theta_{P_{134}}$ | $v_2$ | 4 |
| $\theta_{P_{234}}$ | $v_1$ | 8 |

| edge | normalized volume |
|---|---|
| $\theta_{P_{12}}$ | 1 |
| $\theta_{P_{13}}$ | 1 |
| $\theta_{P_{14}}$ | 2 |
| $\theta_{P_{23}}$ | 2 |
| $\theta_{P_{24}}$ | 1 |
| $\theta_{P_{34}}$ | 1 |

In our last step, we determine the normalized volumes of all line segments $\theta^{\mathrm{supp}} = \mathrm{conv}(s \in \mathrm{supp}(\Delta^{\mathrm{FI}}) \mid s \text{ reaches its minimum } \kappa^{\Delta}(s) \text{ on } \mathrm{vert}([\theta \cap \Delta]))$ corresponding to 1-dimensional faces $\theta \preceq \Delta^{\mathrm{can}} = \Delta$. Therefore, we compute $\kappa^{\Delta}(s)$ for all $s \in \mathrm{supp}(\Delta^{\mathrm{FI}})$ (Equation (2.2)):

$$\begin{aligned}
\kappa^{\Delta}(v_1) &= \min\nolimits_{P\in\mathrm{vert}(\Delta)} \{\langle P, v_1\rangle\} = \min\{2,-2,-2,-2\} = -2,\\
\kappa^{\Delta}(v_2) &= \min\nolimits_{P\in\mathrm{vert}(\Delta)} \{\langle P, v_2\rangle\} = \min\{-1,7,-1,-1\} = -1,\\
\kappa^{\Delta}(v_3) &= \min\nolimits_{P\in\mathrm{vert}(\Delta)} \{\langle P, v_3\rangle\} = \min\{-1,-1,7,-1\} = -1,\\
\kappa^{\Delta}(v_4) &= \min\nolimits_{P\in\mathrm{vert}(\Delta)} \{\langle P, v_4\rangle\} = \min\{-1,-1,-1,3\} = -1
\end{aligned}$$

with $\mathrm{vert}(\Delta) = \{P_1, P_2, P_3, P_4\}$. Using this information, we get

$$\begin{aligned}
&\theta^{\mathrm{supp}}_{P_{12}} = \mathrm{conv}(v_3, v_4),\ \theta^{\mathrm{supp}}_{P_{13}} = \mathrm{conv}(v_2, v_4),\ \theta^{\mathrm{supp}}_{P_{14}} = \mathrm{conv}(v_2, v_3),\\
&\theta^{\mathrm{supp}}_{P_{23}} = \mathrm{conv}(v_1, v_4),\ \theta^{\mathrm{supp}}_{P_{24}} = \mathrm{conv}(v_1, v_3),\ \theta^{\mathrm{supp}}_{P_{34}} = \mathrm{conv}(v_1, v_2)
\end{aligned}$$

with

$$v(\theta_{P_{12}}^{\mathrm{supp}})=1,\ v(\theta_{P_{13}}^{\mathrm{supp}})=1,\ v(\theta_{P_{14}}^{\mathrm{supp}})=4,\ v(\theta_{P_{23}}^{\mathrm{supp}})=1,\ v(\theta_{P_{24}}^{\mathrm{supp}})=1,\ \text{and } v(\theta_{P_{34}}^{\mathrm{supp}})=1.$$

A combination of all data and the fact $\Delta^{\mathrm{can}}=\Delta$ yields

$$\begin{aligned} e_{\mathrm{str}}(Y) &= v(\Delta) - \sum_{\substack{\theta\preceq\Delta^{\mathrm{can}}\\ \dim(\theta)=2}} v([\theta\cap\Delta]) + \sum_{\substack{\theta\preceq\Delta^{\mathrm{can}}\\ \dim(\theta)=1}} v([\theta\cap\Delta])\cdot v(\theta^{\mathrm{supp}}) + \omega \\ &= v(\Delta) - \sum_{\substack{\theta\preceq\Delta^{\mathrm{can}}\\ \dim(\theta)=2}} v(\theta) + \sum_{\substack{\theta\preceq\Delta^{\mathrm{can}}\\ \dim(\theta)=1}} v(\theta)\cdot v(\theta^{\mathrm{supp}}) \\ &= 32-(8+4+4+8)+(1\cdot1+1\cdot1+2\cdot4+2\cdot1+1\cdot1+1\cdot1)=22, \end{aligned}$$

where $\omega=|\{v\in\mathrm{vert}(\Delta^{\mathrm{can}})\,|\,v\notin\mathrm{vert}(\Delta)\}|=0$, *i.e.*, $c_2^{\mathrm{str}}(Y)=22$ and $c_1(Y)^2=2$ (Remark 7.2.3).

**Example 7.2.21.** Let us consider the canonical Fano 3-polytope with ID 547524 (Figure 7.2 and A.3(c), Table A.8, A.9, and A.10), *i.e.*, $\Delta=\mathrm{conv}(P_1,P_2,P_3,P_4)\subseteq M_{\mathbb{R}}$ with $\dim(\Delta^{\mathrm{FI}})=3$, $0\in\mathrm{vert}(\Delta^{\mathrm{FI}})$, and vertices

$$P_1:=(0,2,1),\ P_2:=(-2,-3,-5),\ P_3:=(2,1,1),\ \text{and } P_4:=(0,0,1).$$

The normalized volume $v(\Delta)$ of $\Delta$ equals 24. The Fine interior $\Delta^{\mathrm{FI}}$ of $\Delta$ is given by the convex hull $\mathrm{conv}(F_1,F_2,F_3,F_4)$ with vertices

$$F_1:=(0,0,0),\ F_2:=(0,1/2,0),\ F_3:=(1/3,1/3,0),\ \text{and } F_4:=(-1/3,-1/3,-1).$$

To compute the stringy Euler number $e_{\mathrm{str}}(Y)$ of a canonical model $Y$ corresponding to $\Delta$ with the formula in Conjecture 7.2.2, we have to collect the missing relevant data:

In this case, we get

$$\mathrm{supp}(\Delta^{\mathrm{FI}})=\{S_i\,|\,1\le i\le 10\}$$

with $S_1:=(-1,1,0)$, $S_2:=(-1,2,-1)$, $S_3:=(0,0,-1)$, $S_4:=(0,1,-1)$, $S_5:=(0,2,-1)$, $S_6:=(1,0,-1)$, $S_7:=(1,1,-1)$, $S_8:=(2,0,-1)$, $S_9:=(3,0,-1)$, and $S_{10}:=(-1,-2,2)$, which leads to

$$\Delta^{\mathrm{can}}=\mathrm{conv}(P_1,P_2,P_3,P_4,P_5)$$

with $P_5:=(0,-1,-1)$ (Figure 7.2).

Let us denote the 2-dimensional faces of $\Delta^{\mathrm{can}}$ with $\theta_{P_{ijk}}\preceq\Delta^{\mathrm{can}}$ defined as $\theta_{P_{ijk}}:=\mathrm{conv}(P_i,P_j,P_k)$ $(1\le i<j<k\le5)$ and the 1-dimensional faces of $\Delta^{\mathrm{can}}$ with $\theta_{P_{ij}}\preceq\Delta^{\mathrm{can}}$ defined as $\theta_{P_{ij}}:=\mathrm{conv}(P_i,P_j)$ $(1\le i<j\le5)$. The following two tables list all facets $\theta_{P_{ijk}}$ with $\dim([\theta_{P_{ijk}}\cap\Delta])=2$ and the normalized volumes of $[\theta_{P_{ijk}}\cap\Delta]$ as well as all edges $\theta_{P_{ij}}$ with $\dim([\theta_{P_{ij}}\cap\Delta])=1$ and the normalized volumes of $[\theta_{P_{ij}}\cap\Delta]$ (Figure 7.2):

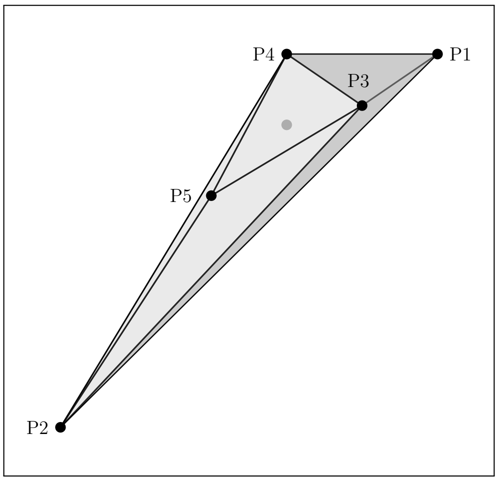



**Figure 7.2: Canonical hull $\Delta^{\mathrm{can}}$ of a Canonical Fano 3-polytope $\Delta$ with $\dim(\Delta^{\mathrm{FI}}) = 3$ (ID 547524).** Shaded faces are occluded. The whole polytope is the canonical hull $\Delta^{\mathrm{can}} = \mathrm{conv}(P_1, P_2, P_3, P_4, P_5)$ of $\Delta = \mathrm{conv}(P_1, P_2, P_3, P_4)$ (grey coloured) with $P_1 = (0, 2, 1)$, $P_2 = (-2, -3, -5)$, $P_3 = (2, 1, 1)$, $P_4 = (0, 0, 1)$, and $P_5 = (0, -1, -1)$. The origin is coloured light grey.

| facet $\theta_{P_{ijk}} \preceq \Delta^{\mathrm{can}}$ with $\dim([\theta_{P_{ijk}} \cap \Delta]) = 2$ | $v([\theta_{P_{ijk}} \cap \Delta])$ |
|---|---|
| $\theta_{P_{134}}$ | 4 |
| $\theta_{P_{124}}$ | 4 |
| $\theta_{P_{123}}$ | 6 |

| edge $\theta_{P_{ij}} \preceq \Delta^{\mathrm{can}}$ with $\dim([\theta_{P_{ij}} \cap \Delta]) = 1$ | $v([\theta_{P_{ij}} \cap \Delta])$ |
|---|---|
| $\theta_{P_{23}}$ | 2 |
| $\theta_{P_{14}}$ | 2 |
| $\theta_{P_{34}}$ | 1 |
| $\theta_{P_{13}}$ | 1 |
| $\theta_{P_{24}}$ | 1 |
| $\theta_{P_{12}}$ | 1 |

In our last step, we determine the normalized volumes of all line segments $\theta^{\mathrm{supp}} = \mathrm{conv}(s \in \mathrm{supp}(\Delta^{\mathrm{FI}}) \,|\, s \text{ reaches its minimum } \kappa^{\Delta}(s) \text{ on } \mathrm{vert}([\theta_{P_{ij}} \cap \Delta]))$ corresponding to 1-dimensional faces $\theta_{P_{ij}} \preceq \Delta^{\mathrm{can}}$. Therefore, we compute $\kappa^{\Delta}(s)$ for all $s \in \mathrm{supp}(\Delta^{\mathrm{FI}})$ (Equation (2.2)):

$$\begin{aligned}
\kappa^{\Delta}(S_1) &= \min_{P \in \mathrm{vert}(\Delta)} \{\langle P, S_1\rangle\} &= \min\{2, -1, -1, 0\} = -1,\\
\kappa^{\Delta}(S_2) &= \min_{P \in \mathrm{vert}(\Delta)} \{\langle P, S_2\rangle\} &= \min\{3, 1, -1, -1\} = -1,\\
\kappa^{\Delta}(S_3) &= \min_{P \in \mathrm{vert}(\Delta)} \{\langle P, S_3\rangle\} &= \min\{-1, 5, -1, -1\} = -1,\\
\kappa^{\Delta}(S_4) &= \min_{P \in \mathrm{vert}(\Delta)} \{\langle P, S_4\rangle\} &= \min\{1, 2, 0, -1\} = -1,\\
\kappa^{\Delta}(S_5) &= \min_{P \in \mathrm{vert}(\Delta)} \{\langle P, S_5\rangle\} &= \min\{3, -1, 1, -1\} = -1,\\
\kappa^{\Delta}(S_6) &= \min_{P \in \mathrm{vert}(\Delta)} \{\langle P, S_6\rangle\} &= \min\{-1, 3, 1, -1\} = -1,\\
\kappa^{\Delta}(S_7) &= \min_{P \in \mathrm{vert}(\Delta)} \{\langle P, S_7\rangle\} &= \min\{1, 0, 2, -1\} = -1,
\end{aligned}$$

$$\begin{aligned}
\kappa^{\Delta}(S_8) &= \min_{P\in \mathrm{vert}(\Delta)} \{\langle P, S_8\rangle\} \ = \min\{-1,1,3,-1\} = -1,\\
\kappa^{\Delta}(S_9) &= \min_{P\in \mathrm{vert}(\Delta)} \{\langle P, S_9\rangle\} \ = \min\{-1,-1,5,-1\} = -1,\\
\kappa^{\Delta}(S_{10}) &= \min_{P\in \mathrm{vert}(\Delta)} \{\langle P, S_{10}\rangle\} = \min\{-2,-2,-2,2\} = -2
\end{aligned}$$

with $\mathrm{vert}(\Delta) = \{P_1, P_2, P_3, P_4\}$. Using this information, we get

$$\begin{aligned}
&\theta^{\mathrm{supp}}_{P_{23}} = \mathrm{conv}(S_1, S_{10}),\ \theta^{\mathrm{supp}}_{P_{14}} = \mathrm{conv}(S_3, S_6, S_8, S_9),\ \theta^{\mathrm{supp}}_{P_{34}} = \mathrm{conv}(S_2, S_3),\\
&\theta^{\mathrm{supp}}_{P_{13}} = \mathrm{conv}(S_3, S_{10}),\ \theta^{\mathrm{supp}}_{P_{24}} = \mathrm{conv}(S_5, S_9),\ \theta^{\mathrm{supp}}_{P_{12}} = \mathrm{conv}(S_9, S_{10})
\end{aligned}$$

with

$$v(\theta^{\mathrm{supp}}_{P_{23}}) = 1,\ v(\theta^{\mathrm{supp}}_{P_{14}}) = 3,\ v(\theta^{\mathrm{supp}}_{P_{34}}) = 1,\ v(\theta^{\mathrm{supp}}_{P_{13}}) = 1,\ v(\theta^{\mathrm{supp}}_{P_{24}}) = 1,\ \text{and } v(\theta^{\mathrm{supp}}_{P_{12}}) = 1.$$

A combination of all data yields

$$\begin{aligned}
e_{\mathrm{str}}(Y) &= v(\Delta) - \sum_{\substack{\theta\preceq\Delta^{\mathrm{can}}\\ \dim(\theta)=2}} \theta_2 + \sum_{\substack{\theta\preceq\Delta^{\mathrm{can}}\\ \dim(\theta)=1}} \theta_1 + \omega\\
&= 24 - (4+4+6) + (1\cdot 1 + 2\cdot 3 + 1\cdot 1 + 2\cdot 1 + 1\cdot 1 + 1\cdot 1) + 1 = 23,
\end{aligned}$$

where $\omega = |\{v \in \mathrm{vert}(\Delta^{\mathrm{can}}) \,|\, v \notin \mathrm{vert}(\Delta)\}| = 1$ (Figure 7.2), *i.e.*, $c_2^{\mathrm{str}}(Y) = 23$ and $c_1(Y)^2 = 1$ (Remark 7.2.3)..

**Remark 7.2.22.** The detailed information about all 49 canonical Fano 3-polytopes with $\dim(\Delta^{\mathrm{FI}}) = 3$ can be found in the Appendix. To be precise, it is listed in Table A.8, A.9, and A.10 and can be viewed in Figure A.3, respectively (cf. Remark 7.1.8).

Chapter 8

# Calabi-Yau Hypersurfaces in Weighted Projective Spaces and Vafa's Formula

In the last chapter, we discuss implications in physics. Therefore, we combine the content of Chapter 6, Section 6.3, where we computed stringy Euler numbers of certain hypersurfaces in a combinatorial way using intrinsic information provided by corresponding underlying polytopes (Theorem 6.3.2 and 6.3.3) with techniques that Vafa has developed based on physicists' reasoning to compute Witten's index by studying the conformal field theory of the Landau-Ginzburg model [Vaf89].

The mentioned connection looks as follows: Let $Y$ be a quasi-smooth Calabi-Yau hypersurface in a well-formed weighted projective space $\mathbb{P}(w_0, w_1, \ldots, w_d)$. We show that the affine hypersurface $Z_S = \{(x_0, \ldots, x_d) \in (\mathbb{C}^*)^{d+1} \mid \sum_{i=0}^d x_i = 0 \text{ and } \prod_{i=0}^d x_i^{w_i} = 1\}$ in the $d$-dimensional algebraic torus $\mathbb{T}^d = \{(x_0, \ldots, x_d) \in (\mathbb{C}^*)^{d+1} \mid \prod_{i=0}^d x_i^{w_i} = 1\}$ is birational to a Calabi-Yau variety $Y^\vee$ that is expected to be the mirror of $Y$. We support this expectation in Theorem 8.2.5 showing that the Witten index $i_{\mathrm{W}}(Y)$ of $Y$ computed via the well-known formula of Vafa [Vaf89] in Equation (8.2) equals, up to sign, the stringy Euler number $e_{\mathrm{str}}(Y^\vee)$ of $Y^\vee$ computed via the combinatorial formula in Theorem 6.3.2. To be precise, we use a topological interpretation of Vafa's formula for Witten's index and treat it as a topological invariant of $Y$ that is the zero locus of a weighted homogeneous polynomial $f$ of degree $\sum_{i=0}^d w_i$ in the weighted projective space $\mathbb{P}(w_0, w_1, \ldots, w_d)$.

## 8.1 Motivation and Mirror Symmetry Constructions

As mentioned in the Introduction at the beginning, mirror symmetry is the motivation of this present thesis. The included basic and extended mirror symmetry tests justify our interest in topological invariants: stringy Hodge numbers and stringy Euler numbers. In particular, the focus on the latter class is motivated by mirror symmetry applied to Calabi-Yau hypersurfaces $Y$ and $Y^\vee$. There are several examples and construction rules for such $Y$ and $Y^\vee$ [Bat98b, BB96, BD96] passing the stringy extended mirror symmetry test

$$h_{\text{str}}^{p,q}(Y) = h_{\text{str}}^{(d-1)-p,q}(Y^\vee) \ \ (0 \leq p, q \leq d-1)$$

and deduce the stringy basic mirror symmetry test

$$(-1)^{d-1} e_{\text{str}}(Y) = e_{\text{str}}(Y^\vee),$$

where $d-1$ is the dimension $\dim(Y)$ of $Y$. These equations for stringy Hodge numbers and stringy Euler numbers are viewed as evidence for mirror symmetry. In particular, stringy Euler numbers have been used for stringy basic topological tests for mirror symmetry in different mirror constructions. Several papers of Batyrev and Borisov have been dedicated to a combinatorial method constructing mirror pairs of Calabi-Yau varieties in an explicit way [Bor93, Bat94, BB96, BB97].

Going back to our setting, *i.e.*, let $Y$ be a quasi-smooth Calabi-Yau hypersurface $Y$ (Definition 2.1.4 and Remark 8.2.2) of degree $w$ in a well-formed weighted projective space $\mathbb{P}(w_0, w_1, \ldots, w_d)$ (Definition 8.2.1 and Remark 8.2.2). If we add the condition that all weights $w_i$ divide $w$ $(0 \leq i \leq d)$, we are dealing with Calabi-Yau hypersurfaces $Y$ of *Fermat-type* . Applying Section 5.4 in [Bat94], $Y$ corresponds to a reflexive simplex $S \subseteq M_{\mathbb{R}}$ obtained as the convex hull of unique lattice points $v_i \in M$ $(0 \leq i \leq d)$ solving the linear relation $\sum_{i=0}^{d} w_i v_i = 0$ (Equation (8.3)).

A starting point for mirror symmetry tests was the notion of reflexive polytopes $\Delta$ (Definition 2.3.13) introduced in [Bat94]. Using the theory of toric varieties, one can prove that the affine hypersurface $Z_\Delta \subseteq \mathbb{T}^d$ defined by $f_\Delta = 0$ is birational to a Calabi-Yau hypersurface $Y$ [Bat94], where $f_\Delta$ is a generic Laurent polynomial with Newton polytope $\Delta$ (Definition 2.3.28). In the same way, one obtains another Calabi-Yau hypersurface $Y^*$ corresponding to the dual reflexive polytope $\Delta^*$. In this case, there exists a purely combinatorial formula [BD96, Corollary 7.10] computing the stringy Euler number $e_{\text{str}}(Y)$ and $e_{\text{str}}(Y^*)$ of the Calabi-Yau variety $Y$ and $Y^*$, respectively, via

$$e_{\text{str}}(Y) = \sum_{\substack{\theta \preceq \Delta \\ 1 \leq \dim(\theta) \leq d-2}} (-1)^{\dim(\theta)-1} v(\theta) \cdot v\left(\theta^*\right),$$

where $v(\cdot)$ denotes the normalized volume of a polytope (Definition 2.3.6). Two alternative independent and more general proofs of this formula can be found in [Bat17, Theorem 4.11] or, together with its generalization to Calabi-Yau complete intersections, in Proposition 6.5.1. This formula, together with the duality between faces of the reflexive polytopes $\Delta$ and $\Delta^*$ (page 22), immediately implies the equality

$$(-1)^{d-1} e_{\text{str}}(Y) = e_{\text{str}}(Y^*), \tag{8.1}$$

*i.e.*, $Y$ and $Y^*$ pass the stringy basic mirror symmetry test. This supports our assumption that $Y$ and $Y^*$ are mirror dual. Applying this mirror construction to the subclass of reflexive simplices $S$ and $S^*$ from above implies that the corresponding birational Calabi-Yau hypersurfaces $Y$ and $Y^\vee$ of $Z_S$ and $Z_{S^*}$, respectively, will pass the stringy basic mirror symmetry test $(-1)^{d-1}e_{\mathrm{str}}(Y) = e_{\mathrm{str}}(Y^\vee)$. Therefore, we found evidence to support the assumption to regard them as mirror dual.

In addition, another phenomenon was discovered by physicists in [KS98a] while computing Hodge numbers of Calabi-Yau fourfolds. First, they considered weight vectors $(w_0, \dots, w_5) \in \mathbb{Z}^6_{>0}$ of degree $w = w_0 + w_1 + \ldots + w_5$ and checked if there exist weighted homogeneous polynomials of degree $w \leq 4{,}000$ such that the constructed hypersurfaces are quasi-smooth and therefore Calabi-Yau fourfolds (Remark 8.2.2). To this end, Kreuzer and Skarke used the support of a computer and have found 525,572 weight vectors with degree $w \leq 4{,}000$ that give rise to Calabi-Yau fourfolds. Secondly, they considered for given weight vectors $(w_0, \dots, w_5) \in \mathbb{Z}^6_{>0}$ general polynomials of degree $w = w_0 + w_1 + \ldots + w_5 \leq 150$. The associated Newton polytopes are given as the convex hull of all lattice points $(a_0, \dots, a_5) \in \mathbb{Z}^6_{>0}$ determined by the exponents occurring in the monomials of the given polynomials. The obtained Newton polytopes give rise to Calabi-Yau fourfolds if they are reflexive. With the support of a computer, this is true for exactly 914,164 weight vectors. In a further step, they computed the Hodge numbers of all 525,572 Calabi-Yau fourfolds with Vafa's formulas [Vaf89]. To compute the Hodge numbers of all 914,164 Calabi-Yau fourfolds, they used Batyrev's formulas [Bat94, BD96]. To get the main statement, it is not necessary to discuss both formulas in detail. The observed phenomenon was that they recognized (without a general explanation) consistency for all 109,308 weight vectors in the overlap. We note that in one dimension less, a hypersurface is quasi-smooth (*i.e.,* a Calabi-Yau threefold) if and only if the Newton polytope is reflexive for a given weight vector. Therefore, there would be no reason to use the computer for search processes twice.

The described phenomenon is explained in Theorem 8.2.5 from a more general point of view, *i.e.*, we allow any dimension $d$ and do not request the Calabi-Yau hypersurface $Y$ to be of Fermat-type, *i.e.* not all $w_i$ have to divide $w$ $(0 \leq i \leq d)$. To be precise, let $Y$ be a quasi-smooth Calabi-Yau hypersurface of degree $w = w_0 + w_1 + \ldots + w_d$ in a $d$-dimensional well-formed weighted projective space $\mathbb{P}(w_0, w_1, \dots, w_d)$ (Definition 8.2.1 and Remark 8.2.2). Using this data, let $Y^\vee$ be a canonical Calabi-Yau model of a $S$-non-degenerate affine hypersurface $Z_S$ in the $d$-dimensional algebraic torus $\mathbb{T}^d$, where $S$ is the $d$-dimensional lattice simplex with vertices $v_0, v_1, \dots, v_d$ satisfying the linear relation $w_0 v_0 + w_1 v_1 + \ldots + w_d v_d = 0$. For such Calabi-Yau varieties the Witten index $i_{\mathrm{W}}(Y)$ of $Y$ computes, up to sign, the stringy Euler number $e_{\mathrm{str}}(Y^\vee)$ of $Y^\vee$. In a way, Theorem 8.2.5 supports our conjecture to regard $Y$ and $Y^\vee$ as mirror dual. But at least, Corollary 8.2.13 provides mathematical evidence for the observations made in [KS98a] by physicists restricted to stringy Euler numbers.

We emphasize that a mathematical proof for the assertion that Vafa's formula computes the stringy Euler number $e_{\mathrm{str}}(Y)$ of $Y$ is still pending. Knowing this would be considered an advancement because we would have a further indication that establishes the above mirror construction. Moreover, we would have new examples to investigate mirror symmetry. There are several papers in the literature investigating various methods adopting the same construction to verify mirror symmetry [Iri11, CIR14]. This construction would require no restrictions on quasi-smooth Calabi-Yau hypersurfaces $Y$

in weighted projective spaces and would be supported by passing the stringy basic mirror symmetry test. In particular, it does not construct the Calabi-Yau mirror variety $Y^\vee$ in an explicit way, but provides an affine hypersurface $Z_S$ in the algebraic torus $\mathbb{T}^d$ that is birational to $Y^\vee$.

Moreover, the received set of suggested mirror symmetry pairs obtained through the proposed construction intersects non-empty with the received sets of possible mirror symmetry pairs obtained through the following two well-known combinatorial constructions: The first one is named after Batyrev [Bat94] and, as explained above, concerns reflexive polytopes $\Delta$ and constructs mirrors by using the dual reflexive polytopes $\Delta^*$. Batyrev's mirror construction and the proposed mirror construction overlap for reflexive simplices. The second one is called the 'Berglund-Hübsch-Krawitz mirror construction' [BH93, Kra10] and considers Calabi-Yau hypersurfaces $Y$ that are birational to affine hypersurfaces $Z \subseteq \mathbb{T}^d$ defined by Laurent polynomials $f(x) = \sum_m a_m x^m$, whose Newton polytopes $\Delta(f) = \operatorname{conv}(m \in M \,|\, a_m \neq 0) \subseteq M_\mathbb{R}$ (Definition 2.3.28) are lattice (not necessarily reflexive) simplices. The mirror duality for the stringy Hodge numbers of Calabi-Yau varieties obtained by the Berglund-Hübsch-Krawitz mirror construction is proved by Chiodo and Ruan [CR10] and Borisov [Bor13]. These two mirror constructions reveal to different classes of hypersurfaces in toric varieties, but they coincide for Calabi-Yau hypersurfaces of Fermat-type, *i.e.*, for reflexive simplices. In summary, this means: The produced sets of mirror symmetry pairs from Batyrev's mirror construction, as well as from the Berglund-Hübsch-Krawitz mirror construction, intersect non-empty with the produced set of mirror symmetry pairs from the proposed mirror construction for reflexive simplices. For this reason, it is natural to expect the existence of a generalization including both as special cases [AP15, Bor13, ACG16, Pum11, BH16].

## 8.2 Witten Indices and Stringy Euler Numbers

The simplest example of a $d$-dimensional weighted projective space is the projective space $\mathbb{P}^d = \mathbb{P}(1, 1, \ldots, 1)$ with the associated graded ring $\mathbb{C}[z_0, \ldots, z_d]$, where each variable $z_i$ has degree 1. In general, weighted projective spaces are defined as follows:

**Definition 8.2.1.** Let $(w_0, w_1, \ldots, w_d) \in \mathbb{Z}^{d+1}_{>0}$ be a vector with $\gcd(w_0, \ldots, w_d) = 1$. Then we call

$$\mathbb{P}(w_0, w_1, \ldots, w_d) := (\mathbb{C}^{d+1} \setminus \{0\})/\sim$$

*weighted projective space* of dimension $d$, where $\sim$ is an equivalence relation such that

$$(a_0, a_1, \ldots, a_d) \sim (b_0, b_1, \ldots, b_d) \Leftrightarrow a_i = \lambda^{w_i} b_i \ \ (0 \leq i \leq d) \text{ for some } \lambda \in \mathbb{C}^*.$$

The associated graded ring is $\mathbb{C}[z_0, \ldots, z_d]$, where $z_i$ has degree $w_i$ $(0 \leq i \leq d)$. A polynomial f is called *weighted homogeneous of degree* $w$ if every monomial $z^\alpha$ appearing in $f$ satisfies $\alpha \cdot (w_0, w_1, \ldots, w_d) = w$ [CLS11, page 53].

**Remark 8.2.2.** Let $(w_0, w_1, \ldots, w_d) \in \mathbb{Z}^{d+1}_{>0}$ be a primitive lattice vector such that for any $i \in I := \{0, 1, \ldots, d\}$ the greatest common divisor of $d$ integers $w_0, \ldots, w_{i-1}, w_{i+1}, \ldots,$

$w_d$ equals 1. In literature, such a vector $(w_0, w_1, \ldots, w_d)$ is called *well-formed*. We consider a well-formed $d$-dimensional weighted projective space $\mathbb{P}(w_0, w_1, \ldots, w_d)$ and assume that it contains a quasi-smooth hypersurface $Y$ of degree $w := \sum_{i=0}^{d} w_i$ constructed through a weighted homogeneous polynomial $f$ of degree $w$. $Y$ is called *quasi-smooth* if the common zeros of all partial derivatives $\frac{\partial}{\partial z_i} f$ $(i \in I)$ is the point $z_0 = z_1 = \ldots = z_d = 0$. An equivalent criterion to check wether a general hypersurface is quasi-smooth can be found in [IF00, Theorem 8.1] using the weights $w_0, \ldots, w_d$. The quasi-smoothness condition ensures that the hypersurface has no singularities in addition to those coming from the singularities of the ambient space [KS98b], where the only singularities of weighted projective spaces are cyclic quotient singularities [CLS11, Definition 11.4.5]. Using Lemma 1.12 in [CG11], every quasi-smooth hypersurface of degree $w$ in a well-formed weighted projective space $\mathbb{P}(w_0, w_1, \ldots, w_d)$ is a Calabi-Yau variety (Definition 2.1.4).

Vafa has developed techniques to compute Witten's index based on physicists reasoning by studying the conformal field theory of the Landau-Ginzburg model [Vaf89]. We treat the *Witten index* $i_{\mathrm{W}}(Y)$ of the quasi-smooth Calabi-Yau hypersurface $Y$ as a topological invariant of $Y$. In this way, we use a topological interpretation of Vafa's formula for Witten's index applied to $Y$ that is the zero locus of the weighted homogeneous polynomial $f$ of degree $w$ in a well-formed weighted projective space $\mathbb{P}(w_0, w_1, \ldots, w_d)$:

$$i_{\mathrm{W}}(Y) = \frac{1}{w} \sum_{l,r=0}^{w-1} \prod_{\substack{0 \leq i \leq d \\ lq_i, rq_i \in \mathbb{Z}}} \left(1 - \frac{1}{q_i}\right), \tag{8.2}$$

where $q_i := \frac{w_i}{w}$ $(i \in I)$ and by definition $\prod_{\substack{0 \leq i \leq d \\ lq_i, rq_i \in \mathbb{Z}}} \left(1 - \frac{1}{q_i}\right) = 1$ if there is no $q_i$ with $lq_i, rq_i \in \mathbb{Z}$.

**Example 8.2.3.** Consider the weighted projective space $\mathbb{P}(1,1,1,1)$ and let $Y_1$ be a quasi-smooth Calabi-Yau hypersurface of degree $w = \sum_{i=0}^{3} 1 = 4$ contained in it with $q_i = \frac{1}{4}$ for all $0 \leq i \leq 3$. Then

$$i_{\mathrm{W}}(Y_1) = \frac{1}{4} \sum_{l,r=0}^{3} \prod_{\substack{0 \leq i \leq 3 \\ lq_i, rq_i \in \mathbb{Z}}} \left(1 - \frac{1}{q_i}\right) = \frac{1}{4} \sum_{l,r=0}^{3} \prod_{\substack{0 \leq i \leq 3 \\ lq_i, rq_i \in \mathbb{Z}}} (1 - 4) = 24.$$

An analogous computation for a quasi-smooth Calabi-Yau hypersurface $Y_2$ of degree $w = 5$ in the weighted projective space $\mathbb{P}(1,1,1,2)$ with $q_i = \frac{1}{5}$ for all $0 \leq i \leq 2$ and $q_3 = \frac{2}{5}$ yields

$$i_{\mathrm{W}}(Y_2) = \frac{1}{5} \sum_{l,r=0}^{4} \prod_{\substack{0 \leq i \leq 4 \\ lq_i, rq_i \in \mathbb{Z}}} \left(1 - \frac{1}{q_i}\right) = \frac{1}{5} \left((1-5)^3 \cdot \left(1 - \frac{5}{2}\right) + 24\right) = 24.$$

Note that $Y_1$ and $Y_2$ are $K3$-surfaces and therefore we already know that the stringy Euler number of $Y_1$ and $Y_2$ equals 24 (Theorem 6.3.1 (proof)), *i.e.*, here we obtain $i_{\mathrm{W}}(Y_1) = e_{\mathrm{str}}(Y_1)$ and $i_{\mathrm{W}}(Y_2) = e_{\mathrm{str}}(Y_2)$ (cf. Corollary 8.2.13).

**Remark 8.2.4.** For a quasi-smooth hypersurface of degree $w$ in a well-formed weighted projective space $\mathbb{P}(w_0, w_1, \ldots, w_4)$ (*i.e.*, $d = 4$) Equation (8.2) reveals the usual Euler number $e(Y')$ of the resolved Calabi-Yau hypersurface $Y'$ in $\mathbb{P}(w_0, w_1, \ldots, w_4)$, where in this case a Calabi-Yau resolution always exists and is constructed in [BRG91].

The following theorem states the main result of this chapter by suggesting a Calabi-Yau hypersurface $Y^\vee$ and showing that $Y$ and $Y^\vee$ meet the equation

$$e_{\mathrm{str}}(Y^\vee) = (-1)^{d-1} i_{\mathrm{W}}(Y),$$

*i.e.*, we can interpret Witten's index of $Y$, up to sign, as the stringy Euler number of $Y^\vee$. For this purpose, we consider the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}^{d+1} \to M \to 0$$

of abelian groups, where the map $\mathbb{Z} \to \mathbb{Z}^{d+1}$ sends 1 to $(w_0, w_1, \ldots, w_d)$ and the lattice vectors $v_0, v_1, \ldots, v_d \in M$ denote the images under the surjective map $\mathbb{Z}^{d+1} \to M$ of the lattice vectors in the standard basis of $\mathbb{Z}^{d+1}$. The dual short exact sequence

$$0 \to N \to \mathbb{Z}^{d+1} \to \mathbb{Z} \to 0$$

identifies the real vector space $N_{\mathbb{R}}$ with a subspace in $\mathbb{R}^{d+1}$ defined by the linear equation $\sum_{i=0}^{d+1} w_i y_i = 0$. We define $S \subseteq M_{\mathbb{R}}$ to be a $d$-dimensional lattice simplex with vertices $v_0, v_1, \ldots, v_d$ satisfying the linear relation

$$w_0 v_0 + w_1 v_1 + \ldots + w_d v_d = 0 \in M. \tag{8.3}$$

**Theorem 8.2.5.** *Let $Y^\vee$ be a canonical Calabi-Yau model* (Definition 2.1.8) *of a $S$-non-degenerate affine hypersurface $Z_S$* (Definition 2.3.28) *in the $d$-dimensional algebraic torus $\mathbb{T}^d$ defined by the equations*

$$x_0 + x_1 + \ldots + x_d = 0 \text{ and } \prod_{i=0}^{d} x_i^{w_i} = 1,$$

*where the lattice simplex $S$ is given as above. Then the stringy Euler number $e_{\mathrm{str}}(Y^\vee)$ of $Y^\vee$ equals*

$$(-1)^{d-1} \frac{1}{w} \sum_{\substack{\emptyset \subseteq J \subseteq I \\ |J| \leq d-1}} (-1)^{|J|} n_J^2 \prod_{j \in J} \frac{1}{q_j} = (-1)^{d-1} i_{\mathrm{W}}(Y),$$

*where $Y$ is a quasi-smooth Calabi-Yau hypersurface of degree $w = \sum_{i=0}^{d} w_i$ in the well-formed weighted projective space $\mathbb{P}(w_0, w_1, \ldots, w_d)$.*

It is important to note that by eliminating one variable, $Z_S$ can also be considered as the zero set of a Laurent polynomial $f_S$ such that $S$ is the Newton polytope of $f_S$ (Definition 2.3.28). The following basic example illustrates this fact:

**Example 8.2.6.** Consider the $d$-dimensional reflexive simplex $S \coloneqq \operatorname{conv}(e_1, \ldots, e_d, -e_1 - \ldots - e_d)$ (Figure 2.3(a) for $d = 3$) and $(w_0, \ldots, w_d) = (1, \ldots, 1)$. Using the notation from Theorem 8.2.5, $Z_S$ is defined by the equations

$$\prod_{i=0}^{d} x_i = 1 \text{ and } x_0 + \ldots + x_d = 0,$$

*i.e.*, $x_0 = \prod_{i=1}^{d} x_i^{-1}$ and therefore $\prod_{i=1}^{d} x_i^{-1} + x_1 + \ldots + x_d = 0$. This leads to the corresponding Laurent polynomial

$$f_S(x) = a_{-e_1 - \ldots - e_d} \prod_{i=1}^{d} x_i^{-1} + a_{e_1} x_1 + \ldots + a_{e_d} x_d$$

with sufficiently general coefficients $a_{-e_1-\ldots-e_d}, a_{e_1}, \ldots, a_{e_d} \in \mathbb{C}$.

The final proof of Theorem 8.2.5 needs some additional propositions, which we will present and prove now:

Let $\mathbb{P}(w_0, w_1, \ldots, w_d)$ be a well-formed weighted projective space with $w = \sum_{i=0}^{d} w_i$ and $C \coloneqq \langle g \rangle$ be the cyclic group of order $w$ whose generator $g$ acts linearly on the affine space $\mathbb{C}^{d+1}$ with coordinates $x_0, x_1, \ldots, x_d$ by the diagonal matrix

$$\operatorname{diag}(e^{2\pi i q_0}, e^{2\pi i q_1}, \ldots, e^{2\pi i q_d}),$$

where $q_i = \frac{w_i}{w}$ $(i \in I)$. For any subset $J \subseteq I = \{0, 1, \ldots, d\}$, we define the cyclic subgroup

$$C_J \coloneqq \left\{ g^l \in C \,\middle|\, l q_j \in \mathbb{Z} \;\; \forall j \in J \right\} \subseteq C.$$

The subgroup $C_J$ consists of those elements $g^l \in C$ that act trivially on the $|J|$-dimensional subspace $L_J \subseteq \mathbb{C}^{d+1}$ defined by the equations $x_i = 0$ for all $i \in I \setminus J$.

**Proposition 8.2.7.** *Let $J = \{i_0, i_1, \ldots, i_k\} \subseteq I$ be a subset of $I$ and set*

$$n_J \coloneqq \gcd(w, w_{i_0}, w_{i_1}, \ldots, w_{i_k}).$$

*Then $n_J$ equals the order $|C_J|$ of the subgroup $C_J \subseteq C$.*

*Proof.* Assume without loss of generality $J$ to be $\{0, \ldots, k\}$ and set $u \coloneqq w/n_J \in \mathbb{N}$. Then one has $u q_j = w_j/n_J \in \mathbb{Z}$ for all $j \in J$, *i.e.*, $g^u \in C_J$. Therefore, the cyclic group $C_J \subseteq C$ contains the cyclic subgroup $\langle g^u \rangle$ of order $n_J$. We set $u_j \coloneqq w_j/n_J \in \mathbb{N}$ for $j \in J$. If $g^l \in C_J$, then $l u_j/u = l q_j \in \mathbb{Z}$ for all $j \in J$. Since the greatest common divisor of $u$ and $u_0, \ldots, u_k$ equals 1, there exist integers $a$ and $a_0, \ldots, a_k$ such that

$$au + \sum_{j \in J} a_j u_j = 1.$$

Therefore, $u$ divides $alu + \sum_{j \in J} a_j l u_j = l$, *i.e.*, $g^l \in \langle g^u \rangle$. $\square$

Using this notation, we can rewrite the formula of Vafa in Equation (8.2):

**Proposition 8.2.8.** *Let $Y$ be a quasi-smooth hypersurface of degree $w = \sum_{i=0}^{d} w_i$ in a well-formed $d$-dimensional weighted projective space $\mathbb{P}(w_0, w_1, ..., w_d)$* (Remark 8.2.2). *Then the Witten index of $Y$ in Vafa's formula is computable as*

$$i_{\mathrm{W}}(Y) = \frac{1}{w} \sum_{\emptyset \subseteq J \subseteq I} (-1)^{|J|}\, n_J^2 \prod_{j \in J} \frac{1}{q_j},$$

*where $q_i = \frac{w_i}{w}$ $(i \in I)$. Moreover, it is sufficient to consider the sum only over all subsets $J \subseteq I$ such that $|J| \leq d-1$,* i.e.,

$$i_{\mathrm{W}}(Y) = \frac{1}{w} \sum_{\substack{\emptyset \subseteq J \subseteq I \\ |J| \leq d-1}} (-1)^{|J|}\, n_J^2 \prod_{j \in J} \frac{1}{q_j}.$$

*Proof.* If $(l, r) = (0, 0)$, then $rq_i, lq_i \in \mathbb{Z}$ for all $i \in I$ and we obtain the product

$$\prod_{i=0}^{d} \left(1 - \frac{1}{q_i}\right) = \sum_{\emptyset \subseteq J \subseteq I} \prod_{j \in J} \left(-\frac{1}{q_j}\right) = \sum_{\emptyset \subseteq J \subseteq I} (-1)^{|J|} \prod_{j \in J} \frac{1}{q_j}$$

that appears as a summand in Vafa's formula (8.2). The same calculation can be done for pairs $(l, r) \neq (0, 0)$: For an integer $s$, set $J_s := \{i \in I \mid sq_i \in \mathbb{Z}\}$. Then

$$\prod_{\substack{0 \leq i \leq d \\ lq_i, rq_i \in \mathbb{Z}}} \left(1 - \frac{1}{q_i}\right) = \prod_{i \in J_l \cap J_r} \left(1 - \frac{1}{q_i}\right) = \sum_{\emptyset \subseteq J \subseteq J_l \cap J_r} (-1)^{|J|} \prod_{j \in J} \frac{1}{q_j}$$

for any pair $(l, r)$. Using Equation (8.2), we obtain

$$\begin{aligned} i_{\mathrm{W}}(Y) &= \frac{1}{w} \sum_{l,r=0}^{w-1} \sum_{\emptyset \subseteq J \subseteq J_l \cap J_r} (-1)^{|J|} \prod_{j \in J} \frac{1}{q_j} = \frac{1}{w} \sum_{\emptyset \subseteq J \subseteq I} \sum_{\substack{0 \leq l,r \leq w-1 \\ g^l, g^r \in C_J}} (-1)^{|J|} \prod_{j \in J} \frac{1}{q_j} \\ &= \frac{1}{w} \sum_{\emptyset \subseteq J \subseteq I} (-1)^{|J|}\, n_J^2 \prod_{j \in J} \frac{1}{q_j}. \end{aligned}$$

Now, we note that $n_J = 1$ if $|J| \in \{d, d+1\}$, since we assume

$$\gcd(w_0, \ldots, w_{i-1}, w_{i+1}, \ldots, w_d) = 1 \ \ \forall i \in I.$$

Using the equality $\sum_{i=0}^{d} q_i = 1$, we obtain

$$(-1)^{d+1} \prod_{i=0}^{d} \frac{1}{q_i} + (-1)^d \sum_{i=0}^{d} \prod_{\substack{j \in I \\ j \neq i}} \frac{1}{q_j} = 0$$

and this implies

$$i_{\mathrm{W}}(Y) = \frac{1}{w} \sum_{\substack{\emptyset \subseteq J \subseteq I \\ |J| \leq d-1}} (-1)^{|J|}\, n_J^2 \prod_{j \in J} \frac{1}{q_j}.$$

□

For the following proposition, let $\mathbb{P}(w_0, w_1, \ldots, w_d)$ be a well-formed weighted projective space and $S$ a lattice simplex given by Equation (8.3). By Definition 2.3.22, $S$ is called almost pseudoreflexive if the polytopes $S$ and $[S^*] = \operatorname{conv}(S^* \cap N) \subseteq N_{\mathbb{R}}$ contain the origin in their interior and are both $d$-dimensional, where $S^* \subseteq N_{\mathbb{R}}$ is the dual simplex of $S \subseteq M_{\mathbb{R}}$ (Definition 2.3.3).

**Proposition 8.2.9.** *If the well-formed weighted projective space $\mathbb{P}(w_0, w_1, \ldots, w_d)$ contains a quasi-smooth Calabi-Yau hypersurface $Y$ of degree $w = \sum_{i=0}^{d} w_i$, then the lattice simplex $S \subseteq M_{\mathbb{R}}$ is almost pseudoreflexive.*

*Proof.* As stated before the proposition, we first show that

$$S = \Big\{ \sum_{i=0}^{d} c_i v_i \,\Big|\, c_i \in \mathbb{R}_{\geq 0} \sum_{i=0}^{d} c_i = 1 \Big\}$$

and $[S^*]$ contain the origin in their interior: The origin $0 \in M$ is an interior lattice point in $S$ because $\frac{1}{w} \sum_{i=0}^{d} w_i v_i = \frac{1}{w} \cdot 0 = 0$ using Equation (8.3), $w_i/w > 0$ for all $0 \leq i \leq d$, and $\sum_{i=0}^{d} \frac{w_i}{w} = 1$.

To show that $[S^*]$ contains the origin $0 \in N$ in its interior, let $\Delta = \operatorname{conv}(m \mid a_m \neq 0) \subseteq \mathbb{R}^{d+1}$ be the Newton polytope (Definition 2.3.28) of the weighted homogeneous polynomial $f = \sum_{m \in \mathbb{Z}^{d+1}} a_m z^m \in \mathbb{C}[z_0, \ldots, z_d]$ of degree $w$ associated with the Calabi-Yau hypersurface $Y$. On the one hand, the geometric genus

$$p_g(Y) = \dim(H^0(Y, \Omega_Y^{\dim(Y)}))$$

of the Calabi-Yau hypersurface $Y$ equals 1 because

$$p_g(Y) = \dim(H^0(Y, \Omega_Y^{\dim(Y)})) = \dim(H^0(Y, \omega_Y)) = \dim(H^0(Y, \mathcal{O}_Y)) = 1.$$

On the other hand, the geometric genus $p_g(Y)$ equals $\left|\Delta^\circ \cap \mathbb{Z}^{d+1}\right|$ (Equation (7.1)). A combination of these two facts yields $\left|\Delta^\circ \cap \mathbb{Z}^{d+1}\right| = 1$.

Furthermore, $\Delta$ is the convex hull of all lattice points in the $d$-dimensional rational simplex

$$S' := \{(y_0, y_1, \ldots, y_d) \in \mathbb{R}^{d+1}_{\geq 0} \,|\, w_0 y_0 + w_1 y_1 + \ldots + w_d y_d = w\}$$

and $(1, \ldots, 1) \in S'$ because $w_0 \cdot 1 + w_1 \cdot 1 + \ldots + w_d \cdot 1 = w_0 + w_1 + \ldots + w_d = w$. Thereby, we get $\Delta^\circ \cap \mathbb{Z}^{d+1} = \{(1, \ldots, 1)\}$ because all other lattice points of $\Delta$ lie on facets of $\Delta$. In addition, the shifted simplex $S' - (1, \ldots, 1)$ is the dual simplex $S^* \subseteq N_{\mathbb{R}}$ to $S$ because

$$S^* = \{(y_0, y_1, \ldots, y_d) \in N_{\mathbb{R}} \,|\, y_i \geq -1 \ \ \forall i \in I\},$$

*i.e.*, $(0, \ldots, 0)$ is an interior lattice point of $S^*$. In particular, $(0, \ldots, 0) \in [S^*]^\circ$.

Moreover, the lattice points in $[S^*]$ correspond to monomials of the weighted homogenous polynomial $f$ defining the quasi-smooth Calabi-Yau hypersurface $Y$. Assume all monomials belong to one hypersurface then the common zeros of all partial derivations $\frac{\partial}{\partial z_i} f$ $(i \in I)$ would not only vanish at the point $(0, \ldots, 0)$. This is a contradiction to our quasi-smoothness condition of $Y$ (Remark 8.2.2) and therefore $[S^*]$ is also $d$-dimensional. $\square$

We proceed investigating the normalized volume (Definition 2.3.6) of the $d$-dimensional lattice simplex $S \subseteq M_{\mathbb{R}}$ with vertices $v_0, v_1, \ldots, v_d$ satisfying the linear relation $w_0 v_0 + w_1 v_1 + \ldots + w_d v_d = 0 \in M$:

**Proposition 8.2.10.** *The $d$-dimensional lattice simplex $S$ from above has integral normalized volume $w$,* i.e.,

$$v(S) = w = w_0 + w_1 + \ldots + w_d.$$

*Moreover, for any (lattice) face $S_J \preceq S$ corresponding to a subset $J \subseteq I$ one has*

$$v(S_J) = n_{\overline{J}},$$

*where $\overline{J} := I \setminus J$ and $n_{\overline{J}}$ is defined in* Proposition 8.2.7.

*Proof.* Consider the lattice $\overline{M} := M \oplus \mathbb{Z}$ together with the lattice vectors $\overline{v}_i := (v_i, 1) \in \overline{M}$ $(0 \leq i \leq d)$. Denote by $\overline{S}$ the $(d+1)$-dimensional lattice simplex with vertex $0 \in \overline{M}$ and vertices $\overline{v}_0, \ldots, \overline{v}_d$.

The (integral) normalized volume $v(\overline{S})$ of $\overline{S}$ equals the index of the sublattice $M' := \langle \overline{v}_i \rangle_{0 \leq i \leq d}$ generated by the lattice vectors $\overline{v}_i$ $(0 \leq i \leq d)$ in $\overline{M}$. Since $M$ is generated by $v_i$ $(0 \leq i \leq d)$, the lattice $\overline{M}$ is generated by $\overline{v}_i$ together with the lattice vector $\overline{v} := (0,1) \in M \oplus \mathbb{Z} = \overline{M}$. So the quotient $\overline{M}/M'$ is a cyclic group generated by $\overline{v} + M'$.

Let $l$ be a positive integer, then one has $l(\overline{v} + M') = M'$ if and only if there exist $l_i \in \mathbb{Z}$ such that $l\overline{v} = \sum_{i=0}^{d} l_i \overline{v}_i$, *i.e.*, $\sum_{i=0}^{d} l_i = l$ and $\sum_{i=0}^{d} l_i v_i = l \cdot 0 = 0 \in M$. On the other hand, any linear relation between the lattice vectors $v_0, v_1, \ldots, v_d$ is a multiple of the integral linear relation $\sum_{i=0}^{d} w_i v_i = 0$, *i.e.*, in particular, $\sum_{i=0}^{d} l_i v_i = \alpha \sum_{i=0}^{d} w_i v_i = 0$ $(\alpha \in \mathbb{Z})$. This shows that $l(\overline{v} + M') = M'$ if and only if $l$ is a multiple of $w$, *i.e.*, $\overline{M}/M' \cong \mathbb{Z}/w\mathbb{Z}$ and $v(S) = v(\overline{S}) = w$.

Analogously, we take any subset $J = \{i_0, \ldots, i_k\} \subseteq I$ and consider the $k$-dimensional face of $\overline{S}$ with vertices $0 \in \overline{M}$ and $\overline{v_{i_0}}, \ldots, \overline{v_{i_k}}$. Then the normalized volume $v(\overline{S}_J)$ of $\overline{S}_J$ equals the index of the sublattice $M'_J$ generated by the lattice vectors $\overline{v_j}$ $(j \in J)$ in the $|J|$-dimensional lattice $\overline{M}_J := \overline{M} \cap (M'_J \otimes \mathbb{Q})$. First of all, we note that the quotient $\overline{M}_J/M'_J$ is a subgroup of the cyclic group $\overline{M}/M' \cong \mathbb{Z}/w\mathbb{Z}$. Indeed, we have a homomorphism $\varphi : \overline{M}_J \to \overline{M}/M'$ obtained from the embedding of $\overline{M}_J$ into $\overline{M}$ and the kernel of $\varphi$ is $\overline{M}_J \cap M' = \overline{M} \cap (M'_J \otimes \mathbb{Q}) \cap M' = M'_J$ because $(M'_J \otimes \mathbb{Q}) \cap M' = M'_J$. Next, we note that an element $l(\overline{v} + M') \in \overline{M}/M'$ belongs to the subgroup $\overline{M}_J/M'_J$ if and only if the coefficients $lq_i$ in the equation

$$l\overline{v} = \sum_{i=0}^{d} lq_i \overline{v}_i$$

are integers for all $i \notin J$. By Proposition 8.2.7, the latter happens if and only if $g^l \in C_{\overline{J}}$. This shows that $\overline{M}_J/M'_J \cong C_{\overline{J}}$ and $v(S_J) = v(\overline{S}_J) = n_{\overline{J}}$. □

We denote by $\sigma^J$ the normal cone in the normal fan $\Sigma^S$ (Definition 2.3.8) corresponding to a face $S_J \preceq S$ of the lattice simplex $S$.

**Proposition 8.2.11.** *Let $J \subseteq I = \{0, \dots, d\}$ be a subset of $I$. Then the normalized volume $v(\sigma^J \cap S^*)$ of the rational polytope $\sigma^J \cap S^*$ is given by*

$$v(\sigma^J \cap S^*) = \frac{n_{\overline{J}}}{w} \prod_{i \in \overline{J}} \frac{1}{q_i},$$

*where $\overline{J} = I \setminus J$ and $q_i = \frac{w_i}{w}$ $(i \in I)$.*

*Proof.* Set $p := (1, \dots, 1) \in \mathbb{Z}^{d+1}$. For simplicity, we assume without loss of generality $J = \{0, \dots, k\}$, *i.e.*, $\overline{J} = \{k+1, \dots, d\}$. We consider the shifted rational polytope $S' := S^* + p \subseteq N_{\mathbb{R}} + p \subseteq \mathbb{R}^{d+1}$. The dual rational face $S^*_J \preceq S^*$ in the shifted simplex $S'$ has dimension $d - k - 1$ and vertices $\frac{1}{q_i} e_i$ $(i \in \overline{J})$ because

$$S' = \{(y_0, y_1, \dots, y_d) \in \mathbb{R}^{d+1}_{\geq 0} \mid w_0 y_0 + w_1 y_1 + \dots + w_d y_d = w\}$$

has vertices $\frac{1}{q_i} e_i$ $(i \in I)$, where $\{e_0, e_1, \dots, e_d\}$ denotes the standard basis of $\mathbb{R}^{d+1}$. Let $\Pi_{\overline{J}}$ be the pyramid with top $0 \in \mathbb{R}^{d+1}$ over the shifted face $S'_J := S^*_J + p \subseteq S^* + p = S'$. Then

$$v(\Pi_{\overline{J}}) = \prod_{i \in \overline{J}} \frac{1}{q_i}.$$

The pyramid $\Pi_{\overline{J}}$ is contained in the subspace generated by $e_{k+1}, \dots, e_d$. The basis of the pyramid is the simplex $S'_J$ that belongs to a hyperplane in the affine linear subspace defined by the equation

$$w_{k+1} x_{k+1} + \dots + w_d x_d = w.$$

The integral distance between this hyperplane and the origin equals $w / n_{\overline{J}}$, where $n_{\overline{J}} = \gcd(w, w_{k+1}, \dots, w_d)$. Therefore, we have

$$v(S^*_J) = v(S'_J) = v(\Pi_{\overline{J}}) \cdot \frac{n_{\overline{J}}}{w} = \frac{n_{\overline{J}}}{w} \prod_{i \in \overline{J}} \frac{1}{q_i}.$$

Since the integral distance from the origin to $S^*_J$ equals 1, we get $v(S^*_J) = v(\sigma^J \cap S^*)$. $\square$

Now, we are in the position to prove the main theorem of this chapter:

*Proof of Theorem 8.2.5.* Using Proposition 8.2.9, we know that $S \subseteq M_{\mathbb{R}}$ is a $d$-dimensional almost pseudoreflexive simplex. Therefore, we apply Theorem 6.3.2 to compute the stringy Euler number of a canonical Calabi-Yau model of a $S$-non-degenerate affine hypersurface $Z_S$ in the $d$-dimensional algebraic torus $\mathbb{T}^d$. Moreover, the faces $\theta \preceq S$ of $S$ are $(|J|-1)$-dimensional simplices $S_J$ parametrized by subsets $J \subseteq I$, where $\dim(S_J) \geq 1$ if and only if $|J| \geq 2$.

The normalized volumes $v(\theta) = v(S_J)$ and $v(\sigma^\theta \cap \Delta^*) = v(\sigma^J \cap S^*)$ have been computed in Proposition 8.2.10 and Proposition 8.2.11, respectively. So we get

$$e_{\mathrm{str}}(Y^\vee) = \sum_{\substack{\emptyset \subseteq J \subseteq I \\ |J| \geq 2}} (-1)^{|J|} n_{\overline{J}} \cdot \frac{n_{\overline{J}}}{w} \prod_{i \in \overline{J}} \frac{1}{q_i} = \frac{1}{w} \sum_{\substack{\emptyset \subseteq J \subseteq I \\ |J| \geq 2}} (-1)^{|J|} n^2_{\overline{J}} \prod_{i \in \overline{J}} \frac{1}{q_i}.$$

Since $|J|+|\overline{J}|=|I|=d+1$, we obtain

$$e_{\text{str}}(Y^{\vee})=(-1)^{d-1}\frac{1}{w}\sum_{\substack{\emptyset\subseteq J\subseteq I\\ |\overline{J}|\leq d-1}}(-1)^{|\overline{J}|}\,n_{\overline{J}}^2\prod_{i\in\overline{J}}\frac{1}{q_i}=(-1)^{d-1}\frac{1}{w}\sum_{\substack{\emptyset\subseteq J\subseteq I\\ |J|\leq d-1}}(-1)^{|J|}n_J^2\prod_{j\in J}\frac{1}{q_j}$$

and

$$\frac{1}{w}\sum_{\substack{\emptyset\subseteq J\subseteq I\\ |J|\leq d-1}}(-1)^{|J|}n_J^2\prod_{j\in J}\frac{1}{q_j}=i_{\text{W}}(Y)$$

by Proposition 8.2.8. □

The following corollaries consider a quasi-smooth Calabi-Yau hypersurface $Y$ of degree $w=\sum_{i=0}^{d}w_i$ in a well-formed weighted projective space $\mathbb{P}(w_0,w_1,\ldots,w_d)$. In addition, we consider a canonical Calabi-Yau model $Y^{\vee}$ of a $S$-non-degenerate affine hypersurface $Z_S$ in the $d$-dimensional algebraic torus $\mathbb{T}^d$ defined by the equations

$$x_0+x_1+\ldots+x_d=0 \text{ and } \prod_{i=0}^{d}x_i^{w_i}=1,$$

where $S\subseteq M_{\mathbb{R}}$ denotes the $d$-dimensional lattice simplex with vertices $v_0,v_1,\ldots,v_d$ satisfying the linear relation $w_0v_0+w_1v_1+\ldots+w_dv_d=0\in M$.

**Corollary 8.2.12.** *Let $Y$ and $Y^{\vee}$ be given as above. In addition, we require that all weights $w_i$ divide $w$ $(0\leq i\leq d)$. Then $Y$ is a Calabi-Yau hypersurface of Fermat-type and $S$ a reflexive simplex* (page 118). *Moreover, the Witten index of $Y$ is computable via*

$$i_{\text{W}}(Y)=(-1)^{d-1}\sum_{\substack{\theta\preceq S\\ 1\leq\dim(\theta)\leq d-2}}(-1)^{\dim(\theta)-1}v(\theta)\cdot v(\theta^*),$$

*where $\sum_{\substack{\theta\preceq S\\ 1\leq\dim(\theta)\leq d-2}}(-1)^{\dim(\theta)-1}v(\theta)\cdot v(\theta^*)$ is the computation formula for the stringy Euler number $e_{\text{str}}(Y^{\vee})$ of $Y^{\vee}$.*

*Proof.* Theorem 8.2.5 yields

$$(-1)^{d-1}i_{\text{W}}(Y)=e_{\text{str}}(Y^{\vee}).$$

Applying [Bat94, Section 5.4], the simplex $S\subseteq M_{\mathbb{R}}$ is reflexive. Using one of three references on page 118, we get the stringy Euler number of $Y^{\vee}$ via

$$e_{\text{str}}(Y^{\vee})=\sum_{\substack{\theta\preceq S\\ 1\leq\dim(\theta)\leq d-2}}(-1)^{\dim(\theta)-1}v(\theta)\cdot v\left(\theta^*\right).$$

A combination of these two facts yields

$$i_{\text{W}}(Y)=(-1)^{d-1}e_{\text{str}}(Y^{\vee})=(-1)^{d-1}\sum_{\substack{\theta\preceq S\\ 1\leq\dim(\theta)\leq d-2}}(-1)^{\dim(\theta)-1}v(\theta)\cdot v\left(\theta^*\right).$$

□

As mentioned in Section 8.1 on page 119, there is so far no general mathematical proof for the assertion that Vafa's formula computes the stringy Euler number of $Y$, *i.e.*,

$$i_{\mathrm{W}}(Y) = e_{\mathrm{str}}(Y). \tag{8.4}$$

Using Theorem 8.2.5, this would imply

$$(-1)^{d-1} e_{\mathrm{str}}(Y) = e_{\mathrm{str}}(Y^{\vee}),$$

which we would be a further indication that $Y$ and $Y^{\vee}$ are mirror dual because they pass the stringy basic mirror symmetry test. Therefore, this would help to establish the predicted mirror construction. For the special case that $Y$ is Calabi-Yau hypersurfaces of Fermat-type, Equation (8.4) is proved as follows:

**Corollary 8.2.13.** *Let $Y$ and $Y^{\vee}$ be given as above. In addition, we require that all weights $w_i$ divide $w$ $(0 \leq i \leq d)$,* i.e., *$Y$ is a Calabi-Yau hypersurface of Fermat-type and $S$ a reflexive simplex. Then the Witten index of $Y$ equals the stringy Euler number of $Y$,* i.e.,

$$i_{\mathrm{W}}(Y) = e_{\mathrm{str}}(Y).$$

*Proof.* Applying Theorem 8.2.5, we obtain

$$(-1)^{d-1} i_{\mathrm{W}}(Y) = e_{\mathrm{str}}(Y^{\vee})$$

and according to the assumption $S$ is reflexive, *i.e.*, $e_{\mathrm{str}}(Y^{\vee}) = (-1)^{d-1} e_{\mathrm{str}}(Y)$ (Equation (8.1)). A combination of both facts yields the desired equation. □

# Appendix A

## Canonical Fano 3-polytopes with $\Delta^{\mathrm{FI}} \neq \{0\}$

## A.1 Canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with $\mathbf{dim}(\boldsymbol{\Delta}^{\mathbf{FI}}) = 1$ and $\mathbf{0} \in \mathbf{vert}(\boldsymbol{\Delta}^{\mathbf{FI}})$

**Table A.1: 9 Canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with $\mathbf{dim}(\boldsymbol{\Delta}^{\mathbf{FI}}) = 1$ and $\mathbf{0} \in \mathbf{vert}(\boldsymbol{\Delta}^{\mathbf{FI}})$.** Table contains: ID from Graded Ring Database corresponding to $\Delta$, vertices $\mathrm{vert}(\Delta)$ of $\Delta$, vertices $\mathrm{vert}(\Delta^{\mathrm{FI}})$ of $\Delta^{\mathrm{FI}}$, and link to pictures of $\Delta$ and $\Delta^{\mathrm{FI}}$.

| ID | $\mathrm{vert}(\Delta)$ | $\mathrm{vert}(\Delta^{\mathrm{FI}})$ | $\Delta$ & $\Delta^{\mathrm{FI}}$ |
|---|---|---|---|
| 547324 | $(2,3,8),(1,0,0),(0,1,0),(-1,-1,-1)$ | $(0,0,0),(1/2,1/2,1)$ | A.1(a) |
| 547323 | $(-1,1,-2),(1,-2,3),(1,0,0),(-2,5,-3)$ | $(0,0,0),(0,2/3,0)$ | A.1(b) |
| 547311 | $(-1,4,2),(-1,-1,0),(0,0,-1),(2,0,1)$ | $(0,0,0),(0,2/3,2/3)$ | A.1(c) |
| 547490 | $(1,2,4),(1,0,0),(1,-2,3),(-1,1,-2)$ | $(0,0,0),(0,1/2,0)$ | A.1(d) |
| 547321 | $(1,-2,3),(0,1,0),(1,0,0),(-6,3,-8)$ | $(0,0,0),(-1/2,1/2,-1)$ | A.1(e) |
| 547305 | $(0,1,0),(1,0,0),(1,2,4),(-4,-6,-7)$ | $(0,0,0),(-2/3,-2/3,-2/3)$ | A.1(f) |
| 547526 | $(1,0,0),(0,1,0),(-2,1,5),(2,-4,-9)$ | $(0,0,0),(2/3,-2/3,-2)$ | A.1(g) |
| 547454 | $(2,1,7),(1,0,0),(0,1,0),(-2,-3,-3)$ | $(0,0,0),(0,0,1/2)$ | A.1(h) |
| 547446 | $(0,1,1),(-6,7,-15),(1,-2,3),(1,0,0)$ | $(0,0,0),(-1/2,1/2,-1)$ | A.1(i) |

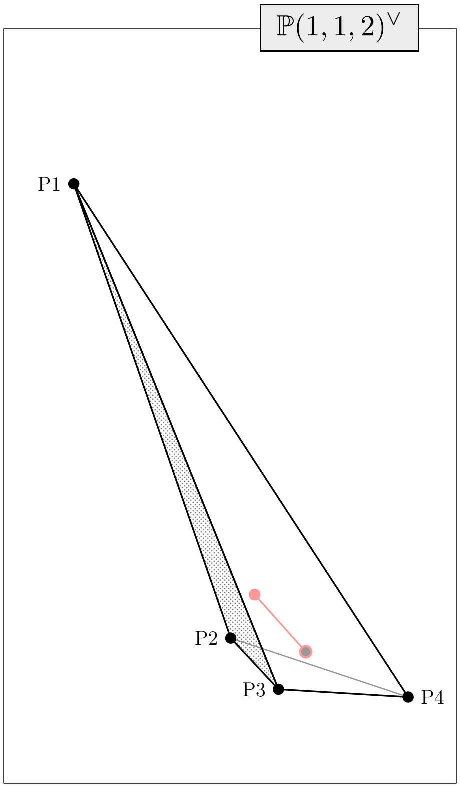
**Table A.2: 9 Canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with $\mathbf{dim}(\boldsymbol{\Delta}^{\mathbf{FI}}) = \mathbf{1}$ and $\mathbf{0} \in \mathbf{vert}(\boldsymbol{\Delta}^{\mathbf{FI}})$.** Table contains: ID from Graded Ring Database corresponding to $\Delta$, unique primitive lattice point $v_\Delta$, vertices $\mathrm{vert}(\theta_{\mathrm{proj}})$ of projected facet $\theta_{\mathrm{proj}} \preceq \Delta$, vertices $\mathrm{vert}(\overline{\Delta})$ of reflexive projection result $\overline{\Delta}$, and link to a picture of the projection result $\overline{\Delta}$.

| ID | $v_\Delta$ | $\mathrm{vert}(\theta_{\mathrm{proj}})$ | $\mathrm{vert}(\overline{\Delta})$ | $\overline{\Delta}$ |
|---|---|---|---|---|
| 547324 | $(1,1,2)$ | $(2,3,8),(1,0,0),(0,1,0)$ | $(3,1),(-1,-1),(-1,1)$ | 7.1(b) |
| 547323 | $(0,1,0)$ | $(1,-2,3),(1,0,0),(-2,5,-3)$ | $(1,3),(1,0),(-1,-3)$ | 7.1(a) |
| 547311 | $(0,1,1)$ | $(-1,4,2),(-1,-1,0),(2,0,1)$ | $(1,3),(1,0),(-1,-3)$ | 7.1(a) |
| 547490 | $(0,1,0)$ | $(1,2,4),(1,0,0),(-1,1,-2)$ | $(3,1),(-1,-1),(-1,1)$ | 7.1(b) |
| 547321 | $(-1,1,-2)$ | $(0,1,0),(1,0,0),(-6,3,-8)$ | $(3,1),(-1,-1),(-1,1)$ | 7.1(b) |
| 547305 | $(-1,-1,-1)$ | $(0,1,0),(1,2,4),(-4,-6,-7)$ | $(1,3),(1,0),(-1,-3)$ | 7.1(a) |
| 547526 | $(1,-1,-3)$ | $(1,0,0),(0,1,0),(2,-4,-9)$ | $(1,3),(1,0),(-1,-3)$ | 7.1(a) |
| 547454 | $(0,0,1)$ | $(2,1,7),(0,1,0),(-2,-3,-3)$ | $(3,1),(-1,-1),(-1,1)$ | 7.1(b) |
| 547446 | $(-1,-1,-2)$ | $(0,1,1),(-6,7,-15),(1,-2,3)$ | $(3,1),(-1,-1),(-1,1)$ | 7.1(b) |

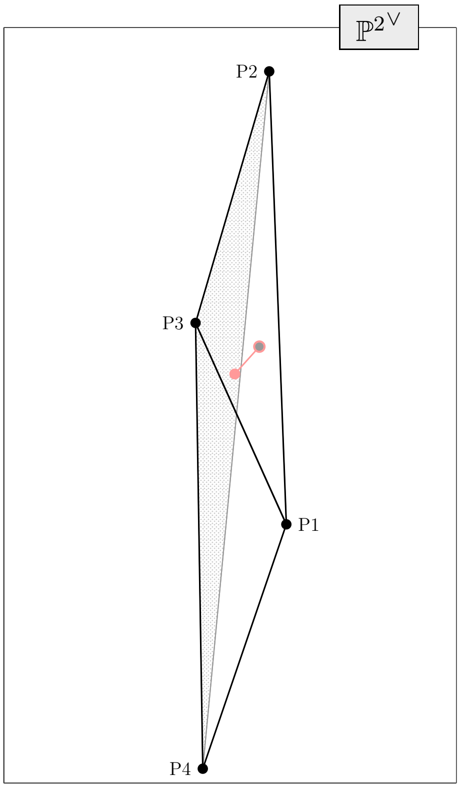
**Table A.3: 9 Canonical Fano 3-polytopes $\Delta$ with $\dim(\Delta^{\mathrm{FI}}) = 1$ and $0 \in \mathrm{vert}(\Delta^{\mathrm{FI}})$.** Table contains: ID from Graded Ring Database corresponding to $\Delta$, $(\langle v_\Delta, v_1\rangle, \ldots, \langle v_\Delta, v_4\rangle)$, elliptic fibration $m$, number of elements $|\mathrm{supp}(\Delta^{\mathrm{FI}})|$ in support of $\Delta^{\mathrm{FI}}$, vertices $\mathrm{vert}(\Delta^{\mathrm{can}})$ of canonical hull $\Delta^{\mathrm{can}}$ of $\Delta$, and stringy Euler number $e_{\mathrm{str}}(Y)$ of a canonical model $Y$ corresponding to $\Delta$.

| ID | $(\langle v_\Delta, v_1\rangle, \ldots, \langle v_\Delta, v_4\rangle)$ | $m$[1] | $\lvert\mathrm{supp}(\Delta^{\mathrm{FI}})\rvert$ | $\mathrm{vert}(\Delta^{\mathrm{can}})$ | $e_{\mathrm{str}}(Y)$ |
|---|---|---|---|---|---|
| 547324 | $(4, 2, 0, -2)$ | $\frac{z_3^2}{z_0^4 z_1^2}$ | 18 | $\mathrm{vert}(\Delta), (0, 1, 4)$ | 24 |
| 547323 | $(-3, 0, 3, 6)$ | $\frac{z_0^3}{z_2^3 z_3^6}$ | 20 | $\mathrm{vert}(\Delta), (-2, 4, -3)$ | 24 |
| 547311 | $(3, 0, -3, 3)$ | $\frac{z_2^3}{z_0^3 z_3^3}$ | 11 | $\mathrm{vert}(\Delta), (-1, 2, 0)$ | 24 |
| 547490 | $(6, 2, -2, 0)$ | $\frac{z_2^2}{z_0^6 z_1^2}$ | 21 | $\mathrm{vert}(\Delta), (1, -1, 4)$ | 24 |
| 547321 | $(-2, 0, 2, 4)$ | $\frac{z_0^2}{z_2^2 z_3^4}$ | 7 | $\mathrm{vert}(\Delta), (1, 0, 1), (0, -3, 4)$ | 24 |
| 547305 | $(0, -3, 3, 3)$ | $\frac{z_1^3}{z_2^3 z_3^3}$ | 6 | $\mathrm{vert}(\Delta), (0, -2, -3), (1, 2, 2)$ | 24 |
| 547526 | $(3, 0, -3, 6)$ | $\frac{z_2^3}{z_0^3 z_3^6}$ | 5 | $\mathrm{vert}(\Delta) \setminus \{(-2, 1, 5)\}$, $(0, 1, 3), (-3, 1, 6)$ | 24 |
| 547454 | $(2, -2, 0, 2)$ | $\frac{z_1^2}{z_0^2 z_3^2}$ | 9 | $\mathrm{vert}(\Delta), (2, 1, 2)$ | 24 |
| 547446 | $(2, 6, 0, -2)$ | $\frac{z_3^2}{z_0^2 z_1^6}$ | 6 | $\mathrm{vert}(\Delta), (1, 0, -1), (1, 0, 3)$ | 24 |

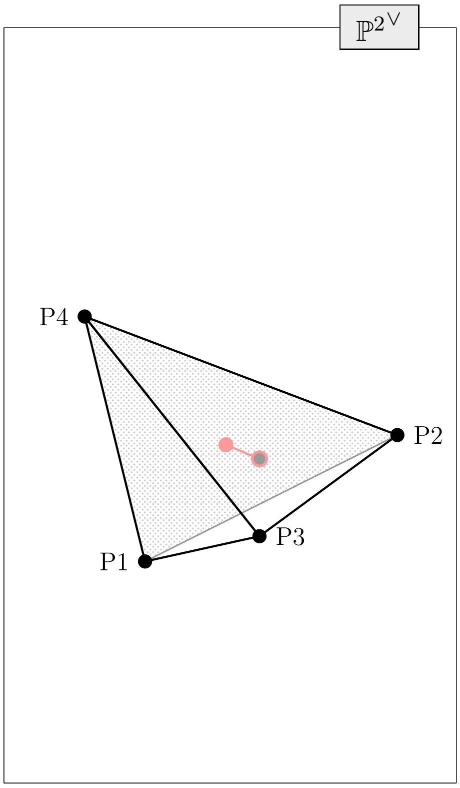
[1] Compared to Table 2 on page 20 in [CG11], we observe mixed up variables $z_0, \ldots, z_3$. The reason for this phenomenon are permuted weights.



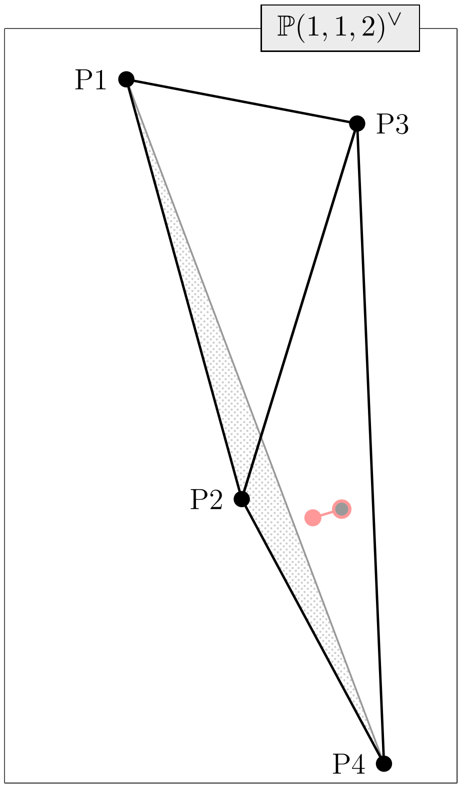

**(a)** ID 547324




**(b)** ID 547323

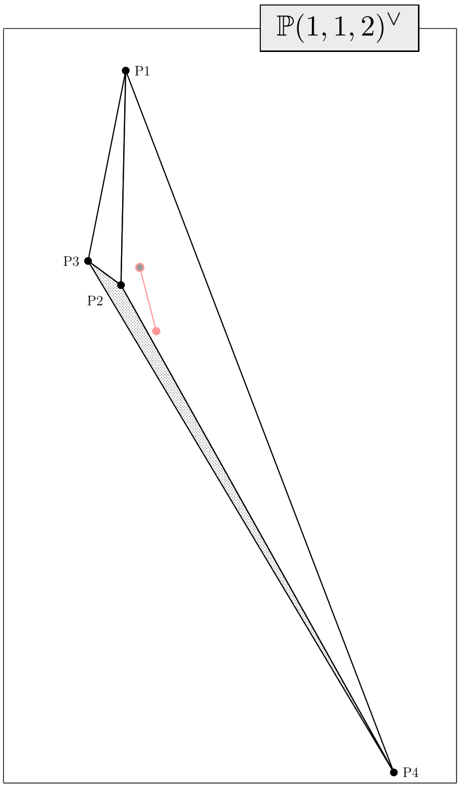



**(c)** ID 547311

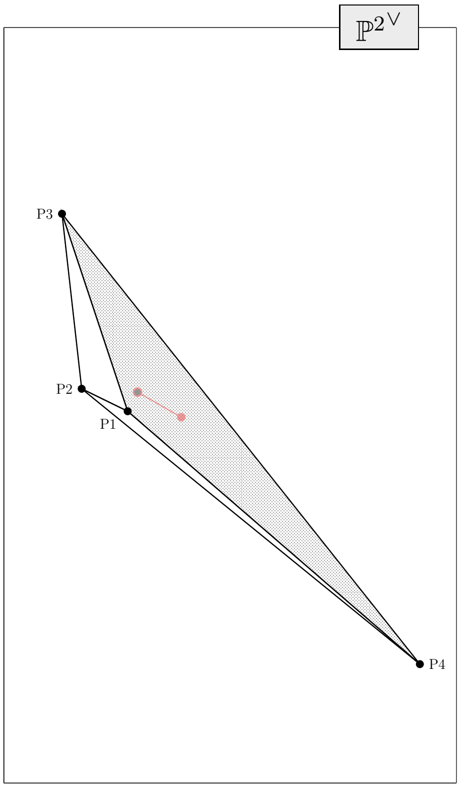



**(d)** ID 547490

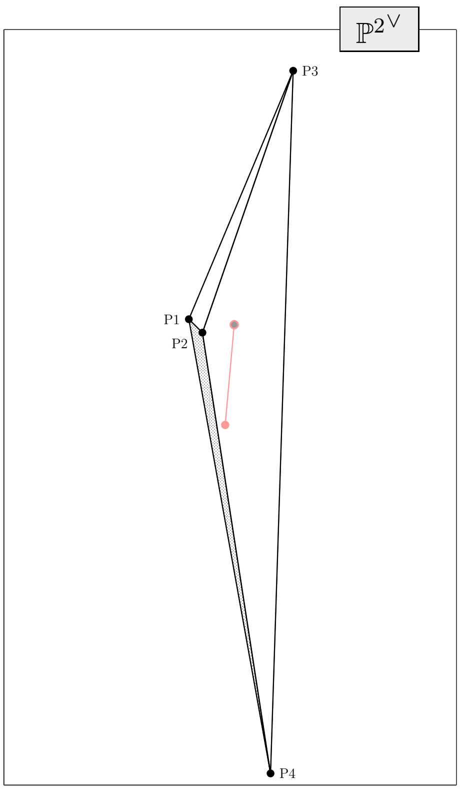
**Figure A.1: 9 Canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with $\mathbf{dim}(\boldsymbol{\Delta}^{\mathbf{FI}}) = \mathbf{1}$ and $\mathbf{0} \in \mathbf{vert}(\boldsymbol{\Delta}^{\mathbf{FI}})$.**



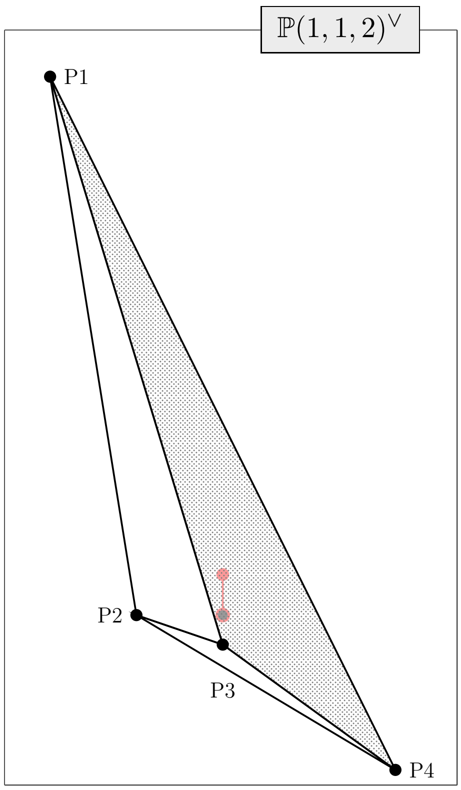

**(e)** ID 547321




**(f)** ID 547305

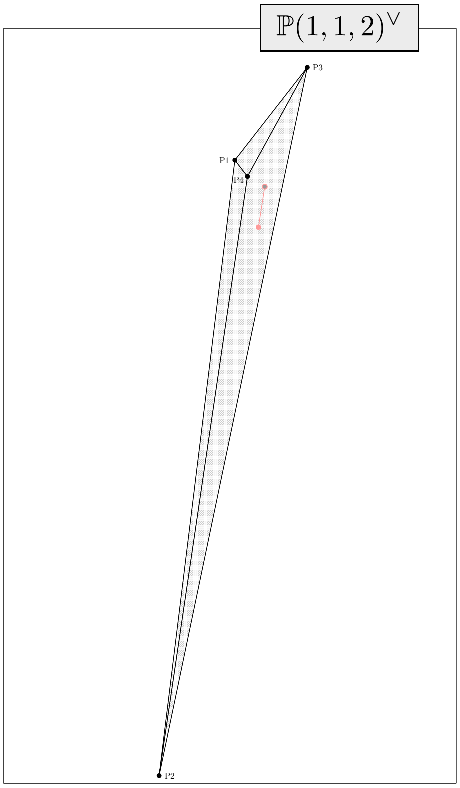



**(g)** ID 547526




**(h)** ID 547454

**Figure A.1: 9 Canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with $\mathbf{dim}(\boldsymbol{\Delta}^{\mathbf{FI}}) = \mathbf{1}$ and $\mathbf{0} \in \mathbf{vert}(\boldsymbol{\Delta}^{\mathbf{FI}})$.**




**(i)** ID 547446

**Figure A.1: 9 Canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with $\mathbf{dim}(\boldsymbol{\Delta}^{\mathbf{FI}}) = \mathbf{1}$ and $\mathbf{0} \in \mathbf{vert}(\boldsymbol{\Delta}^{\mathbf{FI}})$.**

**Remark A.1.1.** In Figure A.1 shaded faces are occluded. The Fine interior is coloured red, the origin grey with a red margin, and the facet that is used to project grey dotted. The surface corresponding to the reflexive polygon (projection result) is boxed top right. The vertices are numbered consecutively as listed in the Graded Ring Database.

## A.2 Canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with $\mathbf{dim}(\boldsymbol{\Delta}^{\mathbf{FI}}) = 1$ and $\mathbf{0} \in (\boldsymbol{\Delta}^{\mathbf{FI}})^\circ$

**Table A.4: 20 Canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with $\mathbf{dim}(\boldsymbol{\Delta}^{\mathbf{FI}}) = 1$ and $\mathbf{0} \in (\boldsymbol{\Delta}^{\mathbf{FI}})^\circ$.** Table contains: ID from Graded Ring Database corresponding to $\Delta$, vertices vert$(\Delta)$ of $\Delta$, and link to pictures of $\Delta$ and $\Delta^{\mathrm{FI}}$.

| ID | vert$(\Delta)$ | $\Delta$ & $\Delta^{\mathrm{FI}}$ |
|---|---|---|
| 547393 | $(0,1,0),(2,1,1),(-2,-3,-5),(2,1,9)$ | A.2(a) |
| 547409 | $(-4,2,9),(1,0,0),(0,1,0),(7,-6,-18)$ | A.2(b) |
| 547461 | $(0,1,0),(2,1,1),(-2,-3,-5),(0,1,4)$ | A.2(c) |
| 544442 | $(1,0,0),(0,1,0),(3,-6,8),(1,-4,4),(-5,6,-12)$ | A.2(d) |
| 544443 | $(-1,-2,0),(3,-6,8),(0,1,0),(1,0,0),(-3,4,-8)$ | A.2(e) |
| 544651 | $(-4,1,-3),(4,-2,3),(0,1,0),(1,-2,3),(-1,1,-3)$ | A.2(f) |
| 544696 | $(5,-4,-15),(1,0,0),(0,1,0),(-4,2,9),(-3,1,6)$ | A.2(g) |
| 544700 | $(-2,-3,-3),(0,1,0),(1,0,0),(-1,-4,-6),(2,5,9)$ | A.2(h) |
| 544749 | $(-6,-5,-8),(0,1,0),(1,0,0),(-2,-1,0),(3,2,4)$ | A.2(i) |
| 520925 | $(0,1,0),(0,0,1),(-2,-1,0),(-2,0,-1),(8,2,3),(-2,-3,-2)$ | A.2(j) |
| 520935 | $(3,4,6),(2,1,2),(-3,-2,-2),(1,0,0),(0,1,0),(-6,-5,-8)$ | A.2(k) |
| 522056 | $(-1,-1,0),(0,1,0),(1,0,0),(-1,-1,-3),(-5,-3,-6),(6,4,9)$ | A.2(l) |
| 522059 | $(2,5,6),(-2,-3,-3),(0,1,0),(1,0,0),(-1,-4,-6),(0,1,3)$ | A.2(m) |
| 522087 | $(1,0,-3),(1,0,0),(0,1,0),(-4,2,9),(-3,1,6),(5,-4,-12)$ | A.2(n) |
| 522682 | $(2,1,4),(-3,-2,-4),(-2,-3,-4),(1,2,4),(1,0,0),(0,1,0)$ | A.2(o) |
| 522684 | $(-2,-1,-4),(3,2,4),(-2,-1,0),(1,0,0),(0,1,0),(-4,-3,-4)$ | A.2(p) |
| 526886 | $(-3,4,-6),(1,0,0),(0,1,0),(3,-6,8),(0,1,-2),(2,-5,6)$ | A.2(q) |
| 439403 | $(1,2,2),(-1,0,0),(-1,1,-1),(1,0,0),(-1,-2,-2),(1,1,3),(1,-3,-1)$ | A.2(r) |
| 275525 | $(4,1,2),(0,1,0),(-2,-1,0),(1,1,2),(-3,-1,-2),(-2,-1,-2),(1,1,0),$ $(1,-1,0)$ | A.2(s) |
| 275528 | $(-1,0,-1),(-3,-2,1),(-2,-1,2),(0,-1,0),(0,1,0),(1,0,1),(2,1,-2),$ $(3,2,-1)$ | A.2(t) |

**Table A.5: 20 Canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with $\mathbf{dim}(\boldsymbol{\Delta}^{\mathbf{FI}}) = \mathbf{1}$ and $\mathbf{0} \in (\boldsymbol{\Delta}^{\mathbf{FI}})^{\circ}$.** Table contains: ID from Graded Ring Database corresponding to $\Delta$, vertices $\mathrm{vert}(\Delta^{\mathrm{FI}})$ of $\Delta^{\mathrm{FI}}$, unique primitive lattice point $v_\Delta$, and link to pictures of $\Delta$ and $\Delta^{\mathrm{FI}}$.

| ID | $\mathrm{vert}(\Delta^{\mathrm{FI}})$ | $v_\Delta$ | $\Delta$ & $\Delta^{\mathrm{FI}}$ |
|---|---|---|---|
| 547393 | $(0,0,-1/2),(0,0,1/2)$ | $(0,0,1)$ | A.2(a) |
| 547409 | $(-2/3,2/3,2),(2/3,-2/3,-2)$ | $(1,-1,-3)$ | A.2(b) |
| 547461 | $(0,0,1/2),(0,0,-1/2)$ | $(0,0,1)$ | A.2(c) |
| 544442 | $(1/2,-1/2,1),(-1/2,1/2,-1)$ | $(1,-1,2)$ | A.2(d) |
| 544443 | $(1/2,-1/2,1),(-1/2,1/2,-1)$ | $(1,-1,2)$ | A.2(e) |
| 544651 | $(2/3,0,0),(-2/3,0,0)$ | $(1,0,0)$ | A.2(f) |
| 544696 | $(-2/3,2/3,2),(2/3,-2/3,-2)$ | $(1,-1,-3)$ | A.2(g) |
| 544700 | $(2/3,4/3,2),(-2/3,-4/3,-2)$ | $(1,2,3)$ | A.2(h) |
| 544749 | $(-1/2,-1/2,-1),(1/2,1/2,1)$ | $(1,1,2)$ | A.2(i) |
| 520925 | $(1,1/2,1/2),(-1,-1/2,-1/2)$ | $(2,1,1)$ | A.2(j) |
| 520935 | $(-1/2,-1/2,-1),(1/2,1/2,1)$ | $(1,1,2)$ | A.2(k) |
| 522056 | $(4/3,2/3,2),(-4/3,-2/3,-2)$ | $(2,1,3)$ | A.2(l) |
| 522059 | $(2/3,4/3,2),(-2/3,-4/3,-2)$ | $(1,2,3)$ | A.2(m) |
| 522087 | $(-2/3,2/3,2),(2/3,-2/3,-2)$ | $(1,-1,-3)$ | A.2(n) |
| 522682 | $(-1/2,-1/2,-1),(1/2,1/2,1)$ | $(1,1,2)$ | A.2(o) |
| 522684 | $(-1/2,-1/2,-1),(1/2,1/2,1)$ | $(1,1,2)$ | A.2(p) |
| 526886 | $(1/2,-1/2,1),(-1/2,1/2,-1)$ | $(1,-1,2)$ | A.2(q) |
| 439403 | $(0,-1/2,-1/2),(0,1/2,1/2)$ | $(0,1,1)$ | A.2(r) |
| 275525 | $(1/2,0,0),(-1/2,0,0)$ | $(1,0,0)$ | A.2(s) |
| 275528 | $(-1/2,-1/2,1/2),(1/2,1/2,-1/2)$ | $(1,1,-1)$ | A.2(t) |

**Table A.6: 20 Canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with $\mathbf{dim}(\boldsymbol{\Delta}^{\mathbf{FI}}) = \mathbf{1}$ and $\mathbf{0} \in (\boldsymbol{\Delta}^{\mathbf{FI}})^{\circ}$.** Table contains: ID from Graded Ring Database corresponding to $\Delta$, unique primitive lattice point $v_{\Delta}$, vertices $\mathrm{vert}(\theta_{\mathrm{proj}})$, $\mathrm{vert}(\theta'_{\mathrm{proj}})$ of projected facets $\theta_{\mathrm{proj}}, \theta'_{\mathrm{proj}} \preceq \Delta$, vertices $\mathrm{vert}(\overline{\Delta})$ of reflexive projection result $\overline{\Delta}$, link to a picture of the projection result $\overline{\Delta}$, and link to pictures of $\Delta$ and $\Delta^{\mathrm{FI}}$.

| ID | $v_{\Delta}$ | $\mathrm{vert}(\theta_{\mathrm{proj}})$ & $\mathrm{vert}(\theta'_{\mathrm{proj}})$ | $\mathrm{vert}(\overline{\Delta})$ | $\overline{\Delta}$ | $\Delta$ & $\Delta^{\mathrm{FI}}$ |
|---|---|---|---|---|---|
| 547393 | $(0,0,1)$ | $(0,1,0),(2,1,1),(-2,-3,-5)$<br>$(0,1,0),(-2,-3,-5),(2,1,9)$ | $(1,0),(1,2),(-3,-2)$ | 7.1(b) | A.2(a) |
| 547409 | $(1,-1,-3)$ | $(-4,2,9),(1,0,0),(0,1,0)$<br>$(-4,2,9),(0,1,0),(7,-6,-18)$ | $(1,-2),(1,1),(-2,1)$ | 7.1(a) | A.2(b) |
| 547461 | $(0,0,1)$ | $(0,1,0),(2,1,1),(-2,-3,-5)$<br>$(2,1,1),(-2,-3,-5),(0,1,4)$ | $(1,0),(1,2),(-3,-2)$ | 7.1(b) | A.2(c) |
| 544442 | $(1,-1,2)$ | $(0,1,0),(1,-4,4),(-5,6,-12)$<br>$(1,0,0),(0,1,0),(3,-6,8)$ | $(-1,1),(1,1),(-1,-3)$ | 7.1(b) | A.2(d) |
| 544443 | $(1,-1,2)$ | $(-1,-2,0),(0,1,0),(-3,4,-8)$<br>$(3,-6,8),(0,1,0),(1,0,0)$ | $(-1,-3),(1,1),(-1,1)$ | 7.1(b) | A.2(e) |
| 544651 | $(1,0,0)$ | $(-4,1,-3),(0,1,0),(1,-2,3)$<br>$(4,-2,3),(0,1,0),(-1,1,-3)$ | $(1,-3),(-2,3),(1,0)$ | 7.1(a) | A.2(f) |
| 544696 | $(1,-1,-3)$ | $(1,0,0),(0,1,0),(-4,2,9)$<br>$(5,-4,-15),(1,0,0),(-3,1,6)$ | $(-2,1),(1,1),(1,-2)$ | 7.1(a) | A.2(g) |
| 544700 | $(1,2,3)$ | $(-2,-3,-3),(0,1,0),(-1,-4,-6)$<br>$(0,1,0),(1,0,0),(2,5,9)$ | $(1,2),(1,-1),(-2,-1)$ | 7.1(a) | A.2(h) |
| 544749 | $(1,1,2)$ | $(-6,-5,-8),(0,1,0),(1,0,0)$<br>$(0,1,0),(-2,-1,0),(3,2,4)$ | $(3,1),(-1,1),(-1,-1)$ | 7.1(b) | A.2(i) |
| 520925 | $(2,1,1)$ | $(-2,-1,0),(-2,0,-1),(-2,-3,-2)$<br>$(0,1,0),(0,0,1),(8,2,3)$ | $(1,-1),(1,1),(-3,1)$ | 7.1(b) | A.2(j) |

**Table A.6: 20 Canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with $\mathbf{dim}(\boldsymbol{\Delta}^{\mathbf{FI}}) = \mathbf{1}$ and $\mathbf{0} \in (\boldsymbol{\Delta}^{\mathbf{FI}})^{\circ}$.** Table contains: ID from Graded Ring Database corresponding to $\Delta$, unique primitive lattice point $v_{\Delta}$, vertices $\mathrm{vert}(\theta_{\mathrm{proj}})$, $\mathrm{vert}(\theta'_{\mathrm{proj}})$ of projected facets $\theta_{\mathrm{proj}}, \theta'_{\mathrm{proj}} \preceq \Delta$, vertices $\mathrm{vert}(\overline{\Delta})$ of reflexive projection result $\overline{\Delta}$, link to a picture of the projection result $\overline{\Delta}$, and link to pictures of $\Delta$ and $\Delta^{\mathrm{FI}}$.

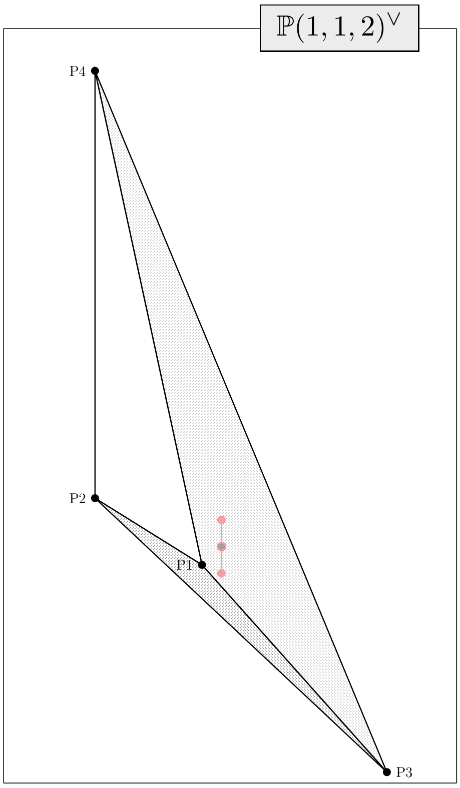
| ID | $v_{\Delta}$ | $\mathrm{vert}(\theta_{\mathrm{proj}})$ & $\mathrm{vert}(\theta'_{\mathrm{proj}})$ | $\mathrm{vert}(\overline{\Delta})$ | $\overline{\Delta}$ | $\Delta$ & $\Delta^{\mathrm{FI}}$ |
|---|---|---|---|---|---|
| 520935 | $(1,1,2)$ | $(1,0,0),(0,1,0),(-6,-5,-8)$<br>$(3,4,6),(2,1,2),(-3,-2,-2)$ | $(-1,1),(-1,-1),(3,1)$ | 7.1(b) | A.2(k) |
| 522056 | $(2,1,3)$ | $(0,1,0),(-1,-1,-3),(-5,-3,-6)$<br>$(-1,-1,0),(1,0,0),(6,4,9)$ | $(-1,2),(2,-1),(-1,-1)$ | 7.1(a) | A.2(l) |
| 522059 | $(1,2,3)$ | $(-2,-3,-3),(0,1,0),(-1,-4,-6)$<br>$(2,5,6),(1,0,0),(0,1,3)$ | $(1,-1),(1,2),(-2,-1)$ | 7.1(a) | A.2(m) |
| 522087 | $(1,-1,-3)$ | $(1,0,0),(0,1,0),(-4,2,9)$<br>$(1,0,-3),(-3,1,6),(5,-4,-12)$ | $(-2,1),(1,1),(1,-2)$ | 7.1(a) | A.2(n) |
| 522682 | $(1,1,2)$ | $(-3,-2,-4),(-2,-3,-4),(1,0,0),(0,1,0)$<br>$(2,1,4),(1,2,4),(1,0,0),(0,1,0)$ | $(1,-1),(1,1),(-1,-1),(-1,1)$ | 7.1(c) | A.2(o) |
| 522684 | $(1,1,2)$ | $(-2,-1,-4),(1,0,0),(-4,-3,-4)$<br>$(3,2,4),(-2,-1,0),(0,1,0)$ | $(-1,1),(-1,-1),(3,1)$ | 7.1(b) | A.2(p) |
| 526886 | $(1,-1,2)$ | $(-3,4,-6),(0,1,-2),(2,-5,6)$<br>$(1,0,0),(0,1,0),(3,-6,8)$ | $(1,1),(-1,1),(-1,-3)$ | 7.1(b) | A.2(q) |
| 439403 | $(0,1,1)$ | $(-1,1,-1),(1,0,0),(-1,-2,-2),(1,-3,-1)$<br>$(1,2,2),(-1,0,0),(-1,1,-1),(1,1,3)$ | $(0,1),(0,-1),(2,-1),(-2,1)$ | 7.1(c) | A.2(r) |
| 275525 | $(1,0,0)$ | $(0,1,0),(-2,-1,0),(1,1,2),(-3,-1,-2)$<br>$(4,1,2),(-2,-1,-2),(1,1,0),(1,-1,0)$ | $(1,2),(1,0),(-1,0),(-1,-2)$ | 7.1(c) | A.2(t) |
| 275528 | $(1,1,-1)$ | $(-3,-2,1),(-2,-1,2),(0,-1,0),(1,0,1)$<br>$(-1,0,-1),(0,1,0),(2,1,-2),(3,2,-1)$ | $(-2,1),(0,1),(0,-1),(2,-1)$ | 7.1(c) | A.2(s) |

**Table A.7: 20 Canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with $\mathbf{dim}(\boldsymbol{\Delta}^{\mathbf{FI}}) = \mathbf{1}$ and $\mathbf{0} \in (\boldsymbol{\Delta}^{\mathbf{FI}})^{\circ}$.** Table contains: ID from Graded Ring Database corresponding to $\Delta$, $(\langle v_\Delta, v_1\rangle, \ldots, \langle v_\Delta, v_f\rangle)$, elliptic fibration $m$, number of elements $|\mathrm{supp}(\Delta^{\mathrm{FI}})|$ in support of $\Delta^{\mathrm{FI}}$, vertices $\mathrm{vert}(\Delta^{\mathrm{can}})$ of canonical hull $\Delta^{\mathrm{can}}$ of $\Delta$, and stringy Euler number $e_{\mathrm{str}}(Y)$ of a canonical model $Y$ corresponding to $\Delta$.

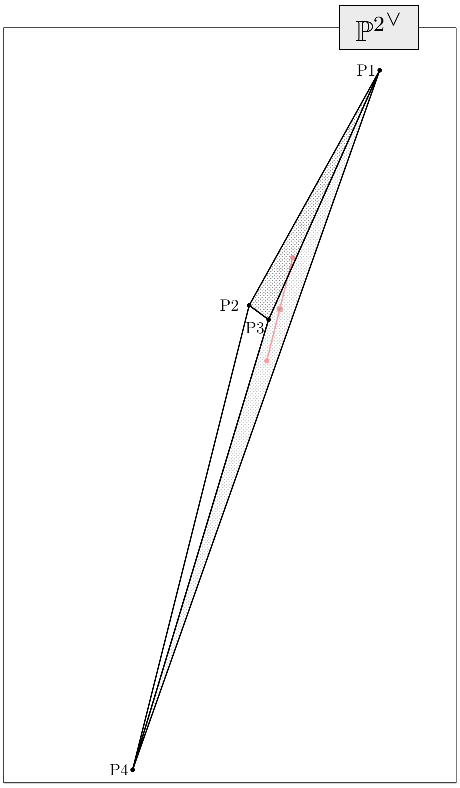
| | $(\langle v_\Delta, v_1\rangle, \ldots, \langle v_\Delta, v_f\rangle)$ | $m$ | $\lvert\mathrm{supp}(\Delta^{\mathrm{FI}})\rvert$ | $\mathrm{vert}(\Delta^{\mathrm{can}})$ | $e_{\mathrm{str}}(Y)$ |
|---|---|---|---|---|---|
| 547393 | $(2,0,0,-2)$ | $\frac{z_3^2}{z_0^2}$ | 6 | $\mathrm{vert}(\Delta)$ | 24 |
| 547409 | $(3,0,0,-3)$ | $\frac{z_3^3}{z_0^3}$ | 5 | $\mathrm{vert}(\Delta)$ | 24 |
| 547461 | $(-2,2,0,0)$ | $\frac{z_0^2}{z_1^2}$ | 6 | $\mathrm{vert}(\Delta)$ | 24 |
| 544442 | $(-2,0,0,0,2)$ | $\frac{z_0^2}{z_4^2}$ | 6 | $\mathrm{vert}(\Delta)$ | 24 |
| 544443 | $(-2,0,0,0,2)$ | $\frac{z_0^2}{z_4^2}$ | 6 | $\mathrm{vert}(\Delta)$ | 24 |
| 544651 | $(-3,0,0,0,3)$ | $\frac{z_0^3}{z_4^3}$ | 5 | $\mathrm{vert}(\Delta)$ | 24 |
| 544696 | $(3,-3,0,0,0)$ | $\frac{z_1^3}{z_0^3}$ | 5 | $\mathrm{vert}(\Delta)$ | 24 |
| 544700 | $(-3,0,0,0,3)$ | $\frac{z_0^3}{z_4^3}$ | 5 | $\mathrm{vert}(\Delta)$ | 24 |
| 544749 | $(2,0,0,0,-2)$ | $\frac{z_4^2}{z_0^2}$ | 6 | $\mathrm{vert}(\Delta)$ | 24 |
| 520925 | $(0,0,-2,0,2)$ | $\frac{z_2^2}{z_4^2}$ | 6 | $\mathrm{vert}(\Delta)$ | 24 |

**Table A.7: 20 Canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with $\mathbf{dim}(\boldsymbol{\Delta}^{\mathbf{FI}}) = \mathbf{1}$ and $\mathbf{0} \in (\boldsymbol{\Delta}^{\mathbf{FI}})^{\circ}$.** Table contains: ID from Graded Ring Database corresponding to $\Delta$, $(\langle v_\Delta, v_1\rangle, \dots, \langle v_\Delta, v_f\rangle)$, elliptic fibration $m$, number of elements $|\mathrm{supp}(\Delta^{\mathrm{FI}})|$ in support of $\Delta^{\mathrm{FI}}$, vertices $\mathrm{vert}(\Delta^{\mathrm{can}})$ of canonical hull $\Delta^{\mathrm{can}}$ of $\Delta$, and stringy Euler number $e_{\mathrm{str}}(Y)$ of a canonical model $Y$ corresponding to $\Delta$.

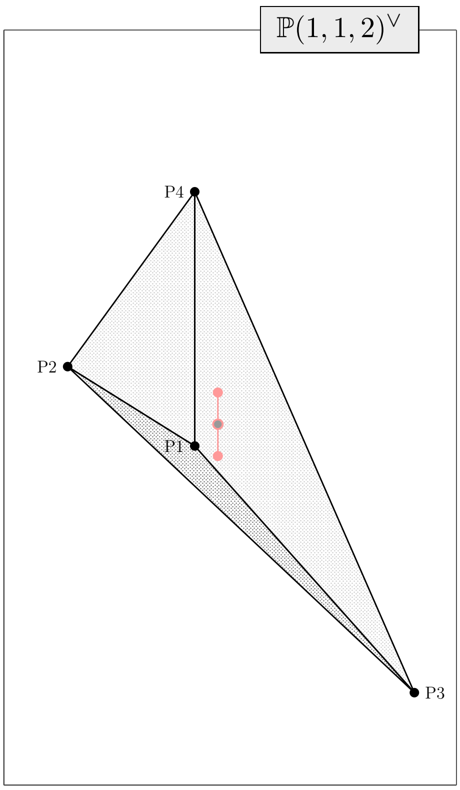
| ID | $(\langle v_\Delta, v_1\rangle, \dots, \langle v_\Delta, v_f\rangle)$ | $m$ | $\lvert\mathrm{supp}(\Delta^{\mathrm{FI}})\rvert$ | $\mathrm{vert}(\Delta^{\mathrm{can}})$ | $e_{\mathrm{str}}(Y)$ |
|---|---|---|---|---|---|
| 520935 | $(2, 0, 0, -2, 0)$ | $\frac{z_3^2}{z_0^2}$ | 6 | $\mathrm{vert}(\Delta)$ | 24 |
| 522056 | $(-3, 0, 0, 3, 0)$ | $\frac{z_0^3}{z_3^3}$ | 5 | $\mathrm{vert}(\Delta)$ | 24 |
| 522059 | $(-3, 0, 0, 0, 3)$ | $\frac{z_0^3}{z_4^3}$ | 5 | $\mathrm{vert}(\Delta)$ | 24 |
| 522087 | $(3, 0, 0, 0, -3)$ | $\frac{z_4^3}{z_0^3}$ | 5 | $\mathrm{vert}(\Delta)$ | 24 |
| 522682 | $(-2, 2, 0, 0, 0)$ | $\frac{z_0^2}{z_1^2}$ | 6 | $\mathrm{vert}(\Delta)$ | 24 |
| 522684 | $(2, 0, 0, 0, -2)$ | $\frac{z_4^2}{z_0^2}$ | 6 | $\mathrm{vert}(\Delta)$ | 24 |
| 526886 | $(-2, 0, 0, 2, 0)$ | $\frac{z_0^2}{z_3^2}$ | 6 | $\mathrm{vert}(\Delta)$ | 24 |
| 439403 | $(2, 0, 0, 0, 0, -2)$ | $\frac{z_5^2}{z_0^2}$ | 6 | $\mathrm{vert}(\Delta)$ | 24 |
| 275525 | $(-2, 0, 0, 0, 0, 2)$ | $\frac{z_0^2}{z_5^2}$ | 6 | $\mathrm{vert}(\Delta)$ | 24 |
| 275528 | $(0, 2, 0, 0, -2, 0)$ | $\frac{z_4^2}{z_1^2}$ | 6 | $\mathrm{vert}(\Delta)$ | 24 |



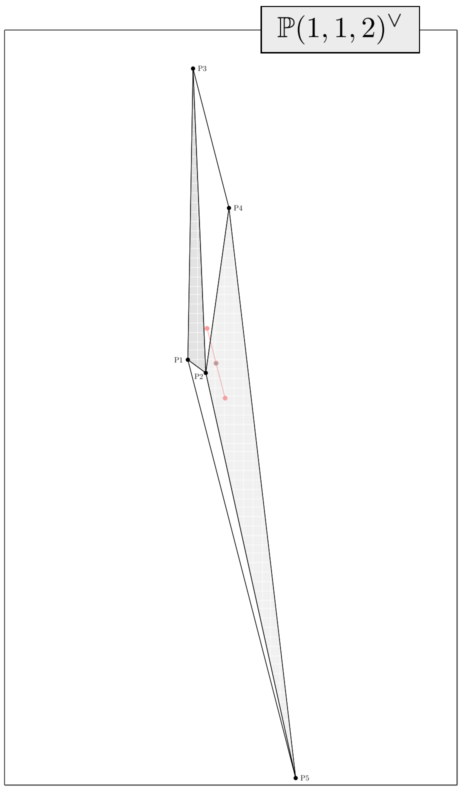

**(a)** ID 547393




**(b)** ID 547409

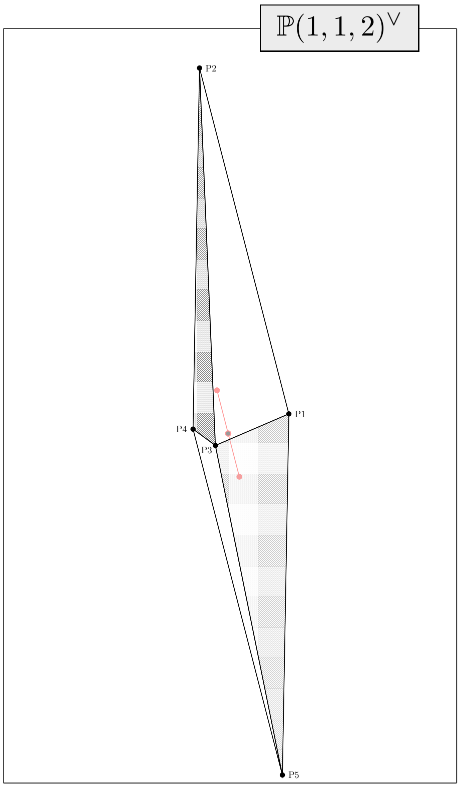



**(c)** ID 547461

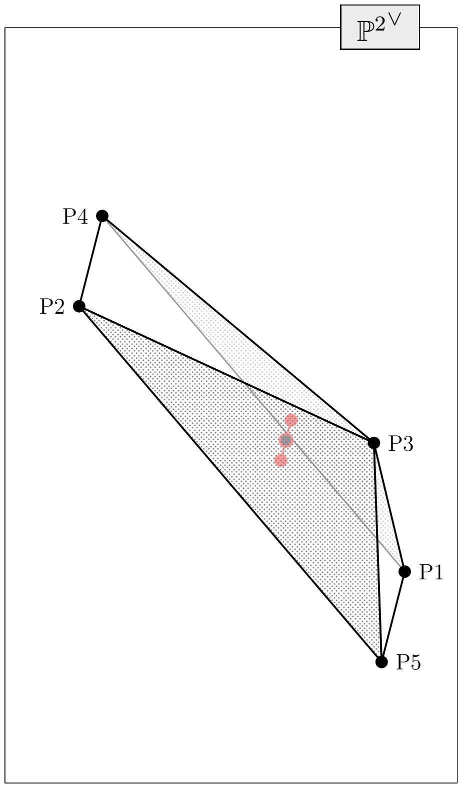



**(d)** ID 544442

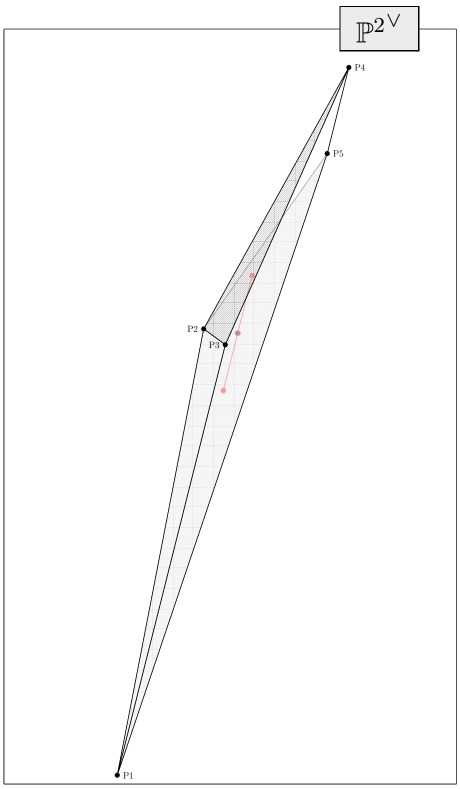
**Figure A.2: 20 Canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with $\dim(\boldsymbol{\Delta}^{\mathrm{FI}}) = 1$ and $\mathbf{0} \in (\boldsymbol{\Delta}^{\mathrm{FI}})^{\circ}$.**



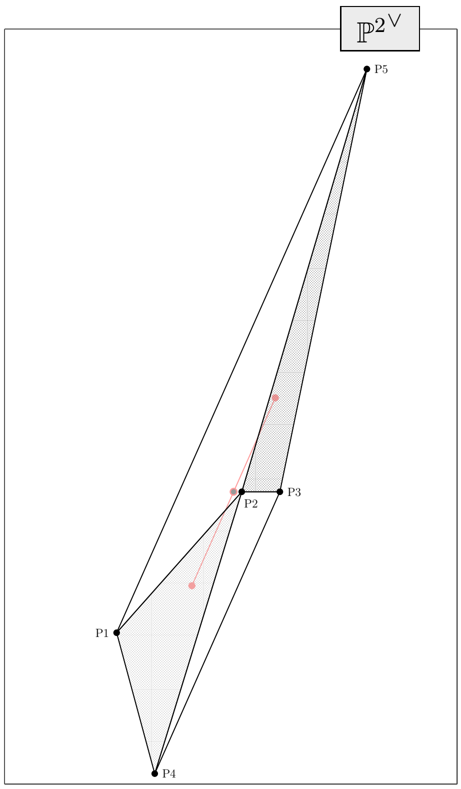

**(e)** ID 544443




**(f)** ID 544651

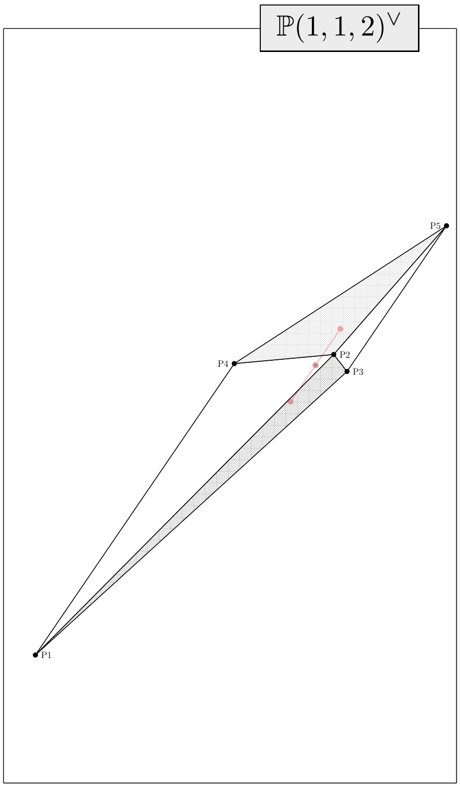



**(g)** ID 544696

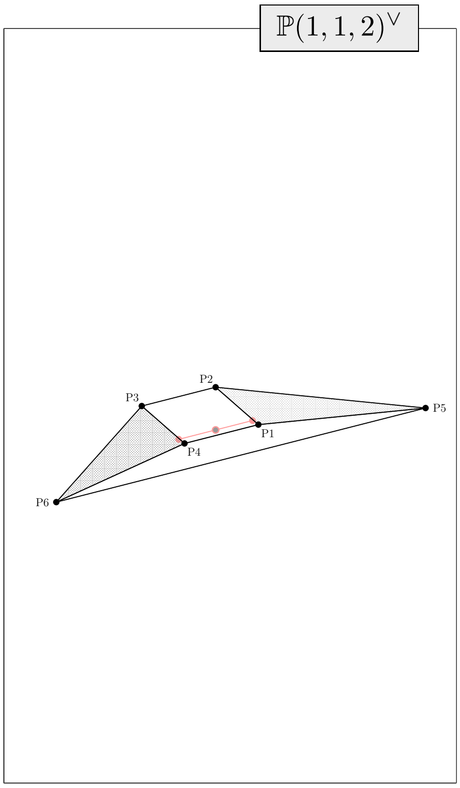



**(h)** ID 544700

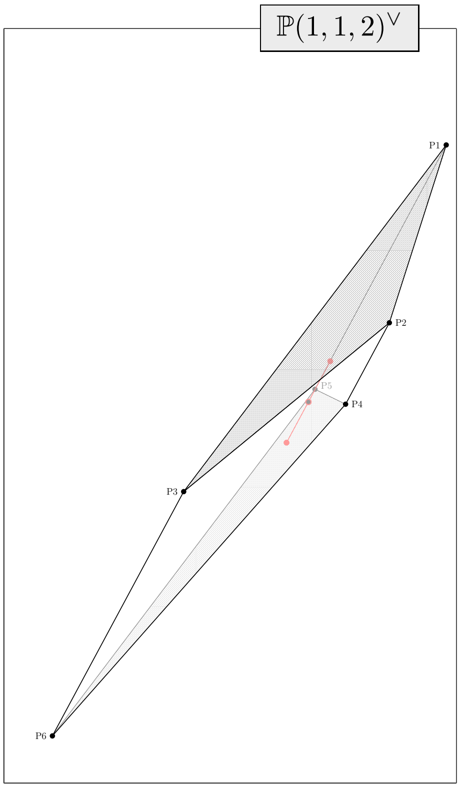
**Figure A.2: 20 Canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with $\dim(\boldsymbol{\Delta}^{\mathbf{FI}}) = 1$ and $\mathbf{0} \in (\boldsymbol{\Delta}^{\mathbf{FI}})^\circ$.**



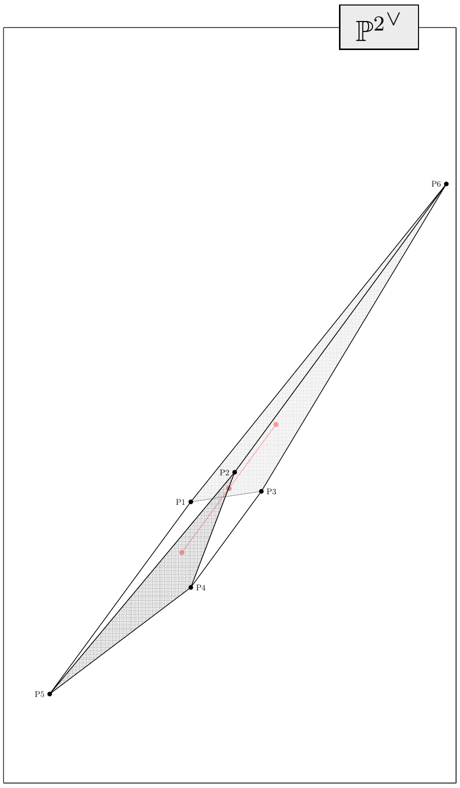

**(i)** ID 544749




**(j)** ID 520925

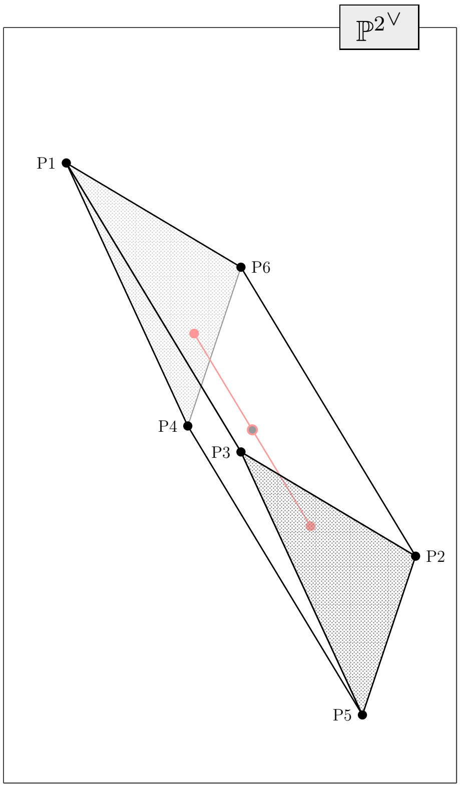



**(k)** ID 520935

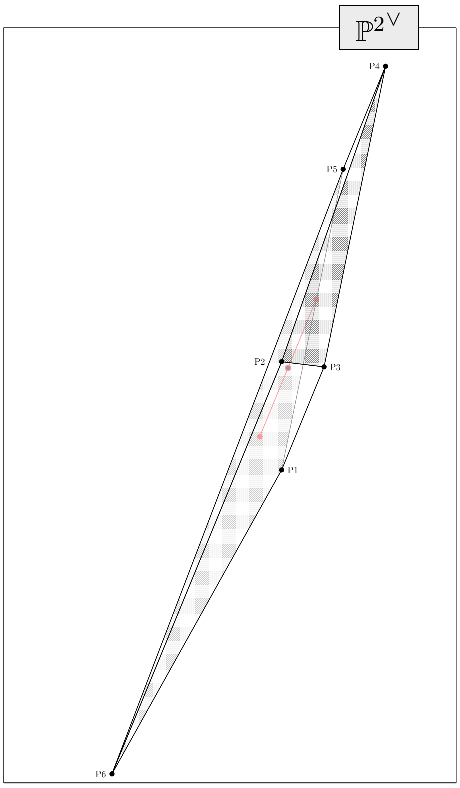



**(l)** ID 522056

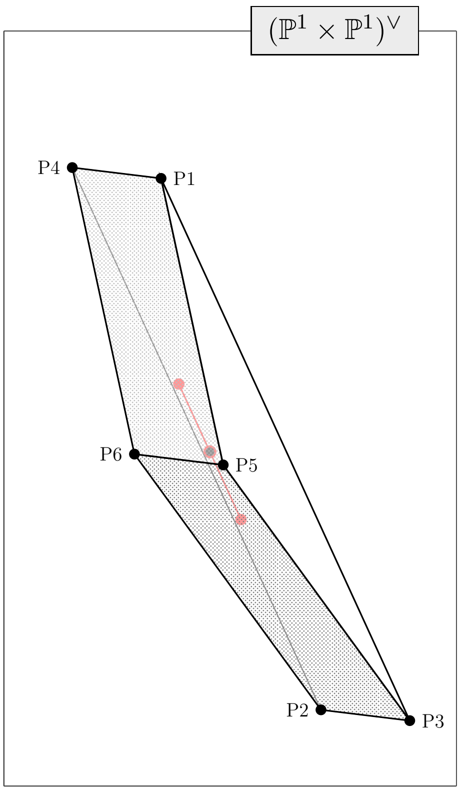
**Figure A.2: 20 Canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with $\mathbf{dim}(\boldsymbol{\Delta}^{\mathbf{FI}}) = \mathbf{1}$ and $\mathbf{0} \in (\boldsymbol{\Delta}^{\mathbf{FI}})^{\circ}$.**



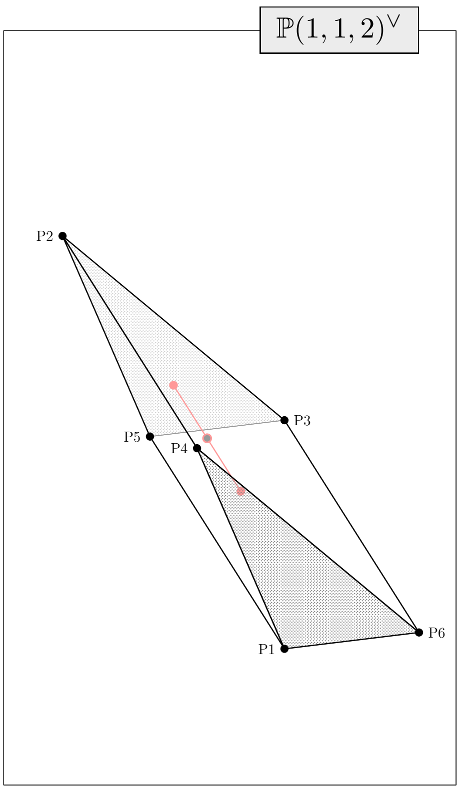

**(m)** ID 522059




**(n)** ID 522087

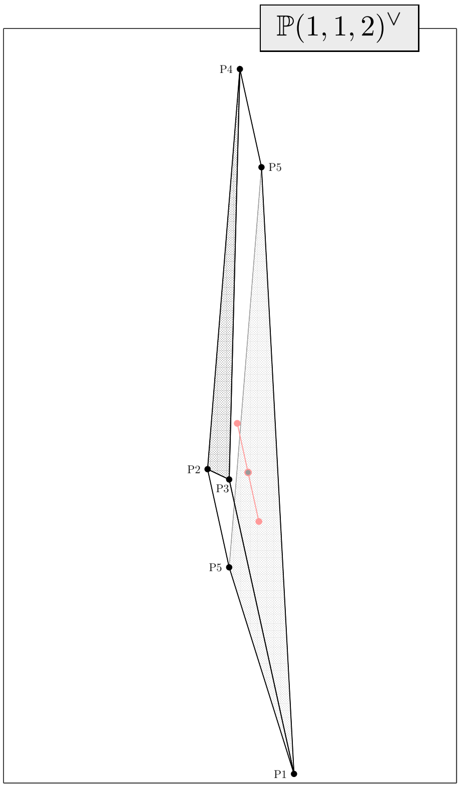



**(o)** ID 522682

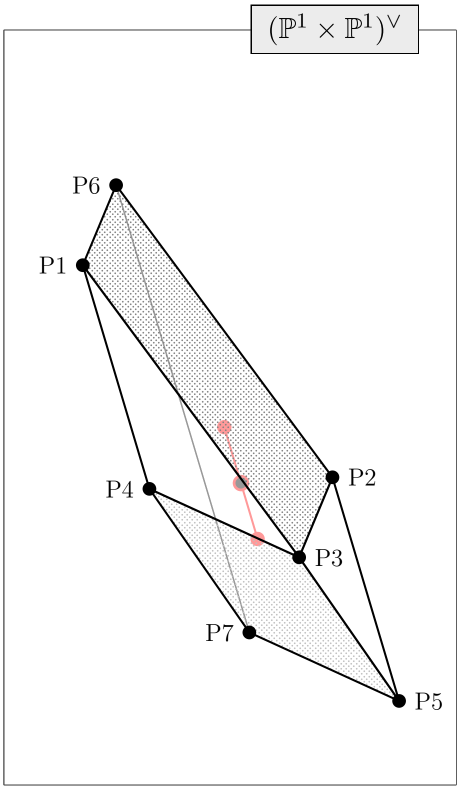



**(p)** ID 522684

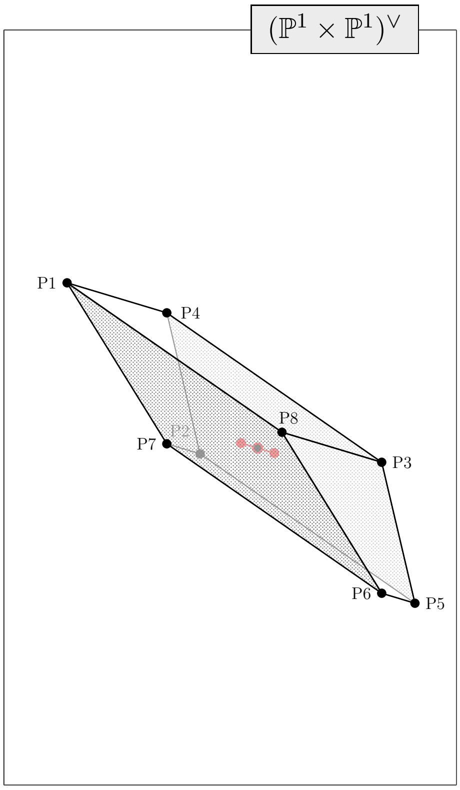
**Figure A.2: 20 Canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with $\dim(\boldsymbol{\Delta}^{\mathrm{FI}}) = 1$ and $\mathbf{0} \in (\boldsymbol{\Delta}^{\mathrm{FI}})^\circ$.**



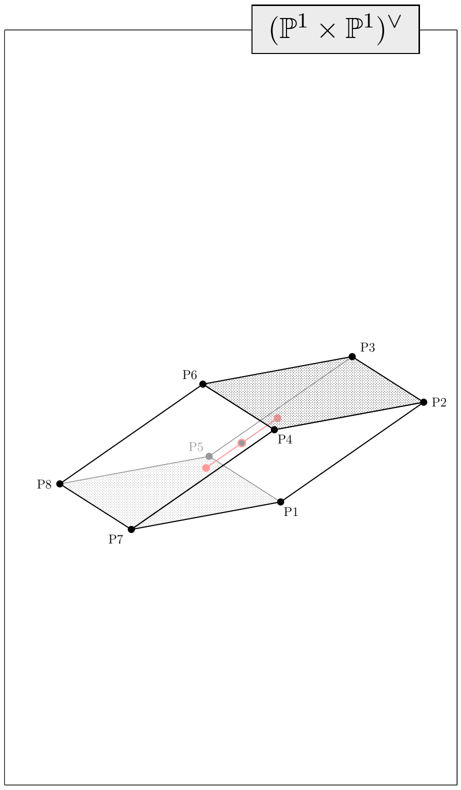

**(q)** ID 526886




**(r)** ID 439403




**(s)** ID 275525




**(t)** ID 275528

**Figure A.2: 20 Canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with $\dim(\boldsymbol{\Delta}^{\mathbf{FI}}) = 1$ and $0 \in (\boldsymbol{\Delta}^{\mathbf{FI}})^{\circ}$.**

**Remark A.2.1.** In Figure A.2 shaded faces are occluded. The Fine interior is coloured red, the origin grey with a red margin, and the facets that are used to project grey dotted. The surface corresponding to the reflexive polygon (projection result) is boxed top right. The vertices are numbered consecutively as listed in the Graded Ring Database.

## A.3 Canonical Fano 3-polytopes Δ with $\dim(\Delta^{\mathbf{FI}}) = 3$ and $0 \in \mathrm{vert}(\Delta^{\mathbf{FI}})$

**Table A.8: 49 Canonical Fano 3-polytopes Δ with $\dim(\Delta^{\mathbf{FI}}) = 3$.** Table contains: ID from Graded Ring Database corresponding to Δ, vertices vert(Δ) of Δ, and link to a picture of Δ.

| ID | vert(Δ) | Δ |
|---|---|---|
| 547444 | $(1,0,0),(-2,-4,-5),(1,2,4),(1,4,2)$ | A.3(a) |
| 547465 | $(-3,-2,-2),(1,0,0),(1,3,1),(1,1,3)$ | A.3(b) |
| 547524 | $(0,2,1),(-2,-3,-5),(2,1,1),(0,0,1)$ | A.3(c) |
| 547525 | $(0,0,1),(0,1,0),(2,1,1),(-2,-5,-7)$ | A.3(d) |
| 545317 | $(-3,4,-6),(0,1,0),(1,0,0),(1,-2,4),(3,-5,6)$ | A.3(e) |
| 545932 | $(0,-1,-1),(1,-1,-3),(-2,1,5),(1,0,0),(1,2,-2)$ | A.3(f) |
| 546013 | $(3,-5,6),(1,-2,4),(1,0,0),(-1,1,-2),(-1,3,-2)$ | A.3(g) |
| 546062 | $(0,1,3),(-2,1,-1),(0,1,0),(1,0,0),(-1,-2,-2)$ | A.3(h) |
| 546070 | $(0,-2,-3),(0,2,1),(-2,-3,-5),(2,1,1),(0,0,1)$ | A.3(i) |
| 546205 | $(1,2,-2),(-1,0,2),(1,0,0),(-2,1,5),(1,-1,-3)$ | A.3(j) |
| 546219 | $(1,1,1),(-3,-2,-2),(1,0,0),(1,3,1),(-1,-1,1)$ | A.3(k) |
| 546663 | $(2,-3,-1),(1,0,0),(0,1,0),(0,0,1),(-2,-3,-3)$ | A.3(l) |
| 546862 | $(1,0,0),(0,1,0),(-2,1,5),(1,-1,-3),(1,2,-2)$ | A.3(m) |
| 546863 | $(-1,-1,1),(1,3,1),(0,0,1),(1,0,0),(-3,-2,-2)$ | A.3(n) |
| 547240 | $(-1,1,-2),(0,1,0),(1,0,0),(1,-2,4),(3,-5,6)$ | A.3(o) |
| 547246 | $(0,-2,-3),(-2,-3,-5),(2,1,1),(0,1,0),(0,0,1)$ | A.3(p) |
| 532384 | $(1,-1,-3),(-2,1,5),(1,0,0),(1,-1,-2),(0,-1,-1),(1,2,-2)$ | A.3(q) |
| 532606 | $(0,-1,2),(-1,-1,0),(0,1,0),(1,0,0),(2,2,-3),(-2,0,-3)$ | A.3(r) |
| 533513 | $(-1,1,2),(1,0,0),(0,1,0),(1,1,2),(-1,-2,-4),(-2,-3,-4)$ | A.3(s) |
| 534667 | $(1,0,3),(-1,-1,-1),(0,1,0),(1,0,0),(-1,-1,0),(5,2,3)$ | A.3(t) |
| 534669 | $(1,3,0),(5,3,2),(-1,-1,-1),(0,0,1),(1,0,0),(-1,-1,0)$ | A.3(u) |
| 534866 | $(-1,-1,-3),(1,0,0),(0,1,0),(1,1,1),(-1,-1,0),(-3,-5,-3)$ | A.3(v) |
| 535952 | $(3,-5,6),(1,-2,4),(1,0,0),(0,1,0),(-1,1,-2),(-1,2,-2)$ | A.3(w) |
| 536013 | $(0,1,1),(0,0,1),(0,1,0),(2,1,1),(-2,-3,-5),(0,-2,-3)$ | A.3(x) |
| 536498 | $(1,2,-2),(1,-1,-2),(1,0,0),(0,1,0),(-2,1,5),(1,-1,-3)$ | A.3(y) |

**Table A.8: 49 Canonical Fano 3-polytopes $\Delta$ with $\dim(\Delta^{\mathrm{FI}}) = 3$.** Table contains: ID from Graded Ring Database corresponding to $\Delta$, vertices $\mathrm{vert}(\Delta)$ of $\Delta$, and link to a picture of $\Delta$.

| ID | vert($\Delta$) | $\Delta$ |
|---|---|---|
| 537834 | $(0,0,1),(1,0,0),(0,1,0),(-2,1,5),(1,-1,-3),(1,2,-2)$ | A.3(z) |
| 538356 | $(-2,-3,-3),(-1,-3,-1),(0,0,1),(0,1,0),(1,0,0),(-1,-1,-3)$ | A.3(aa) |
| 539063 | $(-1,1,-1),(1,1,3),(-3,-2,-2),(1,0,0),(0,1,0),(1,1,2)$ | A.3(ab) |
| 539304 | $(1,0,1),(-3,-1,-2),(1,1,2),(-2,-1,0),(1,0,0),(1,2,0)$ | A.3(ac) |
| 539313 | $(1,-1,-2),(1,1,-1),(-1,2,2),(1,-1,-3),(-2,1,5),(1,0,0)$ | A.3(ad) |
| 540602 | $(0,0,1),(1,0,0),(-2,1,5),(1,-1,-3),(-1,2,2),(1,1,-1)$ | A.3(ae) |
| 540663 | $(1,0,0),(0,1,0),(1,1,2),(-3,-1,-2),(1,1,1),(-3,-2,0)$ | A.3(af) |
| 474457 | $(-1,2,-3),(1,0,2),(0,0,1),(0,1,0),(1,0,0),(-1,-1,0),(-3,-2,-3)$ | A.3(ag) |
| 481575 | $(3,2,4),(-1,-1,-2),(-3,-1,-2),(-2,-1,0),(0,1,0),(1,0,0),(0,0,-1)$ | A.3(ah) |
| 483109 | $(3,0,2),(1,-2,-2),(0,0,-1),(-1,-1,0),(1,1,1),(0,1,0),(-1,0,0)$ | A.3(ai) |
| 490478 | $(1,-1,-2),(1,1,-1),(-1,2,2),(1,-1,-3),(-2,1,5),(1,0,0),(-1,0,2)$ | A.3(aj) |
| 490481 | $(-3,-2,0),(-5,-3,-2),(1,0,0),(0,1,0),(1,1,2),(-1,-1,-1),(2,1,1)$ | A.3(ak) |
| 490485 | $(-1,-1,0),(1,2,0),(1,0,0),(-2,-1,0),(1,1,2),(-3,-1,-2),(1,0,1)$ | A.3(al) |
| 490511 | $(1,0,0),(0,1,0),(-2,-1,0),(1,1,2),(2,1,1),(1,0,1),(-5,-2,-4)$ | A.3(am) |
| 495687 | $(0,0,-1),(1,1,-1),(-1,2,2),(1,-1,-3),(-2,1,5),(1,0,0),(0,0,1)$ | A.3(an) |
| 499287 | $(1,1,1),(-1,-1,-3),(1,0,0),(0,1,0),(0,0,1),(-1,-3,-1),(-2,-3,-3)$ | A.3(ao) |
| 499291 | $(-1,-1,-1),(-1,-1,-3),(1,0,0),(0,1,0),(0,0,1),(-1,-3,-1),(-2,-3,-3)$ | A.3(ap) |
| 499470 | $(1,0,0),(0,1,0),(-2,-1,0),(1,1,2),(0,0,1),(-5,-2,-4),(2,1,1)$ | A.3(aq) |
| 501298 | $(3,-6,8),(-1,1,-2),(1,-2,3),(0,1,0),(1,0,0),(0,1,-1),(3,-5,6)$ | A.3(ar) |
| 501330 | $(1,0,0),(0,1,0),(-2,-1,0),(1,1,2),(1,1,1),(0,0,1),(-5,-2,-4)$ | A.3(as) |
| 354912 | $(3,1,2),(1,0,0),(0,1,0),(-2,-1,0),(1,1,2),(2,1,1),(1,0,1),(-5,-2,-4)$ | A.3(at) |
| 372528 | $(2,1,1),(-1,-1,-1),(1,1,2),(0,1,0),(1,0,0),(-5,-3,-2),(-3,-2,0),(1,1,0)$ | A.3(au) |
| 372973 | $(-5,-2,-4),(1,0,1),(2,1,1),(1,1,2),(-2,-1,0),(0,1,0),(1,0,0),(2,1,2)$ | A.3(av) |
| 388701 | $(1,1,1),(-2,-3,-3),(-1,-3,-1),(0,0,1),(0,1,0),(1,0,0),(-1,-1,-3),$ $(-1,-1,-1)$ | A.3(aw) |

**Table A.9: 49 Canonical Fano 3-polytopes $\Delta$ with $\dim(\Delta^{\mathrm{FI}}) = 3$.** Table contains: ID from Graded Ring Database corresponding to $\Delta$, vertices $\mathrm{vert}(\Delta^{\mathrm{FI}})$ of $\Delta^{\mathrm{FI}}$, and link to a picture of $\Delta$.

| ID | $\mathrm{vert}(\Delta^{\mathrm{FI}})$ | $\Delta$ |
|---|---|---|
| 547444 | $(0,0,0), (-1/2,-1,-3/2), (0,-1/3,-2/3), (0,1/3,-1/3)$ | A.3(a) |
| 547465 | $(0,0,0), (-1,-1/2,-1/2), (0,3/4,1/4), (0,1/4,3/4)$ | A.3(b) |
| 547524 | $(0,0,0), (0,1/2,0), (1/3,1/3,0), (-1/3,-1/3,-1)$ | A.3(c) |
| 547525 | $(0,0,0), (0,0,-1/2), (1/3,0,-1/3), (-1/3,-1,-5/3)$ | A.3(d) |
| 545317 | $(0,0,0), (1,-3/2,2), (2/3,-2/3,1), (1/2,-1/2,1), (2/3,-1,5/3)$ | A.3(e) |
| 545932 | $(0,0,0), (-1/2,1/2,3/2), (0,1/3,2/3), (0,2/3,1/3)$ | A.3(f) |
| 546013 | $(0,0,0), (1,-3/2,2), (0,1/2,0), (1/2,-1/4,1/2), (1/2,-3/4,3/2)$ | A.3(g) |
| 546062 | $(0,0,0), (-1/2,-1/2,-1/2), (-2/3,0,-1/3), (-1/3,0,1/3)$ | A.3(h) |
| 546070 | $(0,0,0), (0,1/2,0), (1/2,1/4,0), (0,-1/2,-1), (-1/2,-3/4,-3/2)$ | A.3(i) |
| 546205 | $(0,0,0), (-1/2,1/2,3/2), (0,1/3,2/3), (0,2/3,1/3)$ | A.3(j) |
| 546219 | $(0,0,0), (-1,-1/2,-1/2), (-1/3,1/3,0), (-2/3,-1/3,0)$ | A.3(k) |
| 546663 | $(0,0,0), (0,-1/2,0), (1/3,-1,-1/3), (-1/3,-1,-2/3)$ | A.3(l) |
| 546862 | $(0,0,0), (-1/2,1/2,3/2), (0,1/3,2/3), (0,2/3,1/3)$ | A.3(m) |
| 546863 | $(0,0,0), (-1,-1/2,-1/2), (-1/3,1/3,0), (-2/3,-1/3,0)$ | A.3(n) |
| 547240 | $(0,0,0), (1,-3/2,2), (2/3,-2/3,1), (1/2,-1/2,1), (2/3,-1,5/3)$ | A.3(o) |
| 547246 | $(0,0,0), (0,0,-1/2), (1/3,0,-1/3), (0,-1/2,-1), (-1/3,-2/3,-4/3)$ | A.3(p) |
| 532384 | $(0,0,0), (-1/2,1/2,3/2), (0,1/3,2/3), (0,2/3,1/3)$ | A.3(q) |
| 532606 | $(0,0,0), (0,1/2,-1/2), (1/3,2/3,-1), (-1/3,1/3,-1)$ | A.3(r) |
| 533513 | $(0,0,0), (-1/2,-1/2,-1), (-1/2,0,0), (-1/3,0,-1/3), (-2/3,-2/3,-1)$ | A.3(s) |
| 534667 | $(0,0,0), (1/2,1/2,1/2), (4/3,2/3,1), (2/3,1/3,1)$ | A.3(t) |
| 534669 | $(0,0,0), (1/2,1/2,1/2), (4/3,1,2/3), (2/3,1,1/3)$ | A.3(u) |
| 534866 | $(0,0,0), (0,-1/2,-1/2), (-1/3,-2/3,-1), (-2/3,-4/3,-1)$ | A.3(v) |
| 535952 | $(0,0,0), (1,-3/2,2), (2/3,-2/3,1), (1/2,-1/2,1), (2/3,-1,5/3)$ | A.3(w) |
| 536013 | $(0,0,0), (0,0,-1/2), (1/3,0,-1/3), (0,-1/2,-1), (-1/3,-2/3,-4/3)$ | A.3(x) |
| 536498 | $(0,0,0), (-1/2,1/2,3/2), (0,1/3,2/3), (0,2/3,1/3)$ | A.3(y) |

**Table A.9: 49 Canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with $\mathbf{dim(\Delta^{FI}) = 3}$.** Table contains: ID from Graded Ring Database corresponding to $\Delta$, vertices $\mathrm{vert}(\Delta^{\mathrm{FI}})$ of $\Delta^{\mathrm{FI}}$, and link to a picture of $\Delta$.

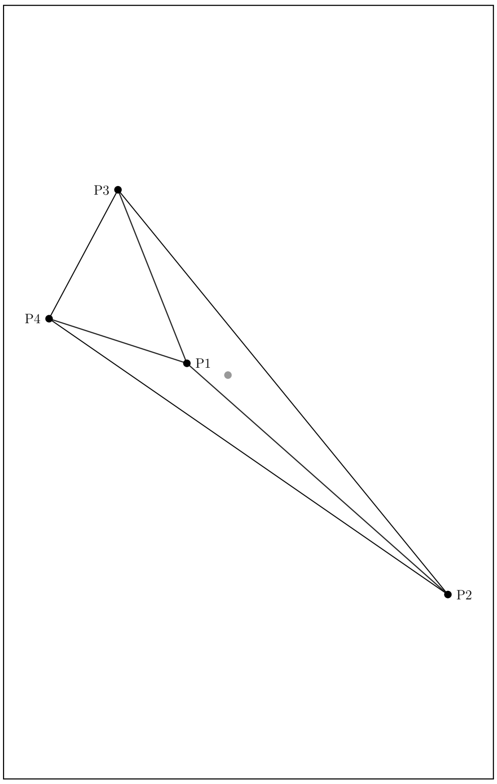
| ID | $\mathrm{vert}(\Delta^{\mathrm{FI}})$ | $\Delta$ |
|---|---|---|
| 537834 | $(0,0,0),(-1/2,1/2,3/2),(0,1/3,2/3),(0,2/3,1/3)$ | A.3(z) |
| 538356 | $(0,0,0),(0,-1/2,-1/2),(-1/3,-2/3,-1),(-1/3,-1,-2/3),(-1/2,-1,-1)$ | A.3(aa) |
| 539063 | $(0,0,0),(-1,-1/2,-1/2),(-2/3,0,-1/3),(-1/3,0,1/3)$ | A.3(ab) |
| 539304 | $(0,0,0),(0,1/2,0),(-1/2,0,0),(0,1/3,1/3),(-2/3,0,-1/3)$ | A.3(ac) |
| 539313 | $(0,0,0),(-1/2,1/2,3/2),(0,1/3,2/3),(0,1/2,1/2),(-1/3,2/3,1)$ | A.3(ad) |
| 540602 | $(0,0,0),(-1/2,1/2,3/2),(0,1/3,2/3),(0,1/2,1/2),(-1/3,2/3,1)$ | A.3(ae) |
| 540663 | $(0,0,0),(-1/2,0,0),(-1,-1/2,0),(-1/3,0,1/3),(-1,-1/3,-1/3)$ | A.3(af) |
| 474457 | $(0,0,0),(0,0,-1/2),(-1/3,1/3,-1),(-2/3,-1/3,-1)$ | A.3(ag) |
| 481575 | $(0,0,0),(-1/2,0,0),(1/2,1/2,1),(0,1/3,1/3),(-1/3,0,1/3)$ | A.3(ah) |
| 483109 | $(0,0,0),(0,-1/2,0),(2/3,-1/3,1/3),(1/3,-2/3,-1/3)$ | A.3(ai) |
| 490478 | $(0,0,0),(-1/2,1/2,3/2),(0,1/3,2/3),(0,1/2,1/2),(-1/3,2/3,1)$ | A.3(aj) |
| 490481 | $(0,0,0),(-1/2,0,0),(-1,-1/2,0),(-1/3,0,1/3),(-4/3,-2/3,-1/3)$ | A.3(ak) |
| 490485 | $(0,0,0),(0,1/2,0),(-1/2,0,0),(0,1/3,1/3),(-2/3,0,-1/3)$ | A.3(al) |
| 490511 | $(0,0,0),(-3/2,-1/2,-1),(-1/2,0,0),(-2/3,0,-1/3),(-1,-1/3,-1/3)$ | A.3(am) |
| 495687 | $(0,0,0),(-1/2,1/2,3/2),(0,1/3,2/3),(0,1/2,1/2),(-1/3,2/3,1)$ | A.3(an) |
| 499287 | $(0,0,0),(0,-1/2,-1/2),(-1/3,-2/3,-1),(-1/3,-1,-2/3),(-1/2,-1,-1)$ | A.3(ao) |
| 499291 | $(0,0,0),(0,-1/2,-1/2),(-1/3,-2/3,-1),(-1/3,-1,-2/3),(-1/2,-1,-1)$ | A.3(ap) |
| 499470 | $(0,0,0),(-3/2,-1/2,-1),(-1/2,0,0),(-2/3,0,-1/3),(-1,-1/3,-1/3)$ | A.3(aq) |
| 501298 | $(0,0,0),(1/2,-1/2,1),(2/3,-2/3,1),(1,-3/2,2),(1,-5/3,7/3)$ | A.3(ar) |
| 501330 | $(0,0,0),(-3/2,-1/2,-1),(-1/2,0,0),(-2/3,0,-1/3),(-1,-1/3,-1/3)$ | A.3(as) |
| 354912 | $(0,0,0),(-3/2,-1/2,-1),(-1/2,0,0),(-2/3,0,-1/3),(-1,-1/3,-1/3)$ | A.3(at) |
| 372528 | $(0,0,0),(-1/2,0,0),(-1,-1/2,0),(-1/3,0,1/3),(-4/3,-2/3,-1/3)$ | A.3(au) |
| 372973 | $(0,0,0),(-3/2,-1/2,-1),(-1/2,0,0),(-2/3,0,-1/3),(-1,-1/3,-1/3)$ | A.3(av) |
| 388701 | $(0,0,0),(0,-1/2,-1/2),(-1/3,-2/3,-1),(-1/3,-1,-2/3),(-1/2,-1,-1)$ | A.3(aw) |

**Table A.10: 49 canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with $\mathbf{dim}(\boldsymbol{\Delta}^{\mathbf{FI}}) = 3$.** Table contains: ID from Graded Ring Database corresponding to $\Delta$, number of elements $|\mathrm{supp}(\Delta^{\mathrm{FI}})|$ in support of $\Delta^{\mathrm{FI}}$, vertices $\mathrm{vert}(\Delta^{\mathrm{can}})$ of canonical hull $\Delta^{\mathrm{can}}$ of $\Delta$, and stringy Euler number $e_{\mathrm{str}}(Y)$ of a canonical model $Y$ corresponding to $\Delta$.

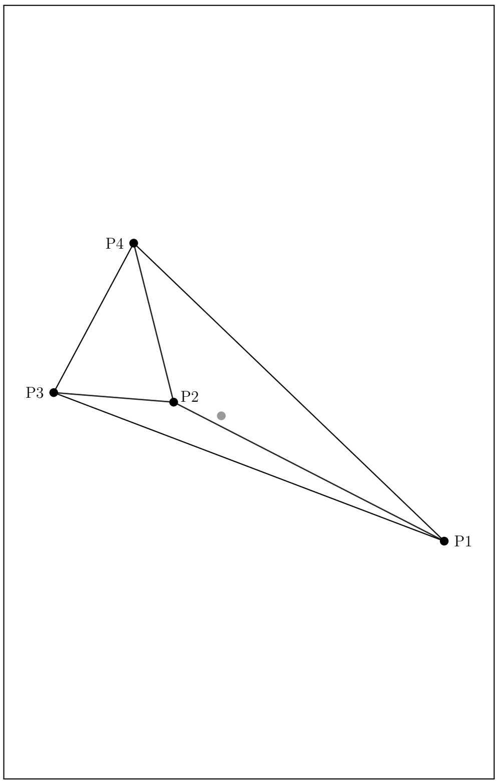
| ID | $\lvert\mathrm{supp}(\Delta^{\mathrm{FI}})\rvert$ | $\mathrm{vert}(\Delta^{\mathrm{can}})$ | $e_{\mathrm{str}}(Y)$ |
|---|---|---|---|
| 547444 | 6 | $\mathrm{vert}(\Delta)$ | 23 |
| 547465 | 9 | $\mathrm{vert}(\Delta)$ | 22 |
| 547524 | 10 | $\mathrm{vert}(\Delta), (0,-1,-1)$ | 23 |
| 547525 | 19 | $\mathrm{vert}(\Delta), (1,1,1), (-1,-2,-3)$ | 23 |
| 545317 | 6 | $\mathrm{vert}(\Delta)$ | 23 |
| 545932 | 8 | $\mathrm{vert}(\Delta), (1,-1,-2), (1,0,-3)$ | 23 |
| 546013 | 9 | $\mathrm{vert}(\Delta)$ | 22 |
| 546062 | 9 | $\mathrm{vert}(\Delta)$ | 23 |
| 546070 | 9 | $\mathrm{vert}(\Delta)$ | 22 |
| 546205 | 13 | $\mathrm{vert}(\Delta)$ | 23 |
| 546219 | 10 | $\mathrm{vert}(\Delta)$ | 23 |
| 546663 | 17 | $\mathrm{vert}(\Delta), (-1,-1,-1)$ | 23 |
| 546862 | 19 | $\mathrm{vert}(\Delta), (0,0,1)$ | 23 |
| 546863 | 10 | $\mathrm{vert}(\Delta) \setminus \{(0,0,1), (1,1,1)\}$ | 23 |
| 547240 | 13 | $\mathrm{vert}(\Delta), (0,1,-1), (0,0,1)$ | 23 |
| 547246 | 16 | $\mathrm{vert}(\Delta), (1,1,1), (-1,-1,-2)$ | 23 |
| 532384 | 8 | $\mathrm{vert}(\Delta), (1,0,-3)$ | 23 |
| 532606 | 8 | $\mathrm{vert}(\Delta), (0,-1,1)$ | 23 |
| 533513 | 8 | $\mathrm{vert}(\Delta), (1,1,1), (0,-1,-1)$ | 23 |
| 534667 | 15 | $\mathrm{vert}(\Delta)$ | 23 |
| 534669 | 12 | $\mathrm{vert}(\Delta)$ | 23 |
| 534866 | 15 | $\mathrm{vert}(\Delta)$ | 23 |
| 535952 | 11 | $\mathrm{vert}(\Delta)$ | 23 |
| 536013 | 12 | $\mathrm{vert}(\Delta)$ | 23 |
| 536498 | 17 | $\mathrm{vert}(\Delta)$ | 23 |

**Table A.10: 49 Canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with $\mathbf{dim}(\boldsymbol{\Delta}^{\mathbf{FI}}) = \mathbf{3}$.** Table contains: ID from Graded Ring Database corresponding to $\Delta$, number of elements $|\mathrm{supp}(\Delta^{\mathrm{FI}})|$ in support of $\Delta^{\mathrm{FI}}$, vertices $\mathrm{vert}(\Delta^{\mathrm{can}})$ of canonical hull $\Delta^{\mathrm{can}}$ of $\Delta$, and stringy Euler number $e_{\mathrm{str}}(Y)$ of a canonical model $Y$ corresponding to $\Delta$.

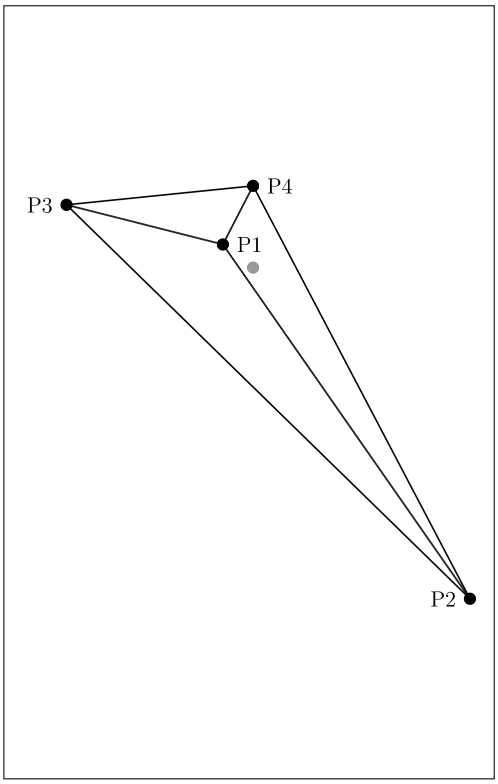
| ID | $\lvert\mathrm{supp}(\Delta^{\mathrm{FI}})\rvert$ | $\mathrm{vert}(\Delta^{\mathrm{can}})$ | $e_{\mathrm{str}}(Y)$ |
|---|---|---|---|
| 537834 | 19 | $\mathrm{vert}(\Delta)$ | 23 |
| 538356 | 17 | $\mathrm{vert}(\Delta), (-1,-1,-1)$ | 23 |
| 539063 | 10 | $\mathrm{vert}(\Delta) \setminus \{(0,1,0),(1,1,2)\}, (1,1,1)$ | 23 |
| 539304 | 11 | $\mathrm{vert}(\Delta), (-2,-1,-1)$ | 23 |
| 539313 | 14 | $\mathrm{vert}(\Delta), (-1,1,2)$ | 23 |
| 540602 | 16 | $\mathrm{vert}(\Delta), (-1,1,2)$ | 23 |
| 540663 | 13 | $\mathrm{vert}(\Delta), (-1,0,-1)$ | 23 |
| 474457 | 8 | $\mathrm{vert}(\Delta)$ | 23 |
| 481575 | 8 | $\mathrm{vert}(\Delta), (-1,-1,-1)$ | 23 |
| 483109 | 11 | $\mathrm{vert}(\Delta)$ | 23 |
| 490478 | 11 | $\mathrm{vert}(\Delta)$ | 23 |
| 490481 | 12 | $\mathrm{vert}(\Delta)$ | 23 |
| 490485 | 10 | $\mathrm{vert}(\Delta)$ | 23 |
| 490511 | 9 | $\mathrm{vert}(\Delta), (2,1,2)$ | 23 |
| 495687 | 14 | $\mathrm{vert}(\Delta)$ | 23 |
| 499287 | 13 | $\mathrm{vert}(\Delta), (-1,-1,-1)$ | 23 |
| 499291 | 17 | $\mathrm{vert}(\Delta)$ | 23 |
| 499470 | 11 | $\mathrm{vert}(\Delta)$ | 23 |
| 501298 | 16 | $\mathrm{vert}(\Delta)$ | 23 |
| 501330 | 13 | $\mathrm{vert}(\Delta)$ | 23 |
| 354912 | 8 | $\mathrm{vert}(\Delta)$ | 23 |
| 372528 | 10 | $\mathrm{vert}(\Delta)$ | 23 |
| 372973 | 9 | $\mathrm{vert}(\Delta)$ | 23 |
| 388701 | 13 | $\mathrm{vert}(\Delta)$ | 23 |



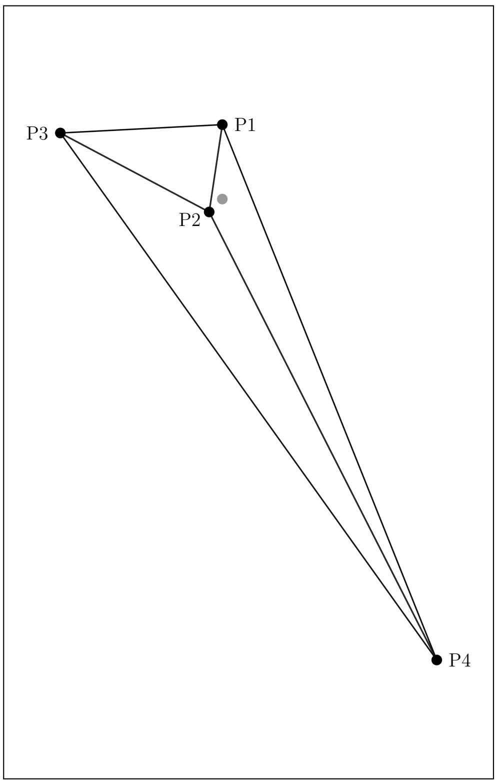

**(a)** ID 547444




**(b)** ID 547465

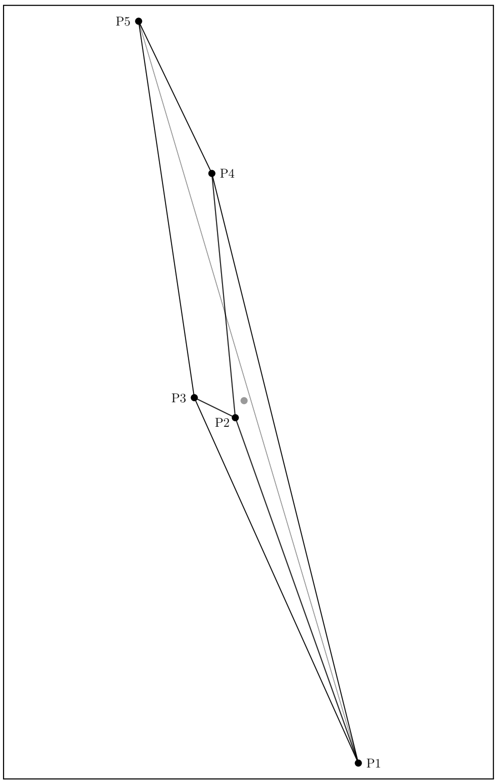



**(c)** ID 547524

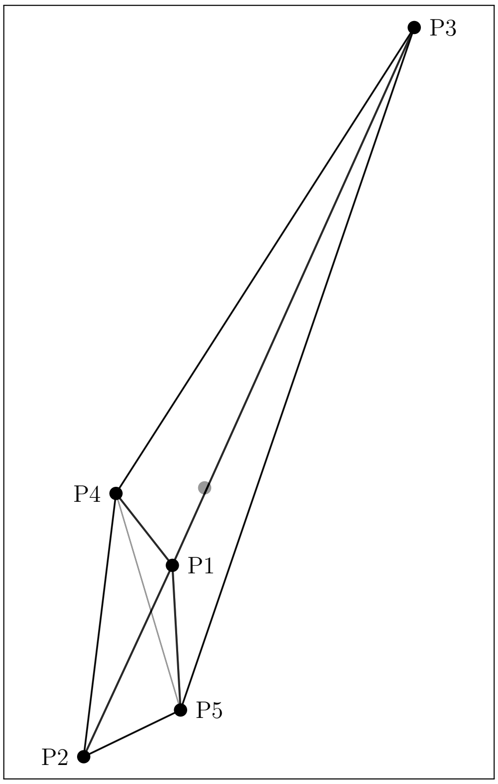



**(d)** ID 547525

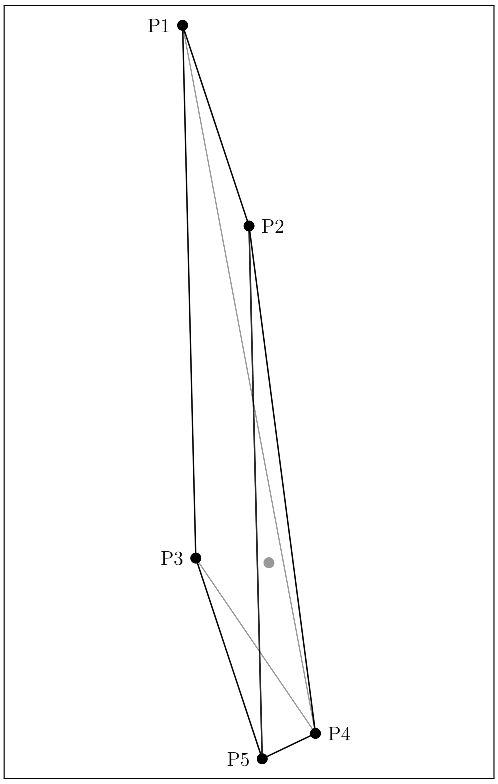
**Figure A.3: 49 Canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with dim$(\boldsymbol{\Delta}^{\mathrm{FI}}) = 3$.**



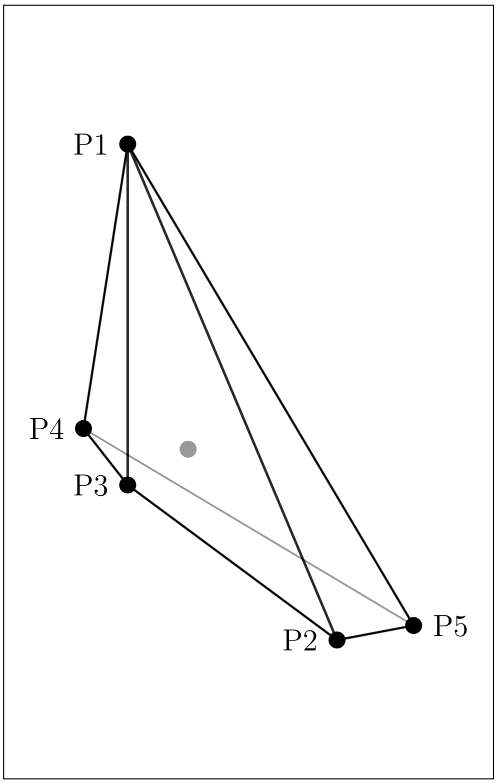

**(e)** ID 545317




**(f)** ID 545932

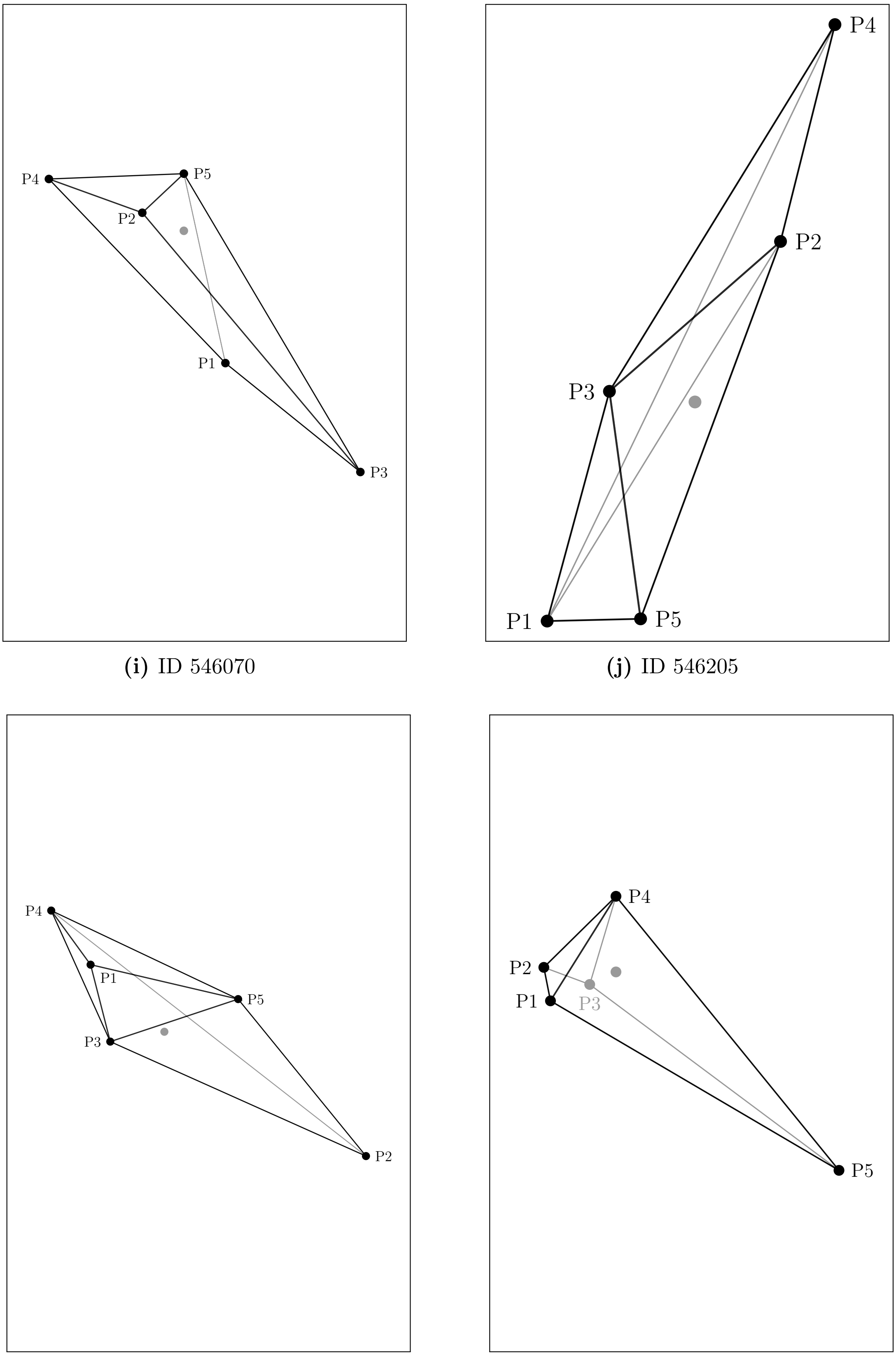



**(g)** ID 546013




**(h)** ID 546062

**Figure A.3: 49 Canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with $\dim(\boldsymbol{\Delta}^{\mathrm{FI}}) = 3$.**



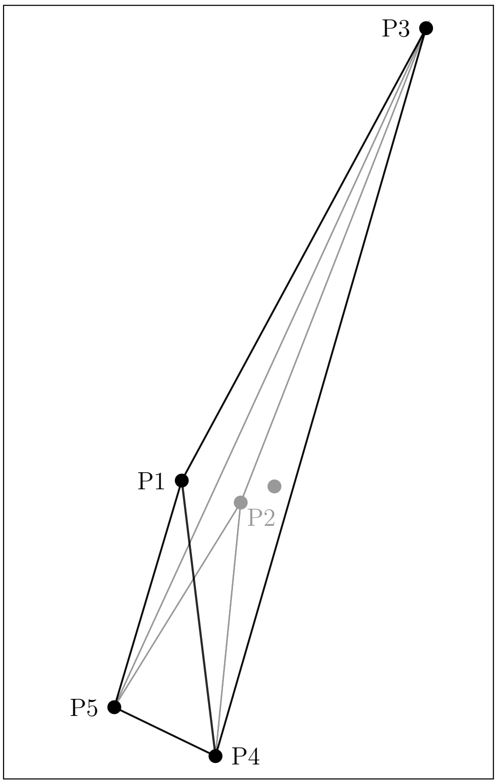

**(i)** ID 546070

**(j)** ID 546205

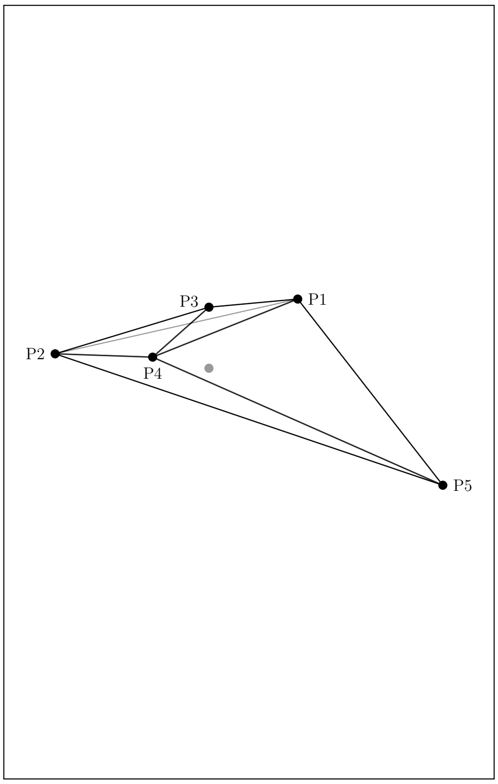
**(k)** ID 546219

**(l)** ID 546663

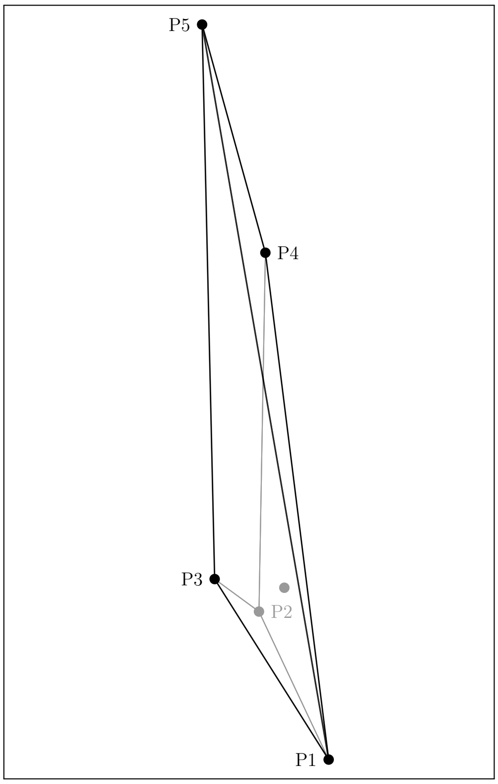
**Figure A.3: 49 Canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with $\dim(\boldsymbol{\Delta}^{\mathbf{FI}}) = 3$.**



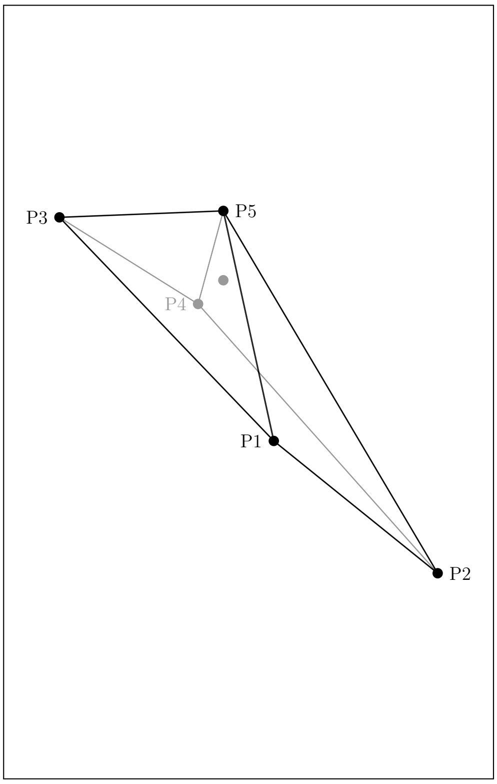

**(m)** ID 546862




**(n)** ID 546863

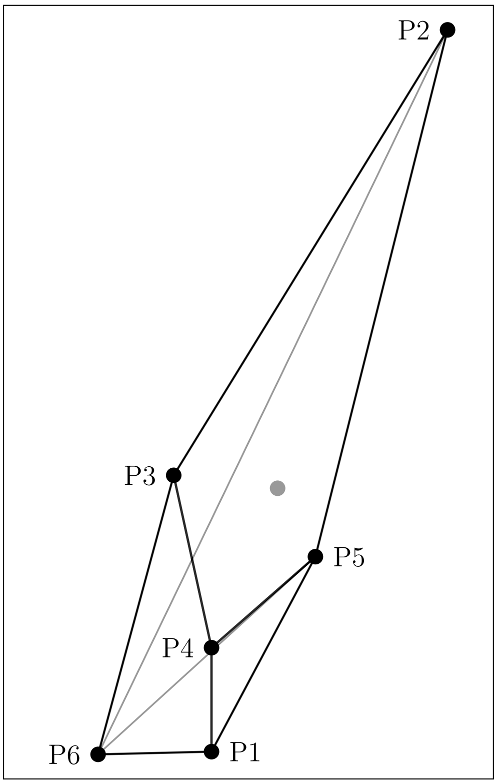



**(o)** ID 547240

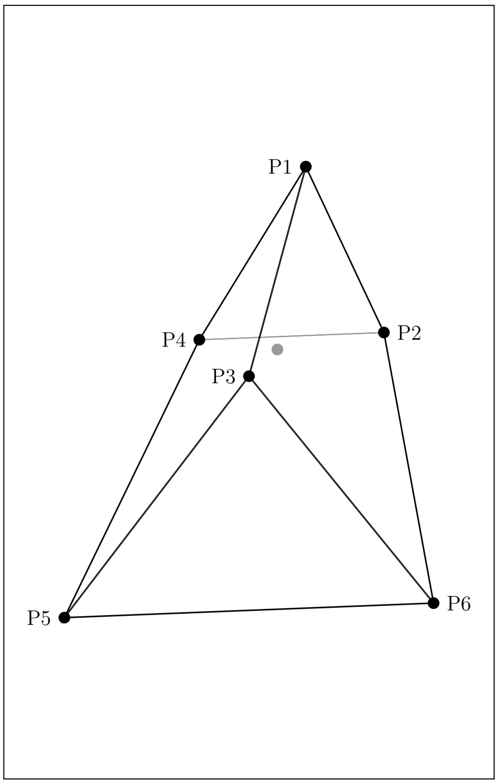



**(p)** ID 547246

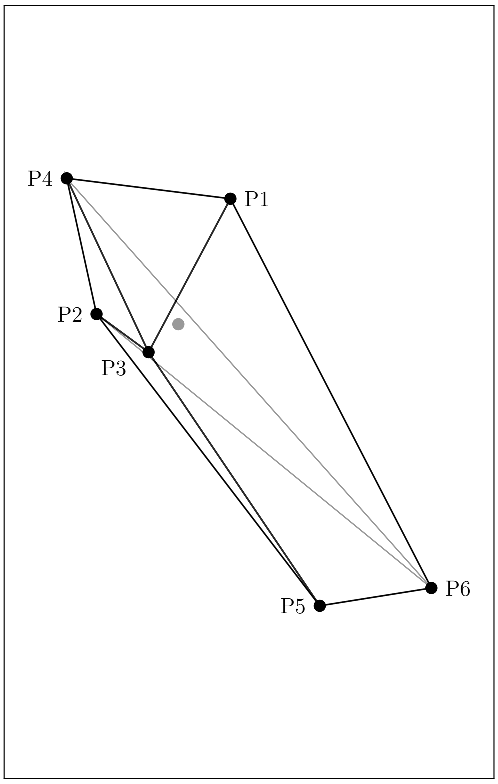
**Figure A.3: 49 Canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with $\mathbf{dim}(\boldsymbol{\Delta}^{\mathbf{FI}}) = 3$.**



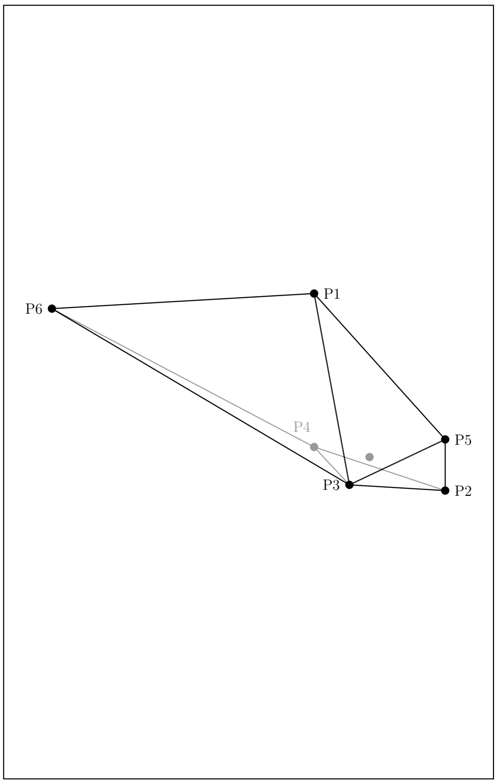

**(q)** ID 532384




**(r)** ID 532606

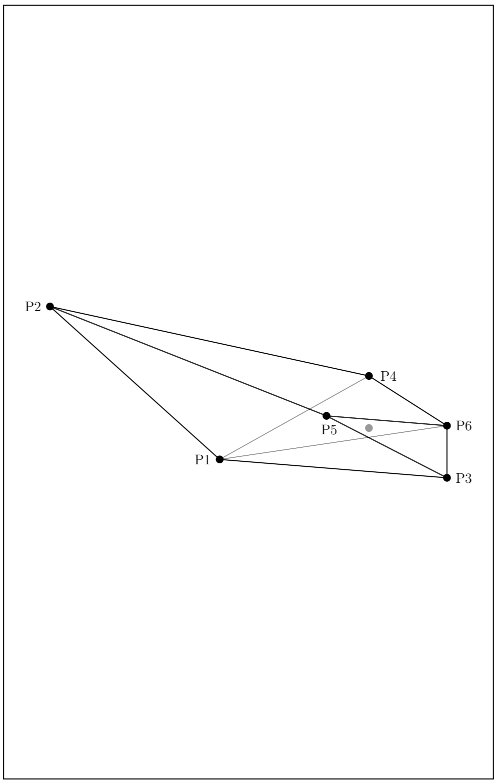



**(s)** ID 533513

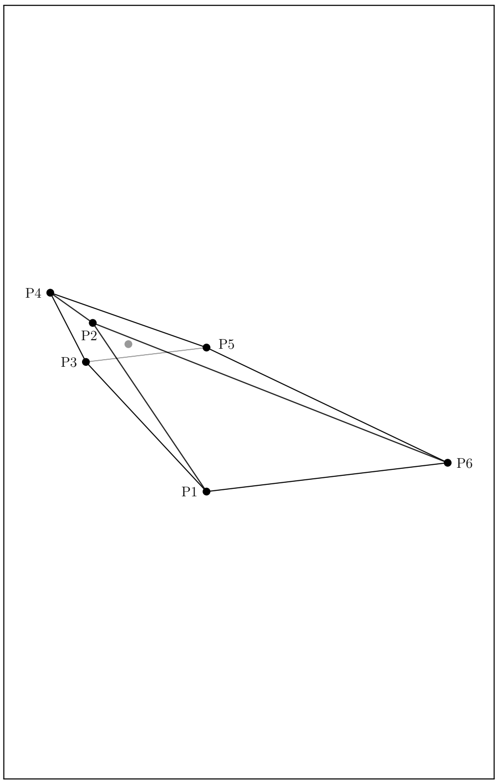



**(t)** ID 534667

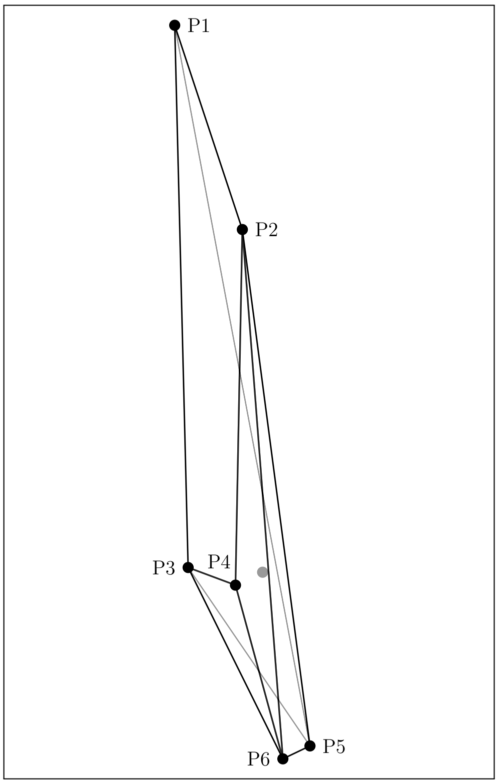
**Figure A.3: 49 Canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with $\dim(\boldsymbol{\Delta}^{\mathrm{FI}}) = 3$.**



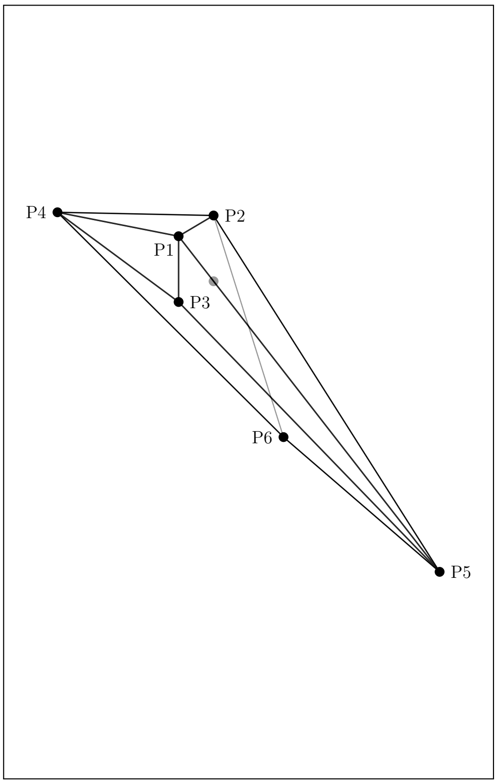

**(u)** ID 534669




**(v)** ID 534866

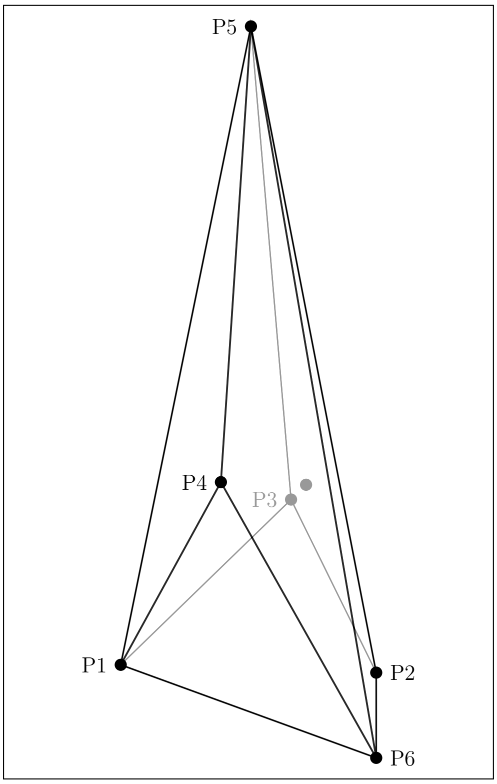



**(w)** ID 535952

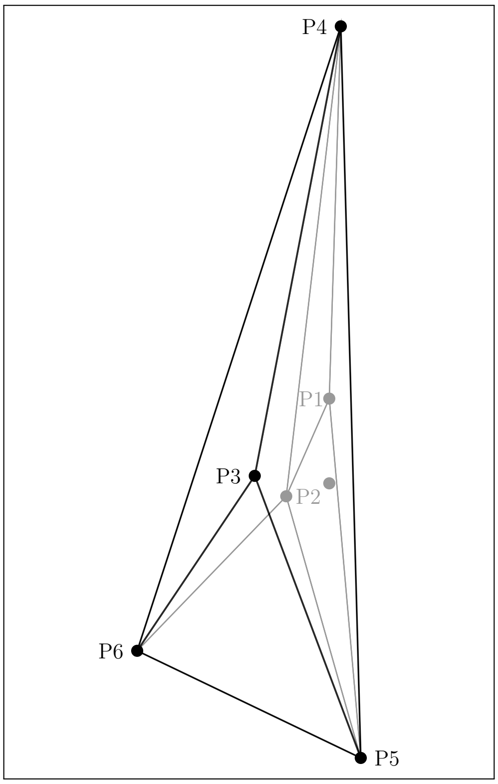



**(x)** ID 536013

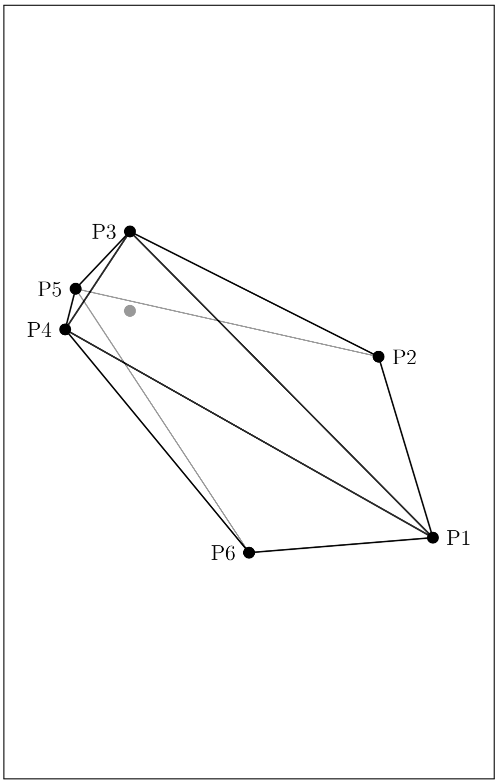
**Figure A.3: 49 Canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with $\mathbf{dim}(\boldsymbol{\Delta}^{\mathbf{FI}}) = 3$.**



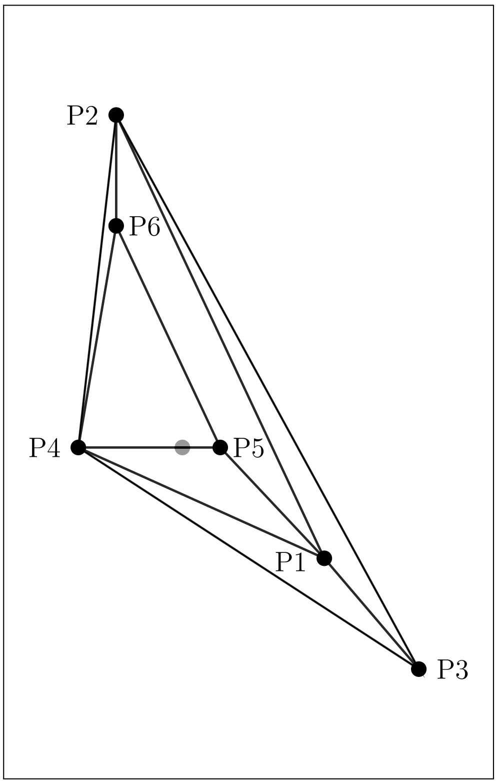

**(y)** ID 536498




**(z)** ID 537834

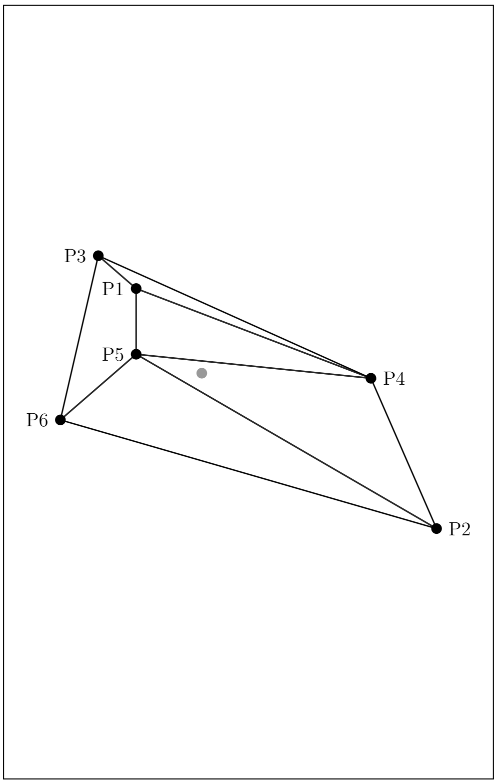



**(aa)** ID 538356

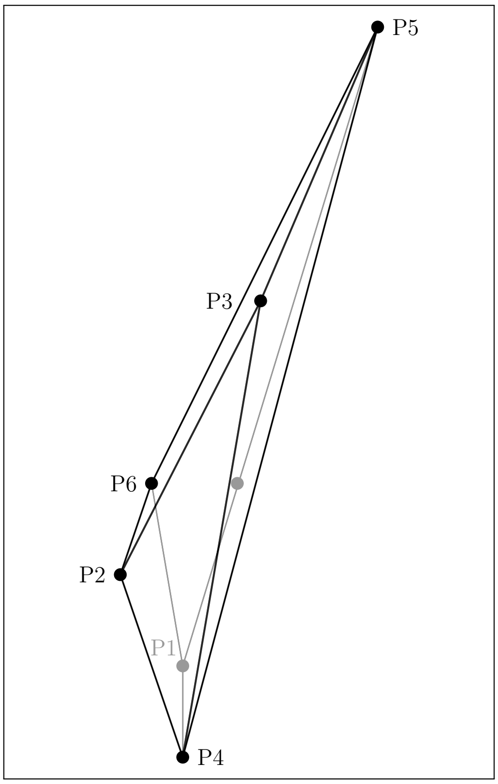



**(ab)** ID 539063

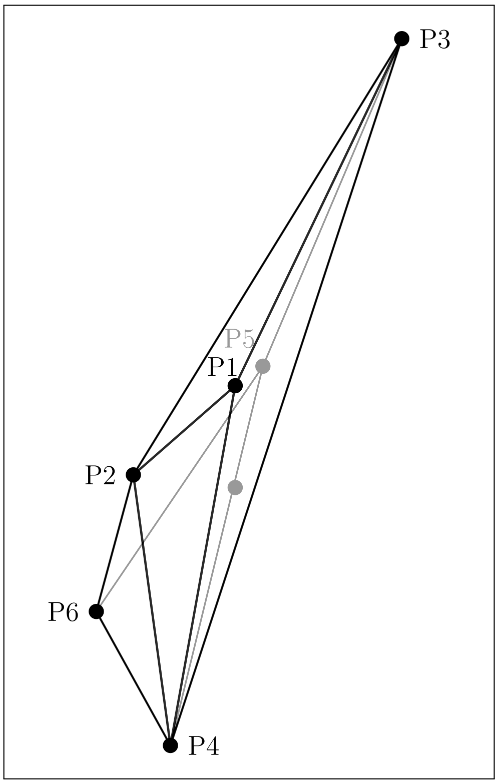
**Figure A.3: 49 Canonical Fano 3-polytopes $\Delta$ with $\dim(\Delta^{\mathrm{FI}}) = 3$.**



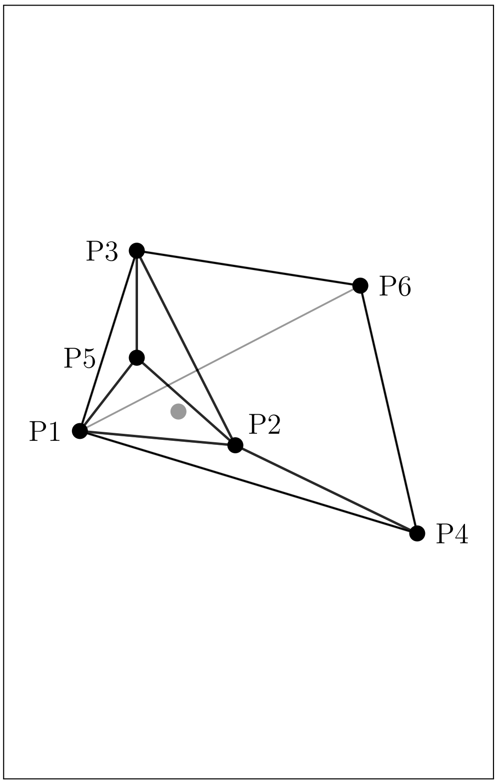

**(ac)** ID 539304




**(ad)** ID 539313

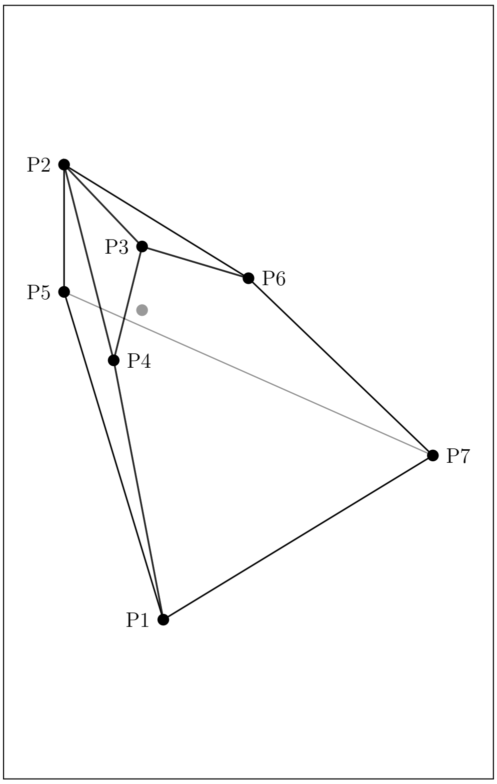



**(ae)** ID 540602

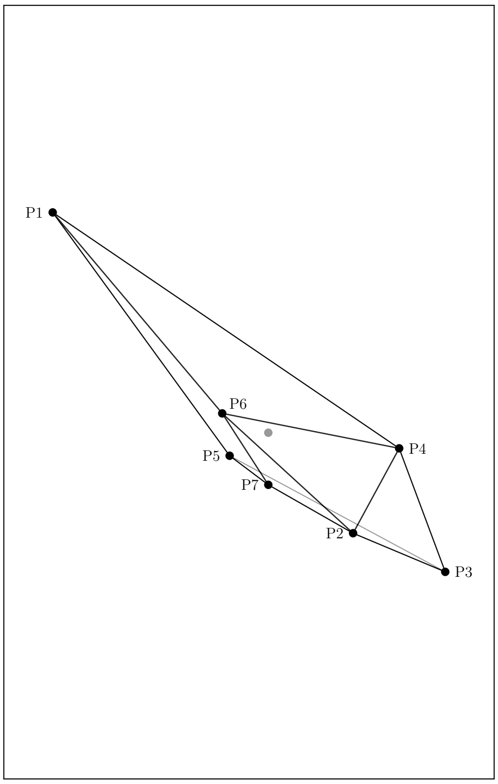



**(af)** ID 540663

**Figure A.3: 49 Canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with $\dim(\boldsymbol{\Delta}^{\mathrm{FI}}) = 3$.**




**(ag)** ID 474457

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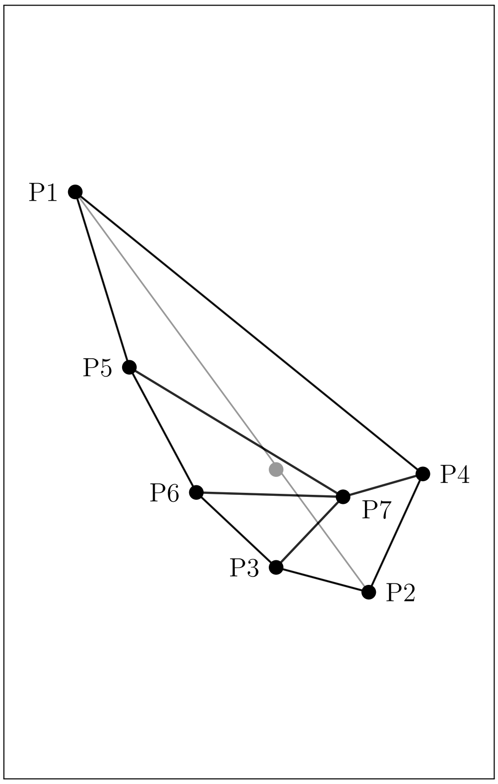
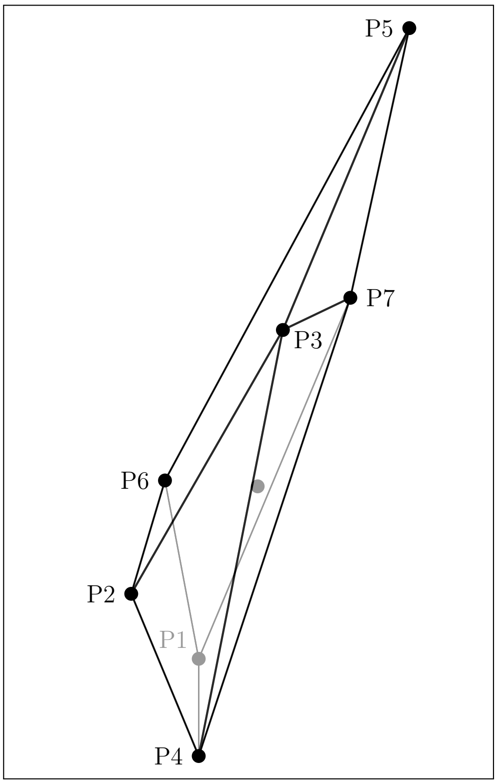
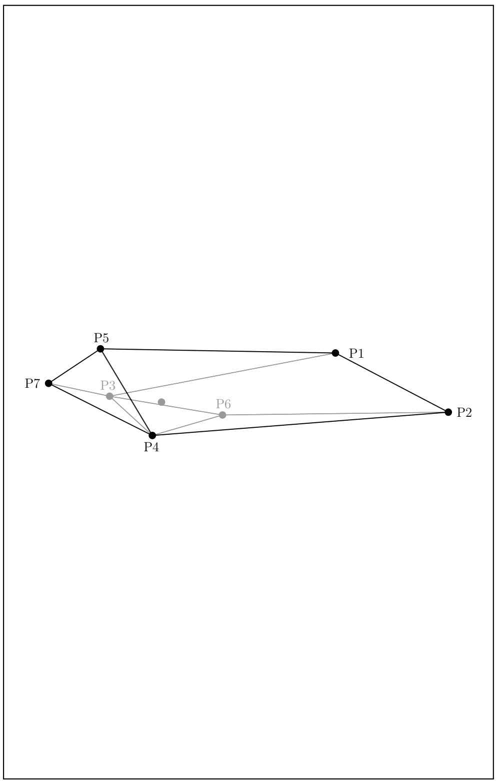



**(ai)** ID 483109

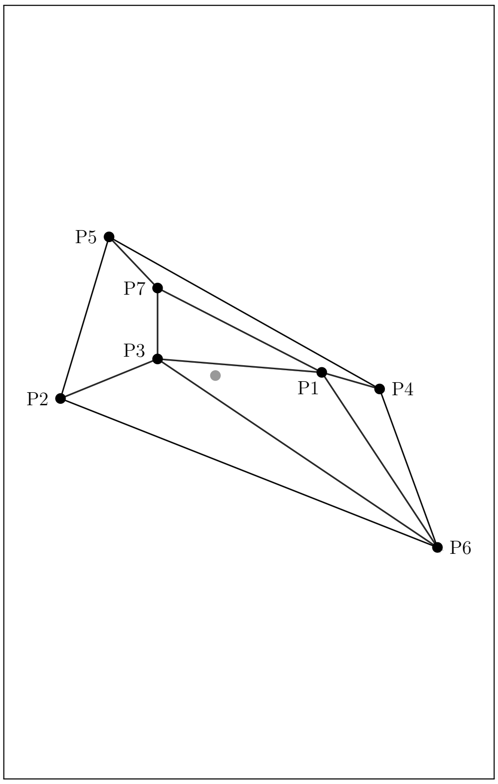



**(aj)** ID 490478

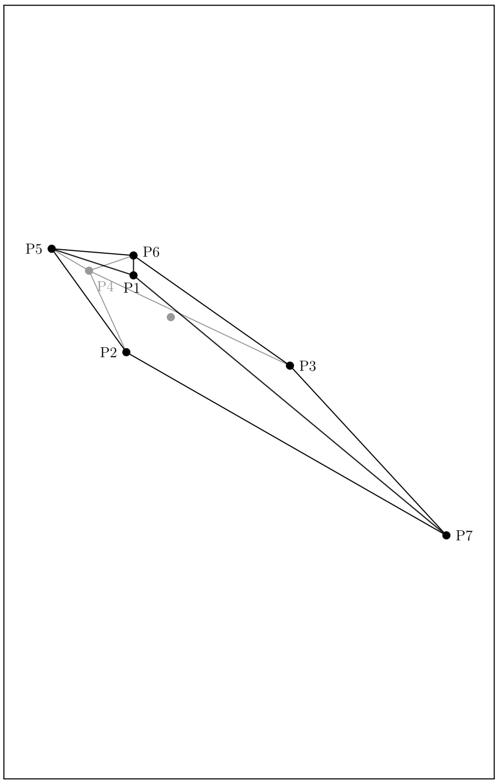
**Figure A.3: 49 Canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with $\dim(\boldsymbol{\Delta}^{\mathbf{FI}}) = 3$.**



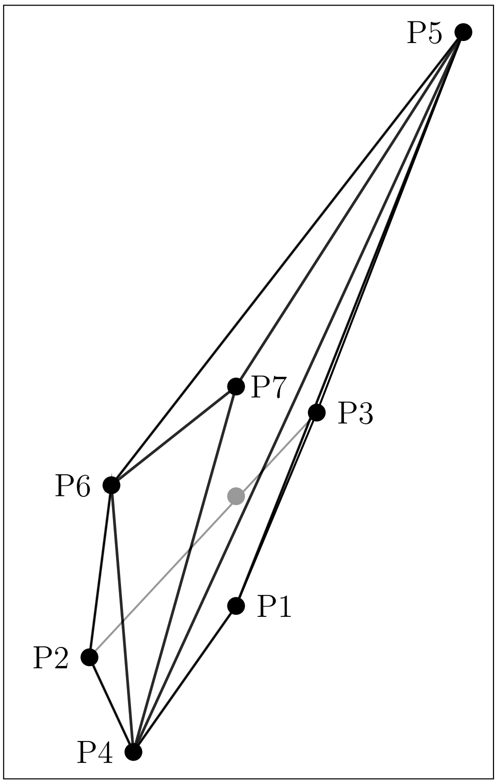

**(ak)** ID 490481




**(al)** ID 490485

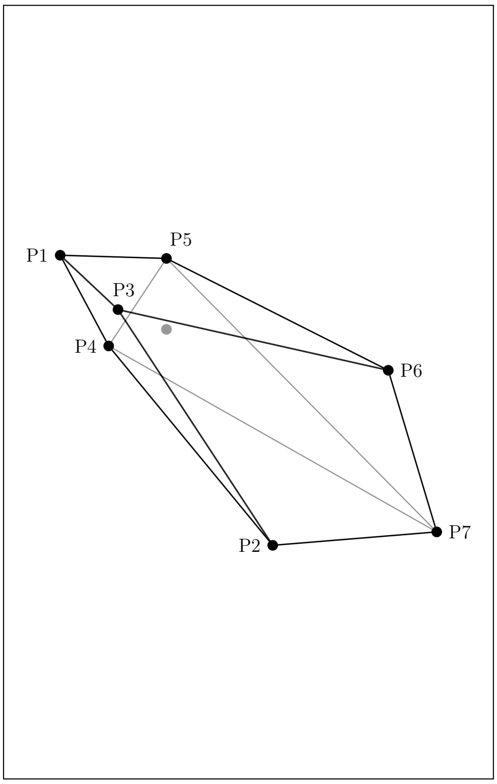



**(am)** ID 490511

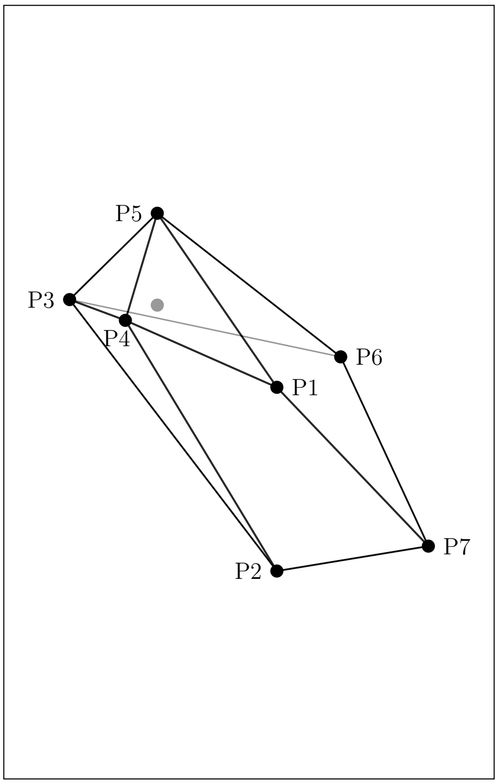



**(an)** ID 495687

**Figure A.3: 49 Canonical Fano 3-polytopes $\Delta$ with $\dim(\Delta^{\mathrm{FI}}) = 3$.**




**(ao)** ID 499287

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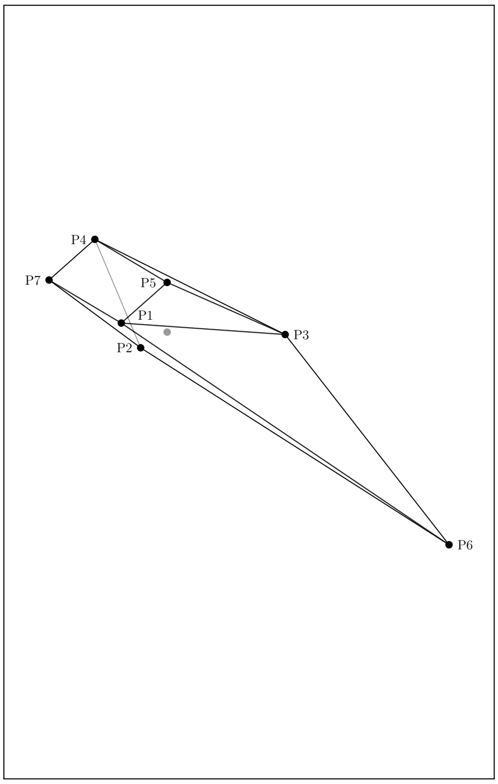
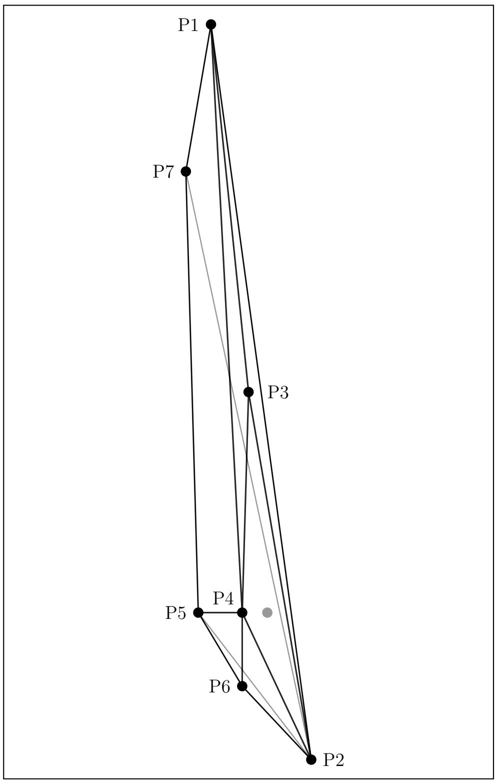
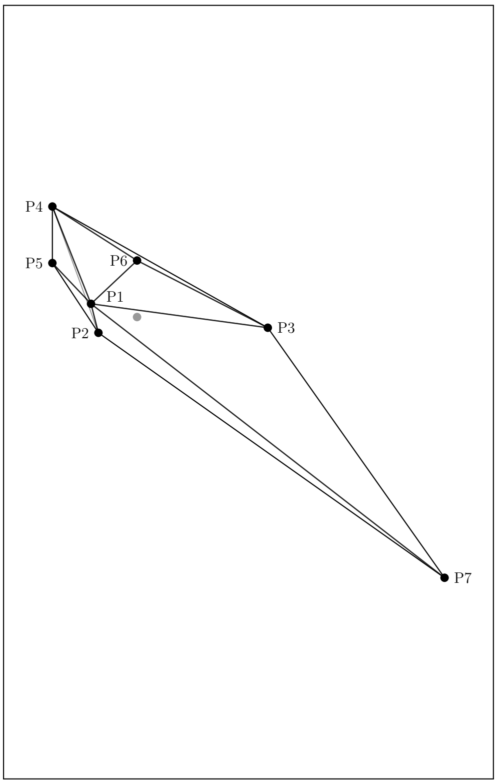



**(aq)** ID 499470

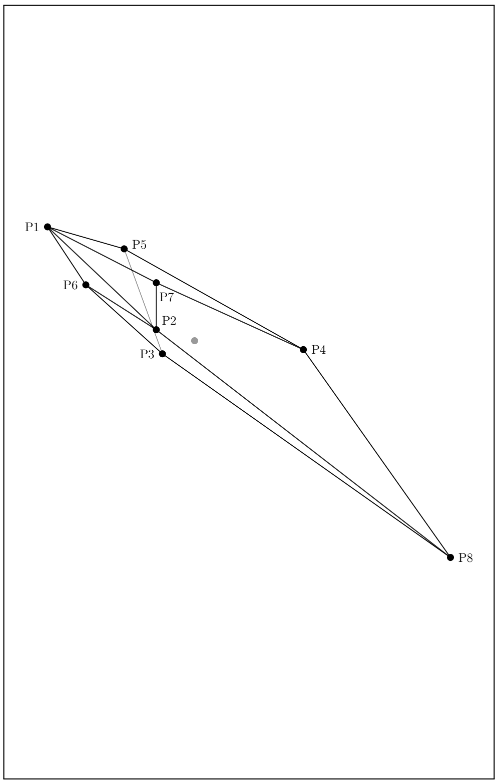



**(ar)** ID 501298

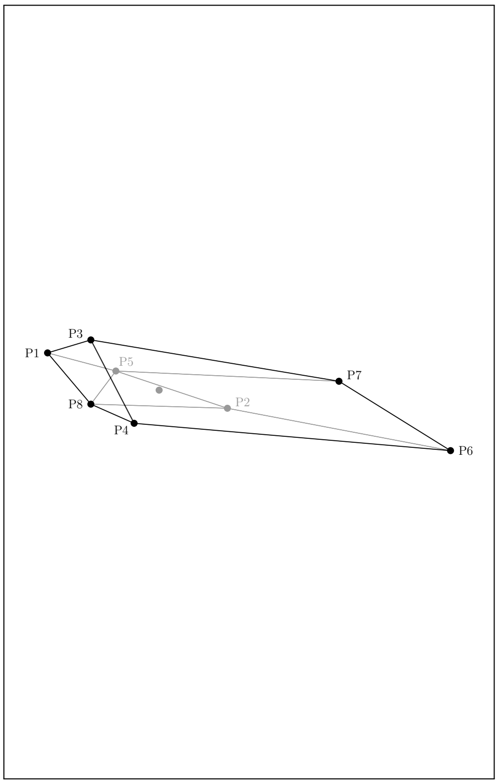
**Figure A.3: 49 Canonical Fano 3-polytopes $\boldsymbol{\Delta}$ with dim($\boldsymbol{\Delta}^{\mathbf{FI}}$) = 3.**



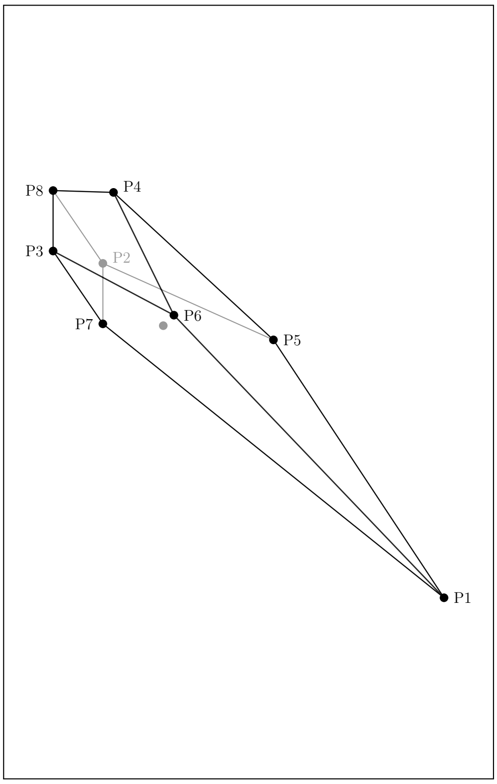

**(as)** ID 501330




**(at)** ID 354912

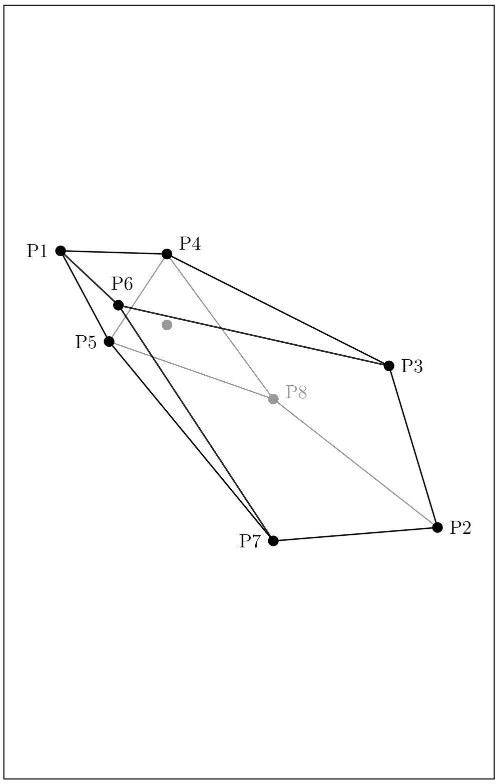



**(au)** ID 372528




**(av)** ID 372973

**Figure A.3: 49 Canonical Fano 3-polytopes $\Delta$ with $\dim(\Delta^{\mathrm{FI}}) = 3$.**




**(aw)** ID 388701

**Figure A.3: 49 Canonical Fano 3-polytopes Δ with dim($\Delta^{\mathrm{FI}}$) = 3.**

**Remark A.3.1.** In Figure A.3 shaded faces are occluded. The origin is coloured grey. The vertices are numbered consecutively as listed in the Graded Ring Database.

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— Karin Schaller

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