# (Real) Tropical Singularities and Bergman Fans

**Dissertation**

an der Mathematisch-Naturwissenschaftlichen Fakultät

der Eberhard Karls Universität Tübingen

zur Erlangung des Grades eines

Doktors der Naturwissenschaften

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"One of the most striking phenomena in ray optics is the formation of caustics: sharp, bright curves to which the light rays are tangential. [...] One of the easiest caustics to observe is that which forms in a cup of coffee [...] Then the reflected rays have as envelope the cusped curve forming the caustic."

Tim Poston & Ian Steward[1]

[1]Poston, Tim and Steward, Ian. 1978. *Catastrophe Theory and its Applications.* Pitman Publishing. Page 247.

# Preface

## Introduction

Tropical geometry is a relatively new and multifaceted field of mathematical research that has deep connections to several other areas of mathematics, e.g. polyhedral geometry, combinatorics and matroid theory. Over the last 15 years tropical geometry has risen to become a widely known theory. Particular attention has been paid to Mikhalkins tropical computations in enumerative geometry of Gromov-Witten invariants of plane curves, which has broad popular appeal and demonstrates the potential of tropical geometry ([**Mik05**]). Since then tropical geometry has become a rich source of mathematical tools and proven itself to be beneficial in a wide spectrum of mathematical areas (e.g. enumerative geometry [**Mik05**], Brill-Noether theory [**CDPR12**], optimization [**ABGJ15**]) and beyond (e.g. economics [**BK16**], biology [**SS04**]).

The root of tropical geometry in this thesis is algebraic geometry — the study of solutions of systems of polynomial equations over a valued field. In this thesis we restrict ourselves to the field $\mathbb{K}_{\mathbb{C}} = \mathbb{C}\{\{t\}\}$ of complex Puiseux series and the field $\mathbb{K}_{\mathbb{R}} = \mathbb{R}\{\{t\}\}$ of real Puiseux series (referred to as *the complex* and *the real case*). We enter the tropical world via *tropicalization*, i.e. an affine algebraic variety degenerates to a polyhedral complex. It is a common approach to solve a mathematical problem by translating it into a more suitable language/change its environment and we consider tropical geometry in this light. Due to this framework tropical geometry is often referred to as the "combinatorial shadow" of algebraic geometry (e.g. [**MR18**], [**MR16**]). The hope is that enough information is kept under tropicalization and that the tropical world offers simpler methods to solve certain problems. This approach poses a lot of questions about the nature of the appearing tropical varieties, which heavily depends on the underlying field, and how algebraic and geometric properties behave under tropicalization. In recent years a lot of effort was made to put tropical geometry on a solid algebraic grounding (e.g. [**MR18**]) to make it a self-contained theory. Several concepts of algebraic geometry have been translated to tropical geometry but this process is far from complete. One of the missing pieces is the concept of a singular tropical variety, something that is not well-established.

### Goals of the Thesis

This thesis aims to contribute to a better understanding of singular tropical hypersurfaces defined by tropical Laurent polynomials with a focus on certain types of singularities. In this thesis we distinguish two situations by means of the underlying field and the specified singularity:

(1) cuspidal plane tropical curves and $k+1$-fold singular tropical hypersurfaces over $\mathbb{K}_{\mathbb{C}}$, and
(2) singular real plane tropical curves and surfaces over $\mathbb{K}_{\mathbb{R}}$

with $k \in \mathbb{N}$. So far there is no intrinsic definition of singular points on tropical hypersurfaces. Due to our approach the following question will be key throughout this thesis:

> Given a tropical hypersurface $T$ defined by a tropical Laurent polynomial and a fixed point $q \in T$, is there an affine algebraic hypersurface $V$ defined by a Laurent polynomial and a singular point $p \in V$ *of certain type* such that $V$ tropicalizes to $T$ and $p$ tropicalizes to $q$?

It seems natural to call a point of a tropical hypersurface *singular* if the answer to the posed question is "yes". This approach suggests to study the set of Laurent polynomials with fixed support $\mathscr{A}$ (given by the columns of a matrix $A \in \mathbb{Z}^{n \times m}$) that provide hypersurfaces with a singularity of specified type. If we restrict ourselves to simple singularities, the variety at hand is defined by the $A$-discriminant ([**GfKZ94**]). The singularities we are going to study impose further constraints in addition to the $A$-discriminant. Consequently, we study (real) tropicalizations of (real) subvarieties of the (real) variety defined by the $A$-discriminant. The following schedule builds the common thread for our investigations:

(1) Determine the (sub-)variety of the $A$-discriminant describing hypersurfaces with the specified type of singularity.
(2) Compute its tropicalization.
(3) Classify the singular tropical hypersurfaces in terms of regular marked subdivisions.

As tropicalization depends on the underlying field, a brief summary of the state of the art with respect to the specified field is as follows:

**The Complex Case**

In the complex case the tropicalization we frequently use degenerates an affine algebraic variety $V \subset \mathbb{K}_{\mathbb{C}}^n$ to a polyhedral complex $\Sigma$ with weights on its maximal dimensional polyhedra so that $\Sigma$ satisfies a so-called *balancing condition*. Indeed, this balancing condition makes the polyhedral complex $\Sigma$ tropical and we call the polyhedral complex associated to $V$ *tropical variety*. The idea of degenerating varieties can be traced back to several sources, in particular George Bergman who studied logarithmic limits of algebraic varieties ([**Ber71**]). Nowadays, the tropicalization of a linear space defined by an ideal whose generating linear forms have coefficients with trivial valuation (referred to as the *constant coefficient case*) is called *Bergman fan* that purely depends on the matroid associated to the linear space ([**Stu02**]). Indeed, the tropicalizations of linear spaces (and the corresponding Bergman fans respectively) are rigorously studied, well-known and appear in several contexts in tropical geometry (e.g. [**Spe08**], [**FR15**]). Bergman fans carry several different fan structures that can be stated in different terms of matroid theory (e.g. [**AK06**], [**FS05**]). Bergman fans can be seen as one of the most basic objects that allow a feasible computation ([**Rin13**]). For principal ideals there is also a solid theoretical foundation. A tropical hypersurface is dual to a regular marked subdivision of the Newton polytope of the defining Laurent polynomial $F$. Hence, tropical hypersurfaces can be studied purely combinatorial by their dual subdivisions (e.g. [**Kal16**], [**Mik05**]). The tropical $A$-discriminant, i.e. the tropicalization of the variety defined by the $A$-discriminant, equals the Minkowski sum of a Bergman fan and a linear space ([**DFS07**]). In this context, the Bergman fan equals the tropicalization of the family of Laurent polynomials with fixed support $\mathscr{A}$ that have a singularity at $\mathbf{1}_n = (1, \ldots, 1)$. The linear space acts as a shift of the singular point to any other torus point. In [**MMS12a**] the tropical $A$-discriminant of plane curves is studied in terms of the secondary fan whose cones are in one-to-one correspondence to regular marked subdivisions of the corresponding Newton polytope. The sequel ([**MMS12b**]) deals with a classification of surfaces in terms of regular marked subdivisions. Both classifications benefit from the well-known facts of Bergman fans and its subdivisions. A more algebraic approach is provided by [**DT12**] who study singular tropical hypersurfaces in terms of Euler derivatives. The investigation of $k+1$-fold singular tropical hypersurfaces in [**DDRP14**] can be considered as a generalization of [**DT12**]. In [**Shu05**] the focus is set on the enumeration of singular curves in toric surfaces. The method is based on *Veros patchworking* and can be applied to other types of singularities, e.g. cusps. However, the condition “passing through a set of points” is a constraint on the set of curves with a cusp and, therefore, no general systematic study of plane tropical curves with a cusp is provided.

### The Real Case

The study of real tropical varieties arising as tropicalizations of affine algebraic varieties defined over $\mathbb{K}_{\mathbb{R}}$ is still in its early stages. Some basic results for the complex case, e.g. Kapranov’s Theorem (cf. Theorem 1.4.3.16), cannot be transferred to the real case as $\mathbb{K}_{\mathbb{R}}$ is not algebraically closed. This circumstance makes the real case even more sophisticated. Beside the degeneration itself the *real tropicalization* of a real affine algebraic variety $V \subset \mathbb{K}_{\mathbb{R}}^n$ comes with signs. These signs impose restrictions that determine the set of polyhedra of the degeneration of $V$ that belong to the real tropical variety. In [**Tab15**] the real tropical $A$-discriminant is examined with a focus on Euler derivatives. However, there is no systematic study of singular real tropical hypersurfaces. The real tropicalization of the $A$-discriminant is based on the fact that, analogue to the complex case, the real tropicalization of a linear space over $\mathbb{K}_{\mathbb{R}}$ purely depends on the associated (oriented) matroid. In [**AKW06**] the positive part of the tropicalization of a linear space is studied.

## Contents of the Thesis

**Chapter 1:** We introduce basic topics that will be fundamental to the further investigations reported, i.e. polyhedral geometry (Section 1.1), unoriented matroids (Section 1.2) as well as oriented matroids (Section 1.3). In particular, we introduce the Bergman fans that play a decisive role in this thesis. We then introduce tropical geometry in Section 1.4 with a focus on tropicalizations of affine algebraic varieties defined over $\mathbb{K}_{\mathbb{C}}$. Afterwards we give a brief introduction to real tropical geometry with a focus on tropicalizations of affine varieties defined over $\mathbb{K}_{\mathbb{R}}$. Finally, we look at singular hypersurfaces (Section 1.6), where the singularity is of one of the mentioned types, and briefly summarize what is known about the $A$-discriminant (cf. Section 1.6.2) in the complex case.

**Chapter 2:** We study the particular case of an ideal generated by linear forms and one further polynomial in the constant coefficient case:

- In Section 2.1 we determine the tropicalization of a hypersurface in a linear space using coordinate projections according to bases of the matroid describing the linear space. First, we prove some results concerning coordinate projections of Bergman fans in Section 2.1.1. The projection of a Bergman fan according to a basis is again a Bergman fan and we define a new fan structure called *induced coarse subdivision*. The advantage of this subdivision is that there is a one-to-one correspondence between top-dimensional cones of the induced coarse subdivision and parts of the Bergman fan of the linear space we started with. After a brief overview concerning the interplay of algebraic and tropical coordinate projections in Section 2.1.2 (based on [**ST08**], [**Gub13**]) we state the main result in Section 2.1.3: the tropicalization of a hypersurface in a linear space is completely determined by the collection of coordinate projections according to bases of the matroid associated to the linear space (Theorem 2.1.3.25). As the proof is of an algorithmic nature, we provide a pseudocode for the general case and consider tropical curves in tropical planes in detail in Section 2.1.4.

- In Section 2.2 we undertake a different approach. In Section 2.2.1 we linearize the ideal with the help of the Veronese map. As the Veronese map commutes with tropicalization (Lemma 2.2.1.2) we can study the Bergman fan arising from the linearized ideal. The fine subdivision of this Bergman fan induces a new subdivision on the Bergman fan we initially started with. In Section 2.2.2 we use a result of [**AN13**] and show that the codimension one skeleton of this subdivision supports any tropical hypersurface in this tropical linear space.

**Chapter 3:** We examine plane tropical curves with a cusp and $k+1$-fold singular tropical hypersurfaces.

- Section 3.1 deals with the study of plane tropical curves with fixed support $\mathscr{A}$ having a cusp. In Section 3.1.1 we show that it is sufficient to study the curves with a cusp fixed at $\mathbf{1}_2$. The further constraint on a singularity to become a cusp is given by a homogeneous polynomial of degree two as we show in Section 3.1.2. This is where Chapter 2 comes into play as we deal with an ideal defined by linear forms and one further polynomial of degree two. We study the ambient linear space (defined by the linear part of the ideal) in Section 3.1.2.1 and the additional polynomial in Section 3.1.2.2. Section 3.1.3 deals with the coordinate projections. These can be determined purely from simple affine relations among the elements of $\mathscr{A}$ (cf. Theorem 3.1.3.11). In Section 3.1.4 we undertake the first steps towards a classification for *generic* supports $\mathscr{A}$ (i.e. no three points of $\mathscr{A}$ are colinear).
- In Section 3.2 we deal with $k+1$-fold singular tropical hypersurfaces. Section 3.2.1 contains the justification to study tropical hypersurfaces with a $k+1$-fold singularity at $\mathbf{1}_n$. In Section 3.2.2 we see that all further constraints additional to the $A$-discriminant are linear, i.e. the tropicalization of the family of Laurent polynomials with a $k+1$-fold singularity at $\mathbf{1}_n$ is a Bergman fan. One approach to this matroid is given by Euler derivatives as we show in Section 3.2.3. Finally, in Section 3.2.4 we take a look at a regular subdivision provided by a point of this Bergman fan.

**Chapter 4:** The last chapter deals with a classification of singular real plane tropical curves and singular real tropical surfaces.

- In Section 4.1 we introduce the theoretical foundation. We define the *signed Bergman fan* which is a collection of polyhedral fans arising from an oriented matroid in Section 4.1.1.

We study some of its polyhedral structures (that are analogous to the fan structures of classical Bergman fans). Then we relate signed Bergman fans and real tropicalizations of linear spaces over $\mathbb{K}_{\mathbb{R}}$ in Section 4.1.2.

- Section 4.2 deals with a classification of singular real plane tropical curves. The foundation of our classification is [**MMS12a**] as real tropical curves inherit their polyhedra from the complex tropical curves. However, there is no theory of duality for real tropical curves and subdivisions. The purpose of Section 4.2.1 is the introduction of terminology for real plane tropical curves, dual *signed regular marked subdivisions* and the *signed secondary fan*. At the end of this section we express the real tropical $A$-discriminant over the *real tropical group*, the underlying algebraic structure of real tropical geometry. In Section 4.2.2 we determine the real tropicalization of the family of real Laurent polynomials with fixed support $\mathscr{A} \subset \mathbb{Z}^2$ that provide real plane curves with a singularity at $\mathbf{1}_2$. Analogous to the complex case, this is the tropicalization of a linear space and we apply the theory developed in Section 4.1. Afterwards, in Section 4.2.3, we undertake a first classification. In Section 4.2.4 we compare the structures of the signed secondary fan and the real tropical $A$-discriminant. It turns out that — in contrast to the complex case — we cannot express the real tropical $A$-discriminant completely in terms of the signed secondary fan. In other words, there are subdivisions where we a priori cannot say whether the dual real plane tropical curve is singular or not. We offer a solution using a *height profile* of the signed subdivision and in Section 4.2.5 we give a criteria in terms of Euler derivatives. In Section 4.2.6 we restrict ourselves to signed regular marked subdivisions without unmarked points, i.e. any point of the Newton polytope is part of the subdivision.
- Section 4.3 deals with a classification of singular real tropical surfaces. In Section 4.3.1 we fix the singularity at $\mathbf{1}_3$ and focus on the sign conditions of the affine relations we get in the signed regular marked subdivisions. In Section 4.3.2 we explain that we face similar problems as occurred in the curve case. Hence, we restrict the classification to *generic* real tropical surfaces (cf. Definition 4.3.2.7, based on [**MMS12b**, Definition 16]). The complete classification is stated in Section 4.3.3.

**Results of the Thesis**

The following list contains the main results achieved in this thesis sorted by chapter. Note that Chapter 1 contains only the preliminaries.

In Chapter 2: “Computation and Characterization of Tropical Hypersurfaces in Tropical Linear Spaces”

- Theorem 2.1.3.25 states that the tropicalization of a hypersurface in a linear space (with constant coefficients) is completely determined by its coordinate projections.
- Algorithm 1 is an explicit algorithm for the reconstruction of a tropical hypersurface in a linear space from a (sufficiently large) set of coordinate projections.
- In Definition 2.2.1.15 we define a new polyhedral structure on a tropicalized linear space that supports any tropicalization of a hypersurface in this linear space. The methods applied offer a new approach to the tropicalization of a hypersurface in a linear space.

- Corollary 2.2.2.31 states that polynomials with minimal support contained in the degree $d$ part of an ideal generated by linear forms and a polynomial of degree $d$ is a tropical basis.

In Chapter 3: "Tropical Hypersurfaces with a Specified Singularity"

- Theorem 3.1.3.11 states that coordinate projections of the tropicalization of the family of Laurent polynomials providing curves with a cusp at $\mathbf{1}_2$ can be determined by the affine relations on the fixed support $\mathscr{A}$.
- Theorem 3.1.4.17 states that if $\mathscr{A}$ is a generic support of cardinality $m$ then the tropicalization of the family of Laurent polynomials that provide a curve with a cusp at $\mathbf{1}_2$ equals the Bergman fan arising from the uniform matroid of rank $m-4$ over $\{1,\dots,m\}$.
- Proposition 3.2.3.15 determines a tropical basis of the family of Laurent polynomials providing a hypersurface with a $k+1$-fold singularity at $\mathbf{1}_2$.
- Proposition 3.2.4.16 gives an algebraic criteria for the existence of a $k+1$-fold singularity in a tropical hypersurface.

In Chapter 4: "Signed Bergman Fans and Real Tropical Singularities"

- Section 4.1 introduces signed Bergman fans associated to oriented matroids.
- Theorem 4.2.6.44 contains a classification of singular real plane tropical curves of maximal dimensional type.
- Theorem 4.3.3.9 contains a classification of generic singular real tropical surfaces of maximal dimensional type.

For theoretical purposes as well as for practical reasons it is often desirable to compute tropicalizations, i.e. gaining a good understanding of the tropicalization itself is of particular importance. As mentioned before, the tropicalization of an algebraic variety defined by an arbitrary ideal is not well understood and even hard to compute (see [**MR16**] for recent developments in this area). An exception is formed by linear spaces. The computation of tropicalizations of linear spaces is an active field of research (e.g. [**HJS16**], [**Rin13**]). Chapter 2 can be considered as a contribution to the family of computable tropical varieties. Currently, there are only a few tools for tropical computations, e.g. GFAN ([**Jen**], TROPLI ([**Rin13**]) and POLYMAKE ([**GJ00**]) using A-TINT ([**Ham14b**]). The algorithm stated in Section 2.1.4 is implemented for SINGULAR ([**DGPS16**]) as the input is an ideal. We use POLYMAKE for polyhedral computations. The necessary library for SINGULAR and the script for POLYMAKE can be found at:

`https://github.com/cjuergens/trophials.git`

## Deutsche Zusammenfassung

Der Aufstieg der tropischen Geometrie zu einer vielbeachteten Theorie gelang in den vergangenen 15 Jahren. Zu ihren vielen Wurzeln zählt ohne Zweifel die algebraische Geometrie, die ideengebend für viele Konzepte der tropischen Geometrie ist. Das Ziel der vorliegenden Arbeit ist es, die Entwicklung der tropischen Geometrie zu einer eigenständigen Theorie zu unterstützen, indem zu einem besseren Verständnis von *singulären tropischen Hyperflächen* beigetragen wird. Wir nennen eine tropische Hyperfläche *singulär*, wenn sie die Tropikalisierung einer singulären algebraischen Hyperfläche ist. Die zu untersuchenden singulären Hyperflächen dieser Arbeit können durch ihre Art und den der Untersuchung zugrundegelegten Körper unterschieden werden: kuspische ebene Kurven und $k+1$-fach singuläre Hyperflächen über den komplexen Puiseux-Reihen $\mathbb{K}_{\mathbb{C}}$ und singuläre ebene Kurven und Flächen über den reellen Puiseux-Reihen $\mathbb{K}_{\mathbb{R}}$. Ausgangspunkt der Untersuchungen bildet die $A$-Diskriminante $\nabla$, die alle Laurent Polynome mit festem Support (gegeben durch die Spalten der Matrix $A \in \mathbb{Z}^{n\times m}$) parametrisiert, die eine singulären Hyperfläche definieren. Wir fokussieren uns auf den linearen Anteil $\nabla_{\mathbf{1}_n}$ der Hyperflächen mit Singularität an $\mathbf{1}_n$.

In Kapitel 2 werden Tropikalisierungen von Hyperflächen $H$, die in einem linearen Raum $L$ enthalten sind, im Konstante-Koeffizienten-Fall über $\mathbb{K}_{\mathbb{C}}$ untersucht. In Abschnitt 2.1 zeigen wir, dass die Tropikalisierung der Hyperfläche $H \subset L$ aus (hinreichend vielen) Koordinatenprojektionen bzgl. Basen des Matroids assoziiert zu $L$ bestimmt werden kann. Der sich aus dem Beweis ergebende Algorithmus ist für Kurven in 2-dimensionalen linearen Räumen in SINGULAR (unter Verwendung von POLYMAKE) implementiert. In Abschnitt 2.2 definieren wir eine Fächerstruktur (in Abhängigkeit einer Zahl $d \in \mathbb{N}$) für die Tropikalisierung des linearen Raumes $L$, deren Kodimension-1-Skelett jede Tropikalisierung einer Hyperfläche $H \subset L$ enthält, die durch ein Polynom vom Grad $d$ definiert ist. Zuletzt geben wir eine tropische Basis für Ideale erzeugt durch Linearformen und ein Polynom an.

In Kapitel 3 untersuchen wir Tropikalisierungen von Untervarietäten von $\nabla_{\mathbf{1}_n}$ über $\mathbb{K}_{\mathbb{C}}$. In Abschnitt 3.1 zeigen wir, das der Anteil $\nabla_{\mathbf{1}_2}^{\mathrm{cusp}}$ der Kurven mit Kuspe an $\mathbf{1}_2$ eine Hyperfläche in $\nabla_{\mathbf{1}_2}$ ist. Unter Verwendung der Resultate aus Kapitel 2 zeigen wir, dass jede Koordinatenprojektion der Tropikalisierung von $\nabla_{\mathbf{1}_2}^{\mathrm{cusp}}$ durch affine Relationen im Support $A$ festgelegt ist. Darauf aufbauend unternehmen wir erste Schritte hinsichtlich einer Klassifikation. In Abschnitt 3.2 betrachten wir die lineare Untervarietät $\nabla_{\mathbf{1}_n}^{k+1}$ der Hyperflächen mit $k+1$-facher Singularität an $\mathbf{1}_n$. Mittels des zu $\nabla_{\mathbf{1}_n}^{k+1}$ assoziierten Matroiden bestimmen wir eine tropische Basis für die Tropikalisierung von $\nabla_{\mathbf{1}_n}^{k+1}$. Zuletzt wird ein tropisch-algebraisches Kriterium für das Vorliegen einer $k+1$-fachen Singularität in einer tropischen Hyperfläche präsentiert.

In Kapitel 4 klassifizieren wir singuläre reelle tropische Kurven und Flächen über $\mathbb{K}_{\mathbb{R}}$. In Abschnitt 4.1 entwickeln wir eine Theorie über vorzeichenbehaftete Bergmanfächer, gegeben durch orientierte Matroide. Diese vorzeichenbehafteten Bergmanfächer realisieren reelle Tropikalisierungen linearer Räume und ermöglichen es die reelle Tropikalisierung des linearen Anteils $\nabla_{\mathbb{R},\mathbf{1}_n}$ der reellen $A$-Diskriminante $\nabla_{\mathbb{R}}$ zu bestimmen. Abschnitt 4.2 enthält die systematische Klassifikation singulärer reeller ebener tropischer Kurven und Abschnitt 4.3 die Klassifikation *generischer* singulärer reeller tropischer Flächen.

**Danksagung**

An erster Stelle möchte ich mich herzlich bei meiner Doktormutter Hannah Markwig bedanken, die mir die Möglichkeit geboten hat, in diesem spannenden und vielseitigen Gebiet eine Dissertation über ein schönes Thema anzufertigen. Vielen Dank für die exzellente Betreuung!
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# Contents

CHAPTER 1

# Preliminaries

In this preliminary chapter we introduce basic concepts and terminology of some fields in mathematics that are fundamental for tropical geometry and come across our research purposes. We omit all proofs (except for Lemma 1.6.1.5) and refer to the literature.

In the first sections (Sections 1.1 to 1.3) the focus is set on polyhedral geometry and (un)oriented matroids. Then we give a brief introduction to tropical geometry. We distinguish two cases: tropical geometry over an algebraically closed field in Section 1.4 followed by real tropical geometry over an real closed field in Section 1.5. At the end we consider singular tropical hypersurfaces in Section 1.6. In this thesis we restrict to two certain fields: in the algebraically closed case we work over the field $\mathbb{K}_{\mathbb{C}}$ of complex Puiseux series and otherwise we work over the field $\mathbb{K}_{\mathbb{R}}$ of real Puiseux series (Definition 1.4.1.2 and Convention 1.4.1.3).

NOTATION 1.1. Let $\mathbb{N}$ ($\mathbb{N}_0$) denote the set of natural numbers (with zero). Fix an integer $n \in \mathbb{N}$. By $[n+1]$ we denote a finite set of $n+1$ elements. If nothing else is mentioned we work with $[n+1] = \{0,\dots,n\}$. We fix the lattice $\Lambda = \mathbb{Z}^n$ of rank $n$ and the real vector space $V := \Lambda \otimes \mathbb{R} \cong \mathbb{R}^n$. The dual lattice is denoted by $(\Lambda)^\vee \cong \mathbb{Z}^n$ and the dual vector space by $(V)^\vee = (\mathbb{R}^n)^\vee$. Furthermore, we write $\mathbf{1}_n = (1,\dots,1) \in \mathbb{R}^n$ and $\mathbf{0}_n = (0,\dots,0) \in \mathbb{R}^n$. For a subset $B \subset [n]$ let $p_B$ denote the coordinate projection of $V$ onto the coordinates indexed by $B$. We call a finite set of points $\mathscr{A} = \{\alpha_1,\dots,\alpha_m\} \subset \mathbb{R}^n$ *point configuration*. A matrix $A \in \mathbb{R}^{n\times m}$ containing in column $i$ the coordinates of $\alpha_i$ is called *matrix representation of* $\mathscr{A}$. Let $\mathbb{K}$ denote a field. The $n$-dimensional affine space over $\mathbb{K}$ is denoted by $\mathbb{A}^n_{\mathbb{K}} = \mathbb{A}^n$, the $n$-dimensional projective space by $\mathbb{P}^n_{\mathbb{K}} = \mathbb{P}^n$ and $T^n_{\mathbb{K}} = T^n = (\mathbb{K}^*)^n$ denotes the torus. The corresponding coordinate rings are denoted by $S = \mathbb{K}[x_1,\dots,x_n]$, $R = \mathbb{K}[x_0,\dots,x_n]$ and $L = \mathbb{K}\left[x_1^\pm,\dots,x_n^\pm\right]$ respectively. Let $W \subset T^n$ be a set. We denote the *vanishing ideal* of $W$ by $\mathbf{I}(W) \subset L$. Conversely, if $I \subset L$ is an ideal we denote the *vanishing set* of $I$ by $\mathscr{V}(I)$. The vanishing ideal (and the vanishing set) is analogously defined for $S$, $R$ (and $\mathbb{A}^n$, $\mathbb{P}^n$). If nothing else is mentioned we fix the graded lexicographic term order $>_{\text{lex}}$. Let $\mathscr{M}_{n,d} \subset \mathbb{N}_0^n$ denote the set of vectors $\alpha \in \mathbb{N}_0^n$ in $n$ variables of length $d = |\alpha| = \sum_i \alpha_i$. This allows us to write degree $d$ monomials as $x^\alpha = x_1^{\alpha_1}\dots x_n^{\alpha_n}$ for $\alpha \in \mathscr{M}_{n,d}$. By abuse of notation we identify $\mathscr{M}_{n,d}$ with the set of monomials in $n$ variables of degree $d$. We adapt this notation for partial derivatives. We call $\frac{\partial^{|\alpha|}}{\partial x^\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}$ the $d$-fold derivative with respect to $\alpha \in \mathscr{M}_{n,d}$. We denote the cardinality of $\mathscr{M}_{n,d}$ by $N_n^d = |\mathscr{M}_{n,d}| = \binom{n-1+d}{d}$.

## 1.1. Polyhedral Geometry

First, we introduce well-known terminology of polyhedral geometry based on [**Zie95**, section 1+7]. We fix an integer $n \in \mathbb{N}$ and use Notation 1.1 throughout this section, i.e. $V = \mathbb{R}^n$ and $\Lambda = \mathbb{Z}^n$. The fundamental objects in this thesis are polyhedral complexes:

### 1.1.1. Polyhedral Complexes

We start with the definition of the integral part of polyhedral geometry:

**Definition 1.1.1.1** (Cones and polyhedrons)**.** A *polyhedron* is a subset $\sigma \subset V$ given by finitely many linear equalities and inequalities, i.e. it is a set of the form

$$\sigma = \{x \in V : a_i(x) \leq b_i,\ i \in [m]\}$$

for some $m \in \mathbb{N}$ and $a_i \in (V)^\vee,\ b_i \in \mathbb{R}$. A set $\sigma \subset V$ is called *cone* if we can describe it as a polyhedron with $b_i = 0$ for all $i \in [m]$.

This is the so-called $\mathscr{H}$-description of polyhedra. A bounded polyhedron is called *polytope*. If $\Gamma$ is a subgroup of $\mathbb{R}$ then we call a polyhedron $\sigma$ $\Gamma$-rational if all entries of $a_i$ and $b_i$ are from $\Gamma$ for all $i \in [m]$. We denote the affine span of $\sigma$ by $\operatorname{aff}(\sigma)$. Let $V_\sigma$ denote the smallest linear space parallel to $\operatorname{aff}(\sigma)$. Let $\Lambda_\sigma := V_\sigma \cap \Lambda$ denote its lattice. The dimension of $\sigma$ is defined by $\dim(\sigma) = \dim(V_\sigma)$. We call a 1-dimensional polyhedron *ray*.

**Remark 1.1.1.2** (Faces of polyhedra)**.** A *face* $\tau$ of a polyhedron $\sigma$ is a set of the form $\sigma \cap H_{a,b}$ with $H_{a,b} = \{x \in V : a(x) = b\}$ for some $a \in (V)^\vee$, $b \in \mathbb{R}$ such that $\sigma$ is contained in one of the halfspaces $H^{\mp}_{a,b} = \{x \in V : a(x) \gtreqless b\}$. By changing an inequality in the description of $\sigma$ to an equality we obtain a proper face. We write $\tau \leq \sigma$ if $\tau$ is a face of $\sigma$ (and $\tau < \sigma$ if $\tau$ is a proper face of $\sigma$). Faces of $\sigma$ of codimension one are called *facets*. If we change an inequality of $\sigma$ to an equality we restrict to the subset of elements of $\sigma$ that maximize this inequality. Hence, a face $\tau < \sigma$ of a polyhedron is the locus where an element $a \in (V)^\vee$ maximizes, i.e. $\tau = \operatorname{face}_w(\sigma) = \{x \in \sigma : w(x) \geq w(y) \forall y \in \sigma\}$ for some $w \in (V)^\vee$. The *relative interior* of $\sigma$ is defined by $\operatorname{relint}(\sigma) = \sigma \setminus \bigcup_{\tau<\sigma} \tau$ and the *boundary* is $\partial\sigma = \sigma \setminus \operatorname{relint}(\sigma)$.

**Remark 1.1.1.3** ($\mathscr{V}$-description)**.** A cone $\sigma \subset V$ can be described equivalently by a positive hull of a finite set of vectors, i.e. there is an element $r \in \mathbb{N}$ and $q_i \in V$ for all $i \in [r]$ such that

$$\sigma = \operatorname{cone}(q_1, \ldots, q_r) = \left\{ \sum_i \lambda_i q_i : \lambda_i \in \mathbb{R}_{\geq 0}, i \in [r] \right\}.$$

This is the $\mathscr{V}$-description for cones. However, polyhedra $\sigma \subset V$ likewise have a $\mathscr{V}$-description, namely as a *Minkowski sum*

$$\sigma = P + C = \{p + c : p \in P,\ c \in C\}$$

of a convex hull of points $P = \operatorname{conv}(p_1, \ldots, p_k)$ and a cone $C = \operatorname{cone}(q_1, \ldots, q_r)$. If a polyhedron $\sigma$ is bounded we have $C = \emptyset$ and $\sigma = P$ is a polytope. Then $\sigma$ is described by the convex hull of vertices and, therefore, this is the $\mathscr{V}$-description for polytopes. By $\operatorname{vert}(P)$ we denote the set of 0-dimensional faces of $P$ such that $P = \operatorname{conv}(\operatorname{vert}(P))$. Often (especially in Chapter 4) polytopes are considered up to *IUA-equivalence*, i.e. up to integral unimodular affine transformations. For more information about $\mathscr{V}$- and $\mathscr{H}$-descriptions (and how to pass from one to another) see [**Zie95**, section 1.1].

A $k$-dimensional cone $\sigma$ is called *simplicial* if $\sigma$ is generated by exactly $k$ vectors $v_1, \ldots, v_k$. In this case we obtain a facet of $\sigma$ by removing one generator $v_i$. The facet is again simplicial. Faces of simplicial cones of codimension $k$ can be obtained by removing a subset of $k$ generators.

**Example 1.1.1.4** (Types of simplices)**.** A polytope $P$ is called *simplex* if it is $n$-dimensional and the convex hull of $n+1$ points. The *standard simplex* $\Delta_n$ is the convex hull of the $n$ unit vectors $e_i$. By $d\cdot\Delta_n$ we denote the $d$-times *stretched standard simplex*, i.e. it is the convex hull of the $n$ vectors $d\cdot e_i$ for $i=1,\dots,n$. Note that $d\cdot\Delta_n$ is contained in the hyperplane $H_{\mathbf{1}_n,d}=\{x\in V:\langle x,(\mathbf{1}_n\rangle=d\}$. The *hypersimplex* is defined by $\Delta_n^d=\operatorname{conv}(\{x\in\{0,1\}^n:\langle\mathbf{1}_n,x\rangle=d\})\subset V$.

**Definition 1.1.1.5** (Polyhedral set)**.** A *polyhedral set* $X$ is a finite union of polyhedra, i.e. there is an element $r\in\mathbb{N}$ and polyhedra $\sigma_i\subset V$ for all $i\in[r]$ such that $X=\bigcup_{i\in[r]}\sigma_i$.

**Definition 1.1.1.6** (Fans and polyhedral complexes)**.** A *polyhedral complex* $\Sigma$ is a finite set of polyhedra satisfying the following conditions:

(1) if $\sigma\in\Sigma$ and $\tau\leq\sigma$ then $\tau\in\Sigma$ and
(2) if $\sigma,\sigma'\in\Sigma$ then $\tau=\sigma\cap\sigma'$ is common face.

A polyhedral complex $\Sigma$ is called *polyhedral fan* if all of its polyhedra $\sigma$ are cones.

The elements of a polyhedral complex $\Sigma$ are also called *cells*. We call $\Sigma$ $\Gamma$-rational if all of its polyhedra are $\Gamma$-rational. The dimension of a polyhedral complex $\Sigma\subset V$ is defined by $\max_{\sigma\in\Sigma}\dim(\sigma)$. The set of all $k$-dimensional polyhedra in $\Sigma$ is denoted by $\Sigma^{(k)}$. Top dimensional cells of $\Sigma$ are called *facets*. Codimension one faces of facets are called *ridges*. A polyhedral complex is called *pure* if all of its facets are equidimensional. We call the underlying polyhedral set $|\Sigma|=\bigcup_{\sigma\in\Sigma}\sigma$ of $\Sigma$ the *support* of $\Sigma$. Note that a polyhedral complex is uniquely defined by its facets. If $|\Sigma|=V$ we call $\Sigma$ *complete*. The biggest linear subspace $L\subset V$ that is contained in all polyhedra $\sigma\subset\Sigma$ is called *lineality space* of $\Sigma$.

**Definition 1.1.1.7** (Polyhedral structures)**.** Let $\Sigma$ be a polyhedral complex and $X=\bigcup_i\sigma_i$ a polyhedral set. If $|\Sigma|=X$ we say that $\Sigma$ gives $X$ a *polyhedral structure*. If we can pick a polyhedral fan $\Sigma$ such that $|\Sigma|=X$ than $X$ has a *fan structure*.

In Section 1.2 we associate a polyhedral fan to a matroid called *Bergman fan* (cf. Definition 1.2.2.11). Its various polyhedral structures play an important role in this thesis. We get to know to some existing (e.g. Section 1.2.2) that prove to be fundamental for our purposes in Chapter 2 (particularly in Section 2.1) and we even introduce a new structure ourself (cf. Section 2.2).

**Definition 1.1.1.8** (Refinements and intersections)**.** Let $\Sigma,\Sigma'$ be two polyhedral complexes.

- The *intersection* of $\Sigma$ and $\Sigma'$ is defined to be $\Sigma\cap\Sigma'=\{\sigma\cap\sigma'|\sigma\in\Sigma,\sigma'\in\Sigma'\}$.
- $\Sigma$ is called *refinement* of $\Sigma'$ if $|\Sigma|=|\Sigma'|$ and each cone $\sigma\in\Sigma$ is contained in a cone $\sigma'\in\Sigma'$.

If $\Sigma$ is a refinement of $\Sigma'$ every polyhedron of $\Sigma'$ is a union of polyhedra of $\Sigma$. If two polyhedral complexes have identical support their intersection is a common refinement, i.e. a refinement of both.

**Remark 1.1.1.9** (Intrinsic polyhedral structure)**.** Note that every polyhedral set $X$ offers an intrinsic polyhedral structure: take all the halfspaces $H^{\mp}_{a_i,b_i}=\{x\in V:a_i(x)\gtreqless b_i\}$ arising from the $\mathscr{H}$-description of a polyhedron $\sigma=\{x\in V:a_i(x)\leq b_i\}$ of $X$ (cf. Remark 1.1.1.2). Let $\mathscr{H}$ denote the polyhedral complex defined by the cones $H^+_{a_i,b_i}$, $H^-_{a_i,b_i}$ and $H^=_{a_i,b_i}$. The intersection of all $\mathscr{H}_j$ (arising from $\sigma_j$ of $X$) is a polyhedral complex $\mathscr{X}$. Any polyhedron $\sigma_j$ of $X$ is a union of cells of $\mathscr{X}$.

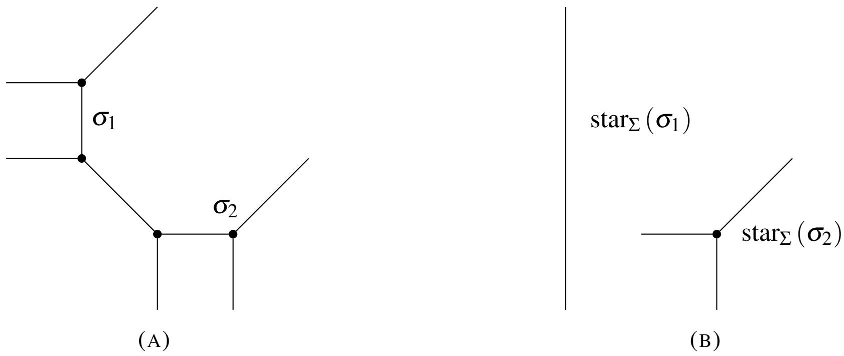



FIGURE 1. A polyhedral complex $\Sigma$, two cells $\sigma_1$, $\sigma_2$ and $\mathrm{star}_\Sigma(\sigma_1)$, $\mathrm{star}_\Sigma(\sigma_2)$.

At last, note that it can be useful to consider a polyhedral complex $\Sigma$ locally around a cell $\tau$. The local adjacencies of a cell $\tau$ in a polyhedral complex $\Sigma$ form a fan called *star* that is defined as follows:

**Definition 1.1.1.10** (Star)**.** Let $\Sigma$ be a polyhedral complex and $\tau \in \Sigma$ be a cell. For $\sigma \in \Sigma$ with $\tau \leq \sigma$ let $\overline{\sigma} = \{\lambda(x-y) : x \in \sigma, y \in \tau, \lambda \in \mathbb{R}_{\geq 0}\}$ denote the cone that is indexed by $\sigma \in \Sigma$. Then the *star* of $\tau$ in $\Sigma$ is the fan defined by

$$\mathrm{star}_\Sigma(\tau) = \{\overline{\sigma} : \sigma \in \Sigma \text{ and } \tau \leq \sigma\}.$$

### 1.1.2. The Balancing Condition

We introduce the balancing condition for polyhedral complexes. As we will see in Section 1.4, the tropicalization of any algebraic variety $V$ over $\mathbb{K}_\mathbb{C}$ (cf. Definition 1.4.1.2) is balanced.

**Definition 1.1.2.11** (Weighted polyhedral complex)**.** By assigning weights to all facets of a pure $k$-dimensional polyhedral complex in $\Sigma \subset V$ we obtain a *weighted polyhedral complex* $(\Sigma, \omega)$ where $\omega$ is the *weight function*:

$$\omega : \Sigma^{(k)} \longrightarrow \mathbb{Z}, \quad \sigma \longmapsto \omega(\sigma).$$

Let $\sigma \in \Sigma^{(k)}$ be a top dimensional cell, i.e. a facet, and let $\tau < \sigma$ be a ridge. By definition, there is a linear form $a \in (V)^\vee$ whose maximal locus on $\sigma$ is $\tau$ (cf. Remark 1.1.1.2). Then there is a unique generator $u_{\sigma/\tau}$ of $\Lambda_\sigma / \Lambda_\tau \cong \mathbb{Z}$ such that $a(u_{\sigma/\tau}) < 0$. We call $u_{\sigma/\tau}$ the *primitive normal vector* of $\sigma$ with respect to $\tau$. Note that $u_{\sigma/\tau}$ is called primitive since $\Lambda_\sigma / \Lambda_\tau = \mathbb{Z} u_{\sigma/\tau}$.

**Definition 1.1.2.12** (Balancing condition)**.** Let $(\Sigma, \omega)$ be a pure weighted polyhedral complex of dimension $k$. We call $(\Sigma, \omega)$ *balanced* if all ridges $\tau \in \Sigma^{(k-1)}$ satisfy the so-called *balancing condition*:

$$\sum_{\substack{\sigma \in \Sigma: \\ \tau < \sigma}} \omega(\sigma) u_{\sigma/\tau} = 0 \in V/V_\tau.$$

Notice that we can assign weight 0 to a facet $\sigma$ of a pure dimensional polyhedral complex $\Sigma$. With a view to the balancing condition only the non-zero part of $\mathscr{X} = (\Sigma, \omega)$ is essential. Therefore, the support of a weighted polyhedral complex $\mathscr{X} = (\Sigma, \omega)$ equals the polyhedral set provided by the facets with non-zero weight, i.e. $|\mathscr{X}| := \bigcup_{\sigma \in \Sigma^{(\dim(\Sigma))} : \omega(\sigma) \neq 0} \sigma$.

**Remark 1.1.2.13** (Weighted polyhedral structure)**.** Let $\mathscr{X} = (\Sigma, \omega)$ be a weighted balanced polyhedral complex and let $X$ be a polyhedral set. If we have $|\mathscr{X}| = X$ then we call $\mathscr{X}$ *weighted polyhedral structure* of $X$. Two weighted (balanced) polyhedral structures, $(\Sigma, \omega)$ and $(\Sigma', \omega')$, are called *equivalent* if they have same support and their weight functions $\omega$ and $\omega'$ coincide on the common refinement $\Sigma \cap \Sigma'$. Often, weighted balanced polyhedral complexes are considered up to equivalence. If nothing else is mentioned we stick to this convention.

**Remark 1.1.2.14** (Local balancing condition)**.** The local picture around a ridge $\tau \in \Sigma^{(\dim(\Sigma)-1)}$ of a pure balanced weighted polyhedral complex $(\Sigma, \omega)$ is given by $\operatorname{star}_\Sigma(\tau)$ that is a pure balanced weighted fan. In detail if we define weights for $\operatorname{star}_\Sigma(\tau)$ by $\omega'(\overline{\sigma}) = \omega(\sigma)$ for all $\overline{\sigma} \in \operatorname{star}_\Sigma(\tau)$ then $(\operatorname{star}_\Sigma(\tau), \omega')$ is a pure balanced weighted polyhedral fan. Moreover, a pure polyhedral complex $(\Sigma, \omega)$ is balanced if and only if $\operatorname{star}_\Sigma(\tau)$ is balanced for all $\tau \in \Sigma^{(\dim(\Sigma)-1)}$.

**Remark 1.1.2.15** (Sums of weighted balanced polyhedral complexes)**.** It can be shown that weighted balanced polyhedral complexes in $V$ (in particular, in any weighted balanced polyhedral complex) form an abelian group (for details see e.g. [**Rau09**]). We skip details and restrict to the basic ideas. Let $\mathscr{X} = (\Sigma, \omega)$ and $\mathscr{X}' = (\Sigma', \omega')$ be two weighted polyhedral complexes. Intuitively, we tend to define the weighted polyhedral complex $\mathscr{X} + \mathscr{X}'$ called *sum* by taking the union $|\mathscr{X}| \cup |\mathscr{X}'|$ as underlying polyhedral set and $\omega + \omega'$ (properly defined on $|\mathscr{X}| \cup |\mathscr{X}'|$) as weight function. As $|\mathscr{X}| \cup |\mathscr{X}'|$ is a polyhedral set, it is not necessarily a polyhedral complex. However, one can achieve a polyhedral structure on $|X| \cup |X'|$ (using methods sketched in Remark 1.1.1.9) with weight function $\omega + \omega'$.

### 1.1.3. Normal Fans

Now, we associate a complete fan called *normal fan* to a polytope. For more information see [**Zie95**, chapter 7, section 1]. This concept is crucial for tropical hypersurfaces (cf. Proposition 1.4.2.9). Let $P$ be a polytope and $F < P$ a face. We define the *normal cone* of $F$ in $P$ by

$$N_{F,P} = \overline{\left\{w \in (V)^\vee : F = \operatorname{face}_w(P)\right\}}.$$

If $w \in N_{F,P}$ then elements $x \in F$ satisfy $w(x) \geq w(y)$ for all $y \in P$, i.e. $w$ “maximizes” on $F$. The following statement is an immediate consequence:

**Corollary 1.1.3.16.** *Let $P$ be a full-dimensional polytope and $F < P$ a face. Then,*

$$N_{F,P} = \operatorname{cone}\left(u_Q : Q < P \textit{ is a facet and } F \leq Q\right),$$

*where $u_Q$ denotes the unique normal vector of the facet $Q \subset P$ maximizing on $Q$.*

Essentially, normal cones fit together to a complete fan:

**Definition 1.1.3.17.** The *normal fan* $\mathscr{N}_P$ of a polytope $P \subset V$ is the collection of cones $N_{F,P}$ where $F$ varies over the faces of $P$.

In the literature a normal fan of a polytope defined this way is also called “outer” normal fan. This is due to the choice “maximizing”. If $P$ is not full-dimensional then all $N_{F,P}$ contain the lineality space $L$ orthogonal to $V_P$. Often, we consider $\mathscr{N}_P \big/ L$. There is a dimension-reversal correspondence between a polytope $P \subset V$ and its normal fan $\mathscr{N}_P$. A $k$-dimensional face $F$ of a $n$-dimensional polytope $P$ is dual to a $(n-k)$-dimensional normal cone of the normal fan $\mathscr{N}_P$.

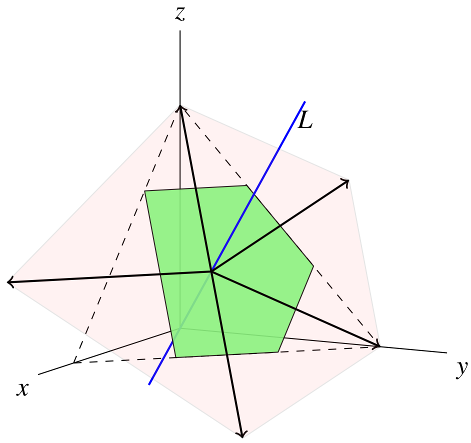



FIGURE 2. Normal fan $\mathscr{N}_P$ of $P$ (green colored) with lineality space $L$ (blue colored).

**Example 1.1.3.18.** Let $P = \mathrm{conv}((2,1,0),(1,0,2),(0,1,2),(0,2,1),(1,2,0)) \subset \mathbb{R}^3$ be a polytope contained in the plane $H_{\mathbf{1}_3,3}$. The polytope is sketched green in Figure 2. The normal fan $\mathscr{N}_P$ is complete and contains a lineality space $L = \langle \mathbf{1}_3 \rangle \subset \mathbb{R}^3$ drawn in blue. Therefore, we portrayed the normal fan in $L^\perp$ by its cone generators that span 2-dimensional cones shaded lightly red.

### 1.1.4. Subdivisions and Secondary Fans

At last, we take a look at subdivisions of polytopes arising from point configurations and then repeat some definitions concerning the secondary fan. The standard reference concerning this topic is [**GfKZ94**]. Let us start with subdivisions. We summarize basics of ([**GfKZ94**, chapter 7, section 2, A.]). A *marked polytope* is a tuple $(P,\mathscr{A})$ consisting of a polytope $P \subset V$ and a subset of the lattice points $\mathscr{A} \subset P \cap \Lambda$ containing $\mathrm{vert}(P)$, i.e. $P = \mathrm{conv}(\mathscr{A})$.

**Definition 1.1.4.19** (Marked subdivision)**.** Let $I \subset \mathbb{N}$ be an index set. A *marked subdivision* of a polytope $P \subset V$ is a collection of marked polytopes $S = \{(P_i,\mathscr{A}_i) : i \in I\}$ such that

- $P = \bigcup_{i\in I} P_i$,
- $P_i \cap P_j$ is a face of both for all $i \in I$,
- $\mathscr{A}_i \subset P \cap \Lambda$ for all $i \in I$, and
- $\mathscr{A}_i \cap (P_i \cap P_j) = \mathscr{A}_j \cap (P_i \cap P_j)$ for all $i \in I$.

We call the collection of all $P_i$ without the markings $\mathscr{A}_i$ the *type* of a marked subdivision $S$ and we write $\partial S = \partial P$ for the boundary of the subdivision.

It is not required that $\bigcup_i \mathscr{A}_i = P \cap \Lambda$. Sometimes we start with a point configuration $\mathscr{A} \subset \Lambda$ and consider the polytope $P = \mathrm{conv}(\mathscr{A})$. If $S = \{(P_i,\mathscr{A}_i) : i \in I\}$ is a subdivision of $P$ we also like to know which points of $\mathscr{A} \subset P$ are contained in the $\mathscr{A}_i$ of the subdivision $S$. To be more precise we distinguish between *marked points* and *unmarked points* of $\Lambda$. We call $\alpha_i \in \mathscr{A}$ *marked* if there is a $j \in I$ such that $\alpha_i \in \mathscr{A}_j$ and *unmarked* otherwise ([**MMS12a**, section 2]). It is possible to obtain a subdivision in a very geometric manner (e.g. [**GfKZ94**, chapter 7, section 2, example 2.2]):

**Definition 1.1.4.20** (Regular subdivision)**.** Let $P \subset V$ be a full-dimensional polytope and $\mathscr{A} \subset P \cap \Lambda$ a subset of the lattice points of $P$ containing $\mathrm{vert}(P)$, i.e. $P = \mathrm{conv}(\mathscr{A})$. For $w \in \mathbb{R}^{|\mathscr{A}|}$ consider

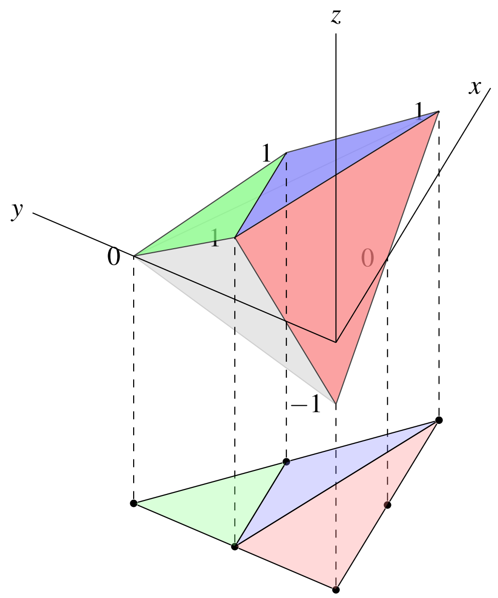



FIGURE 3. A polytope $P = \operatorname{conv}((0,0),(1,0),(0,1),(2,0),(1,1),(0,2))$, shifted according to $w = (-1,0,1,1,1,0)$. The regular subdivision $S_w$ of $P$ induced by $w$ is sketched below.

the polytope $P_w = \operatorname{conv}((\alpha_1, w_1), \ldots, (\alpha_k, w_k)) \subset V \times \mathbb{R}$. We call $P_w$ *shifted polytope* or *shift* of $P$ induced by $w$. The *upper facets* of $P_w$ are facets $F \subset P_w$ whose corresponding normal vector $u_F$ is a generator of $N_{F,P_w}$ with positive last coordinate. We denote the set of all upper facets by $P_w^+$. The coordinate projection $p_{[n]}(P_w^+)$ of the upper facets induces a marked subdivision denoted by $S_w = \{(P_i, \mathscr{A}_i) : i \in [m]\}$ with $\bigcup_i P_i = P$ and we call it *regular subdivision* of $P$ induced by $w$. An element $\alpha_i \in \mathscr{A}$ is marked if $(\alpha_i, w_i)$ is contained in one of the upper facets of $P_w$.

**Example 1.1.4.21.** Consider the 2 dimensional polygon $P = \operatorname{conv}(\mathscr{A})$ obtained from

$$\mathscr{A} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\} \subset \mathbb{Z}^2.$$

We lift by the vector $w = (-1,0,1,1,1,0) \in \mathbb{R}^6$. The polytope $P_w$ is sketched in Figure 3. The upper facets $P_w^+$ are colorized whereas all other facets are grey. The coordinate projections $p_{[2]}(P_w^+) \subset \mathbb{R}^2$ are sketched below. All points of $\mathscr{A} = \bigcup_i \mathscr{A}_i$ are marked since their lifts are contained in the upper facets (see dashed lines).

If all polytopes of a regular subdivision are simplicial we call the regular subdivision a *regular triangulation*. Two elements $w, w' \in \mathbb{R}^{|\mathscr{A}|}$ are equivalent if and only if they induce the same regular marked subdivision of $P = \operatorname{conv}(\mathscr{A})$, i.e.

$$w \sim w' \quad \Longleftrightarrow \quad S_w = S_{w'}.$$

The equivalence class defined by $w \in \mathbb{R}^{|\mathscr{A}|}$ carries a cone structure. Therefore, we denote the equivalence class of $w$ by $\sigma(w) \subset \mathbb{R}^{|\mathscr{A}|}$. Since any $w \in \mathbb{R}^{|\mathscr{A}|}$ provides a regular marked subdivision we conclude:

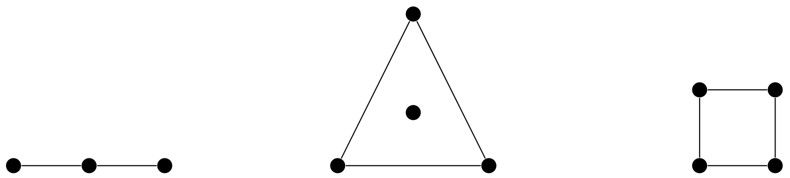



FIGURE 4. Circuits in the plane $\mathbb{R}^2$.

**Definition 1.1.4.22** (Secondary fan)**.** Let $\mathscr{A} \subset \Lambda$ be a finite set and $P = \operatorname{conv}(\mathscr{A})$ be a polytope. The polyhedral set consisting of cones $\sigma(w)$ where $w \in \mathbb{R}^{|\mathscr{A}|}$ varies is a complete fan $\mathrm{Sec}_{\mathscr{A}}$ called *secondary fan* of $P$.

There is a helpful result stating that the dimension of $\sigma(w) \subset \mathrm{Sec}_{\mathscr{A}}$ is determined by $S_{w'}$ with $w' \in \mathrm{relint}(\sigma(w))$. To see this let

$$L_{\mathscr{A}} = \left\{ w \in \mathbb{R}^{|\mathscr{A}|} : \sum_i w_i \alpha_i = 0,\ \sum_i w_i = 0 \right\}$$

denote the space of affine relations among the elements of $\mathscr{A}$. By $L_{\mathscr{A}_j}$ we restrict to affine relations among $\mathscr{A}_j$, i.e.

$$L_{\mathscr{A}_j} = \left\{ w \in L_{\mathscr{A}} : w_i = 0\ \forall i \text{ with } \alpha_i \notin \mathscr{A}_j \right\}.$$

Let $L_S$ denote the sum $\sum_i L_{\mathscr{A}_i}$.

**Lemma 1.1.4.23** ([**GfKZ94**, Corollary 2.7])**.** *Let $\mathscr{A} \subset \Lambda$ be a finite set and $P = \operatorname{conv}(\mathscr{A})$ the associated polytope. Let $S_w = \{(P_i, \mathscr{A}_i) : i \in I\}$ be a regular subdivision of $P$ induced by $w \in \mathbb{R}^{|\mathscr{A}|}$. Then* $\operatorname{codim}(\sigma(w)) = \dim(L_{S_w})$. *In other words, the codimension of a cone in the secondary fan equals the dimension of the space of affine relations among the vertices of marked polytopes forming the regular subdivision that corresponds to the cone in the secondary fan.*

Hence, top-dimensional cones in the secondary fan correspond to regular triangulations where each simplex of the triangulation contains only the vertices as marked points. Codimension one cones correspond to regular subdivisions with exactly one marked polytope that contains a *circuit* ([**GfKZ94**, chapter 7, section 1, B.]):

**Definition 1.1.4.24** (Circuit)**.** A finite collection of points $Z \subset V$ is called *circuit* if $Z$ is affinely dependent and any proper subset is affinely independent.

This definition has its origin in matroid theory (see Section 1.2 and Section 1.3, particularly vector matroids). A circuit can be obtained by a simplex and one additional point that is either contained in the interior of the simplex or not. See Figure 4 for a classification of circuits in the plane $\mathbb{R}^2$.

## 1.2. Matroids and Bergman Fans

Matroid theory abstracts and generalizes the notion of linear independency and is heavily influenced by graph theory and linear algebra. Essentially, matroids capture combinatorial data and, therefore, arise in several fields in mathematics, e.g. tropical geometry ([**Stu02**], [**Ham15**]), combinatorial optimization ([**KV12**]), cryptography ([**BD90**]). The comprehensive reference for matroid theory is [**Oxl11**]. In Section 1.2.1 we briefly summarize basic terminology of matroid theory. There are several (but equivalent) ways to associate a fan to a matroid. This is the subject of Section 1.2.2.

### 1.2.1. Matroids

As defined in Notation 1.1, let $[n] = \{1,\ldots,n\}$ denote the natural numbers from 1 to $n$. We begin with the definition of a matroid ([**Oxl11**, section 1.1]):

**Definition 1.2.1.1** (Matroid)**.** A matroid $M$ is an ordered pair $([n], \mathscr{I})$ consisting of the *ground set* $[n]$ and a collection $\mathscr{I}$ of subsets of $[n]$ called *independent sets* satisfying

(I1) $\emptyset \in \mathscr{I}$
(I2) If $I \in \mathscr{I}$ and $I' \subseteq I$ then $I' \in \mathscr{I}$.
(I3) If $I_1, I_2 \in \mathscr{I}$ and $|I_1| < |I_2|$ then there is an element $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\} \in \mathscr{I}$.

Inclusion-maximal independent sets $I \in \mathscr{I}$ are called *bases*. We denote the collection of bases by $\mathscr{B}$. We call a dependent subset $C \subseteq [n]$ *circuit* if all of its proper subsets are independent. We denote the collection of circuits by $\mathscr{C}$. There are axiomatic systems for both, bases and circuits, such that a matroid $M = ([n], \mathscr{I})$ can be defined equivalently by its bases $\mathscr{B}$ ([**Oxl11**, section 1.2]) or circuits $\mathscr{C}$ ([**Oxl11**, section 1.1]). One benefit of bases is efficiency: it is easier to list all maximal independent sets instead of all independent sets. We switch descriptions in the following whenever it simplifies matters. Note that the matroid-theoretical definition of a circuit coincides with the definition of circuits in affine space (cf. Definition 1.1.4.24). It is possible to construct matroids from other matroids, e.g. by restriction:

**Definition 1.2.1.2** (Restricted matroid, [**Oxl11**, section 1.3])**.** Let $M = ([n], \mathscr{I})$ be a matroid defined by its independent sets $\mathscr{I}$ and $X \subseteq [n]$ a subset. Let $\mathscr{I}\big|_X = \{I \subseteq X : I \in \mathscr{I}\}$ denote the set of independent sets contained in $X$. Then $M\big|_X = \left(X, \mathscr{I}\big|_X\right)$ is a matroid called *restriction of $M$ to $X$*.

The following definitions can be found in [**Oxl11**, section 1.3 + 1.4].

**Definition 1.2.1.3** (Rank, closure and flats)**.** Let $M = ([n], \mathscr{B})$ be a matroid and $B \subset [n]$ a set. The *rank* of $B$ is defined by the size of a basis of $M\big|_B$. For a subset $X \subseteq [n]$ we define its *closure* by $\mathrm{cl}(X) = \{x \in [n] : \mathrm{rank}(X \cup \{x\}) = \mathrm{rank}(X)\}$. A *flat* is a subset $F \subseteq [n]$ satisfying $F = \mathrm{cl}(F)$.

Since all bases $B \in \mathscr{B}$ of a matroid $M$ are equicardinal ([**Oxl11**, Lemma 1.2.1]) the rank is well-defined. From the bases $\mathscr{B}$ of $M$ we get another matroid:

**Definition 1.2.1.4** (Dual matroid)**.** Let $M = ([n], \mathscr{B})$ be a matroid defined by a collection of $\mathscr{B}$. Let $\mathscr{B}' = \{[n] \setminus B : B \in \mathscr{B}\}$ be a collection ob subsets of $[n]$. Then $M' = ([n], \mathscr{B}')$ is called *dual matroid*.

See [**Oxl11**, Theorem 2.1.1] for a proof that the dual matroid $M'$ of $M$ is indeed a matroid.

**Remark 1.2.1.5** (Lattice of flats)**.** The flats of a matroid $M$ form a *lattice* $\mathscr{L}$, i.e. the set of flats is a partially ordered set (by inclusion) and two flats $X, Y$ of $M$ have a least upper bound $X \vee Y = \mathrm{cl}(X \cup Y)$ called *join* and a greatest lower bound $X \wedge Y = X \cap Y$ ([**Oxl11**, Lemma 1.7.3]) called *meet*. A *sublattice* is a subset of $\mathscr{L}$ that satisfies the lattice axioms and inherits the join- and meet-operations from $\mathscr{L}$.

Most of the matroids appearing in this thesis arise from a point configuration. Therefore, *vector matroids* obtained from point configurations are our illustrative model.

**Remark 1.2.1.6** (Matroids from point configurations)**.** Let $\mathscr{A} = \{a_1, \dots, a_m\} \subset \mathbb{R}^n$ be a point configuration spanning $\mathbb{R}^n$ and let $A \in \mathbb{R}^{n \times m}$ be the matrix representation of $\mathscr{A}$. We associate a matroid to $A$ as follows: let $E = [m]$ be the set of column labels of $A$. For $I_r = \{i_1, \dots, i_r\} \subseteq E$ let $A_{I_r} = \begin{bmatrix} \alpha_{i_1} & \cdots & \alpha_{i_r} \end{bmatrix}$ be the matrix containing the columns of $A$ indexed by $I_r$. Let $\mathscr{I}$ be the collection of subsets $I \subseteq E$ such that $\operatorname{rank}(A_{I_r}) = r$. Then $M[A] = (E, \mathscr{I})$ is a matroid ([**Oxl11**, Proposition 1.1.1]) called *vector matroid* whose collection of independent sets is $\mathscr{I}$. Its circuits are elements of $\ker(A)$ with minimal support. This way we can associate matroids to linear spaces. If $\mathbb{K}$ is a field and $M$ a matroid and $A \in \mathbb{K}^{m \times n}$ a matrix such that $M = M[A]$ then we call $M$ *realizable*.

**Example 1.2.1.7.** Consider the point configuration $\mathscr{A} = \{\alpha_1, \dots, \alpha_6\} \subset \mathbb{R}^3$ forming the columns of the matrix $A$ in Equation (1):

$$A = \begin{bmatrix} 1 & 0 & 0 & -2 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}. \tag{1}$$

The ground set of $M[A]$ is $E = [6]$ and the bases $\mathscr{B}$ of $M[A]$ are the bases of $\mathbb{R}^3$ consisting of elements $\alpha_i \in \mathscr{A}$, i.e. in our example we have

$$\mathscr{B} = \{B \subseteq E : |B| = 3\} \setminus \big\{\{1,2,4\}, \{1,2,6\}, \{1,3,4\}, \{1,4,5\}, \{1,4,6\}, \{2,4,6\}\big\}.$$

Hence, $\operatorname{rank}(M[A]) = |B| = 3$.

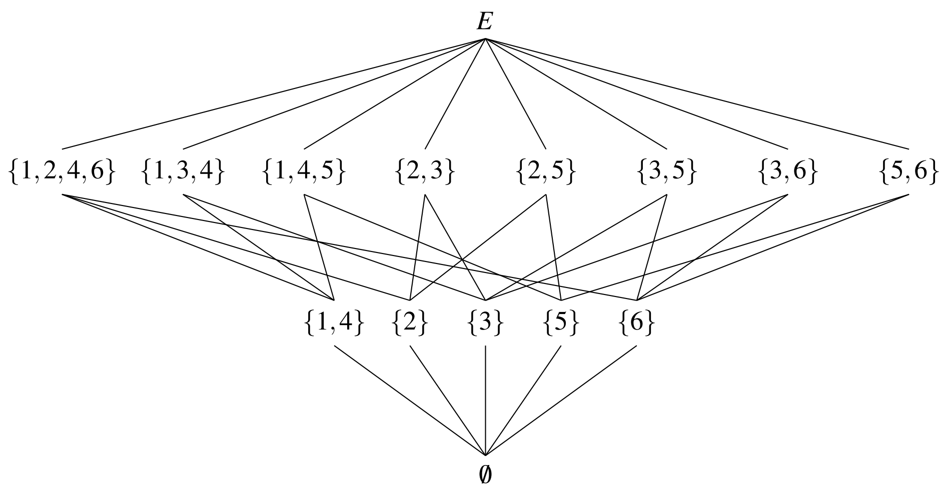



FIGURE 5. Lattice of flats of $M$, cf. Example 1.2.1.7.

**Definition 1.2.1.8** (Matroid polytope)**.** Let $A \subseteq [n]$ be a set. We call $e_A = \sum_{i \in A} e_i$ the *incidence vector of* $A$. For a matroid $M$ of rank $k$ on the ground set $[n]$ with bases $\mathscr{B}$ we call the convex hull $P_M := \operatorname{conv}(e_B : B \in \mathscr{B}) \subset V$ *matroid polytope*.

The incidence vector $e_B$ of a basis $B$ of a matroid $M$ of rank $k$ has $k$ ones and $n-k$ zeros. Thus, the matroid polytope $P_M$ of $M$ is contained in the hypersimplex $\Delta_n^k \subset \mathbb{R}^n$. Also note that every edge $e$ of $P_M$ is a parallel translate of $e_i - e_j$ for some $i, j \in [n]$. The incident vertices of $e$ correspond to bases $B, B' \in \mathscr{B}$ that satisfy the "basis exchange axiom": if $B, B' \in \mathscr{B}$ and $x \in B \setminus B'$ then there exists $y \in B' \setminus B$ such that $(B \setminus \{x\}) \cup \{y\} \in \mathscr{B}$ (see [**Oxl11**, Lemma 1.2.2]). Indeed, this even characterizes matroids and matroid polytopes likewise:

THEOREM 1.2.1.9 ([**GGMS87**, Theorem 4.1]). *Let $\mathscr{B}$ be a collection of k-subsets $B = \{b_1, \ldots, b_k\}$ of $[n]$ and let $P$ be the convex hull of incidence vectors* $\operatorname{conv}(e_B : B \in \mathscr{B})$. *Then $M = (\mathscr{B}, [n])$ is a matroid with bases $\mathscr{B}$ if and only if every edge of $P$ is parallel to an edge $e_i - e_j$ for some $i, j \in [n]$.*

### 1.2.2. Bergman Fans

We follow ([**AK06**], [**FS05**], [**MS15**]) and introduce *Bergman fans* associated to matroids. Let $M$ be a matroid of rank $k$ on the ground set $[n]$. An element $w \in \mathbb{R}^n$ can be considered as a weight function on the ground set $[n]$. For any circuit $C \in \mathscr{C}$ we call $\operatorname{in}_w(C) = \{i \in C : w_i = \max_j \{w_j : j \in C\}\}$ the *initial circuit* of $C$ with respect to $w$.

**Definition 1.2.2.10.** Let $M = (\mathscr{C}, [n])$ be a matroid with circuits $\mathscr{C}$ and $w \in \mathbb{R}^n$. Let $\operatorname{in}_w(\mathscr{C})$ denote the collection of inclusion-minimal initial circuits of $M$. Then $M_w = (\operatorname{in}_w(\mathscr{C}), [n])$ is called *initial matroid*.

We associate a fan to $M$ as follows:

**Definition 1.2.2.11** (Bergman fan)**.** Let $M$ be a matroid on the ground set $[n]$ of rank $k$. An element $a \in [n]$ is called *loop* if it is not contained in any basis of $M$. We call

$$B(M) = \{w \in \mathbb{R}^n : M_w \text{ contains no loop}\}$$

the *Bergman fan* (or *matroid fan*) associated to $M$.

**Remark 1.2.2.12.** In Section 1.4.5 we associate a matroid $M$ to a linear ideal $I \subset \mathbb{C}\left[x_1^{\pm}, \ldots, x_n^{\pm}\right]$. In this sense, the initial matroid $M_w$ of $M$ is the associated matroid to the initial ideal $\operatorname{in}_w(I)$ of $I$ with respect to $w \in \mathbb{R}^n$ (see Section 1.4, in particular Definition 1.4.3.14). In this context note that a loop can be defined equivalently as a 1-element circuit.

We briefly justify the nomenclature "fan". For $w \in \mathbb{R}^n$ let $P_{M'}$ denote the face of $P_M$ minimized by $w$. A face of $P_M$ is a matroid polytope (Theorem 1.2.1.9). Hence, $M'$ is a matroid and $P_{M'}$ is the convex hull of vertices of $P_M$ minimized by $w$. These vertices correspond to the bases of $M_w$ ([**MS15**, Proposition 4.2.10]) and, therefore, $M' = M_w$ is a matroid according to Definition 1.2.1.1. The loop-freeness of $M_w$ translates to the condition that the union of bases corresponding to the vertices of $P_{M_w}$ equals the ground set $[n]$. Hence, $B(M)$ is a subfan of the inverted normal fan of $P_M$:

**Corollary 1.2.2.13.** *Let $M$ be a matroid of rank $k$ on $[n]$ and $\mathscr{B}$ its collection of bases. Then:*

$$B(M) = -\left\{ \sigma \in \mathscr{N}_{P_M}^{(k)} : \bigcup_{\substack{B \in \mathscr{B}: \\ e_B \in P(\sigma)}} B = [n] \right\}$$

*where $P(\sigma)$ denotes the face of $P_M$ dual to $\sigma$.*

Hence, $B(M)$ is a polyhedral set that obtains a polyhedral fan structure from the normal fan of $P_M$. Moreover, we see that $B(M)$ is pure-dimensional and inherits its dimension from the rank of $M$ (this follows from other descriptions of $B(M)$, e.g. Theorem 1.2.3.17). Since $M_w$ contains the bases that are minimized by $w$, we can define $M_w$ equivalently by the bases $B \in \mathscr{B}$ with minimal $w$-weight $\sum_{i \in B} w_i$. We denote the cone corresponding to an initial matroid $M_w$ by $\sigma(M_w)$.

### 1.2.3. Fine and Coarse Subdivision of Bergman Fans

There are several possible fan structures on a Bergman fan $B(M)$ (e.g. [**FS05**], [**Rin13**]). Two polyhedral fan structures of Bergman fans $B(M)$ are well-known and of significant importance in this thesis: the *coarse* and *fine subdivision* (cf. [**AK06**]). First, we focus on the fine subdivision given by *weight classes*. We use weight classes frequently in this thesis.

**Definition 1.2.3.14** (Flag of subsets)**.** Consider the ground set $[n]$ for some $n \in \mathbb{N}$ and fix an element $w \in \mathbb{R}^n$. The *flag of subsets* of $w$, denoted by $\mathscr{F}(w) = (F_1, \dots, F_k)$, is a chain of subsets

$$F_0 = \emptyset \subsetneq F_1 \subsetneq \ldots \subsetneq F_k = [n]$$

such that $w$ is constant on $F_i \setminus F_{i-1}$ and $w\big|_{F_i \setminus F_{i-1}} < w\big|_{F_{i+1} \setminus F_i}$ for all $i$. We call $k$ the *length* of $\mathscr{F}$. The *weight class* of a flag of subsets $\mathscr{F} = \mathscr{F}(w)$ is the set $\sigma_{\mathscr{F}} = \{w' \in \mathbb{R}^n : \mathscr{F}(w') = \mathscr{F}\}$.

If $\mathscr{F}$ is a flag of flats of $M$ we denote it by $\mathscr{F} \lhd M$. Note that the length is bounded by the rank of $M$.

**Example 1.2.3.15.** We continue Example 1.2.1.7. The lattice of flats of $M[A]$ is shown in Figure 5. We consider a part of it by exemplary specifying a flag of flats. As a start let $F_1 = \mathrm{cl}(\{1\}) = \{1,4\}$ be the first non-trivial flat of rank one. We obtain a flat of dimension two by adding an element outside of $F_1$, e.g. $F_2 = \mathrm{cl}(F_1 \cup \{2\}) = \{1,2,4,6\}$. Since $\mathrm{rank}(M[A]) = 3$ there is a unique flat $F_3 = E$ of rank 3. The resulting flag of flats is

$$\emptyset \subset F_1 = \{1,4\} \subset F_2 = \{1,2,4,6\} \subset E.$$

**Remark 1.2.3.16** ($\mathscr{V}$-description of weight classes)**.** Let $\mathscr{F} = \mathscr{F}(w)$ denote the flag of sets of $w \in \mathbb{R}^n$ of length $k$ on $[n]$. The $\mathscr{V}$-description of the weight class $\sigma_{\mathscr{F}}$ equals the cone over the incident vectors of flats of $\mathscr{F}$, i.e.

$$\sigma_{\mathscr{F}} = \{\lambda_1(-e_{F_1}) + \ldots + \lambda_k(-e_{F_k}) : \lambda_i \in \mathbb{R}_{\geq 0} \ \forall i\}$$

with $e_{F_i} = \sum_{j \in F_i} e_j$ for $F_i \in \mathscr{F}$.

THEOREM 1.2.3.17 ([**AK06**, Theorem 1]). *Let $M$ be a matroid of rank $k$ on the ground set $[n]$. Then $B(M)$ is the union of all weight classes of flag of flats of $M$, i.e.*

$$B(M) = \bigcup_{\mathscr{F} : \mathscr{F} \lhd M} \sigma_{\mathscr{F}}$$

With Corollary 1.2.2.13 and Theorem 1.2.3.17 we made the acquaintance of two fan subdivisions.

**Definition 1.2.3.18** ([**AK06**])**.** Let $M$ be a matroid on $[n]$. Let $\mathscr{F} \lhd M$ be a flag of flats and $w \in \sigma_{\mathscr{F}}$.

- The *fine subdivision* of $B(M)$ is given by weight classes $\sigma_{\mathscr{F}}$. A vector $w' \in \mathbb{R}^n$ belongs to the weight class $\sigma_{\mathscr{F}}$ if and only if $\mathscr{F} = \mathscr{F}(w')$.

- The *coarse subdivision* of $B(M)$ is given by $M_w$-equivalence classes. A vectors $w' \in \mathbb{R}^n$ belongs to the cone $\sigma(M_w)$ if and only if $M_w = M_{w'}$.

The coarse subdivision is the coarsest possible fan structure of $B(M)$ ([**Ham14a**, Proposition 3.4.1]). In the following we briefly describe the relationship of the fine and coarse subdivision of $B(M)$.

**Remark 1.2.3.19** (Greedy algorithm, [**Oxl11, AK06**]). Let $\sigma_{\mathscr{F}}$ be a weight class defined by a flag of flats $\mathscr{F} = (F_1, \dots, F_k) \lhd M$ of a matroid $M$. Given an element $w \in \sigma_{\mathscr{F}}$ we can compute the $w$-minimal bases defining $M_w$ using a *Greedy algorithm*: define $B = \emptyset$. Then for $i = 1$ up to $k$ pick $\operatorname{rank}(F_i) - \operatorname{rank}(F_{i-1})$ elements of $F_i \setminus F_{i-1}$ until $B$ is saturated to a basis of $F_i$. The collection of all possible outputs of the Greedy algorithm applied to $\mathscr{F}$ equals the set of $w$-minimal bases $B$ of $M_w$.

Note that $M_w$ only depends on the flag $\mathscr{F} = \mathscr{F}(w)$, i.e. we can also write $M_w = M_{\mathscr{F}}$. We call two flags $\mathscr{F} = (F_i)_{i\in I}$, $\mathscr{F}' = (F_i')_{i\in I'} \lhd M$ *adjacent* if they differ only in one rank, i.e. $I' = I$ and there exists $k \in I$ such that $F_j = F_j'$ for all $j \neq k$ and $F_k \neq F_k'$. A *diamond poset* is a lattice of rank two with unique top and bottom elements and two rank one elements. The weight classes forming a cone $\sigma(M_{\mathscr{F}})$ are characterized as follows:

THEOREM 1.2.3.20 ([**AK06**, Theorem 2]). *Let $M$ be a matroid and $\mathscr{F}, \mathscr{F}' \lhd M$ two adjacent flags that differ only in rank $i$. Then the following statement are equivalent:*

$$M_{\mathscr{F}} = M_{\mathscr{F}'} \quad \Leftrightarrow \quad M_{\mathscr{F}} = M_{\mathscr{F}-F_i} \quad \Leftrightarrow \quad F_i \cup F_i' = F_{i+1} \quad \Leftrightarrow \quad [F_{i-1}, F_{i+1}] \text{ is a diamond poset.}$$

Thus $\sigma(M_w)$ is refined by all weight classes $\sigma_{\mathscr{F}}$ satisfying $M_{\mathscr{F}(w)} = M_{\mathscr{F}}$. At last we mention another description of $B(M)$ that is quite natural for tropical geometry (cf. Definition 1.4.2.6).

**Proposition 1.2.3.21** ([**FS05**, Proposition 2.5]). *Let $M$ be a matroid of rank $k$ on the ground set $[n]$ and $\mathscr{C}$ its collection of circuits. Then:*

$$B(M) = \{w \in \mathbb{R}^n : \text{ the maximum } \max_{i\in C}\{w_i\} \text{ is attained at least twice } \forall C \in \mathscr{C}\}.$$

See [**Rin13**] for more information about the fan subdivision induced by the circuits of a matroid.

## 1.3. Oriented Matroids

*Oriented matroids* arise as well as matroids from combinatorial data but keep track of further additional information called *orientation* (also referred as *signs*). For instance, a digraph provides an oriented matroid based on the *signed circuits* of the graph: a signed circuit records, beside the arcs themselves forming the undirected circuit, the orientation of each arc in the directed circuit. In case of a vector matroid we may retain the signs of circuits. However, the last case requires that we work over an ordered field.

Among the several various possibilities to introduce oriented matroids, we focus on oriented vector matroids emerging from point configurations. We adapt and enlarge the terminology introduced in Section 1.2 and follow the standard reference [**BLVS$^+$99**]. Throughout this section we fix the ground set $E = [n]$ for some $n \in \mathbb{N}$.

### 1.3.1. Oriented Matroids

Before we start with basics of oriented matroids note the following:

**Convention 1.3.1.1.** In contrast to unoriented matroids (Section 1.2) we have to work over an ordered field. If nothing else is mentioned we work over the field $\mathbb{R}$ of real numbers.

In order to define oriented matroids we need some terminology concerning *signed sets*:

**Definition 1.3.1.2** (Signed sets)**.** A *signed set* $X$ of $E$ is a subset $\underline{X} \subseteq E$ with a partition $(X^+, X^-)$ of $\underline{X}$ where $X^+$ is the set of *positive elements* of $X$ and $X^-$ is the set of *negative elements* of $X$. Hence, $\underline{X} = X^+ \cup X^-$ is the *support* of $X$ and $|X|$ denotes the cardinality of $|\underline{X}|$.

Additionally, we define $X^0 = E \setminus \underline{X}$. We notice that if $X$ is a signed set then $-X$ is also a signed set via $(-X)^+ = X^-$ and $(-X)^- = X^+$. By convention, we write $i$ for $i \in X^+$ and $\bar{i}$ for $i \in X^-$. Now, we define an *oriented matroid*:

**Definition 1.3.1.3** (Oriented matroid)**.** An ordered pair $(E, \mathscr{C})$ consisting of a ground set $E$ and a collection $\mathscr{C}$ of signed sets $C$ of $E$ is an *oriented matroid* $M$ if and only if the following conditions are satisfied:

(C0) $\emptyset \notin \mathscr{C}$
(C1) $\mathscr{C} = -\mathscr{C}$
(C2) $\forall X, Y \in \mathscr{C}:$ if $\underline{X} \subseteq \underline{Y}$, then $X = Y$ or $X = -Y$
(C3) $\forall X, Y \in \mathscr{C}$ with $X \neq -Y$ and $e \in X^+ \cap Y^-$ there is a $Z \in \mathscr{C}$ such that
- $Z^+ \subseteq (X^+ \cup Y^+) \setminus \{e\}$
- $Z^- \subseteq (X^- \cup Y^-) \setminus \{e\}$

Conditions (C0) to (C3) are called *circuit axioms* of oriented matroids and the elements of $\mathscr{C}$ are called *signed circuits*.

**Remark 1.3.1.4** (Link to matroids)**.** If we forget about signs the circuit axioms (C0) - (C3) reduce to the circuit axioms in classical matroid theory ([**Oxl11**, chapter 1, section 1.1]). The proof of equivalence of basis axioms and circuit axioms for matroids is quite simpler than for oriented matroids. Due to the brevity of this introduction we skip details and refer the reader to the standard reference. The collection of *signed circuit supports* $\underline{\mathscr{C}} = \{\underline{C} : C \in \mathscr{C}\}$ forms a collection of circuits of a matroid $\underline{M}$ called the *underlying matroid of* $M$. Hence, an oriented matroid $M$ inherits properties of its underlying matroid $\underline{M}$ (e.g. its rank).

The following definition can be found in [**BLVS**$^+$**99**, Section 3.1].

**Definition 1.3.1.5** (Reorientation)**.** Let $M = (E, \mathscr{C})$ be an oriented matroid. For $A \subseteq E$ we define a *reorientation of* $M$ with respect to $A$ as follows: for each signed circuit $C \in \mathscr{C}$ we define the reoriented signed circuit ${}_{-A}C$ by $({}_{-A}C)^+ = (C^+ \setminus A) \cup (C^- \cap A)$ and $({}_{-A}C)^- = (C^- \setminus A) \cup (C^+ \cap A)$. By ${}_{-A}\mathscr{C}$ we denote the set of reoriented signed circuits. The oriented matroid given by ${}_{-A}\mathscr{C}$ is denoted by ${}_{-A}M$.

**Remark 1.3.1.6** (Oriented matroids of point configurations)**.** Let $\mathscr{A} = \{\alpha_1, \ldots, \alpha_m\} \subset \mathbb{R}^n$ be a point configuration and $A \in \mathbb{R}^{n \times m}$ the matrix representation. We associate an oriented matroid to $\mathscr{A}$ in

| | | | |
|---|---|---|---|
| $12\overline{3}$ | $124$ | $125$ | $34$ |
| $135$ | $1\overline{4}5$ | $\overline{2}35$ | $\overline{2}45.$ |

TABLE 1. Signed circuits of $M[A]$ of Example 1.3.1.7.

a similar fashion as for unoriented matroids (cf. Remark 1.2.1.6). We fix the ground set $[m]$. A minimal linear dependence among elements of $\mathscr{A}$ can be expressed by an element $\lambda \in \ker(A)$, i.e.

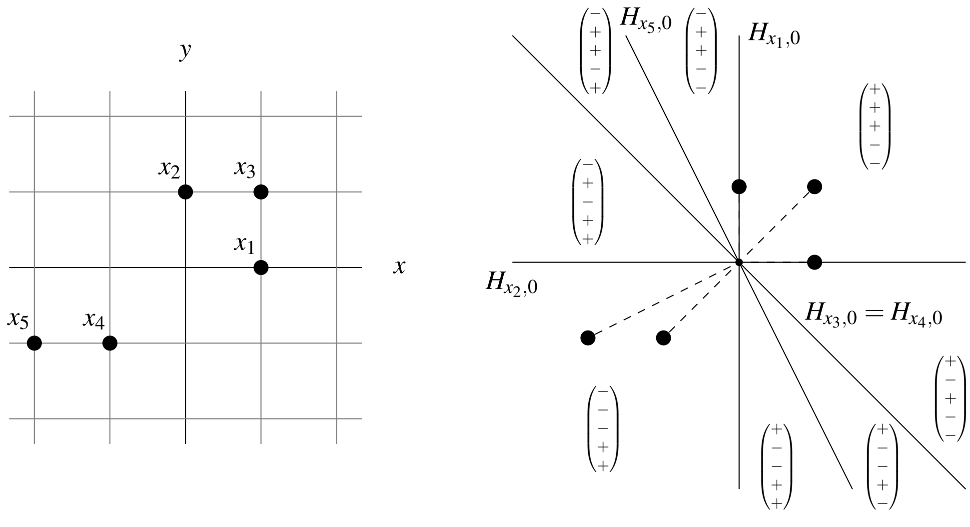
$$A\lambda = \sum_i \lambda_i \alpha_i = 0.$$

Hence, $\underline{C} := \{i \in [m] : \lambda_i \neq 0\}$ forms an (unoriented) circuit of the unoriented vector matroid associated to $A$. The partition of $\underline{C}$ into the subsets $C^{\pm} := \{i \in [m] : \lambda_i \gtrless 0\}$ provides a signed circuit. The collection of signed circuits (also denoted by $\mathscr{C}$) forms an oriented matroid denoted by $M[A] = ([m], \mathscr{C})$ called *oriented vector matroid* ([**BLVS⁺99**, §3, section 2]). In contrast to the unoriented case the proof is more involved due to axiom (C3).

**Example 1.3.1.7.** Consider the point configuration $\mathscr{A} = \{\alpha_1, \ldots, \alpha_5\} \subset \mathbb{R}^2$ illustrated by the columns of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & -1 & -2 \\ 0 & 1 & 1 & -1 & -1 \end{pmatrix}.$$

The point configuration is shown in Figure 6 (A). The signed circuits of $M[A]$ are listed in Table 1.




(A) Point configuration $\mathscr{A} \subset \mathbb{R}^2$.

(B) Topes and covectors of $M[A]$.

FIGURE 6. Point configuration and topes of Example 1.3.1.7.

### 1.3.2. Oriented Vector Matroids and Covectors

Another approach to oriented matroids is provided by *covectors*. Even though the concept of covectors allows to define an oriented matroid with an axiomatic system equivalent to Definition 1.3.1.3, we assume that, from now on, oriented matroids always arise from point configurations $\mathscr{A} \subset \mathbb{R}^n$, i.e. we consider oriented vector matroids. We prepare the ground:

**Definition 1.3.2.8** (Sign vector)**.** An element $s \in \mathscr{S} := \{+,-,0\}^n \cong \{1,-1,0\}^n$ is called *sign vector*. A sign vector $s \in \mathscr{S}$ with $s_i \in \{\pm\}$ for all $i$ is called *pure sign vector*.

**Remark 1.3.2.9** (Sign vectors and signed sets)**.** Signed sets $X$ of $E$ and sign vectors $s \in \mathscr{S}$ can be identified via

$$s_e = \begin{cases} + \text{ if } e \in X^+ \\ - \text{ if } e \in X^- \\ 0 \text{ if } e \in X^0 \end{cases} \quad \forall e \in E$$

and vice versa. We denote the associated elements to $X$ and $s$ by $s(X)$ and $X(s)$ respectively. Via this identification we define $s^+ = X^+$ and $s^-, s^0$ analogously.

**Remark 1.3.2.10.** We frequently make use of the following: for $a, b \in \mathscr{S}$ two sign vectors let $a \cdot b$ denote the sign vector obtained by multiplying $a$ and $b$ componentwise, i.e. $(a \cdot b)_i = a_i b_i$. We define a partial order on $\mathscr{S}$ by

$$a \subseteq b \qquad :\Longleftrightarrow \qquad a^+ \subseteq b^+ \text{ and } a^- \subseteq b^-.$$

The previous remark allows to understand reorientation via sign vectors:

**Remark 1.3.2.11** (Reorientation via sign vectors I)**.** Let $M$ be an oriented matroid on the ground set $E$. Consider a subset $A \subseteq E$. It defines a signed set $\tilde{A}$ by $\tilde{A}^- = A$ and $\tilde{A}^+ = E \setminus A^-$. Using the identification in Remark 1.3.2.9, $\tilde{A}$ provides a sign vector $s(\tilde{A})$ defined by $s(\tilde{A})_e = \pm$ if and only $e \in \tilde{A}^{\pm}$. Multiplying a sign vector $v \in \mathscr{S}$ with $s(\tilde{A})$ can be understood as "changing signs of $v$ at coordinates indexed by $A$". Let $C \in \mathscr{C}$ be a signed circuit and let $s_C = s(C)$ denote the sign vector associated to $C$. Then the reoriented signed circuit ${}_{-A}C$ equals the signed circuit $C\left(s(\tilde{A}) \cdot s_C\right)$. By abuse of notation we write $s(\tilde{A}) = {}_{-A}s$ to indicate that we want to "change signs at $A$".

**Definition 1.3.2.12** (Covector)**.** Let $\mathscr{A} \subset \mathbb{R}^n$ be a point configuration and $M[A]$ the oriented vector matroid associated to the matrix representation $A \in \mathbb{R}^{n \times m}$ of $\mathscr{A}$. A sign vector $s \in \mathscr{S}$ is called *covector* of $M[A]$ if there is an element $y \in (\mathbb{R}^n)^\vee$ such that

$$s = \left(\operatorname{sign}(y(\alpha_1)), \ldots, \operatorname{sign}(y(\alpha_m))\right).$$

The set of covectors of $M[A]$ is denoted by $\mathscr{L}_{M[A]} \subseteq \mathscr{S}$.

**Remark 1.3.2.13** (Topes)**.** In other words, we interpret elements $y \in (\mathbb{R}^n)^\vee$ as linear functionals on the point configuration $\mathscr{A}$. A matroid $M$ can be defined equivalently by the set $\mathscr{L}_M$ of covectors ([**BLVS**$^+$**99**, chapter 4, §1]). If $v \in \mathscr{L}_M$ is pure then $v$ is called *tope*. We denote the set of topes by $\mathscr{Y}$. Thus, $(\mathbb{R}^n)^\vee$ is subdivided into cells where each full dimensional cell is indexed by a tope of $M$. The concept is closely related to hyperplane arrangements defined by the $\alpha_i \in \mathscr{A}$.

**Example 1.3.2.14.** We continue Example 1.3.1.7. In Figure 6 the subdivision of $\left(\mathbb{R}^2\right)^\vee$ is sketched. The dashed lines correspond to linear subspaces defined by the $\alpha_i \in \mathscr{A}$. All topes are labelled by their corresponding pure sign vectors.

**Remark 1.3.2.15** (Reorientation via sign vectors II)**.** Let $M$ be an oriented matroid and $B \subset E$ a subset of the ground set. In Remark 1.3.2.11 we considered the reoriented matroid $_{-B}M$ in terms of sign vectors. As covectors are sign vectors, the reorientation of a covector $v \in \mathscr{L}_M$ with respect to $B \subset E$ means switching signs at $B$, i.e. $v \in \mathscr{L}_M$ translates to $_{-B}s \cdot v \in \mathscr{L}_{-BM}$ of $_{-B}M$ (cf. [**BLVS$^+$99**, Lemma 4.18] and Remark 1.3.2.10).

## 1.4. Tropical (Algebraic) Geometry

Tropical geometry is the marital gathering of a lot of fields in mathematical research like algebraic geometry, combinatorics and polyhedral geometry. The tropical objects occurring in this thesis arise as tropicalizations of classical algebraic varieties. In this section we explain the *tropicalization* process of classical algebraic varieties and present two essential statements concerning tropicalizations: the *Fundamental Theorem* and the *Structure Theorem*. In the end we focus on tropicalizations of linear varieties. We stick close to [**MS15**] which is a broad reference for tropical geometry, particularly from a computational point of view.

### 1.4.1. Puiseux Series and the Tropical Semiring

We distinguish two situations in this thesis: tropical geometry arising from algebraic geometry over an algebraically closed field and real tropical geometry arising from real algebraic geometry over a real ordered field. We deal with the latter case in Section 1.5. Overall, we work with *valued fields*:

**Definition 1.4.1.1** (Valued field)**.** Let $\mathbb{K}$ be a field and let $\mathbb{K}^*$ denote the units in $\mathbb{K}$. A *valuation* on $\mathbb{K}$ is a map $\mathrm{val} : \mathbb{K} \to \mathbb{R} \cup \{\infty\}$ satisfying

(1) $\mathrm{val}(a) = \infty$ if and only if $a = 0$,
(2) $\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$, and
(3) $\mathrm{val}(a+b) \geq \min\{\mathrm{val}(a), \mathrm{val}(b)\}$ for all $a, b \in \mathbb{K}^*$.

A field $\mathbb{K}$ with a valuation val is called *valued field*.

The image of val restricted to $\mathbb{K}^*$ is an additive subgroup of $\mathbb{R}$ called *value group of* val and we denote it by $\Gamma_{\mathrm{val}}$. The elements of $\mathbb{K}$ with non-negative valuation form a local ring $H$ that contains the unique maximal ideal $\mathbf{m} = \{x \in \mathbb{K} : \mathrm{val}(x) > 0\}$. The quotient $\Bbbk = H/\mathbf{m}$ is called *residue field of* $\mathbb{K}$. Note that every field has a trivial valuation $\mathrm{val} = 0$. In this thesis, the figurehead of a field with non-trivial valuation is the field of Puiseux series:

**Definition 1.4.1.2** (Puiseux series)**.** Let $\mathbb{K}$ be a field. The field of *Puiseux series* over $\mathbb{K}$, denoted by $\mathbb{K}\{\{t\}\}$, consists of elements of the form $a = \sum_{k=i}^{\infty} a_k t^{\frac{k}{n}} \in \mathbb{K}\{\{t\}\}$, i.e. there are $i \in \mathbb{Z}$, $n \in \mathbb{N}$ such that $a \in \mathbb{K}\{\{t\}\}$ is a formal power series in the indeterminate $t$ such that $a_k \in \mathbb{K}$ for all $k \geq i$ and the powers of $t$ have the common denominator $n$. We fix a valuation

$$\mathrm{val} : \mathbb{K}\{\{t\}\}^* \longrightarrow \mathbb{R},\ a = \sum_{k=i}^{\infty} a_k t^{\frac{k}{n}} \longmapsto \mathrm{val}(a) = \min\left\{\frac{i}{n} : a_i \neq 0\right\}$$

Hence, the valuation of an element $a \in (\mathbb{K}\{\{t\}\})^*$ is the lowest exponent of $t$ that appears in the series. For $a = 0 \in \mathbb{K}\{\{t\}\}$ we define $\mathrm{val}\,(a) = \infty$.

It is a well-known fact that $\mathbb{K}\{\{t\}\}$ is an algebraically closed field if $\mathbb{K}$ is an algebraically closed field of characteristic zero ([**MS15**, Theorem 2.1.5]). For the remaining parts of this thesis we use the following

**Convention 1.4.1.3.** By $\mathbb{K}_{\mathbb{C}} = \mathbb{C}\{\{t\}\}$ we denote the field of *complex Puiseux series* and the field of *real Puiseux series* is denoted by $\mathbb{K}_{\mathbb{R}} = \mathbb{R}\{\{t\}\}$. In the remaining part of this thesis we always work over one of these fields, i.e. $\mathbb{K} \in \{\mathbb{K}_{\mathbb{C}}, \mathbb{K}_{\mathbb{R}}\}$.

Now, we define the underlying algebraic structure of our tropical objects:

**Definition 1.4.1.4** (Tropical semiring)**.** We call $\mathbb{T} = (\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ *tropical semiring*. The underlying set is formed by the real numbers $\mathbb{R}$ with an additional element $\{-\infty\}$. The operations of $\mathbb{T}$ are the *tropical addition* $\oplus$ and the *tropical multiplication* $\odot$, defined as follows:

$$x \oplus y = \max\{x, y\} \qquad \text{and} \qquad x \odot y = x + y.$$

We see that "$-\infty$" is the neutral element for $\oplus$ and "0" is the neutral element for $\odot$. Note that there is no additive inverse. Therefore, the structure is called semiring.

The main objects of the thesis are called *tropical varieties* that arise as *tropicalizations* of classical varieties. In this section we introduce tropical geometry over $\mathbb{K} = \mathbb{K}_{\mathbb{C}}$, i.e. we stick to Convention 1.4.1.3 and work over the field of complex Puiseux series. We proceed as follows: in Section 1.4.2 we begin with hypersurfaces $\mathscr{V}(\langle F \rangle) \subset T^n$ defined by Laurent polynomials $F \in L = \mathbb{K}\left[x_1^{\pm}, \dots, x_n^{\pm}\right]$ due to their key role and explore their polyhedral structure. Then we move on to varieties $\mathscr{V}(I) \subset T^n$ defined by ideals $I \subset \mathbb{K}\left[x_1^{\pm}, \dots, x_n^{\pm}\right]$ in Section 1.4.3, reveal their polyhedral structure and summarize main results. At last, we focus on two special cases that arise frequently in this thesis: first, the *constant coefficient case* in Section 1.4.4, dealing with ideals $I$ having a generating set in $\mathbb{C}\left[x_1^{\pm}, \dots, x_n^{\pm}\right]$, and second, linearly generated ideals in the constant coefficient case in Section 1.4.5.

### 1.4.2. Tropical Hypersurfaces

Recall that we have $\mathbb{K} = \mathbb{K}_{\mathbb{C}}$. Throughout this section let $X = \mathscr{V}(I) \subset T^n = T^n_{\mathbb{K}}$ be a hypersurface defined by the principal ideal $I = \langle F \rangle \subset \mathbb{K}\left[x_1^{\pm}, \dots, x_n^{\pm}\right]$. Let $\mathscr{A} = \{\alpha_1, \dots, \alpha_m\} \subset \mathbb{Z}^n$ be the support of $F$, i.e.

$$F = \sum_i a_i x^{\alpha_i} \in \mathbb{K}\left[x_1^{\pm}, \dots, x_n^{\pm}\right]. \tag{2}$$

**Definition 1.4.2.5** (Tropical polynomials)**.** A *tropical polynomial* $f$ with support $\mathscr{A} = \{\alpha_1, \dots, \alpha_m\} \subset \mathbb{Z}^n$ is a finite sum of tropical monomials with coefficients in $\mathbb{T}$, i.e.

$$f = \bigoplus_i p_i \odot w^{\odot \alpha_i} = \max_i \{p_i + \langle \alpha_i, w \rangle\} \in \mathbb{T}[w_1, \dots, w_n].$$

The *tropicalization* of a polynomial $F \in \mathbb{K}\left[x_1^{\pm}, \dots, x_n^{\pm}\right]$ as above (cf. Equation (2)) is defined by

$$\operatorname{trop}(F) = \bigoplus_i \left(-\operatorname{val}(a_i) \odot w^{\odot \alpha_i}\right) = \max_i \{-\operatorname{val}(a_i) + \langle \alpha_i, w \rangle\}. \tag{3}$$

In order to pass from a classical polynomial to its tropical dependant we just have to exchange the classical addition and multiplication with the tropical operations and replace all coefficients with their negative valuation. A tropical polynomial $f \in \mathbb{T}[w_1,\dots,w_n]$ attains the maximum for some generic $w \in \mathbb{R}^n$ at a single term of $f$. Tropical geometry is concerned with the "pathological" situation where the maximum of $f$ is attained at two or more terms:

**Definition 1.4.2.6** (Tropical hypersurface)**.** Let $f = \max_i\{p_i + \langle \alpha_i, w\rangle\} \in \mathbb{T}[w_1,\dots,w_n]$ be a tropical polynomial with support $\mathscr{A} = \{\alpha_1,\dots,\alpha_m\} \subset \mathbb{Z}^n$. The *tropical hypersurface* given by $f$ is the set

$$\mathscr{T}(f) = \{w \in \mathbb{R}^n : \exists i,j \in [m],\ i \neq j \text{ such that } f(w) = p_i + \langle \alpha_i, w\rangle = p_j + \langle \alpha_j, w\rangle\}$$

Since tropical polynomials $f$ may arise as tropicalizations of Laurent polynomials the definition of the tropicalization of a hypersurface suggests itself:

**Definition 1.4.2.7** (Tropicalization)**.** Let $X = \mathscr{V}(F) \subset T^n$ be a hypersurface defined by a Laurent polynomial $F \in \mathbb{K}\left[x_1^{\pm},\dots,x_n^{\pm}\right]$. The *tropicalization of* $X$ is defined by $\operatorname{trop}(X) = \mathscr{T}(\operatorname{trop}(F))$.

By definition a tropical hypersurface is piecewise affine linear. For $n = 2$ and $f \in \mathbb{K}[x^{\pm}, y^{\pm}]$ we call $\mathscr{T}(f) \subset \mathbb{R}^2$ *plane tropical curve*. The points of a tropical hypersurface correspond to a balance of two or more terms of its defining polynomial. We illustrate this with the help of the *Newton polytope*:

**Definition 1.4.2.8** (Newton polytope)**.** Let $F = \sum_i a_i x^{\alpha_i} \in \mathbb{K}\left[x_1^{\pm},\dots,x_n^{\pm}\right]$ be a Laurent polynomial with support $\mathscr{A} \subset \mathbb{Z}^n$. We call $\operatorname{Newt}(F) = \operatorname{conv}(\alpha_i : a_i \neq 0) \subset \mathbb{R}^n$ *Newton polytope* of $F$.

The following proposition equips the tropical hypersurface $\mathscr{T}(\operatorname{trop}(F))$ with a polyhedral structure:

**Proposition 1.4.2.9** ([**MS15**, Proposition 3.1.6])**.** *Let* $F = \sum_i a_i x^{\alpha_i} \in \mathbb{K}\left[x_1^{\pm},\dots,x_n^{\pm}\right]$ *be a Laurent polynomial with support* $\mathscr{A} = \{\alpha_1,\dots,\alpha_m\} \subset \mathbb{Z}^n$*. The tropical hypersurface* $\mathscr{T}(\operatorname{trop}(F))$ *is the support of a pure* $\Gamma_{\mathrm{val}}$*-rational polyhedral complex of dimension* $n-1$ *in* $\mathbb{R}^n$*. It is the* $(n-1)$*-skeleton of the polyhedral complex dual to the regular subdivision of the Newton polytope* $\operatorname{Newt}(F)$ *of* $F = \sum_i a_i x^{\alpha_i}$ *given by the weights* $-\operatorname{val}(a_i)_{i\in[m]}$ *on the lattice points* $\mathscr{A}$ *in* $\operatorname{Newt}(F)$*.*

For a plane curve $\mathscr{V}(F) = X \subset T^2$ defined by $F = \sum_i a_i x^{\alpha_i}$ with $\mathscr{A} = \{\alpha_1,\dots,\alpha_m\} \subset \mathbb{Z}^2$ the situation is particularly nice and we can sketch the plane tropical curve $\mathscr{T}(\operatorname{trop}(F))$ easily: let $S = \{(P_i, \mathscr{A}_i) : i \in I\}$ be the regular subdivision of $\operatorname{Newt}(F)$ induced by $-(\operatorname{val}(a_i))_{i\in[m]} \in \mathbb{R}^{|\mathscr{A}|}$. The two dimensional polytopes $P_i$ correspond to vertices of $\mathscr{T}(\operatorname{trop}(F))$, edges of $P_i$ correspond to (maybe unbounded) edges in $\mathscr{T}(\operatorname{trop}(F))$ and vertices ( = zero dimensional faces) of $P_i$ correspond to areas in $\mathbb{R}^2$ where $\operatorname{trop}(F)$ attains its maximum at the unique term corresponding to the vertex of $P_i$. We obtain the coordinate of a vertex of $\mathscr{T}(\operatorname{trop}(F))$ by equalizing the tropical vertex equations of the dual polygon $P_i$ in $S$. The edges of $\mathscr{T}(\operatorname{trop}(F))$ are perpendicular to their dual counterparts in $\operatorname{Newt}(F)$.

**Example 1.4.2.10.** Consider $F = 2t + x_1 + t^{-1}x_2 + 3t^{-1}x_1^2 - t^{-1}x_1x_2 + 9x_2^2 \in \mathbb{K}\left[x_1^{\pm}, x_2^{\pm}\right]$ having support

$$\mathscr{A} = \left\{ \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix} \right\}.$$

The vector containing the valuations of the coefficients equals $w = -\left(\operatorname{val}(a_i)_{i\in[6]}\right) = (-1,0,1,1,1,0)$. Hence, $P = \operatorname{conv}(\mathscr{A})$ and $P_w$ coincide with the polytopes depicted in Example 1.1.4.21. We determine $\mathscr{T}(\operatorname{trop}(F))$ using Proposition 1.4.2.9. At first, we like to know the coordinates of the vertex in

$\mathscr{T}(\mathrm{trop}(F))$ dual to the marked polygon $P_1 = \mathrm{conv}(\mathscr{A}_1)$ with vertices $\mathscr{A}_1 = \{(0,1),(1,1),(0,2)\}$. Note that $\mathrm{trop}(F)$ is the maximum of affine linear functions indexed by $\mathscr{A}$. Hence, $\mathrm{trop}(F)(w)$ attains its maximum at the monomials of $\mathscr{A}_1$ if and only if $w$ satisfies Equation (4).

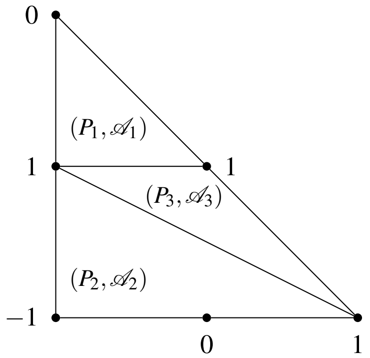
$$1+x_2 = 1+x_1+x_2 = 2x_2 \quad \Rightarrow \quad x_1 = 0 \wedge x_2 = 1 \tag{4}$$

$$1+x_2 = 1+x_1+x_2 \geq 2x_2 \quad \Rightarrow \quad x_1 = 0 \wedge x_2 \leq 1 \tag{5}$$

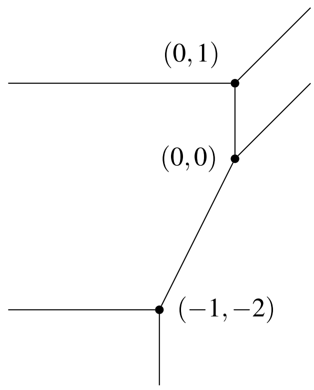
$$1+x_2 = 2x_2 \geq 1+x_1+x_2 \quad \Rightarrow \quad x_1 \leq 0 \wedge x_2 = 1 \tag{6}$$

$$1+x_1+x_2 = 2x_2 \geq 1+x_2 \quad \Rightarrow \quad x_1 \geq 0 \wedge x_2 \geq 1 \wedge 1+x_1-x_2 = 0 \tag{7}$$

Second, we identify edges of $\mathscr{T}(\mathrm{trop}(F))$ dual to edges of $P_1$. The one dimensional polyhedra of $\mathscr{T}(\mathrm{trop}(F))$ are perpendicular to edges of $P_1$. Hence, we pick vertices of an edge of $P_1$ and equalize the corresponding affine linear functions. Equation (5) and Equation (6) describe the vertical and horizontal edges starting at $(0,1)$. Equation (7) is equivalent to the polyhedron $(0,1)+\lambda(1,1)$ for all $\lambda \in \mathbb{R}_{\geq 0}$. The edge coming from Equation (5) is bounded by $(0,0)$. By proceeding this way for all $P_i$, we obtain the plane tropical curve $\mathscr{T}(\mathrm{trop}(F))$ shown in Figure 7.




(A) The regular subdivision $S$ of $\mathrm{Newt}(F)$ induced by $w = -\mathrm{val}((a_i)_{i\in\mathscr{A}})$.




(B) The plane tropical curve $\mathscr{T}(\mathrm{trop}(F)) \subset \mathbb{R}^2$ dual to $S$.

FIGURE 7. The subdivided Newton polytope and the dual plane tropical curve to $F = 2t + x_1 + t^{-1}x_2 + 3t^{-1}x_1^2 - t^{-1}x_1x_2 + 9x_2^2 \in \mathbb{K}\left[x_1^{\pm}, x_2^{\pm}\right]$.

If all coefficients of $F$ have zero valuation the regular subdivision of $\mathrm{Newt}(F)$ induced by the vector $\mathbf{0}_m = -(\mathrm{val}(a_i))_{i\in[m]}$ is simply the Newton polytope $\mathrm{Newt}(F)$.

**Corollary 1.4.2.11** ([**MS15**, Proposition 3.1.10])**.** *Let $F \in \mathbb{K}\left[x_1^{\pm},\ldots,x_n^{\pm}\right]$ be a Laurent polynomial whose coefficients have zero valuation. Then $\mathscr{T}(\mathrm{trop}(F)) \subset \mathbb{R}^n$ is the support of the $(n-1)$-skeleton of the normal fan of* $\mathrm{Newt}(F)$.

**Example 1.4.2.12.** Consider the homogeneous polynomial

$$F = 3x_1^2x_2 - 4x_1x_3^2 + 11x_2x_3^2 - 5x_2^2x_3 + x_1x_2^2 \in \mathbb{K}\left[x_1^{\pm}, x_2^{\pm}, x_3^{\pm}\right]$$

of degree 3. The two dimensional Newton polytope $\mathrm{Newt}(F)$ is contained in $H_3$. Note that $\mathscr{N}_{\mathrm{Newt}(F)}$ contains the lineality space $L$ generated by $\mathbf{1}_3$ and we can consider the tropical hypersurface in $\mathbb{R}^3/\mathbf{1}_3$. Hence, $\mathscr{T}(\mathrm{trop}(F)) \subset \mathbb{R}^3/\mathbf{1}_3$ is the one-dimensional skeleton of $\mathscr{N}_{\mathrm{Newt}(F)}/\mathbf{1}_{\mathbf{1}_3}$, i.e. it is a plane tropical curve. The normal fan is shown in Figure 2.

### 1.4.3. Tropical Varieties

Let $I \subset \mathbb{K}\left[x_1^{\pm},\dots,x_n^{\pm}\right]$ be an arbitrary ideal in the Laurent polynomials over $\mathbb{K}=\mathbb{K}_{\mathbb{C}}$. In the classical world we have $\mathscr{V}(I)=\bigcap_{F\in I}\mathscr{V}(F)$ and we define the *tropical variety* equivalently:

**Definition 1.4.3.13** (Tropical variety)**.** Let $I \subset \mathbb{K}\left[x_1^{\pm},\dots,x_n^{\pm}\right]$ be an ideal and $X=\mathscr{V}(I)\subset T^n$ the associated variety. The *tropicalization* of $X$ is defined by

$$\operatorname{trop}(X)=\bigcap_{F\in I}\mathscr{T}(\operatorname{trop}(F))$$

and we call $\operatorname{trop}(\mathscr{V}(I))$ a *tropical variety*.

Proposition 1.4.2.9 indicates the polyhedral nature of tropical hypersurfaces. So far, tropical varieties $\operatorname{trop}(\mathscr{V}(I))$ given by ideals $I \subset \mathbb{K}\left[x_1^{\pm},\dots,x_n^{\pm}\right]$ are subsets of $\mathbb{R}^n$. The used approach does not reveal the polyhedral structure of tropical varieties. Therefore, we need *initial ideals*:

**Definition 1.4.3.14** (Initial ideals)**.** Let $\mathscr{A}=\{\alpha_1,\dots,\alpha_m\}\subset\mathbb{Z}^n$ be a finite point configuration and let $F=\sum_i a_i x^{\alpha_i}\in\mathbb{K}\left[x_1^{\pm},\dots,x_n^{\pm}\right]$ be a Laurent polynomial with support $\mathscr{A}$. For $w\in\mathbb{R}^n$ fixed and $\operatorname{trop}(F)(w)=W$ we define the *initial form* of $F$ with respect to $w$ by

$$\operatorname{in}_w(F)=\sum_{\substack{i\in[m]:\\ -\operatorname{val}(a_i)+\langle w,\alpha_i\rangle=W}}\overline{a_i t^{-\operatorname{val}(a_i)}}x^{\alpha_i}\in\Bbbk\left[x_1^{\pm},\dots,x_n^{\pm}\right],$$

where $\overline{a_i t^{-\operatorname{val}(a_i)}}\in\Bbbk$, the residue field of $\mathbb{K}$. The *initial ideal* of an ideal $\mathscr{I}\subset\mathbb{K}\left[x_1^{\pm},\dots,x_n^{\pm}\right]$ with respect to $w\in\mathbb{R}^n$ is defined by

$$\operatorname{in}_w(\mathscr{I})=\langle\operatorname{in}_w(f):f\in\mathscr{I}\rangle\subset\Bbbk\left[x_1^{\pm},\dots,x_n^{\pm}\right].$$

A helpful result is the following

**Lemma 1.4.3.15** ([**MS15**, Lemma 2.6.2])**.** *Let $\mathscr{I}\subset\mathbb{K}\left[x_1^{\pm},\dots,x_n^{\pm}\right]$ be an ideal and fix $w\in\mathbb{R}^n$. Then:*

- *For any $g\in\operatorname{in}_w(\mathscr{I})$ exists $f\in\mathscr{I}$ such that $g=\operatorname{in}_w(f)$.*
- *If $f,g\in\mathbb{K}\left[x_1^{\pm},\dots,x_n^{\pm}\right]$ then $\operatorname{in}_w(fg)=\operatorname{in}_w(f)\operatorname{in}_w(g)$.*
- *The initial ideal $\operatorname{in}_w(\mathscr{I})$ contains a monomial if and only if there is a polynomial $f\in\mathscr{I}$ such that $\operatorname{in}_w(f)$ is a monomial.*

Note that the first statement implies the third. Lemma 1.4.3.15 essentially proves the first equality in the *Fundamental Theorem* ([**MS15**, Theorem 3.2.3]):

THEOREM 1.4.3.16 (Fundamental Theorem). *Let $I\subset\mathbb{K}\left[x_1^{\pm},\dots,x_n^{\pm}\right]$ be an ideal and $X=\mathscr{V}(I)\subset T^n$ the associated variety. Then*

$$\begin{aligned}\operatorname{trop}(\mathscr{V}(I))&=\{w\in\mathbb{R}^n:\operatorname{in}_w(\mathscr{I})\neq\langle1\rangle\}\\&=\overline{\{(-\operatorname{val}(x_1),\dots,-\operatorname{val}(x_n))\in\mathbb{R}^n:x\in\mathscr{V}(I)\}},\end{aligned}$$

*where we take the closure with respect to the standard topology.*

The special case of hypersurfaces is called *Kapranov's Theorem* ([**EKL06**]) which serves as a base point for the proof of the statement with arbitrary codimension. The first equality exposes the polyhedral structure of $\operatorname{trop}(\mathscr{V}(\mathscr{I}))$: it is a subcomplex of the Gröbner complex of $\mathscr{I}$. For more information see [**MS15**, section 2.5 + 3.2]. The second equality reveals the degenerative character.

**Remark 1.4.3.17** (Tropicalization map)**.** The Fundamental Theorem allows to define the tropicalization of an algebraic variety $X$ as the closure of the image of val. This is a quite common approach (e.g. [**RGST05**], [**MMS12a**], [**AN13**]) and we do so for real tropical varieties in Section 1.5. Hence, we can define the tropicalization map $\mathrm{trop} = -\mathrm{val}$.

By definition the tropicalization of $X = \mathscr{V}(\mathscr{I})$ equals the intersection of all tropical hypersurfaces $\mathscr{T}(\mathrm{trop}(F))$ with $F \in \mathscr{I}$. However, in classical algebraic geometry, it suffices to intersect a generating set of $\mathscr{I}$. The restriction to a finite subset $T \subset \mathscr{I}$ providing $\mathrm{trop}(\mathscr{V}(\mathscr{I}))$ would be very profitable:

**Definition 1.4.3.18** (Tropical basis)**.** Let $I \subset \mathbb{K}\left[x_1^{\pm}, \ldots, x_n^{\pm}\right]$ be an ideal in the Laurent polynomials. A finite set $T = \{F_1, \ldots, F_k\}$ of $I$ is called *tropical basis* if $\mathrm{trop}(\mathscr{V}(I)) = \bigcap_i \mathscr{T}(\mathrm{trop}(F_i))$.

Note that a tropical basis does not generate the ideal $I$ in general. This is not a big issue since a tropical basis can be saturated with generators of $I$. However, in contrast to classical algebraic geometry, arbitrary generating set of $I$ do not provide tropical bases. If $\langle F_1, \ldots, F_k \rangle = I$ is an arbitrary generating set then $\bigcap_{i=1}^k \mathscr{T}(\mathrm{trop}(F_i))$ is called *tropical prevariety* ([**BJS$^+$07**]) since it does not necessarily coincide with $\mathrm{trop}(\mathscr{V}(I))$. Apparently, a tropical basis offers an approach for the computation of the tropical variety. Besides, we can think of a tropical basis as a witness: if $T$ is a tropical basis then $w \notin \mathrm{trop}(\mathscr{V}(\mathscr{I}))$ if and only if there is an element $f \in T$ such that $w \notin \mathscr{T}(f)$. With regard to Lemma 1.4.3.15 (c) and the Fundamental Theorem we have:

**Corollary 1.4.3.19.** *Let $\mathscr{I} \subset \mathbb{K}\left[x_1^{\pm}, \ldots, x_n^{\pm}\right]$ be an ideal and let $T = \{F_1, \ldots, F_m\} \subset \mathscr{I}$ be a tropical basis for $\mathrm{trop}(\mathscr{V}(\mathscr{I}))$. Fix an element $w \in \mathbb{R}^n$. Then $\mathrm{in}_w(\mathscr{I})$ contains a monomial if and only if there is $i \in [m]$ such that $\mathrm{in}_w(F_i)$ is a monomial.*

In case of principal ideals the generator is a tropical basis:

**Lemma 1.4.3.20.** *If $\mathscr{I} = \langle F \rangle \subset \mathbb{K}\left[x_1^{\pm}, \ldots, x_n^{\pm}\right]$ is a principal ideal then $\{F\}$ is a tropical basis.*

**Remark 1.4.3.21** (Tropical bases)**.** Tropical bases are an important field of study in tropical geometry and topic of ongoing research (e.g. [**JS18**]). Knowledge about a tropical basis of a tropical variety can be understood as a first step in order to understand the tropical variety. An important result is that every ideal $I \subset \mathbb{K}\left[x_1^{\pm}, \ldots, x_n^{\pm}\right]$ has a finite tropical basis ([**BJS$^+$07**] for the constant coefficient case, [**HT09**], [**MS15**, Proposition 2.6.6]) and it is even possible to augment any generating set of an ideal to a "small" tropical basis ([**HT09**]). Beside the tropical bases that are known and presented in this preliminary chapter, we identify tropical bases for certain types of ideals in the constant coefficient case in this thesis. For instance, in Chapter 2 we identify a tropical basis for an ideal generated by linear forms and a polynomial. In Chapter 3 we determine a tropical basis of the ideal describing the parameter space for plane tropical curves with a $k+1$-fold singularity. In Chapter 4 we use a known tropical basis of linear spaces to classify singular real plane tropical curves and singular real tropical surfaces. In particular, tropicalizations of linear spaces are remarkably well-known ([**Stu02**]), cf. Section 1.4.5.

Recall that the Fundamental Theorem 1.4.3.16 gives a first hint to the polyhedral nature of tropical varieties. Essentially, a tropical variety supports a polyhedral structure that fulfills the so-called *balancing condition* (cf. Section 1.1.2). To get an idea consider Figure 7. The (weighted) sum of rays incident to a vertex is zero. This reminds us of mechanical equilibrium, i.e. the net force on the vertex is zero. This is summarized in a result called *Structure Theorem* ([**MS15**, Theorem 3.3.5]):

THEOREM 1.4.3.22 (Structure Theorem). *Let $I \subset \mathbb{K}\left[x_1^{\pm},\ldots,x_n^{\pm}\right]$ be an ideal such that $X = \mathscr{V}(I) \subset T^n$ is an irreducible variety of dimension k. Then* $\operatorname{trop}(\mathscr{V}(I)) \subset \mathbb{R}^n$ *is the support of a pure balanced weighted $\Gamma_{\mathrm{val}}$-rational polyhedral complex of dimension k.*

It needs some work to "balance" a tropical variety $\operatorname{trop}(X)$ arising from an irreducible variety $X = \mathscr{V}(I) \subset T^n$ with $\operatorname{codim}(X) > 1$. For instance, the polyhedral structure is not canonically defined. Furthermore, we need the correct weights — arising intrinsically from $I$ — on the facets of the polyhedral structure. In the following, we show how to assign weights to facets of a tropical hypersurface arising from a Laurent polynomial. We refer to [**MS15**, chapter 3, §3 + §4] for a complete description.

Let $F = \sum_i a_i x^{\alpha_i} \subset \mathbb{K}\left[x_1^{\pm},\ldots,x_n^{\pm}\right]$ be a Laurent polynomial with support $\mathscr{A} \subset \mathbb{Z}^n$ of size $m$ and let $P = \operatorname{Newt}(F) \subset \mathbb{R}^n$ be the associated Newton polytope. Due to Proposition 1.4.2.9 the coefficient vector $w = (-\operatorname{val}(a_i))_{i\in[m]}$ induces a regular subdivision $S = \{(P_i,\mathscr{A}_i) : i \in I\}$ on $P$ that is dual to the polyhedral complex $\mathscr{T}(\operatorname{trop}(F))$. Recall that a facet $\sigma \in \mathscr{T}(\operatorname{trop}(F))$ corresponds to an edge $e(\sigma) \in P_i$ for some $i \in I$. We assign a weight $\omega(\sigma)$ to $\sigma$ that is the edge length of $e(\sigma)$ in $P_i$, i.e. the number of lattice points minus one on $e(\sigma)$.

**Corollary 1.4.3.23.** *The tropical hypersurface* $\operatorname{trop}(X) \subset \mathbb{R}^n$ *of $X = \mathscr{V}(\mathscr{I})$ defined by the principal ideal $\mathscr{I} = \langle F\rangle \subset \mathbb{K}\left[x_1^{\pm},\ldots,x_n^{\pm}\right]$ is balanced with the weights defined by the edges lengths in the regular subdivision S induced by the valuated coefficients of F.*

**Example 1.4.3.24.** We continue with Example 1.4.2.10. The edge $\sigma \subset \mathscr{T}(\operatorname{trop}(F))$ dual to the edge $e(\sigma) = \operatorname{conv}((0,0),(1,0),(2,0)) \subset P_2$ obtains weight two. All other edges $\sigma \neq \sigma' \subset \mathscr{T}(\operatorname{trop}(F))$ obtain weight one. We check the balancing condition at all ridges $\tau \subset \mathscr{T}(\operatorname{trop}(F))$:

$$\begin{pmatrix}-1\\0\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}0\\-1\end{pmatrix} = 0 \in \mathbb{R}^2/V_\tau \text{ with } \tau = (0,1),$$
$$\begin{pmatrix}-1\\-2\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix} = 0 \in \mathbb{R}^2/V_\tau \text{ with } \tau = (0,0),$$
$$\begin{pmatrix}-1\\0\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix} + (2)\cdot\begin{pmatrix}0\\-1\end{pmatrix} = 0 \in \mathbb{R}^2/V_\tau \text{ with } \tau = (-1,-2).$$

### 1.4.4. The Constant Coefficient Case

In this part we examine the special case of ideals $J = \langle F_1,\ldots,F_s\rangle \subset \mathbb{K}\left[x_1^{\pm},\ldots,x_n^{\pm}\right]$ generated by elements $F_j \in \Bbbk\left[x_1^{\pm},\ldots,x_n^{\pm}\right]$. This leads to the theory of tropical geometry over an underlying field with trivial valuation. Note that $\Bbbk = \mathbb{C}$, considered with trivial valuation, is a valued subfield of $(\mathbb{C}\{\{t\}\},\operatorname{val})$ (cf. Definition 1.4.1.2) with $\operatorname{val}\big|_{\mathbb{C}} = 0$.

We take one step back and fix the underlying field $\Bbbk$ with trivial valuation. In order to introduce tropical geometry over $\Bbbk$ we revise the definitions introduced so far. Fortunately, we can use the identical basic definitions concerning tropicalizations, i.e. we use Definition 1.4.2.5, Definition 1.4.2.6, Definition 1.4.2.7 and Definition 1.4.3.13 with respect to $\Bbbk$ and $\operatorname{val}(a) = 0$ for all $a \in \Bbbk$. As the generators of $J$ are elements of $\Bbbk\left[x_1^{\pm},\dots,x_n^{\pm}\right]$ they define an ideal $I=\langle F_1,\dots,F_s\rangle\subset\Bbbk\left[x_1^{\pm},\dots,x_n^{\pm}\right]$ and, therefore, a variety $\mathscr{V}(I)\subset(\Bbbk^*)^n$. The following theorem establishes tropical geometry over fields with trivial valuation, emanating from tropical geometry over valued fields:

THEOREM 1.4.4.25 ([**MS15**, Theorem 3.2.4]). *Let $\mathbb{K}=\mathbb{C}\{\{t\}\}$ be the field of complex Puiseux series with residue field $\Bbbk=\mathbb{C}$. Let $X=\mathscr{V}(I)\subset(\Bbbk^*)^n$ be a variety defined by an ideal $I\subset\Bbbk\left[x_1^{\pm},\dots,x_n^{\pm}\right]$ and let $Y=\mathscr{V}(J)$ denote its extension to $T^n$, the $n$-dimensional torus over $\mathbb{K}$, defined by the ideal $J=I\cdot\mathbb{K}\left[x_1^{\pm},\dots,x_n^{\pm}\right]$. Then:*

$$\operatorname{trop}(Y)=\operatorname{trop}(X).$$

Essentially, the theorem states that, with regard to the tropical variety, it does not matter whether we think of $I=\langle F_1,\dots,F_s\rangle$ as an ideal in $\mathbb{K}\left[x_1^{\pm},\dots,x_n^{\pm}\right]$ or $\Bbbk\left[x_1^{\pm},\dots,x_n^{\pm}\right]$: the tropical variety $\operatorname{trop}(\mathscr{V}(I))$ is identical. As the ideal $J$ is generated by elements $F_j\in\Bbbk\left[x_1^{\pm},\dots,x_n^{\pm}\right]$ and, therefore, all coefficients of generators $F_j$ have trivial valuation, we call this situation the *constant coefficient case*. According to Proposition 1.4.2.9, $\operatorname{trop}(\mathscr{V}(F))$ is a fan for all $F\in\mathbb{C}\left[x_1^{\pm},\dots,x_n^{\pm}\right]$. Hence, any tropical variety $\operatorname{trop}(\mathscr{V}(\langle F_1,\dots,F_s\rangle))$ with $\{F_1,\dots,F_s\}\subset\mathbb{C}\left[x_1^{\pm},\dots,x_n^{\pm}\right]$ is a fan. In particular, it is a subfan of the Gröbner fan of $I$.

NOTATION 1.2. In the following we say that $I\subset\mathbb{K}\left[x_1^{\pm},\dots,x_n^{\pm}\right]$ is an ideal *with constant coefficients* if there is a generating set $\{F_1,\dots,F_s\}\subset\Bbbk\left[x_1^{\pm},\dots,x_n^{\pm}\right]$ of $I$.

### 1.4.5. Tropical Linear Spaces

A well-known and rigorously studied class of ideals in the constant coefficient case (cf. Section 1.4.4) is given by linear generated ideals $\mathscr{I}\subset\Bbbk\left[x_1^{\pm},\dots,x_n^{\pm}\right]$. Tropicalizations of linear spaces $\mathscr{V}(\mathscr{I})\subset T^n$ defined by linear ideals $\mathscr{I}\subset\Bbbk\left[x_1^{\pm},\dots,x_n^{\pm}\right]$ play a key role in this thesis.

**Definition 1.4.5.26** (Coefficient matrix)**.** Let $>$ be a term order on $\mathbb{K}[x_0,\dots,x_n]$ such that we have $x_{s_1}>\ldots>x_{s_n}$. Let $\mathscr{I}$ be an ideal generated by $n-k$ linear forms $l_i=\sum_j a_{ij}x_j\in\mathbb{K}[x_0,\dots,x_n]$. We call the matrix $A(\mathscr{I})\in\mathbb{K}^{n-k\times n}$ containing the coefficient of $x_{s_j}$ of linear form $l_i$ in column $j$ and row $i$ the *coefficient matrix of $\mathscr{I}=\langle l_1,\dots,l_{n-k}\rangle\subset\mathbb{K}[x_0,\dots,x_n]$*.

The coefficient matrix $A$ of a linear ideal $\mathscr{I}=\langle l_1,\dots,l_{n-k}\rangle$ satisfies $\mathscr{V}(\mathscr{I})=\ker(A)\subset T^n$. As $\ker(A)$ is a linear space we can associate a vector matroid to it. Therefore, we use

**Definition 1.4.5.27** (Gale dual)**.** Let $A\in\mathbb{K}^{n-k\times n}$ be a matrix of rank $n-k$ for some $n,k\in\mathbb{N}$ with $n>k$. A matrix $G\in\mathbb{K}^{k\times n}$ such that the rows of $G$ form a basis of $\ker(A)$ is called *Gale dual* of $A$.

**Remark 1.4.5.28** (Gale dual I)**.** Let $\mathscr{I}\subset\mathbb{K}\left[x_1^{\pm},\dots,x_n^{\pm}\right]$ be an ideal generated by linear forms with constant coefficients and let $A=A(\mathscr{I})$ denote its coefficient matrix. For every Gale dual $G$ of $A$ holds $\ker(A)=\operatorname{Im}\left(G^{\top}\right)$. We call $M(\mathscr{I}):=M[G]$ the *associated matroid to $\mathscr{I}$* which is the vector matroid arising from the point configuration given by the columns of $G$. The matroid $M=M(\mathscr{I})$ is uniquely determined by the set of circuits $\mathscr{C}$ (cf. Section 1.2). A circuit $C\in\mathscr{C}$ of $M$ provides a minimal linear dependency among the columns of the Gale dual $G$. Equivalently, a circuit $C$ of $M$ corresponds to a linear form in $\mathscr{I}$ with inclusion-minimal support. In terms of the coefficient matrix $A$, a circuit $C$ of $M$ corresponds to an element in the row space of $A$ with inclusion-minimal support. Moreover, $M[G]$ is dual to $M[A]$ due to [**Oxl11**, Theorem 2.2.8]. Note that $\dim(\mathscr{V}(\mathscr{I})) = \operatorname{rank}(M(\mathscr{I}))$ and $M(\mathscr{I})$ is independent of the choice of $G$.

It is known that tropicalizations of linear spaces only depend on the matroid associated to the linear ideal ([**Stu02**, §9.3], see also [**MS15**, chapter 4] for a comprehensive disquisition of this topic):

**Proposition 1.4.5.29** (Tropicalization of linear spaces)**.** *Let $\mathscr{I} = \langle l_1, \ldots, l_{n-k} \rangle \subset \mathbb{K}\left[x_1^{\pm}, \ldots, x_n^{\pm}\right]$ be an ideal generated by linear forms $l_i = \sum_j a_{ij} x_j \in \Bbbk\left[x_1^{\pm}, \ldots, x_n^{\pm}\right]$. Let $X = \mathscr{V}(\mathscr{I}) \subset T^n$ denote the linear space defined by $\mathscr{I}$ and $M(\mathscr{I})$ the matroid associated to $\mathscr{I}$. Then* $\operatorname{trop}(X) = |B(M(\mathscr{I}))|$, *i.e. $B(M(\mathscr{I}))$ gives* $\operatorname{trop}(X)$ *a Bergman fan structure. By setting $\omega(\sigma) := 1$ for all full dimensional cones $\sigma \subset B(M(\mathscr{I}))$ the Bergman fan carries a weighted balanced fan structure.*

We refer the reader to [**Fra12**, Proposition 3.1.10] for a proof showing that $(B(M(\mathscr{I})), \omega)$ with $\omega(\sigma) = 1$ for all $\sigma \subset B(M(\mathscr{I}))$ satisfies the balancing condition.

**Remark 1.4.5.30** (Gale dual II)**.** Based on [**Oxl11**, section 2] we give a construction guidance for particularly nice Gale duals that we use a lot in this thesis. Let $\mathscr{I} = \langle l_1, \ldots, l_{n-k} \rangle \subset \mathbb{K}\left[x_1^{\pm}, \ldots, x_n^{\pm}\right]$ denote the ideal generated by linear forms $l_i = \sum_j a_{ij} x_j$ with constant coefficients. Suppose $B$ is a basis of $M = M(\mathscr{I})$ and w.l.o.g. $B = \{n-k+1, \ldots, n\}$. Then $B^{\complement} = \{1, \ldots, n-k\}$ is a basis of $M[A]$ and we can find a set of linear generators of $\mathscr{I}$ such that the coefficient matrix $A = A(\mathscr{I})$ is of the form

$$A(\mathscr{I}) = \begin{array}{c} \begin{array}{cc} B^{\complement} & \quad B \end{array} \\ \left( \begin{array}{c|c} \mathbb{E}_{n-k} & A_B \end{array} \right) \end{array}$$

where $A_B \in \mathbb{K}^{n-k \times k}$. We can read off a Gale dual immediately:

$$G = \begin{array}{c} \begin{array}{cc} B^{\complement} & \quad B \end{array} \\ \left( \begin{array}{cc} -(A_B)^{\top} & \mathbb{E}_k \end{array} \right) \end{array}.$$

As $AG^{\top} = 0$, $G$ is indeed a Gale dual. For more details we refer to Section 2.1.2 in Chapter 2.

**Definition 1.4.5.31** (General tropical linear spaces)**.** Let $I \subset \mathbb{K}\left[x_1^{\pm}, \ldots, x_n^{\pm}\right]$ be a linear ideal with constant coefficients that defines a $k$-dimensional linear space $X = \mathscr{V}(\mathscr{I}) \subset T^n$. We call $X$ generic if $M(\mathscr{I}) = U_{k,n}$ is a uniform matroid of rank $k$ on the ground set $[n]$. For generic linear spaces $X \subset T^n$ we write $\operatorname{trop}(X) = L_n^k$ and call $L_n^k$ *general tropical linear space*.

## 1.5. Real Tropical Geometry

Real tropical geometry is a rather new and growing field of research, even for tropical geometry. In this thesis we restrict ourselves to the field of real Puiseux series $\mathbb{R}\{\{t\}\}$ (cf. Section 1.5.1), i.e. Puiseux series with real coefficients. This introduction is based on [**Tab15**].

### 1.5.1. Real Puiseux Series and the Real Tropical Group

In this section we work over the non-archimedian valued field of real Puiseux series $\mathbb{K}_{\mathbb{R}} = \mathbb{R}\{\{t\}\}$ (cf. Definition 1.4.1.2). We denote the torus of $(\mathbb{K}_{\mathbb{R}})^n$ by $T_{\mathbb{R}}^n$. We revise the definition of Puiseux series with respect to real coefficients: analogously to complex Puiseux series, an element $a \in \mathbb{R}\{\{t\}\}$

is a power series in the indeterminate $t$ with rational exponents that have a common denominator but the coefficients are real valued, i.e. there exist $i \in \mathbb{Z}$, $n \in \mathbb{N}$ and $a_k \in \mathbb{R}$ for all $k \geq i$ such that

$$a = \sum_{k=i}^{\infty} a_k t^{\frac{k}{n}} \in \mathbb{R}\{\{t\}\}. \tag{8}$$

We define the valuation $\mathrm{val} : \mathbb{R}\{\{t\}\}^* \longmapsto \mathbb{R}$ exactly as for $\mathbb{C}\{\{t\}\}$, i.e. the valuation of $0 \neq a \in \mathbb{R}\{\{t\}\}$ is the lowest exponent of $t$ appearing in $a$. For example, the element in Equation (8) has $\mathrm{val}(a) = \frac{i}{n}$ if $a_i \neq 0$. A first — and not surprising — difference to $\mathbb{C}\{\{t\}\}$ is that $\mathbb{R}\{\{t\}\}$ is not algebraically closed. However, $\mathbb{R}\{\{t\}\}$ is an ordered field via $a > 0$ if and only if the coefficient of $t^{\mathrm{val}(a)}$ is positive.

**Definition 1.5.1.1** (Sign function)**.** Let $a \in \mathbb{R}\{\{t\}\}$ be a real Puiseux series. We define a *sign function*

$$s : \mathbb{R}\{\{t\}\}^* \longrightarrow \{-1, 1\}, \quad s(a) = \begin{cases} 1 \text{ if } a > 0, \\ -1 \text{ if } a < 0. \end{cases}$$

**Convention 1.5.1.2.** In the following we work over $\mathbb{K}_{\mathbb{R}} = \mathbb{R}\{\{t\}\}$ and refer to it as *real tropical geometry* or the *real case*. We refer to $\mathbb{K}_{\mathbb{C}} = \mathbb{C}\{\{t\}\}$ as *the complex case*.

The underlying algebraic structure of our real tropical objects is defined as follows:

**Definition 1.5.1.3.** The tuple $\mathbb{TR} = (\{-1,1\} \times \mathbb{R}, \odot_{\mathbb{R}})$ is called *real tropical group*. The composition of $(a,x), (b,y) \in \mathbb{TR}$ via $\odot_{\mathbb{R}}$ is defined by

$$(a,x) \odot_{\mathbb{R}} (b,y) = (ab, x+y).$$

In contrast to $\mathbb{T}$ there is no reasonable "addition" in $\mathbb{TR}$ as it is not clear how to "add" signs.

### 1.5.2. Real Tropicalization

We revise some definitions of Section 1.4. First, we fix the notion of a *real tropicalization*:

**Definition 1.5.2.4** (Real tropicalization)**.** The *real tropicalization* is defined by

$$\begin{aligned} \mathrm{trop}_{\mathbb{R}} : {\mathbb{K}_{\mathbb{R}}}^* &\longrightarrow \mathbb{TR} = \{-1,1\} \times \mathbb{R} \\ x &\longmapsto (s(x), -\mathrm{val}(x)) \end{aligned}$$

where $s$ is the sign function and val the valuation of $\mathbb{K}_{\mathbb{R}}$. For $y = (a,b) \in \mathbb{TR}$ we call $a$ the sign of $y$ and $b$ the modulus of $y$. By abuse of notation we refer to the first component of $y$ by $s(y)$ and to the second component of $y$ by $|y|$.

In comparison to the tropicalization in Section 1.4 (in particular Remark 1.4.3.17), real tropicalization takes signs into account. Consequently, we cannot extend the real tropicalization to non-torus points as the sign function $s$ is only defined on ${\mathbb{K}_{\mathbb{R}}}^*$.

**Definition 1.5.2.5** (Real tropicalized variety)**.** Let $X = \mathscr{V}(I) \subset T^n_{\mathbb{R}}$ be an algebraic variety defined by $I \subset \mathbb{K}_{\mathbb{R}}\left[x_1^{\pm}, \ldots, x_n^{\pm}\right]$. The *real tropicalization* of $X$ is the closure of the set $\{\mathrm{trop}_{\mathbb{R}}(x) : x \in X\}$ in $\mathbb{TR}^n$ and is denoted by $\mathrm{trop}_{\mathbb{R}}(X)$.

**Remark 1.5.2.6.** The Fundamental Theorem (cf. Theorem 1.4.3.16) allows to define the tropicalization of $X = \mathscr{V}(I) \subset T^n$ with $I \subset \mathbb{C}\{\{t\}\}\left[x_1^{\pm}, \ldots, x_n^{\pm}\right]$ identically, i.e. $\mathrm{cl}\left(\mathrm{Im}(-\mathrm{val}\big|_X)\right) = \mathrm{trop}(X)$.

**Definition 1.5.2.7** (Real tropical polynomial)**.** Let $\mathscr{A} = \{\alpha_1, \dots, \alpha_m\} \subset \mathbb{Z}^n$ be a finite set. A *real tropical polynomial* $f \in \mathbb{TR}\left[w_1^{\pm}, \dots, w_n^{\pm}\right]$ with support $\mathscr{A}$ is a formal sum of products of real tropical monomials with respect to $\mathscr{A}$ and coefficients in $\mathbb{TR}$, i.e.

$$f = \bigoplus_i p_i w^{\alpha_i}.$$

A real tropical polynomial $f$ yields a piecewise affine linear function by restricting to the modulus of $f$, i.e. $|f| : \mathbb{TR}^n \longrightarrow \mathbb{R}$ is the map defined by $|f|(w) = \max_i \{|p_i| + \langle \alpha_i, |w| \rangle\}$.

As in the complex case we can translate Laurent polynomials to real tropical polynomials:

**Definition 1.5.2.8** (Real tropicalized polynomials)**.** Let $\mathscr{A} = \{\alpha_1, \dots, \alpha_m\} \subset \mathbb{Z}^n$ be the support of a real Laurent polynomial $F = \sum_i a_i x^{\alpha_i} \in \mathbb{K}_{\mathbb{R}}\left[x_1^{\pm}, \dots, x_n^{\pm}\right]$. The *real tropicalization of* $F$ is the real tropical polynomial defined by

$$f = \operatorname{trop}_{\mathbb{R}}(F) := \bigoplus_i \left(s(a_i), -\operatorname{val}(a_i)\right) w^{\alpha_i} \in \mathbb{TR}\left[w_1^{\pm}, \dots, w_n^{\pm}\right].$$

**Definition 1.5.2.9** (Real tropical hypersurface)**.** Let $f \in \mathbb{TR}\left[w_1^{\pm}, \dots, w_n^{\pm}\right]$ be a real tropical polynomial with support $\mathscr{A} = \{\alpha_1, \dots, \alpha_m\} \subset \mathbb{Z}^n$. We define the *real tropical hypersurface* $\mathscr{T}_{\mathbb{R}}(f)$ defined by $f$ as follows: $y = (y_1, \dots, y_n) \in \mathscr{T}_{\mathbb{R}}(f)$ if and only if there are two monomials $\alpha_i, \alpha_j \in \mathscr{A}$ such that $|f|$ attains the maximum at $\alpha_i$ and $\alpha_j$, i.e.

$$|p_i| + \langle \alpha_i, |y| \rangle = |p_j| + \langle \alpha_j, |y| \rangle \geq |p_k| + \langle \alpha_k, |y| \rangle \quad \forall k \neq i, j,$$

and the signs at $\alpha_i$ and $\alpha_j$ are opposite, i.e.

$$s(p_i) \prod_{l=1}^{n} s(y_l)^{(\alpha_i)_l} \neq s(p_j) \prod_{l=1}^{n} s(y_l)^{(\alpha_j)_l}.$$

As stated in Remark 1.5.2.12 there is no analog to Kapranov's Theorem (cf. Theorem 1.4.3.16) of the complex case for the real tropical world:

**Example 1.5.2.10** ([**Tab15**, example 3.2])**.** Consider $F = x^2 - x + 1 \in \mathbb{K}_{\mathbb{R}}[x]$. Then $\mathscr{V}(F) = \emptyset \subset T_{\mathbb{R}}$ but $\mathscr{T}_{\mathbb{R}}(\operatorname{trop}_{\mathbb{R}}(F)) = \{(+,0)\}$, i.e. $\operatorname{trop}_{\mathbb{R}}(\mathscr{V}(F)) \subsetneq \mathscr{T}_{\mathbb{R}}(\operatorname{trop}_{\mathbb{R}}(F))$.

**Definition 1.5.2.11** (Real tropical basis)**.** Let $I \subset \mathbb{K}_{\mathbb{R}}\left[x_1^{\pm}, \dots, x_n^{\pm}\right]$ be an ideal. A *real tropical basis of* $I$ is a finite set $\{F_1, \dots, F_k\}$ of $I$ such that

$$\operatorname{trop}_{\mathbb{R}}(\mathscr{V}(I)) = \bigcap_{i=1}^{k} \mathscr{T}_{\mathbb{R}}(\operatorname{trop}_{\mathbb{R}}(F_i)).$$

Example 1.5.2.10 shows that the generator of a principle ideal is not necessarily a tropical basis (in contrast to the complex case, cf. Lemma 1.4.3.20).

**Remark 1.5.2.12** (Tropical geometry over $\mathbb{R}\{\{t\}\}$)**.** As the preceding introduction has shown, there are some significant differences to tropical geometry over $\mathbb{C}\{\{t\}\}$ as introduced in Section 1.4. First, $\mathbb{R}\{\{t\}\}$ is not algebraically closed since $\mathbb{R}$ is not algebraically closed. With a view towards classical algebraic geometry we may expect difficulties due to the invalidity of the Nullstellensatz. To make things worse, Kapranov's Theorem (Theorem 1.4.3.16 restricted to hypersurfaces) does not hold (cf. Lemma 1.4.3.20, see also Example 1.5.2.10). There is a demand for an answer why one should consider $\mathbb{R}\{\{t\}\}$ as a good underlying field. For us Tarski's principle serves as approval: any first-order sentence over $\mathbb{R}\{\{t\}\}$ is true if and only if it is true over $\mathbb{R}$.

## 1.6. Singular Tropical Hypersurfaces

The theory of *singularities* of algebraic varieties is inseparably connected to the concept of *tangent spaces* of algebraic varieties. The archetype of a tangent space is almost certainly the *tangent line*. Intuitively, a tangent $T$ is a line that "touches" a curve $C$ *locally* in a point $x$. Geometrically, we may think of "touching" in the following way: consider a *secant line* $L_{a,b}$ that intersects $C$ in two points $a,b \in C$. Pick two sequences of points on $C$ that move from $a$ and $b$ to $x \in C$. Let $T = \lim_{a,b\to x} L_{a,b}$ denote the limit of the secants. If $T$ uniquely exists we can think of $T$ as a linear approximation of the curve $C$ at $x$ and we call $T$ the tangent at $C$ in $x$. *Singularities* can be understood as the critical points on curves where this linear-approximation-attempt fails. For more information see e.g. [**CLO07**], [**Fis94**].

In this thesis we deal with affine algebraic hypersurfaces with singularities. Curves with singularities play a key role. All problems we face are related to one of the following situations:

I. Curves $C \subset \mathbb{A}^2$ with a *cusp*.
II. Hypersurfaces $V \subset \mathbb{A}^n$ with a *k+1-fold singularity* ($k \in \mathbb{N}$).

All terms in this motivational introduction are explained precisely in the remaining part of this section. Thereby we prioritise curves. We have three parts: first, we briefly summarize the concept of a tangent space to an affine algebraic hypersurface $V \subset \mathbb{A}^n$. See [**CLO07**], [**GfKZ94**], or [**Har77**],[**Per08**] for more general and intrinsic definitions. Then, we introduce two types of singularities for curves $C \subset \mathbb{A}^2$: nodes and cusps. Finally, we introduce $k+1$-fold singularities for hypersurfaces $V \subset \mathbb{A}^n$.

### 1.6.1. $k+1$-fold Singular Hypersurfaces and Cuspidal Curves

Let $f \in \mathbb{K}[x_1,\ldots,x_n]$ be a non-constant polynomial. The Taylor expansion of $f$ at a point $p$ can be written as $f = f_0 + \ldots f_m$ where $f_i$ denotes the degree-$i$-part of the expansion. The linear part $f_1$ earns our closer attention and we call it *differential* of $f$ at $p$:

$$\mathrm{d}_p(f) = \sum_j \frac{\partial f}{\partial x_j}(p)\,(x_j - p_j)\,.$$

**Definition 1.6.1.1** (Tangent space)**.** Let $V \subset \mathbb{A}^n$ be a hypersurface, $p \in V$ and $\mathbf{I}(V) = \langle f \rangle$. The *tangent space to $V$ at $p$* is the variety

$$T_p(V) = \mathscr{V}\left(\mathrm{d}_p(f)\right)$$

Note that $T_p(V)$ is a translate of a linear subspace of $\mathbb{A}^n$ via the coordinate transformation $X_i = x_i - p_i$. Furthermore, for hypersurfaces $V$ with $\mathbf{I}(V) = \langle f \rangle$ we have $\dim_p(V) = n-1$ for all $p \in V$ ([**CLO07**, section 9, §4, Prop. 2]).

**Definition 1.6.1.2** (Singularity)**.** Let $V \subset \mathbb{A}^n$ be a hypersurface and $\mathbf{I}(V) = \langle f \rangle$. We call a point $p \in V$ *singular* if $\dim(T_p(V)) \neq n-1$.

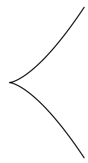


FIGURE 8. Cuspidal curve defined by $F(x,y) = y^2 - x^3$.

If $p \in V$ is not singular we call $p$ *smooth*. Due to Definition 1.6.1.1 the tangent space $T_p(V)$ at $p \in V$ is given by the linear equation $\mathrm{d}_p(f) = 0$. Therefore, we call

$$\mathrm{Sing}_f = \mathscr{V}\left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) \tag{9}$$

the *singular locus of* $V$. Let $C \subset \mathbb{A}^2$ be a curve and $\mathbf{I}(C) = \langle f \rangle$ with $d = \deg(f)$. Consider again the Taylor expansion $f = f_0 + \ldots + f_d$ of $f$ in some point $p \in C$. We call $\mathrm{ord}_p(f) = \min_k\{f_k \neq 0\}$ the *order* of $f$ in $p$. We can reformulate Definition 1.6.1.2:

$$\begin{aligned} \mathrm{ord}_p(f) > 1 \quad &\Leftrightarrow \quad \text{C is singular in p} \\ \mathrm{ord}_p(f) = 1 \quad &\Leftrightarrow \quad \text{C is smooth in p.} \end{aligned}$$

Let $m = \mathrm{ord}_p(f)$ be the order of $f$ in $p$. Now we make use of the fact that a homogeneous polynomial in two variables over a algebraically closed field can be factorized, i.e. we can write

$$f_m(x_1, x_2) = \prod_{j=1}^{k} \left(a_j(x_1 - p_1) + b_j(x_2 - p_2)\right)^{k_j} \tag{10}$$

**Definition 1.6.1.3** (Tangents in singular points)**.** In the situation stated above we call

$$T_j = \mathscr{V}\left(a_j(x_1 - p_1) + b_j(x_2 - p_2)\right)$$

a *tangent line* at $C$ in $p$ of multiplicity $k_j$.

This generalizes Definition 1.6.1.1 for curves since for $\mathrm{ord}_p(f) = 1$ we get $T = \mathscr{V}(\mathrm{d}_p(f))$ as tangents. If $\mathrm{ord}_p(f) \geq 2$ and all occurring tangents have multiplicity one we call $p$ an *ordinary multiple point*. See Example 1.6.1.6 for an example. In case of $\mathrm{ord}_p(f) = 2$ and two different tangents we call $p$ a *node*. If $\mathrm{ord}_p(f) = 2$ and one tangent with multiplicity 2 we call $p$ a cusp [**GfKZ94**, chapter 1]:

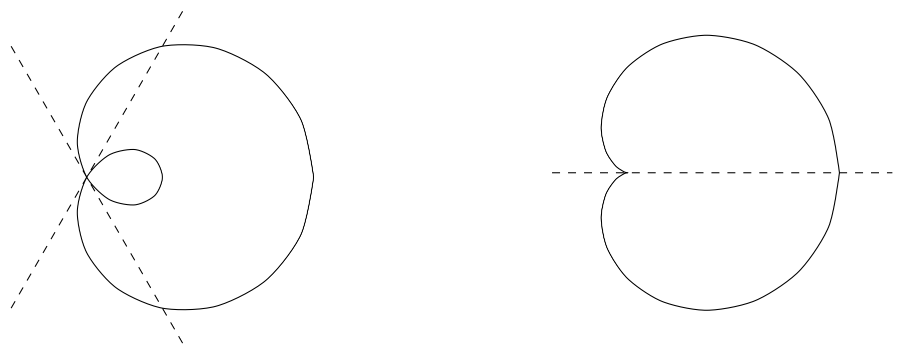
**Definition 1.6.1.4** (Cusp)**.** Let $C \subset \mathbb{A}^2$ be a curve, $\mathbf{I}(C) = \langle f \rangle$ and $p \in C$. Let $\Psi(x_1, x_2) = 2f_2(x_1, x_2)$ be the degree-2 part of the Taylor expansion of $f$ in $p$. We call $p \in \mathrm{Sing}_f$ a *cusp* if $\Psi(x_1, x_2) = L^2 \neq 0$ for a line $L$.

From an algebraic point of view there is a more pleasant way to describe the set of cusps of $C$.

**Lemma 1.6.1.5.** *Let $C \subset \mathbb{A}^2$ be a curve, $\mathbf{I}(C) = \langle f \rangle$, $p \in \mathrm{Sing}_f$ and $H_f$ the Hessian matrix of $f$. The point $p$ is a cusp in $C$ if and only if* $\det(H_f)(p) = 0$.

PROOF. Suppose $\Psi(x_1, x_2) = L^2$ with $L = a(x_1 - p_1) + b(x_2 - p_2)$ (Definition 1.6.1.4). Then

$$L^2 = a^2(x_1 - p_1)^2 + 2ab(x_1 - p_1)(x_2 - p_2) + b^2(x_2 - p_2)^2$$



(A) $C_{ab}$ with $a = 2$, $b = 1$.　　(B) $C_{ab}$ with $a = 2$, $b = 2$.

FIGURE 9. Limaçons (cf. Example 1.6.1.6) with $a \geq b > 0$.

and by comparing coefficients we get $a^2 = \frac{\partial^2 f}{\partial x_1^2}(p)$, $b^2 = \frac{\partial^2 f}{\partial x_2^2}(p)$ and $ab = \frac{\partial^2 f}{\partial x_1 \partial x_2}(p)$. This holds if and only if $\det\big(H_f(p)\big) = 0$. □

Consequently, we can describe the set of cuspidal singularities of $C$ as an algebraic set and we call

$$\text{c-Sing}_f = \mathscr{V}\left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}, \det\big(H_f\big)\right) \tag{11}$$

the *cuspidal locus* of $C$.

**Example 1.6.1.6** (Limaçon)**.** As an illustrating example we consider the family of real plane algebraic curves $C_{ab}$ defined by

$$F_{ab}(x,y) = (x^2 + y^2 - ax)^2 - b^2(x^2 + y^2). \tag{12}$$

The curve $C_{ab}$ is known as *limaçon*. We consider the case $a \geq b > 0$. Then the curve $C_{ab}$ is singular at $(0,0)$. Moreover, we have $\det\big(H_{F_{ab}}\big)\big|_{(0,0)} = -4a^2b^2 + 4b^4$. Due to the assumptions on the values of $a$ and $b$ we conclude that the origin is a cusp if and only if $a = b$. If $b$ varies the cusp evolves to an inner loop and there is a crossing at $(0,0)$. If $b$ converges to 0 the inner loop clings to the outer circle.

We go back to hypersurfaces $V \subset \mathbb{A}^n$ with $n \geq 3$. The approach of the previous part based on the factorization of homogeneous polynomials in two variables over an algebraically closed field (see Equation (10)), i.e. we can not generalize this process for $n > 2$. Consider the limaçon (cf. Example 1.6.1.6), in detail for $a = 2$ and $b = 1$ (cf. Figure 9a). Note that the singularity is the origin. The vanishing locus of the terms of lowest total degree of $F_{21}$ form a good approximation of $V$ at $p$, e.g. $(F_{21})_2 = 3x^2 - y^2 = (\sqrt{3}x - y)(\sqrt{3}x + y)$. Hence, $p = (0,0)$ is a node. This observation can be considered as the starting point for a non-linear approximation of an affine algebraic variety by an object called *tangent cone* $C_p(V)$ (see [**CLO07**, chapter 9, §7]). The tangent cone is the translate of an affine cone of a projective variety ([**CLO07**, chapter 9, §7, Proposition 3]) and it coincides with $T_p(V)$ for all smooth points $p$ ([**CLO07**, chapter 9, §7, Corollary 9]). If $V$ is a hypersurface, $p$ the origin and $\mathbf{I}(V) = \langle f \rangle$ then $f_m$ with $m = \operatorname{ord}_p(f)$ is the sum of terms of lowest total degree. Furthermore, we have $C_p(V) = \mathscr{V}(f_m)$ [**CLO07**, chapter 9, §7, p. 527], i.e. the tangent cone is defined by a polynomial of degree $m$. All partial derivatives of $f$ in $p$ up to order $m-1$ vanish. This leads to the following

**Definition 1.6.1.7** ($k+1$-fold singularity)**.** Let $V \subset \mathbb{A}^n$ be an affine hypersurface, $\mathbf{I}(V) = \langle f \rangle$ and $p \in V$ a point. We call $p$ a $k+1$-*fold singularity* of $V$ for some $k \in \mathbb{N}$ if all partial derivatives of $f$ in $p$ up to order $k$ vanish, i.e. $\forall d \in \{1,\dots,k\}$ and $\forall \alpha \in \mathscr{M}_{n,d}$ holds $\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(p) = 0$.

Note that 2-fold singularities are singularities according to Definition 1.6.1.2). Analogous to (Equation (9)) and (Equation (11)) we call

$$\mathrm{Sing}_f^{k+1} = \mathscr{V}\left(f, \frac{\partial^{|\alpha|} f}{\partial x^\alpha} = 0 : \alpha \in \mathscr{M}_d, \text{ for } d \in \{1,\dots,k\}\right) \tag{13}$$

the $k+1$-*fold singular locus* of $V$.

### 1.6.2. Singular Tropical Hypersurfaces and the Tropical $A$-Discriminant

The ground field of this chapter is the field $\mathbb{K}_\mathbb{C} = \mathbb{C}\{\{t\}\}$ of complex Puiseux series. First, we need to specify when a tropical hypersurface is singular. The definition is not standardized. However, as tropical hypersurfaces may arise as tropicalizations, it seems natural to call a tropical hypersurface singular if there is a algebraic hypersurface tropicalizing to it.

**Definition 1.6.2.8** (Singular tropical hypersurface)**.** Let $f = \bigoplus_{\alpha \in \mathscr{A}} b_\alpha \odot w^{\odot \alpha} \in \mathbb{T}[w_1,\dots,w_n]$ be a tropical polynomial with support $\mathscr{A} \subset \mathbb{Z}^n$. A point $q \in \mathscr{T}(f)$ is called *singular* if there is a Laurent polynomial $F = \sum_\alpha a_\alpha x^\alpha \in \mathbb{K}\left[x_1^\pm,\dots,x_n^\pm\right]$ and a point $p \in \mathrm{Sing}_F$ tropicalizing to it, i.e. $\mathrm{trop}(F) = f$ and $\mathrm{trop}(p) = -\mathrm{val}(p) = q$. In this case we call $\mathscr{T}(f)$ *singular tropical hypersurface*.

We keep to this philosophy for all kinds of singularities appearing in this thesis, e.g we call $q \in \mathscr{T}(f)$ cusp if there is a Laurent polynomial $F$ and a cusp $p \in \mathscr{V}(F)$ satisfying $\mathrm{trop}(F) = f$ and $\mathrm{trop}(p) = q$. The next step is to introduce the tropical discriminant.

NOTATION 1.3. From now on, $\mathscr{A} = \{\alpha_1,\dots,\alpha_m\} \subset \mathbb{Z}^n$ denotes a finite point configuration of cardinality $m$. By $F = \sum_i y_i x^{\alpha_i} \in \mathbb{K}[y_1,\dots,y_m]\left[x_1^\pm,\dots,x_n^\pm\right]$ we denote a generic Laurent polynomial with support $\mathscr{A}$ that is linear in the coefficients $y_i$. Moreover, $F_a = F_a(x) = \sum_i a_i x^{\alpha_i} \in \mathbb{K}\left[x_1^\pm,\dots,x_n^\pm\right]$ denotes the Laurent polynomial with support $\mathscr{A}$ obtained from $F$ with fixed coefficients $a \in (\mathbb{K}^*)^m$ and $F(p) = \sum_i y_i p^\alpha \in \mathbb{K}[y_1,\dots,y_m]$ denotes the linear form obtained from $F$ evaluated at $p \in (\mathbb{K}^*)^n$. The matrix representation of the point configuration $\mathscr{A}$ is denoted by $A \in \mathbb{Z}^{n \times m}$. In addition to that we call $\mathscr{A}' = \{1\} \times \mathscr{A}$ the *shifted* point configuration obtained from $\mathscr{A}$ by adding a new first coordinate equal to one. Analogously, we write $A' \in \mathbb{Z}^{n+1 \times m}$ for the matrix representation of the shift $\mathscr{A}'$ of $\mathscr{A}$.

Suppose $F_a$ defines a singular hypersurface with $a \in T^m$. For $\lambda \in \mathbb{K}^*$ we have $\lambda F_a = F_{\lambda a}$. So, $\lambda F_a$ and $F_{\lambda a}$ define the identical hypersurface. Therefore, let $\nabla$ denote the projective variety of Laurent polynomials $F_a$ with support $\mathscr{A}$ that define a singular hypersurface:

$$\nabla = \{a \in \mathbb{P}(T^m) : \mathscr{V}(F_a) \text{ is singular}\}. \tag{14}$$

The variety $\nabla$ was studied in [**GfKZ94**]. The toric variety $X_{A'}$ offers an approach to $\nabla$. By $X_{A'}$ we denote the Zariski closure of the image of the monomial map

$$\Psi_{\mathscr{A}'} : T^{n+1} \longrightarrow \mathbb{P}^{m-1}, \qquad t = (t_0, \dots, t_n) \longmapsto \left(t^{\alpha'_1} : \dots : t^{\alpha'_m}\right). \tag{15}$$

In detail, $X_{A'} \subset \mathbb{P}^{m-1}$ is a projective toric variety. Note that $X_A$, the projective toric variety arising from $\mathscr{A}$, equals $X_{A'}$. The dual variety $(X_{A'})^\vee \subset \left(\mathbb{P}^{m-1}\right)^\vee$ is the set of hyperplanes tangent to $X_{A'}$ at a regular point. Equivalently, we can think of $(X_{A'})^\vee$ as the family of singular hypersurfaces defined by a polynomial $F_a = \sum_i a_i x^{\alpha_i}$ with support $\mathscr{A}$ and $a \in \mathbb{P}(T^m)$, i.e. $(X_{A'})^\vee = \nabla$. If $(X_{A'})^\vee$ has codimension one there is a unique polynomial $\Delta_{A'} \in \mathbb{Z}[y_i : i = 1, \dots, m]$ such that $(X_{A'})^\vee = \mathscr{V}(\Delta_{A'})$. This polynomial is called *A-discriminant*. The tropical version $\operatorname{trop}(\nabla)$ of the $A$-discriminant is object of recent study (e.g. [**DFS07**], [**DT12**], [**MMS12a**]). We explore a valuable description of $\operatorname{trop}(\nabla)$ by means of the Horn uniformization (according to [**DFS07**], [**MMS12a**]). The subject of study in [**DFS07**] is the tropicalization of varieties defined by linear forms in monomials. In detail let $f : \mathbb{K}^m \dashrightarrow \mathbb{K}^s$ denote a rational map that factors as a linear map $U : \mathbb{K}^m \to \mathbb{K}^r$ followed by a Laurent monomial map $V : \mathbb{K}^r \to \mathbb{K}^s$ such that

$$f_i(x_1, \dots, x_m) = \prod_{k=1}^{r} \left( \sum_{j=1}^{m} (U)_{kj} x_j \right)^{(V)_{ik}}$$

is the $i$-th component of $f$ with $i \in \{1, \dots, s\}$. Let $Y_{UV}$ denote the Zariski closure of the image of $f$. In [**DFS07**] the authors examined tropicalizations of varieties of the form $Y_{UV}$. We state the main result that proves beneficial not only for the $A$-discriminant:

THEOREM 1.6.2.9 ([**DFS07**, Theorem 3.1]). *The tropical variety* $\operatorname{trop}(Y_{UV})$ *equals the image of the Bergman fan* $\operatorname{trop}(\operatorname{Im}(U))$ *under the linear map specified by* $V$.

Varieties parametrized by rational maps $f$ cover some special cases we have to deal with in this thesis. For instance, $V = \mathbb{E}_r$ or $U = \mathbb{E}_m$. In the first case the rational map $f$ is a linear map (cf. Section 1.4.5) whereas in the second case we get a monomial map defining a toric variety (cf. Section 2.2.1). In order to use Theorem 1.6.2.9 for the $A$-discriminant we need to express $(X_{A'})^\vee$ as the image of a rational map. This is where *Horn uniformization* comes into play:

**Remark 1.6.2.10** (Horn uniformization)**.** Due to ([**DFS07**, Proposition 4.1]) the dual variety $(X_{A'})^\vee$ equals the closure of the image of the map:

$$\phi_{A'} : \mathbb{P}\left(\ker(A')\right) \times T^{n+1} \longrightarrow \mathbb{P}(T^m), \quad (u, p) \longmapsto \left(u_1 p^{\alpha'_1} : \dots : u_m p^{\alpha'_m}\right) = u \cdot \psi_{\mathscr{A}'}(p).$$

In other words, $\phi_{A'}$ parameterizes the dual variety $(X_{A'})^\vee$ and makes Theorem 1.6.2.9 applicable.

With suitable choices of $V$ and $U$ it can be shown that $\operatorname{trop}\left((X_{A'})^\vee\right)$ equals the Minkowski sum of the Bergman fan of $\mathbb{P}(ker(A'))$ and the row space of $(A')$:

THEOREM 1.6.2.11 ([**DFS07**, Theorem 1.1]). *Let* $\mathscr{A} \subset \mathbb{Z}^n$ *be the support of cardinality* $m$ *of a generic Laurent polynomial* $F = \sum_i y_i x^{\alpha_i} \in \mathbb{K}[y_1, \dots, y_m]\left[x_1^{\pm}, \dots, x_n^{\pm}\right]$. *Then*

$$\operatorname{trop}(\nabla) = \operatorname{trop}\left(\ker(A')\right) \oplus \operatorname{rowspace}\left(A'\right).$$

In the following we highlight the benefits of Horn uniformization. The decomposition of $\operatorname{trop}(\nabla)$ has an interesting consequence for the underlying singular hypersurface:

**Remark 1.6.2.12** (Singular hypersurfaces with a singularity at $\mathbf{1}_n$)**.** As before $\mathscr{A} \subset \mathbb{Z}^n$ denotes the support of cardinality $m$ of a generic Laurent polynomial $F = \sum_i y_i x^{\alpha_i} \in \mathbb{K}[y_1, \dots, y_m]\left[x_1^{\pm}, \dots, x_n^{\pm}\right]$ and $A' \in \mathbb{Z}^{n+1 \times m}$ denotes the matrix representation of $\mathscr{A}'$. Let $\nabla_{\mathbf{1}_n} \subset \nabla$ denote the subset of Laurent polynomials with support $\mathscr{A}$ that have a singularity at $\mathbf{1}_n$, i.e.

$$\nabla_{\mathbf{1}_n} = \left\{ a \in \mathbb{P}(T^m) : F_a(\mathbf{1}_n) = 0,\ \frac{\partial F_a}{\partial x_i}(\mathbf{1}_n) = 0\ \forall i = 1, \dots, n \right\} \tag{16}$$

Since $F(\mathbf{1}_n) = \sum_i y_i$ and $\frac{\partial F}{\partial x_j}(\mathbf{1}_n) = \sum_i (\alpha_i)_j y_i$ are homogeneous linear forms for $j = 1, \dots, n$, the family hypersurfaces with a singularity at $\mathbf{1}_n$, defined by Laurent polynomials with support $\mathscr{A}$, is a linear space:

$$\nabla_{\mathbf{1}_n}(\mathscr{A}) = \ker(A') \subset \mathbb{P}(T^m)\,. \tag{17}$$

For $a \in \nabla$ we know that there is a singular point $p \in \mathscr{V}(F_a)$. A simple calculation shows:

$$F_a \text{ is singular at } p \qquad \Longleftrightarrow \qquad F_{a \cdot \psi_{\mathscr{A}}(p)} \text{ is singular at } \mathbf{1}_n. \tag{18}$$

So $a \cdot \psi_{\mathscr{A}}(p) \in \ker(A')$. Let $-\mathrm{val}(a) = b$ and $-\mathrm{val}(p) = q$ denote the valuations. The point $a \cdot \psi_{\mathscr{A}}(p)$ tropicalizes to $b + (A')^\top q$, i.e. $b = -\mathrm{val}(a)$ is shifted by an element of rowspace$(A')$. As a consequence we can study $\mathrm{trop}(\nabla)$ by studying $\mathrm{trop}(\nabla_{\mathbf{1}_n}) = \mathrm{trop}(\ker(A'))$.

CHAPTER 2

# Computation and Characterization of Tropical Hypersurfaces in Tropical Linear Spaces

In this chapter we present two independent methods to compute the tropicalization of a hypersurface contained in a linear space in the constant coefficient case. We work over the ground field $\mathbb{K} = \mathbb{K}_{\mathbb{C}}$.

NOTATION 2.1. Let $\mathscr{I} \subset R_{n+1} = \mathbb{K}[x_0, \ldots, x_n]$ be a linear ideal with constant coefficients (cf. Section 1.4.4). Let $X = \mathscr{V}(\mathscr{I}) \subset \mathbb{P}^n$ denote the linear space and $B(M) = \mathrm{trop}(X) \subset \mathbb{R}^{n+1}$ its tropicalization where $M$ is the matroid associated to $\mathscr{I}$ (cf. Section 1.4.5). Moreover, let $F \in R_{n+1}$ be a homogeneous polynomial of degree $d$. Then $\mathscr{V}(\mathscr{I} + \langle F \rangle)$ is a hypersurface in $X$. The tropicalization of $\mathscr{V}(\mathscr{I} + \langle F \rangle)$ is denoted by $\mathscr{T}$.

In Section 2.1 we determine the tropical variety $\mathscr{T}$ with the help of coordinate projections $p_B : B(M) \to \mathbb{R}_B$ according to bases $B$ of $M$. We are confronted with two major tasks: first, we need to know the projections and second, how to reconstruct from projections. Sections 2.1.1 to 2.1.3 form the building blocks of this section. In Section 2.1.1 we focus on the ambient Bergman fan trop$(X)$ of $\mathscr{T}$ and prove some facts about coordinate projections of Bergman fans and injectivity. In particular, for a given basis $B$ of a matroid $M$ we determine a subset of cones of the coarse subdivision of $B(M)$ so that $p_B$ restricted to this subset is bijective (Lemma 2.1.1.5). We show that this subset is uniquely determined by $B$ (Lemma 2.1.1.7). This way we relate the coarse subdivision of the ambient tropical linear space trop$(X)$ of $\mathscr{T}$ with injective coordinate projections and define a new subdivision on the image fan (Definition 2.1.1.6). In Section 2.1.2 we give a brief overview on the interplay of coordinate projections and tropicalizations. Afterwards, in Section 2.1.3, we turn our attention to the actual reconstruction, which heavily relies on our results of Section 2.1.1 concerning the injectivity of coordinate projections. We reveal an essential ordering on the cones of the image of $\mathscr{T}$ under $p_B$ (Lemma 2.1.3.24). This ordering is deeply connected to the fan structure of the ambient Bergman fan of $\mathscr{T}$. It leads immediately to the main result (Theorem 2.1.3.25): the tropicalization $\mathscr{T}$ is completely determined by all of its coordinate projections, i.e. $\mathscr{T}$ can be reconstructed in rank$(M) = k+1$ steps from various (but sufficiently many) of its coordinate projections. The proof of the theorem is constructive and allows an implementation (cf. algorithm 1). For curves the situation is particularly nice (cf. Section 2.1.4). We discuss an implementation for tropical curves in 2-dimensional tropical linear spaces that uses the computer algebra software SINGULAR ([**DGPS16**]) and POLYMAKE ([**GJ00**]). The necessary library and script can be found on the following website:

`https://github.com/cjuergens/trophials.git`

In Section 2.2 we reveal a fan subdivision of the ambient tropical linear space $B(M)$ whose codimension one skeleton supports any tropical hypersurfaces $\mathscr{T} = \mathrm{trop}(\mathscr{V}(\mathscr{I} + \langle F \rangle))$ for any choice of $F \in R_{n+1}$ of degree $d$. The basic idea of this section is to perform a linearization with the Veronese map $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$ of degree $d$ with $N := N^d_{n+1} - 1$. In Section 2.2.1 we begin with the linear ideal $\mathscr{I}$. Consider the degree-d part $\mathscr{I}^{(d)}$ of the linear ideal $\mathscr{I}$. The linear counterpart of $\mathscr{I}^{(d)}$ in $R_{N+1}$ defines a matroid $M^d$ and a Bergman fan $B\left(M^d\right)$. We show that the intersection of the tropicalizations of $\mathrm{Im}(\nu_d)$ and $B\left(M^d\right)$ provide $\mathscr{T}$ (Proposition 2.2.1.13). Moreover, we determine the weight classes of $B\left(M^d\right)$ that contribute full-dimensionally to this intersection. The preimages of these weight classes under $\mathrm{trop}(\nu_d)$, a linear isomorphism, form the degree-d subdivision of $\mathrm{trop}(X)$ (Lemma 2.2.1.23). In Section 2.2.2 we examine $\mathscr{T} = \mathrm{trop}(\mathscr{V}(\mathscr{I} + \langle F \rangle))$ with $F \in R_{n+1}$. We likewise consider the linear counterpart of $\mathscr{I} + \langle F \rangle$ in $R_{N+1}$. The associated Bergman fan is a subfan of the codimension one skeleton of the Bergman fan $B\left(M^d\right)$, i.e. $\mathscr{T}$ is a subfan of the codimension one skeleton of $\mathrm{trop}(X)$ with respect to its degree-d subdivision. The benefit of the linearization is accompanied by some challenges, e.g. we agree to an exchange of high degree ($d$) and a small number of variables ($n+1$) with low degree (1) and a huge number of variables $\binom{n+d}{d}$. However, this approach may admit some further studies in other areas, e.g. a better understanding of the parameter space of plane tropical curves with a cusp (cf. Section 3.1).

## 2.1. Tropicalization via Projections

In this section we stick to the Notation 2.1. The goal is to develop a technique to reconstruct $\mathscr{T}$ from a collection of coordinate projections. We proceed as follows: Section 2.1.1 serves as a preparation. The focus is set on coordinate projections of Bergman fans $B(M)$ according to bases of the matroid $M$. In Section 2.1.2 we deal with coordinate projections of $\mathscr{T}$ (accomplished by results of [**Bir16**] based on [**Gub13**], [**ST08**]). In Section 2.1.3 the reconstruction of $\mathscr{T}$ from all these projections is in the spotlight. In Section 2.1.4 we concentrate on 1-dimensional curves in 2-dimensional tropical linear spaces.

### 2.1.1. Coordinate Projections of Bergman Fans

For the rest of this section let $M$ be a matroid of rank $k+1$ on the ground set $[n+1]$. Whenever we work with a Bergman fan $B(M)$ associated to $M$ we consider it up to a pure weighted balanced polyhedral fan structure. We specify a fan structure whenever it is necessary.

The following introduction of coordinate projections of Bergman fans $B(M)$ leans on [**BGS17**]. By $p_B : B(M) \to \mathbb{R}^{k+1}$ we denote the coordinate projection of $B(M)$ onto the coordinates indexed by $B \subset [n+1]$. To keep the coordinates in mind we also write $\mathbb{R}_B = \mathbb{R}^{k+1}$. A subset $B \subset [n+1]$ provides a *restricted matroid* $M\big|_B$ of $M$ (cf. Definition 1.2.1.2). If $F \subset M$ is a flat then $F \cap B$ is a flat in $M\big|_B$ ([**Oxl11**, Proposition 3.3.1]). A flag of flats $\mathscr{F} = (F_1, \dots, F_{k+1}) \lhd M$ yields a chain of sets

$$F_1 \cap B \subseteq F_2 \cap B \subseteq \dots, F_k \cap B \subseteq F_{k+1} \cap B$$

of $M\big|_B$ by intersecting the flats of $\mathscr{F}$ with $B$. However, some of the $F_i \cap B$ might be equal or empty. By $(\mathscr{F} \cap B)$ we denote the flag of flats in $M\big|_B$ obtained by the collection of sets $\{F_1 \cap B, \dots, F_{k+1} \cap B\}$ with repeated and empty sets deleted. Vice versa, a flag $\mathscr{F}' \lhd M\big|_B$ provides a flag in $M$ by applying the closure operator to all flats of $\mathscr{F}'$. We denote the obtained flag of flats by $\mathrm{cl}(\mathscr{F}')$. Now, we equip a flag of flats with a rank according to [**BGS17**, Definition 3.3]:

**Definition 2.1.1.1** (Rank of a flag of flats)**.** Let $\mathscr{F} = (F_1,\dots,F_m) \lhd M$ be a flag of flats of a matroid $M$. The *rank of* $\mathscr{F}$ is defined by

$$\operatorname{rank}(\mathscr{F}) = \sum_i \operatorname{rank}(F_i).$$

The next lemma adapts methods of [**BGS17**, Lemma 3.2] and [**BGS17**, Lemma 3.4] to our situation:

**Lemma 2.1.1.2.** *Let $B \in \mathscr{B}$ be a basis of the matroid $M$ of rank $k+1$, defined on the ground set $[n+1]$, and let $p_B : B(M) \to \mathbb{R}_B$ denote the coordinate projection. Let $\mathscr{F} \lhd M$ be a flag of flats. Then:*

(a) *$p_B$ maps $\sigma_{\mathscr{F}} \subset B(M)$ surjectively to $\sigma_{\mathscr{F}\cap B} \subset B\left(M\big|_B\right)$.*
(b) *$p_B\big|_{\sigma_{\mathscr{F}}}$ is bijective if and only if $\mathscr{F}$ and $\mathscr{F}\cap B$ have same length.*
(c) *$p_B$ maps $B(M)$ surjectively to $B\left(M\big|_B\right)$ and $|B\left(M\big|_B\right)| = \mathbb{R}_B$.*
(d) *Let $\mathscr{F} \lhd M$ be a flag of flats and $B \in M_{\mathscr{F}}$. If there is another flag $\mathscr{F}' \lhd M$, $\mathscr{F} \neq \mathscr{F}'$, such that $p_B(\sigma_{\mathscr{F}}) = p_B(\sigma_{\mathscr{F}'})$ then $\operatorname{rank}(\mathscr{F}) < \operatorname{rank}(\mathscr{F}')$.*

PROOF. Let $\mathscr{F} = (F_1,\dots,F_q) \lhd M$ be a flag of flats. To prove (a) recall the $\mathscr{V}$-description of $\sigma_{\mathscr{F}}$ (cf. Remark 1.2.3.16): $\sigma_{\mathscr{F}} = \langle -e_{F_i} | i = 1,\dots,q\rangle_{\mathbb{R}_{\geq 0}}$. Hence, $p_B(e_{F_i}) = e_{F_i\cap B}$ since we restrict to the coordinates indexed by $B$. Hence, $p_B(\sigma_{\mathscr{F}}) = \sigma_{\mathscr{F}\cap B} \subset B\left(M\big|_B\right)$ for all flags of flats $\mathscr{F} \lhd M$. For (b) note that $p_B\big|_{\sigma_{\mathscr{F}}}$ is a linear map. $p_B\big|_{\sigma_{\mathscr{F}}}$ is injective if and only if $\sigma_{\mathscr{F}}$ and $\sigma_{\mathscr{F}\cap B}$ have same dimension, i.e. $\mathscr{F}$ and $\mathscr{F}\cap B$ have same length. Statement (c) holds since any flag of flats $\mathscr{F}' = (F'_1,\dots,F'_q) \lhd M\big|_B$ provides a flag of flats $\mathscr{F} = \operatorname{cl}(\mathscr{F}') = (\operatorname{cl}(F'_1),\dots,\operatorname{cl}(F'_q)) \lhd M$ via the closure operator and $\mathscr{F}\cap B = \mathscr{F}'$. Hence, $p_B$ maps $B(M)$ surjectively to $B\left(M\big|_B\right)$. Note that $\mathbb{R}_B$ is an irreducible fan ([**GKM09**, Example 2.18]). Hence, $|B\left(M\big|_B\right)|$ equals $\mathbb{R}_B$ if and only if $\dim\left(B\left(M\big|_B\right)\right) = |B|$. Equivalently, $\operatorname{rank}\left(M\big|_B\right) = |B|$ if and only if $B$ is a basis of $M$. To see the last statement let $\mathscr{F} = (F_1,\dots,F_q)$ be a flag of flats with $B \in M_{\mathscr{F}}$ and $\mathscr{F}' = (G_1,\dots,G_p)$ be another flag in $M$ such that $\mathscr{F}\cap B = \mathscr{F}'\cap B$. Hence, for all $i \in \{1,\dots,q\}$ we have $h_i \in \{1,\dots,p\}$ such that $F_i \cap B = G_{h_i} \cap B$. We conclude:

$$\begin{aligned}\operatorname{rank}(\mathscr{F}) = \operatorname{rank}(\mathscr{F}\cap B) &= \sum_{i=1}^{q} \operatorname{rank}(F_i\cap B) = \sum_{i=1}^{q} \operatorname{rank}\left(G_{h_i}\cap B\right)\\ &\leq \sum_{i=1}^{q} \operatorname{rank}\left(G_{h_i}\right) \leq \sum_{i=1}^{p} \operatorname{rank}(G_i) = \operatorname{rank}\left(\mathscr{F}'\right).\end{aligned} \tag{19}$$

The first equation holds since $B \in M_{\mathscr{F}}$ (cf. Lemma 2.1.1.5). The equality $\operatorname{rank}(\mathscr{F}') = \operatorname{rank}(\mathscr{F})$ implies $q = p$ and $\operatorname{rank}(G_i) = \operatorname{rank}(G_i\cap B)$ for all $i$. Then $G_i$ and $F_i$ contain the basis $F_i\cap B$ of $F_i$ and additionally $\operatorname{rank}(G_i) = \operatorname{rank}(G_i\cap B) = \operatorname{rank}(F_i\cap B) = \operatorname{rank}(F_i)$ for all $i$, i.e. $G_i = F_i$ $\forall i$. Consequently, $\mathscr{F}' = \mathscr{F}$. □

**Remark 2.1.1.3** (Subdivisions of $M\big|_B$)**.** For any basis $B$ of $M$, $M\big|_B$ is a uniform matroid $U_{k+1,k+1}$, i.e. $M\big|_B$ has one basis $B$. Hence, any $w \in \mathbb{R}_B$ minimizes the basis $B$ and there is exactly one equivalence class $M_{w'}$ that is equal to $\mathbb{R}_B$. However, there is a fine fan structure. $B\left(M\big|_B\right)$ with $B = \{b_1,\dots,b_{k+1}\}$ consists of all weight classes $w_{b_{\eta(1)}} \leq w_{b_{\eta(2)}} \leq \dots \leq w_{b_{\eta(k+1)}}$ with $\eta \in S(B)$ a permutation on $k+1 = \operatorname{rank}(M)$ elements.

The next step is an extension of some parts of Lemma 2.1.1.2 to the coarse subdivision of $B(M)$ and, moreover, statements about the injectivity of $p_B$ in relation to the coarse subdivision. In particular, we determine the locus of injectivity of $p_B$ on $B(M)$ for a fixed basis $B$ of $M$. That allows to define a fan structure on $B\left(M\big|_B\right)$ that respects injectivity of $p_B$. Lemma 2.1.1.2 states that $p_B$ maps $B(M)$ surjectively to $\mathbb{R}_B$. We show that almost every point of $\mathbb{R}_B$ has a unique preimage in $B(M)$. The exception is a codimension one fan contained in $B\left(M\big|_B\right)$.

In the following we make use of the local adjacencies of cones. Recall that the neighborhood of a cone in a fan can be described by its star (cf. Definition 1.1.1.10) and, in case of Bergman fans, by the lattice of flats (cf. Definition 1.2.1.2):

**Remark 2.1.1.4** (Star-partition-correspondence)**.** Let $\mathscr{F} \lhd M$ be a flag of maximal length, i.e. its flats are indexed by rank, and let $\mathscr{F}_{\hat{i}} = \mathscr{F} - F_i$ denote the flag obtained from $\mathscr{F}$ by removing the rank $i$ flat $F_i$. The star of $\sigma_{\mathscr{F}_{\hat{i}}}$, denoted by $\operatorname{star}_{B(M)}\left(\sigma_{\mathscr{F}_{\hat{i}}}\right)$, is the fan indexed by all full dimensional weight classes that have $\sigma_{\mathscr{F}_{\hat{i}}}$ as a face (cf. Definition 1.1.1.10). Since $\mathscr{F}_{\hat{i}}$ misses a flat of rank $i$ we obtain all weight classes containing $\sigma_{\mathscr{F}_{\hat{i}}}$ as a face by adding a rank $i$ flat of the sublattice $[F_{i-1}, F_{i+1}]$ of $\mathscr{L}$. These rank $i$ flats are in one-to-one correspondence to the partition $F_{i+1} \setminus F_{i-1} = \bigsqcup_{j=1}^{s} \bar{F}_{ij}$ with $s \geq 2$ such that $F_{ij} := F_{i-1} \cup \bar{F}_{ij}$ is a flat of rank $i$ in $M$ for all $j$ ([**Oxl11**, Corollary 1.4.7]). Hence, $\sigma_{\mathscr{F}_{\hat{i}}+F_{ij}} \leq \sigma_{\mathscr{F}}$ form the weight classes indexing $\operatorname{star}_{B(M)}\left(\sigma_{\mathscr{F}_{\hat{i}}}\right)$, i.e.

$$\operatorname{star}_{B(M)}\left(\sigma_{\mathscr{F}_{\hat{i}}}\right) = \bigcup_{F_{ij} \in [F_{i+1}, F_{i-1}]} \overline{\sigma}_{\mathscr{F}+F_{ij}}.$$

**Lemma 2.1.1.5.** *Let $M$ be a matroid on $[n+1]$ of rank $k+1$. Let $\mathscr{F} = (F_1, \ldots, F_m) \lhd M$ be a flag of flats. Let $B \in \mathscr{B}$ be a basis of $M$. By $p_B : B(M) \to \mathbb{R}_B$ we denote the associated coordinate projection. Moreover, we denote the fan of cones $\sigma(M_{\mathscr{F}})$ with $B \in M_{\mathscr{F}}$ by $B(M)\big|_B$. Then:*

(a) *If $B \in M_{\mathscr{F}}$ then $\operatorname{cl}(\mathscr{F} \cap B) = \mathscr{F}$ and $p_B\big|_{\sigma(M_{\mathscr{F}})}$ is bijective.*
(b) *If $B \in M_{\mathscr{F}}$ then $p_B(\sigma(M_{\mathscr{F}})) = \bigcup_{i \in I} \sigma_{\mathscr{F}'_i}$ where $I$ indexes all weight classes $\sigma_{\mathscr{F}'_i}$ in $B\left(M\big|_B\right)$ satisfying $M_{\operatorname{cl}\left(\mathscr{F}'_i\right)} = M_{\mathscr{F}}$.*
(c) *$p_B$ maps $B(M)\big|_B$ injectively and surjectively to $B\left(M\big|_B\right)$.*

Proof. For (a) note that $B \in M_{\mathscr{F}}$ means that $B$ can be obtained by the Greedy algorithm (cf. Remark 1.2.3.19) iteratively, i.e. it is possible to pick $\operatorname{rank}(F_i) - \operatorname{rank}(F_{i-1})$ elements of $B \cap (F_i \setminus F_{i-1})$ forming a basis of $F_i$ (together with the basis of $F_{i-1}$ obtained in the previous step). Hence, $F_i \cap B$ is a basis of $F_i$, i.e. $\operatorname{cl}(F_i \cap B) = F_i$. In particular, $\operatorname{cl}(\mathscr{F} \cap B) = \mathscr{F}$. Furthermore, all $F_i \cap B$ are non-empty and pairwise different. Therefore, $\mathscr{F} \cap B = (F_1 \cap B, \ldots, F_m \cap B)$ and $\mathscr{F} = (F_1, \ldots, F_m)$ have identical length. By Lemma 2.1.1.2 (b), $p_B\big|_{\sigma_{\mathscr{F}}}$ is bijective. Now, we extend this bijection to the coarse subdivision: there are flags of flats $\mathscr{F}_i \lhd M$ such that $\sigma(M_{\mathscr{F}}) = \bigcup_i \sigma_{\mathscr{F}_i}$ (cf. Section 1.2.3). Additionally, we have $B \in M_{\mathscr{F}} = M_{\mathscr{F}_i}$ for all $i$ (cf. Theorem 1.2.3.20). Thus, $p_B\big|_{\sigma_{\mathscr{F}_i}}$ is bijective for all $i$. Since $p_B$ is linear on $\langle \sigma(M_{\mathscr{F}}) \rangle$ we conclude $p_B\big|_{\sigma(M_{\mathscr{F}})}$ is bijective.

For (b) let $\sigma(M_{\mathscr{F}})$ be a cone in $B(M)$ with $B \in M_{\mathscr{F}}$. We have $\sigma(M_{\mathscr{F}}) = \bigcup_i \sigma_{\mathscr{F}_i}$ for $\mathscr{F}_i \lhd M$ satisfying $M_{\mathscr{F}} = M_{\mathscr{F}_i}$ for all $i$. Using (a) $B \in M_{\mathscr{F}}$ implies that $p_B\big|_{\sigma(M_{\mathscr{F}})}$ is bijective. Consequently, $p_B\big|_{\sigma_{\mathscr{F}_i}}$ is bijective for all $i$ and $p_B(\sigma_{\mathscr{F}_i}) = \sigma_{\mathscr{F}_i \cap B}$. Since $B \in M_{\mathscr{F}_i}$ we know that $F_i \cap B$ is a basis of $F_i$. Hence, $\operatorname{cl}(\mathscr{F}_i \cap B) = \mathscr{F}_i$ and consequently $M_{\operatorname{cl}(\mathscr{F}_i \cap B)} = M_{\mathscr{F}_i} = M_{\mathscr{F}}$. This shows "$\subseteq$". On the other hand, let $\{\mathscr{F}'_i : i \in I\}$ be the set of flags of flats such that $M_{\operatorname{cl}\left(\mathscr{F}'_i\right)} = M_{\mathscr{F}}\ \forall i \in I$. Hence, $\sigma_{\operatorname{cl}\left(\mathscr{F}'_i\right)} \subset \sigma(M_{\mathscr{F}})$ and since $B \in M_{\mathscr{F}}$ the map $p_B\big|_{\sigma(M_{\mathscr{F}})}$ is bijective, i.e. $p_B(\sigma_{\operatorname{cl}\left(\mathscr{F}'_i\right)}) = \sigma_{\mathscr{F}_i} \subset p_B(\sigma(M_{\mathscr{F}}))$.

For (c) let $\sigma_{\mathscr{F}'} \subset B\left(M\big|_B\right)$ be any weight class. Then the flag $\mathscr{F} = \operatorname{cl}(\mathscr{F}') \lhd M$ satisfies $\mathscr{F} \cap B = \mathscr{F}'$ and $B \in M_{\mathscr{F}}$. Thus $p_B(\sigma_{\mathscr{F}}) = \sigma_{\mathscr{F}'}$ and $\sigma_{\mathscr{F}} \subset B(M)\big|_B$ and $p_B$ maps $B(M)\big|_B$ surjectively to $B\left(M\big|_B\right)$.

Now, we show that $p_B$ is also injective on $B(M)\big|_B$. By (a) $p_B$ is injective on $\sigma_{\mathscr{F}} \subset B(M)\big|_B$ since $B \in M_{\mathscr{F}}$. Suppose there is another flag $\mathscr{G} \lhd M$ with $B \in M_{\mathscr{G}}$ such that $p_B(\sigma_{\mathscr{G}}) = p_B(\sigma_{\mathscr{F}})$. Then $\sigma_{\mathscr{F}\cap B} = \sigma_{\mathscr{G}\cap B}$. Since $B \in M_{\mathscr{F}}$ and $B \in M_{\mathscr{G}}$ and $\mathscr{G} \cap B = \mathscr{F} \cap B$ we conclude

$$\mathscr{G} = \mathrm{cl}(\mathscr{G} \cap B) = \mathrm{cl}(\mathscr{F} \cap B) = \mathscr{F}$$

due to (a). Thus $\sigma_{\mathscr{F}} = \sigma_{\mathscr{G}}$, i.e. $p_B$ is injective on $B(M)\big|_B$. □

We briefly explore the value of Lemma 2.1.1.5: statement (a) extends a bijection $p_B\big|_{\sigma_{\mathscr{F}}}$ to the cone $\sigma(M_{\mathscr{F}})$. Statement (b) can be used to define a "coarse" subdivision on $B\left(M\big|_B\right)$ by merging weight classes whose preimages under $p_B$ belong to the same cone $\sigma(M_{\mathscr{F}})$ (cf. Definition 2.1.1.6 below). The obtained subdivision on $B\left(M\big|_B\right)$ is the image of the coarse subdivision of $B(M)\big|_B$ (see also Remark 2.1.1.3). Due to statement (c) the projections of all cones $\sigma(M_{\mathscr{F}})$ with $B \in M_{\mathscr{F}}$ equals $B\left(M\big|_B\right)$.

**Definition 2.1.1.6** (Induced coarse subdivision)**.** Let $M$ be a matroid. Let $p_B : B(M) \to B\left(M\big|_B\right)$ denote the coordinate projection according to a basis $B$ of $M$. The images of all maximal cones of the coarse subdivision of $B(M)\big|_B^{(k+1)}$ under $p_B$ form a fan structure on $B\left(M\big|_B\right)$ called *induced coarse subdivision*. Moreover, we define $\bar{\mathscr{T}}_B = \mathrm{Im}\left(p_B\big|_{B(M)^{\mathrm{codim}(1)}}\right)$.

**Lemma 2.1.1.7.** *Let $M$ be a matroid on $[n+1]$ of rank $k+1$. Let $B \in \mathscr{B}$ be a basis of $M$ and $B(M)\big|_B = \{\sigma(M_{\mathscr{F}}) : \mathscr{F} \lhd M \text{ with } B \in M_{\mathscr{F}}\}$. As before, let $p_B : B(M) \to \mathbb{R}_B$ denote the coordinate projection according to $B$. Then:*

*(a) Consider any weight class $\sigma_{\mathscr{F}'} \subset B\left(M\big|_B\right)^{(k+1)}$. Let $\mathscr{F} = \mathrm{cl}(\mathscr{F}') \lhd M$ denote the closure. Then $\sigma_{\mathscr{F}} \subset B(M)\big|_B$ is the unique weight class of $B(M)$ that is mapped to $\sigma_{\mathscr{F}'}$ under $p_B$, i.e. every point of $\sigma_{\mathscr{F}'}$ has a unique preimage in $B(M)$.*

*(b) $B(M) \setminus B(M)\big|_B^{(k+1)}$ is mapped to $B\left(M\big|_B\right)^{(\mathrm{codim}(1))}$ with respect to the induced coarse subdivision of $B\left(M\big|_B\right)$. Moreover, the image of $B(M) \setminus B(M)\big|_B^{(k+1)}$ equals $\mathrm{Im}\left(p_B\big|_{B(M)^{\mathrm{codim}(1)}}\right)$ with respect to the coarse subdivision of $B(M)$.*

*(c) Let $\mathscr{F}'_{\hat{i}} = \left(F'_1, \dots, F'_{i-1}, F'_{i+1}, \dots, F'_{k+1}\right) \lhd M\big|_B$ be a flag of flats of length $k$ indexed by rank without a flat of rank $i$ and $\mathscr{F}_{\hat{i}} = (F_1, \dots, F_{k+1}) = \mathrm{cl}\left(\mathscr{F}'_{\hat{i}}\right) \lhd M$. Then:*

$$\sigma_{\mathscr{F}'_{\hat{i}}} \subset \mathrm{Im}\left(p_b\big|_{B(M)^{\mathrm{codim}(1)}}\right) \quad \Leftrightarrow \quad [F_{i+1}, F_{i-1}] \text{ is not a diamond poset.}$$

Proof. For (a) note that $p_B$ is injective on $B(M)\big|_B$ due to Lemma 2.1.1.5 (c). Suppose there is a weight class $\sigma_{\mathscr{G}} \subset B(M) \setminus B(M)\big|_B$ such that $p_B(\sigma_{\mathscr{G}}) = p_B(\sigma_{\mathscr{F}})$. Thus $\mathscr{G} \neq \mathscr{F}$ and Lemma 2.1.1.2 (d) implies that $\mathrm{rank}(\mathscr{F}) < \mathrm{rank}(\mathscr{G})$ but this contradicts the maximality of $\mathrm{rank}(\mathscr{F})$.
For (b) consider a weight class $\sigma_{\mathscr{F}} \subset B(M)^{(k+1)} \setminus B(M)\big|_B^{(k+1)}$, i.e. $\mathscr{F} \lhd M$ is maximal and $B \notin M_{\mathscr{F}}$. According to Lemma 2.1.3.19 this implies $\mathscr{F} \cap B$ is not maximal, i.e. $p_B(\sigma_{\mathscr{F}}) = \sigma_{\mathscr{F}\cap B}$ is not $k+1$-dimensional. Thus $p_B$ is not injective on $\sigma_{\mathscr{F}}$ and $\sigma_{\mathscr{F}\cap B} \subset B\left(M\big|_B\right)^{(\mathrm{codim}(1))}$.
For (c) note that $\mathscr{F}_{\hat{i}} \cap B = \mathscr{F}'_{\hat{i}}$ with $B \in M_{\mathscr{F}_{\hat{i}}}$. Moreover, we indexed $\mathscr{F}_{\hat{i}}$ by rank, i.e. $\mathscr{F}_{\hat{i}}$ has length $k$ and misses a flat of rank $i$. Thus we have (w.l.o.g.) $F'_j = \{b_1, \dots, b_j\}$ for all $j \neq i$. From $\mathscr{F}_{\hat{i}}$ we obtain two adjacent flags of flats of maximal length: since $F'_{i+1} \setminus F'_{i-1} = \{b_i, b_{i+1}\}$ there are flags of flats $\mathscr{F}_1 = \mathscr{F}'_{\hat{i}} + \left(F'_{i-1} \cup \{b_i\}\right)$ and $\mathscr{F}_2 = \mathscr{F}'_{\hat{i}} + \left(F'_{i-1} \cup \{b_{i+1}\}\right)$ in $M\big|_B$ that differ only in rank $i$.

Now suppose $\sigma_{\mathscr{F}_{\hat{i}}'} \subset \operatorname{Im}\left(p_B\big|_{B(M)^{\mathrm{codim}(1)}}\right)$. Then $M_{\mathrm{cl}(\mathscr{F}_1)} \neq M_{\mathrm{cl}(\mathscr{F}_1)}$ — otherwise $p_B$ would be injective on $\sigma\left(M_{\mathscr{F}_{\hat{i}}}\right) = \sigma\left(M_{\mathrm{cl}(\mathscr{F}_1)}\right) = \sigma\left(M_{\mathrm{cl}(\mathscr{F}_2)}\right)$ due to Lemma 2.1.1.5 (b) and (c), see also Theorem 1.2.3.20. Hence, $\mathrm{cl}(\mathscr{F}_1)$ and $\mathrm{cl}(\mathscr{F}_2)$ are adjacent as they differ only in rank $i$, and the inequality $M_{\mathrm{cl}(\mathscr{F}_1)} \neq M_{\mathrm{cl}(\mathscr{F}_2)}$ is equivalent to $[F_{i-1}, F_{i+1}]$ not being a diamond poset (Theorem 1.2.3.20). Vice versa suppose $[F_{i-1}, F_{i+1}]$ is not a diamond poset. By Theorem 1.2.3.20 this is equivalent to $M_{\mathrm{cl}(\mathscr{F}_1)} \neq M_{\mathrm{cl}(\mathscr{F}_2)}$. Hence, $\sigma_{\mathscr{F}_{\hat{i}}} \subset B(M)^{(\mathrm{codim}(1))}$ and $p_B\left(\sigma_{\mathscr{F}_{\hat{i}}}\right) = \sigma_{\mathscr{F}_{\hat{i}}'} \subset \operatorname{Im}\left(p_B\big|_{B(M)^{\mathrm{codim}(1)}}\right)$. □

The core statements of Lemma 2.1.1.7 are (a) and (b): due to Lemma 2.1.1.5 the projection $p_B$ is injective on $B(M)\big|_B^{(k+1)}$, the full dimensional cones of $B(M)$ with $B$ as minimal basis. Statement (a) tells us that to any top-dimensional cone $\sigma$ of the induced coarse subdivision corresponds a unique top-dimensional cone in $B(M)\big|_B^{(k+1)}$ and this is the only cone of $B(M)$ mapped to $\sigma$. This will be beneficial with regard to the reconstruction of a hypersurface in $B(M)$ from all its projections. Due to (b) the residual part is mapped to the image of the codimension one skeleton of $B(M)$ and, therefore, can be seen as the critical locus of the projection $p_B$. Statement (c) has an algorithmic benefit: it tells us a way to check whether a weight class of $B\left(M\big|_B\right)$ belongs to the non-injectivity locus.

**Example 2.1.1.8.** Consider the rank 3 matroid $M$ on the ground set $E = [5]$ associated to the matrix

$$G = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Fix the basis $B = \{3,4,5\}$. Then $M\big|_B$ is the uniform matroid $U_{3,3}$. Hence, $B\left(M\big|_B\right) = L_3^3 = \mathbb{R}^3$ due to Lemma 2.1.1.2 (c). We consider $B\left(M\big|_B\right) \subseteq \mathbb{R}^3/\mathbf{1}_3 \cong \mathbb{R}^2$ by considering representatives with last coordinate equal to zero and we use the labels shown in Figure 10 (A). We examine the induced coarse subdivision of $B\left(M\big|_B\right)$. There are six weight classes $\sigma_x$ labeled by $x \in \{I, II, III, IV, V, VI\}$. As a start consider $\sigma_x$, its flag of flats $\mathscr{F}_x$ in $M\big|_B$ and the lift $\mathrm{cl}(\mathscr{F}_x)$ in $M$ for $x = I, II$:

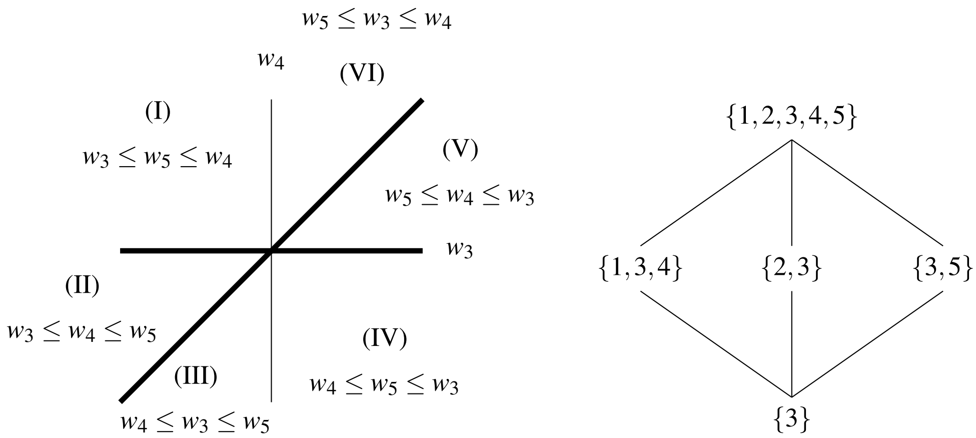
$$\begin{aligned} & \mathscr{F}_I : \{3\} \subsetneq \{3,5\} \subsetneq \{3,4,5\} && \text{and} && \mathscr{F}_{II} : \{3\} \subsetneq \{3,4\} \subsetneq \{3,4,5\}, \\ \Rightarrow\quad & \mathrm{cl}(\mathscr{F}_I) : \{3\} \subsetneq \{3,5\} \subsetneq \{1,2,3,4,5\} && \text{and} && \mathrm{cl}(\mathscr{F}_{II}) : \{3\} \subsetneq \{1,3,4\} \subsetneq \{1,2,3,4,5\}, \\ \Rightarrow\quad & M_{\mathrm{cl}(\mathscr{F}_I)} = \{\{3,5,a\} : a \in \{1,2,4\}\} && \text{and} && M_{\mathrm{cl}(\mathscr{F}_{II})} = \{\{3,b,c\} : b \in \{1,4\}, c \in \{2,5\}\}. \end{aligned}$$

Consequently, $M_{\mathrm{cl}(\mathscr{F}_I)} \neq M_{\mathrm{cl}(\mathscr{F}_{II})}$ and the face $\sigma_{\mathscr{F}_I} \cap \sigma_{\mathscr{F}_{II}}$ described by $\mathscr{F}_I - F_2 = (\{3\}, \{3,4,5\})$ is contained in $\operatorname{Im}\left(p_B\big|_{B(M)^{\mathrm{codim}(1)}}\right)$. Indeed, the sublattice $[\mathrm{cl}(3), \mathrm{cl}(3,4,5)]$ of the lattice of flats of $M$ is not a diamond poset (see Figure 10 (B)). Note that for $\sigma_{VI}$ we have

$$\begin{aligned} \mathscr{F}_{VI} : &\quad \{5\} \subsetneq \{3,5\} \subsetneq \{3,4,5\} \\ \Rightarrow\quad \mathrm{cl}(\mathscr{F}_{VI}) : &\quad \{5\} \subsetneq \{3,5\} \subsetneq \{1,2,3,4,5\} \\ \Rightarrow\quad M_{\mathrm{cl}(\mathscr{F}_{VI})} &= \{\{3,5,a\} : a \in \{1,2,4\}\}. \end{aligned}$$

Hence, $M_{\mathrm{cl}(\mathscr{F}_{VI})} = M_{\mathrm{cl}(\mathscr{F}_I)}$ and $\sigma_I \cup \sigma_{VI} = p_B\left(\sigma(M_{\mathrm{cl}(\mathscr{F}_I)})\right)$.

Due to Lemma 2.1.1.5 and Lemma 2.1.1.7 the induced coarse subdivision lends itself to define a inverse map on its top-dimensional cones:




(A) $B\left(M\big|_B\right)$ for $B = \{3,4,5\}$ with its coarse subdivision.

(B) The sublattice $[\mathrm{cl}(\{3\}), \mathrm{cl}(\{3,4,5\})]$.

FIGURE 10. $L_3^3$ and a sublattice of the lattice of flats arising from $M[G]$ of Example 2.1.1.8.

**Definition 2.1.1.9** (Lifting map)**.** Let $M$ be a matroid and $p_B : B(M) \to B\left(M\big|_B\right)$ the coordinate projection according to the basis $B$ of $M$. Let $\sigma_{\mathscr{F}'} \subset B\left(M\big|_B\right)$ be a weight class. The map

$$h_B : B\left(M\big|_B\right) \to B(M)\big|_B,\ \sigma_{\mathscr{F}'} \mapsto \sigma_{\mathrm{cl}(\mathscr{F}')}$$

is called *lift* and is the inverse map of $p_B\big|_{B(M)\big|_B}$.

**Remark 2.1.1.10** (Properties of lifts $h_B$)**.** As stated in Remark 2.1.1.3, $B\left(M\big|_B\right) = B\left(U_{k+1,k+1}\right)$ is a complete fan, i.e. $h_B$ is defined on $\mathbb{R}^{k+1}$. We get the lift of an arbitrary cone $\sigma \subset \mathbb{R}^{k+1}$ by assembling the lifts of its refinement by the fine subdivision. Therefore, it is sufficient to describe the lift of a cone $\sigma \subset \sigma_{\mathscr{F}}$ with $\mathscr{F} \lhd M\big|_B$. The lift $h_B(\sigma)$ is obtained from $\mathrm{cl}(\mathscr{F})$: we adjoin the missing coordinates indexed by $B^{\complement}$ to the generators of $\sigma$ in its $\mathscr{V}$-description with values specified by $\mathrm{cl}(\mathscr{F})$, cf. Definition 1.2.3.14. Recall that $p_B$ is injective on cones of $B(M)\big|_B$. If $\sigma_{\mathscr{F}} \subset B\left(M\big|_B\right)$ is contained in a top-dimensional cone of the induced coarse subdivision then $h_B(\sigma_{\mathscr{F}}) = \sigma_{\mathrm{cl}(\mathscr{F})}$ is the unique preimage of $\sigma_{\mathscr{F}}$ under $p_B$ (Lemma 2.1.1.7). Consequently, there is a unique correspondence between cones of $B\left(M\big|_B\right) \setminus \bar{\mathscr{T}}_B$ and $B(M)\big|_B^{(k+1)}$.

**Example 2.1.1.11.** As an illustrating example consider the tropical linear space $X = B(U_{3,4})\big/\mathbf{1}_4$ shown in Figure 11. We restrict to representatives with last coordinate equal to zero. Let $\Sigma$ denote the 1-dimensional subfan in $X$ defined by the rays

$$\sigma_1 = \begin{bmatrix}-1\\-1\\0\end{bmatrix}, \sigma_2 = \begin{bmatrix}-1\\0\\-2\end{bmatrix}, \sigma_3 = \begin{bmatrix}-2\\0\\-1\end{bmatrix}, \sigma_4 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \sigma_5 = \begin{bmatrix}1\\0\\1\end{bmatrix}. \tag{20}$$

The top-dimensional cones of $X$ form the coarse subdivision. The red rays illustrate $\Sigma$ in $X$. All projections of rays of $\Sigma$ are colored as follows: the green rays are contained in top-dimensional cones of the induced coarse subdivision, i.e. their lifts under $h_B$ belong to $\Sigma$. The blue rays in all projections illustrate rays that are the image of a top-dimensional cone of the coarse subdivision of

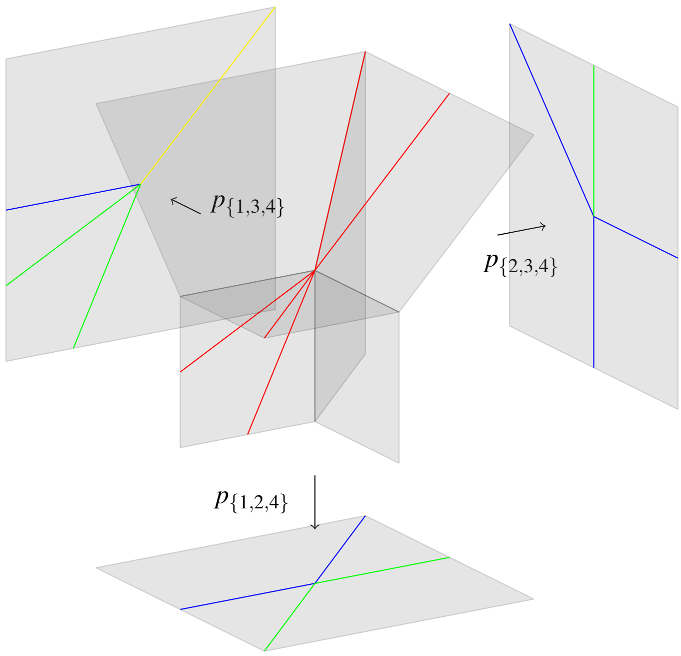



FIGURE 11. Coordinate projections of a 1-dimensional fan $\Sigma$ contained in a general tropical linear space $L_4^3 = B(U_{3,4})$. The coloring is explained in Example 2.1.1.11.

$X$. Hence, the lift of such a ray is not necessarily a ray of $\Sigma$. The lift of the yellow ray is $\sigma_4 \in \Sigma$ but the projection of $\sigma_5$ is also the yellow ray. Note that $\Sigma$ can be balanced. However, if $\Sigma$ is the tropicalization of a hypersurface in a linear space is examined in [**Bir16**], cf. Remark 2.1.1.12.

**Remark 2.1.1.12** (Realizability of fans contained in Bergman fans)**.** We partially rely on results of [**Bir16**] (building on [**Gub13**] and [**ST08**]), in which coordinate projections are used to attack the relative tropical inverse problem, i.e. the task to find for a given balanced fan $\Sigma$ in a tropicalized linear space $B(M)$ an ideal whose tropicalization yields $\Sigma$. Contrary, our approach aims at computing tropicalizations for given ideals, however, we can rely on similar tools involving coordinate projections.

At last we focus on the lattices involved in coordinate projections:

**Lemma 2.1.1.13.** *Let $M$ be a matroid of rank $k+1$ on the ground set $[n+1]$ and let $B$ be a basis of $M$. Let $\sigma_{\mathscr{F}} \subset B(M)$ be the weight class associated to $\mathscr{F} \lhd M$ with $B \in M_{\mathscr{F}}$. Then $p_B|_{\sigma_{\mathscr{F}}}$ is a bijection of lattices. In particular, $\left[\Lambda_{\sigma_{\mathscr{F}\cap B}} : p_B\left(\Lambda_{\sigma_{\mathscr{F}}}\right)\right] = 1$.*

PROOF. The lattice of $B(M)$ is $\Lambda = \mathbb{Z}^{n+1}$. The lattice of $B\left(M|_B\right)$ is the lattice $\mathbb{Z}_B = p_B(\Lambda)$, the restriction of $\Lambda$ to coordinates indexed by $B$. Consider any weight class $\sigma_{\mathscr{F}}$ with $\mathscr{F} \lhd M$ such that $B \in M_{\mathscr{F}}$. As $B \in M_{\mathscr{F}}$ the map $p_B|_{\sigma_{\mathscr{F}}}$ is a bijection (Lemma 2.1.1.5 (a)). Additionally, $p_B|_{\sigma_{\mathscr{F}}}$ is a bijection of lattices: the lattice $\Lambda_{\sigma_{\mathscr{F}}}$ is generated by all incidence vectors $e_{F_i}$, $F_i \in \mathscr{F}$ (cf. Remark 1.2.3.16). The image cone $p_B(\sigma_{\mathscr{F}}) = \sigma_{\mathscr{F}\cap B}$ is generated by all $e_{F_i \cap B}$, $F_i \in \mathscr{F}$ (cf. Lemma 2.1.1.2). That way $p_B|_{\sigma_{\mathscr{F}}}$ provides a unique correspondence between lattice points of $\Lambda_{\sigma_{\mathscr{F}}}$ and $p_B\left(\Lambda_{\sigma_{\mathscr{F}}}\right) = \Lambda_{\sigma_{\mathscr{F}\cap B}}$. We conclude that $\left[\Lambda_{\sigma_{\mathscr{F}\cap B}} : p_B\left(\Lambda_{\sigma_{\mathscr{F}}}\right)\right] = 1$. $\square$

### 2.1.2. Algebraic and Tropical Coordinate Projections

Recall the superior goal of this section: we like to retrieve $\mathscr{T}$ from all of its coordinate projections. For that purpose we need to know the projections. In this part we introduce the algebraic set up and study the interplay of algebraic and tropical coordinate projections.
The first challenge, the projection of $\mathscr{T}$ according to a basis $B$ of $M(\mathscr{I})$, is accomplished with a statement of [**Bir16**] (based on [**ST08**], [**Gub13**]): in a nutshell, tropicalizations commute with projections. The outcome of this is that the projection of a tropical hypersurface contained in a linear space ends up as the tropicalization of a variety defined by a single polynomial. This allows a feasible computation of the push forward since the defining polynomial of the hypersurface can be obtained by elimination. All details are summarized in the beginning of the section and we follow [**Bir16**]. We renew notations 2.1:

NOTATION 2.2. In the following $\mathscr{I} = \langle l_0, \ldots, l_{n-k-1} \rangle \subset R_{n+1} = \mathbb{K}[x_0, \ldots, x_n]$ denotes an ideal generated by linear forms $l_i = \sum_j a_{ij} x_j$ with constant coefficients (cf. Section 1.4.4), i.e. $l_i \in \Bbbk[x_0, \ldots, x_n]$ for all $i$. Let $M := M(\mathscr{I})$ denote the matroid of rank $k+1$ associated to $\mathscr{I}$ and let $\mathscr{B}$ be its set of bases. The linear ideal $\mathscr{I}$ defines a linear subspace $X = \mathscr{V}(\mathscr{I}) \subset \mathbb{P}^n$. Let $B = \{b_0, \ldots, b_k\}$ be a basis of $M$. Let $\mathbb{R}_B$ denote the restriction of $\mathbb{R}^{n+1}$ to the coordinates indexed by $B$. The restriction of $R_{n+1}$ to the variables indexed by $B$ is denoted by $R_B := \mathbb{K}\left[x_{b_0}, \ldots, x_{b_k}\right]$. In the following $F \in \Bbbk[x_0, \ldots, x_n]$ denotes a homogeneous polynomial with constant coefficients.

Consider the coordinate projection that maps $X$ to the projective space $\mathbb{P}^k$ with coordinates indexed by $B$. Since $B$ is a basis of $M = M(\mathscr{I})$ the map is an isomorphism inducing an isomorphism of rings:

$$\Phi_B : R_B \longrightarrow R_{n+1}/\mathscr{I},\ x_i \longmapsto x_i + \mathscr{I}.$$

We focus on the inverse isomorphism $\pi_B = (\Phi_B)^{-1}$ that is defined by

$$\pi_B : R_{n+1}/\mathscr{I} \longrightarrow R_B,\ \bar{x}_i \longmapsto \sum_j c_{i,b_j} x_{b_j}$$

with $C := \left(c_{i,b_j}\right) \in K^{n+1 \times k+1}$ uniquely determined. The basis $B$ implies that we can express variables indexed by $B^{\complement}$ by a linear combination of variables indexed by $B$, i.e. $x_i - \sum_{j=1}^{k+1} c_{i,b_j} x_{b_j} \in \mathscr{I}$ for all $i$. The variables indexed by the basis $B$ remain identical whereas the variables indexed by $B^{\complement}$ are substituted by the linear relations obtained from $\mathscr{I}$. For $F \in R_{n+1}$ we write $F_B := \pi_B(\bar{F}) \in R_B$. Algebraically, we replace $x_i$ in $F$ by $\sum_{j=1}^{k+1} c_{i,b_j} x_{b_j}$ for all $i$.

**Definition 2.1.2.14** (Relative Newton polytope)**.** Let $F \in R_{n+1}$ be a homogeneous polynomial of degree $d$. Let $\mathscr{B} = \{B_1, \ldots, B_m\}$ denote a set of bases of the matroid $M$ associated to the linear ideal $\mathscr{I}$. We call $\mathscr{N}_{\mathscr{B}} = \{N_1, \ldots, N_m\}$ the *relative Newton polytope of $F$ with respect to $\mathscr{B}$* where $N_i = \operatorname{Newt}(F_{B_i}) \subset \mathbb{R}_{B_i}$ denotes the Newton polytope of $F_{B_i} = \pi_{B_i}(\bar{F}) \in R_{B_i}$ according to $B_i$.

As $F$ is homogeneous of degree $d$ all polynomials $F_{B_i}$ are likewise homogeneous of degree $d$. Thus $N_i \subset d \cdot \Delta_{k+1}$ for any basis $B_i = \{b_{i_1}, \ldots, b_{i_{k+1}}\}$ of $M$.

NOTATION 2.3. We write $\mathscr{T} = \operatorname{trop}(\mathscr{V}(\mathscr{I} + \langle F \rangle))$ and $\mathscr{T}(N_i) = \mathscr{T}_{B_i} = \operatorname{trop}(\mathscr{V}(F_{B_i}))$.

In the remaining part we establish a connection between $\mathscr{T}$ and $\mathscr{T}_B$. For this purpose the concept of push forwards of fans is essential. We follow [**GKM09**] and introduce push forwards in terms of Section 1.1. Let $\Sigma \subset V = \mathbb{R}^n$, $\Sigma' \subset V' = \mathbb{R}^k$ be two fans with lattices $\Lambda = \mathbb{Z}^n$, $\Lambda' = \mathbb{Z}^k$.

**Definition 2.1.2.15** (Morphism of fans and push forwards)**.** Let $f : \Lambda \longrightarrow \Lambda'$ be a $\mathbb{Z}$-linear map of lattices. $f$ is called *morphism* if $f$ is a map from $|\Sigma| \subset V$ to $|\Sigma'| \subset V'$ and $\mathbb{Z}$-linear. The *image fan* $f_*(\Sigma)$ is obtained by picking a suitable refinement (cf. [**GKM09**, Construction 2.24]) of $\Sigma$ such that the set of cones $f_*(\Sigma) := \{f(\sigma) | \sigma \subset \Sigma \text{ and } f \text{ injective on } \sigma\}$ forms a fan in $V'$. If $\Sigma$ is a weighted fan with weight function $\omega$ then each maximal dimensional cone $\sigma' \in f_*(X)$ is defined to have a weight

$$\omega_{f_*(\Sigma)}(\sigma') := \sum_{\substack{\sigma \in \Sigma: \\ f(\sigma) = \sigma'}} \omega(\sigma) \cdot \left[\Lambda'_{\sigma'} : f(\Lambda_\sigma)\right].$$

If $(\Sigma, \omega)$ is a tropical fan (i.e. $(\Sigma, \omega)$ satisfies the balancing condition, cf. Definition 1.1.2.12), then the image fan $\left(f_*(\Sigma), \omega_{f_*(\Sigma)}\right)$ is a tropical fan ([**GKM09**, Proposition 2.25]). The following theorem is a building block for our purpose of a reconstruction of $\mathscr{T}$ from all of its coordinate projections $\mathscr{T}_B$. It is stated in [**Bir16**, Theorem 3.4] (and is based on [**ST08**], [**Gub13**]):

THEOREM 2.1.2.16. *We work with notations 2.2 and 2.3, i.e. let $X = \mathscr{V}(\mathscr{I}) \subset \mathbb{P}^n$ be a linear space defined by a linear ideal $\mathscr{I} \subset R_{n+1}$ with constant coefficients and $B$ a basis of the associated matroid $M$. Then*

$$\operatorname{trop}(\mathscr{V}(F_B)) = (p_B)_*(\mathscr{T})$$

*for every homogeneous polynomial $F \in R_{n+1}$ that is not divisible by a monomial.*

Thus, projections commute with tropicalizations and $(p_B)_*(\mathscr{T})$ occurs as the tropical hypersurface of $F_B$ arising from $F$ with variables indexed by $B^{\complement}$ eliminated. Since $\mathscr{T}_B = \operatorname{trop}(\mathscr{V}(F_B)) \subset B\left(M\big|_B\right)$, we can apply results developed in Section 2.1.1. This is the subject of study in the following section.

### 2.1.3. The Reconstruction of $\mathscr{T}$ from all $\mathscr{T}_B$

Now we turn our attention to the actual reconstruction process. As before the linear ideal $\mathscr{I}$ and the homogeneous polynomial $F \in R_{n+1}$ of degree $d$ are given. We assume that $F$ is not divisible my a monomial. The goal is to reconstruct $\mathscr{T}$ from the tropical hypersurfaces $\mathscr{T}_B$ with $B \in \mathscr{B}$. Even though $\mathscr{T}_B$ is easily obtained from the Newton polytope of $F_B$ (Theorem 2.1.2.16) there are some issues we are confronted with: both the tropical hypersurface $\mathscr{T}_B$ and the critical locus of $p_B$, $\bar{\mathscr{T}}_B$, are $k$-dimensional. Consequently, a cone $\sigma \subset \mathscr{T}_B$ that is contained in $\bar{\mathscr{T}}_B$ may be the image of different cones $\sigma_i \subset \mathscr{T}$ satisfying $p_B(\sigma_i) = \sigma$ for all $i$ (see also the Definition 2.1.2.15 of push forwards). Furthermore, each $\mathscr{T}_B$, arising as a subfan of the normal fan of $N_i$, carries the coarsest possible fan structure which not necessarily reflects the fan structure of $\mathscr{T}$.

**Convention 2.1.3.17** (Fixation of fan structures)**.** As $\mathscr{T}$ is contained in $B(M)$ we can consider $\mathscr{T}$ refined by the fine subdivision of $B(M)$. Analogously we consider $\mathscr{T}_B$ refined by the fine subdivision of $B\left(M\big|_B\right)$. We consider the ambient spaces $B(M)$ and $B\left(M\big|_B\right)$ with their "coarse" subdivisions, i.e. $B(M)^{(k+1)}$ is the set of top-dimensional cones of the coarse subdivision. The fan $B\left(M\big|_B\right)$ is considered with the induced coarse subdivision.

We continue using Notation 2.2 and state a basic observation concerning $\mathscr{T}$. Note that we have $\dim(B(M)) = k+1$ and, therefore, $\dim(\mathscr{T}) = k$. Then, for all $\sigma \subset \mathscr{T}^{(k)}$ one of the following conditions hold:

- There is a maximal flag $\mathscr{F} \lhd M$ such that $\sigma \subset \sigma_{\mathscr{F}} \subset \sigma(M_{\mathscr{F}}) \subset B(M)^{(k+1)}$.
- There is a flag $\mathscr{F}_{\hat{i}} \lhd M$ of length $k$ such that $\sigma \subset \sigma_{\mathscr{F}_{\hat{i}}} \subset \sigma\left(M_{\mathscr{F}_{\hat{i}}}\right) \subset B(M)^{\operatorname{codim}(1)}$.

We summarize this classification of top-dimensional cones of $\mathscr{T}$ in the following

**Corollary 2.1.3.18.** *Let $\mathscr{T}$ denote the tropicalization of $\mathscr{V}(\mathscr{I} + \langle F \rangle)$ where $\mathscr{I} \subset R_{n+1}$ is a linear ideal and $F \in R_{n+1}$ a homogeneous polynomial of degree $d$. Let $M$ denote the matroid associated to $\mathscr{I}$. Furthermore,* $\operatorname{rank}(M) = \dim(\mathscr{V}(\mathscr{I})) = k+1$ *so that* $\dim(\mathscr{T}) = k$. *We consider $\mathscr{T}$ refined by the fine subdivision of $B(M)$. By $\mathscr{F}_{\hat{j}} \lhd M$ we denote a flag of length $k$ missing a flat of rank $j$. Then we get a partition $\mathscr{T} = \bigsqcup_{i=0}^{k} C_i$ where $C_0$ and $C_i$ are weighted fans defined by*

$$C_0 = \left\{\sigma \subset \mathscr{T}^{(k)} : \exists \mathscr{F} \lhd M \text{ maximal such that } \sigma \subset \sigma_{\mathscr{F}} \subset \sigma(M_{\mathscr{F}}) \subset B(M)^{(k+1)}\right\} \text{ and}$$

$$C_i = \left\{\sigma \subset \mathscr{T}^{(k)} : \exists \mathscr{F}_{\hat{i}} \lhd M \text{ of length } k \text{ such that } \sigma \subset \sigma_{\mathscr{F}_{\hat{i}}} \subset \sigma\left(M_{\mathscr{F}_{\hat{i}}}\right) \subset B(M)^{\operatorname{codim}(1)}\right\}.$$

*for $i = 1, \ldots, k$.*

As a subfan of $\mathscr{T}$ the polyhedral set $C_i$ becomes a weighted fan for $0 = 1, \ldots, k$. It is clear from the definition that $C_0$ plays a special role in the partition. For $\tau \in C_0$ the weight class $\sigma_{\mathscr{F}}$ containing $\tau$ is defined by a maximal flag $\mathscr{F} \lhd M$ whereas $\tau \in C_j$ for any $j \in \{1, \ldots, k\}$ is contained in a weight class $\sigma_{\mathscr{F}_{\hat{j}}}$ defined by a flag $\mathscr{F}_{\hat{j}} \lhd M$ missing a flat of rank $j$.

**Lemma 2.1.3.19.** *Let $M$ be a matroid on $[n+1]$ of rank $k+1$ and let $B$ be a basis. Then:*

- *If $\mathscr{F} \lhd M$ is maximal then: $\mathscr{F} \cap B$ is maximal if and only if $B \in M_{\mathscr{F}}$.*
- *If $\mathscr{F}_{\hat{i}} \lhd M$ misses a flat of rank $i$ then: $\mathscr{F}_{\hat{i}} \cap B$ misses a flat of rank $i$ if and only if $B \in M_{\mathscr{F}_{\hat{i}}}$.*

PROOF. Both "if" directions follow immediately with the Greedy algorithm: if $B \in M_{\mathscr{G}}$ for a flag $\mathscr{G} = (G_1, \ldots, G_p) \lhd M$ indexed by rank then $G_i \cap B$ is a basis of $G_i$, i.e. $\operatorname{rank}(G_i) = \operatorname{rank}(G_i \cap B)$ and, therefore, $\mathscr{G}$ and $\mathscr{G} \cap B$, as the case may be, miss flats of identical rank.
Now we show the "only-if" directions separately. First, let $\mathscr{F} \lhd M$ be maximal, i.e. $F_i \in \mathscr{F}$ has rank $i$. Suppose $\mathscr{F} \cap B \lhd M\big|_B$ is maximal. Thus $F_i \cap B$ has rank $i$ in $M\big|_B$. Since $F_i \cap B \subset B$ there is an ordering on $[k+1]$ such that $F_i \cap B = \{b_{j_1}, \ldots, b_{j_i}\}$ for all $i$. Hence, $F_i \cap B \setminus F_{i-1} \cap B = \{b_{j_i}\}$, i.e. $B$ can be obtained by the Greedy algorithm 1.2.3.19. Thus $B \in M_{\mathscr{F}}$.
Consider $\mathscr{F}_{\hat{i}} \lhd M$, indexed by rank. Suppose $\mathscr{F}_{\hat{i}} \cap B$ misses a flat of rank $i$. In general, we have $\operatorname{rank}(F_j \cap B) \leq \operatorname{rank}(F_j)$. Consequently, for $\mathscr{F}_{\hat{i}} \cap B$, we have $\operatorname{rank}(F_j \cap B) = \operatorname{rank}(F_j) = j$ for all $j \neq i$. Thus $F_j \cap B$ is a basis for $F_j$ and, therefore, $B$ can be obtained by the Greedy algorithm applied to $\mathscr{F}_{\hat{i}}$, i.e. $B \in M_{\mathscr{F}_{\hat{i}}}$. □

**Remark 2.1.3.20** ($M_{\mathscr{F}}$ for $\mathscr{F} \lhd M$ maximal)**.** Notice that Lemma 2.1.1.5 (a) and Lemma 2.1.3.19 have interesting consequences for weight classes $\sigma_{\mathscr{F}}$ defined by $\mathscr{F} \lhd M$ maximal. The map $p_B\big|_{\sigma_{\mathscr{F}}}$ is a bijection if and only if $B \in M_{\mathscr{F}}$. This equivalence does not hold for arbitrary flags, particularly not for flags that miss a flat of certain rank (cf. Remark 2.1.3.21). We can think of $M_{\mathscr{F}}$ as the set of bases that project $\sigma(M_{\mathscr{F}})$ bijectively.

**Remark 2.1.3.21** ($M_{\mathscr{F}_{\hat{j}}}$ for $\mathscr{F}_{\hat{j}} \lhd M$ missing a flat of rank $j$)**.** Consider a flag $\mathscr{F}_{\hat{j}} \lhd M$ missing a flat of rank $j$. Let $B$ be a basis such that $p_B\big|_{\sigma_{\mathscr{F}_{\hat{j}}}}$ is a bijection, i.e. $\mathscr{F}_{\hat{j}}$ and $\mathscr{F}_{\hat{j}} \cap B$ have same length. Equivalently, $\mathscr{F}_{\hat{j}} \cap B$ misses a flat of rank $i \geq j$. According to Lemma 2.1.3.19, $\mathscr{F}_{\hat{j}} \cap B$ misses a flat of rank $i = j$ if and only if $B \in M_{\mathscr{F}_{\hat{j}}}$. Hence, we can think of $M_{\mathscr{F}_{\hat{j}}}$ as the set of bases that map $\sigma_{\mathscr{F}_{\hat{j}}}$ bijectively <u>and</u> preserve the rank of the missing flat.

Now we show that $\mathscr{T}$ is uniquely determined by all of its coordinate projections $\mathscr{T}_B$. Thereby we make use of the partition of $\mathscr{T}$ (cf. Corollary 2.1.3.18). In particular, part 2.1.3.1 deals with the reconstruction of $C_0$ and part 2.1.3.2 deals with $C_i$ for $i = 1, \dots, k$. In part 2.1.3.3 we state the main result of this section.

### 2.1.3.1. Cones of $\mathscr{T}_B \setminus \bar{\mathscr{T}}_B$

As a start consider the cones of $\mathscr{T}_B \setminus \bar{\mathscr{T}}_B$. Recall that $\mathscr{T}_B \setminus \bar{\mathscr{T}}_B$ is contained in the top-dimensional part of $B\left(M\big|_B\right)$ with respect to the induced coarse subdivision. The coarse subdivision is well-understood due to Lemma 2.1.1.5 and Lemma 2.1.1.7, i.e. we can take first steps towards the reconstruction of $\mathscr{T}$ from all $\mathscr{T}_B$. We stick to Convention 2.1.3.17, i.e. we consider $\mathscr{T}_B$ with its fine subdivision. We define the set

$$D_0 = \{h_B(\sigma) : \exists B \in \mathscr{B} \text{ such that } \sigma \subset \mathscr{T}_B \setminus \bar{\mathscr{T}}_B\}. \tag{21}$$

We define a weight on $\tau = h_B(\sigma) \in D_0$ by setting

$$\omega(\tau) = \omega_{\mathscr{T}_B}(\sigma). \tag{22}$$

In the following lemma we show that $C_0$ and $D_0$ have equal support and $\omega_{\mathscr{T}}\big|_{C_0} = \omega\big|_{D_0}$:

**Lemma 2.1.3.22.** *The weighted fan $C_0$, defined in Corollary 2.1.3.18, and the weighted fan $D_0$, defined in Equations* (21) *to* (22)*, coincide.*

PROOF. At first we only consider the supports $|C_0|$ and $|D_0|$. In the end we show that the weights defined on $C_0$ and $D_0$ coincide.
Consider $\tau \in D_0$. By definition there is a basis $B$ and a cone $\sigma \subset \mathscr{T}_B \setminus \bar{\mathscr{T}}_B$ such that $\tau = h_B(\sigma)$. This, in turn, implies the existence of a maximal flag $\mathscr{F} \lhd M$ with $B \in M_{\mathscr{F}}$ such that

$$\sigma \subset \sigma_{\mathscr{F}\cap B} \subset p_B\left(\sigma\left(M_{\mathscr{F}}\right)\right) \subset B\left(M\big|_B\right)^{(k+1)}.$$

We have $\mathscr{T}_B = (p_B)_*(\mathscr{T})$ (Theorem 2.1.2.16). Then $\omega_{\mathscr{T}_B}(\sigma) \neq 0$ (cf. Definition 2.1.2.15) implies that there is at least one cone $\tau' \in \mathscr{T}$ such that $p_B(\tau') = \sigma$. Note that $p_B(\tau) = p_B(h_B(\sigma)) = \sigma$. We know that $p_B(B(M)\big|_B) = B\left(M\big|_B\right)$ and $p_B$ is injective on $(B(M)\big|_B)^{(k+1)}$ (Lemma 2.1.1.5). Moreover, $\sigma_{\mathscr{F}}$ is the unique weight class of $B(M)$ mapped to $\sigma_{\mathscr{F}\cap B}$ (Lemma 2.1.1.7). Hence, $\tau' = h_B(\sigma) = \tau \subset \sigma_{\mathscr{F}} \subset \sigma(M_{\mathscr{F}}) \subset (B(M)\big|_B)^{(k+1)}$ with $\mathscr{F} \lhd M$ maximal, i.e. $h_B(\sigma) = \tau \subset \mathscr{T}^{(k)}$ and, therefore, $h_B(\sigma) = \tau \in C_0$.
Vice versa, consider $\tau \in C_0$. By definition $\tau \subset \mathscr{T}^{(k)}$ and there is a maximal flag $\mathscr{F} \lhd M$ such that $\tau \subset \sigma_{\mathscr{F}} \subset \sigma(M_{\mathscr{F}}) \subset B(M)^{(k+1)}$. We know that $p_B$ is injective on $\sigma(M_{\mathscr{F}}) \subset (B(M)\big|_B)^{(k+1)}$ for any $B \in M_{\mathscr{F}}$ (Lemma 2.1.1.5), i.e. $\mathscr{F} \cap B$ is maximal (this also follows with Lemma 2.1.3.19). Hence, $p_B(\tau) \subset \sigma_{\mathscr{F}\cap B} \subset B\left(M\big|_B\right)^{(k+1)}$. From Lemma 2.1.1.13 and $\omega_{\mathscr{T}}(\tau) > 0$ we conclude that $\omega_{\mathscr{T}_B}(p_B(\tau)) > 0$, i.e. $p_B(\tau) \subset \mathscr{T}_B \setminus \bar{\mathscr{T}}_B$ for all $B \in M_{\mathscr{F}}$. This implies $h_B(p_B(\tau)) = \tau \in D_0$.

It remains to show that the weights $\omega$ and $\omega_{\mathscr{T}}$ coincide on $C_0 = D_0$. Take any $\tau \in D_0$. Due to the definition of $D_0$ there is a basis $B$ and $\sigma \subset \mathscr{T}_B \setminus \bar{\mathscr{T}}_B$ such that $\tau = h_B(\sigma)$. Let $\mathscr{F}' \lhd M\big|_B$ be the maximal flag such that $\sigma \subset \sigma_{\mathscr{F}'}$. Let $\mathscr{F} \lhd M$ denote the closure of $\mathscr{F}'$. Then $\tau \subset \sigma_{\mathscr{F}} \subset \sigma(M_{\mathscr{F}})$ and $B \in M_{\mathscr{F}}$, i.e. the map $p_B$ is injective on $\sigma(M_{\mathscr{F}})$ (Lemma 2.1.1.5). Note that $p_B(\tau) = \sigma$. From $\sigma \subset \mathscr{T}_B \setminus \bar{\mathscr{T}}_B$ we get $0 \neq \omega_{\mathscr{T}_B}(\sigma)$. Hence, by the definition of the push forward (cf. Definition 2.1.2.15), there must be at least one cone $\tau' \in \mathscr{T}$ such that $p_B(\tau') = \sigma$. From Lemma 2.1.1.7 we conclude that $\tau' = \tau$, i.e. there is only $\tau \in C_0 \subset \mathscr{T}$ such that $p_B(\tau) = \sigma \subset \mathscr{T}_B$. Accordingly, the weight of $\sigma$ in $\mathscr{T}_B$ equals

$$0 \neq \omega_{\mathscr{T}_B}(\sigma) = \omega_{\mathscr{T}}(\tau)\left[\Lambda_\sigma : p_B(\Lambda_\tau)\right].$$

In particular, $p_B\big|_{\sigma_{\mathscr{F}}}$ is a bijection between the lattices $\Lambda_{\sigma_{\mathscr{F}}}$ and $\Lambda_{\sigma_{\mathscr{F}\cap B}}$ (Lemma 2.1.1.13). Hence, we get $\left[\Lambda_{\sigma_{\mathscr{F}\cap B}} : p_B\left(\Lambda_{\sigma_{\mathscr{F}}}\right)\right] = 1$. This likewise holds for $\tau = h_B(\sigma) \subset \sigma_{\mathscr{F}}$: $\left[\Lambda_\sigma : p_B(\Lambda_\tau)\right] = 1$. Recall that we defined $\omega(\tau) = \omega_{\mathscr{T}_B}(\sigma)$. From the equation above we get $\omega_{\mathscr{T}_B}(\sigma) = \omega_{\mathscr{T}}(\tau)$, i.e. $\omega(\tau) = \omega_{\mathscr{T}}(\tau)$ as desired. □

It turns out that $C_0$ can be obtained from all projections $\mathscr{T}_B$ of $\mathscr{T}$. By definition $C_0$ contains all top-dimensional cones of $\mathscr{T}$ that are contained in $B(M)^{(k+1)}$. We get $\mathscr{T} - C_0 = \sum_{j=1}^k C_j \subset B(M)^{\mathrm{codim}(1)}$ due to Corollary 2.1.3.18. Consider its push forward, i.e.

$$(p_B)_*(\mathscr{T} - C_0) = (p_B)_*(\mathscr{T}) - (p_B)_*(C_0) = \mathscr{T}_B - (p_B)_*(C_0) = (p_B)_*\left(\sum_{j=1}^k C_j\right) \subset \bar{\mathscr{T}}_B. \tag{23}$$

Consequently, $(p_B)_*(\mathscr{T} - C_0)$ is the residue of $\mathscr{T}_B$ after removing all images of cones of $\mathscr{T}$ that are contained in cones of $B(M)^{(k+1)}$. Unfortunately, $\bar{\mathscr{T}}_B$ equals the image of $B(M) \setminus B(M)\big|_B^{(k+1)}$, the critical locus of $p_B$ (Lemma 2.1.1.7). At this point we like to stress again that cones of $C_0$ are contained in cones $\sigma(M_{\mathscr{F}})$ where $\mathscr{F} \lhd M$ can be chosen maximal. The situation changes for cones of $C_i$ with $i \neq 0$: there, a weight class containing a cone of $C_i$ is defined by a flag $\mathscr{F}_{\hat{i}}$ of length $k$, i.e. a flat of rank $i$ is missing. The push forwards of $\mathscr{T} - C_0$ only depends on cones of $C_i$ for $i = 1, \dots, k$ (cf. Equation (23)).

#### 2.1.3.2. Cones of $(p_B)_*(\mathscr{T} - C_0) \subset \bar{\mathscr{T}}_B$

Given a cone $\sigma \subset (p_B)_*(\mathscr{T} - C_0)$ we face the following decision problem arising from the fact that $\sigma \subset \bar{\mathscr{T}}_B$: is $h_B(\sigma)$ a cone of $\mathscr{T}$, or is $\sigma$ the image of a cone $\tau \in \mathscr{T}$ different from $h_B(\sigma)$? We know that $\sigma \subset \sigma_{\mathscr{F}_{\hat{i}}}$ for a flag $\mathscr{F}_{\hat{i}} \lhd M\big|_B$ of length $k$ missing one flat of rank $i$. However, the equality

$$(p_B)_*(\mathscr{T} - C_0) = (p_B)_*\left(\sum_{j=1}^k C_j\right)$$

reveals that we only deal with projections of cones contained in weight classes defined by flags missing exactly one flat.

**Remark 2.1.3.23** (Projection of codimension one weight classes)**.** Consider a flag $\mathscr{F}_{\hat{i}} \lhd M$ missing a flat of rank $i$, i.e. $\mathscr{F}_{\hat{i}} = (F_1, \dots, F_{i-1}, F_{i+1}, \dots, F_{k+1})$. Let $B$ be a basis such that $\mathscr{F}_{\hat{i}}$ and $\mathscr{F}_{\hat{i}} \cap B$ have identical length, i.e. $p_B\big|_{\sigma_{\mathscr{F}_{\hat{i}}}}$ is injective (Lemma 2.1.1.2 (a)). As $\mathscr{F}_{\hat{i}}$ and $\mathscr{F}_{\hat{i}} \cap B$ have same length, $\mathscr{F}_{\hat{i}} \cap B \lhd M\big|_B$ misses only one flat of rank $l$. Note that $\mathrm{rank}(F_j \cap B) \leq \mathrm{rank}(F_j)$, i.e. the rank of the missing flat in $\mathscr{F}_{\hat{i}} \cap B$ satisfies $l \geq i$. According to Lemma 2.1.3.19 we have $l = i$ if and only if $B \in M_{\mathscr{F}_{\hat{i}}}$.

Remark 2.1.3.23 (based on Lemma 2.1.3.19) approves the following hypothesis: if the image of a weight class $\sigma_{\mathscr{G}}$ under $p_B$ equals $\sigma_{\mathscr{F}_{\hat{i}}} \subset B\left(M\big|_B\right)$ we expect that $\mathscr{G}$ misses at most one flat of rank $j \leq i$. This hypothesis fits into the results we have so far: for a given flag $\mathscr{F} \triangleleft M$, Lemma 2.1.1.2 (d) does not tell us any details about the actual form of a flag $\mathscr{G} \triangleleft M$ satisfying $\mathscr{G} \neq \mathscr{F}$ and $p_B(\sigma_{\mathscr{F}}) = p_B(\sigma_{\mathscr{G}})$ for a basis $B$. The conclusion is that $\operatorname{rank}(\mathscr{G}) > \operatorname{rank}(\mathscr{F})$. This inequality facilitates the following characterization:

**Lemma 2.1.3.24.** *Let $M$ be a rank $k+1$ matroid on the ground set $[n+1]$ and let $\mathscr{B}$ denote the set of basis of $M$. Pick any basis $B \in \mathscr{B}$. Let $\mathscr{F}'_{\hat{i}} \triangleleft M\big|_B$ be a flag of flats of length $k$, i.e. $\mathscr{F}'_{\hat{i}} = (F'_1, \ldots, F'_{k+1})$ indexed by rank with a flat of rank $i$ missing. By $\operatorname{cl}\left(\mathscr{F}'_{\hat{i}}\right) = \mathscr{F}_{\hat{i}} = (F_1, \ldots, F_{k+1}) \triangleleft M$ we denote the flag of flats in $M$ of length $k$ that is likewise indexed by rank and misses a rank $i$ flat. With a suitable order on the elements of $B$ we have*

$$F_j \cap B = \{b_1, \ldots, b_j\} \quad \forall j \neq i. \tag{24}$$

*As stated in Remark 2.1.1.4 we get $F_{i+1} \setminus F_{i-1} = \sqcup_j \bar{F}_{ij}$ from the sublattice $[F_{i-1}, F_{i+1}]$. If there exists a flag $\mathscr{G} = (G_1, \ldots, G_p) \triangleleft M$ such that $\mathscr{F}_{\hat{i}} \neq \mathscr{G}$ and $p_B(\sigma_{\mathscr{G}}) = p_B(\sigma_{\mathscr{F}_{\hat{i}}})$ then $\mathscr{G}$ satisfies the following conditions:*

(a) *$\mathscr{G}$ can be indexed by rank and either $\mathscr{G} = \mathscr{G}_{\max} = (G_1, \ldots, G_{k+1})$ has maximal length $k+1$ or $\mathscr{G} = \mathscr{G}_{\hat{q}} = \left(G_1, \ldots, G_{q-1}, G_{q+1}, \ldots, G_{k+1}\right)$ has length $k$ and misses a flat of rank $q \leq i-1$.*
(b) *Using (a) it holds $G_l = F_l$ for all $l \geq i+1$.*
(c) *If $\mathscr{G}$ misses a flat of rank $q \in \{1, \ldots, i-1\}$ it holds*
   - *$G_l = F_l$ for all $1 \leq l \leq q-1$, and*
   - *$G_l = \operatorname{cl}(F_{l-1} + \{x\})$ for all $q+1 \leq l \leq i$ and an element $x \in \bar{F}_{ij}$ for some $j$ and $x \notin B$.*
(d) *If $\mathscr{G}$ is maximal there is an element $q \in \{1, \ldots, i-1\}$ such that*
   - *$G_l = F_l$ for all $1 \leq l \leq q$, and*
   - *$G_l = \operatorname{cl}(F_{l-1} + \{x\})$ for all $q+1 \leq l \leq i$ and an element $x \in \bar{F}_{ij}$ for some $j$ and $x \notin B$.*

PROOF. Let $\mathscr{G} = (G_1, \ldots, G_p) \triangleleft M$ be a flag of flats such that $p_B(\sigma_{\mathscr{G}}) = p_B\left(\sigma_{\mathscr{F}_{\hat{i}}}\right)$. Since $\mathscr{F}_{\hat{i}}$ only misses a rank $i$ flat we have

$$\operatorname{rank}(\mathscr{F}_{\hat{i}}) = \left(\sum_{j=1}^{k+1} j\right) - i = \frac{(k+1)(k+2)}{2} - i.$$

Suppose we reindex the flats of $\mathscr{G}$ by rank. Moreover, suppose that $\mathscr{G}$ misses $m$ flats of rank $t_1, \ldots, t_m$. Then we have

$$\operatorname{rank}(\mathscr{G}) = \left(\sum_{j=1}^{k+1} j\right) - \sum_{j=1}^{m} t_j = \frac{(k+1)(k+2)}{2} - \sum_{j=1}^{m} t_j.$$

By Lemma 2.1.1.2 (d) we know that $\operatorname{rank}(\mathscr{G}) > \operatorname{rank}(\mathscr{F}_{\hat{i}})$. This implies

$$\operatorname{rank}(\mathscr{G}) > \operatorname{rank}(\mathscr{F}_{\hat{i}}) \quad \Longleftrightarrow \quad i > \sum_{j=1}^{m} t_j.$$

We conclude that $t_j < i$ for all $j \in \{1, \ldots, m\}$. Thus $\mathscr{G}$ is saturated with flats of rank $l \in \{i, \ldots, k+1\}$ and if $\mathscr{G}$ misses a flat then $\mathscr{G}$ only misses flats of rank lower than $i$.

Note that $F_j \cap B = \{b_1, \dots, b_j\}$ is a basis for $F_j$ for all $j \neq i$. The equality $p_B(\sigma_{\mathscr{G}}) = p_B(\sigma_{\mathscr{F}_{\hat{i}}})$ is equivalent to $\mathscr{F}_{\hat{i}} \cap B = \mathscr{G} \cap B$. This yields the following condition obtained from the proof of part (d) of Lemma 2.1.1.2:

(25) For all $j \neq i$ exists an element $h_j \in \{1, \dots, p\}$ such that $F_j \cap B = G_{h_j} \cap B$.

Thus $\forall j \neq i$ it holds: $F_j \cap B = G_{h_j} \cap B = \{b_1, \dots, b_j\}$. For all $1 \leq r < s \leq k$ we have $G_{h_r} \subsetneq G_{h_s}$, i.e. $\operatorname{rank}(G_{h_r}) \lneq \operatorname{rank}(G_{h_s})$. Thus $G_{h_1} \subsetneq \dots \subsetneq G_{h_{k+1}}$ is a flag of length $k$ that consists of pairwise distinct flats of $\mathscr{G}$, i.e. for $\mathscr{G}$ holds $k \leq p$. As $\operatorname{rank}(M) = k+1$ we conclude $k \leq p \leq k+1$, i.e. $\mathscr{G}$ has length $k$ or $k+1$. This means $m \in \{0, 1\}$ for the constraints of the missing flats of $\mathscr{G}$ (see above). Hence, $\mathscr{G}$ is maximal or misses a flat of rank $q \leq i-1$. This shows (a).

Now, we focus on flats of $\mathscr{G}$ having rank greater or equal to $i+1$. Recall from (a) that $\mathscr{G}$ is saturated with flats of rank $i+1$ to $k+1$, i.e. there are $k+1-i$ flats involved. By assumption $\mathscr{F}_{\hat{i}}$ also contains a flat of rank $l$ for each $l \in \{i+1, \dots, k+1\}$. Due to the statement in Equation (25) and Equation (24) it holds that $j = \operatorname{rank}(F_j) = \operatorname{rank}(F_j \cap B) = \operatorname{rank}(G_{h_j} \cap B) \leq \operatorname{rank}(G_{h_j})$ for each $j \in \{i+1, \dots, k+1\}$. For the boundaries we get $i+1 \leq \operatorname{rank}(G_{h_{i+1}})$ and $\operatorname{rank}(G_{h_{k+1}}) \leq k+1$. Hence, the flats of the chain $G_{h_{i+1}} \subsetneq \dots \subsetneq G_{h_{k+1}}$ are flats of $\mathscr{G}$ with rank between $i+1$ and $k+1$. Since $\operatorname{rank}(G_{h_j}) \lneq \operatorname{rank}(G_{h_{j+1}})$ for all $j \in \{i+1, \dots, k\}$ the chain contains $k+1-i$ different flats $G_{h_j}$ of $\mathscr{G}$, one for each rank between $i+1$ and $k+1$. Consequently, $\operatorname{rank}(G_{h_j}) = j$ for all $j \in \{i+1, \dots, k+1\}$, i.e. $h_j = j$ and $G_{h_j} = G_j$ for $i+1 \leq j \leq k+1$. From the condition stated in Equation (25) we get $F_j \cap B = G_j \cap B = \{b_1, \dots, b_j\}$. As $\operatorname{rank}(G_j) = j$ for $i+1 \leq j \leq k+1$ we get $F_j = G_j$ for $j \in \{i+1, \dots, k+1\}$. This shows (b).

At last we concentrate on flats of $\mathscr{G}$ with rank lower or equal to $i$. From (a) we know that $\mathscr{G}$ has either $i$ or $i-1$ flats of rank lower or equal to $i$. The flag $\mathscr{F}_{\hat{i}}$ has $i-1$ flats of rank lower or equal to $i$ since it misses a rank $i$ flat. Thus condition stated in Equation (25) translates to: for all $j \in \{1, \dots, i-1\}$ exists $h_j \in \{1, \dots, i\}$ such that $F_j \cap B = G_{h_j} \cap B$. Hence, $G_{h_1} \subsetneq \dots \subsetneq G_{h_{i-1}}$ has length $i-1$ and is part of the flats of $\mathscr{G}$ with ranks from 1 to $i$. Thus $G_{h_1} \subsetneq \dots \subsetneq G_{h_{i-1}}$ differs from the flats of $\mathscr{G}$ with rank lower or equal to $i$ by at most one flat, in dependence of the length of $\mathscr{G}$.

- Suppose $\mathscr{G}$ has length $k$, i.e. there is an element $q \in \{1, \dots, i-1\}$ such that $\mathscr{G}$ misses a flat of rank $q$ (cf. part (a)). Then $\mathscr{G}$ contains $i-1$ flats of rank lower or equal to $i$. Since $G_{h_1} \subsetneq \dots \subsetneq G_{h_{i-1}}$ has length $i-1$ and is part of the flats of $\mathscr{G}$ with ranks from 1 to $i$ the flags coincide, i.e. we get $h_j = j$ for all $j \leq q-1$ and $h_j = j+1$ for all $j \geq q$.
- Suppose $\mathscr{G}$ is maximal, i.e. there are $i$ flats of rank lower or equal to $i$. Then there is an element $q \in \{1, \dots, i-1\}$ such that $G_q \cap B = G_{q+1} \cap B$ since the $i$ flats of $\mathscr{G}$ of rank lower or equal to $i$ provide $i-1$ flats in $\mathscr{G} \cap B = \mathscr{F}_{\hat{i}} \cap B$ of rank lower or equal to $i-1$. As $G_q$ and $G_{q+1}$ provide the identical flat in $\mathscr{G} \cap B$ we can choose (w.l.o.g.) $h_j \neq q$ for all $j \in \{1, \dots, i-1\}$. In other words, we get $h_j = j$ for all $j \leq q-1$ and $h_j = j+1$ for all $j \geq q$.

We continue with $\mathscr{G}$ missing a flat of rank $q \in \{1, \dots, i-1\}$. For $j \leq q-1$ we have $h_j = j$, i.e. $G_{h_j} = G_j$ has rank $j$. Due to condition stated in Equation (25) we get

$$F_j \cap B = G_{h_j} \cap B = G_j \cap B = \{b_1, \dots, b_j\}.$$

Hence, we have $F_j = G_j$ for all $j \leq q-1$. For $j \geq q$ we have $h_j = j+1$ and, therefore, we get $F_j \cap B = G_{h_j} \cap B = G_{j+1} \cap B = \{b_1, \ldots, b_j\}$. Recall that $G_{j+1}$ has rank $j+1$. Hence, for each $j \geq q$ there must be an element $x_j \in [n+1]$ independent of $b_1, \ldots, b_j$ such that $G_{j+1} = \mathrm{cl}(b_1, \ldots, b_j, x_j)$. For $j = q$ we have $G_{q+1} = \mathrm{cl}(b_1, \ldots, b_q, x_q)$, i.e. $x_q$ and $b_1, \ldots, b_q$ are independent. As $b_{q+1} \notin G_{q+1}$ we know that $b_{q+1}$ and $b_1, \ldots, b_q, x_q$ are independent. Thus $\{b_1, \ldots, b_q, b_{q+1}, x_q\}$ has rank $q+2$ and $\{b_1, \ldots, b_q, b_{q+1}, x_q\} \subset G_{q+2} = \mathrm{cl}\left(b_1, \ldots, b_{q+1}, x_{q+1}\right)$, i.e. $G_{q+2} = \mathrm{cl}\left(b_1, \ldots, b_q, b_{q+1}, x_q\right)$. In other words, we can replace $x_{q+1}$ by $x_q$. By successively repeating this argument we conclude $x_q = \ldots = x_{i-1} =: x$. For $j = i-1$ we have $G_{h_{i-1}} = G_i = \mathrm{cl}(b_1, \ldots, b_{i-1}, x)$. The rank of $G_i$ is $i$ and $G_i \subset G_{i+1} = \mathrm{cl}(b_1, \ldots, b_{i+1}) = F_{i+1}$ (cf. part (b)), i.e. we see that

$$F_{i-1} = \mathrm{cl}(b_1, \ldots, b_{i-1}) \subsetneq G_i = \mathrm{cl}(b_1, \ldots, b_{i-1}, x) \subsetneq F_{i+1} = \mathrm{cl}(b_1, \ldots, b_{i+1}).$$

Consequently, $G_i$ is a flat of rank $i$ of the sublattice $[F_{i-1}, F_{i+1}]$ of the lattice of flats, i.e. $G_i = F_{il}$ for some $l$ (cf. Remark 2.1.1.4). In particular, we get $x \in \bar{F}_{il}$ for some $l$. This shows part (c).
In both cases, $\mathscr{G}$ maximal or missing a flat of rank $q \in \{1, \ldots, i-1\}$, we have the identical constraints on the indices $h_j$ with $1 \leq j \leq i-1$. In case of a maximal flag $\mathscr{G}$ the indices $h_j$ skip $q$. Since $G_q \cap B = G_{q+1} \cap B = \{b_1, \ldots, b_q\}$ and $\mathrm{rank}(G_q) = q$ it follows that $G_q = \mathrm{cl}(b_1, \ldots, b_q) = F_q$. This shows (d). $\square$

Lemma 2.1.3.24 unravels the above-mentioned decision problem for flags $\mathscr{F}_{\hat{i}} \lhd M\big|_B$ of length $k$: if $p_B(\sigma_{\mathscr{G}}) = p_B(\sigma_{\mathrm{cl}(\mathscr{F}_{\hat{i}})})$ for a flag $\mathscr{G} \lhd M$ then $\mathscr{G}$ misses at most one flat of rank $j < i$.

#### 2.1.3.3. The Reconstruction

Before stating the main result we summarize substantial results of the previous parts that shed light on the coordinate projections $\mathscr{T}_B$. According to Lemma 2.1.3.19 for any $\tau \in C_i$ and $i \in \{0, \ldots, k\}$ we find a basis $B$ such that $p_B(\tau) \subset \mathscr{T}_B$ is full-dimensional. In addition, the length as well as the rank $i$ of the missing flat of the flag $\mathscr{F} \lhd M$ such that $\tau \subset \sigma_{\mathscr{F}}$ is preserved under $p_B$, i.e. $\mathscr{F} \cap B$ and $\mathscr{F}$ have identical length and miss a flat of rank $i$. Lemma 2.1.3.24 states that all flags $\mathscr{G} \lhd M$ satisfying $\mathscr{F} \cap B = \mathscr{G} \cap B$ miss at most one flat of rank $j < i$. This has interesting consequences: consider any $\sigma \subset (\mathscr{T}_B)^{(k)}$. According to the weight formula of push forwards we have

$$\omega_{\mathscr{T}_B}(\sigma) = \sum_{\substack{\tau \in \mathscr{T}: \\ p_B(\tau) = \sigma}} \omega_{\mathscr{T}}(\tau) \left[\Lambda_\sigma : p_B(\Lambda_\tau)\right].$$

Suppose $\sigma \subset \sigma_{\mathscr{F}_{\hat{i}}}$ for a flag $\mathscr{F}_{\hat{i}} \lhd M\big|_B$ of length $k$ missing a flat of rank $i$. According to our explanations above, the weight formula can be rewritten as

$$\omega_{\mathscr{T}_B}(\sigma) = \sum_{j=0}^{i} \sum_{\substack{\tau \in C_j: \\ p_B(\tau) = \sigma}} \omega_{\mathscr{T}}(\tau) \left[\Lambda_\sigma : p_B(\Lambda_\tau)\right].$$

The only flag in $M$ that preserves length and the rank of the missing flat $i$ under $p_B$ is $\mathrm{cl}\left(\mathscr{F}_{\hat{i}}\right) \lhd M$ (Lemma 2.1.3.19 and Lemma 2.1.3.24). Now we like to know if there is an element $\tau \in C_i$ such that $p_B(\tau) = \sigma$. Surprisingly, we can deduce this information from $C_0, \ldots, C_{i-1}$ and $\mathscr{T}_B$ because

$$\omega_{\mathscr{T}_B}(\sigma) - \sum_{j=0}^{i-1} \sum_{\substack{\tau \in C_j: \\ p_B(\tau) = \sigma}} \omega_{\mathscr{T}}(\tau) \left[\Lambda_\sigma : p_B(\Lambda_\tau)\right] \begin{cases} = 0 \text{ if } \nexists \tau \in C_i \text{ with } p_B(\tau) = \sigma, \\ \neq 0 \text{ if } \exists \tau \in C_i \text{ with } p_B(\tau) = \sigma. \end{cases}$$

This is the basic idea of the proof of the main result of this section. We can reconstruct $\mathscr{T}$ from all $\mathscr{T}_B$ by a successive reconstruction of the weighted fans $C_i$, initiating with $i = 0$.

THEOREM 2.1.3.25. *Let $F \in R_{n+1}$ be a homogeneous polynomial not divisible by any monomial and let $\mathscr{I} \subset R_{n+1}$ be a linear ideal. Let $M$ denote the matroid associated to $\mathscr{I}$. Then, the tropical variety $\mathscr{T} = \operatorname{trop}(\mathscr{V}(\mathscr{I} + \langle F \rangle))$ is completely determined by all coordinate projections $\mathscr{T}_B = (p_B)_*(\mathscr{T})$ according to bases $B$ of $M$.*

PROOF. According to Corollary 2.1.3.18 we have $\mathscr{T} = \sqcup_{i=0}^{k} C_i$. We reconstruct all $C_i$ from all $\mathscr{T}_B$ recursively in $k+1$ steps. Therefore, we use a recursion that proceeds as follows: in each step we define

$$\mathscr{T}_B^j = (p_B)_* \left( \mathscr{T} - \sum_{s=0}^{j-1} D_s \right) \tag{26}$$

for all bases $B$ called *$j$-th residue of $\mathscr{T}_B$*. For $j = 0$ we set $\mathscr{T}_B^0 = \mathscr{T}_B$. Then we determine

$$D_j = \left\{ h_B(\sigma) : \exists B \text{ and } \mathscr{F}'_{\hat{j}} \triangleleft M\big|_B \text{ such that } \sigma \subset \sigma_{\mathscr{F}'_{\hat{j}}} \text{ and } \sigma \subset \left( \mathscr{T}_B^j \right)^{(k)} \right\}. \tag{27}$$

The polyhedral set $D_j$ becomes a weighted fan by defining weights $\omega(\tau) = \omega_{\mathscr{T}_B^j}(\sigma)$ for $\tau = h_B(\sigma) \in D_j$.

Our claim is that $D_j = C_j$ for $j = 0, \ldots, k$. We prove the correctness of the recursion by induction on $j$, i.e. we show $C_j = D_j$ for all $j$ and $\omega\big|_{D_j} = \omega_{\mathscr{T}}\big|_{C_j}$. Before stating the actual proof let us highlight the essential idea of the recursion: $\mathscr{T}_B^j$ is the residue of $\mathscr{T}_B$ after removing all images of cones of $\mathscr{T}$ that are contained in weight classes defined by flags $\mathscr{F}_{\hat{s}} \triangleleft M$ with $s < j$. Then, according to Lemma 2.1.3.24, the cones of the $j$-th residue of $\mathscr{T}_B$ contained in weight classes defined by flags of flats $\mathscr{F}_{\hat{j}} \triangleleft M\big|_B$ are necessarily images of cones in $C_j$. These are precisely the cones collected by $D_j$.

The start of the induction is $j = 0$. By Lemma 2.1.3.22 we know that $C_0 = D_0$ and $\omega\big|_{D_0} = \omega_{\mathscr{T}}\big|_{C_0}$. We continue with the induction step. The proof is split into two parts: firstly we show that $C_j$ and $D_j$ have equal support. In the second part we show that the weights coincide.

Suppose there is some $j \leq k-1$ such that we have $C_s = D_s$ for all $s \leq j-1$. We show $C_j = D_j$. By the induction hypothesis we know that $\mathscr{T} - \sum_{s=0}^{j-1} D_s = \mathscr{T} - \sum_{s=0}^{j-1} C_s = \sum_{s=j}^{k} C_k$. We consider the $j$-th residue of $\mathscr{T}_B$:

$$\begin{aligned}
\mathscr{T}_B^j = (p_B)_* \left( \mathscr{T} - \sum_{s=0}^{j-1} D_s \right) &= \mathscr{T}_B - \sum_{s=0}^{j-1} (p_B)_*(D_s) \\
= (p_B)_* \left( \mathscr{T} - \sum_{s=0}^{j-1} C_s \right) & \\
= (p_B)_* \left( \sum_{s=j}^{k} C_s \right). &
\end{aligned}$$

The first line reveals that $\mathscr{T}_B^j$ can be computed purely from $\mathscr{T}_B$ and all $D_s$ with $s \leq j-1$. The last line follows from the induction hypothesis, i.e. $D_s = C_s$ for $s \leq j-1$, and Corollary 2.1.3.18. In particular, it serves as a justification of the name "residue" since $\mathscr{T}_B^j$ is the push forward of the remaining part $\sum_{s=j}^{k} C_s \subset \mathscr{T}$.

Suppose there is a cone $\tau \in C_j$. By definition there is a flag $\mathscr{F}_{\hat{j}} \lhd M$ of length $k$ and missing a flat of rank $j$ such that $\tau \subset \sigma_{\mathscr{F}_{\hat{j}}} \subset B(M)^{\mathrm{codim}(1)}$. Pick any $B \in M_{\mathscr{F}_{\hat{j}}}$. Then $\mathscr{F}'_{\hat{j}} = \mathscr{F}_{\hat{j}} \cap B \lhd M\big|_B$ is a flag of length $k$ missing a flat of rank $j$ (Lemma 2.1.3.19). Moreover, $p_B\big|_{\sigma_{\mathscr{F}_{\hat{j}}}}$ is bijective due to $B \in M_{\mathscr{F}_{\hat{j}}}$ (Lemma 2.1.1.5 (a)). For the image of $\tau$ under $p_B$, $\sigma = p_B(\tau)$, we have $\sigma \subset (\mathscr{T}_B^j)^{(k)}$ and $\sigma \subset \sigma_{\mathscr{F}'_{\hat{j}}}$ where $\mathscr{F}'_{\hat{j}} \lhd M\big|_B$ is a flag of length $k$ missing a rank $j$ flat. Hence, $h_B(\sigma) \in D_j$.
Vice versa, consider a cone $\tau \in D_j$. By definition there is a basis $B$ and a flag $\mathscr{F}'_{\hat{j}} \in M\big|_B$ of length $k$ missing a flat of rank $j$ such that $\sigma \subset (\mathscr{T}_B^j)^{(k)}$ and $\sigma \subset \sigma_{\mathscr{F}'_{\hat{j}}}$. In particular, this implies $\omega_{\mathscr{T}_B^j}(\sigma) \neq 0$. The closure of $\mathscr{F}'_{\hat{j}}$, denoted by $\mathscr{F}_{\hat{j}}$, likewise has length $k$ and misses a flat of rank $j$. Lemma 2.1.3.24 states that any flag $\mathscr{G} \lhd M$ with $\mathscr{G} \neq \mathscr{F}_{\hat{j}}$ and $p_B(\sigma_{\mathscr{F}_{\hat{j}}}) = p_B(\sigma_{\mathscr{G}})$ is either maximal or misses a flat of rank $l < j$. Consequently, if $\sigma$ is the image of a cone $\tau'$ of $\mathscr{T}$ with $\tau' \subset \sigma_{\mathscr{G}}$ then $\tau' \in C_l$ for some $l \leq j-1$. This, in turn, implies that $\tau' \not\subset \sum_{s=j}^{k} C_s$. Since $\omega_{\mathscr{T}_B^j}(\sigma) \neq 0$ there must be a cone in $\sum_{s=j}^{k} C_s$ mapped to $\sigma$. The residual possibility is $h_B(\sigma) = \tau \in C_j$.
It remains to verify the weights. Take any $\tau = h_B(\sigma) \in D_j$. By definition we have $\sigma \subset (\mathscr{T}_B^j)^{(k)}$, contained in a weight class $\sigma_{\mathscr{F}'_{\hat{j}}}$ defined by a flag $\mathscr{F}'_{\hat{j}} \lhd M\big|_B$ of length $k$ missing a flat of rank $j$. We know that $\mathscr{F}_{\hat{j}} = \mathrm{cl}\left(\mathscr{F}'_{\hat{j}}\right) \lhd M$ also has length $k$ and misses flat of rank $j$. Let us take a look at the weight of $\sigma$ using the description of $\mathscr{T}_B^j$ stated above:

$$0 \neq \omega_{\mathscr{T}_B^j}(\sigma) = \sum_{s=j}^{k} \sum_{\substack{\tau \in C_s: \\ p_B(\tau)=\sigma}} \omega_{\mathscr{T}}(\tau)\left[\Lambda_\sigma : p_B(\Lambda_\tau)\right] = \omega_{\mathscr{T}}(\tau)\left[\Lambda_\sigma : p_B(\Lambda_\tau)\right].$$

The last equality holds since $\tau \in D_j = C_j$ and Lemma 2.1.3.24: any weight class $\sigma_{\mathscr{G}} \subset B(M)$ satisfying $\mathscr{G} \neq \mathscr{F}_{\hat{j}}$ and $p_B(\sigma_{\mathscr{F}_{\hat{j}}}) = p_B(\sigma_{\mathscr{G}})$ is either maximal or misses a flat of rank $l \leq j-1$. Thus any cone $\tau' \neq \tau$ mapped to $\sigma$ is contained in a weight class $\sigma_{\mathscr{G}}$ such that $\mathscr{G} \neq \mathscr{F}_{\hat{j}}$ is either maximal (i.e. $\tau' \in C_0$) or misses a flat of rank $l \leq j-1$ (i.e. $\tau' \in C_l$). Since $\tau \in C_j$ is the only cone in $C_j$ mapped to $\sigma$ we get:

$$\omega_{\mathscr{T}_B^j}(\sigma) = \omega_{\mathscr{T}}(\tau)\left[\Lambda_\sigma : p_B(\Lambda_\tau)\right]. \tag{28}$$

From Lemma 2.1.1.13 we know that $\left[\Lambda_{\sigma_{\mathscr{F}\cap B}} : p_B\left(\Lambda_{\sigma_{\mathscr{F}}}\right)\right] = 1$. This likewise holds for cones $\tau \subset \sigma_{\mathscr{F}}$ such that $p_B(\tau) = \sigma$, i.e. we get $[\Lambda_\sigma : p_B(\Lambda_\tau)] = 1$ in Equation (28). Hence, $\omega_{\mathscr{T}_B^j}(\sigma) = \omega_{\mathscr{T}}(\tau)$ which leads to $\omega(\tau) = \omega_{\mathscr{T}}(\tau)$ for $\tau \in D_j$. $\square$

**Remark 2.1.3.26.** So far we make use of all bases $B$ of $M$ in Theorem 2.1.3.25. It is an interesting open problem to determine subsets of the bases of $M$ that are sufficient for the recovery of $\mathscr{T}$.

The proof of Theorem 2.1.3.25 is constructive and is of an algorithmic nature. We can state the proof in the shape of a pseudocode as shown in Algorithm 1.

**Remark 2.1.3.27** (Implementability)**.** We briefly discuss the implementability of the suggested Algorithm 1 in POLYMAKE. The algorithm heavily relies on push forwards and sums of tropical fans (both in row 10 within the computation of $\mathscr{T}_B^j$, see also Equation (26) and Equation (27)). Generally, the computation of a push forward or a sum of tropical fans necessitates to refine fans several times. Naively, cones of the result (i.e. the image of two or more cones, or the sum of cones respectively) may overlap and it is necessary to ensure that the result is a fan again. In case of push forwards this

**Algorithm 1** Tropicalization of $\mathscr{V}(\mathscr{I}+\langle F\rangle)$, constant coefficients.

**Input:** linear ideal $\mathscr{I}\subset R_{n+1}$, homogeneous polynomial $F\in R_{n+1}$ not divisible by monomials.

compute matroid $M=M(\mathscr{I})$, its bases $\mathscr{B}$ and $\mathscr{T}_B=\operatorname{trop}(\mathscr{V}(F_B))$ for all $B\in\mathscr{B}$.
consider $\mathscr{T}_B$ with fine subdivision, define $k=\operatorname{rank}(M)-1$, $D_0:=\emptyset$, $\omega:=0$ and $\mathscr{T}_B^0=\mathscr{T}_B$.
**for all** $B\in\mathscr{B}$ **do**
    **for all** $\sigma\subset\mathscr{T}_B^{(k)}\setminus\bar{\mathscr{T}}_B$ **do**
        Add $\tau=h_B(\sigma)$ to $D_0$, define $\omega(\tau)=\omega_{\mathscr{T}_B}(\sigma)$.
    **end for**
**end for**
make $D_0$ a weighted fan.
**for** $j\in\{1,\dots,k\}$ **do**
    compute $\mathscr{T}_B^j=\mathscr{T}_B^{j-1}-(p_B)_*\left(D_{j-1}\right)$ for all $B$ and define $D_j:=\emptyset$.
    **for all** $B\in\mathscr{B}$ **do**
        **for all** $\sigma\subset(\mathscr{T}_B^j)^{(k)}$ **do**
            **if** $\exists\mathscr{F}_{\hat{j}}\lhd M\big|_B:\sigma\subset\sigma_{\mathscr{F}_{\hat{j}}}$ **then**
                Add $\tau=h_B(\sigma)$ to $D_j$, define $\omega(\tau)=\omega_{\mathscr{T}_B^j}(\sigma)$.
            **end if**
        **end for**
    **end for**
    make $D_j$ a weighted fan.
**end for**

**Output:** $\left(\bigcup_j D_j,\omega\right)$.

is achieved by refining the initial fan of the map (cf. [**GKM09**, Construction 2.24]). In our case we have a map $p_B$ for each basis $B$ of the matroid $M$, i.e. in each round of the recursion it is necessary to compute a refinement that suffices for all $p_B$ simultaneously. A priori, this necessitates the continual computation of refinements which are computational intense. So far there are no practical algorithms implemented in POLYMAKE to compute arbitrary push forwards or sums of tropical cycles due to the explained complications. This expels the algorithm from a general straightforward-implementation using implemented algorithms in POLYMAKE. If the tropical linear space has dimension two then the problems caused by overlapping disperse. The contained tropical hypersurface is 1-dimensional, i.e. we deal with rays. Hence, proper overlapping does not happen: either rays are equal or not. Moreover, the group index simplifies to a greatest common divisor computation. For details we refer to Section 2.1.4.

**Remark 2.1.3.28** (Divisible polynomials)**.** If we consider a linear ideal $\mathscr{I}\subset R_{n+1}$ with constant coefficients and a homogeneous polynomial $F\in R_{n+1}$ that is divisible by a monomial then we cannot proceed as in the proof of Theorem 2.1.3.25. In particular, the push forward of $\mathscr{T}$ does not necessarily coincide with $\operatorname{trop}(\mathscr{V}(F_B))$. Hence, we cannot compute the tropicalization of the projective variety $\mathscr{V}(\mathscr{I}+\langle F\rangle)$. However, if we consider $\mathscr{V}(\mathscr{I}+\langle F\rangle)\subset T^{n+1}$ then we can divide $F$ by the problematic monomial $x^\beta$ and continue with the resulting polynomial.

**Remark 2.1.3.29** (Non-constant coefficient case)**.** Theorem 2.1.3.25 and the resulting algorithm 1 can be extended to polynomials $F\in R_{n+1}$ with non-constant coefficients. Basically, the proof of the theorem (algorithm respectively) is based on properties of the ambient tropical linear space. In particular, the fine subdivision plays the key role. However, there is no reason to restrict the lift $h_B$ (cf.

Definition 2.1.1.9) solely to cones contained in weight classes. Please note that Theorem 2.1.2.16 likewise holds for non-constant polynomials ([**Bir16**]).

### 2.1.4. Tropical Curves in 2-Dimensional Tropical Linear Spaces

In this part we present an algorithm that computes the tropicalization of an algebraic curve contained in a 2-dimensional linear space. It is an adapted version of algorithm 1 tailored to the special circumstances of curves in 2-dimensional linear spaces. Before going into details, we outline some aspects of curves and their ambient linear spaces and highlight their computational assets.
In the following we show why, in contrast to the general case, curves in 2-dimensional linear spaces are computationally manageable (cf. Remark 2.1.3.27). As tropical curves are 1-dimensional polyhedral fans their cones are simplicial, i.e. each cone can be described by one generator (cf. part 2.1.4.1). This allows improvements and simplifications in computations occurring in the algorithmic solution presented in Section 2.1.3, e.g. the computation of projections, lattice indices and refinements (cf. parts 2.1.4.2, 2.1.4.3). Moreover, the developed algorithm 1 works intensely with the fine subdivision of Bergman fans. Here, we only deal with matroids of rank 3. This leads to a small number of codimension one weight classes (cf. part 2.1.4.4). All efforts lead to an algorithm designed for curves (cf. algorithm 2) that is available for SINGULAR and uses POLYMAKE for polyhedral computations (cf. introduction of this chapter). We fix the set up for the rest of this section:

NOTATION 2.4. We adapt Notations 2.1 to 2.3. Fix some integer $n \in \mathbb{N}$, $n \geq 3$. We work with a linear ideal $\mathscr{I} = \langle l_0, \ldots, l_{n-3} \rangle \subset R_{n+1}$ generated by linear forms $l_i = \sum_j a_{ij} x_j$ and let $F \in R_{n+1}$ be a homogeneous polynomial of degree $d$. Recall that the polynomial $F$ as well as all linear forms $l_i$ have constant coefficients. The linear space $X = \mathscr{V}(\mathscr{I}) \subset \mathbb{P}^n$ is 2-dimensional, $\mathscr{V}(\mathscr{I} + \langle F \rangle) \subset \mathbb{P}^n$ is 1-dimensional, i.e. it is a curve. The matroid $M = M(\mathscr{I})$ associated to the ideal $\mathscr{I}$ has rank 3. In the following we consider tropicalizations in $\mathbb{R}^{n+1} / \mathbf{1}_{n+1}$: $\operatorname{trop}(X) = B(M) \subset \mathbb{R}^{n+1} / \mathbf{1}_{n+1}$ is a tropical linear space, $\dim(B(M)) = \operatorname{rank}(M) - 1 = 2$ and $\mathscr{T} = \operatorname{trop}(\mathscr{V}(\mathscr{I} + \langle F \rangle)) \subset \operatorname{trop}(X)$ is a tropical curve. As before $(p_B)_* (\mathscr{T}) = \mathscr{T}_B$ denotes the push forward of $\mathscr{T}$ with regard to the basis $B$ of $M$. We denote the weight function of $\mathscr{T}$ by $\omega_{\mathscr{T}}$ and the weight function of the $j$-th residue of $\mathscr{T}_B$ by $\omega_{\mathscr{T}_B^j}$.

Moreover, we make use of the following convention:

**Convention 2.1.4.30.** We identify $\mathbb{R}^n$ with $\mathbb{R}^{n+1} / \mathbf{1}_{n+1}$ via the isomorphism that maps $(x_1, \ldots, x_n)$ to $(x_1, \ldots, x_n, 0)$. Moreover, we assume that $F$ is not divisible by a monomial.

With Remark 2.1.3.27 in mind we proceed by examining the trouble spots of Algorithm 1 for the curve-case. We fix the basis $B = \{0, 1, 2\} \subset [n+1]$. Let us briefly take a look at the tropical curve $\mathscr{T} \subset \mathbb{R}^{n+1} / \mathbf{1}_{n+1}$.

#### 2.1.4.1. Data Describing Curves

As $\mathscr{T} \subset \mathbb{R}^n$ is a curve it consists of finitely many 1-dimensional cones $\sigma_i$ called rays that we index by $I \subset \mathbb{N}$. The curve $\mathscr{T}$ is a pointed fan and the only ridge $\tau \subset \mathscr{T}$ is the origin, i.e. the rays $\sigma_i$ are attached to the vertex $V_\tau = \{0\}$. The primitive normal vector $u_{\sigma_i/\tau} \in \Lambda_{\sigma_i} / \Lambda_\tau = \Lambda_{\sigma_i}$ generates $\sigma_i$ and has a unique representative $v_{\sigma_i/\tau} \in \Lambda = \mathbb{Z}^{n+1}$ with last coordinate equal to zero (cf. Convention 2.1.4.30). This representative satisfies $\gcd(v_{\sigma_i/\tau}) = 1$ since $u_{\sigma_i/\tau}$ is a primitive generator of

$\Lambda_{\sigma_i} / \Lambda_\tau$. In the following we denote this unique representative of $u_{\sigma_i/\tau}$ by $v_i$. Consequently, $v_i$ and $\mathbf{1}_{n+1}$ are primitive generators of $\sigma_i$ when considered in $\mathbb{R}^{n+1}$. We conclude that the weight function $\omega_{\mathscr{T}}$ together with the collection of generators $v_i$ of $\sigma_i$ with $i \in I$ describe the tropical curve $\mathscr{T}$ uniquely. The remaining parts of this section deal with the computation of this data.

#### 2.1.4.2. Projections of Rays

We turn our attention to the coordinate projections of the curve $\mathscr{T}$ according to the basis $B = \{0,1,2\}$ of $M$. We have $|B\left(M\big|_B\right)| = \mathbb{R}^3 \big/ \mathbf{1}_3$, i.e. the Bergman fan is 2-dimensional. Thus, $\mathscr{T}_B = \operatorname{trop}\left(\mathscr{V}\left(F_B\right)\right)$ is a tropical curve. According to the definition of push forwards (cf. Definition 2.1.2.15) the weight of $\sigma' \subset (\mathscr{T}_B)^{(1)}$ is defined by the the equation

$$\omega_{\mathscr{T}_B}\left(\sigma'\right) = \sum_{\substack{\sigma \subset \mathscr{T}: \\ p_B(\sigma) = \sigma'}} \omega_{\mathscr{T}}\left(\sigma\right)\left[\Lambda_{\sigma'} : p_B\left(\Lambda_\sigma\right)\right].$$

Let us determine each summand of the weight individually. Thereby we focus on the lattice index. For $\sigma' \subset \mathscr{T}_B$, $\sigma \subset \mathscr{T}$ rays such that $p_B(\sigma) = \sigma'$ we denote the unique representatives with last coordinate equal to zero by $v'$ and $v$. By abuse of notation, let $p_B(v) \in \mathbb{R}^3$ denote the representative of $p_B(v) \in \mathbb{R}^3 \big/ \mathbf{1}_3$ with last coordinate equal to zero. We have $V_{\sigma'} = \langle v' \rangle = \langle p_B(v) \rangle \in \mathbb{R}^3 \big/ \mathbf{1}_3$ since we deal with one dimensional cones. As $p_B(v) \in \mathbb{Z}^3 \big/ \mathbf{1}_3$ we have $p_B(v) \in \Lambda_{\sigma'}$. We know that $v'$ generates $\mathbb{Z} \cong \Lambda_{\sigma'}$ whereas $p_B(v)$ generates a subgroup of $\mathbb{Z}$. In particular, we have

$$\gcd(p_B(v))v' = p_B(v) \in \mathbb{Z}^3 \big/ \mathbf{1}_3.$$

To see this note that there is some $\lambda \in \mathbb{R}$ such that $v' = \lambda p_B(v)$. Then,

$$1 = \gcd\left(v'\right) = \gcd\left(\lambda p_B(v)\right) = \lambda \gcd(p_B(v)),$$

as $v'$ is a primitive, and $\frac{1}{\lambda} = \gcd\left(p_B(v)\right)$. Hence, the lattice index simplifies to a greatest common divisor calculation. This allows to reformulate the summand arising from $\sigma \subset \mathscr{T}$ contributing to the weight of $\sigma' \subset \mathscr{T}_B$ as $\omega_{\mathscr{T}}(\sigma)\gcd\left(p_B(v)\right)$. In total we get

$$\omega_{\mathscr{T}_B}\left(\sigma'\right) = \sum_{\substack{i \in I: \\ p_B(v_i) = \gcd(p_B(v_i))v'}} \omega_{\mathscr{T}}\left(\sigma_i\right)\gcd\left(p_B(v_i)\right).$$

#### 2.1.4.3. Cones of $\mathscr{T}$ Contributing to $\mathscr{T}_B$

With regard to the last part it would be beneficial to know the set of $\sigma_i \subset \mathscr{T}$ contributing to $\mathscr{T}_B$ for a fixed basis $B$. Let $v_i \in \mathbb{Z}^{n+1}$ be the unique representative of the primitive normal vector of a ray $\sigma_i \subset \mathscr{T}$. Then, either $p_B(v) = (v_0, v_1, v_2) = \mathbf{0}_3 \in \mathbb{Z}^3 \big/ \mathbf{1}_3$ or $p_B(v) = (v_0, v_1, v_2) \neq \mathbf{0}_3 \in \mathbb{Z}^3 \big/ \mathbf{1}_3$. For the latter case, let $\sigma' = \operatorname{cone}(p_B(v))$ denote the one dimensional cone. From $\omega_{\mathscr{T}}(\sigma_i) > 0$ for all $i \in I$ and the fact that lattice indices take values in $\mathbb{N}$ we know that $\sigma' = \operatorname{cone}(p_B(v)) \subset (\mathscr{T}_B)^{(1)}$ with $\omega_{\mathscr{T}_B^j}(\sigma') > 0$. Accordingly, the projection of a ray of $\mathscr{T}$ is a ray of $\mathscr{T}_B$ or equals the origin in $\mathbb{R}^3 \big/ \mathbf{1}_3$. This makes refinements redundant that are necessary for push forwards in the general case (cf. Definition 2.1.2.15). As the $j$-th residue of $\mathscr{T}_B$ can be computed from the residue $\mathscr{T}_B^{j-1}$ of the previous iteration and $D_j$ (cf. row 10 in algorithm 1) we obtain a feasible algorithm for the computation of the $j$-th residue $\mathscr{T}_B^j$. Lastly, note that it is easy to check whether $\sigma \subset \mathscr{T}$ contributes to $\mathscr{T}_B$. Let $v$ be the generating element of $\sigma$. Either the three coordinates of $v$ indexed by $B$ coincide or not.

#### 2.1.4.4. The Image of the Codimension One Skeleton of $B(M)$

First, note that $|B(M|_B)| = \mathbb{R}^3/\mathbf{1}_3 \cong \mathbb{R}^2$, i.e. $B(M|_B)$ is a complete fan supported on weight classes of the form

$$x_i \leq x_j \leq x_k \tag{29}$$

with pairwise distinct $i,j,k \in \{0,1,2\}$ and $x_2 = 0$. The codimension one weight classes of $B(M|_B)$ arise from full dimensional weight classes (cf. Equation (29)) by forcing a single inequality to an equality. For the weight class described by Equation (29) we have only two choices that correspond to flags of flats missing a flat of certain rank:

$$x_i = x_j \leq x_k \quad \Leftrightarrow \quad \emptyset \subsetneq \{i,j\} \subsetneq \{i,j,k\} = B, \text{ and} \tag{30}$$

$$x_i \leq x_j = x_k \quad \Leftrightarrow \quad \emptyset \subsetneq \{i\} \subsetneq \{i,j,k\} = B. \tag{31}$$

Note that the flag shown in Equation (30) misses a flat of rank one whereas flag in Equation (31) misses a flat of rank two. Let $\mathscr{F}_{\hat{1}}, \mathscr{F}_{\hat{2}} \triangleleft M|_B$ denote indetermined flags of the form shown in Equation (30) and Equation (31). Using the $\mathscr{V}$-description we see that $\sigma_{\mathscr{F}_{\hat{1}}} = \operatorname{cone}(-e_i - e_j) \in \mathbb{R}^3/\mathbf{1}_3$ and $\sigma_{\mathscr{F}_{\hat{2}}} = \operatorname{cone}(-e_i) \in \mathbb{R}^3/\mathbf{1}_3$. Hence, the codimension one weight classes of $B(M|_B)$ are rays. In particular, we observe that the generators have two coordinates with identical values. As $\mathscr{T}_B$ also consists of rays we see that a ray of $\mathscr{T}_B$ either completely belongs to $\bar{\mathscr{T}}_B$ or not — again, refinements are needless. Moreover, the $\mathscr{H}$-description of the rays (i.e. the weight classes) significantly simplifies the process of identifying rays of $\mathscr{T}_B$ that potentially belong to $\bar{\mathscr{T}}_B$. Let $v' \in \mathbb{R}^3/\mathbf{1}_3$ generate $\sigma' \subset \mathscr{T}_B$. If $\sigma' \subset \mathscr{T}_B \cap \bar{\mathscr{T}}_B$ then $v'$ has to coordinates with equal value different to the value of the remaining coordinate. Otherwise, $\sigma' \neq \sigma_{\mathscr{F}_{\hat{1}}}, \sigma_{\mathscr{F}_{\hat{2}}}$ and, therefore, cannot be in the codimension one skeleton of $B(M|_B)$. Equivalently, if the coordinates of $v'$ are pairwise different then $\sigma' \subset \mathscr{T}_B \setminus \bar{\mathscr{T}}_B$. This way we identify all rays contained in the relative interior of full dimensional weight classes of $B(M|_B)$ (cf. line 4 to 6 in Algorithm 1). The other way around is more interesting for our purpose: if $\sigma' \subset \mathscr{T}_B$ is a ray and likewise a codimension one weight class of the form $\sigma' = \sigma_{\mathscr{F}_{\hat{s}}}$ with $s \in \{1,2\}$ (see Equation (30) and Equation (31)) then we need to decide whether $\sigma' \subset \bar{\mathscr{T}}_B$ or not. According to Lemma 2.1.1.7 (c) we need to check whether $[\operatorname{cl}(F_{s-1}), \operatorname{cl}(F_{s+1})]$ is a diamond poset for $\mathscr{F}_{\hat{s}} \in \{\mathscr{F}_{\hat{1}}, \mathscr{F}_{\hat{2}}\}$. For $\mathscr{F}_{\hat{1}}$ this means $F_0 = \emptyset$ and, therefore, we have to look for flats of rank one in $\mathscr{L}$ that are contained $\operatorname{cl}(F_2) = \operatorname{cl}(\{i,j\})$. For $\mathscr{F}_{\hat{2}}$ we have $\operatorname{cl}(F_1) = \operatorname{cl}(\{i\})$, i.e. we have to look for flats of rank two that contain $\{i\}$. This is sufficient since all rank two flats containing $\{i\}$ are themselves contained in $\operatorname{cl}(F_3) = \operatorname{cl}(\{i,j,k\}) = \operatorname{cl}(B) = [n+1]$. For $\mathscr{F}_{\hat{1}}$ as well as for $\mathscr{F}_{\hat{2}}$ holds: the search for flats with the prescribed constraints yields three or more results if and only if $\sigma' \subset \mathscr{T}_B \cap \bar{\mathscr{T}}_B$ (cf. Lemma 2.1.1.7 (c)).

**Remark 2.1.4.31** (Implementability in the curve-case)**.** From parts 2.1.4.2 to 2.1.4.4 we see that the main obstacles that prevent an implementation for arbitrary hypersurfaces in linear spaces are vincible in the curve-case. Algorithm 2 shows an pseudocode of a possible implementation. So far we paid no attention to fast implementations. A straight forward improvement worth mentioning is the following: suppose the algorithm is in the $j$-th round of the recursion ($j \in \{0,1,2\}$) and we consider $\mathscr{T}_B^j$. Let $\mathscr{F}_{\hat{0}} \triangleleft M|_B$ denote a flag of full length whereas $\mathscr{F}_{\hat{s}} \in \{\mathscr{F}_{\hat{1}}, \mathscr{F}_{\hat{2}}\}$ (cf. the flags in Equation (30) and Equation (31)). If $\sigma' \subset \mathscr{T}_B^j$ is a ray that is contained in the weight class $\sigma_{\mathscr{F}_{\hat{j}}}$ then we add $\tau = h_B(\sigma')$ to $D_j$. Instead of continuing with other bases $B' \neq B$ in the $j$-th round we can perform an "update" to all $\mathscr{T}^j_{B'}$ for all $B' \neq B$. In detail, we project the currently computed ray $\tau$ with $p_{B'}$ to $\mathbb{R}_{B'}$ and subtract the projection from $\mathscr{T}^j_{B'}$, i.e. we compute $\mathscr{T}^j_{B'} - (p_{B'})_*(\tau)$ for all $B' \neq B$. This prevents that we have to deal with images of $\tau$ in other residues $\mathscr{T}^j_{B'}$ again. Recall that we see a full dimensional image of $\tau$ contained in a weight class $\sigma_{\mathscr{F}_j} \lhd M\big|_{B'}$ in every $\mathscr{T}_{B'}$ with $B' \in M_{\mathrm{cl}\left(\mathscr{F}_j\right)}$ (cf. Lemma 2.1.3.19). Consequently, we compute $\tau$ only once.

**Remark 2.1.4.32** (Description of TROPHIALS.LIB)**.** An implementation of Algorithm 2 can be found in TROPHIALS.LIB written for SINGULAR that uses the script TROPHIALS_PMSCRIPT written for POLYMAKE. In order to compute the **trop**icalization of a **h**ypersurface **in a l**inear **s**pace, proceed as in the following application example showing the commands for the SINGULAR-shell:

```
> LIB ``trophials.lib'';
> ring r = 0,(x,y,z,a),dp;
> ideal I = 13x + 17y - 5z - 89a;
> poly f = xy3-7xz2a + 13xa3 - 91xz2a - 15xy2z + 9yza2 + y2z2 + z4 + a3z;
> ideal J = I+f;
> list T = trophials_tropicalize(J);
```

LISTING 1. SINGULAR-example using TROPHIALS.LIB.

The output of SINGULAR with respect to Listing 1 is as follows:

```
Trop(V(I))
----------
Rays:
0,1, 0, 1,
0,0, 0, -1,
0,0, -1,0,
0,-1,0, 0,
0,1, 1, 1
Lineality:
1,1,1,1
Weights:
1,4,3,4,3
```

LISTING 2. Output of SINGULAR-example, cf. Listing 1

We briefly describe how the scripts work. First, the procedure TROPHIALS_TROPICALIZE(J) checks the input, i.e. is the ideal homogeneous, has it correct dimension, is $f$ divisible by a monomial an so on. If $f$ is divisible by a monomial $x^\beta$ for some $\beta \in \mathbb{Z}^2$, the procedure computes the tropicalization of $\mathscr{V}(I + \langle f' \rangle) \subset T^n$ where $f' = x^{-\beta} f$ and tells the user. Then the procedure computes the matroid-data, the relative Newton polytope of $f'$ with respect to the linear ideal $I$ (which is the linear part of $J$) and saves the data on the hard drive. In particular, each Newton polytope of the relative Newton polytope is saved in a individual file called "polymake_data_XY" where XY is the consecutive numbering of all bases of the matroid. The matroid-data is saved in "polymake_data_matroid". Then SINGULAR calls polymake together with the script file TROPHIALS_PMSCRIPT. Basically, the polymake-script

| Example with / Runtime | | using | |
|---|---|---|---|
| n | d | GFAN | TROPHIALS |
| 4 | 4 | 107 ms | 1 s 890 ms |
| 9 | 4 | 15 s 523 ms | 5 s 790 ms |
| 12 | 4 | 4 min 58 s | 13 s 950 ms |
| 15 | 4 | $\sim$ 75 min | 35 s 200 ms |
| 4 | 6 | 121 ms | 2 s 130 ms |
| 9 | 6 | 19 s 721 ms | 7 s 250 ms |
| 12 | 6 | 3 min 29 s | 15 s 970 ms |
| 15 | 6 | $\sim$ 75 min | 37 s 30 ms |

TABLE 2. Runtime measurements of GFAN and TROPHIALS.LIB.

accomplishes the remaining part of Algorithm 2. The data is saved in "singular_data", loaded by SINGULAR and written to the output. The procedure checks all imposed requirements on the ideal and writes some information to the output, e.g. whether the matroid is uniform.

The fact that the output is a 1-dimensional fan makes a list of maximal cones redundant. The command TROPHIALS_TROPICALIZE returns a list containing the considered ideal (as it may change due to divisibility), the rays, weights and lineality space. The library contains additional procedures, e.g. a conversion procedure that translates an ideal and the ring for GFAN. Table 2 contains runtime measurements of TROPHIALS_TROPICALIZE() and GFAN with randomly chosen ideals. Here, the runtime measurement of GFAN refers to the command GFAN_TROPICALTRAVERSE only. Moreover, TROPHIALS checks the input first and then computes the data whereas GFAN immediately starts its computations.

## 2.2. The Degree-d Subdivision

Let $X \subset \mathbb{P}^n$ be a linear space defined by a linear ideal $\mathscr{I} \subset R_{n+1}$ and let $F \in R_{n+1}$ be a homogeneous polynomial of degree $d$, each with constant coefficients. A priori we do not know how the hypersurface $\mathscr{T} = \operatorname{trop}(\mathscr{V}(\mathscr{I} + \langle F \rangle))$ is located in $\operatorname{trop}(X)$, i.e. we do not have a clue about the form of the cones of $\mathscr{T}$. In this section we equip tropical linear spaces $\operatorname{trop}(X)$, arising as tropicalizations of linear spaces $X \subset \mathbb{P}^n$, with a subdivision (depending on a number $d \in \mathbb{N}$) whose codimension one skeleton supports tropical hypersurfaces of the form $\operatorname{trop}(\mathscr{V}(\mathscr{I} + \langle F \rangle))$ for arbitrary polynomials $F \in R_{n+1}$ of degree $d$. First, we enhance Notation 2.1 and 2.2:

NOTATION 2.5. Recall that $\mathscr{M}_{n+1,d}$ denotes the set of monomials of degree $d$ in $n+1$ variables (cf. Notation 1.1). We identify $\mathscr{M}_{n+1,d} = \mathbb{Z}^{n+1} \cap d \cdot \Delta_{n+1}$ and consider $\mathscr{M}_{n+1,d}$ ordered lexicographically. The number of monomials of degree $d$ in $n+1$ variables is denoted by $N^d_{n+1} = \binom{n+d}{d}$ and $N := N^d_{n+1} - 1$ for the fixed integers $n, d$. For a subset $B \subset [n+1]$ we identify the monomials of degree $d$ in the variables indexed by $B$ by $B^d$, i.e. $B^d = \{x \in \mathbb{Z}^{n+1} \cap d \cdot \Delta_{n+1} : \sum_{i \in B} x_i = d\} \subset \mathscr{M}_{n+1,d}$. Thus for $x \in B^d$ we have $x_i = 0$ for all $i \notin B$. The identification can be established by the bijective map

$$\chi : B^d \longrightarrow \{x \in \mathbb{Z}^{n+1} \cap d \cdot \Delta_{n+1} : \sum_{i \in B} x_i = d\},\ (i_1, \ldots, i_d) \longmapsto \sum_{j=1}^{d} e_{i_j}. \tag{32}$$

If nothing else is mentioned we consider $B^d \subset \mathbb{Z}^{n+1}$.

**Algorithm 2** Tropicalization of curves in 2-dimensional linear spaces.

**Input:** linear ideal $\mathscr{I} = \langle l_0, \ldots, l_{n-3} \rangle \subset R_{n+1}$, homogeneous polynomial $F \in R_{n+1}$.
check input, e.g. dimensions, divisibility of $F$.
compute matroid $M = M(\mathscr{I})$, its bases $\mathscr{B}$ and $\mathscr{T}_B = \operatorname{trop}(\mathscr{V}(F_B))$ for all $B \in \mathscr{B}$.
define $D_0 := \emptyset$ and $\omega := 0$, $\mathscr{T}_B^0 = \mathscr{T}_B$.
**for all** $B = \{i, j, k\} \in \mathscr{B}$ **do**
  **for all** $\sigma = \operatorname{cone}(v) \subset \mathscr{T}_B^{(1)} \setminus \bar{\mathscr{T}}_B$ **do**
    Add $\tau = h_B(\sigma)$ to $D_0$, define $\omega(\tau) = \omega_{\mathscr{T}_B}(\sigma)$.
    perform update with $\tau$ to all $\mathscr{T}_{B'}$ with $B' \neq B$.
  **end for**
**end for**
make $D_0$ a weighted fan.
compute $\mathscr{T}_B^1 = \mathscr{T}_B - (p_B)_* (D_0)$ for all $B$ and define $D_1 := \emptyset$.
**for all** $B = \{i, j, k\} \in \mathscr{B}$ **do**
  **for all** $\sigma = \operatorname{cone}(v) \subset \mathscr{T}_B^1$ such that $\exists \eta \in S_3$ and $v_{\eta(i)} = v_{\eta(j)} \lneq v_{\eta(k)}$ **do**
    Add $\tau = h_B(\sigma)$ to $D_1$, define $\omega(\tau) = \omega_{\mathscr{T}_B^1}(\sigma)$.
    perform update with $\tau$ to all $\mathscr{T}_{B'}^1$ with $B' \neq B$.
  **end for**
**end for**
make $D_1$ a weighted fan.
compute $\mathscr{T}_B^2 = \mathscr{T}_B^1 - (p_B)_* (D_1)$ for all $B$ and define $D_2 := \emptyset$.
**for all** $B = \{i, j, k\} \in \mathscr{B}$ **do**
  **for all** $\sigma = \operatorname{cone}(v) \subset \mathscr{T}_B^2$ such that $\exists \eta \in S_3$ and $v_{\eta(i)} \leq v_{\eta(j)} = v_{\eta(k)}$ **do**
    Add $\tau = h_B(\sigma)$ to $D_2$, define $\omega(\tau) = \omega_{\mathscr{T}_B^2}(\sigma)$.
    perform update with $\tau$ to all $\mathscr{T}_{B'}^2$ with $B' \neq B$.
  **end for**
**end for**
**Output:** $(D_0 \cup D_1 \cup D_2, \omega)$.

### 2.2.1. The Degree-d Subdivision of a Tropical Linear Space

Given an integer $d \in \mathbb{N}$ we equip $\operatorname{trop}(X)$ with a subdivision called degree-d subdivision that is natural in the following sense: if $F \in R_{n+1}$ is a homogeneous polynomial of degree $d$ then $\mathscr{T}$ lives in the codimension one skeleton of the degree-d subdivision of $\operatorname{trop}(X)$. As explained in Section 1.4.5 the tropicalization of a linear space, e.g. $X = \mathscr{V}(\mathscr{I})$, carries a fine subdivision given by the weight classes. All linear subspaces (of codimension $m$) of $X$ arise as subfans of the (codimension $m$) skeleton of the Bergman fan with respect to the fine subdivision. We adapt this for a hypersurface $\mathscr{V}(F)$, defined by a homogeneous polynomial $F \in R_{n+1}$, contained in $\mathscr{V}(\mathscr{I})$. The degree-d subdivision of $\operatorname{trop}(X)$ whose codimension one skeleton is the frame where $\mathscr{T}(\mathscr{I} + F)$ lives in. Recall that linear subspaces of $X$ yield more linear constraints on the variables. As $F$ has degree $d$ we deal with non-linear constraints. The basic idea of this section is to *linearize* the ideal $\mathscr{I} + \langle F \rangle$ by using the Veronese map.

**Definition 2.2.1.1** (Veronese map)**.** Fix two integers $n, d \in \mathbb{N}$. The *Veronese map of degree d* is the homeomorphism defined by

$$\begin{aligned} \nu_d : \mathbb{P}^n &\longrightarrow \mathbb{P}^N \\ x = (x_0 : \ldots : x_n) &\longmapsto (x^{\alpha_0} : \ldots : x^{\alpha_N}) =: (y_0 : \ldots : y_N), \end{aligned}$$

with $\alpha_i \in \mathscr{M}_{n+1,d}$. The closure of $\operatorname{Im}(\nu_d)$ is a toric variety called *Veronese variety*. We denote its defining ideal called *Veronese ideal* by $\mathscr{J}$. It is generated by the set $\mathscr{U}$ of all binomials $y_i \cdot y_j - y_k \cdot y_l$ such that $\alpha_i + \alpha_j = \alpha_k + \alpha_l$ with $\alpha_i, \alpha_j, \alpha_k, \alpha_l \in \mathscr{M}_{n+1,d}$.

Let $V$ denote the matrix representation of $\mathscr{M}_{n+1,d}$. As the Veronese map is a monomial map its tropical version, obtained from $\nu_d$ by tropicalizing componentwise (cf. Section 1.4.2), is the transposition of $V$, i.e. $\operatorname{trop}(\nu_d) = V^\top$, whose rows are the elements $\alpha \in \mathscr{M}_{n+1,d}$. With regard to the binomial generators of $\mathscr{J}$ a short computation shows

$$w \in \mathscr{T}(\operatorname{trop}(y_i \cdot y_j - y_k \cdot y_l)) \iff \langle e_i + e_j - (e_k + e_l), w \rangle = 0. \tag{33}$$

In other words, the tropical hypersurface arising from a generator of $\mathscr{J}$ is a proper hyperplane. Writing all linear equations obtained from the tropicalizations of the binomial generators of $\mathscr{J}$ in a matrix $L$ we see $\operatorname{trop}(\mathscr{V}(\mathscr{J}))$ is a linear space given by the kernel of $L$. The following lemma establishes the connection between $L$ and $\operatorname{trop}(\nu_d)$:

**Lemma 2.2.1.2.** *Fix integers $n, d \in \mathbb{N}$. Let $\mathscr{J} \subset R_{N+1}$ denote the Veronese ideal. Then, we have* $\operatorname{Im}(\operatorname{trop}(\nu_d)) = \operatorname{trop}(\operatorname{Im}(\nu_d))$ *and, moreover,* $\operatorname{Im}(V^\top) = \ker(L)$, *i.e. the Veronese map commutes with tropicalization (cf. Figure 12). In particular:*

$$\widetilde{w} \in \operatorname{trop}(\mathscr{V}(\mathscr{J})) \subset \mathbb{R}^{N+1} \iff \exists\, w \in \mathbb{R}^{n+1} : (\widetilde{w})_\alpha = \langle \alpha, w \rangle \;\forall \alpha \in \mathscr{M}_{n+1,d}.$$

PROOF. The first statement follows from Theorem 1.6.2.9 with $U = \mathbb{E}_{n+1}$ and $V = \operatorname{trop}(\nu_d)$. For the second equality consider the set of generators

$$\mathscr{U} = \{g_1, \ldots, g_M\} = \{y_i \cdot y_j - y_k \cdot y_l : \exists \alpha_i, \alpha_j, \alpha_k, \alpha_l \in \mathscr{M}_{n+1,d} \text{ such that } \alpha_i + \alpha_j = \alpha_k + \alpha_l\}$$

of the Veronese ideal $\mathscr{J}$. We have $\operatorname{trop}(\mathscr{V}(\mathscr{J})) \subseteq \bigcap_s \operatorname{trop}(\mathscr{V}(g_s)) = \ker(L)$ due to the definition of $L$ (see above). We know that $\ker(L) \cong \mathbb{R}^{n+1}$, i.e. it is a proper linear space. Since tropicalization preserves dimension we conclude that $\operatorname{trop}(\mathscr{V}(\mathscr{J})) = \ker(L)$ since $\mathbb{R}^{n+1}$ is irreducible (cf. [**GKM09**, Example 2.18]). Let $\operatorname{trop}(\nu_d) = V^\top \in \mathbb{Z}^{N+1 \times n+1}$ denote the tropical Veronese map. Note that $\operatorname{rank}(V^\top) = n+1$. Consider an element $\operatorname{trop}(\nu_d)(w) = \tilde{w}$. The coordinate of $\tilde{w}$ indexed by $\alpha \in \mathscr{M}_{n+1,d}$ is given by $\left(V^\top w\right)_\alpha = \langle w, \alpha \rangle$. Pick any generator $g_h = y_i y_j - y_k y_l \in \{g_1, \ldots, g_M\}$. According to Equation (33) we have $\tilde{w} \in \operatorname{trop}(\mathscr{V}(g_h))$ if and only if $\langle \tilde{w}, e_i + e_j - e_k - e_l \rangle = 0$. Equivalently,

$$\tilde{w}_i + \tilde{w}_j - \tilde{w}_k - \tilde{w}_l = \langle w, \alpha_i \rangle + \langle w, \alpha_j \rangle - \langle w, \alpha_k \rangle - \langle w, \alpha_l \rangle = \langle w, \alpha_i + \alpha_j - \alpha_k - \alpha_l \rangle = 0.$$

Hence, $\operatorname{Im}(\operatorname{trop}(\nu_d)) \subseteq \ker(L) = \operatorname{trop}(\mathscr{V}(\mathscr{J}))$ and since $\operatorname{Im}(\operatorname{trop}(\nu_d))$ is a $n+1$-dimensional linear space we get the equality $\operatorname{Im}(V^\top) = \ker(L)$. □

**Remark 2.2.1.3.** There is a more general statement for monomial maps: let $\Phi : T^n \longrightarrow T^m$ be a monomial map of tori. Let $\overline{\Phi(X)}$ denote the Zariski closure of the image of a subvariety $X \subset T^n$. Then $\operatorname{trop}\left(\overline{\Phi(X)}\right) = \operatorname{trop}(\Phi)(\operatorname{trop}(X))$ ([**MS15**, Corollary 3.2.13]). Note that $X$ can be any subvariety of $T^n$, in contrast to Theorem 1.6.2.9 where $X$ is linear.

$$
\begin{tikzcd}
\mathbb{P}^n \arrow[r, "\nu_d"] \arrow[d, "\mathrm{trop}"'] & \mathrm{Im}(\nu_d) \subset \mathbb{P}^N \arrow[d, "\mathrm{trop}"] \\
\mathbb{R}^{n+1} \arrow[r, "\mathrm{trop}(\nu_d)"'] & \mathrm{Im}(V^\top) = \ker(L) \subset \mathbb{R}^{N+1}
\end{tikzcd}
$$

FIGURE 12. The Veronese map commutes with tropicalization.

**Remark 2.2.1.4** (Hilbert function)**.** Let $I \subset R_{n+1}$ be a homogeneous ideal. By $R_{n+1}^{(d)}$ and $I^{(d)}$ we denote the degree $d$ part of $R_{n+1}$ and $I$. The *Hilbert function* of $I$ is defined by

$$
\mathrm{h}_I : \mathbb{N} \longrightarrow \mathbb{N}, \quad r \longmapsto \dim_{\mathbb{K}} \left( \mathbb{K}[x_0, \ldots, x_n]^{(r)} \Big/ I^{(r)} \right).
$$

It is well-known that for every homogeneous ideal $I$ there exists some $r_0 \in \mathbb{N}$ and a polynomial $p_I$ called *Hilbert polynomial* with rational coefficients such that $\mathrm{h}_I(r) = p_I(r)$ for $r \geq r_0$.

The following lemma determines the dimension of $\mathscr{I}^{(d)}$ with the help of the Hilbert function of $\mathscr{I}$:

**Lemma 2.2.1.5.** *The degree $d$ part of the linear ideal $\mathscr{I} = \langle l_i : i \in \{0, \ldots, m-1\} \rangle$ (cf. Notation 2.2) has dimension $N_{n+1}^d - N_{n-m+1}^d = \sum_{j=0}^{m-1} N_{n+1-j}^{d-1}$ with $m = n-k$.*

PROOF. By definition of the Hilbert function,

$$
\begin{aligned}
\mathrm{h}_{\mathscr{I}}(d) &= \dim_K \left( R_{n+1}^{(d)} / \mathscr{I}^{(d)} \right) = \dim_K \left( R_{n+1}^{(d)} \right) - \dim_K \left( \mathscr{I}^{(d)} \right) \\
\Rightarrow \qquad \dim_K \left( \mathscr{I}^{(d)} \right) &= \dim_K \left( R_{n+1}^{(d)} \right) - \dim_K \left( (R_{n+1}/\mathscr{I})^{(d)} \right) \\
&= \dim_K \left( R_{n+1}^{(d)} \right) - \dim_K \left( R_{n+1-m}^{(d)} \right) = N_{n+1}^d - N_{n+1-m}^d
\end{aligned}
$$

Furthermore, we note that

$$
\sum_{j=0}^{m-1} N_{n+1-j}^{d-1} = \sum_{j=0}^{m-1} \binom{(n+1-j)-1+(d-1)}{d-1} = N_{n+1}^d - N_{n+1-m}^d
$$

by using the formula $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$. □

In the following we work in $\mathbb{P}^n$ ($R_{n+1}$ respectively) and $\mathbb{P}^N$ ($R_{N+1}$ respectively).

**Convention 2.2.1.6** (Total orders)**.** In the following we work with a fixed basis $B = \{b_0, \ldots, b_k\}$ of $M$ unless otherwise agreed. Its complement is denoted by $B^\complement = \{b'_0, \ldots, b'_{m-1}\}$ with $m = n-k$. We fix a total order $>$ on the variables of $R_{n+1}$ by

$$
x_{b'_0} > \ldots > x_{b'_{m-1}} > x_{b_0} > \ldots x_{b_k}.
$$

This allows to write the linear forms $l_i$ generating $\mathscr{I}$ as $l_i = x_{b'_i} + \sum_{j \in B} a_{i,j} x_j$ for $i \in \{0, \ldots, m-1\}$. Based on this total order we consider $R_{n+1}$ with graded lexicographic order $>_{\mathrm{lex}}$, e.g. firstly we compare degrees and secondly due to the lexicographic order. To keep notations as simple as possible we assume that $B^\complement = \{0, \ldots, m-1\}$ and $B = \{m, \ldots, n\}$.

**Remark 2.2.1.7** (Coefficient matrix of $\mathscr{I}$)**.** Recall that $M$ denotes the matroid associated to the ideal $\mathscr{I} \subset R_{n+1}$. Then the coefficient matrix $A(\mathscr{I})$ (cf. Definition 1.4.5.26) can be written as

$$
\begin{aligned}
A(\mathscr{I}) &= \begin{array}{c|c}
 & \begin{array}{cccccc} x_{b'_0} & \dots & x_{b'_{m-1}} & x_{b_0} & \dots & x_{b_k} \end{array} \\
\begin{array}{c} x_{b'_0} \\ \\ \vdots \\ \\ x_{b'_{m-1}} \end{array} & \begin{pmatrix} 1 & & & & \\ & & & & \\ & \ddots & & & A_B \\ & & & & \\ & & 1 & & \end{pmatrix}
\end{array} \\
&= \begin{bmatrix} \mathbb{E}_m & A_B \end{bmatrix} \in \mathbb{K}^{m \times n+1}.
\end{aligned}
$$

where $\mathbb{E}_m$ denotes the unit matrix of size $m$ whose columns correspond to $B^{\complement}$ and $A_B \in \mathbb{R}^{m \times n+1-m}$ the matrix whose columns correspond to the basis $B$. Due to Remark 1.4.5.30 we get the Gale dual

$$
G = \begin{bmatrix} -\left(A_B^{\top}\right) & \mathbb{E}_{k+1} \end{bmatrix} \in \mathbb{K}^{k+1 \times n+1}. \tag{34}
$$

**Remark 2.2.1.8** (Coefficient matrix of $\mathscr{I}^{(d)}$)**.** Consider the decomposition $\mathscr{M}_{n+1,d} = \left(B^d\right)^{\complement} \sqcup B^d$ into the set $B^d$ of monomials in the variables purely indexed by the basis $B$ and the remaining monomials $\left(B^d\right)^{\complement} = \mathscr{M}_{n+1,d} \setminus B^d$. Due to the fixed total order on $R_{n+1}$ (cf. Convention 2.2.1.6) we have $x^{\alpha} > x^{\beta}$ for all pairs $(\alpha, \beta) \in \left(B^d\right)^{\complement} \times B^d$. Let $\mathscr{I}^d \subset R_{n+1}$ denote the ideal generated by the following elements: for a fixed $j \in \{0, \dots, m-1\}$ we multiply the linear generator $l_j$ of $\mathscr{I}$ with all monomials of degree $d-1$ in the variables $x_{b'_j}, \dots, x_{b'_{m-1}}, x_{b_0} \dots, x_{b_k}$. We do this for all $j$, i.e. we multiply $l_0$ with $N_{n+1}^{d-1}$ monomials, $l_1$ with $N_n^{d-1}$ monomials and so on. The linear form $l_{m-1}$ is multiplied with $N_{n-m+2}^{d-1}$ monomials. Then we arrange the generators of $\mathscr{I}^d$ according to the leading monomial, e.g. $x^{\alpha} l_i$ has leading monomial $x^{\alpha + e_{b'_i}}$ since $x_{b'_i} > x_{b_j}$ for all $j \in B$ and $\alpha$ having only non-zero entries at coordinates greater or equal to $b'_i$. This allows to write the coefficient matrix of $\mathscr{I}^d$ in the following form:

$$
\begin{aligned}
A(\mathscr{I}^d) &= \begin{array}{c|c}
 & \begin{array}{cccccc} N_{n+1}^{d-1} & N_n^{d-1} & N_{n-1}^{d-1} & \dots & N_{n-m+2}^{d-1} & N_{k+1}^{d} \end{array} \\
\begin{array}{c} N_{n+1}^{d-1} \\ N_n^{d-1} \\ N_{n-1}^{d-1} \\ \vdots \\ N_{n-m+2}^{d-1} \end{array} & \begin{pmatrix}
\mathbb{U}_{N_{n+1}^{d-1}} & * & * & * & * & \\
 & \mathbb{U}_{N_n^{d-1}} & * & * & \vdots & \\
 & & \mathbb{U}_{N_{n-1}^{d-1}} & * & * & A' \\
 & & & \ddots & * & \\
 & & & & \mathbb{U}_{N_{n-m+2}^{d-1}} &
\end{pmatrix}
\end{array} \\
&= \begin{bmatrix} \mathbb{U}_{N_{n+1}^d - N_{n-m}^d} & A' \end{bmatrix}.
\end{aligned}
$$

where $\mathbb{U}_k$ denotes an upper right matrix of size $k$ with ones on the diagonal and $A' \in \mathbb{R}^{N_{n+1}^d - N_{n-m+1}^d \times N_{k+1}^d}$. Note that the columns are arranged due to the total order defined in Convention 2.2.1.6. This also holds for the rows since the generators of $\mathscr{I}^d$ have degree $d$. By construction all upper right matrices $\mathbb{U}_k$ form a single upper right square matrix with $N_{n+1}^d - N_{n-m+1}^d$ rows/columns (cf. Lemma 2.2.1.5). Hence, $\operatorname{rank}\left(A(\mathscr{I}^d)\right) = N_{n+1}^d - N_{n-m+1}^d$. We have $\dim_K \left(\mathscr{I}^{(d)}\right) = N_{n+1}^d - N_{n-m+1}^d$

(cf. Lemma 2.2.1.5) and moreover, all generators of $\mathscr{I}^d$ are contained in $\mathscr{I}^{(d)}$. Hence, the rows of $A(\mathscr{I}^d)$ generate the degree $d$ part of $\mathscr{I}^{(d)}$ and we can use $A\left(\mathscr{I}^d\right)$ as an equivalent coefficient matrix to $A\left(\mathscr{I}^{(d)}\right)$. By performing row operations on $A(\mathscr{I}^d)$ we obtain an equivalent matrix,

$$A^d = \begin{bmatrix} \mathbb{E}_{N^d_{n+1}-N^d_{n-m+1}} & A_{B^d} \end{bmatrix} \in \mathbb{K}^{N^d_{n+1}-N^d_{n-m+1} \times N+1}, \tag{35}$$

with $A_{B^d} \in \mathbb{K}^{N^d_{n+1}-N^d_{n-m} \times N^d_{k+1}}$. Note that the first $N^d_{n+1} - N^d_{n-m+1}$ columns correspond to $\left(B^d\right)^\complement$.

**Definition 2.2.1.9.** We identify $R_{N+1} \cong \mathbb{K}\left[y_\alpha | \alpha \in \mathscr{M}_{n+1,d}\right]$ and define $\mathscr{I}^d_{\text{lin}}$ to be the linear ideal in $R_{N+1}$ obtained from $\mathscr{I}^d$ by exchanging $x^\alpha$ with $y_\alpha$. The ideal $\mathscr{I}^d_{\text{lin}}$ provides a matroid $M^d = M\left(\mathscr{I}^d_{\text{lin}}\right)$.

Usually, we consider the matroid associated to $\mathscr{I}$ via a representation as a vector matroid arising from a Gale dual (cf. Remark 1.4.5.28). The linear generators of $\mathscr{I}^d_{\text{lin}}$ correspond to the rows of $A^d$. Hence, $M^d$ is a vector matroid on a Gale dual $G^d$ of $A^d$, i.e. $M\left[G^d\right] = M^d$. We can immediately deduce:

**Corollary 2.2.1.10.** *If $B$ is a basis of $M = M(\mathscr{I})$ then $B^d$ is a basis of $M^d = M\left(\mathscr{I}^d_{\text{lin}}\right)$.*

In order to use the Gale dual construction explained in Remark 1.4.5.30 we need to know the exact form of $A_{B^d}$. Additionally, the knowledge of the exact form of $A_{B^d}$ allows to compute the elimination of variables indexed by $B^\complement$ in a polynomial $F$ (Corollary 2.2.2.29) in terms of linear algebra. Recall from the linear case that a basis $B$ of $M$ allows to express variables indexed by $B^\complement$ (see also Section 2.1.2). Due to this fact we can achieve a coefficient matrix $A$ as shown in Remark 2.2.1.7. An analogous statement holds for the coefficient matrix $A^d$:

**Lemma 2.2.1.11.** *Let $B$ be a basis of $M$. Fix an element $\alpha \in \left(B^d\right)^\complement$. We define $s = s(\alpha) = \sum_{j \in B^\complement} (\alpha)_j$ and $k_i = \min\{j \in B^\complement : \left(\alpha - \sum_{l=1}^{i-1} e_{k_l}\right)_j \neq 0\}$ for $i = 1, \dots, s$. We think of $k_i$ as the "i-th smallest index" of $\alpha$ contained in $B^\complement$ and $s(\alpha)$ as a "discrepancy" of $\alpha$ to $B$. We assume $B^\complement = \{0, \dots, m-1\}$ and $B = \{m, \dots, n+1\}$ such that the index $i$ of linear form $l_i$ indicates its leading monomial, i.e.*

$$l_i = x_{b'_i} + \sum_{j \in B} a_{i,j} x_j = x_i + \sum_{j \in B} a_{i,j} x_j$$

*for $i \in \{0, \dots, m-1\}$. The row of $A\left(\mathscr{I}^d_{\text{lin}}\right)$ indexed by $\alpha \in \left(B^d\right)^\complement$ corresponds to the product $x^{\alpha - e_{k_1}} l_{k_1}$. Then the row of $A^d$ indexed by the monomial $\alpha$ corresponds to the polynomial*

$$l_\alpha = x^\alpha + (-1)^{s-1} \sum_{j_1, \dots, j_s \in B} \left( \prod_{l=1}^{s} a_{k_l, j_l} x_{j_l} \right) x^{\alpha - \sum_{l=1}^{s} e_{k_l}}.$$

Proof. We get $A^d$ from $A\left(\mathscr{I}^d\right)$ by performing row operations. The proof elucidates the details. We show the statement by induction on $s = s(\alpha)$. For the start we set $s = 1$. Consider $\alpha \in \left(B^d\right)^\complement$ with exactly one non-zero coordinate indexed by $B^\complement$. This is $k = \min\{j \in B^\complement : (\alpha)_j \neq 0\}$. Hence, we get

$$x^{\alpha - e_k} l_k = x^{\alpha - e_k} \left( x_k + \sum_{j \in B} a_{k,j} x_j \right) = x^\alpha + \sum_{j \in B} a_{k,j} x_j x^{\alpha - e_k}.$$

Note that $\alpha - e_k + e_j \in B^d$ for all $j \in B$, i.e. we have the desired form.

For the induction step suppose the statement holds for all $\alpha$ with $s(\alpha) = s - 1$. Let $\alpha \in \left(B^d\right)^\complement$ be an element such that $s(\alpha) = s$. Consider $k_1 = \min\{j \in B^{\complement} : (\alpha)_j \neq 0\}$. Then, the row of $A\left(\mathscr{I}^d\right)$ indexed by $\alpha - e_{k_1}$ corresponds to the polynomial

$$x^{\alpha - e_{k_1}} l_{k_1} = x^{\alpha} + \sum_{j \in B} a_{k_1,j} x_j x^{\alpha - e_{k_1}}. \tag{36}$$

For $\alpha - e_{k_1}$ we have $s(\alpha - e_{k_1}) = s(\alpha) - 1$. Consider $k_2 = \min\{j \in B^{\complement} : \left(\alpha - e_{k_1}\right)_j \neq 0\}$. Then $x^{\alpha - e_{k_1} + e_j}$ is the leading monomial of $x^{(\alpha - e_{k_1} + e_j) - e_{k_2}} l_{k_2}$. Since $s(\alpha - e_{k_1} + e_j) = s - 1$ we can make use of the induction hypothesis, i.e. we have

$$x^{(\alpha - e_{k_1} + e_j) - e_{k_2}} l_{k_2} = x^{\alpha - e_{k_1} + e_j} + (-1)^{s(\alpha - e_{k_1} + e_j) - 1} \sum_{i_2, \ldots, i_s \in B} \left( \prod_{l=2}^{s} a_{k_l, i_l} x_{i_l} \right) x^{(\alpha - e_{k_1} + e_j) - \sum_{l=2}^{s} e_{k_l}}. \tag{37}$$

Hence, we can reduce the sum on the right in Equation (36):

$$\begin{aligned}
& x^{\alpha - e_{k_1}} l_{k_1} - \sum_{j \in B} a_{k_1,j} \left( x^{(\alpha - e_{k_1} + e_j) - e_{k_2}} l_{k_2} \right) \\
=& x^{\alpha - e_{k_1}} l_{k_1} - \sum_{j \in B} a_{k_1,j} \left( x^{(\alpha - e_{k_1} + e_j)} + (-1)^{s(\alpha - e_{k_1} + e_j) - 1} \sum_{i_2, \ldots, i_s \in B} \left( \prod_{l=2}^{s} a_{k_l, i_l} x_{i_l} \right) x^{(\alpha - e_{k_1} + e_j) - \sum_{l=2}^{s} e_{k_l}} \right) \\
=& x^{\alpha} - \sum_{j \in B} a_{k_1,j} (-1)^{s(\alpha - e_{k_1} + e_j) - 1} \left( \sum_{i_2, \ldots, i_s \in B} \left( \prod_{l=2}^{s} a_{k_l, i_l} x_{i_l} \right) x^{(\alpha - e_{k_l} + e_j) - \sum_{l=2}^{s} e_{k_l}} \right) \\
=& x^{\alpha} + (-1)^{s(\alpha) - 1} \sum_{j \in B} \sum_{i_2, \ldots, i_s \in B} a_{k_1,j} \left( \prod_{l=2}^{s} a_{k_l, i_l} x_{i_l} \right) x^{(\alpha - e_{k_1} + e_j) - \sum_{l=2}^{s} e_{k_l}} \\
=& x^{\alpha} + (-1)^{s(\alpha) - 1} \sum_{i_1, \ldots, i_s \in B} \left( \prod_{l=1}^{s} a_{k_l, i_l} x_{i_l} \right) x^{\alpha - \sum_{l=1}^{s} e_{k_l}},
\end{aligned}$$

with $j$ renamed by $i_1$ in the last step. □

With Lemma 2.2.1.11 we can write down the entries of $A_{B^d}$ explicitly. For that purpose consider an element $\alpha \in \left(B^d\right)^{\complement}$, in particular $\alpha' = \alpha - \sum_{l=1}^{s(\alpha)} e_{k_l}$. The question at hand is what elements $\beta \in B^d$ we can get by saturating $\alpha'$. It is not hard to see that there are elements $i_1, \ldots, i_{s(\alpha)} \in B$ such that $\alpha' + \sum_{l=1}^{s(\alpha)} e_{i_l} = \beta$ if and only if $(\beta - \alpha')_j \geq 0$ for all $j \in B$.

**Corollary 2.2.1.12.** *Fix two elements $\alpha \in \left(B^d\right)^{\complement}$ and $\beta \in B^d$. Define $s = s(\alpha) = \sum_{j \in \left(B^d\right)^{\complement}} (\alpha)_j$ and $k_i = \min\{j \in B^{\complement} : \left(\alpha - \sum_{l=1}^{i-1} e_{k_l}\right)_j \neq 0\}$ for $i = 1, \ldots, s(\alpha)$ and, moreover, $\alpha' = \alpha - \sum_{l=1}^{s} e_{k_l}$. Then the entry of $A^d$ in the row indexed by $\alpha$ and the column indexed by $\beta$ equals*

$$\left(A^d\right)_{\alpha,\beta} = \begin{cases} (-1)^{s-1} \sum_{\substack{i_1, \ldots, i_s \in B: \\ x^{\alpha'} \prod_l x_{i_l} = x^{\beta}}} \left( \prod_{l=1}^{s} a_{k_l, i_l} \right) \text{ if } (\beta - \alpha')_j \geq 0 \forall j \in B, \text{ or} \\ 0 \text{ if } \exists j \in B \text{ such that } (\beta - (\alpha')_j < 0. \end{cases}$$

For the rest of this section we write $M$ for $M(\mathscr{I})$ and $M^d$ for $M(\mathscr{I}^d_{\text{lin}})$. Theorem 1.6.2.9 (Remark 2.2.1.3 respectively) states that $\operatorname{trop}(\nu_d(\mathscr{V}(\mathscr{I}))) = (\operatorname{trop}(\nu_d)) B(M)$. The next lemma shows that the Bergman fan $B\left(M^d\right) \subset \mathbb{R}^{N+1}$ restricted to $\operatorname{trop}(\mathscr{V}(\mathscr{J}))$ equals $(\operatorname{trop}(\nu_d)) B(M)$:

**Proposition 2.2.1.13.** *Let $\mathscr{I} \subset R_{n+1}$ be a linear ideal and $M$ the matroid associated to the ideal $\mathscr{I}$. Let $\mathscr{I}^d_{\text{lin}} \subset R_{N+1}$ denote the linear counterpart of the degree $d$ part of $\mathscr{I}$, $M^d$ the matroid associated* *to the ideal $\mathscr{I}^d_{\text{lin}}$. Then:*

$$|B(M)| \cong \left|B\left(M^d\right)\right| \cap \operatorname{trop}(\mathscr{V}(\mathscr{J})).$$

PROOF. The tropicalizations of $\mathscr{V}(\mathscr{I})$ and $\mathscr{V}\left(\mathscr{I}^d_{\text{lin}}\right)$ are supported on $B(M)$ and $B\left(M^d\right)$. Recall that the support of $\operatorname{trop}(\mathscr{V}(\mathscr{J}))$ is a proper linear space in $\mathbb{R}^{N+1}$ (cf. Lemma 2.2.1.2). Also note that $|B(M)| = \operatorname{trop}(\mathscr{V}(\mathscr{I})) = \operatorname{trop}\left(\mathscr{V}\left(\mathscr{I}^{(d)}\right)\right)$ because of the equality $\mathscr{V}(\mathscr{I}) = \mathscr{V}\left(\mathscr{I}^{(d)}\right)$. Then:

$$\begin{aligned}
\operatorname{trop}\left(\mathscr{V}\left(\mathscr{I}^{(d)}\right)\right) &= \left\{w \in \mathbb{R}^{n+1} \,\middle|\, w \in \mathscr{T}(g)\, \forall g \in \mathscr{I}^{(d)}\right\} \\
&= \left\{w \in \mathbb{R}^{n+1} \,\middle|\, w \in \mathscr{T}(g)\, \forall g \in \operatorname{rowspace}\left(A^d\right)\right\} \\
&= \left\{w \in \mathbb{R}^{n+1} \,\middle|\, V^\top w \in \mathscr{T}(g)\, \forall g \in \mathscr{I}^d_{\text{lin}}\right\} \\
&\cong \left\{\widetilde{w} \in \mathbb{R}^{N+1} \,\middle|\, \widetilde{w} \in \operatorname{trop}(\mathscr{V}(\mathscr{J})) \text{ and } \widetilde{w} \in \mathscr{T}(g)\, \forall g \in \mathscr{I}^d_{\text{lin}}\right\} \\
&= |B\left(M^d\right)| \cap \operatorname{trop}(\mathscr{V}(\mathscr{J})).
\end{aligned}$$

□

**Remark 2.2.1.14.** The map $\operatorname{trop}(\nu_d) = V^\top$ is a $\mathbb{Z}$-linear morphism of fans. With regard to push forwards (cf. Definition 2.1.2.15) we just discovered a description of its image fan in terms of Bergman fans, i.e. $\operatorname{trop}(\nu_d)_*(B(M)) = B\left(M^d\right) \cap \operatorname{trop}(\mathscr{V}(\mathscr{J}))$. Moreover, let $\mathscr{C}^d$ denote the circuits of $M^d$ and recall that $\mathscr{U}$ denotes the binomials generating the Veronese ideal $\mathscr{J}$. Then $(\operatorname{trop}(\nu_d))B(M) = \bigcap_{f \in \mathscr{C}^d \cup \mathscr{U}} \mathscr{T}(f)$.

In turns out we can study $B(M) = \operatorname{trop}(\mathscr{V}(\mathscr{I}))$ by studying the Bergman fan $B\left(M^d\right)$. Since $B\left(M^d\right)$ carries a fine subdivision we can pull it back to $\operatorname{trop}(\mathscr{V}(\mathscr{I}))$ via $\operatorname{trop}(\nu_d) = V^\top$:

**Definition 2.2.1.15** (degree-d subdivision)**.** The degree-d subdivision of $\operatorname{trop}(\mathscr{V}(\mathscr{I}))$ is induced by weight classes of $B\left(M^d\right)$: two elements $w, w' \in \operatorname{trop}(\mathscr{V}(\mathscr{I}))$ belong to the same cone of the degree-d subdivision if and only if $V^\top w$ and $V^\top w'$ belong to the same weight class in $B\left(M^d\right)$.

At first glance the Bergman fan of $M^d$ has much more weight classes than $B(M)$. Also, many of them may intersect $\operatorname{trop}(\mathscr{V}(\mathscr{J}))$ only in $\{0\}$. We like to know the fraction of relevant weight classes, i.e. full dimensional weight classes of $B\left(M^d\right)$ that have top-dimensional intersection with $\operatorname{trop}(\mathscr{V}(\mathscr{J}))$. These can be characterized with the help of an additional fan called *total fan*:

**Definition 2.2.1.16** (Total fan)**.** Let $\mathscr{A} \subset \Lambda$ be a finite set of lattice points. By $\mathscr{N}^T_{\mathscr{A}}$ we denote the common refinement of all normal fans $\pm\mathscr{N}_{P'}$ of polytopes $P' \subset V$ with vertices contained in $\mathscr{A}$. We call $\mathscr{N}^T_{\mathscr{A}}$ the *total fan* of $\mathscr{A}$.

According to Corollary 2.2.1.10 the basis $B$ of $M$ provides the basis $B^d$ of $M^d$. We identify these basis with lattice points of standard simplices:

**Remark 2.2.1.17** ($B$ and $B^d$)**.** We identify $B = \{b_0, \ldots, b_k\} \cong \Delta_{k+1} \cap \Lambda \subset \mathbb{R}_B$ and due to Notation 2.5, $B^d = \{\beta_1, \ldots, \beta_{N^d_{k+1}}\} \cong d \cdot \Delta_{k+1} \cap \Lambda \subset \mathbb{R}_B$. Note that $d \cdot (\Delta_{k+1} \cap \Lambda) \subset d \cdot \Delta_{k+1} \cap \Lambda$ is the set of vertices of $d \cdot \Delta_{k+1}$. Since stretching a polytope does not change the normal fan we conclude that $\mathscr{N}^T_B$ is refined by $\mathscr{N}^T_{B^d}$. Since $k \cdot \Delta_{n+1} \subset H_{\mathbf{1}_{n+1},k} = \{x \in \mathbb{R}^{n+1} : \langle \mathbf{1}_{n+1}, x\rangle = k\}$ we can mod out the lineality space generated by $\mathbf{1}_{n+1} \in \mathbb{R}^{n+1}$. If we consider total fans in $\mathbb{R}^{n+1}/\mathbf{1}_{n+1}$ in the following we do so by considering representatives with last coordinate equal to zero (cf. Convention 2.1.4.30).

**Remark 2.2.1.18** (1:1 correspondence of cones in total fans and total orders)**.** A full dimensional cone $\sigma$ of the total fan $\mathcal{N}_{B^d}^T \subseteq \mathbb{R}^k$ corresponds uniquely to a total order on $B^d$. To be more precise, consider the hyperplane $H_{w,b} = \{x \in \mathbb{R}^k : \langle w, x\rangle = b\}$ (cf. Remark 1.1.1.2) defined by an element $w \in \operatorname{relint}(\sigma)$ of a full dimensional cone $\sigma \subset \mathcal{N}_{B^d}^T$ and $b \in \mathbb{R}$. We can think of $b$ as a parameter. By starting with $b \ll -d$ and increasing the value $b$ we obtain a unique total order on $B^d$ since $H_{w,b}$ "moves" in direction of $w$ through $B^d \subset \mathbb{R}^k$. The order is precisely the sequence of the elements of $B^d$ passing through the hyperplane $H_{w,b}$ as $b$ varies, i.e. we obtain a chain of inequalities

$$\langle w, \beta_{\nu(1)}\rangle \lneq \ldots \lneq \langle w, \beta_{\nu(N_{k+1}^d)}\rangle \text{ with } \nu \in S_{N_{k+1}^d}. \tag{38}$$

Let $S(B^d) \subset S_{N_{k+1}^d}$ denote the subset of permutations arising this way. The subset $S(B^d)$ allows us to index full dimensional cones of $\mathcal{N}_{B^d}^T$. In contrast to that, any element $\eta \in S_{k+1}$ provides a full dimensional cone $\sigma \subset \mathcal{N}_B^T$. Note that $w \in \operatorname{relint}(\sigma)$ gives a chain of inequalities (a total order on $B$ respectively)

$$w_{b_{\eta(1)}} \leq \ldots \leq w_{b_{\eta(k+1)}} \iff \langle w, e_{b_{\eta(1)}}\rangle \leq \ldots \leq \langle w, e_{b_{\eta(k+1)}}\rangle \tag{39}$$

$$\iff \langle w, d\cdot e_{b_{\eta(1)}}\rangle \leq \ldots \leq \langle w, d\cdot e_{b_{\eta(k+1)}}\rangle \tag{40}$$

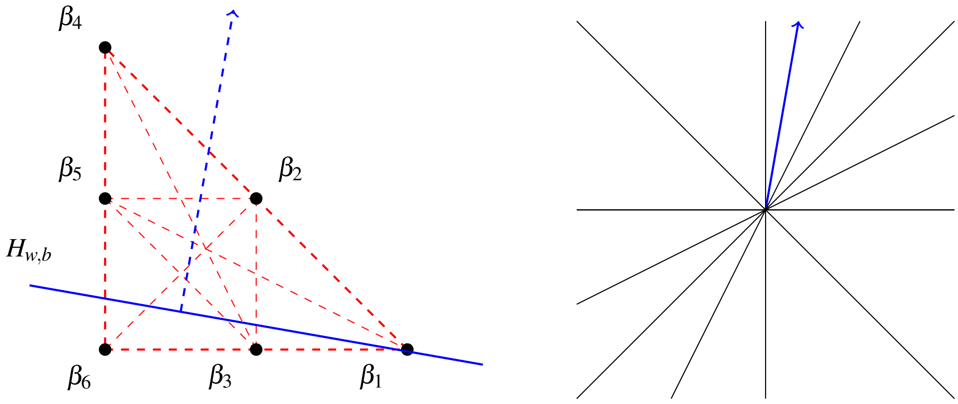
for some $\eta \in S_{k+1}$. To keep track of the basis $B$ we write $S(B) = S_{k+1}$ and index cones $\sigma = \sigma_\eta^T \subset \mathcal{N}_B^T$ by elements $\eta \in S(B)$. We emphasize again that cones of $\mathcal{N}_B^T$ are indexed by $S(B) = S_{k+1}$ since any permutation gives a full dimensional cone (cf. Equation (39)). On the contrary, $S(B^d) \subsetneq S_{N_{k+1}^d}$. We think of $S(B^d)$ as the total orders on $B^d$ "realized" geometrically by the total fan $\mathcal{N}_{B^d}^T$.

**Example 2.2.1.19.** Consider $B = \{1,2,3\}$ and $B^2 = \{\beta_1, \ldots, \beta_6\} \cong 2\cdot\Delta_3 \cap \mathbb{Z}^3$. $B^2$ and the total fan $\mathcal{N}_{B^2}^T$ are shown in Figure 13. We projected to $\mathbb{R}^2$ by forgetting the last coordinate. The blue dashed vector $w \in \sigma_\eta^T \subset \mathcal{N}_{B^2}^T$ induces the total order

$$\begin{aligned}
&\langle w, \beta_6\rangle \leq \langle w, \beta_3\rangle \leq \langle w, \beta_1\rangle \leq \langle w, \beta_5\rangle \leq \langle w, \beta_2\rangle \leq \langle w, \beta_4\rangle \\
\iff\ & 2w_3 \leq w_1 + w_3 \leq 2w_1 \leq w_2 + w_3 \leq w_1 + w_2 \leq 2w_2 \\
\iff\ & w_3 \leq w_1 \leq w_2 \text{ and } 2w_2 \leq w_1 + w_2,\ w_1 + w_3 \leq 2w_2,\ 2w_1 \leq w_2 + w_3
\end{aligned}$$

**Remark 2.2.1.20.** We distinguish top-dimensional cones of $\mathcal{N}_B^T$ and $\mathcal{N}_{B^d}^T$ by their indices, i.e. $\sigma_\eta^T$ with $\eta \in S(B^d)$ is a cone in $\mathcal{N}_{B^d}^T$ and $\sigma_\nu^T$ with $\nu \in S(B)$ is a cone in $\mathcal{N}_B^T$. The sets $S(B^d)$ and $S(B)$ are related as follows: let $\eta \in S(B)$ be a total order on $B$. The elements $d\cdot e_{b_{\eta(i)}} \in B^d$ form the vertices of $d\cdot\Delta_{k+1}$. We say $\nu \in S(B^d)$ *induces* $\eta \in S(B)$ and write $\nu \rightharpoonup \eta$ if the cone $\sigma_\nu^T \subset \mathcal{N}_{B^d}^T$ is contained in $\sigma_\eta^T \subset \mathcal{N}_B^T$. In other words, the total order given by $w \in \operatorname{relint}(\sigma_\nu^T)$ restricted to the vertices of $d\cdot\Delta_{k+1} \cong B^d$ agrees with the total order on $B$ induced by $w' \in \operatorname{relint}(\sigma_\eta^T)$ (cf. Equation (39) and Equation (40)).

An immediate consequence is the following




(A) $B^2 \cong 2 \cdot \Delta_3 \cap \Lambda$.    (B) The total fan $\mathcal{N}^T_{B^2} \subset \mathbb{R}^2$.

FIGURE 13. $B^2$ and the total fan. Note that affine hyperplanes parallel to dashed red lines do not provide a proper total order.

**Corollary 2.2.1.21.** *Let $B$ be a basis of $M$ and $\sigma^T_\eta \subset \mathcal{N}^T_B$ with $\eta \in S(B)$. Then:*

$$\sigma^T_\eta = \bigcup_{\substack{\nu \in S(B^d): \\ \nu \rightharpoonup \eta}} \sigma^T_\nu .$$

We turn back to $B(M)$ and $B(M^d)$. The goal is to relate the fine subdivision of $B(M^d)$ with $B(M)$. In the following we work with a fixed basis $B$ of $M$. According to Lemma 2.1.1.5 (c) there is a bijection between $B(M)\big|_B$ and $B\left(M\big|_B\right)$. We also know that $B\left(M\big|_B\right)$ is a complete fan (Lemma 2.1.1.2) whose support equals $\mathbb{R}_B$. Consider a top dimensional weight class $\sigma_{\mathscr{F}} \subset B\left(M\big|_B\right)$ defined by a maximal flag of flats $\mathscr{F} \lhd M\big|_B$. Then $\sigma_{\mathscr{F}}$ is a weight class of the form

$$w_{b_{\eta(1)}} \leq \ldots \leq w_{b_{\eta(k+1)}} \tag{41}$$

for some $\eta \in S(B)$. In particular, $\sigma_{\mathscr{F}} = \sigma^T_\eta \subset \mathcal{N}^T_B$. We see that $B\left(M\big|_B\right) = \mathcal{N}^T_B$.

**Definition 2.2.1.22.** Fix a basis $B$ of the matroid $M$. For $\eta \in S_{k+1}$ let $\sigma_\eta \subset B(M)\big|_B$ denote the weight class such that $p_B(\sigma_\eta) = \sigma^T_\eta \subset \mathcal{N}^T_B$. For $\nu \in S_{N^d_{k+1}}$ let $\sigma_\nu \subset B(M)\big|_B$ denote the cone such that $p_B(\sigma_\nu) = \sigma^T_\nu \subset \mathcal{N}^T_{B^d}$.

Note that $\sigma_\eta \subset B(M)\big|_B$ is a proper weight class if $\eta \in S(B) = S_{k+1}$ (cf. Equation (41)) whereas $\sigma_\nu$ is a (maybe empty) cone properly contained in $\sigma_\eta$ if $\nu \in S_{N^d_{k+1}}$ and $\nu \rightharpoonup \eta$. Consequently, we have

$$B(M)\big|_B = \bigcup_{\eta \in S(B)} \sigma_\eta .$$

Due to Corollary 2.2.1.21 the cone $\sigma^T_\eta$ with $\eta \in S(B)$ is the union of $\sigma^T_\nu$ with $\nu \in S(B^d)$ and $\nu \rightharpoonup \eta$. Hence, we get

$$B(M)\big|_B = \bigcup_{\nu \in S(B^d)} \sigma_\nu .$$

NOTATION 2.6. An element $\nu \in S_{N^d_{k+1}}$ indexes both a cone $\sigma_\nu \subset B(M)\big|_B$ as well as a weight class $\sigma_\nu \subset B\left(M^d\right)\big|_{B^d}$. In oder to distinguish these cones we write $\tilde{\sigma}_\nu$ for the weight class $\sigma_\nu \subset B\left(M^d\right)\big|_{B^d}$.

The following lemma establishes a connection between $\sigma_\nu$ and $\tilde{\sigma}_\nu$ for $\nu \in S(B^d)$:

**Lemma 2.2.1.23.** *Let $M$ be the matroid associated to a linear ideal $\mathscr{I}$ and let $M^d$ denote the matroid associated to $\mathscr{I}^d_{\text{lin}}$. Let $B$ be a basis of $M$ and $\nu \in S(B^d)$. Then $\sigma_\nu \cong \tilde{\sigma}_\nu \cap \operatorname{trop}(\mathscr{V}(\mathscr{J}))$ and $\tilde{\sigma}_\nu \subset B\left(M^d\right)\big|_{B^d}$.*

PROOF. The goal is to show that $V^\top \sigma_\nu = \tilde{\sigma}_\nu \cap \operatorname{trop}(\mathscr{V}(\mathscr{J}))$. Then the statement follows since $\operatorname{trop}(\nu_d) = V^\top$ is a linear isomorphism.
Consider any $w \in \sigma_\nu$. By definition $p_B(w)$ satisfies $\langle p_B(w), \beta_{\nu(1)}\rangle \leq \ldots \leq \langle p_B(w), \beta_{\nu(N^d_{k+1})}\rangle$ where $\{\beta_1, \ldots, \beta_{N^d_{k+1}}\} = B^d \subset \mathbb{R}_B$. Equivalently, $w$ satisfies $\langle w, \beta_{\nu(1)}\rangle \leq \ldots \leq \langle w, \beta_{\nu(N^d_{k+1})}\rangle$ if we consider $B^d$ in $\mathscr{M}_{n+1,d}$. Note that coordinates of $\beta_i \in B^d \subset \mathscr{M}_{n+1,d}$ indexed by $\alpha \in B^{\complement}$ are zero. Consequently, $V^\top w = \tilde{w} \in B\left(M^d\right)$ satisfies $\tilde{w}_{\beta_{\nu(1)}} \leq \ldots \leq \tilde{w}_{\beta_{\nu(N^d_{k+1})}}$ (cf. Lemma 2.2.1.2). These inequalities describe $\sigma_\nu^T \subset \mathscr{N}^T_{B^d} \cong \mathbb{R}_{B^d}$. Due to Corollary 2.2.1.10, $B^d$ is a basis, i.e. $\tilde{\sigma}_\nu \subset B\left(M^d\right)\big|_{B^d}$ is the corresponding top-dimensional weight class such that $p_{B^d}(\tilde{\sigma}_\nu) = \sigma_\nu^T$. Moreover, $p_{B^d}\big|_{\tilde{\sigma}_\nu}$ is a bijection (Lemma 2.1.1.5 (a)). Thus $\tilde{w} \in \tilde{\sigma}_\nu$ and since $\tilde{w} \in \operatorname{trop}(\mathscr{V}(\mathscr{J}))$ we get $\tilde{w} \in \tilde{\sigma}_\nu \cap \operatorname{trop}(\mathscr{V}(\mathscr{J}))$ and $\tilde{\sigma}_\nu \subset B\left(M^d\right)\big|_{B^d}$. Vice versa, consider an element $\tilde{w} \in \tilde{\sigma}_\nu \cap \operatorname{trop}(\mathscr{V}(\mathscr{J}))$. We have $\nu \in S(B^d)$, i.e. all elements $\tilde{w}' \in \tilde{\sigma}_\nu$ satisfy the inequalities $\tilde{w}'_{\beta_{\nu(1)}} \leq \ldots \leq \tilde{w}'_{\beta_{\nu(N^d_{k+1})}}$. Since $B$ is a basis of $M$, $B^d$ is a basis of $M^d$ (Corollary 2.2.1.10). Hence, $\tilde{\sigma}_\nu \subset B\left(M^d\right)\big|_{B^d}$ and $\tilde{\sigma}_\nu$ is top-dimensional. Due to $\tilde{w} \in \operatorname{trop}(\mathscr{V}(\mathscr{J})) \cap B\left(M^d\right)$ there is an element $w \in B(M)$ such that $V^\top w = \tilde{w}$ (Lemma 2.2.1.2). Hence, $w$ satisfies $\langle w, \beta_{\nu(1)}\rangle \leq \ldots \leq \langle w, \beta_{\nu(N^d_{k+1})}\rangle$. Equivalently, $\langle p_B(w), \beta_{\nu(1)}\rangle \leq \ldots \leq \langle p_B(w), \beta_{\nu(N^d_{k+1})}\rangle$. We conclude that $w \in \sigma_\nu$. □

Consider a basis $B$ of $M$ and consider $B^d = \{\beta_1, \ldots, \beta_{N^d_{k+1}}\} \subset \mathbb{R}_B$ (cf. Remark 2.2.1.17). The inequalities

$$\langle w, \beta_{\nu(1)}\rangle \leq \ldots \leq \langle w, \beta_{\nu(N^d_{k+1})}\rangle$$

with $\nu \in S_{N^d_{k+1}}$ provide a top-dimensional cone in $\mathbb{R}_B$ if and only if $\nu \in S(B^d)$. Consequently, $\sigma_\nu \subset B(M)\big|_B^{(k+1)}$ if and only if $\nu \in S(B^d)$. However, the inequalities

$$\tilde{w}_{\beta_{\nu(1)}} \leq \ldots \leq \tilde{w}_{\beta_{\nu(N^d_{k+1})}}$$

define a top-dimensional weight class in $B\left(M^d\right)\big|_{B^d}$ for any $\nu \in S_{N^d_{k+1}}$. With regard to Lemma 2.2.1.23 the sets $S(B^d)$ arising from bases $B$ of $M$ characterize the top-dimensional weight classes of $B\left(M^d\right)$ that contribute to $B\left(M^d\right) \cap \operatorname{trop}(\mathscr{V}(\mathscr{J}))$:

**Corollary 2.2.1.24.** *Let $\nu \in S_{N^d_{k+1}}$ be a permutation, $B$ be a basis of $M$ and let $\tilde{\sigma}_\nu \subset B\left(M^d\right)\big|_{B^d}$ be the weight class defined by $\nu$. Then $\tilde{\sigma}_\nu \cap \operatorname{trop}(\mathscr{V}(\mathscr{J}))$ is top-dimensional if and only if $\nu \in S(B^d)$. In other words, the degree-$d$ subdivision of $B(M)$ is given by the cones $\sigma_\nu \subset B(M)\big|_B$ with $\nu \in S(B^d)$.*

**Example 2.2.1.25.** Consider the linear form $l = x_1 + x_2 + x_3 + x_4 \in \mathbb{K}[x_1, x_2, x_3, x_4]$, the hyperplane $X = \mathscr{V}(l) \subseteq \mathbb{P}^3$ and its tropicalization $\operatorname{trop}(\mathscr{V}(\langle l\rangle))/\mathbf{1}_4 \subset \mathbb{R}^3$. We exemplify the fine subdivision and its refinement by the degree-2 subdivision with the help of a cone in the $\langle w_1, w_2\rangle$-plane. First, we focus on the fine subdivision. The following illustrates the coefficient matrix of $\langle l\rangle$ and, with regard to the basis $B=\{1,2,3\}\in\mathscr{B}$, a suitable Gale dual of $A(\langle l\rangle)$:

$$A(\langle l\rangle)=\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \end{array} \quad \text{and} \quad G=\begin{array}{c} \begin{array}{cccc} x_4 & x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

So, $M=M(\langle l\rangle)=U_{3,4}$ is a uniform matroid. Figure 14a contains an illustration of $B(M)/\mathbf{1}_4$ (with representatives having last coordinate equal to zero). The cones of $B(M)$ inside the $\langle w_1,w_2\rangle$-plane are described by the inequalities

$$w_1\le w_2\le w_3=w_4=0 \text{ and} \tag{42}$$

$$w_2\le w_1\le w_3=w_4=0. \tag{43}$$

$B^2$ and its total fan are depicted in Figure 13a where we use the total order (cf. Convention 2.2.1.6) initially fixed on $[n+1]$, i.e. $\beta_1=(2,0,0),\beta_2=(1,1,0),\beta_3=(1,0,1),\beta_4=(0,2,0),\beta_5=(0,1,1)$ and $\beta_6=(0,0,2)$. The elements $\beta_1,\beta_4$ and $\beta_6$ correspond to the vertices of $2\cdot\Delta_3$. We can identify them with $B$ such that

$$w_1\le w_2\le w_3 \quad\Leftrightarrow\quad \langle w,\beta_1\rangle\le\langle w,\beta_4\rangle\le\langle w,\beta_6\rangle.$$

Hence, we can write $\sigma_\eta$ with $\eta=\mathrm{id}\in S(B)$ for the weight class described by the Equation (42). Now we turn our attention to the degree-2 subdivision of $\sigma_\eta$. According to Lemma 2.2.1.23 we seek for $\nu\in S(B^d)$ such that $\nu\rightharpoonup\eta$. There are $\nu_1=\mathrm{id}\in S(B^d)$ providing the cone described by

$$\langle w,\beta_1\rangle\le\langle w,\beta_2\rangle\le\langle w,\beta_3\rangle\le\langle w,\beta_4\rangle\le\langle w,\beta_5\rangle\le\langle w,\beta_6\rangle$$

and $\nu_2=(3\ 4)\in S(B^d)$ that provides the cone described by

$$\langle w,\beta_1\rangle\le\langle w,\beta_2\rangle\le\langle w,\beta_4\rangle\le\langle w,\beta_3\rangle\le\langle w,\beta_5\rangle\le\langle w,\beta_6\rangle.$$

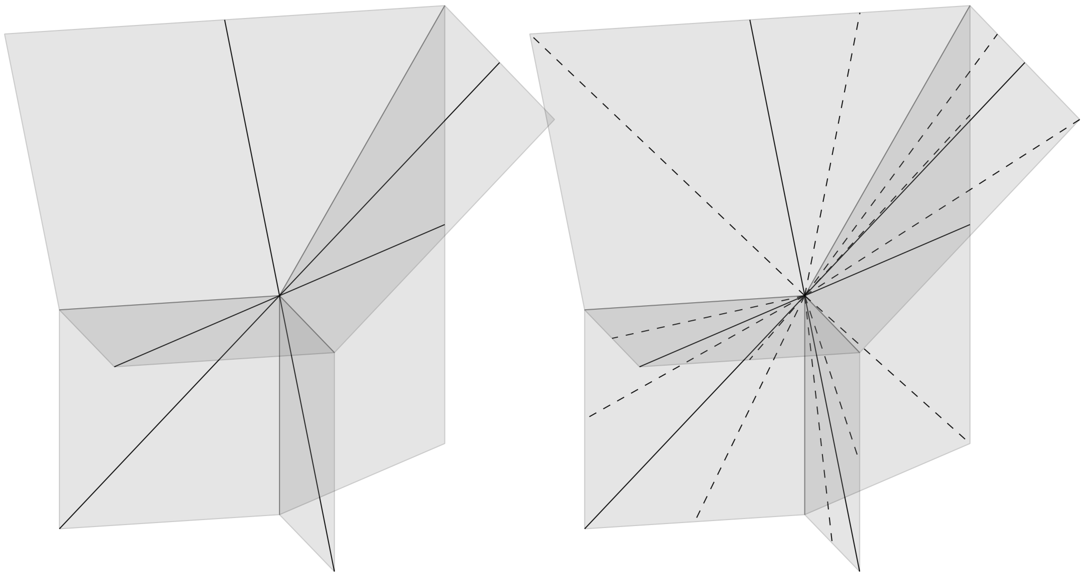
If we consider the cones $\sigma_{\nu_i}\subset B\left(M^2\right)\big|_{B^2}$ with $i=1,2$ we obtain the following flats of flats:

$$\mathscr{F}_{\nu_1}:\quad \{11\}\subset\{11,12\}\subset\{11,12,13\}\subset\{11,12,13,22\}\subset\{11,12,13,22,23\}\subset B^d \text{ and}$$

$$\mathscr{F}_{\nu_2}:\quad \{11\}\subset\{11,12\}\subset\{11,12,22\}\subset\{11,12,22,13\}\subset\{11,12,22,13,23\}\subset B^d.$$

We see that $\mathscr{F}_{\nu_1}$ and $\mathscr{F}_{\nu_2}$ differ only in rank 3. At last we verify that the closures of these flags provide the degree-2 subdivision of $w_1\le w_2\le w_3=0$. Consider $\langle l\rangle^2=\langle x_i\cdot l : i=1,\dots,4\rangle\subset\mathbb{K}[x_1,x_2,x_3,x_4]$. Due to Lemma 2.2.1.11 we can write down the row reduced coefficient matrix of $\langle l\rangle^2$ with respect to the basis $B^2=\{1,2,3\}^2$:

$$A\left(\langle l\rangle^2\right)=\begin{array}{c} \begin{array}{cccccccccc} x_4x_4 & x_1x_4 & x_2x_4 & x_3x_4 & x_1x_1 & x_1x_2 & x_1x_3 & x_2x_2 & x_2x_3 & x_3x_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -2 & -2 & -1 & -2 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$



(A) $\operatorname{trop}(\mathscr{V}(\langle\sum_i x_i\rangle)) \subset \mathbb{R}^4/\mathbf{1}_4$ with its fine subdivision.

(B) $\operatorname{trop}(\mathscr{V}(\langle\sum_i x_i\rangle) \subset \mathbb{R}^4/\mathbf{1}_4$ with the degree-2 subdivision. Each cone of the fine subdivision is subdivided by two cones.

FIGURE 14. $\operatorname{trop}(\mathscr{V}(\sum_i x_i))\,\mathbb{R}^4/\mathbf{1}_4$ with its fine and degree-2 subdivision.

A suitable Gale dual is

$$G\left(\langle l\rangle^2\right) = \begin{array}{c} \\ x_1x_1 \\ x_1x_2 \\ x_1x_3 \\ x_2x_2 \\ x_2x_3 \\ x_3x_3 \end{array} \begin{array}{c} \begin{array}{cccccccccc} x_4x_4 & x_1x_4 & x_2x_4 & x_3x_4 & x_1x_1 & x_1x_2 & x_1x_3 & x_2x_2 & x_2x_3 & x_3x_3 \end{array} \\ \begin{pmatrix} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

From this we can read off the closures of $\mathscr{F}_{v_i}$ for $i \in \{1,2\}$:

$$\operatorname{cl}(\mathscr{F}_{v_1}):\ \{11\} \subset \{11,12\} \subset \{11,12,13,14\} \subset \{11,12,13,14,22\} \subset \{11,12,13,14,22,23,24\} \subset B^d,$$

$$\operatorname{cl}(\mathscr{F}_{v_2}):\ \{11\} \subset \{11,12\} \subset \{11,12,22\} \subset \{11,12,13,14,22\} \subset \{11,12,13,14,22,23,24\} \subset B^d.$$

We are interested in $\tilde{\sigma}_{v_i} \cap \operatorname{trop}(\mathscr{V}(\mathscr{J}))$. By exchanging $\tilde{w}_\alpha = \langle w, \alpha\rangle$ for all $\alpha \in \mathscr{M}_{n+1,d}$ (Lemma 2.2.1.2) we obtain the corresponding cones $\sigma_{v_i}$ in $B(M)$ (Lemma 2.2.1.23):

$$2w_1 \le w_1 + w_2 \le w_1 + w_3 = w_1 + w_4 \le 2w_2 \le w_2 + w_3 = w_2 + w_4 \le 2w_3 = w_3 + w_4 = 2w_4 \text{ and}$$

$$2w_1 \le w_1 + w_2 \le 2w_2 \le w_1 + w_3 = w_1 + w_4 \le w_2 + w_3 = w_2 + w_4 \le 2w_3 = w_3 + w_4 = 2w_4,$$

or in a more pleasant form (omitting $w_4$ since $w_3 = w_4$):

$$p_B(\sigma_{v_1}) = \sigma_{v_1}^T = \{w \in \mathbb{R}^3 : w_1 \le w_2 \le w_3,\ w_1 + w_3 \le 2w_2\} \text{ and}$$

$$p_B(\sigma_{v_2}) = \sigma_{v_2}^T = \{w \in \mathbb{R}^3 : w_1 \le w_2 \le w_3,\ 2w_2 \le w_1 + w_3\}.$$

### 2.2.2. The Codimension One Skeleton of the Degree-d Subdivision

Now we investigate what happens if we add a homogeneous polynomial with constant coefficients to the generating set of the linear ideal, i.e. the set up for this section is as follows:

NOTATION 2.7. We stick to Notation 2.1 and extend it by the ideal $\mathscr{I} + \langle F \rangle$ consisting of the ideal $\mathscr{I} = \langle l_0, \dots, l_{n-k-1} \rangle \subset R_{n+1}$ generated by linear forms as before and $F \in R_{n+1}$ a homogeneous polynomial of degree $d$. Recall that we are in the constant coefficient case.

A building block for this section is the following

THEOREM 2.2.2.26 ([**AN13**],Theorem 3.7). *Let $I$ be a saturated homogeneous ideal with Hilbert polynomial $p_I$. Then there is a tropical basis in $I$ consisting of polynomials of degree not greater than the Gotzmann number $m_0$ of $p_I$.*

We briefly introduce the Gotzmann number ([**AN13**, section 2.1 and 2.2]). A *numerical polynomial* is a rational polynomial with values in $\mathbb{Z}$ for large enough integer arguments. A well-known numerical polynomial is the Hilbert polynomial of a homogeneous ideal. It can be shown that every numerical polynomial of degree $s$ can be written as

$$p_I(r) = \mathscr{P}(m_0, \dots, m_s; r) = \sum_{k=0}^{s} \binom{r+k}{k+1} - \binom{r+k-m_k}{k+1}$$

with $m_0, \dots, m_s \in \mathbb{Z}$ and $m_s \neq 0$ ([**AN13**]). The value $m_0$ is called *Gotzmann number*. We use Theorem 2.2.2.26 to prove the following:

**Proposition 2.2.2.27.** *Let $I = \mathscr{I} + \langle F \rangle$ be a homogeneous ideal generated by a linear ideal $\mathscr{I}$ and a homogeneous polynomial $F$ of degree $d$. Then the degree $d$ part $I^{(d)}$ of $I$ contains a tropical basis.*

PROOF. The ideal $I = \langle l_i, F \,|\, i = 0, \dots, n-k-1 \rangle$ is saturated. We need to show that the Gotzmann number according to $I$ is $d = \deg(F)$. Then, the statement follows with Theorem 2.2.2.26. Recall Remark 2.2.1.4: the Hilbert function is defined by

$$\mathrm{h}_I : \mathbb{Z} \longrightarrow \mathbb{Z}, \quad r \longmapsto \dim_{\mathbb{K}} \left( \mathbb{K}[x_0, \dots, x_n]^{(r)} \Big/ I^{(r)} \right).$$

Let $p_I$ be the Hilbert polynomial associated to $I$ and $r_0 \in \mathbb{N}$ such that $\mathrm{h}_I(r) = p_I(r)$ for $r \geq r_0$. The Hilbert polynomial $p_I$ is a numerical polynomial $\mathscr{P}$, i.e. there are integers $m_0, \dots, m_s \in \mathbb{Z}$ with $m_s \neq 0$ such that

$$p_I(r) = \mathscr{P}(m_0, \dots, m_s; r) = \sum_{k=0}^{s} \binom{r+k}{k+1} - \binom{r+k-m_k}{k+1}$$

Note that for principal ideals $\langle g \rangle \subset \mathbb{K}[x_0, \dots, x_n]$ we have $m_0 = \ldots = m_{n-1} = \deg(g)$ (cf. [**AN13**, examples in section 2.1]). Again $\bar{F}$ denotes the coset of $F$ after modding out all linear forms $l_i$, i.e. $\bar{F} \in \mathbb{K}[x_0, \dots, x_n] / \mathscr{I}$. It holds:

$$\mathrm{h}_I(r) = \dim_{\mathbb{K}} \left( \mathbb{K}[x_0, \dots, x_n]^{(r)} \Big/ I^{(r)} \right) = \dim_{\mathbb{K}} \left( \mathbb{K}[x_0, \dots, x_k]^{(r)} \Big/ \langle \bar{F} \rangle^{(r)} \right) = h_{\langle \bar{F} \rangle}(r).$$

Hence, there exists $r_0$ such that for all $r \geq r_0$ we have $\mathrm{h}_I(r) = h_{\langle \bar{F} \rangle}(r) = p_{\langle \bar{F} \rangle}(r) = \mathscr{P}(m_0, \dots, m_{k-1}; r)$ with $m_0 = \ldots = m_{k-1} = \deg(\bar{F})$. Hence, the Gotzmann number equals $d = \deg(F) = \deg(\bar{F})$. □

In Section 2.2.1 we determined the degree $d$ part of $\mathscr{I}$, which is the linear part of $I$. By adding $F$ to $\mathscr{I}$ we adjoin one further generator of degree $d$. As one might expect the dimension of the degree

$d$ part of $I$ equals the dimension of the degree $d$ part of $\mathscr{I}$ plus one. To see this we have to adapt notations from Section 2.2.1 for the new setup involving $F$:

NOTATION 2.8. We fix a basis $B = \{b_0, \dots, b_k\}$ of $M = M(\mathscr{I})$ and we denote the complement by $B^{\complement} = [n+1] \setminus B = \{b'_0, \dots, b'_{m-1}\}$ with $m = n-k$. Again we write $\mathscr{I}^d$ for the ideal generated by all linear forms $l_i = x_{b'_i} + \sum_{j \in B} a_{i,j} x_j$ for $i = 0, \dots, m-1$ multiplied with all monomials of degree $d-1$ in the variables $x_{b'_i}, \dots, x_{b'_m}, x_{b_0}, \dots, x_{b_k}$ for all $i$. Contrary to Section 2.2.1 we define the ideal $I^d$ generated by $\mathscr{I}^d$ and $F$. Exchanging monomials $x^\alpha$ by variables $y_\alpha$ for $\alpha \in \mathscr{M}_{n+1,d}$ (cf. Definition 2.2.1.9) we obtain a linear ideal $I^d_{\text{lin}} \subset R_{N+1}$ with $N = N^d_{n+1} - 1$ whose coefficient matrix is of the form

$$A\left(I^d_{\text{lin}}\right) = \begin{bmatrix} A\left(\mathscr{I}^d_{\text{lin}}\right) \\ A(F) \end{bmatrix}$$

By construction, the coefficient matrix of $I^d_{\text{lin}}$ coincides with the coefficient matrix of $\mathscr{I}^d_{\text{lin}}$ as defined in Section 2.2.1 except for the last line. Only the last row containing the coefficients of the generator $F$ is new. We have seen that performing row reduction on $A\left(\mathscr{I}^d_{\text{lin}}\right)$ leads to a matrix

$$A^d = \begin{bmatrix} \mathbb{E}_{N^d_{n+1} - N^d_{n-m+1}} & A_{B^d} \end{bmatrix}.$$

According to Lemma 2.2.1.11 we know the exact form of $A^d$. Hence, by performing row operations on $A\left(I^d\right)$, we obtain an equivalent coefficient matrix of the following form:

$$A^d_F = \begin{pmatrix} A^d \\ F_B \end{pmatrix}.$$

Note that $F_B = \pi_B(\bar{F})$ for $\bar{F} \in R_{n+1}/\mathscr{I} \cong R_B$ (cf. Section 2.1.2).

Similar to the case of $\mathscr{I}$ (cf. Section 2.2.1) the polynomials corresponding to the rows of $A^d_F$ generate the degree $d$ part of $I^{(d)}$:

**Corollary 2.2.2.28.** *Let $\mathscr{I} = \langle l_0, \dots, l_{n-k-1} \rangle \subset R_{n+1}$ be an ideal generated by linear forms and $F \in R_{n+1}$ a homogeneous polynomial of degree $d$. Then the dimension of the degree $d$ part of $I = \mathscr{I} + \langle F \rangle$ equals $\dim_{\mathbb{K}}\left(\mathscr{I}^{(d)}\right)$ plus one, i.e.*

$$\dim_{\mathbb{K}}\left(I^d\right) = N^d_{n+1} - N^d_{n+1-m} + 1$$

PROOF. From the Hilbert function we get

$$\begin{aligned}
\mathrm{h}_I(r) &= \dim_{\mathbb{K}}\left(\left(\mathbb{K}[x_0, \dots, x_n]/I\right)^{(r)}\right) \\
&= \dim_{\mathbb{K}}\left(\left(\mathbb{K}[x_0, \dots, x_k]/\langle \bar{F} \rangle\right)^{(r)}\right) \\
&= \dim_{\mathbb{K}}\left(\mathbb{K}[x_0, \dots, x_k]^{(r)}\right) - \dim_{\mathbb{K}}\left(\langle \bar{F} \rangle^{(r)}\right) \\
\implies \quad \dim_{\mathbb{K}}\left(I^{(r)}\right) &= \dim_{\mathbb{K}}\left(\mathbb{K}[x_0, \dots, x_n]^{(r)}\right) - \dim_{\mathbb{K}}\left(\mathbb{K}[x_0, \dots, x_k]^{(r)}\right) + \dim_{\mathbb{K}}\left(\langle \bar{F} \rangle^{(r)}\right) \\
&= N^d_{n+1} - N^d_{n+1-m} + 1 \text{ for } d = r.
\end{aligned}$$

□

Consequently, the polynomials corresponding to the rows of $A_F^d$ generate the degree $d$ part of $I^{(d)}$. At this point we see that we can compute $F_B$ since we know the exact form of $A^d$:

**Corollary 2.2.2.29.** *We work with the set up defined in notations 2.8. We write the homogeneous polynomial $F$ of degree $d$ in the form $F = \sum_{\alpha\in(B^d)^\complement} c_\alpha x^\alpha + \sum_{\alpha\in B^d} c_\alpha x^\alpha$ where $B$ is a basis of $M$. As $\left(B^d\right)^\complement$ indexes the rows of $A^d$ we can deduce $F_B$ from $A^d$ and $F$ as follows:*

$$F_B = F - \sum_{\alpha\in(B^d)^\complement} c_\alpha l_\alpha.$$

PROOF. Note that $l_\alpha$ is a polynomial of degree $d$. The leading monomial has an exponent in $\left(B^d\right)^\complement$, all remaining monomials have exponents purely in $B^d$. Consequently, $F - \sum_{\alpha\in(B^d)^\complement} c_\alpha l_\alpha$ purely contains monomials with exponents in $B^d$. □

Due to Corollary 2.2.2.28 we know that the row space of $A_F^d$ equals the degree $d$ part of $I$. From Proposition 2.2.2.27 we know that the degree $d$ part of $I$ contains a tropical basis. Both statements combined provide an adapted version of Proposition 2.2.1.13:

**Proposition 2.2.2.30.** *Let $\mathscr{I} = \langle l_0, \ldots, l_{n-k-1}\rangle \subset R_{n+1}$ be an ideal generated by linear forms and $F \in R_{n+1}$ a homogeneous polynomial of degree $d$. Let $M_F^d$ denote the matroid associated to $I_{\text{lin}}^d$. Then:*

$$\operatorname{trop}(\mathscr{V}(I)) \cong B\left(M_F^d\right) \cap \operatorname{trop}(\mathscr{V}(\mathscr{J})).$$

PROOF. According to Proposition 2.2.2.27 the degree $d$ part of $I$ contains a tropical basis, i.e. we have $\operatorname{trop}(\mathscr{V}(I)) = \bigcap_{g\in I^{(d)}} \operatorname{trop}(\mathscr{V}(g))$. Also recall that the linear part of a linear ideal contains a tropical basis. Using this we conclude:

$$\begin{aligned}
\operatorname{trop}(\mathscr{V}(I)) &= \bigcap_{g\in I^{(d)}} \mathscr{T}(g) \\
&= \bigcap_{g\in \text{rowspace}(A_F^d)} \mathscr{T}(g) \\
&= \{w \in \mathbb{R}^{n+1} : V^\top w \in \mathscr{T}(g) \forall g \in (I_{\text{lin}}^d)^{(1)}\} \\
&\cong \{\tilde{w} \in \mathbb{R}^{N+1} : \tilde{w} \in \operatorname{trop}(\mathscr{V}(\mathscr{J})) \text{ and } \tilde{w} \in \mathscr{T}(g) \forall g \in (I_{\text{lin}}^d)^{(1)}\} \\
&= \operatorname{trop}(\mathscr{V}(\mathscr{J})) \cap B\left(M_F^d\right).
\end{aligned}$$

□

We like to point out that $I$ is non-linear. However, we can describe the tropical variety associated to $I$ by the Bergman fan $B\left(M_F^d\right)$ arising from $M_F^d$, the matroid associated to $I_{\text{lin}}^d$. In particular, we know a tropical basis for $B\left(M_F^d\right)$, that is the set of circuits of $M_F^d$.

**Corollary 2.2.2.31.** *Let $I = \mathscr{I} + \langle F\rangle \subset R_{n+1}$ be an ideal with constant coefficients where $F \in R_{n+1}$ is a homogeneous polynomial of degree $d$ and $\mathscr{I} \subset R_{n+1}$ a linear ideal. Then the set of polynomials in $I^{(d)}$ with minimal support forms a tropical basis for $I$.*

**Remark 2.2.2.32.** At first sight Proposition 2.2.2.30 yields a practical method to compute $\operatorname{trop}(\mathscr{V}(I))$. As $I_{\text{lin}}^d \subset R_{N+1}$ is linear, the tropicalization is completely determined by $M_F^d$, the matroid associated to $I_{\text{lin}}^d$ (cf. Section 1.4.5). As Bergman fans arising from matroids are well-understood this method also provides an theoretical approach to the variety $\operatorname{trop}(\mathscr{V}(I))$. There are several software packages capable of computing tropical linear spaces with constant coefficients, e.g. SINGULAR ([**DGPS16**]), POLYMAKE ([**GJ00**]) and GFAN ([**Jen**]). Moreover, we do not need to compute the whole Bergman fan $B\left(M_F^d\right)$ as Lemma 2.2.1.23 restricts to the relevant parts of $B\left(M_F^d\right)$. However, there are concomitants to accept: obviously, for both theoretical and practical purposes, we exchange a problem with a small number of variables ($n$) and higher degree ($d$) for a large number of variables $\left(\binom{n+d}{d}\right)$. Although $\operatorname{trop}(\mathscr{V}(\mathscr{J}))$ is a proper linear space we have to compute the intersection with $B\left(M_F^d\right)$. Alternatively, one could use Corollary 2.2.2.31: the circuits of $M_F^d$ provide a tropical basis of $I$. We only have to exchange $y_\alpha$ with $x^\alpha$ in all circuits for all $\alpha \in \mathscr{M}_{n+1,d}$. Presumably, the number of circuits of $M_F^d$ that we have to compute is large in comparison to the number of circuits of $M$. Then we have to intersect all tropical hypersurfaces obtained this way from the circuits of $M_F^d$.

CHAPTER 3

# Tropical Hypersurfaces with a Specified Singularity

During the last few years singular tropical hypersurfaces captured a lot of attention (e.g. [**DFS07**], [**DT12**], [**MMS12a**], [**Tak17**]). Primarily, the focus was set on hypersurfaces with the simplest type of singularities, i.e. hypersurfaces containing points where the differential vanishes (cf. Definition 1.6.1.1).

The family $\nabla$ of singular hypersurfaces defined by Laurent polynomials with fixed support $\mathscr{A} \subset \mathbb{Z}^n$ defined over $\mathbb{K}_{\mathbb{C}}$ and its tropicalization has been introduced in Section 1.6.2. In this chapter, we explore tropicalizations of two subfamilies of $\nabla$: we examine the family $\nabla^{\text{cusp}}$ of *plane curves with a cusp* ($n=2$) in Section 3.1 and the family $\nabla^{k+1}$ of $k+1$-*fold singular hypersurfaces* ($n \geq 2$) in Section 3.2. The superior goal is not only a better understanding of tropical hypersurfaces with singularities of a type mentioned above but also their classification. Recall that tropical hypersurfaces are dual to regular subdivisions of $\operatorname{Newt}(\operatorname{conv}(\mathscr{A}))$. Therefore, a possibility to accomplish this goal is to study the relationship between the secondary fan $\operatorname{Sec}_{\mathscr{A}}$ associated to the point configuration $\mathscr{A}$ and the tropicalization $\operatorname{trop}(X)$ of the family at hand.

We fix the ground field $\mathbb{K} = \mathbb{K}_{\mathbb{C}}$. The examinations in Sections 3.1 and 3.2 proceed according to a similar pattern. In order to give a brief summary we fix notations (based on Notation 1.3) valid for the entire chapter and specified at the beginning of each section:

NOTATION 3.1. Let $F = \sum_i y_i x^{\alpha_i} \in \mathbb{K}[y_1, \ldots, y_m]\left[x_1^{\pm}, \ldots, x_n^{\pm}\right]$ denote a generic Laurent polynomial with fixed support $\mathscr{A} = \{\alpha_1, \ldots, \alpha_m\} \subset \mathbb{Z}^n$ of cardinality $m$. The neutral element of the torus $T^n = (\mathbb{K}^*)^n$ is denoted by $\mathbf{1}_n = (1, \ldots, 1)$. Recall that an element $a \in T^m$ provides a Laurent polynomial $F_a = \sum_i a_i x^{\alpha_i} \in \mathbb{K}\left[x_1^{\pm}, \ldots, x_n^{\pm}\right]$ whereas $F(p) = \sum_i y_i p^{\alpha_i} \in \mathbb{K}[y_1, \ldots, y_m]$ denotes the linear form obtained from $F$ evaluated at $p = (p_1, \ldots, p_n) \in T^n$. In the following we write

$$\mathscr{I} = \langle F(\mathbf{1}_n), \frac{\partial F}{\partial x_i}(\mathbf{1}_n) : i = 1, \ldots, n \rangle. \tag{44}$$

for the ideal generated by $F$ and all of its partial derivatives $\frac{\partial F}{\partial x_i}$ for $i = 1, \ldots, n$ evaluated at $\mathbf{1}_n$. Let $A \in \mathbb{Z}^{n \times m}$ be the matrix representation of the point configuration $\mathscr{A}$. The coefficient matrix of $\mathscr{I}$ equals

$$A' = \begin{bmatrix} 1 & \cdots & 1 \\ \alpha_1 & \cdots & \alpha_m \end{bmatrix} = \begin{bmatrix} \mathbf{1}_m^\top \\ A \end{bmatrix} \in \mathbb{Z}^{n+1 \times m}, \tag{45}$$

It is the matrix representation of the shift $\mathscr{A}' = \{1\} \times \mathscr{A}$ of $\mathscr{A}$ into $\mathbb{R}^3$.

A priori, we describe the philosophy that is leading through the sections. It has its origin in the study of the tropical $A$-discriminant: consider the family $\nabla$ of Laurent polynomials with fixed support

$\mathscr{A} \subset \mathbb{Z}^n$ providing singular hypersurfaces. Horn uniformization (Remark 1.6.2.10) revealed that

$$\nabla = \{a \in \mathbb{P}(T^m) : \mathscr{V}(F_a) \text{ is singular}\} = \ker(A') \cdot \mathrm{Im}(\psi_{\mathscr{A}}),$$

i.e. each point of $\nabla$ is a product of an element in $\ker(A')$ and $\mathrm{Im}(\psi_{\mathscr{A}})$. Note that

$$\ker(A') = \nabla_{\mathbf{1}_n} = \{a \in \mathbb{P}(T^m) : F_a(\mathbf{1}_n) = 0, \frac{\partial F_a}{\partial x_i}(\mathbf{1}_n) = 0 \; \forall i\} = \mathscr{V}(\mathscr{I}). \tag{46}$$

Tropically, Horn uniformization yields $\mathrm{trop}(\nabla) = \mathrm{trop}(\ker(A')) \oplus \mathrm{rowspace}(A')$ (Theorem 1.6.2.11). This decomposition means that $\mathrm{trop}(\nabla)$ can be studied by fixing the singularity at $\mathbf{1}_n$ (cf. Equation (46)). The choice of a different point $\mathbf{1}_n \neq p \in T^n$ for the fixed singularity translates to a shift of a point $a \in \mathrm{trop}(\nabla_{\mathbf{1}_n})$ by $A'^{\top}$ (cf. Remark 1.6.2.12).

The tropical varieties $\mathrm{trop}(\nabla^{\mathrm{cusp}})$ and $\mathrm{trop}(\nabla^{k+1})$ can be studied similarly to $\mathrm{trop}(\nabla)$: we can fix the singular point at $\mathbf{1}$ since adapted versions of Horn uniformization work as well (cf. Remark 3.1.1.1 and Remark 3.2.1.4). In order to determine the tropicalizations, the algebraic equations describing $\nabla^{\mathrm{cusp}}_{\mathbf{1}}$ and $\nabla^{k+1}_{\mathbf{1}}$ in $\nabla_{\mathbf{1}}$ are required. Even though the approach to study hypersurfaces with a singularity at $\mathbf{1}$ is similar in both sections it turns out that, algebraically, $\nabla^{\mathrm{cusp}}_{\mathbf{1}}$ is a hypersurface in $\nabla_{\mathbf{1}}$ (cf. Section 3.1.2) whereas $\nabla^{k+1}_{\mathbf{1}}$ is a proper linear subspace of $\nabla_{\mathbf{1}}$ (cf. Section 3.2.2). In the endeavor to study $\mathrm{trop}(\nabla^{\mathrm{cusp}}_{\mathbf{1}})$ we make use of the methods developed in Chapter 2 (in particular Section 2.1) and study the relative Newton polytope of the defining ideal (Section 3.1.3). Since $\nabla^{k+1}_{\mathbf{1}}$ is defined by a linear ideal we focus on the associated matroid as it determines the tropicalization $\mathrm{trop}(\nabla^{k+1}_{\mathbf{1}})$ completely (cf. Section 3.2.3).

## 3.1. First Steps towards Plane Tropical Curves with a Cusp

In this section we examine plane tropical curves with a cusp. We adapt the notations for this case:

NOTATION 3.2. We use Notation 3.1 with $n = 2$, i.e. we consider a generic bivariate Laurent polynomial $F = \sum_i y_i x^{\alpha_i} \in \mathbb{K}[y_1, \ldots, y_m][x^{\pm}, y^{\pm}]$ with support $\mathscr{A} \subset \mathbb{Z}^2$. We abbreviate the polynomial ring forming the coefficients by $R = \mathbb{K}[y_1, \ldots, y_m]$. We write $x^{\alpha_i} = x^{\alpha_{i,x}} y^{\alpha_{i,y}}$ for all $\alpha_i = (\alpha_{i,x}, \alpha_{i,y}) \in \mathscr{A}$. Furthermore, we write $|\alpha_i, \alpha_j| = \det \begin{bmatrix} \alpha_i & \alpha_j \end{bmatrix}$ for the determinant of the $2 \times 2$ matrix having $\alpha_i$ and $\alpha_j$ as columns.

In Section 3.1.1 we justify why it is sufficient to study the family of plane curves with a cusp at $\mathbf{1}_2 = (1, 1)$. In Section 3.1.2 we identify the algebraic relations describing $\nabla^{\mathrm{cusp}}_{\mathbf{1}_2}$. We show that the vanishing ideal $\mathscr{J}$ of $\nabla^{\mathrm{cusp}}_{\mathbf{1}_2}$ is generated by linear forms (describing $\nabla_{\mathbf{1}_2}$) and a single homogeneous polynomial $D$ of degree 2. In order to apply results of Chapter 2 we examine the ambient linear space $\nabla$ and its associated matroid in Section 3.1.2.1. Then we focus on the polynomial $D$ in Section 3.1.2.2. In Section 3.1.3 we show that the relative Newton polytope of $D$ with respect to $M$ is completely determined by the affine geometry of the support $\mathscr{A}$. For this purpose we use the software package SINGULAR. In Section 3.1.4 we take first steps towards a classification of plane tropical curves with a cusp.

Let us take a look at the main results. Recall that $\nabla_{\mathbf{1}_2} = \mathscr{V}(\mathscr{I})$ is linear and $M = (E, \mathscr{B})$ the associated matroid to $\mathscr{I}$. Theorem 2.1.3.25 states that we can recover $\mathrm{trop}(\nabla^{\mathrm{cusp}}_{\mathbf{1}_2}) = \mathrm{trop}(\mathscr{V}(\mathscr{I} + \langle D \rangle))$ from all of its coordinate projections with respect to bases $B$ of $M$. According to Theorem 2.1.2.16

we have $(p_B)_*(\mathrm{trop}(\mathscr{V}(\mathscr{I}+\langle D\rangle))) = \mathrm{trop}(\mathscr{V}(D_B))$ where $D_B = \pi(\bar{D}) \in R_B$. The tropical hypersurface $\mathrm{trop}(\mathscr{V}(D_B))$ is dual to the Newton polytope of $D_B$ (Proposition 1.4.2.9). Thus, we can determine $\mathrm{trop}(\nabla^{\mathrm{cusp}}_{\mathbf{1}_2})$ provided we know the relative Newton polytope $\mathscr{N}_{\mathscr{B}}$ of $D$. The main result of this section, Theorem 3.1.3.11, states that $\mathrm{Newt}(D_B)$ depends on affine relations of points of $\mathscr{A}$:

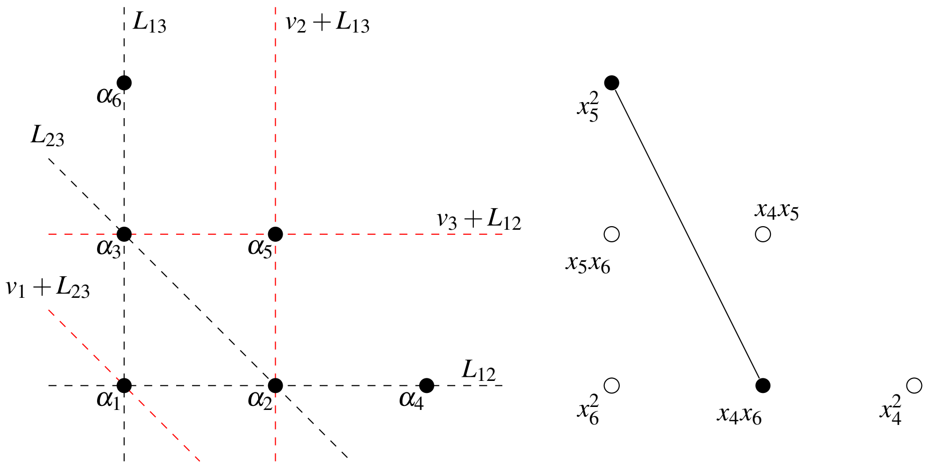
THEOREM. *Let $\mathscr{A} \subset \mathbb{Z}^2$ be a point configuration, $M$ the associated matroid to the linear ideal $\mathscr{I}$ providing $\mathscr{V}(\mathscr{I}) = \nabla_{\mathbf{1}_2}$. Consider any basis $B$ of $M$. Let $L_{ij} \subset \mathbb{R}^2$ denote the line passing through $\alpha_i$ and $\alpha_j$ for $i,j \in B^{\complement}$. Moreover, let $v_i + L_{jk}$ denote the line parallel to $L_{jk}$ passing through $\alpha_i$ for all pairwise distinct $i,j,k \in B^{\complement}$. For $i,j \in B$ we have:*

*(1) $2e_i \in \mathrm{Newt}(D_B)$ if and only if $\alpha_i \notin L_{rs}$ for pairwise distinct $r,s \in B^{\complement}$.*
*(2) Suppose $\alpha_i \in L_{rs}$ for some $r,s \in B^{\complement}$. Then $e_i + e_j \in \mathrm{Newt}(D_B)$ if and only if $\alpha_j \notin L_{rs}$ and $\alpha_j \notin v_t + L_{rs}$ with $t = B^{\complement} \setminus \{r,s\}$.*

To get an idea, we give an example. Consider the point configuration

$$\mathscr{A} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}. \tag{47}$$

The maximal affine independent set $\{\alpha_1, \alpha_2 \alpha_3\} \subset \mathscr{A}$ provides the basis $B = E \setminus \{1,2,3\}$ of $M$, see Figure 15a. We sketch $\mathrm{Newt}(D_B) \subset \mathbb{R}_B$ in $\mathbb{R}^2$ by forgetting the last coordinate, cf. Figure 15b.




(A) The point configuration $\mathscr{A}$, the lines $L_{12}, L_{13}, L_{23}$ and $v_1 + L_{23}, v_2 + L_{13}, v_3 + L_{12}$.

(B) The Newton polytope of $D_B$ according to the basis $B = E \setminus \{1,2,3\} = \{4,5,6\}$.

FIGURE 15. The point configuration $\mathscr{A}$, cf. Equation (47), and the Newton polytope of $D_{\{4,5,6\}}$.

First, note that $\mathrm{Newt}(D_B) \subset 2 \cdot \Delta_{|E|-3}$, i.e. the vertices of $\mathrm{Newt}(D_B)$ are of the form $e_i + e_j \in \mathbb{R}_B$ with $i,j \in B$. For $i = j$ the main theorem states that we have $2e_i \in \mathrm{Newt}(D_B)$ if and only if the point $\alpha_i$ does not lie on one of the lines $L_{12}, L_{1,3}, L_{2,3}$. This is true for $i = 5$ whereas $\alpha_4 \in L_{12}$ and $\alpha_6 \in L_{13}$, cf. the black dashed lines in Figure 15a. Hence, the first statement of the theorem covers the vertices of $2 \cdot \Delta_{|E|-3}$. The second statement covers all intermediate points — if necessary: suppose $2e_i, 2e_j \in \mathrm{Newt}(D_B)$ for some $i,j \in B$. Then $e_i + e_j \in \mathrm{Newt}(D_B)$ anyway. Thus, suppose that $2e_i \notin \mathrm{Newt}(D_B)$, w.l.o.g. $\alpha_i \in L_{12}$. For $j \in B \setminus \{i\}$ the theorem states: $e_i + e_j \in \mathrm{Newt}(D_B)$ if and only if $\alpha_j$ is not contained in $L_{12}$ or the parallel line to $L_{12}$ passing trough $\alpha_3$. For $i = 4$ we have $\alpha_4 \in L_{12}$, i.e. $2e_4 \notin \mathrm{Newt}(D_B)$. Then, for $j = 5 \neq i$, we have $\alpha_5 \in v_3 + L_{12}$, i.e. $e_4 + e_5 \notin \mathrm{Newt}(D_B)$. Similarly, $e_5 + e_6 \notin \mathrm{Newt}(D_B)$ since $\alpha_6 \in L_{13}$ and $\alpha_5 \in v_2 + L_{13}$. However, $e_4 + e_6 \in \mathrm{Newt}(D_B)$ as $\alpha_6 \notin v_3 + L_{12}$.

Suppose we have $\mathrm{Newt}(D_B) = 2 \cdot \Delta_{|E|-3}$ for all bases $B$ of $M$. Equivalently, for all bases $B$ of $M$ we have $\alpha_i \notin L_{rs}$ for all $i \in B$ and pairwise distinct $r, s \in B^{\complement}$. In other words, no three points of $\mathscr{A}$ are colinear. In this case we call $\mathscr{A}$ *generic* (cf. Corollary 3.1.3.14). The second main result concerns generic supports and is geared towards a classification:

THEOREM. *Let $\mathscr{A} \subset \mathbb{Z}^2$ be a generic point configuration of cardinality $m$. Then we have*

$$\mathrm{trop}(\nabla^{cusp}_{\mathbf{1}_2}) = B(U_{m-4,m}).$$

Thus $\mathrm{trop}(\nabla^{\mathrm{cusp}}_{\mathbf{1}_2})$ carries a Bergman fan structure arising from a uniform matroid $U_{m-4,m}$ if $\mathscr{A}$ is generic. All top-dimensional weight classes are defined by inequalities of the form

$$\sigma_{\mathscr{F}}: \quad w_{i_1} \leq w_{i_2} \leq \ldots \leq w_{i_{m-4}} = w_{i_{m-3}} = w_{i_{m-2}} = w_{i_{m-1}} = w_{i_m}. \tag{48}$$

In the regular subdivision of $P = \mathrm{conv}(\mathscr{A})$ induced by an element $w$ contained in the weight class $\sigma_{\mathscr{F}}$ described by Equation (48) we see a pentagon formed by the points $\{\alpha_{i_{m-4}}, \ldots, \alpha_{i_m}\}$ — by assumption no three points of $\mathscr{A}$ are colinear — getting the highest weights $w_{i_{m-4}} = \ldots = w_{i_m}$.

### 3.1.1. Plane Tropical Curves with a Cusp at $\mathbf{1}_2$

Recall the definition of a cuspidal singularity introduced in Section 1.6: let $C = \mathscr{V}(F_a) \subset T^2$ be a curve defined by a Laurent polynomial $F_a \in \mathbb{K}[x^{\pm}, y^{\pm}]$ with $a \in \nabla$ and $p \in \mathrm{Sing}_{F_a}$. The point $p \in C$ is a cusp, i.e. $p \in \text{c-Sing}_{F_a}$, if and only if the determinant of the Hessian of $F_a$ vanishes at $p$ (Lemma 1.6.1.5). The primary goal of this section is to understand the tropicalization of the family of plane tropical curves with a cusp:

$$\nabla^{\mathrm{cusp}} = \{a \in \mathbb{P}(T^m) : \mathscr{V}(F_a) \text{ has a cusp.}\} = \{a \in \mathbb{P}(T^m) : \text{c-Sing}_{F_a} \neq \emptyset\}.$$

For simple singularities the Horn uniformization proved successful (cf. Remark 1.6.2.10). This is also true for plane tropical curves with a cusp, i.e. it is sufficient to study $\nabla^{\mathrm{cusp}}_{\mathbf{1}_2}$:

**Remark 3.1.1.1** (Cusps at $p = \mathbf{1}_2$)**.** We proceed as in the Remark 1.6.2.10 where the Horn uniformization for singular hypersurfaces is explained. We already know that

$$F_a = \sum_{\alpha \in \mathscr{A}} a_\alpha x^\alpha \text{ is singular at } p \quad \Longleftrightarrow \quad F_{a \cdot \Psi_{\mathscr{A}}(p)} = \sum_{\alpha \in \mathscr{A}} a_\alpha p^\alpha x^\alpha \text{ is singular at } \mathbf{1}_2.$$

Recall that $\Psi_{\mathscr{A}}$ denotes the monomial map associated to $\mathscr{A}$. Asking for $p$ to be a cusp requires $p$ to be a singularity of $\mathscr{V}(F_a)$ and additionally $\det(H_{F_a})(p) = 0$. Fortunately, the further condition,

$\det(H_{F_a}) = 0$, blends in the Horn uniformization as well. Provided $p \in \mathscr{V}(F_a)$ is a cusp:

$$\det(H_{F_a})(p) = \left[\left(\frac{\partial^2 F_a}{\partial x^2}\right)\left(\frac{\partial^2 F_a}{\partial y^2}\right) - \left(\frac{\partial^2 F_a}{\partial x \partial y}\right)^2\right](p) = 0$$

$$\Leftrightarrow \left[\sum_{(i,j)\in[m]^2} \alpha_{i,x}(\alpha_{i,x}-1)\,\alpha_{j,y}(\alpha_{j,y}-1)\,a_i a_j x^{\alpha_i+\alpha_j-2e_1-2e_2} - \alpha_{i,x}\alpha_{i,y}\alpha_{j,x}\alpha_{j,y}a_i a_j x^{\alpha_i+\alpha_j-2e_1-2e_2}\right](p) = 0$$

$$\Leftrightarrow \left[\sum_{(i,j)\in[m]^2} \alpha_{i,x}(\alpha_{i,x}-1)\,\alpha_{j,y}(\alpha_{j,y}-1)\,a_i a_j x^{\alpha_i+\alpha_j} - \alpha_{i,x}\alpha_{i,y}\alpha_{j,x}\alpha_{j,y}a_i a_j x^{\alpha_i+\alpha_j}\right](p) = 0$$

The last equivalence holds since $p \in T^2$ and, therefore, $p^{2(e_1+e_2)} \neq 0$, i.e. we can multiply (divide respectively) with $p^{2(e_1+e_2)}$.

$$\Leftrightarrow \left[\sum_{(i,j)\in[m]^2} \alpha_{i,x}(\alpha_{i,x}-1)\,\alpha_{j,y}(\alpha_{j,y}-1)\,a_i a_j p^{\alpha_i+\alpha_j}x^{\alpha_i+\alpha_j} - \alpha_{i,x}\alpha_{i,y}\alpha_{j,x}\alpha_{j,y}a_i a_j p^{\alpha_i+\alpha_j}x^{\alpha_i+\alpha_j}\right](\mathbf{1}_2) = 0$$

$$\Leftrightarrow \left[\sum_{(i,j)\in[m]^2} \alpha_{i,x}(\alpha_{i,x}-1)\,\alpha_{j,y}(\alpha_{j,y}-1)\,a_i a_j p^{\alpha_i+\alpha_j}x^{\alpha_i+\alpha_j-2e_1-2e_2}\right.$$

$$\left. - \alpha_{i,x}\alpha_{i,y}\alpha_{j,x}\alpha_{j,y}a_i a_j p^{\alpha_i+\alpha_j}x^{\alpha_i+\alpha_j-2e_1-2e_2}\right](\mathbf{1}_2) = 0$$

$$\Leftrightarrow \left[\left(\frac{\partial^2 F_{a\cdot\Psi_{\mathscr{A}}(p)}}{\partial x^2}\right)\left(\frac{\partial^2 F_{a\cdot\Psi_{\mathscr{A}}(p)}}{\partial y^2}\right) - \left(\frac{\partial^2 F_{a\cdot\Psi_{\mathscr{A}}(p)}}{\partial x \partial y}\right)^2\right](\mathbf{1}_2) = \det\left(H_{F_{a\cdot\Psi_{\mathscr{A}}(p)}}\right)(\mathbf{1}_2).$$

From Remark 3.1.1.1 we immediately conclude

**Corollary 3.1.1.2.** *Let $\mathscr{A} \subset \mathbb{Z}^2$ be a finite point configuration of cardinality $m$ and let $A' \in \mathbb{Z}^{3\times m}$ denote its shifted matrix representation. Then:*

$$\operatorname{trop}(\nabla^{cusp})) = \operatorname{trop}\left(\nabla^{cusp}_{\mathbf{1}_2}\right) \oplus \operatorname{rowspace}(A').$$

Corollary 3.1.1.2 attests that $\operatorname{trop}(\nabla^{\text{cusp}})$ can be recovered from $\operatorname{trop}(\nabla^{\text{cusp}}_{\mathbf{1}_2})$. Thus, we focus on $\operatorname{trop}(\nabla^{\text{cusp}}_{\mathbf{1}_2})$ in the following.

### 3.1.2. The Vanishing Ideal of $\nabla^{\mathbf{cusp}}_{\mathbf{1}_2}$

In this section we prepare the ground for the tropicalization of

$$\nabla^{\text{cusp}}_{\mathbf{1}_2} = \{a \in \mathbb{P}(T^m) : \mathscr{V}(F_a) \text{ has a cusp at } \mathbf{1}_2\}.$$

We aim not only for the tropicalization itself but for deeper combinatorial insights. In order to study $\operatorname{trop}(\nabla^{\text{cusp}}_{\mathbf{1}_2})$ we examine the defining ideal of $\nabla^{\text{cusp}}_{\mathbf{1}_2}$. We enhance our notations by the following:

NOTATION 3.3. We define

$$\mathscr{I} := \langle F(\mathbf{1}_2),\, \frac{\partial F}{\partial x}(\mathbf{1}_2),\, \frac{\partial F}{\partial y}(\mathbf{1}_2)\rangle,\; D := \det(H_F)(\mathbf{1}_2) \text{ and } \mathscr{J} := \mathscr{I} + \langle D\rangle \subset \mathbb{K}[y_1,\ldots,y_m].$$

By $M$ we denote the associated matroid to the linear part $\mathscr{I}$ of $\mathscr{J}$.

Consequently, $\nabla^{\text{cusp}}_{\mathbf{1}_2} = \mathscr{V}(\mathscr{J})$ and $\nabla_{\mathbf{1}_2} = \mathscr{V}(\mathscr{I}) = \ker(A')$ is the linear space that parametrizes all Laurent polynomials with fixed support $\mathscr{A}$ that provide a plane tropical curves with a simple singularity at $\mathbf{1}_2$. This topic was studied in [**MMS12a**]. We observe that $\nabla^{\text{cusp}}_{\mathbf{1}_2} = \mathscr{V}(\mathscr{J})$ is a hypersurface in the linear space $\nabla_{\mathbf{1}_2} = \mathscr{V}(\mathscr{I}) = \ker(A')$. This circumstance suggests to apply the methods introduced in Section 2.1 of Chapter 2. There we examined the tropicalization of a hypersurface contained in a linear space from all coordinate projections. The presented algorithm heavily depends on the ambient linear space $\nabla_{\mathbf{1}_2}$, in particular on the matroid $M$. Here we focus on the fine subdivision of $B(M) = \operatorname{trop}(\ker(A'))$. We treat this in part 3.1.2.1. Furthermore, it is necessary to know the coordinate projections of $\nabla^{\text{cusp}}_{\mathbf{1}_2}$. In detail, we need to know $D_B = \pi(\bar{D})$ of $\bar{D} \in R_m/\mathscr{I}$ according to the bases $B$ of $M$ (cf. Theorem 2.1.2.16). Additionally, it is required that $D$ is not divisible by a monomial. We focus on that topic in Section 3.1.2.2.

#### 3.1.2.1. The Ambient Linear Space $\nabla_{\mathbf{1}_2} = \ker(A')$

As a first step we need to identify the matroid $M$ associated to the ambient linear space $\mathscr{V}(\mathscr{I})$ of $\nabla^{\text{cusp}}_{\mathbf{1}_2}$. This is one of the easier tasks of the upcoming challenges: its defining ideal $\mathscr{I}$ describes $\nabla_{\mathbf{1}_2}$, i.e. $\mathscr{V}(\mathscr{I}) = \nabla_{\mathbf{1}_2} = \ker(A')$. Recall that $M$ denotes the matroid associated to $\mathscr{I}$. To get a better notion of $\nabla_{\mathbf{1}_2}$ we focus on $M$. We know that $M = M[G]$ where $G$ is any Gale dual to $A'$ (cf. Remark 1.4.5.28). We summarize some useful facts about $A'$ (and $G$ respectively) (see also [**MMS12a**, Construction 3.2 and Remark 3.3]):

**Lemma 3.1.2.3.** *Let $\mathscr{A} = \{\alpha_1, \dots, \alpha_m\} \subset \mathbb{Z}^2$ be the fixed point configuration and $A' \in \mathbb{Z}^{3\times m}$ the matrix representation of $\mathscr{A}'$, the shift of $\mathscr{A}$ into $\mathbb{R}^3$ (cf. Notation 3.1). Let $\{\alpha_1, \alpha_2, \alpha_3\}$ be an affine basis. Recall that $|\alpha_i, \alpha_j|$ denotes the determinant of the minor of $A$ given by the columns $i, j$. Then:*

(a) *$q := |\alpha_1, \alpha_2| - |\alpha_1, \alpha_3| + |\alpha_2, \alpha_3| \neq 0$, and*

(b) *by performing row operations on $A'$ we get an equivalent matrix $\tilde{A}' = \begin{bmatrix} \mathbb{E}_3 & \bar{A}' \end{bmatrix}$ with $\left(\bar{A}'\right)_j = \begin{bmatrix} a_j & b_j & c_j \end{bmatrix}^\top$ such that*

$$\begin{aligned} a_j &= \frac{1}{q}\left(|\alpha_2, \alpha_3| - |\alpha_2, \alpha_j| + |\alpha_3, \alpha_j|\right), \\ b_j &= \frac{1}{q}\left(-|\alpha_1, \alpha_3| + |\alpha_1, \alpha_j| - |\alpha_3, \alpha_j|\right), \\ c_j &= \frac{1}{q}\left(|\alpha_1, \alpha_2| - |\alpha_1, \alpha_j| + |\alpha_2, \alpha_j|\right) \text{ and} \\ \alpha_j &= a_j\alpha_1 + b_j\alpha_2 + c_j\alpha_3 \text{ with } a_j + b_j + c_j = 1 \end{aligned}$$

*for $j = 4, \dots, m$.*

Proof. For (a) note that $\{\alpha_1, \alpha_2, \alpha_3\} \subset \mathbb{R}^2$ is a linear dependent set, i.e. there is a vector $(0,0) \neq (\lambda, \mu) \in \mathbb{R}^2$ such that $\alpha_3 = \lambda\alpha_1 + \mu\alpha_2$. Then $q = |\alpha_1, \alpha_2|(1 - \lambda - \mu)$ by substitution of $\alpha_3$. It remains to show that $\lambda + \mu \neq 1$. This immediately follows since $\alpha_1, \alpha_2, \alpha_3$ are affinely independent.
For (b) we equip $\mathbb{R}^3$ with coordinates $(t, x, y)$. Notice that the point configuration $\mathscr{A}'$ lives in the plane defined by $t = 1$. Since $\alpha_1, \alpha_2, \alpha_3$ are affinely independent we can pick these three vectors as pivots for Gaussian elimination. The Gaussian elimination is an affine transformation moving the affine basis to the standard basis $e_1, e_2, e_3$. The affine basis $e_1, e_2, e_3$ spans the plane given by the equation $t + x + y = 1$. Moreover, $\bar{A}'$ contains the coordinates of $\alpha_j$ in column $j$ that likewise satisfy the plane equation. The expressions of $a_j, b_j$ and $c_j$ are obtained via Gauss elimination. $\square$

#### 3.1.2.2. The Polynomial $D = \det(H_F)(\mathbf{1}_2)$

In this part we take a closer look at the additional generator $D = \det(H_F)(\mathbf{1}_2) \in R = \mathbb{K}[y_1,\dots,y_m]$ that differs $\mathscr{J}$ from $\mathscr{I}$. Recall from Section 2.2.2 that we know how to obtain $D_B = \pi(\bar{D}) \in R_B$, provided that we know $D$ and $A^d$ (with $d = \deg(D)$) in detail (Corollary 2.2.2.29). First, we seek for a detailed description of $D$. As one might guess, $D$ shapes up as a homogeneous polynomial of degree two:

**Lemma 3.1.2.4.** *Let $F = \sum_i y_i x^{\alpha_i} \in \mathbb{K}[y_1,\dots,y_m][x^{\pm}, y^{\pm}]$ be a generic Laurent polynomial with support $\mathscr{A} = \{\alpha_1,\dots,\alpha_m\} \subset \mathbb{Z}^2$. Then $D = \det(H_F)(\mathbf{1}_2) \in \mathbb{Z}[y_i : i = 1,\dots,m]$ is a homogeneous polynomial of degree two and the coefficient of $y_iy_j$ in $\det(H_F)(\mathbf{1}_2)$ has the following form:*

*$i = j$:* $\operatorname{coef}_D\left(y_i^2\right) = \alpha_{i,x}\alpha_{i,y}\det\begin{bmatrix}\alpha_i - e_1 & \alpha_i - e_2\end{bmatrix} = \alpha_{i,x}\alpha_{i,y}(1-\alpha_{i,x}-\alpha_{i,y})$.

*$i \neq j$:* $\operatorname{coef}_D(y_iy_j) = \alpha_{i,x}\alpha_{j,y}\det\begin{bmatrix}\alpha_i - e_1 & \alpha_j - e_2\end{bmatrix} + \alpha_{j,x}\alpha_{i,y}\det\begin{bmatrix}\alpha_j - e_1 & \alpha_i - e_2\end{bmatrix}$.

PROOF. We have

$$\det(H_F)(\mathbf{1}_2) = \left[\left(\frac{\partial^2 F}{\partial x^2}\right)\left(\frac{\partial^2 F}{\partial y^2}\right) - \left(\frac{\partial^2 F}{\partial x \partial y}\right)^2\right](\mathbf{1}_2)$$

$$= \left(\sum_i \alpha_{i,x}(\alpha_{i,x}-1)y_i\right)\left(\sum_j \alpha_{j,y}(\alpha_{j,y}-1)y_j\right) - \left(\sum_i \alpha_{i,x}\alpha_{i,y}y_i\right)\left(\sum_j \alpha_{j,x}\alpha_{j,y}y_j\right). \tag{49}$$

Observe that each bracket in Equation (49) is a linear form in the $y_i$, i.e.the products yield terms of degree two in the variables $y_i$. Now we fix two indices $i, j \in \{1,\dots,m\}$. Suppose $i = j$ then each bracket in the expression shown in Equation (49) contains exactly one monomial with $y_i$ such that

$$\begin{aligned}\operatorname{coef}_D\left(y_i^2\right) &= \alpha_{i,x}(\alpha_{i,x}-1)\alpha_{i,y}(\alpha_{i,y}-1) - \alpha_{i,x}\alpha_{i,y}\alpha_{i,x}\alpha_{i,y}\\ &= \alpha_{i,x}\alpha_{i,y}((\alpha_{i,x}-1)(\alpha_{i,y}-1) - \alpha_{i,x}\alpha_{iy})\\ &= \alpha_{i,x}\alpha_{i,y}\det\begin{bmatrix}\alpha_i - e_1 & \alpha_i - e_2\end{bmatrix} = \alpha_{i,x}\alpha_{i,y}(1-\alpha_{i,x}-\alpha_{i,y}).\end{aligned}$$

If $i \neq j$ each bracket in the expression shown in Equation (49) contains a monomial with $y_i$ and $y_j$, i.e. the product of two brackets yields two terms with $y_iy_j$, i.e.

$$\begin{aligned}\operatorname{coef}_D(y_iy_j) &= \alpha_{i,x}(\alpha_{i,x}-1)\alpha_{j,y}(\alpha_{j,y}-1) + \alpha_{j,x}(\alpha_{j,x}-1)\alpha_{i,y}(\alpha_{i,y}-1)\\ &\quad - \alpha_{i,x}\alpha_{i,y}\alpha_{j,x}\alpha_{j,y} - \alpha_{j,x}\alpha_{j,y}\alpha_{i,x}\alpha_{i,y}\\ &= \alpha_{i,x}\alpha_{j,y}\det\begin{bmatrix}\alpha_i - e_1 & \alpha_j - e_2\end{bmatrix} + \alpha_{j,x}\alpha_{i,y}\det\begin{bmatrix}\alpha_j - e_1 & \alpha_i - e_2\end{bmatrix}.\end{aligned}$$

□

Corollary 2.2.2.29 states that $D_B$ depends (among others) on the exact form of $D$. With the help of Lemma 3.1.2.4 we can write down $D$ explicitly since we know the support $\mathscr{A}$. However, $D_B$ describes the push forward if $D$ is not divisible by a monomial. Now, we examine conditions on the support that guarantee that $D$ is not divisible by a monomial.

**Lemma 3.1.2.5.** *Let $F = \sum_i y_i x^{\alpha_i}$ be a generic bivariate Laurent polynomial with support $\mathscr{A} \subset \mathbb{Z}^2$ as introduced in Notation 3.2. Let $L_1, L_2$ and $L_3$ be the lines each defined by two of the points $(0,0), (1,0), (0,1)$. If $|\mathscr{A} \setminus \bigcup_i L_i| \geq 2$ then $D = \det(H_F)(\mathbf{1}_2)$ is not divisible by a monomial.*

PROOF. According to Lemma 3.1.2.4, the polynomial $D$ is homogeneous and has degree two. Suppose $D$ is divisible by a variable, i.e. $D = y_i l$ for some $i \in [m]$ and a linear form $l \in R_m$. Hence, $D$ contains no monomial of the form $y_k^2$ for all $k \neq i$. Therefore, the coefficient of $y_k^2$ is zero for all $k \neq i$. Due to Lemma 3.1.2.4 we have

$$\operatorname{coef}_D\left(y_k^2\right) = \alpha_{k,x}\alpha_{k,y}\left(1 - \alpha_{k,x} - \alpha_{k,y}\right).$$

Consequently, $\operatorname{coef}_D\left(y_k^2\right) = 0$ if and only if $\alpha_k \in \bigcup_i L_i$. We conclude that, if $D = y_i l$ as above, then $\alpha_k \in \bigcup_i L_i$ for all $k \neq i$. Hence $|\mathscr{A} \setminus \bigcup_i L_i| \leq 1$. Note that $y_i^2$ may appear in $D = y_i l$. □

The previous lemma gives a simple geometric criteria in terms of the support. However, if we have $\mathscr{A} \subset \operatorname{cone}(e_1, e_2) + (1,1)$ then $D$ is not divisible by a monomial. The following remark explains how to take advantage of this:

**Remark 3.1.2.6.** Recall that we consider $F_a \in \mathbb{K}[x^{\pm}, y^{\pm}]$ with fixed support $\mathscr{A} \subset \mathbb{Z}^2$. Hence, $F_a$ is equivalent to any $x^{\beta} F_a$, i.e. $F_a$ has a cusp at $p \in T^2$ if and only if $x^{\beta} F_a$ has a cusp at $p \in T^2$. Note that the support of $x^{\beta} F_a$ is the Minkowski sum $\beta + \mathscr{A}$. Therefore, we can shift an arbitrary support $\mathscr{A} \subset \mathbb{Z}^2$ into the are $(1,1) + \operatorname{cone}(e_1, e_2)$. The obtained support guarantees that $D$ is not divisible by a variable.

### 3.1.3. Relative Newton Polytope of $D = \det(H_F)(\mathbf{1}_2)$

Suppose we know all coordinate projections $(p_B)_*(\operatorname{trop}(\nabla_{\mathbf{1}_2}^{\text{cusp}}))$ according to bases $B$ of $M$. Then Theorem 2.1.3.25 states that we can recover $\operatorname{trop}(\nabla_{\mathbf{1}_2}^{\text{cusp}})$. Hence, there is an essential need for the coordinate projections of $\operatorname{trop}(\nabla_{\mathbf{1}_2}^{\text{cusp}})$ with respect to bases $B \in \mathscr{B}$. Theorem 2.1.2.16 offers an solution. We prepare the ground and work with the following

**Convention 3.1.3.7.** In this section we assume that $\{\alpha_1, \alpha_2, \alpha_3\}$ forms an affine basis of $\mathscr{A}$. Then we can express the linear generators of $\mathscr{I}$ as

$$l_1 = y_1 + \sum_{j \in B} a_{1,j} y_j, \; l_2 = y_2 + \sum_{j \in B} a_{2,j} y_j \text{ and } l_3 = y_3 + \sum_{j \in B} a_{3,j} y_j$$

as $B = [m] \setminus \{1,2,3\}$ is a basis of $M$. In particular, the coefficient matrix of $\mathscr{I}$ (cf. Notation 3.3) can be written as

$$A = \begin{bmatrix} \mathbb{E}_3 & \bar{A} \end{bmatrix}$$

where $\left(\bar{A}\right)_{i,j} = a_{i,j}$ with $i \in \{1,2,3\}$ and $j \in B$. With regard to Lemma 3.1.2.3 we have $\sum_k a_{k,j} = 1$ and $\alpha_j = a_{1,j}\alpha_1 + a_{2,j}\alpha_2 + a_{3,j}\alpha_3$ for all $j \in B$.

**Remark 3.1.3.8** (Coordinate projections)**.** We recall the coordinate projections (cf. Section 2.1.2). We write $R = \mathbb{K}[y_1, \ldots, y_m]$. Again, $\pi_B : R/\mathscr{I} \longrightarrow R_B$ denotes the projection to the polynomial ring with coordinates indexed by $B$ and $D_B = \pi(\bar{D})$. We have $(p_B)_*(\operatorname{trop}(\nabla_{\mathbf{1}_2}^{\text{cusp}})) = \operatorname{trop}(\mathscr{V}(D_B))$ for any basis $B$ of $M$ if $D$ is not divisible by a monomial (Theorem 2.1.2.16).

The tropical hypersurface $\operatorname{trop}(\mathscr{V}(D_B))$ purely depends on the Newton polytope $\operatorname{Newt}(D_B)$ (Proposition 1.4.2.9). Therefore, we need to know the relative Newton polytope $\mathscr{N}_{\mathscr{B}}$ of $D$ with respect to the bases $\mathscr{B}$ of $M$. Essentially, we need to know which coefficients of $D_B$ are (non-)zero:

**Proposition 3.1.3.9.** *According to Notation 3.1, let $F = \sum_i y_i x^{\alpha_i} \in \mathbb{K}[y_1, \ldots, y_m][x^{\pm}, y^{\pm}]$ be a generic Laurent polynomial with support $\mathscr{A} = \{\alpha_1, \ldots, \alpha_m\} \subset \mathbb{Z}^2$. The matroid associated to the linear* *ideal* $\mathscr{I} = \langle F(\mathbf{1}_2), \frac{\partial F}{\partial x}(\mathbf{1}_2), \frac{\partial F}{\partial y}(\mathbf{1}_2)\rangle \subset \mathbb{K}[y_1,\dots,y_m]$ *is denoted by* $M$. *We use Convention 3.1.3.7, i.e.* $B = [m]\setminus\{1,2,3\}$ *is a basis of* $M$. *Let* $D_B \in R_B$ *denote the image of* $\bar{D} \in R_m/\mathscr{I}$ *under* $\pi_B$ *(cf. Remark 3.1.3.8). Moreover, we write* $q = |\alpha_1,\alpha_2| - |\alpha_1,\alpha_3| + |\alpha_2,\alpha_3|$. *Then:*

(a) *For* $i \in B$ *we have* $\operatorname{coef}_{D_B}\left(y_i^2\right) = a_{1,i}a_{2,i}a_{3,i}q^2(-1)$.

(b) *Suppose* $\operatorname{coef}_{D_B}\left(y_i^2\right) = 0$ *(w.l.o.g.* $a_{1,i} = 0$*) for some* $i \in B$. *Then, for* $j \in B\setminus\{i\}$, *we have* $\operatorname{coef}_{D_B}(y_iy_j) = a_{2,i}a_{3,i}a_{1,j}(a_{1,j}-1)q^2$.

PROOF. First, we outline the basic idea of the proof that uses results of other sections, in particular Section 2.2.2. The first task is to determine $D_B$. We accomplish this using methods of Section 2.2.1. Recall that $A^2 = A\left(\mathscr{I}^2\right)$ denotes the coefficient matrix of the ideal generated by the shifted linear forms $l_i$ up to degree 2 (cf. the construction explained in Remark 2.2.1.8). As $D$ is a homogeneous polynomial of degree 2 (Lemma 3.1.2.4) we can consider the coefficient matrix

$$A_D^2 = \begin{bmatrix} A^2 \\ D \end{bmatrix}$$

of $\mathscr{I}^2 + D$. Since $B$ is a basis of $M$ we know that $B^2$ is a basis of $M^2 = M\left(\mathscr{I}^2\right)$, the matroid associated to the row space of $A^2$. Hence, we can get an equivalent matrix

$$A_D^2 = \begin{bmatrix} \mathbb{E}_{N_m^2 - N_{m-3}^2} & A_{B^2} \\ & D \end{bmatrix}.$$

The columns of $A_{B^2}$ are indexed by elements of $B^2 \subset \mathscr{M}_{m,2}$, i.e. the polynomials corresponding to the rows of $A_D^2$, except for the last one, have a leading monomial in $(B^2)^{\complement}$ and apart from that they have monomials solely in $B^2$. Hence, we can obtain $D_B$ by performing Gauss elimination on the coefficient matrix $A_D^2$ (cf. Corollary 2.2.2.29). The second task is to determine the coefficient of a monomial $y_iy_j$ of $D_B$ explicitly. Corollary 2.2.1.12 describes $A_{B^2}$ in detail. It states that entries of $A_{B^2}$ are given in terms of the coefficients of the linear forms $l_i$. The coefficient of $y_iy_j$ in $D_B$ for $i,j \in B$ is a linear combination of entries of the column of $A_D^2$ indexed by $\beta = e_i + e_j \in B^2$. We show that it is an expression of elements $a_{rs}$ with $r \in \{1,2,3\}$, $s \in \{i,j\}$ and $\alpha_1,\alpha_2,\alpha_3$. At last we use the computer algebra system SINGULAR to compute the claimed expressions.
We have $B = [m]\setminus\{1,2,3\}$ and $B^{\complement} = \{1,2,3\}$. Thus, $(B^2)^{\complement} = \{e_i + e_j | i \in B^{\complement}, j \in [m]\}$ is the complement of $B^2 \subset \mathscr{M}_{m,2}$. Now we use methods developed in Section 2.2. Due to Lemma 3.1.2.4, the polynomial $D = \det(H_F)(\mathbf{1}_2)$ is homogeneous of degree two and its coefficients are uniquely determined. Recall the following definitions (adapted for this proof, i.e. $d = 2$) from Lemma 2.2.1.11: for an element $\alpha \in \left(B^2\right)^{\complement}$ we define $s = s(\alpha) = \sum_{j\in B^{\complement}}(\alpha)_j$ and $k_i = \min\{j \in B^{\complement} : \left(\alpha - \sum_{l=1}^{i-1} e_{k_l}\right)_j \neq 0\}$ for $i = 1,\dots,s$. Note that $s \in \{1,2\}$ and $k_i \in B^{\complement}$ for all $i$. We write $d_\alpha = \operatorname{coef}_D(y^\alpha)$ for all $\alpha \in \mathscr{M}_{m,2}$ such that

$$D = \det(H_F)(\mathbf{1}_2) = \sum_{\alpha\in\mathscr{M}_{m,2}\setminus B^2} d_\alpha y^\alpha + \sum_{\alpha\in B^2} d_\alpha y^\alpha.$$

The polynomial $l_\alpha$ has leading monomial $y^\alpha$ and all other monomials have exponents in $B^2$. By Corollary 2.2.2.29 we obtain $D_B$ from $D$ by subtracting $d_\alpha l_\alpha$ for all $\alpha \in \left(B^2\right)^{\complement} = \mathscr{M}_{m,2}\setminus B^2$ from $D$.

$$D_B = \sum_{\alpha\in\mathscr{M}_{m,2}\setminus B^2} d_\alpha(y^\alpha - l_\alpha) + \sum_{\alpha\in B^2} d_\alpha y^\alpha \in R_B. \tag{50}$$

Now, we determine the coefficient of $y^\beta$ in $D_B$ with $\beta = e_i + e_j \in B^2$ for some $i, j \in B$. We define $\alpha' = \alpha - \sum_{l=1}^{s(\alpha)} e_{k_l}$ for $\alpha \in \left(B^2\right)^{\complement}$. Let $A^2$ denote the coefficient matrix of $\mathscr{I}^2$ (cf. Remark 2.2.1.8). The linear combination of $D$ and all $l_\alpha$ correspond to row operations on $A^2_D$. In Corollary 2.2.1.12 we examined the exact form of $A^2$, in particular the minor indexed by elements of $B^2$. We conclude from Equation (50) that the coefficient of $y^\beta$ in $D_B$ equals the coefficient of $y^\beta$ in $D$ (i.e. $d_\beta$) minus the coefficients of $y^\beta$ in $l_\alpha$ for all $\alpha \in (B^2)^{\complement}$, multiplied by $d_\alpha$ each. In particular, the coefficient of $y^\beta$ in $D_B$ is only influenced by the coefficients of $y^\beta$ in $l_\alpha$, provided it is non-zero. Since the coefficient of $y^\beta$ in $l_\alpha$ equals $(A^2)_{\alpha,\beta}$ we get:

$$\begin{aligned}
\operatorname{coef}_{D_B}\left(y^\beta\right) &= d_\beta - \sum_{\alpha \in \left(B^2\right)^{\complement}} \left(A^d\right)_{\alpha,\beta} d_\alpha \\
&= d_\beta - \sum_{\substack{\alpha \in \left(B^2\right)^{\complement}: \\ \left(\beta - \alpha'\right)_j \geq 0 \forall j \in B}} \left(A^d\right)_{\alpha,\beta} d_\alpha \\
&= d_\beta - \sum_{\substack{\alpha \in \left(B^2\right)^{\complement}: \\ \left(\beta - \alpha'\right)_j \geq 0 \forall j \in B}} (-1)^{s(\alpha)-1} \sum_{\substack{i_1, \dots, i_{s(\alpha)} \in B: \\ x^{\alpha'} \prod_l x_{i_l} = x^\beta}} \left( \prod_{l=1}^{s(\alpha)} a_{k_l, i_l} \right) d_\alpha
\end{aligned}$$

Suppose $i = j$, i.e. $\beta = 2e_i$ for some $i \in B$. Then $(\beta - \alpha')_k = (2e_i - \alpha')_k \geq 0$ for all $k \in B$ is equivalent to $(\alpha)_k = 0$ for all $k \in B \setminus \{i\}$. Consequently, we sum over all $\alpha \in (B^2)^{\complement}$ with $\alpha_k = 0$ for all $k \in B \setminus \{i\}$. Thus, if $(\alpha)_l \neq 0$ then $l \in B^{\complement} \cup \{i\} = \{1,2,3,i\}$ and we conclude that for any $\alpha \in (B^2)^{\complement}$ appearing in the sum exist $r \in B^{\complement} = \{1,2,3\}$ and $s \in \{1,2,3,i\}$ such that we can write $\alpha = e_r + e_s$. In order to satisfy $\alpha' + \sum_{l=1}^{s(\alpha)} e_{i_l} = \beta$ we conclude that $i_l = i$ for all $l$. Consequently:

$$\operatorname{coef}_{D_B}\left(y_i^2\right) = d_\beta - \sum_{\substack{\alpha \in \left(B^2\right)^{\complement}: \\ \alpha_j = 0 \forall j \in B \setminus \{i\}}} (-1)^{s(\alpha)-1} \left( \prod_{l=1}^{s(\alpha)} a_{k_l, i} \right) d_\alpha$$

Note that $a_{k_l,i} \in \{a_{1,i}, a_{2,i}, a_{3,i}\}$ since $k_l \in B^{\complement} = \{1,2,3\}$ for all $l$. Due to Lemma 3.1.2.3, these elements satisfy $\alpha_i = a_{1,i}\alpha_1 + a_{2,i}\alpha_2 + a_{3,i}\alpha_3$. Again, we focus on the $d_\alpha$ that appear in the sum of the last equation. We just saw that any $\alpha \in (B^2)^{\complement}$ appearing in the sum is of the form $\alpha = e_r + e_s$ with $r \in B^{\complement} = \{1,2,3\}$ and $s \in B^{\complement} \cup \{i\} = \{1,2,3,i\}$. Then, according to Lemma 3.1.2.4, $d_\alpha$ is an expression in $\alpha_r, \alpha_s \in \mathscr{A}$. Consequently, every $d_\alpha$ appearing in the sum is an expression in $\alpha_1, \alpha_2, \alpha_3, \alpha_i \in \mathscr{A}$. We substitute $\alpha_i = a_{1,i}\alpha_1 + a_{2,i}\alpha_2 + a_{3,i}\alpha_3$. Thus every $d_\alpha$ that appears in the sum can be expressed by $a_{1,i}, a_{2,i}, a_{3,i}$ and $\alpha_1, \alpha_2, \alpha_3$. In total the coefficient of $y^\beta$ is an expression in $a_{1,i}, a_{2,i}, a_{3,i}$ and $\alpha_1, \alpha_2, \alpha_3$. With the help of the computer algebra system SINGULAR we obtain the desired form. See appendix A for a script.

The proof for (b) works analogously. Suppose $a_{1,i} = 0$. Then, we get $\alpha_i = a_{2,i}\alpha_2 + a_{3,i}\alpha_3$ and $a_{2,i} = 1 - a_{3,i}$ according to Lemma 3.1.2.3. Consider $\beta = e_i + e_j$ for some $j \in B \setminus \{i\}$. Note that $(\beta - \alpha')_l = (e_i + e_j - \alpha')_l \geq 0$ for all $l \in B$ is equivalent to $(\alpha)_l = 0$ for all $l \in B \setminus \{i, j\}$. Thus, if $\alpha_l \neq 0$ we have $l \in B^{\complement} \cup \{i, j\}$. As $\alpha \in (B^2)^{\complement}$ we have $|\alpha| = 2$ and there is (at least) one element $r \in B^{\complement} = \{1,2,3\}$ such that $(\alpha)_r \neq 0$. As $|\alpha| = 2$ there is an element $s \in B^{\complement} \cup \{i, j\} = \{1,2,3,i,j\}$

such that $\alpha = e_r + e_s$. Hence, the elements $d_\alpha$, appearing in the expression describing the coefficient of $y_i y_j$, arising from $\alpha = e_r + e_s$ satisfies $r \in B^\complement = \{1,2,3\}$ and $s \in B^\complement \cup \{i,j\} = \{1,2,3,i,j\}$. Then, according to Proposition 3.1.3.9, $d_\alpha$ is an expression in $\alpha_r, \alpha_s \in \mathscr{A}$. We substitute $\alpha_i$ and $\alpha_j$ by their affine linear combinations. Hence, every $d_\alpha$ appearing in the expression describing the coefficient of $y_i y_j$ is an expression in $a_{2,i}, a_{3,i}, a_{1,j}, a_{2,j}, a_{3,j}$ and $\alpha_1, \alpha_2, \alpha_3$. In order to satisfy $\alpha' + \sum_{l=1}^{s(\alpha)} e_{i_l} = \beta$ we conclude that $i_l \in \{i,j\}$ for all $l$. Hence, $a_{k_l,i_l} \in \{a_{1,i}, a_{2,i}, a_{3,i}, a_{1,j}, a_{2,j}, a_{3,j}\}$. Again we use SINGULAR to obtain the desired form of $\operatorname{coef}_{D_B}(y_i y_j)$. $\square$

**Remark 3.1.3.10** (Newton polytope of $D_B$)**.** Lemma 3.1.2.4 states that the polynomials $D$ and $D_B$ are homogeneous of degree 2, i.e. $\operatorname{Newt}(D_B) \subset 2 \cdot \Delta_{m-3} \subset \mathbb{R}_B$. The Newton polytope of $D_B$ is defined by the convex hull of all points $e_i + e_j$, provided that the coefficient of $y_i y_j$ in $D_B$ is non-zero (Definition 1.4.2.8). With Proposition 3.1.3.9 we can partially give answers for a subset of monomials of $D_B$. The monomial $y_i^2$ appears in $D_B$ if and only if $a_{1,i} a_{2,i} a_{3,i} \neq 0$. Suppose the monomial $y_i^2$ does not appear in $D_B$ since (w.l.o.g.) $a_{1,i} = 0$. Recall from Lemma 3.1.2.3 that we have $\alpha_i = a_{1,i}\alpha_1 + a_{2,i}\alpha_2 + a_{3,i}\alpha_3$. Since $a_{1,i} = 0$ we have $\alpha_i = a_{2,i}\alpha_2 + a_{3,i}\alpha_3$ and $a_{2,i} + a_{3,i} = 1$. Now suppose that additionally (w.l.o.g.) $a_{2,i} = 0$. Then $\alpha_i = a_{3,i}\alpha_3$ and since $a_{3,i} = 1$ we have $\alpha_i = \alpha_3$. We conclude that for all $i \in B$ at most one element of $\{a_{1,i}, a_{2,i}, a_{3,i}\}$ is zero. Let us continue with $a_{1,i} = 0$. As $a_{2,i}, a_{3,i} \neq 0$ we conclude that $y_i y_j$ appears in $D_B$ for $j \in B \setminus \{i\}$ if and only if $a_{1,j} \neq 0, 1$. As the elements $a_{i,j}$ with $i \in B^\complement = \{1,2,3\}$ and $j \in B$ are the coordinates of $\alpha_i$ in the affine basis $\alpha_1, \alpha_2, \alpha_3$ we can reformulate Proposition 3.1.3.9 geometrically in terms of the affine geometry of $\mathscr{A}$.

THEOREM 3.1.3.11. *We stick to Notation 3.1 and Convention 3.1.3.7, i.e. consider a generic Laurent polynomial $F = \sum_i y_i x^{\alpha_i} \in \mathbb{K}[y_1, \dots, y_m][x^\pm, y^\pm]$ with support $\mathscr{A} = \{\alpha_1, \dots, \alpha_m\} \subset \mathbb{Z}^2$. Let $M$ be the matroid associated to $\mathscr{I} = \langle F(\mathbf{1}_2), \frac{\partial F}{\partial x}(\mathbf{1}_2), \frac{\partial F}{\partial y}(\mathbf{1}_2)\rangle \subset \mathbb{K}[y_1, \dots, y_m]$. According to Convention 3.1.3.7, $B = [m] \setminus \{1,2,3\}$ is a basis of $M$. Moreover, let $D = \det(H_F)(\mathbf{1}_2)$ be the determinant of the Hessian matrix and we denote the image of $\bar{D} \in R_m/\mathscr{I}$ under $\pi_B$ by $D_B \in R_B$ (cf. Remark 3.1.3.8). Each 2-element subset $\{\alpha_i, \alpha_j\}$ of the affine basis defines an affine line $L_{ij} = \{x \in \mathbb{R}^2 : \exists(\lambda, \mu) \in H_{\mathbf{1}_2,1} \subset \mathbb{R}^2$ such that $x = \lambda\alpha_i + \mu\alpha_j\}$ for $i, j \in \{1,2,3\}$ passing through $\alpha_i$ and $\alpha_j$. Then the vertices of the Newton polytope of $D_B$, $\operatorname{vert}(\operatorname{Newt}(D_B)) \subset \mathbb{Z}_B$, can be read from $\mathscr{A}$ as follows:*

(a) *$2e_i \in \operatorname{vert}(\operatorname{Newt}(D_B)) \subset \mathbb{R}_B$ for $i \in B$ if and only if $\alpha_i \notin L_{rs}$ for $r, s \in \{1,2,3\}$ and $r \neq s$.*

(b) *Suppose we have $\alpha_i \in L_{rs}$ for a pair $r, s \in \{1,2,3\}$ with $r \neq s$ and $i \in B$. Then we have $2(e_i + e_j) \in \operatorname{vert}(\operatorname{Newt}(D_B)) \subset \mathbb{R}_B$ for $j \in B \setminus \{i\}$ if and only if $\alpha_j \notin L_{rs}$ and $\alpha_j \notin v_t + L_{rs}$ where $v_t + L_{rs}$ is the affine line parallel to $L_{rs}$ through $\alpha_t$ with $t \in \{1,2,3\} \setminus \{r,s\}$.*

PROOF. First, statement (a). Note that $2e_i \in \operatorname{vert}(\operatorname{Newt}(D_B))$ if and only if $\operatorname{coef}_{D_B}(y_i^2) \neq 0$. Recall that $\alpha_i = a_{1,i}\alpha_1 + a_{2,i}\alpha_2 + a_{3,i}\alpha_3$ and $a_{1,i} + a_{2,i} + a_{3,i} = 1$ for all $i \in B$. From Proposition 3.1.3.9 (a) we know that $\operatorname{coef}_{D_B}(y_i^2) = a_{1,i}a_{2,i}a_{3,i}q^2(-1)$. Since $q \neq 0$ (Lemma 3.1.2.3) we have $\operatorname{coef}_{D_B}(y_i^2) = 0$ if and only if either $a_{1,i} = 0$ or $a_{2,i} = 0$ or $a_{3,i} = 0$. Notice that at most one coefficient is equal to zero — otherwise (w.l.o.g. $a_{1,i} = a_{2,i} = 0$) we have $\alpha_i = a_{3,i}\alpha_3$ and $a_{1,i} + a_{2,i} + a_{3,i} = a_{3,i} = 1$, i.e. $\alpha_i = \alpha_3$. Suppose $\operatorname{coef}_{D_B}(y_i^2) = 0$. W.l.o.g. we consider the case $a_{1,i} = 0$. Then $\alpha_i = a_{2,i}\alpha_2 + a_{3,i}\alpha_3 \in L_{23}$ since $a_{2,i} + a_{3,i} = 1$. Hence, $\operatorname{coef}_{D_B}(y_i^2) = 0$ implies $\alpha_i \in L_{rs}$ for some $r, s \in \{1,2,3\}$ with $r \neq s$.

Vice versa, $\alpha_i \in L_{rs}$ for some $r,s \in \{1,2,3\}$ with $r \neq s$ implies that there is $(a,b) \in H_{\mathbf{1}_2,1}$ such that $\alpha_i = a\alpha_r + b\alpha_s$ and $a+b=1$. Suppose (w.l.o.g.) $r=1, s=2$. Thus $\alpha_i = a\alpha_1 + b\alpha_2$ is the representation of $\alpha_i$ with the affine basis $\alpha_1, \alpha_2, \alpha_3$. This means $a_{1,i} = 0$ and $a_{2,i} = a$, $a_{3,i} = b$. From Proposition 3.1.3.9 we know that $\mathrm{coef}_{D_B}\left(y_i^2\right) = a_{1,i}a_{2,i}a_{3,i}q^2(-1)$. As $a_{1,i} = 0$ we get $\mathrm{coef}_{D_B}\left(y_i^2\right) = 0$. The proof of statement (b) is similar. By assumption we have $\alpha_i \in L_{rs}$ for some $r,s \in \{1,2,3\}$, $r \neq s$. W.l.o.g. $r=2, s=3$ and $a_{1,i} = 0$ such that $\alpha_i \in L_{23}$. Then $e_i + e_j \in \mathrm{vert}\left(\mathrm{Newt}\left(D_B\right)\right)$ if and only if $\mathrm{coef}_{D_B}\left(y_iy_j\right) \neq 0$. Provided that $\alpha_i \in L_{23}$, Proposition 3.1.3.9 states $\mathrm{coef}_{D_B}\left(y_iy_j\right) = a_{2,i}a_{3,i}a_{1,j}(a_{1,j} - 1)q^2$ for $j \in B \setminus \{i\}$. In part (a) we have seen that $a_{2,i}, a_{3,i} \neq 0$ if $a_{1,i} = 0$. Thus $\mathrm{coef}_{D_B}\left(y_iy_j\right) = 0$ if and only if either $a_{1,j} = 0$ or $a_{1,j} = 1$. First, note that $a_{1,j} = 0$ is equivalent to $\alpha_j \in L_{23}$. Second, $a_{1,j} = 1$ is equivalent to $a_{2,j} + a_{3,j} = 0$ and $\alpha_j = \alpha_1 + a_{2,j}\alpha_2 + a_{3,j}\alpha_3 = \alpha_1 + a_{2,j}(\alpha_2 - \alpha_3) \in v_1 + L_{23}$. $\square$

**Remark 3.1.3.12.** Note that Theorem 3.1.3.11 (Proposition 3.1.3.9 respectively) do not provide evidence for all monomials of $D_B$. For instance, assume that $2e_i$ and $2e_j$ belong to $\mathrm{vert}\left(\mathrm{Newt}\left(D_B\right)\right)$. Then $e_i + e_j$ is a lattice point of $\mathrm{Newt}\left(D_B\right)$ but we have no evidence whether $\mathrm{coef}_{D_B}\left(x_ix_j\right)$ is zero or not. Fortunately, the set of lattice points obtained from Theorem 3.1.3.11 is sufficient, i.e. its convex hull equals $\mathrm{Newt}\left(D_B\right)$. To see this note that $\mathrm{Newt}\left(D_B\right) \subset 2 \cdot \Delta_{m-3}$, i.e. the lattice points in $\mathbb{Z}_B \cap 2 \cdot \Delta_{m-3}$ are of the form $e_i + e_j$ with $i,j \in B$. It is irrelevant whether $\mathrm{coef}_{D_B}\left(y_iy_j\right) \neq 0$ if we have $\mathrm{coef}_{D_B}\left(y_i^2\right) \neq 0$ and $\mathrm{coef}_{D_B}\left(y_j^2\right) \neq 0$. If (w.l.o.g.) $\mathrm{coef}_{D_B}\left(y_i^2\right) = 0$ then Theorem 3.1.3.11 helps to decide for $\mathrm{coef}_{D_B}\left(y_iy_j\right)$. However, the corner points of $2 \cdot \Delta_{m-3}$ can be examined with Theorem 3.1.3.11. This way we can consider all edges of $2 \cdot \Delta_{m-3}$ step by step.

Theorem 3.1.3.11 has some simple consequences:

**Corollary 3.1.3.13.** *Let $\{\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}\} \subset \mathscr{A} = \{\alpha_1, \ldots, \alpha_m\}$ be an affine basis providing the basis $B = [m] \setminus \{i_1, i_2, i_3\}$ of $M$. Then* $\mathrm{Newt}\left(D_B\right) = 2 \cdot \Delta_{m-3}$ *if and only if $\alpha_i$ does not lie on $L_{rs}$ for all $i \in B$ and all pairwise distinct $r,s \in \{i_1, i_2, i_3\}$.*

PROOF. We have $\mathrm{Newt}\left(D_B\right) = 2 \cdot \Delta_{m-3}$ if and only if $2e_i \in \mathrm{vert}\left(\mathrm{Newt}\left(D_B\right)\right)$ for all $i \in B$. Equivalently, $a_{1,i}, a_{2,i}, a_{3,i} \neq 0$ for all $i \in B$. By Proposition 3.1.3.9 this is equivalent to $\alpha_i \notin L_{rs}$ for all $i \in B$ and all pairwise distinct $r,s \in \{i_1, i_2, i_3\}$. In other words, $\alpha_i$ does not lie on $L_{rs}$ for all $i \in B$ and all pairwise distinct $r,s \in \{i_1, i_2, i_3\}$ if and only if $\mathrm{Newt}\left(D_B\right) = 2 \cdot \Delta_{m-3}$. $\square$

This can be generalized to the full support $\mathscr{A}$:

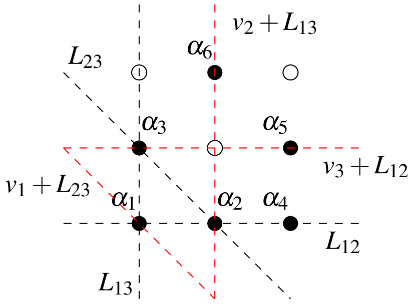
**Corollary 3.1.3.14.** *Let $\mathscr{A} \subset \mathbb{Z}^2$ be a point configuration. Then $\mathscr{A}$ is generic, i.e. no three points of $\mathscr{A}$ are colinear, if and only if* $\mathrm{Newt}\left(D_B\right) = 2 \cdot \Delta_{m-3}$ *for all bases $B$ of the matroid $M$.*

**Example 3.1.3.15.** Consider the point configuration

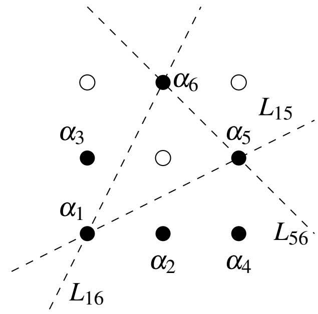
$$\mathscr{A} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}.$$

The support is illustrated in Figure 16a by black points. Now we determine the relative Newton polytope $\mathscr{N}_{\mathscr{B}}$ of $D$. According to Theorem 3.1.3.11, $\mathrm{Newt}\left(D_B\right)$ can be read from $\mathscr{A}$ for any basis $B$ of $M$. Using Corollary 3.1.3.13 we can list all matroid bases providing $\mathrm{Newt}\left(D_B\right) = 2 \cdot \Delta_3$, namely

$$\left\{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},\{1,3,4\},\{1,4,5\},\{1,4,6\},\{2,3,4\},\{2,4,5\},\{2,4,6\}\right\}.$$




(A) $B = \{4,5,6\}$ and (dashed) lines $L_{12}, L_{13}, L_{23}$ (colored black) and $v_1 + L_{23}, v_2 + L_{13}, v_3 + L_{12}$ (colored red).




(B) For $B = \{2,3,4\}$ we have $\{\alpha_2, \alpha_3, \alpha_4\} \cap L_{pq} = \emptyset$ for all $p, q \in \{1,5,6\}$.

FIGURE 16. Support $\mathscr{A}$ of Example 3.1.3.15 and lines $L_{ij}$ with $i, j \in B^{\complement}$.

For instance consider the affine basis $\{\alpha_1, \alpha_5, \alpha_6\}$ and the lines $L_{15}, L_{16}, L_{56}$ illustrated in Figure 16b. This choice yields the basis $B = \{2,3,4\}$. We see that $L_{ij} \cap \mathscr{A} = \{\alpha_i, \alpha_j\}$ for all pairwise distinct $i, j \in \{1,5,6\}$. By Corollary 3.1.3.13 we conclude that $\operatorname{Newt}(D_B) = 2 \cdot \Delta_3$. The Newton polytopes of all $D_B$ arising from remaining bases of $M$ can be deduced from $\mathscr{A}$ with Theorem 3.1.3.11, see Figure 17. For example, consider the affine basis indexed by $\{1,2,3\}$ (see Figure 16a) providing the basis $B = \{4,5,6\}$. The situation is sketched in Figure 16a. We see that $\alpha_4 \in L_{12}$ and, therefore, $a_{3,4} = 0$. Consequently, $2e_4 \notin \operatorname{vert}\left(\operatorname{Newt}\left(D_{\{4,5,6\}}\right)\right)$. Since $\alpha_5$ and $\alpha_6$ do not lie on any of the lines $L_{12}, L_{13}$ and $L_{23}$ we have $2e_5, 2e_6 \in \operatorname{vert}\left(\operatorname{Newt}\left(D_{\{4,5,6\}}\right)\right)$. Since $a_{3,4} = 0$ we have $\operatorname{coef}_{D_B}(y_4 y_j) = a_{1,4} a_{2,4} a_{3,j} (a_{3,j} - 1) q^2$ for $j \in \{5,6\}$. Note that $\alpha_5 \in v_3 + L_{12}$, i.e. $a_{3,5} = 1$.

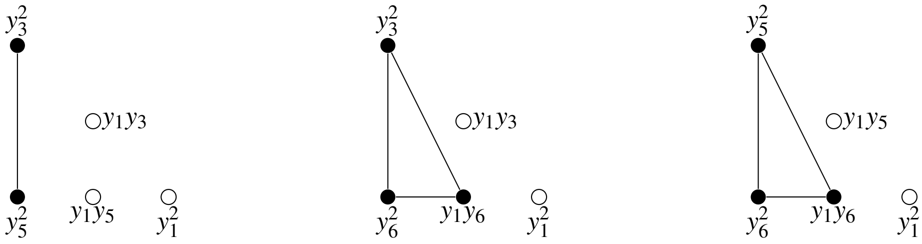
### 3.1.4. First Steps towards a Classification of Plane Tropical Curves with a Cusp

In this part we venture first steps towards a classification of plane tropical curves with a cusp. In Section 3.1.3 we determined $D_B$ for a basis $B$ of $M$. Note that $\operatorname{trop}(\mathscr{V}(D_B))$ determines the push forward of $\operatorname{trop}(\nabla^{\text{cusp}})$ if $D$ is not divisible by a monomial. All results in Section 3.1.3 concerning the Newton polytope of $D_B$ only depend on affine relations in $\mathscr{A}$, i.e. the results are invariant under shifts of $\mathscr{A}$ in the plane. Hence, if we pick a support $\mathscr{A} \subset \mathbb{Z}^2$ and $D$ is divisible by a monomial, we work with the translation of $\mathscr{A}$ into the area $(1,1) + \operatorname{cone}(e_1, e_2)$. See Remark 3.1.2.6 for further details.

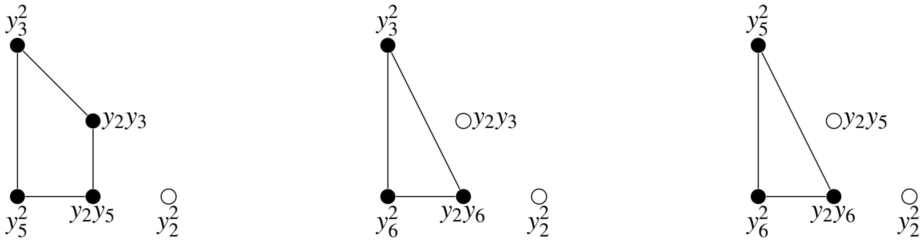
In the first instance we consider *generic* point configurations $\mathscr{A} \subset \mathbb{Z}^2$ that have no three colinear points. Then, the ambient tropical linear space $\operatorname{trop}\left(\nabla_{\mathbf{1}_2}\right)$ as well as the tropical hypersurfaces $\operatorname{trop}(\mathscr{V}(D_B))$ defined by $D_B$ become *general*.

**Lemma 3.1.4.16.** *Let $U_{k+1,n+1}$ denote the uniform matroid of rank $k+1$ on $n+1$ elements, $F \in R_{n+1}$ a homogeneous polynomial of degree $d$ such that $\operatorname{Newt}(F) = d \cdot \Delta_{n+1}$ and $\mathscr{A} \subset \mathbb{Z}^{n+1}$ a generic point configuration of size $m$.*

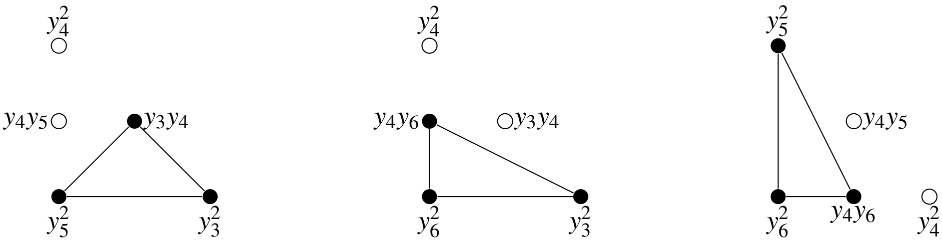
- *If $k \neq n$ then, with respect to the coarse subdivision, $B\left(U_{k+1,n+1}\right)^{\operatorname{codim}(1)} = B\left(U_{k,n+1}\right)$.*
- $|\operatorname{trop}(\mathscr{V}(F))| = |B(U_{n,n+1})|$.
- $\operatorname{trop}\left(\nabla_{\mathbf{1}_2}\right) = L_m^{m-3} = B\left(U_{m-3,m}\right)$.




(A) Newt $\left(D_{\{1,3,5\}}\right)$. (B) Newt $\left(D_{\{1,3,6\}}\right)$. (C) Newt $\left(D_{\{1,5,6\}}\right)$.




(D) Newt $\left(D_{\{2,3,5\}}\right)$. (E) Newt $\left(D_{\{2,3,6\}}\right)$. (F) Newt $\left(D_{\{2,5,6\}}\right)$.




(G) Newt $\left(D_{\{3,4,5\}}\right)$. (H) Newt $\left(D_{\{3,4,6\}}\right)$. (I) Newt $\left(D_{\{4,5,6\}}\right)$.

FIGURE 17. The Newton polytopes $\mathrm{Newt}(D_B)$ for all bases $B$ such that $\mathrm{Newt}(D_B) \neq 2 \cdot \Delta_3$.

PROOF. For (a) let $k$ be strictly smaller than $n$. The lattice of flats $\mathscr{L}\left(U_{k+1,n+1}\right)$ consists of flats $F_{i_j}$ denoting the $j$-th flat of rank $i$. In $U_{k+1,n+1}$ every circuit contains $k+2$ elements, i.e. we have $|F_{i_j}| = i$ for all $i \neq k+1$ and $|F_{k+1}| = n+1$. We are interested in all codimension one weight classes that belong to $B\left(U_{k+1,n+1}\right)^{\mathrm{codim}(1)}$ with respect to the coarse subdivision. Consider any flag of flats $\mathscr{F}_{\hat{i}} = (F_1, \dots, F_{k+1}) \lhd U_{k+1,n+1}$ indexed by rank and missing a rank $i$ flat. The codimension one weight class $\sigma_{\mathscr{F}_{\hat{i}}}$ belongs to the codimension one skeleton if and only if $[F_{i-1}, F_{i+1}]$ is not a diamond poset. Equivalently the flag $\mathscr{F}_{\hat{i}} \lhd U_{k+1,n+1}$ misses a flat of rank $k$. To see this note that $|F_{i+1} \setminus F_{i-1}| = 2$ for all $i \leq k-1$. The critical index is $i = k$. Then $|F_{k+1} \setminus F_{k-1}| > 2$ if and only if $k < n$.

For (b) note that $d \cdot \left(\Delta_{n+1} \cap \mathbb{Z}^{n+1}\right) = \mathrm{vert}(d \cdot \Delta_{n+1})$ for $d, n \in \mathbb{N}$. However, the points of $\Delta_{n+1}$ are the support of the linear form $\sum_i x_i$. Consequently, $d \cdot \mathrm{Newt}\left(\sum_{i=1}^{n+1} x_i\right) = \mathrm{Newt}(F)$ and we get $\mathscr{N}_{\mathrm{Newt}(F)} = \mathscr{N}_{\mathrm{Newt}(\sum_i x_i)}$, i.e. the tropical hypersurfaces share the identical support. Additionally, we saw that $\mathrm{trop}(\mathscr{V}(\sum_i x_i)) = B(U_{n,n+1})$. Consequently, $|\mathrm{trop}(\mathscr{V}(F))| = |B(U_{n,n+1})|$, and by changing

weights to $d$ the statement follows. In few words, the shape $\Delta_{n+1}$ provides the support of the tropical hypersurface and the stretching factor $d$ affects the weights.
For (c) let $\mathscr{A}$ be generic so that any three points of $\mathscr{A}$ form an affine basis. Consequently, all subsets $B \subset [m]$ of size $m-3$ form a basis of $M[G]$, the vector matroid associated to a Gale dual $G$ of $A'$. Hence, $M[G] = U_{m-3,m}$ verifying the last statement. □

THEOREM 3.1.4.17. *Based on Notation 3.2, let $\mathscr{A} \subset \mathbb{Z}^2$ be a generic point configuration of cardinality m. Let $\mathscr{I} \subset \mathbb{K}[y_1, \dots, y_m]$ denote the linear ideal such that $\nabla_{\mathbf{1}_2} = \mathscr{V}(\mathscr{I})$ and by M we denote the matroid associated to $\mathscr{I}$. Then we have* $\operatorname{trop}\left(\nabla_{\mathbf{1}_2}^{cusp}\right) = B(M)^{(\operatorname{codim}(1))}$ *with respect to the coarse subdivision.*

PROOF. As $\mathscr{A}$ is a generic point configuration we know that $M(\mathscr{I}) = U_{m-3,m}$ since any three elements of $\mathscr{A}$ form an affine basis (Lemma 3.1.4.16). Moreover, we have $\operatorname{Newt}(D_B) = 2 \cdot \Delta_{m-3}$ for all bases $B$ (Corollary 3.1.3.14). Consequently, we have $|\operatorname{trop}(\mathscr{V}(D_B))| = |B(U_{m-4,m-3})|$ for all bases $B$ of $U_{m-3,m}$ by Lemma 3.1.4.16. Hence, $U_{m-3,m}$ is the underlying matroid of the ambient tropical linear space $\operatorname{trop}(\nabla_{\mathbf{1}_2})$ whereas $U_{m-4,m-3}$ describes the projections $\operatorname{trop}(\mathscr{V}(D_B))$. Also note that $B(U_{m-3,m})^{\operatorname{codim}(1)} = B(U_{m-4,m})$ (Lemma 3.1.4.16). From now on we write $\mathscr{T}_B = \operatorname{trop}(\mathscr{V}(D_B))$ and $\mathscr{T} = \operatorname{trop}(\mathscr{V}(\mathscr{I} + \langle D \rangle)) = \operatorname{trop}(\nabla_{\mathbf{1}_2}^{\text{cusp}})$ (cf. Notation 3.3) as we did in Section 2.1.2. We want to show that $\mathscr{T} = B(U_{m-4,m})$. If we show $(p_B)_*(\mathscr{T}) = (p_B)_*(B(U_{m-4,m}))$ for all bases $B$ of $U_{m-3,m}$ the statement follows since the reconstruction is unique.
We have $(p_B)_*(\mathscr{T}) = \operatorname{trop}(\mathscr{V}(D_B))$ due to Theorem 2.1.2.16 and $\operatorname{trop}(\mathscr{V}(D_B)) = B(U_{m-4,m-3})$. Now let us turn to $(p_B)_*(B(U_{m-4,m}))$. The coordinate projections we have to take into consideration arise from bases of $U_{m-3,m}$. Take any basis $B$ of $U_{m-3,m}$. Then $U_{m-4,m}\big|_B = U_{m-4,m-3}$ and, therefore, $p_B(B(U_{m-4,m})) = B(U_{m-4,m-3})$. Thus $\mathscr{T} = B(U_{m-4,m})$. As $B(U_{k,n})^{(\operatorname{codim}(1))} = B(U_{k-1,n})$ with respect to the coarse subdivision (Lemma 3.1.4.16) the statement follows. □

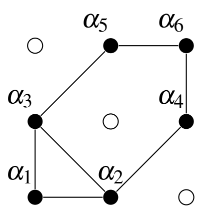
**Example 3.1.4.18.** Consider the generic point configuration

$$\mathscr{A} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$$

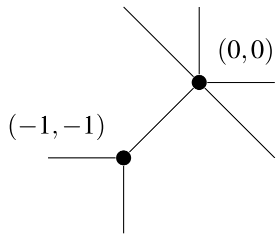
as shown in Figure 18a. From Theorem 3.1.4.17 we conclude $\operatorname{trop}\left(\nabla_{\mathbf{1}_2}^{\text{cusp}}\right) = B(U_{2,6})$. So, top-dimensional weight classes can be described by inequalities of the form

$$x_{i_1} \leq x_{i_2} = x_{i_3} = x_{i_4} = x_{i_5} = x_{i_6}.$$

Let $\sigma_{\mathscr{F}}$ denote the weight class described by the inequalities above. Any $w \in \operatorname{relint}(\sigma_{\mathscr{F}})$ provides five of the six elements of $\mathscr{A}$ with highest height. The remaining element receives a lower height so that the shifted polytope $\operatorname{conv}(\mathscr{A})_w$ contains the upper face formed by the five points shifted by the highest heights. The regular subdivision of $\operatorname{conv}(\mathscr{A})$ induced by $w$ contains a pentagon. The cusp at $p = \mathbf{1}_2$ tropicalizes to the vertex $q = (0,0)$ dual to the pentagon because all heights at these points are equal. Figure 18a illustrates a regular subdivision of $\mathscr{A}$ induced by $w = (0,1,1,1,1,1)$. The dual tropical curve $C$ is shown in Figure 18b. We see that $q = (0,0) \in C$ is a 5-valent vertex of $C$. Moreover, $\operatorname{trop}\left(\nabla_{\mathbf{1}_2}^{\text{cusp}}\right) = B(U_{m-4,m})$ is a proper subfan of the secondary fan $\operatorname{Sec}_{\mathscr{A}}$.




(A) Generic support $\mathscr{A}$ of Example 3.1.4.18 and a subdivision defined by the height vector $w = (0,1,1,1,1,1)$.




(B) Plane tropical curve with coefficient vector $w = (0,1,1,1,1,1)$.

FIGURE 18. A plane tropical curve with a cusp at $\mathbf{0}_2$.

## 3.2. $k+1$-fold Singular Tropical Hypersurfaces

In this section we study tropical hypersurfaces with a $k+1$-fold singularity. We proceed as follows: In Section 3.2.1 we show that we can recover $\nabla^{k+1}$ from $\nabla^{k+1}_{\mathbf{1}_n}$, i.e. it is sufficient to study hypersurfaces with a $k+1$-fold singularity at $\mathbf{1}_n$. In Section 3.2.2 we examine the vanishing ideal and the associated matroid of the linear space $\nabla^{k+1}_{\mathbf{1}_n}$. In Section 3.2.3 we study $\nabla^{k+1}_{\mathbf{1}_n}$ in terms of Euler derivatives similar to [**DT12**] who covered the case $k=1$. In particular, we discover a tropical basis for $\nabla^{k+1}_{\mathbf{1}_n}$. In Section 3.2.4 we give tropical conditions characterizing tropical hypersurfaces with a $k+1$-fold singularity (Proposition 3.2.4.16). In the end we take a brief look at the relationship of $\nabla^{k+1}$ and $\mathrm{Sec}_{\mathscr{A}}$.

### 3.2.1. $k+1$-fold Singular Hypersurfaces

We study the family of Laurent polynomials that define a $k+1$-fold singularity. We start with notations 3.1 and expand it stepwise. Let $F = \sum_i y_i x^{\alpha_i} \in \mathbb{K}[y_1,\ldots,y_m]\left[x_1^{\pm},\ldots,x_n^{\pm}\right]$ be a generic Laurent polynomial with fixed support $\mathscr{A} \subset \mathbb{Z}^n$ of cardinality $m$. Recall that, by Definition 1.6.1.7, $p \in \mathrm{Sing}^{k+1}_{F_a}$ if and only if $F_a$ and all partial derivatives of $F_a$ up to order $k$ vanish at $p$. Hence, we study

$$\nabla^{k+1} = \{a \in \mathbb{P}(T^m) : \mathscr{V}(F_a) \text{ contains a } k+1\text{-fold singularity.}\} = \{a \in \mathbb{P}(T^m) : \mathrm{Sing}^{k+1}_{F_a} \neq \emptyset\}.$$

Before going into details note the following:

**Remark 3.2.1.1** (Connection to [**DDRP14**]). A building block of [**DDRP14**] is the study of higher order dual varieties of projective toric varieties. There is a natural generalization of the dual variety $X_A^\vee$ of $X_A$: the $k$-th dual variety $X_{A,k}^\vee$ is the Zariski closure of the set of hyperplanes tangent to $X$ to the order $k$ at a regular point ([**Pie83**]). In the following sections we present results based on generalizations of [**DT12**]. Since all stated results concerning $k+1$-fold singular tropical hypersurfaces coincide with achievements in section 5 of [**DDRP14**] this section constitutes a rediscovery and all presented results were obtained independently.

We enhance our notations about multiindices:

NOTATION 3.4 (Multiindices). In Notation 1.1 we defined $\mathscr{M}_{n+1,d}$ as the set of monomials in $n+1$ variables and degree $d$. Equivalently, it consists of multiindices $\alpha = (\alpha_0,\ldots,\alpha_n) \in \mathbb{N}_0^{n+1}$ of length $|\alpha| = \sum_i \alpha_i = d$. The cardinality of $\mathscr{M}_{n+1,d}$ is denoted by $N^d_{n+1} = \binom{n+d}{d}$. Via dehomogenization

$\mathscr{M}_{n+1,d}$ can be understood as the set of multiindices in $n$ variables up to length $d$. To distinguish we denote the set of multiindices $\alpha$ in $n$ coordinates of length $k_1 \leq |\alpha| \leq k_2$ by $\mathscr{M}(k_1,k_2)$. If $k_1$ is neglected $\mathscr{M}(k_2) = \cup_{i=0}^{k_2} \mathscr{M}_{n,i}$ refers to the set of multiindices up to length $k_2$. Here, we use an "increasing" graded lexicographic order whenever we need an order on the elements of $\mathscr{M}(k)$. More precisely, we consider $\mathscr{M}(k)$ ordered increasingly by length (degree respectively) in the first instance and for constant length by the lexicographic order.

To get a simplified notation for the derivatives of $F$ with respect to $\beta \in \mathscr{M}(d)$ we introduce the *falling factorial* (see e.g. [**Ste50**], [**Knu92**]):

**Definition 3.2.1.2** (Falling factorials)**.** For $x \in \mathbb{Z}$ and $n \in \mathbb{N}_0$ we define

$$(x)_n = \begin{cases} \prod_{j=0}^{n-1} (x-j) & \text{if } n \neq 0, \\ 1 & \text{if } n = 0. \end{cases}$$

**Example 3.2.1.3.** Consider $x = 7$ and $n = 3$. Then $(7)_3 = 7(7-1)(7-2)$ which is the coefficient of $\frac{\partial^3 y^7}{\partial y^3}$. More generally if $\alpha \in \mathscr{M}_{n+1,d}$ and $\beta \in \mathscr{M}(d)$ we get

$$\frac{\partial^{|\beta|} x^\alpha}{\partial x^\beta} = \prod_{\substack{\beta_i \in \beta: \\ \beta_i \neq 0}} \prod_{s=0}^{\beta_i - 1} (\alpha_l - s) x^{\alpha - \beta} = \prod_i (\alpha_i)_{\beta_i} x^{\alpha - \beta}.$$

Now we show that it suffices to study hypersurfaces with a $k+1$-fold singularity fixed at $\mathbf{1}_n$:

**Remark 3.2.1.4** ($k+1$-fold singularities at $p = \mathbf{1}_n$)**.** Pick an element $a \in \nabla^{k+1}$ and consider the Laurent polynomial $F_a \in \mathbb{K}\left[x_1^{\pm}, \ldots, x_n^{\pm}\right]$. By definition there is a point $p \in T^n$ such that $F_a$ and all partial derivatives of $F_a$ up to order $k$ vanish at $p$. Pick any $\beta \in \mathscr{M}(k)$. Then:

$$\begin{aligned} \frac{\partial^{|\beta|} F_a}{\partial x^\beta}(p) = 0 \quad &\Leftrightarrow \quad x^\beta \frac{\partial^{|\beta|} F_a}{\partial x^\beta}(p) = 0 \\ &\Leftrightarrow \quad \sum_i a_i \left( \prod_j (\alpha_{i,j})_{\beta_j} \right) x^{\alpha_i}(p) = 0 \\ &\Leftrightarrow \quad \sum_i a_i \left( \prod_j (\alpha_{i,j})_{\beta_j} \right) p^{\alpha_i} x^{\alpha_i}(\mathbf{1}_n) = 0 \\ &\Leftrightarrow \quad \frac{\partial^{|\beta|} F_{a \cdot \psi_{\mathscr{A}}(p)}}{\partial x^\beta}(\mathbf{1}_n) = 0. \end{aligned}$$

We conclude that $F_a$ has a $k+1$-fold singularity at $p \in \mathscr{V}(F_a)$ if and only if $F_{a \cdot \psi_{\mathscr{A}}(p)}$ has a $k+1$-fold singularity at $\mathbf{1}_n$. Thus $k+1$-fold singularities blend into the Horn uniformization as well.

Remark 3.2.1.4 implies the following

**Corollary 3.2.1.5.** *Let $\mathscr{A} \subset \mathbb{Z}^n$ be a finite point configuration of cardinality $m$ and let $A' \in \mathbb{Z}^{n+1 \times m}$ denote its shifted matrix representation. Then:*

$$\operatorname{trop}\left(\nabla^{k+1}\right) = \operatorname{trop}\left(\nabla^{k+1}_{\mathbf{1}_n}\right) \oplus \operatorname{rowspace}\left(A'\right).$$

Due to Corollary 3.2.1.5 we concentrate on $\nabla^{k+1}_{\mathbf{1}_n}$ and its tropicalization.

### 3.2.2. The Vanishing Ideal of $\nabla_{\mathbf{1}_n}^{k+1}$

The investigation of $k+1$-fold singular hypersurfaces and its tropicalizations follows a similar pattern to the case $k=1$ (cf. Section 1.6.2). Notations 3.1 are specified to $k=1$, i.e. the first step is an extension of notations for arbitrary $k \in \mathbb{N}$. Note that $\frac{\partial^{|\beta|} F}{\partial x^\beta}(\mathbf{1}_n) = \sum_i y_i \frac{\partial^{|\beta|}}{\partial x^\beta} x^{\alpha_i}(\mathbf{1}_n) \in \mathbb{K}[y_1, \ldots, y_m]$ is a linear form for any $\beta \in \mathscr{M}(k)$. Moreover, $\beta = (0, \ldots, 0) = \mathbf{0}_n \in \mathscr{M}(k)$ such that $\frac{\partial^{|\mathbf{0}_n|} F}{\partial x^{\mathbf{0}_n}}(\mathbf{1}_n) = F(\mathbf{1}_n)$. Hence, we study

$$\mathscr{I}_k = \langle \frac{\partial^{|\beta|} F}{\partial x^\beta}(\mathbf{1}_n) : \beta \in \mathscr{M}(k) \rangle$$

in the following, which is a straight-forward generalization of $\mathscr{I}$ as defined in notations 3.1. In particular, if we consider $k=1$ and restrict to $\mathscr{M}(1)$ we get $\mathscr{I}_1 = \mathscr{I}$, i.e. $\nabla_{\mathbf{1}_n}^2 = \nabla_{\mathbf{1}_n}$. As shown above $\mathscr{I}_k$ is a linear ideal generated by $F$ and all partial derivatives of $F$ up to order $k$ evaluated at $\mathbf{1}_n$. Let $A'_k$ denote the coefficient matrix of $\mathscr{I}_k$. To sum it up,

$$\nabla_{\mathbf{1}_n}^{k+1} = \mathscr{V}(\mathscr{I}_k) = \ker(A'_k).$$

Accordingly, the tropicalization $\operatorname{trop}\left(\nabla_{\mathbf{1}_n}^{k+1}\right)$ equals the tropicalization of $\ker\left(A'_k\right)$ and it only depends on the matroid $M_k := M(\mathscr{I}_k)$ associated to $\mathscr{I}_k$. The matroid $M_k$ can be described by its set of circuits, i.e. the set of linear forms in the row space of $A'_k$ with minimal support (cf. Definition 1.2.1.1 and ensuing comments). The circuits form a tropical basis (cf. Definition 1.4.3.18) and, therefore, the identification of circuits of $M_k$ is the first goal in this section (Proposition 3.2.3.15).
In order to understand the matroid $M_k$ we seek for a better description of $A'_k$. Thereby falling factorials simplify notation remarkably:

**Definition 3.2.2.6** (Lifted support)**.** Fix a finite set $\mathscr{A} \subset \mathbb{Z}^n$ and an integer $d \in \mathbb{N}$. For $\beta \in \mathscr{M}_{n,d}$ fixed we define the map $\theta_\beta : \mathscr{A} \longrightarrow \mathbb{Z}$ by $\theta_\beta(\alpha) = \prod_i (\alpha_i)_{\beta_i}$. For a fixed $k \in \mathbb{N}$ we call the image of the map $\theta_k : \mathscr{A} \longrightarrow \mathbb{Z}^{N_{n+1}^k - 1}$ defined by

$$\theta_k(\alpha) = \left(\theta_\beta(\alpha)\right)_{\beta \in \mathscr{M}(1,k)}$$

the *lift of* $\mathscr{A}$ and we denote it by $\mathscr{A}_k = \{\theta_k(\alpha_1), \ldots, \theta_k(\alpha_m)\}$. Let $\theta'_k : \mathscr{A} \longrightarrow \mathbb{Z}^{N_{n+1}^k}$ denote the map defined by $\theta'_k(\alpha) = \left(\theta_\beta(\alpha)\right)_{\beta \in \mathscr{M}(k)}$.

Note that $\theta_k$ and $\theta'_k$ only differ by one component corresponding to $\mathbf{0}_n \in \mathscr{M}(k)$.

**Remark 3.2.2.7** (Lifted and/or shifted supports)**.** Let $\mathscr{A} \subset \mathbb{Z}^n$ be the support of a Laurent polynomial $F \in \mathbb{K}\left[x_1^\pm, \ldots, x_n^\pm\right]$ of cardinality $m$. As before $A \in \mathbb{Z}^{n \times m}$ denotes the matrix representation of $\mathscr{A}$, i.e. column $i$ of $A$ contains the coordinates of $\alpha_i$, and $\mathscr{A}' = \{1\} \times \mathscr{A}$ is the *shifted* point configuration obtained from $\mathscr{A}$ by adding a new first coordinate equal to one. We write

$$A' = \begin{bmatrix} \mathbf{1}_m^\top \\ A \end{bmatrix}$$

for the matrix representation of the shift $\mathscr{A}'$ and we call it *shifted* matrix representation. We adopt this for $A_k$ and $\mathscr{A}_k$, e.g. $A'_k$ denotes the shift of $A_k$ which in turn is the matrix representation of the lift $\mathscr{A}_k$ of $\mathscr{A}$. Note that we obtain the lift of $\mathscr{A}$ via $\theta_k$ and the shifted lift via $\theta'_k$, i.e. the accent always indicates a shift whereas the index indicates a lift.

**Example 3.2.2.8.** We consider the lift to $k=2$ of the point configuration

$$\mathscr{A}=\left\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}2\\0\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}0\\2\end{bmatrix},\begin{bmatrix}2\\1\end{bmatrix}\right\}.$$

Hence, $\theta_2 : \mathscr{A} \longrightarrow \mathbb{Z}^5$ since $\sum_{d=1}^{2} N_2^d = 2+3=5$. We lift with respect to $\mathscr{M}(1,2)=\mathscr{M}_{2,1}\cup\mathscr{M}_{2,2}$ where $\mathscr{M}_{2,1}=\left\{(1,0)^\top,(0,1)^\top\right\}$ and $\mathscr{M}_{2,2}=\left\{(2,0)^\top,(1,1)^\top,(0,2)^\top\right\}$. The lifted support is

$$\mathscr{A}_2=\left\{\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}2\\0\\2\\0\\0\end{bmatrix},\begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\2\\0\\0\\2\end{bmatrix},\begin{bmatrix}2\\1\\2\\2\\0\end{bmatrix}\right\}.$$

We describe the lift of $\alpha_7=(2,1)^\top$ in detail. Here, we write $\alpha_7=(\alpha_{7,x},\alpha_{7,y})$ and omit the transposition symbol. Then:

$$\begin{aligned}
\theta_2\left((2,1)\right) &= \Bigg(\underbrace{\theta_{(1,0)}\left((2,1)\right),\theta_{(0,1)}\left((2,1)\right)}_{\beta\in\mathscr{M}_{2,1}},\underbrace{\theta_{(2,0)}\left((2,1)\right),\theta_{(1,1)}\left((2,1)\right),\theta_{(0,2)}\left((2,1)\right)}_{\beta\in\mathscr{M}_{2,2}}\Bigg)\\
&= \Bigg(\prod_{s=0}^{\beta_1-1}(\alpha_{7,x}-s),\prod_{s=0}^{\beta_2-1}(\alpha_{7,y}-s),\\
&\quad \prod_{s=0}^{\beta_1-1}(\alpha_{7,x}-s),\prod_{s=0}^{\beta_1-1}(\alpha_{7,x}-s)\prod_{s=0}^{\beta_2-1}(\alpha_{7,y}-s),\prod_{s=0}^{\beta_2-1}(\alpha_{7,y}-s)\Bigg)\\
&= (2\cdot 1, 1\cdot 1, (2\cdot 1)\cdot 1, 2\cdot 1, 1\cdot 0).
\end{aligned}$$

The shifted lift $\mathscr{A}_k'$ arising from $\mathscr{A}$ helps to write down the coefficient matrix of $\mathscr{I}_k$:

$$A_k'=\begin{bmatrix}F(\mathbf{1}_n)\\ \frac{\partial^{|\beta|}F}{\partial x^\beta}(\mathbf{1}_n)\end{bmatrix}_{\beta\in\mathscr{M}(1,k)}=\begin{bmatrix}1&\cdots&1\\ \theta_k(\alpha_1)&\cdots&\theta_k(\alpha_m)\end{bmatrix}=\begin{bmatrix}\theta_k'(\alpha_1)&\cdots&\theta_k'(\alpha_m)\end{bmatrix}.$$

The rows of $A_k'$ (and $A_k$) are ordered by the (increasing) graded lexicographic order. The first row contains only ones and corresponds to $F(\mathbf{1}_n)$ ($\beta=\mathbf{0}_n$ respectively). By definition $\theta_{\mathbf{0}_n}(\alpha_i)=1$ for all $i$. Each $\beta\in\mathscr{M}(1,k)$ provides a row in $A_k'$ corresponding to the partial derivative of $F$ with respect to $\beta$ evaluated at $\mathbf{1}_n$.

### 3.2.3. The Tropicalization of $\nabla_{\mathbf{1}_n}^{k+1}$ via Euler Derivatives

Now, we define *Euler derivatives* similar to [**DT12**] but for higher degree. Let $L$ be an integral affine function on $\mathbb{R}^{|\mathscr{M}(1,k)|}$, i.e. $L(y)=\langle\lambda,y\rangle+\nu$ with $\lambda=\left(\lambda_\beta\right)_{\beta\in\mathscr{M}(1,k)}\in\mathbb{Z}^{|\mathscr{M}(1,k)|}$, $\nu\in\mathbb{Z}$. We define the *Euler derivative* of a Laurent polynomial $F$ due to $L$ as follows:

**Definition 3.2.3.9** (Euler derivative)**.** Let $F=\sum_i a_i x^{\alpha_i}\in\mathbb{K}\left[x_1^{\pm},\ldots,x_n^{\pm}\right]$ be a Laurent polynomial with support $\mathscr{A}\subset\mathbb{Z}^n$ and let $L(y)=\langle\lambda,y\rangle+\nu\in\mathbb{Z}\left[y_\beta:\beta\in\mathscr{M}(1,k)\right]$ be an integral affine function. The Euler derivative of $F$ with respect to $L$ is

$$\frac{\partial F}{\partial L}=\sum_{\beta\in\mathscr{M}(1,k)}\lambda_\beta x^\beta\frac{\partial^{|\beta|}F}{\partial x^\beta}+\nu F$$

**Lemma 3.2.3.10.** *Let $F = \sum_i a_i x^{\alpha_i}$ be a Laurent polynomial with support $\mathscr{A} \subset \mathbb{Z}^n$ and consider the integral affine function $L(y) = \langle \lambda, y \rangle + \nu \in \mathbb{Z}\left[y_\beta : \beta \in \mathscr{M}(1,k)\right]$ defined on $\mathbb{R}^{|\mathscr{M}(1,k)|}$. Then:*

(a) $\frac{\partial F}{\partial L} = \sum_i L(\theta_k(\alpha_i))\, a_i x^{\alpha_i}$.
(b) *If $\beta \in \mathscr{M}_{n,k}$ then $\theta_\beta(\alpha) \in \mathbb{Z}[\alpha_1, \ldots, \alpha_n]$ has degree $k$.*
(c) *For fixed $\alpha \in \mathscr{A}$ and $\beta \in \mathscr{M}(1,k)$ we have $\theta_\beta(\alpha) \neq 0$ if and only if for all $j$ holds either $\alpha_j < 0$ or $\alpha_j \geq \beta_j$.*

PROOF. We take a closer look at $\frac{\partial F}{\partial L}$. Using the Definition 3.2.2.6 of a lifted point configuration $\mathscr{A}_k$ the Euler derivative reduces to

$$\begin{aligned}
\frac{\partial F}{\partial L} &= \sum_{\beta \in \mathscr{M}(1,k)} \lambda_\beta x^\beta \frac{\partial^{|\beta|} F}{\partial x^\beta} + \nu F \\
&= \sum_{\beta \in \mathscr{M}(1,k)} \lambda_\beta x^\beta \frac{\partial^{|\beta|}}{\partial x^\beta} \left( \sum_i a_i x^{\alpha_i} \right) + \nu \left( \sum_i a_i x^{\alpha_i} \right) \\
&= \sum_i a_i \left[ \sum_{\beta \in \mathscr{M}(1,k)} \lambda_\beta x^\beta \frac{\partial^{|\beta|}}{\partial x^\beta} (x^{\alpha_i}) + \nu (x^{\alpha_i}) \right] \\
&= \sum_i a_i \left[ \sum_{\beta \in \mathscr{M}(1,k)} \lambda_\beta x_1^{\beta_1} \cdot \ldots \cdot x_d^{\beta_d} \cdot \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \ldots \partial x_d^{\beta_d}} (x^{\alpha_i}) + \nu (x^{\alpha_i}) \right] \\
&= \sum_i a_i \left[ \sum_{\beta \in \mathscr{M}(1,k)} \lambda_\beta \cdot \prod_i (\alpha_i)_{\beta_i} \cdot x^{\alpha_i} + \nu \cdot x^{\alpha_i} \right] \\
&= \sum_i a_i \left( \sum_{\beta \in \mathscr{M}(1,k)} \lambda_\beta \theta_\beta(\alpha_i) + \nu \right) x^{\alpha_i} \\
&= \sum_i L(\theta_k(\alpha_i))\, a_i x^{\alpha_i}.
\end{aligned}$$

To see the second statement consider $\theta_\beta(\alpha) = \prod_i (\alpha_i)_{\beta_i} = \prod_{\substack{\beta_l \in \beta: \\ \beta_l \neq 0}} \prod_{s=0}^{\beta_l - 1} (\alpha_l - s)$ with $\beta \in \mathscr{M}_{n,k}$ as a polynomial in the indeterminates $\alpha = (\alpha_1, \ldots, \alpha_n)$. The length of $\beta$ is $|\beta| = \sum_i \beta_i = k$. The product $\prod_{s=0}^{\beta_l - 1} (\alpha_l - s)$ has $\beta_l$ factors. Hence, the product over all $l$ such that $\beta_l \neq 0$ has $\sum_l \beta_l = k$ factors, i.e. $\theta_\beta(\alpha)$ contains a monomial of degree $k$.
Fix $\beta \in \mathscr{M}_{n,d}$ for some integer $d \in \{1, \ldots, k\}$. $\theta_\beta(\alpha) \neq 0$ if and only if $\prod_{s=0}^{\beta_l - 1} (\alpha_l - s) \neq 0$ for all $l$ such that $\beta_l \neq 0$. Equivalently $\theta_\beta(\alpha) \neq 0$ if and only if $0 \notin \{\alpha_l, \alpha_l - 1, \ldots, \alpha_l - (\beta_l - 1)\}$ for all $l$ such that $\beta_l \neq 0$. Note that $\alpha_l \in \mathbb{Z}$ whereas $\beta_l \in \mathbb{N}$. Thus $\theta_\beta(\alpha) \neq 0$ if and only if $\alpha_l < 0$ or $\alpha_l \geq \beta_l$. □

**Remark 3.2.3.11** (Euler derivatives and circuits of $M_k$)**.** Lemma 3.2.3.10 (a) is not surprising with regard to the matroid $M_k$ associated to $\mathscr{I}_k$. By definition an Euler derivative is a linear combination of the linear generators of $\mathscr{I}_k$ with respect to $(\lambda, \nu)$. Equivalently, an Euler derivative corresponds to an element in the row space of $A'_k$ with respect to $(\lambda, \nu)$. If we focus on a single component in the row space of $A'_k$ we see that the $i$-th component of the linear combination in the row space of $A'_k$ equals the scalar product of $\theta'_k(\alpha_i) = (1, \theta_k(\alpha_i))$ and $(\nu, \lambda)$. This, by definition, is $L$ evaluated at $\theta_k(\alpha_i)$. This observation is a first allusion to the circuits of $M_k$ (cf. Remark 1.4.5.28).

**Definition 3.2.3.12.** Let $f = \bigoplus_i p_i \odot w^{\odot \alpha_i} \in \mathbb{T}[w_1,\dots,w_n]$ be a tropical polynomial with support $\mathscr{A}$ and let $L = \langle y, \lambda \rangle + \nu \in \mathbb{Z}\left[y_\beta : \beta \in \mathscr{M}(1,k)\right]$ denote an integral affine function. Let $\mathscr{A}_k$ denote the lift of $\mathscr{A}$. Then the Euler derivative of $f$ with respect $L$ is defined by

$$\frac{\partial f}{\partial L} = \bigoplus_{\substack{\alpha \in \mathscr{A}_k: \\ L(\alpha) \neq 0}} p_\alpha \odot w^{\odot \alpha}.$$

As one expects the Euler derivative is compatible/commutes with tropicalization:

**Lemma 3.2.3.13.** *Let $f = \bigoplus_i p_i \odot w^{\odot \alpha_i} \in \mathbb{T}[w_1,\dots,w_n]$ be a tropical polynomial with support $\mathscr{A}$ and let $L = \langle y, \lambda \rangle + \nu \in \mathbb{Z}\left[y_\beta : \beta \in \mathscr{M}(1,k)\right]$ denote an integral affine function. Then*

$$\frac{\partial f}{\partial L} = \operatorname{trop}\left(\frac{\partial F}{\partial L}\right)$$

*for every Laurent polynomial $F = \sum_i a_i x^{\alpha_i}$ with support $\mathscr{A}$ that tropicalizes to $f$.*

Proof. The proof we give here is analogous to the proof of Lemma 2.4 in [**DT12**]. Take any $F$ with support $\mathscr{A}$ that satisfies $\operatorname{trop}(F) = f$. We have $\frac{\partial F}{\partial L} = \sum_i L(\theta_k(\alpha_i))\, a_i x^{\alpha_i}$ due to Lemma 3.2.3.10 (a). The valuation of a coefficient is $\operatorname{val}(L(\theta_k(\alpha_i))\, a_i) = \operatorname{val}(L(\theta_k(\alpha_i))) + \operatorname{val}(a_i)$. The value group has characteristic zero. Hence, we have $\operatorname{val}(L(\theta_k(\alpha_i))) = 0$ if $L(\theta_k(\alpha_i)) \neq 0$ and $\operatorname{val}(L(\theta_k(\alpha_i))) = \infty$ otherwise. The statement follows by tropicalizing $\sum_i L(\theta_k(\alpha_i))\, a_i x^{\alpha_i}$. $\square$

Let $L(y) = \langle y, \lambda \rangle + \nu \in \mathbb{Z}\left[y_\beta : \beta \in \mathscr{M}(1,k)\right]$ be an integer affine linear form. Due to Lemma 3.2.3.10 each component of $\theta_k(\alpha)$ is a polynomial in $\alpha$. Let us consider the component of $\theta_k(\alpha)$ indexed by the multiindex $\beta$ of length $k$ in detail:

$$\theta_\beta(\alpha) = \prod_{\substack{\beta_l \in \beta: \\ \beta_l \neq 0}} \prod_{s=0}^{\beta_l - 1} (\alpha_l - s)$$

has degree $k$ when considered as a polynomial in the variables $\alpha = (\alpha_1,\dots,\alpha_n)$ (Lemma 3.2.3.10). If we substitute $y_\beta = \theta_\beta(x_1,\dots,x_n)$ for all $\beta \in \mathscr{M}(1,k)$, the linear form $L \in \mathbb{Z}\left[y_\beta : \beta \in \mathscr{M}(1,k)\right]$ provides an integer polynomial $\tilde{L} \in \mathbb{Z}[x_1,\dots,x_n]$ of degree $k$:

$$L(y) = \sum_{\beta \in \mathscr{M}(1,k)} \lambda_\beta y_\beta + \nu = \sum_{\beta \in \mathscr{M}(1,k)} \lambda_\beta \prod_{\substack{\beta_l \in \beta: \\ \beta_l \neq 0}} \prod_{s=0}^{\beta_l - 1} (x_l - s) + \nu =: \tilde{L}(x) \tag{51}$$

We say that $\tilde{L}$ *is the associated polynomial to* $L$. The Euler derivative of a Laurent polynomial $F$ can be written in two ways,

$$\frac{\partial F}{\partial L} = \sum_i L(\theta_k(\alpha_i))\, a_i x^{\alpha_i} \quad \text{and} \quad \frac{\partial F}{\partial \tilde{L}} = \sum_i \tilde{L}(\alpha_i) a_i x^{\alpha_i}.$$

The next lemma shows that we have the choice: either we consider the lifted point configuration $\mathscr{A}_k$ and affine integer linear forms $L$ or we consider $\mathscr{A}$ and integer polynomials $\tilde{L}$ of degree at most $k$:

**Lemma 3.2.3.14.** *Let $I \subset [m]$ be a subset and $\mathscr{A} \in \mathbb{Z}^n$ a finite subset of cardinality $m$. Any integral affine linear form $L \in \mathbb{Z}\left[y_\beta : \beta \in \mathscr{M}(1,k)\right]$ vanishing on $\theta_k(\alpha_i) \in \mathscr{A}_k$ for all $i \in I$ corresponds uniquely to an integer polynomial $\tilde{L} \in \mathbb{Z}[x_1,\dots,x_n]$ of degree at most $k$ vanishing on $\alpha_i$ for all $i \in I$.*

PROOF. Let $N = N^k_{n+1} - 1$ denote the cardinality of $\mathscr{M}(1,k)$. Both an affine integer linear form $L(y) = \langle y, \lambda \rangle + \nu \in \mathbb{Z}\left[y_\beta : \beta \in \mathscr{M}(1,k)\right]$ with support $\Delta_n \cup \{0\}$ as well as a polynomial $F$ in $n$ variables of degree at most $k$ with full support, i.e. $\operatorname{supp}(F) = \mathscr{M}(0,k)$, have $N+1$ terms.
Let $(\lambda, \nu) \in \mathbb{Z}^N \times \mathbb{Z}$ denote the tuple consisting of the coefficient vector of the degree one terms of $L$ and the constant term, i.e. the linear form can be written as $L(y) = \langle y, \lambda \rangle + \nu \in \mathbb{Z}\left[y_\beta : \beta \in \mathscr{M}(1,k)\right]$.
Let $F = \sum_{\gamma \in \mathscr{M}(1,k)} \mu_\gamma x^\gamma + \eta \in \mathbb{Z}[x_1, \ldots, x_n]$ be a polynomial of degree at most $k$ with coefficient vector $(\mu, \eta) \in \mathbb{Z}^N \times \mathbb{Z}$. If we substitute

$$y_\beta = y_\beta(x_1, \ldots, x_n) = \theta_\beta(x_1, \ldots, x_n) = \prod_{\substack{\beta_l \in \beta: \\ \beta_l \neq 0}} \prod_{s=0}^{\beta_l - 1} (x_l - s)$$

in $L$ then the expansion, denoted by $\tilde{L}$, is a polynomial of the form

$$\tilde{L}(x_1, \ldots, x_n) = \sum_{\gamma \in \mathscr{M}(1,k)} \mu_\gamma x^\gamma + \eta = F.$$

The coefficients $\mu_\gamma$ are $\mathbb{Z}$-linear combinations of $\lambda_\beta$ for certain $\beta \in \mathscr{M}(1,d)$ with $1 \leq d \leq k$. To get an idea which $\beta \in \mathscr{M}(1,d)$ contribute to $\mu_\gamma$ fix an element $\gamma \in \mathscr{M}(1,k)$ and consider the term of $L$ indexed by $\beta \in \mathscr{M}(1,d)$:

$$y_\beta(x_1, \ldots, x_n) = \theta_\beta(x_1, \ldots, x_n) = \prod_{\substack{\beta_l \in \beta: \\ \beta_l \neq 0}} \prod_{s=0}^{\beta_l - 1} (x_l - s).$$

If $|\beta| = d$ the product above provides terms of degree $1 \leq t \leq d$. Thus:

(1) If $\gamma_i = 0$ for some $i \in [n]$ then it is necessary for $\beta$ to satisfy $\beta_i = 0$.
(2) If $\gamma_i > 0$ for some $i \in [n]$ then it is necessary for $\beta$ to satisfy $\beta_i \geq \gamma_i$.

For (1) note that $\beta_i \neq 0$ implies that all monomials in the expansion of $y_\beta$ contain the factor $x_i$. If $\gamma_i = 0$ no monomial in $y_\beta$ coincides with $x^\gamma$ because $\beta_i \neq 0$ whereas $\gamma_i = 0$. For (2) note that $\beta_i < \gamma_i$ implies that the expansion of $y_\beta$ contains no monomial with $x_i$ to the power $\gamma_i$. Thus none of these monomials contributes to $x^\gamma$. If $\beta_i \geq \gamma_i$ the expansion of $y_\beta$ contains monomials with $x_i$ to the power $\gamma_i$.
The coefficients $(\lambda, \nu)$ of $L$ and $(\mu, \eta)$ of $F$ are related by a linear map $\psi : \mathbb{Z}^{N+1} \longrightarrow \mathbb{Z}^{N+1}$. We like to get an idea of the shape of the associated matrix transformation $K \in \mathbb{Z}^{N+1 \times N+1}$ with the intention to show that $K$ is an unimodular transformation. Before note that $\mathbf{0}_n \notin \mathscr{M}(1,k)$. Since $\deg\left(y_\beta\right) > 0$ for all $\beta \in \mathscr{M}(1,k)$ the constant term $\eta$ in $L$ remains unaffected by both the substitution and expansion. Therefore, we can restrict to a map $\psi : \mathbb{Z}^N \mapsto \mathbb{Z}^N$ mapping $\lambda$ to $\mu$.
Both columns and rows of $K$ are indexed by $\mathscr{M}(1,k)$. Here we make an exception on the ordering: we index $K$ by decreasing degree. We order elements of equal degree with the lexicographic ordering. Let $\gamma \in \mathscr{M}(1,k)$ be arbitrary and $|\gamma| = d$. The row of $K$ indexed by $\gamma$ provides the $\gamma$-th component $\mu_\gamma$ of the image of $\lambda$. An element $\delta \in \mathscr{M}(1,k)$ such that the expansion $y_\delta$ provides a monomial $x^\gamma$ satisfies $\delta_i \geq \gamma_i$ if $\gamma_i > 0$ and $\delta_j = 0$ if $\gamma_j = 0$. Hence, $\delta \geq \gamma$. Furthermore, $\delta = \gamma$ contributes $x^\gamma$ to $F$, i.e. $(K)_{\gamma,\gamma} = 1$. We conclude that $K$ is an triangular matrix with ones on the diagonal. Thus, $\det(K) = 1$ and since $K$ has integer entries $K$ is an unimodular matrix providing an isomorphism $\psi : \mathbb{Z}^N \longrightarrow \mathbb{Z}^N$ (see e.g. [**KV12**, Theorem 5.8]). $\square$

Let $L(y) = \langle y, \lambda \rangle + \nu \in \mathbb{Z}\left[y_\beta : \beta \in \mathscr{M}(1,k)\right]$ be an affine integral linear form that vanishes on a maximal subset of $\mathscr{A}_k$. By a scalar multiplication we can achieve that gcd $\left((\lambda_\beta)_{\beta \in \mathscr{M}(1,k)}\right) = 1$. Thus we can define the set $\Lambda(\mathscr{A}_k)$ of affine integral linear forms $L$ that vanish on a maximal subset of $\mathscr{A}_k$ and satisfy gcd $\left((\lambda_\beta)_{\beta \in \mathscr{M}(1,k)}\right) = 1$. Since $\mathscr{A}_k$ is finite the set $\Lambda(\mathscr{A}_k)$ is finite. We define the set

$$P^k_{\mathbf{1}_n}(\mathscr{A}) := \left\{ \sum_i \tilde{L}(\alpha_i) y_i : L \in \Lambda(\mathscr{A}_k) \right\} \subset \mathbb{K}[y_1, \ldots, y_m]. \tag{52}$$

The finiteness of $\Lambda(\mathscr{A}_k)$ implies the finiteness of $P^k_{\mathbf{1}_n}(\mathscr{A})$. We now return to the tropicalization of $\nabla^{k+1}_{\mathbf{1}_n}$ in which the set $P^k_{\mathbf{1}_n}(\mathscr{A})$ proves beneficial. The following statement generalizes Proposition 2.7 in [**DT12**]:

**Proposition 3.2.3.15.** *The set $P^k_{\mathbf{1}_n}(\mathscr{A})$ is a tropical basis for* $\operatorname{trop}(\nabla^{k+1}_{\mathbf{1}_n})$.

PROOF. Let $F = \sum_i y_i x^{\alpha_i} \in \mathbb{K}[y_1, \ldots, y_m][x_1, \ldots, x_n]$ be a generic polynomial with support $\mathscr{A}$. We have $\nabla^{k+1}_{\mathbf{1}_n} = \mathscr{V}(\mathscr{I}_k)$ where $\mathscr{I}_k = \langle \frac{\partial^{|\beta|} F}{\partial x^\beta}(\mathbf{1}_n) : \beta \in \mathscr{M}(k) \rangle$ such that

$$\operatorname{trop}\left(\nabla^{k+1}_{\mathbf{1}_n}\right) = \operatorname{trop}\left(\ker(A'_k)\right) = B(M_k)$$

where $M_k = M(\mathscr{I}_k)$ denotes the matroid associated to $\mathscr{I}_k$. An Euler derivative of $F$, evaluated at $\mathbf{1}_n$, is an integer linear combination of $F$ and partial derivatives of $F$ up to order $k$ evaluated at $\mathbf{1}_n$, i.e. Euler derivatives correspond to elements in the row space of $A'_k$. Elements in the row space correspond to linear forms vanishing on $\nabla^{k+1}_{\mathbf{1}_n}$. Moreover, the set of linear forms with minimal support that vanish on $\nabla^{k+1}_{\mathbf{1}_n}$ form a tropical basis. The generators of $\mathscr{I}_k$ are integral linear forms, i.e. it is sufficient to consider $\mathbb{Z}$-linear combinations of the generators. Therefore, consider the integral affine linear form $L(y) = \langle y, \lambda \rangle + \beta \in \mathbb{Z}\left[y_\beta : \beta \in \mathscr{M}(1,k)\right]$. Then, with the help of Lemma 3.2.3.10 (a), the Euler derivative of $F$ with respect to $L$ is

$$\begin{aligned}\frac{\partial F}{\partial L}(\mathbf{1}_n) &= \sum_{\beta \in \mathscr{M}(1,k)} \lambda_\beta x^\beta \frac{\partial^{|\beta|} F}{\partial x^\beta}(\mathbf{1}_n) + \beta F(\mathbf{1}_n) \\ &= \sum_i \tilde{L}(\alpha_i) \cdot y_i.\end{aligned}$$

The polynomial $\tilde{L}$ occurs as as decision maker, i.e. the coefficient of $y_i$ is zero if and only if $\tilde{L}(\alpha_i)$ is equal to zero. As discussed before, $\tilde{L} \in \mathbb{Z}[x_1, \ldots, x_n]$ is a polynomial of degree at most $k$ with integer coefficients that is associated to $L$. Hence, $\frac{\partial F}{\partial L}(\mathbf{1}_n) \in \mathbb{Z}[y_1, \ldots, y_n]$ is a linear form with minimal support if and only if $\tilde{L}(\alpha_i) = 0$ holds for a maximal number of lattice points $\alpha_i \in \mathscr{A}$. Due to Lemma 3.2.3.14 for each $L \in \Lambda(\mathscr{A})$ exists an unique associated polynomial to $\tilde{L}$. By definition the set $\Lambda(\mathscr{A})$ contains all affine integer polynomials such that $\mathscr{A} \setminus \mathscr{V}\left(\tilde{L}\right)$ is minimal, i.e. $P^k_{\mathbf{1}_n}(\mathscr{A})$ is a tropical basis for $B(M_k)$. □

### 3.2.4. $k+1$-fold Singular Tropical Hypersurface: a Case Study

In the first instance we like to decide whether a given tropical polynomial contains a $k+1$-fold singularity. Euler derivatives enable to give an analogous statement to the classical one, i.e. a point of a tropical hypersurface is a singular point if and only if all Euler derivative vanish:

**Proposition 3.2.4.16.** *Let $f = \bigoplus_i p_i \odot w^{\alpha_i} \in \mathbb{T}[w_1,\dots,w_n]$ be a tropical polynomial with support $\mathscr{A} \subset \mathbb{Z}^n$ and let $q \in \mathscr{T}(f)$ be a point in the tropical hypersurface. Then $q$ is a $k+1$-fold singularity of $\mathscr{T}(f)$ if and only if $q \in \bigcap_L \mathscr{T}\left(\frac{\partial f}{\partial L}\right)$ for all integer polynomials $L \in \mathbb{Z}[m_i : i = 1,\dots,m]$ of degree at most $k$.*

For similar proofs see [**DT12**, Theorem 2.9], [**DDRP14**, Theorem 5.6].

Proof. $\Rightarrow$: Let $q \in \mathscr{T}(f)$ be a $k+1$-fold singularity. By Definition 1.6.2.8 there exist $a \in \nabla^{k+1}$ such that $b \in \mathscr{V}(F_a)$ is a $k+1$-fold singularity with $-\mathrm{val}(b) = q$ and $-\mathrm{val}(a) = p$ satisfying $\mathrm{trop}(F_a) = f$. Then $\frac{\partial^{|\beta|} F_a}{\partial x^\beta}(b) = 0$ for all $\beta \in \mathscr{M}(k)$. Hence, $\frac{\partial F_a}{\partial L}(b) = 0$ for all integer polynomials $L \in \mathbb{Z}[x_1,\dots,x_n]$ of degree at most $k$ and consequently $-\mathrm{val}(b) = q \in \mathscr{T}\left(\mathrm{trop}\left(\frac{\partial F_a}{\partial L}\right)\right) = \mathscr{T}\left(\frac{\partial f}{\partial L}\right)$ for all $L$.
$\Leftarrow$: Suppose $q \in \bigcap_L \mathscr{T}\left(\frac{\partial f}{\partial L}\right)$ where the intersection goes over all integer polynomials $L \in \mathbb{Z}[x_1,\dots,x_n]$ of degree at most $k$. Note that

$$\frac{\partial f}{\partial L}(q) = \bigoplus_{i:L(\alpha_i)\neq 0} p_i \odot q^{\alpha_i} = \max_{i:L(\alpha_i)\neq 0} \{p_i + \langle \alpha_i, q\rangle\}$$

attains the maximum at least twice for any $L$. Consequently, the tropical linear forms arising from $P^k_{\mathbf{1}_n}(\mathscr{A})$ attain the maximum at least twice at $s := p + A^\top q$, i.e. $s \in \mathrm{trop}\left(\nabla^{k+1}_{\mathbf{1}_n}\right)$. Using the Fundamental Theorem 1.4.3.16, there exists an element $r \in \ker(A'_k)$ such that $-\mathrm{val}(r) = s$. Moreover, $\mathscr{V}(F_r)$ has a $k+1$-fold singularity at $\mathbf{1}_n$. Pick an element $t \in T^n$ such that $\mathrm{val}(t) = q$. Then $r \cdot \psi_{\mathscr{A}}(t) \in \nabla^{k+1}$ and it is easy to see that $F_{r\cdot\psi_{\mathscr{A}}(t)}$ has a $k+1$-fold singularity at $t^{-1}$. Furthermore, we have

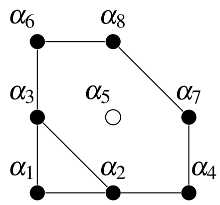
$$\begin{aligned}
\mathrm{trop}\left(F_{r\cdot\psi_{\mathscr{A}}(t)}\right) &= \max_i \{-\mathrm{val}(r_i \cdot \psi_{\mathscr{A}}(t)_i) + \langle w, \alpha_i\rangle\} \\
&= \max_i \{-\mathrm{val}(r_i) - \mathrm{val}(t^{\alpha_i}) + \langle w, \alpha_i\rangle\} \\
&= \max_i \{s_i - \mathrm{val}(t^{\alpha_i}) + \langle w, \alpha_i\rangle\} \\
&= \max_i \{p_i + \langle \alpha_i, q\rangle - \langle \alpha_i, q\rangle + \langle w, \alpha_i\rangle\} \\
&= \max_i \{p_i + \langle w, \alpha_i\rangle\} \\
&= f
\end{aligned}$$

and we see that $\mathscr{T}(f)$ has a $k+1$-fold singularity at $-\mathrm{val}(t^{-1}) = q$. □

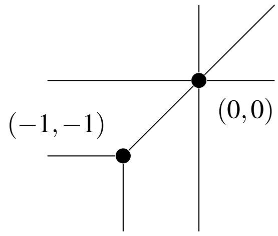
In an earlier version of [**DDRP14**] it was stated that $\mathrm{trop}\left(\nabla^{k+1}\right)$ is a subfan of the secondary fan $\mathrm{Sec}_{\mathscr{A}}$ of the point configuration $\mathscr{A}$. Unfortunately, this is wrong as the following example shows:

**Example 3.2.4.17.** Let $F = \sum_i y_i x^{\alpha_i} \in \mathbb{K}[y_1,\dots,y_m][x^\pm, y^\pm]$ be a generic bivariate Laurent polynomial with support

$$\mathscr{A} := \left\{ \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix} \right\} \subset \mathbb{Z}^2.$$




(A) The regular subdivision $S_w$ induced by $w=(-1,0,0,0,-1,0,0,0)$.




(B) Plane tropical curve with coefficient vector $w=(0,1,1,1,1,1)$.

FIGURE 19. 3-fold singular plane tropical curve with support $\mathscr{A}$, cf. Example 3.2.4.17.

Then $\nabla^3_{\mathbf{1}_2}$ equals the kernel of the coefficient matrix of $A_2'$.

(53)

$$A_2' = \begin{bmatrix} 1&1&1&1&1&1&1&1\\ 0&1&0&2&1&0&2&1\\ 0&0&1&0&1&2&1&2\\ 0&0&0&2&0&0&2&0\\ 0&0&0&0&1&0&2&2\\ 0&0&0&0&0&2&0&2 \end{bmatrix} \in \mathbb{Z}^{6\times 8} \text{ and } G_2 = \begin{bmatrix} -1&2&1&-1&-2&0&1&0\\ -1&1&2&0&-2&-1&0&1 \end{bmatrix}.$$

The matrix $A_2'$ has full rank and, therefore, $\dim(\ker(A_2'))=2$. Consider the weight class $\sigma_{\mathscr{F}}$ provided by the $\mathscr{F}=(\{1,5\},\{1,2,3,4,5,6,7,8\})\triangleleft M_2$. The defining inequalities are

$$w_1=w_5\leq w_2=w_3=w_4=w_6=w_7=w_8.$$

The regular marked subdivision $S_w$ induced by $w\in\sigma_{\mathscr{F}}$ is shown in Figure 19a. Let $\sigma(w)$ be the cone of $\mathrm{Sec}_{\mathscr{A}}$ corresponding to regular subdivisions equal to $S_w$. There are two marked polytopes $(P_1,\mathscr{A}_1)$ and $(P_2,\mathscr{A}_2)$. Let the first one be the triangle formed by $\alpha_1,\alpha_2,\alpha_3$ and the second one the polygon given by the remaining six points. Then $\mathrm{codim}(\sigma(w))=\dim(L_{S_w})=3$ (Lemma 1.1.4.23), i.e. $\dim(\sigma(w))=5$. On the contrary, $\sigma_{\mathscr{F}}\oplus\mathrm{rowspace}(A)$ is 4-dimensional in $\mathrm{trop}\left(\nabla^{k+1}\right)$, which contains the 3-dimensional lineality space spanned by $A^\top$ and $\mathbf{1}_m$. Hence, there is a dimension gap. Note that $\alpha_5$ is not marked since $(\alpha_5,w_5)$ is not contained in any upper face of the shifted polytope $P_w$. Hence, $\alpha_5$ is not involved in the affine relations in $S_w$. Nevertheless, $w_5$ is related to $w_1$ by the equation $w_1=w_5$. Consequently, $\sigma_{\mathscr{F}}\oplus\mathrm{rowspace}(A)$ is properly contained in $\sigma(w)$ due to this equation. We can slightly modify the value at $w_5$, e.g. consider $w'=w\pm\varepsilon e_5$ with $\varepsilon\ll 1$. Then $S_{w'}=S_w$ but $w'\notin\sigma_{\mathscr{F}}\oplus\mathrm{rowspace}(A)$. So, $w_1=w_5$ subdivides the cone $\sigma(w)$ into two proper non-empty cones. We conclude that $\mathrm{trop}(\nabla^{k+1})$ is not a proper subfan of $\mathrm{Sec}_{\mathscr{A}}$.

CHAPTER 4

# Signed Bergman Fans and Real Tropical Singularities

Real tropical geometry is a field of current study (e.g. [**Tab15**], [**MMS12b**]). Particular attention was paid to the positive part of a real tropical variety ([**SW05**]). Thereby oriented matroids arise naturally when considering the positive part of real linear tropical varieties ([**AKW06**]).

This chapter is organized as follows: in Section 4.1 we introduce signed Bergman fans associated to oriented matroids. Signed Bergman fans realize real tropicalizations of real linear spaces. In the subsequent sections we apply the theory developed in Section 4.1. In Section 4.2 we study singular real plane tropical curves defined by real Laurent polynomials with fixed support $\mathscr{A} \subset \mathbb{Z}^2$. The main result is a characterization of real plane tropical curves of maximal dimensional type (cf. Definition 4.2.6.41) with a singularity in a fixed point. In Section 4.3 we study singular real tropical surfaces. A focus is set on *generic* (cf. Definition 4.3.2.7) singular real tropical surfaces of maximal dimensional type. Signed Bergman fans play a key role in these classifications as they make the real tropical discriminant accessible. Throughout this chapter we fix the ground field $\mathbb{K}_{\mathbb{R}} = \mathbb{R}\{\{t\}\}$ (cf. Definition 1.4.1.2).

## 4.1. Signed Bergman Fans and Real Tropical Linear Spaces

In Section 4.1.1 we develop a theory of signed Bergman fans associated to oriented matroids. Thereby classical Bergman fans associated to unoriented matroids serve as an archetype. After enlarging the terminology introduced in Section 1.3 we define signed Bergman fans and prove some basic properties. Section 4.1.2 deals with the tropicalization of real linear spaces.

### 4.1.1. Signed Bergman Fans

For the rest of this introductory section let $M$ denote an oriented matroid on the ground set $E = [n]$ of rank $k$. If nothing else is mentioned we consider all matroids (and its properties) with signs, i.e. we do not write “oriented” explicitly. Often, oriented matroids are considered “up to reorientation” (cf. Definition 1.3.1.5). We explicitly refuse this approach and enrich the terminology with regard to signs.
Recall from Remark 1.3.2.9 that a signed circuit $C \in \mathscr{C}$ can be uniquely identified with a sign vector $s_C = s(C) \in \{\pm, 0\}^E$ via the identification

$$(s_C)_e = \pm \quad \Longleftrightarrow \quad e \in C^{\pm} \quad \text{and} \quad (s_C)_e = 0 \quad \Longleftrightarrow \quad e \in C^0 \; \forall e \in C.$$

This identification allows to formulate the following

**Definition 4.1.1.1** (s-acyclic matroid)**.** Let $M$ be an oriented matroid and $s \in \mathscr{S}$ a pure sign vector. We call the oriented matroid $M$ *s-acyclic* if there is no circuit $C \in \mathscr{C}$ with $s_C \subseteq s$.

**Remark 4.1.1.2.** Notice that $s_C \subseteq s$ is equivalent to $s_C^+ \subseteq s^+$ and $s_C^- \subseteq s^-$ (cf. Remark 1.3.2.10), i.e. for each $e \in E$ we have $(s_c \cdot s)_e \in \{0, +\}$. This allows to reformulate the definition in terms of sign vectors as follows: let $s \in \mathscr{S}$ be a pure sign vector. Then $M$ is $s$-acyclic if $s_C \cdot s \nsubseteq (+)^n$ for all $C \in \mathscr{C}$.

**Remark 4.1.1.3.** The situation with $s = (+)^n$ fixed was studied in [**AKW06**], [**BLVS$^+$99**]. An oriented matroid $M$ is called *acyclic* if there is no all-positive circuit in $M$. This case is covered by Definition 4.1.1.1.

**Lemma 4.1.1.4.** *Let $M$ be a matroid on $E = [n]$, $A \subseteq E$ a subset and $s = {}_{-A}s \in \mathscr{S}$ the pure sign vector associated to $A$ for reorientation, i.e. $s^- = A$ and $s^+ = E \setminus A$. Then:*

$$M \text{ is } s\text{-acyclic} \iff {}_{-A}M \text{ is acyclic} \iff s \text{ is a tope of } M$$

PROOF. We start with the first equivalence. Due to Remark 4.1.1.2, $M$ is $s$-acyclic if and only if $s_C \cdot s \nsubseteq (+)^n$ for all $C \in \mathscr{C}$. Due to Remark 1.3.2.11, the sign vector of the reoriented circuit ${}_{-A}C$ is $s \cdot s_C = {}_{-A}s \cdot s_C$. All circuits of ${}_{-A}M$ are of the form ${}_{-A}C$ for all $C \in \mathscr{C}$. Thus, $M$ is $s$-acyclic is equivalent to ${}_{-A}C$ is not all-positive for all $C \in \mathscr{C}$. Equivalently, ${}_{-A}M$ is acyclic.
Before proving the second equivalence note that, by [**BLVS$^+$99**, Proposition 3.4.8], an oriented matroid $M$ is acyclic if and only if $(+)^n$ is a tope. Let us figure out how this statement behaves under reorientation. According to Remark 1.3.2.15, covectors of ${}_{-A}M$ are obtained by multiplying the covectors of $M$ with $s = {}_{-A}s$. Hence, the above equivalence translates to ${}_{-A}M$ is acyclic if and only if $s \cdot (+)^n$ is a tope. To see the second equivalence suppose that ${}_{-A}M$ is acyclic. Equivalently $(+)^n$ is a tope of ${}_{-A}M$. Applying $s$ for reorientation means that $s \cdot (+)^n = s$ is a tope of ${}_{-A}({}_{-A}M) = M$. □

**Remark 4.1.1.5** (Oriented initial matroids)**.** Oriented initial matroids are identically defined as unoriented initial matroids (cf. Definition 1.2.2.10). In short we consider an element $w \in \mathbb{R}^n$ as a weight function on $E$. The *signed initial circuit* of a signed circuit $C \in \mathscr{C}$ is defined by the sets of positive/negative elements $\mathrm{in}_w(C)^{\pm} = \{j \in C^{\pm} | w_j = \max_{i \in \underline{C}}\{w_i\}\}$. In particular, for a signed circuit $C \in \mathscr{C}$ we have $\underline{\mathrm{in}_w(C)} = \mathrm{in}_w(\underline{C})$. The collection of inclusion-minimal signed initial circuits $\mathrm{in}_w(C)$ for all $C \in \mathscr{C}$ is denoted by $\mathrm{in}_w(\mathscr{C})$. The oriented initial matroid $M_w$ is defined by the signed initial circuits $\mathrm{in}_w(\mathscr{C})$ (i.e. it is an oriented matroid according to Definition 1.3.1.3, see [**AKW06**, Proposition 2.3] for a proof that $M_w$ satisfies the oriented matroid axioms). Notice that the underlying initial circuit of an signed initial circuit is invariant to reorientation.

**Remark 4.1.1.6** (Flags of subsets and weight classes)**.** Analogously to the unoriented case (cf. Definition 1.2.3.14), let $\mathscr{F}(w)$ denote the flag of subsets $\emptyset \subset F_1 \subset F_2 \subset \ldots F_k \subset E$ for a fixed $w \in \mathbb{R}^n$ such that $w$ is constant on $F_{i+1} \setminus F_i$ and $w_{F_i \setminus F_{i-1}} < w_{F_{i+1} \setminus F_i}$. The weight class of $w$ is the set of $v \in \mathbb{R}^n$ such that $\mathscr{F}(w) = \mathscr{F}(v)$. Since $M_w$ depends only on the flag $w$ is in we also refer to this initial matroid as $M_{\mathscr{F}}$.

**Definition 4.1.1.7** (s-flat/flag)**.** Let $M$ be an oriented matroid on $E$ and $s \in \mathscr{L}_M$ a tope. We call a flat $F$ of $M$ an *s-flat* if there is a covector $v \in \mathscr{L}_M$ such that $v \subseteq s$ and $F = v^0$. We call a flag of flats $\mathscr{F}$ an $s$-flag if all flats of $\mathscr{F}$ are $s$-flats. Moreover, we define $F_{i,j} = F_i \setminus F_j$ for all $0 \leq j \leq i \leq k$ where $F_0 = \emptyset$ (cf. Definition 1.2.3.14).

**Remark 4.1.1.8.** Suppose $\mathscr{F} = (F_1, \ldots, F_k) \lhd M$ is an $s$-flag of an oriented matroid for a given pure sign vector $s \in \{\pm\}^m$. Let $v_1, \ldots, v_k \in \mathscr{L}_M$ be the set of covectors such that $v_i^0 = F_i$ and $v_i \subseteq s$.

As $F_i \subset F_{i+1}$ and $v_i \subseteq s$ for all $i$, we have $v_{i+1} \subseteq v_i$ and, moreover, $v_i$ coincides with $v_{i+1}$ at all coordinates where $v_{i+1}$ is non-zero. Thus $s$-flags correspond to chains of covectors ordered by "$\subseteq$" (cf. Remark 1.3.2.10) ending with $s$. Also, note that $v_i \neq 0$ at $F_{k,i}$ and $v_i$ differs from $v_{i+1}$ at $F_{i+1,i}$.

**Remark 4.1.1.9** (Big face lattice)**.** Due to [**BLVS$^+$99**, Proposition 4.1.13] the collection of flats of $\underline{M}$ equals the collection of zero sets $v^0$ of covectors $v \in \mathscr{L}_M$. The covectors $\mathscr{L}_M$ of $M$, equipped with the induced partial order $\subseteq$ of $\mathscr{S}$ (cf. Remark 1.3.2.10) and bottom/top elements $\hat{0}/\hat{1}$, form a lattice $\mathscr{F}_{\text{big}}(M) = \left(\mathscr{L}_M \cup \left\{\hat{0}, \hat{1}\right\}, \subseteq\right)$ called the *big face lattice* of $M$. Let $\mathscr{F}_s$ denote the sublattice of $\mathscr{F}_{\text{big}}$ of all $s$-flats. We call $\mathscr{F}_s$ the *s-lattice* (of $\mathscr{F}_{\text{big}}$).

**Example 4.1.1.10.** Recall the oriented vector matroid $M[A]$ defined in Example 1.3.1.7 given by

$$A = \begin{pmatrix} 1 & 0 & 1 & -1 & -2 \\ 0 & 1 & 1 & -1 & -1 \end{pmatrix}. \tag{54}$$

The point configuration $\mathscr{A}$ defined by the columns of $A$ is shown in Figure 6a. The subdivision of $\left(\mathbb{R}^2\right)^\vee$ into topes of $M[A]$ is shown in Figure 6b. Consider the chains of covectors

$$\mathscr{S}: \quad v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \subseteq v_2 = \begin{bmatrix} - \\ + \\ 0 \\ 0 \\ + \end{bmatrix} \subseteq v_3 = \begin{bmatrix} - \\ + \\ + \\ - \\ + \end{bmatrix}, \quad \mathscr{S}': \quad v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \subseteq v_2 = \begin{bmatrix} - \\ + \\ 0 \\ 0 \\ + \end{bmatrix} \subseteq v_3' = \begin{bmatrix} - \\ + \\ - \\ + \\ + \end{bmatrix}. \tag{55}$$

Let $\mathscr{F} = (v_1^0, v_2^0, v_3^0)$ and $\mathscr{F}' = (v_1^0, v_2^0, v_3^{0'})$ denote the flags of flats arising from the zero sets of the covectors of $\mathscr{S}$ and $\mathscr{S}'$, cf. Equation (55). Note that

$$\mathscr{F} = \mathscr{F}' = (F_1, F_2, F_3) \text{ with } F_1 = E,\ F_2 = \{3,4\} \text{ and } F_3 = \emptyset,$$

whereas $\mathscr{S} \neq \mathscr{S}'$. If we pick $s = v_3 \in \{\pm\}^5$ (or $s' = v_3' \in \{\pm\}^5$) then $\mathscr{F}$ is an $s$-flag ($s'$-flag respectively). For a tope $s \in \mathscr{Y}$, the $s$-flats correspond to faces of the cell indexed by $s$ in $\left(\mathbb{R}^2\right)^\vee$. All $s$-flags of $M$ correspond to collections of faces of the cell indexed by $s$ in $\left(\mathbb{R}^2\right)^\vee$ ordered by inclusion.

**Definition 4.1.1.11** (Signed Bergman fan)**.** The *signed Bergman fan* of an oriented matroid $M$ on the ground set $E$ with respect to the pure sign vector $s \in \mathscr{S}$ is

$$\mathscr{B}^s(M) := \{w \in \mathbb{R}^n \,|M_w \text{ is } s\text{-acyclic}\}$$

**Remark 4.1.1.12.** The signed Bergman fan $\mathscr{B}^s(M)$ with respect to $s = (+)^n$ equals the *positive Bergman fan* $\mathscr{B}^+(M)$ of [**AKW06**]. There, a covector $v \in \mathscr{L}_M$ is called *positive* if $v^- = \emptyset$ and a flat is positive if it is a $(+)^n$-flat.

For the positive Bergman fan, i.e. $s = (+)^n$, there is the following theorem:

THEOREM 4.1.1.13 ([**AKW06**, Theorem 3.4]). *Given an oriented matroid $M$ and $w \in \mathbb{R}^n$ which corresponds to a flag $\mathscr{F} = \mathscr{F}(w)$, the following are equivalent:*

1. *$\mathscr{M}_{\mathscr{F}}$ is acyclic.*
2. *For each signed circuit $C$ of $M$,* $\text{in}_w(C)$ *contains a positive and negative element of $C$.*
3. *$\mathscr{F}$ is a flag of positive flats of $M$.*

Now we generalize Theorem 4.1.1.13 for any pure sign vector $s \in \mathscr{S}$:

THEOREM 4.1.1.14. *Let $M$ be an oriented matroid on $E$, $s \in \mathscr{S}$ a pure sign vector, $w \in \mathbb{R}^n$ and $\mathscr{F} = \mathscr{F}(w)$ the corresponding flag. The following are equivalent:*

1. *$M_{\mathscr{F}}$ is $s$-acyclic.*
2. *For all $C \in \mathscr{C}$, $C$ attains its maximum at $(s \cdot s_C)^+$ and $(s \cdot s_C)^-$.*
3. *$\mathscr{F}$ is a $s$-flag of flats of $M$.*

PROOF. At first we set $A = s^-$, i.e. we reorientate with respect to $A$. Note that reorienting and initializing a matroid $M$ commutes, i.e. for $A \subseteq E$ and $w \in \mathbb{R}^n$ we have

$$_{-A}(M_{\mathscr{F}}) = (_{-A}M)_{\mathscr{F}}.$$

The reason for this is that $w$ picks the indices of a (signed) circuit $C$ with $w_i$ maximal independently from the signs. Circuits of the underlying matroid $\underline{M}$ remain invariant under reorientation. Also note that covectors $v \in \mathscr{L}_M$ translate to covectors $s \cdot v \in \mathscr{S}_{_{-A}M}$ of $_{-A}M$ ([**BLVS**$^+$**99**, section 3 and Lemma 4.18]) and finally, $_{-A}(_{-A}M) = M$ and $s \cdot (s \cdot v) = v$.
For 1. $\Rightarrow$ 2. suppose $M_{\mathscr{F}}$ is $s$-acyclic. Thus $_{-A}(M_{\mathscr{F}}) = (_{-A}M)_{\mathscr{F}}$ is acyclic (cf. Lemma 4.1.1.4) and we have $s(\mathrm{in}_w(_{-A}C)) \nsubseteq (+)^n$ for all circuits $C \in \mathscr{C}$. Since reorientation commutes with initialization we have $_{-A}s \cdot s(\mathrm{in}_w(C)) = s \cdot s(\mathrm{in}_w(C)) \nsubseteq (+)^n$ for all $C \in \mathscr{C}$. Consequently, for all $C \in \mathscr{C}$ exist $e, f \in E$ such that (w.l.o.g.) $(s)_e(s(\mathrm{in}_w(C)))_e = -$ and $(s)_f(s(\mathrm{in}_w(C)))_f = +$. Since $\mathrm{in}_w(C) \subseteq C$ we conclude that for all $C \in \mathscr{C}$ exist $e, f \in E$ such that $(s)_e(s(C))_e = -$ and $(s)_f(s(C))_f = +$. In other words, for all $C \in \mathscr{C}$ holds that $C$ attains its maximum at $(s \cdot s(C))^+$ and $(s \cdot s(C))^-$.
Vice versa, suppose $C$ attains its maximum at $(s \cdot s(C))^+$ and $(s \cdot s(C))^-$ for all $C \in \mathscr{C}$. Recall that $s \cdot s_C = {_{-A}s} \cdot s_C$. We conclude that $(_{-A}s \cdot s(\mathrm{in}_w(C)))^{\pm} \neq \emptyset$ for all $C \in \mathscr{C}$. Hence, we have $s \cdot s(\mathrm{in}_w(C)) = s(\mathrm{in}_w(_{-A}C)) \nsubseteq (+)^n$ for all $C \in \mathscr{C}$, i.e. $(_{-A}M)_{\mathscr{F}}$ is acyclic. Lemma 4.1.1.4 implies that $M_{\mathscr{F}}$ is $s$-acyclic.
For 1. $\Rightarrow$ 3. suppose that $M_{\mathscr{F}}$ is $s$-acyclic. Hence, $(_{-A}M)_{\mathscr{F}}$ is acyclic (Lemma 4.1.1.4). Thus $\mathscr{F}$ is a positive flag of flats of $_{-A}M$ (Theorem 4.1.1.13). Let $\mathscr{F} = (F_1, \dots, F_l)$ be the flag of flats and $\{v_1, \dots, v_l\} \subset \mathscr{L}_{_{-A}M}$ the covectors such that $v_i^0 = F_i$ for $1 \leq i \leq l$. According to Remark 1.3.2.15, $\{s \cdot v_1, \dots, s \cdot v_l\} \subset \mathscr{L}_M$ is a set of covectors satisfying $(s \cdot v_i)^0 = F_i$ for $1 \leq i \leq l$. To see this note that $s$ is a pure sign vector, i.e. $s^0 = \emptyset$. Thus $(s \cdot v_i)_e = 0$ if and only if $(v_i)_e = 0$. Moreover, $v_i \subset (+)^n$, i.e. $v_i^- = \emptyset$ for $1 \leq i \leq l$. Thus $s_i = (s \cdot v_i) \subseteq s$ and, therefore, $\{s_1, \dots, s_l\} = \{s \cdot v_1, \dots, s \cdot v_l\} \subset \mathscr{L}_M$ is a set of covectors such that $\mathscr{F}$ is a $s$-flag.
Vice versa, let $\mathscr{F}$ be a $s$-flag, i.e. $\mathscr{F} = (F_1, \dots, F_l)$ and there is a set $\{v_1, \dots, v_l\} \subset \mathscr{L}_M$ such that $F_i = v_i^0$ and $v_i \subseteq s$. Remark 1.3.2.15 implies that $\{s \cdot v_1, \dots, s \cdot v_l\} = \{s_1, \dots, s_l\} \subset \mathscr{L}_{_{-A}M}$ is a set of covectors such that $s_i^0 = (s \cdot v_i)^0 = v_i^0$ and $s_i \subset (+)^n$ for $1 \leq i \leq l$. Hence, $\mathscr{F}$ is a positive flag of $_{-A}M$. Due to Theorem 4.1.1.13 this implies $(_{-A}M)_{\mathscr{F}}$ is acyclic. By Lemma 4.1.1.4 this is equivalent to $M_{\mathscr{F}}$ is $s$-acyclic. $\square$

**Corollary 4.1.1.15.** *Let $M$ be an oriented matroid and let $\mathscr{Y}$ denote its set of topes (cf. Remark 1.3.2.13). Then:*

$$s \notin \mathscr{Y} \quad \Rightarrow \quad \mathscr{B}^s(M) = \emptyset.$$

PROOF. Assume $\mathscr{B}^s(M) \neq \emptyset$. Then there is an element $w \in \mathscr{B}^s(M)$ such that $M_w$ is $s$-acyclic. By Theorem 4.1.1.14 we know that $\mathscr{F}(w)$ is a $s$-flag of $M$. Hence, $s \in \mathscr{Y}$. $\square$

From Theorem 4.1.1.14 we can immediately deduce the following

**Corollary 4.1.1.16.** *Let $M$ be an oriented matroid. Then:*

$$\mathscr{B}^s(M) = \bigcup_{\substack{\mathscr{F} \lhd \underline{M}: \\ \mathscr{F} \text{ is an } s\text{-flag}}} \sigma_{\mathscr{F}}.$$

### 4.1.2. Real Tropical Linear Spaces

Recall that $\mathbb{K}_{\mathbb{R}} = \mathbb{R}\{\{t\}\}$ is the ground field. We examine the special case of real tropicalizations of linear subspaces $V = \mathscr{V}(\mathscr{I}) \subset T^n_{\mathbb{R}}$ defined by ideals $\mathscr{I} = \langle l_1, \ldots, l_{n-k} \rangle \subset \mathbb{K}_{\mathbb{R}}\left[x_1^{\pm}, \ldots, x_n^{\pm}\right]$ generated by linear forms $l_i = \sum_j a_{ij} x_j$.

**Remark 4.1.2.17** (Oriented matroids from linear ideals)**.** The linear forms $\{l_1, \ldots, l_{n-k}\}$ generating the linear ideal $\mathscr{I} \subset \mathbb{K}_{\mathbb{R}}\left[x_1^{\pm}, \ldots, x_n^{\pm}\right]$ can be written as

$$l_i = \sum_{j \in J_1} a_{ij} x_j - \left( \sum_{j \in J_2} a_{ij} x_j \right) \tag{56}$$

where $J_1, J_2$ are disjoint sets and $a_{ij} > 0$ for all $i, j$. We work in the constant coefficient case, i.e. $\mathrm{val}(a_j) = \mathrm{val}(b_j) = 0$. The linear form, written as in Equation (56), provides a signed circuit $C_i$ by defining $C_i^+ = J_1$ and $C_i^- = J_2$. All signed circuits obtained this way form an oriented matroid denoted by $M = M(\mathscr{I})$. Similar to the complex case we call $M(\mathscr{I})$ the *matroid associated to* $\mathscr{I}$ (cf. Section 1.4.5, in particular Remark 1.4.5.28). The tropicalization of $l_i$ equals

$$\mathrm{trop}_{\mathbb{R}}(l_i) = \left( \bigoplus_{j \in J_1} 0^+ w_j \right) \oplus \left( \bigoplus_{j \in J_2} 0^- w_j \right). \tag{57}$$

Note that we can determine $\mathrm{trop}_{\mathbb{R}}(l_i)$ uniquely from $C_i$. Thus, we refer to the real tropical linear form obtained from a signed circuit $C$ by $l_C$.

The underlying matroid $\underline{M}$ of $M$ determines the classical tropical linear space $\mathrm{trop}(\mathscr{V}(\mathscr{I}))$ (cf. Section 1.4.5, see also [**Stu02**]). In the real case an analogous statement holds for the oriented matroid $M$ with regard to $\mathrm{trop}_{\mathbb{R}}(\mathscr{V}(\mathscr{I}))$:

THEOREM 4.1.2.18 ([**Tab15**, Theorem 3.14]). *Let $\mathscr{I} \subset \mathbb{K}_{\mathbb{R}}\left[x_1^{\pm}, \ldots, x_n^{\pm}\right]$ be a linear ideal with constant coefficients and $M$ the associated oriented matroid with signed circuits $\mathscr{C}$. Then:*

$$\mathrm{trop}_{\mathbb{R}}(\mathscr{V}(\mathscr{I})) = \bigcap_{C \in \mathscr{C}} \mathscr{T}_{\mathbb{R}}(l_C).$$

This theorem also holds in the non-constant coefficient case. The positive part of $\mathrm{trop}_{\mathbb{R}}(\mathscr{V}(\mathscr{I}))$ is well-understood ([**AKW06**, Proposition 4.1]). For $\mathscr{I} \subset \mathbb{K}_{\mathbb{R}}\left[x_1^{\pm}, \ldots, x_n^{\pm}\right]$ a linear ideal with constant coefficients and $M$ the associated oriented matroid we have $\mathrm{trop}_{\mathbb{R}}(\mathscr{V}(\mathscr{I})) \cap ((+)^n \times \mathbb{R}^n) = \mathscr{B}^+(M)$. We generalize this statement for arbitrary pure sign vectors $s \in \mathscr{S}$:

THEOREM 4.1.2.19. *Let $\mathscr{I} \subset \mathbb{K}_{\mathbb{R}}\left[x_1^{\pm}, \ldots, x_n^{\pm}\right]$ be a linear ideal with constant coefficients and $M$ the associated oriented matroid with signed circuits $\mathscr{C}$. Let $s \in \mathscr{S}$ be a pure sign vector. Then:*

$$\mathrm{trop}_{\mathbb{R}}(\mathscr{V}(\mathscr{I})) \cap (s \times \mathbb{R}^n) = \mathscr{B}^s(M).$$

PROOF. The circuits $\mathscr{C}$ form a tropical basis of $\mathrm{trop}_{\mathbb{R}}(\mathscr{V}(\mathscr{I}))$ (Theorem 4.1.2.18). Therefore, we show

$$\left(\bigcap_{C\in\mathscr{C}} \mathscr{T}_{\mathbb{R}}(l_C)\right) \cap (s \times \mathbb{R}^n) = \mathscr{B}^s(M)$$

for arbitrary pure sign vectors $s \in \mathscr{S}$. Recall that $l_C$ is the real tropical linear form obtained uniquely from $C$ (cf. Remark 4.1.2.17). We write it in its simplest form, i.e. $l_C = \sum_{j\in\underline{C}} p_j w_j \in \mathbb{TR}[w_1,\dots,w_n]$ with $p_j \in \{0^{\pm}\}$ for all $j \in \underline{C}$. Furthermore, recall that we denoted the sign vector obtained from $C$ by $s_C$. Here we get $(s_C)_j = s(p_j) \in \{\pm\}$ for all $j \in \underline{C}$ and $(s_C) = 0$ for all $j \notin \underline{C}$. Now suppose $w \in (\bigcap_{C\in\mathscr{C}} \mathscr{T}_{\mathbb{R}}(l_C)) \cap (s\times\mathbb{R}^n)$. The sign vector of $w$ is $s$, in detail $s(w_i) = s_i$ for all $i$. As a consequence we identify the modulus of $w$ with $w$. Since $w \in \bigcap_{C\in\mathscr{C}} \mathscr{T}_{\mathbb{R}}(l_C)$ it follows that, by definition, for all circuits $C \in \mathscr{C}$ exist $i, j \in \underline{C}$ such that $s(p_i)s(w_i) \neq s(p_j)s(w_j)$ and $w_i = w_j \geq w_k \forall k$. Equivalently, for all circuits $C \in \mathscr{C}$ exist $i, j \in \underline{C}$ such that (w.l.o.g.) $i \in (s\cdot s_C)^-$, $j \in (s\cdot s_C)^+$ and $w_i = w_j \geq w_k \forall k$. This is precisely statement 2 of Theorem 4.1.1.14, i.e. $M_w$ is $s$-acyclic. $\square$

## 4.2. Singular Real Plane Tropical Curves

This section deals with the classification of singular real plane tropical curves. Foundation pillar of our approach is the real tropical discriminant. We adapt notations for the real case:

NOTATION 4.1. Let $\Delta \subset \mathbb{Z}^2$ be a convex lattice polygon and $\mathscr{A} = \Delta \cap \mathbb{Z}^2 = \{\alpha_1,\dots,\alpha_m\}$ the set of lattice points of $\Delta$. For $\alpha \in \mathscr{A}$ we write $\alpha = (\alpha_x, \alpha_y)$. We revise Notation 3.1 for $n = 2$ and the real case, i.e. we work over $\mathbb{K}_{\mathbb{R}} = \mathbb{R}\{\{t\}\}$ and $T^m_{\mathbb{R}} = (\mathbb{K}_{\mathbb{R}}{}^*)^m$. Consider a generic bivariate real Laurent polynomial $F = \sum_i y_i x^{\alpha_i} \in \mathbb{K}_{\mathbb{R}}[y_1,\dots,y_m][x^{\pm}, y^{\pm}]$ that is linear in the coefficients. We write $R = \mathbb{K}_{\mathbb{R}}[y_1,\dots,y_m]$ for the polynomial ring forming the coefficients. By $F_a = \sum_i a_i x^{\alpha_i} \in \mathbb{K}_{\mathbb{R}}[x^{\pm}, y^{\pm}]$ we denote the Laurent polynomial obtained from $F$ with fixed coefficients $a = (a_1,\dots,a_m) \in T^m_{\mathbb{R}}$ and by $F(p) = \sum_i y_i p^{\alpha_i} \in \mathbb{K}_{\mathbb{R}}[y_1,\dots,y_m]$ we denote the polynomial obtained from $F$ by evaluating at $p \in T^2_{\mathbb{R}}$. We study the family of real Laurent polynomials that provide a singular real plane curves:

$$\nabla_{\mathbb{R}} = \{a \in \mathbb{P}(T^m_{\mathbb{R}}) : \mathscr{V}(F_a) \text{ is singular.}\}. \tag{58}$$

In the following we write

$$\mathscr{I} = \langle F(\mathbf{1}_2),\, \frac{\partial F}{\partial x}(\mathbf{1}_2),\, \frac{\partial F}{\partial y}(\mathbf{1}_2)\rangle \subset R. \tag{59}$$

for the ideal generated by $F$ and its partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial x}$ evaluated at $\mathbf{1}_2$. Let $A \in \mathbb{Z}^{2\times m}$ be the matrix representation of the point configuration $\mathscr{A}$. The coefficient matrix of $\mathscr{I}$ equals

$$A' = \begin{bmatrix} 1 & \cdots & 1 \\ \alpha_1 & \cdots & \alpha_m \end{bmatrix} = \begin{bmatrix} \mathbf{1}_m^{\top} \\ A \end{bmatrix} \in \mathbb{Z}^{3\times m}, \tag{60}$$

It is the matrix representation of the shift $\mathscr{A}' = \{1\} \times \mathscr{A}$ of $\mathscr{A}$ into $\mathbb{R}^3$. By $\psi_{\mathscr{A}} : T^2_{\mathbb{R}} \to T^m_{\mathbb{R}}$ we denote the monomial map according to $\mathscr{A}$. By $p_B : \mathbb{R}^n \to \mathbb{R}_B$ we denote the coordinate projection onto the coordinates indexed by $B$. For $a \in \mathbb{R}$ we write $a^{\pm}$ for the element $(\pm, a) \in \mathbb{TR}$ and we abbreviate $(+,a)^m$ by $(a^+)^m$. Moreover, we write $\mathbf{0}^{\pm}_m = (0^{\pm})^m$ and $\mathbf{1}^{\pm}_m = (1^{\pm})^m$.

In this section we proceed as follows: in Section 4.2.1 we summarize basic concepts of real plane tropical curves and enhance notations. We introduce the signed secondary fan, i.e. the parameter space of real tropical Laurent polynomials with fixed support. We show that there is a duality between real plane tropical curves and signed regular marked subdivisions which is similar to the duality in the complex case (cf. Remark 4.2.1.5). In Section 4.2.2 we focus on real plane tropical curves with a singularity at $\mathbf{1}_2$: $\nabla_{\mathbb{R},\mathbf{1}_2}$ is a linear space that is completely determined by an oriented matroid. We use the results of Section 4.1.2 to tropicalize $\nabla_{\mathbb{R},\mathbf{1}_2}$. We obtain a description of the top-dimensional weight classes of $\operatorname{trop}_{\mathbb{R}}\left(\nabla_{\mathbb{R},\mathbf{1}_2}\right)$ that allows to establish a connection between the signed secondary fan of $\mathscr{A}$ and the real tropical discriminant associated to $\mathscr{A}$. In Section 4.2.3 we investigate the signed regular marked subdivisions obtained from top-dimensional weight classes of $\operatorname{trop}_{\mathbb{R}}(\nabla_{\mathbb{R},\mathbf{1}_2})$. In Section 4.2.4 we compare the structures of the signed secondary fan and the real tropical discriminant. We show that — in contrast to the complex case — we cannot predict wether a given real plane tropical curve is singular purely from its signed subdivision. Therefore, we provide solutions for these curves in Section 4.2.5, e.g. using Euler derivatives. At last, we classify all singular real plane tropical curves of maximal dimensional type in Section 4.2.6.

Singular plane tropical curves over an algebraically closed field (e.g. $\mathbb{K}_{\mathbb{C}}$) were studied and characterized in [**MMS12a**]. We adapt their methods for the real case and point out the differences owed to the signs.

### 4.2.1. Real Plane Tropical Curves and the Signed Secondary Fan

In this section we introduce the signed secondary fan and examine its relationship to real plane tropical curves. To begin with we exploit basics of real tropical Laurent polynomials. Therefore, consider a real tropical Laurent polynomial $f = \bigoplus_i p_i w^{\alpha_i} \in \mathbb{TR}\left[w_1^{\pm}, w_2^{\pm}\right]$ with support $\mathscr{A}$. The real tropical Laurent polynomial $f$ provides a piecewise affine linear function $|f|(w) = \max_i\{|p_i| + \langle |w|, \alpha_i\rangle\}$ (cf. Definition 1.5.2.7) called *modulus of $f$*. Basically, $|f|$ forgets about the signs. The tropical curve defined by $|f|$ is dual to the regular marked subdivision of $\Delta = \operatorname{Newt}(f)$ (cf. Definition 1.1.4.20, Proposition 1.4.2.9) with heights $|p|$. The first goal of this section is a similar statement about duality of real tropical hypersurfaces and signed regular marked subdivisions.

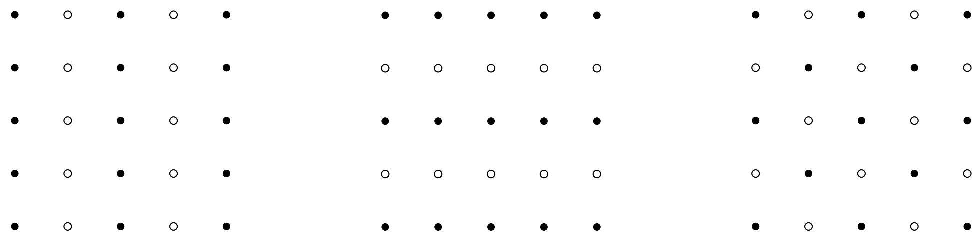
**Definition 4.2.1.1** (Signed marked subdivision)**.** A *signed marked polytope* $(P, Q, s_Q)$ consists of a marked polytope $(P, Q)$ and a sign vector $s_Q \in \{\pm\}^{|Q|}$ such that $\alpha \in Q$ has a sign $s_\alpha$. A *signed marked subdivision* is a set of signed marked polytopes, $T = \{(P_i, Q_i, s_{Q_i}) : i = 1, \ldots, k\}$, satisfying

- the collection of marked polytopes $(P_i, Q_i)$ with $i \in [k]$ forms a marked subdivision, and
- the signs of marked polytopes are compatible: $p_{Q_i \cap Q_j}(s_{Q_i}) = p_{Q_i \cap Q_j}(s_{Q_j})$ for all $i, j \in [k]$.

If we forget about signs we obtain a marked subdivision denoted by $|T|$. As for marked subdivisions, we call the collection of $P_i$ without markings and signs the *type* of $T$. The boundary $\partial T$ of $T$ is $\partial|T|$.

An element $(s, u) \in \{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$ provides a signed regular marked subdivision $T$: we get a regular marked subdivision $|T| = \{(P_i, Q_i) : i = 1, \ldots, k\}$ of $\Delta$ by the modulus $u \in \mathrm{Sec}_{\mathscr{A}}$ (Definition 1.1.4.20) and equip the vertices with the signs according to $s$, i.e. $\alpha \in Q_i$ gets the sign $s_\alpha$.

**Remark 4.2.1.2** (Klein group)**.** Note that we have$\{\pm\}^2 = \{(+,+), (+,-), (-,+), (-,-)\}$. This 4-element set forms the *Klein group* $\mathbb{Z}_2 \times \mathbb{Z}_2$. In general, the set of pure sign vectors $\{\pm\}^m$ forms a



(A) $\{(a,b) : a \equiv 0 \mod 2\}$. (B) $\{(a,b) : b \equiv 0 \mod 2\}$. (C) $\{(a,b) : a+b \equiv 0 \mod 2\}$.

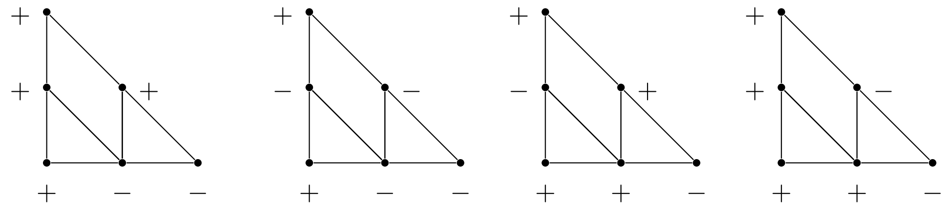
FIGURE 20. Sublattices of $\mathbb{Z}^2$ corresponding to elements of the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

multiplicative group. If we identify $\{\pm\} \cong \{1,-1\}$ we can think of $\psi_{\mathscr{A}} : \{\pm\}^2 \to \{\pm\}^m$ as a group homomorphism where $\psi_{\mathscr{A}}$ denotes the monomial map according to $\mathscr{A}$ (see e.g. Equation (15)). Let $G = \{\psi_{\mathscr{A}}(v) : v \in \{\pm\}^2\} \subset \{\pm\}^m$ denote the image of $\mathbb{Z}_2 \times \mathbb{Z}_2$. We have $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ if and only if the cardinality of $G$ is 4. In the following we refer to $\{\pm\}^2$ as the Klein group. We can associate sublattices of $\mathbb{Z}^2$ to elements of the Klein group: to $v \in \mathbb{Z}_2 \times \mathbb{Z}_2$ we associate the sublattice $\{p \in \mathbb{Z}^2 : v^p = +\}$ (cf. Figure 20).

**Remark 4.2.1.3** (Signed regular marked subdivisions and the Klein group)**.** Consider the signed regular marked subdivision $T$ obtained from $(s,u) \in \{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$. Let $v \in \{\pm\}^2$ be arbitrary. We denote the signed regular marked subdivision defined by the element $(s \cdot \psi_{\mathscr{A}}(v), u) \in \{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$ by $T_v$. Note that $T_v$ consists of the identical marked polytopes as $T$, i.e. $|T_v| = |T|$. The sign vector containing the signs of marked points of $T_v$ equals the sign vector of $T$ multiplied with $\psi_{\mathscr{A}}(v)$. In particular, we have $T = T_{(+,+)}$. Note that we get $T_v$ from $T$ by changing signs at all points contained in the sublattice corresponding to $v$ (cf. Remark 4.2.1.2 and Figure 21). We define an equivalence relation on $\{\pm\}^m$: two elements $s, s' \in \{\pm\}^m$ are called *sign equivalent* or (*G-equivalent*) if and only if there exists an element $v \in \mathbb{Z}_2 \times \mathbb{Z}_2$ such that $s = \psi_{\mathscr{A}}(v) \cdot s'$. Equivalently, we call $s, s' \in \{\pm\}^m$ sign equivalent if and only if $s = s' \in \{\pm\}^m / G$.

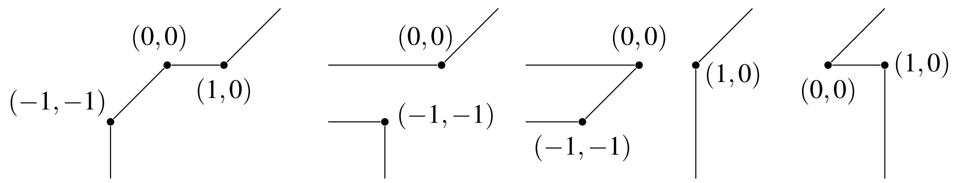
**Remark 4.2.1.4** (Charts of real plane tropical curves)**.** An element $(s,u) \in \{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$ defines a real tropical polynomial $f = \bigoplus_{\alpha \in \mathscr{A}} (s_\alpha, u_\alpha) w^\alpha \in \mathbb{TR}\left[w_1^\pm, w_2^\pm\right]$ and, therefore, a real plane tropical curve $C_f = \mathscr{T}_{\mathbb{R}}(f) \subset \mathbb{TR}^2$ (Definition 1.5.2.9). A point $w \in C_f$ has two signs, $s(w) = v \in \{\pm\}^2$, according to its two coordinates. We denote the copy of $\mathbb{R}^2$ containing the elements $w \in \mathbb{TR}^2$ with $s(w) = v$ by $\mathbb{R}^2_v$ and call it *v-chart* of $\mathbb{TR}^2$. By $C_{f,v}$ we denote the part of $C_f$ containing the solutions $w \in C_f$ with $s(w) = v$ and call $C_{f,v}$ *v-chart of* $C_f$ (or simply *v-chart*). With regard to Definition 1.5.2.9, $C_{f,v}$ contains the elements $w \in \mathbb{R}^2_v$ such that the maximum $|f|(w)$ is attained at two monomials $\alpha, \alpha' \in \mathscr{A}$ where the signs differ, i.e. $s_\alpha v^\alpha \neq s_{\alpha'} v^{\alpha'}$.

**Remark 4.2.1.5** (Duality)**.** An element $(s,u) \in \{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$ provides a real tropical Laurent polynomial $f = \bigoplus_{\alpha \in \mathscr{A}} (s_\alpha, u_\alpha) w^\alpha \in \mathbb{TR}\left[w_1^\pm, w_2^\pm\right]$ and, therefore, a real plane tropical curve $C_f$. Moreover, $(s,u)$ induces a signed regular marked subdivision $T$ of $\Delta$. In the following we explain the duality of the curve $C_f$ and the signed regular marked subdivisions $T_v$ arising from $(s,u) \in \{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$: first, note that $|T|$ is the regular marked subdivision of $\Delta$ according to the modulus $u$, i.e. $\mathscr{T}(|f|)$ is dual to $|T|$ in the usual sense. In particular, vertices of $\mathscr{T}(|f|)$ correspond to marked polygons and each edge $e$ of $\mathscr{T}(|f|)$ corresponds to an edge $E$ of a polygon. Dual edges are perpendicular. An



(A) $T_{(+,+)}$ (B) $T_{(+,-)}$ (C) $T_{(-,-)}$ (D) $T_{(-,+)}$

FIGURE 21. Signed regular marked subdivisions $T_v$ with $v \in \{\pm\}^2$.




(A) $C_{f,(+,+)}$ (B) $C_{f,(+,-)}$ (C) $C_{f,(-,-)}$ (D) $C_{f,(-,+)}$

FIGURE 22. Real plane tropical curve $\mathscr{T}_{\mathbb{R}}(f)$.

edge $e$ of $\mathscr{T}(|f|)$ is unbounded if and only if the dual edge $E$ is contained in the boundary of $T$. Now consider $\mathbb{R}^2_v$ indexed by $v \in \{\pm\}^2$. Then $T_v$ is the signed regular marked subdivision obtained from $T$ by adjusting signs with $\psi_{\mathscr{A}}$ (cf. Remark 4.2.1.3): if $(P_i, Q_i, s_{Q_i})$ is a signed marked polygon of $T$ then $(P_i, Q_i, s_{Q_i} \cdot \psi_{Q_i}(v))$ is a signed marked polygon of $T_v$ where $(s_{Q_i}\psi_{Q_i}(v))_\alpha = (s_{Q_i})_\alpha v^\alpha$ for each $\alpha \in Q_i$. Moreover, $C_{f,v}$ is dual to $T_v$ with respect to signs. In detail, $C_{f,v}$ contains a vertex dual to a signed marked polygon $(P_i, Q_i, s_{Q_i} \cdot \psi_{Q_i}(v))$ if and only if there are two elements $\alpha, \alpha' \in Q_i$ such that $(s_{Q_i} \cdot \psi_{Q_i}(v))_\alpha \neq (s_{Q_i} \cdot \psi_{Q_i}(v))_{\alpha'}$. Please note that this is precisely the sign condition of Definition 1.5.2.9. There is a (bounded or unbounded) edge $e$ in $C_{f,v}$ if and only if there is a dual edge $E$ contained in a signed marked polytope $(P_i, Q_i, s_{Q_i} \cdot \psi_{Q_i}(v))$ such that there are two elements $\alpha, \alpha' \in E \cap Q_i$ such that $(s_{Q_i} \cdot \psi_{Q_i}(v))_\alpha \neq (s_{Q_i} \cdot \psi_{Q_i}(v))_{\alpha'}$. Thus $C_{f,v}$ contains those polyhedra of $\mathscr{T}(|f|)$ that are dual to signed marked polytopes of $T_v$ with at least two vertices having different signs.

**Example 4.2.1.6.** Consider the point configuration $\mathscr{A} = p_{\{1,2\}}(2 \cdot \Delta_3) \cap \mathbb{Z}^2 = \{\alpha_1, \ldots, \alpha_6\}$ and the element $(s,u) \in \{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$ with $s = (+,-,+,-,+,+)$ and $u = (-1,0,0,-1,0,0)$. Thus we consider the real tropical polynomial $f = -1^+ \oplus 0^- x \oplus 0^+ y \oplus -1^- x^2 \oplus 0^+ xy \oplus 0^+ y^2$. The signed regular marked subdivision $T$ of $\Delta$ induced by $(s,u)$ is shown in Figure 21. Note that $T = T_{(+,+)}$. The real tropical hypersurface $C_f \subset \mathbb{TR}$ is shown in Figure 22. Note that charts of $\mathscr{T}_{\mathbb{R}}(f)$ are not necessarily connected or balanced.

**Remark 4.2.1.7** (Duality in the real/complex case)**.** The duality in the complex case explained in Remark 4.2.1.5 implies that we can deduce the type of the subdivision of a given complex plane tropical curve. In the real case this is not true, e.g. there might be signed marked polytopes in the subdivision without any influence/evidence in the real plane tropical curve.

**Remark 4.2.1.8** (Switching signs)**.** An element $(s,u) \in \{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$ defines both a polynomial $f = \bigoplus_{\alpha \in \mathscr{A}} (s_\alpha, u_\alpha) w^\alpha \in \mathbb{TR}\left[w_1^\pm, w_2^\pm\right]$ as well as a set of signed regular marked subdivisions $T_v$

with $v \in \{\pm\}^2$ (Remark 4.2.1.3 ). Consider $(s', u) \in \{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$ such that $s$ is sign equivalent to $s'$. It defines a real plane tropical curve $C_{f'}$ arising from the real tropical polynomial $f' = \bigoplus_{\alpha \in \mathscr{A}} (s'_\alpha, u_\alpha) w^\alpha \in \mathbb{TR}\left[w_1^\pm, w_2^\pm\right]$. $C_f$ and $C_{f'}$ are related as follows: since $s'$ is sign equivalent to $s$ there is an element $v \in \{\pm\}^2$ such that $s' = \psi_{\mathscr{A}}(v)s$. Thus, the signed regular marked subdivision $T' = T'_{(+,+)}$ defined by $(s', u)$ equals $T_v$, the signed regular marked subdivision obtained from $(s, u)$ with signs switched according to $v$. Since we index charts by elements of $\{\pm\}^2$ we conclude: the charts of $C_{f'}$ and $C_f$ are equal up to a permutation. This means that $T'_{v'}$ equals $T_{v \cdot v'}$ for all $v' \in \{\pm\}^2$.

**Definition 4.2.1.9** (Signed secondary fan)**.** We call $\{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$ *signed secondary fan* of the point configuration $\mathscr{A} \subset \mathbb{Z}^2$. By abuse of notation we write $s(w)$ for the sign vector of $w \in \{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$ and $|w|$ for the modulus (cf. the notations defined for $\mathbb{TR}$ in Section 1.5.1). We define an equivalence relation on $\{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$. Two elements $w, w' \in \{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$ are called equivalent if and only if $s(w)$ is sign equivalent to $s(w')$ and the regular marked subdivisions of $|w|$ and $|w'|$ are equal. Equivalently, $w, w' \in \{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$ are called equivalent if and only if $s(w) = s(w') \in \{\pm\}^m / G$ and $\sigma(|w|) = \sigma(|w'|) \subset \mathrm{Sec}_{\mathscr{A}}$ (cf. Definition 1.1.4.22).

**Remark 4.2.1.10** (Equivalence classes of $\{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$)**.** As explained in Section 1.1.4 equivalence classes of regular marked subdivisions form cones $\sigma_T$ providing $\mathrm{Sec}_{\mathscr{A}}$ with a fan structure. As $\mathbb{TR}$ has no reasonable addition (cf. Definition 1.5.1.3) the signed secondary fan $\{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$ does not carry a fan structure. We denote the equivalence class of an element $(s, u) \in \{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$ by $s \times \sigma_T$ where $S_u = T$ (cf. Definition 1.1.4.20). The representatives of $s \times \sigma_T$ form the set $G \cdot s \times \sigma_T$ where $G \cdot s$ denotes the orbit of $s$ and $\sigma_T$ is the cone such that $S_u = T$. We define the dimension of $s \times \sigma_T$ by $\dim(\sigma_T)$.

We end this section with an observation concerning the real tropical discriminant.

**Remark 4.2.1.11** (Singularities at $p = \mathbf{1}_2$)**.** By definition, for any $a \in \nabla_{\mathbb{R}}$ exists a singularity $p \in \mathscr{V}(F_a)$. It is not hard to see that

$$F_a = \sum_{\alpha \in \mathscr{A}} a_\alpha x^\alpha \text{ is singular at } p \quad \Leftrightarrow \quad F_{a \cdot \psi_{\mathscr{A}}(p)} = \sum_{\alpha \in \mathscr{A}} a_\alpha p^\alpha x^\alpha \text{ is singular at } \mathbf{1}_2.$$

In the complex case we have seen that $\mathrm{trop}(\psi_{\mathscr{A}}) = A^\top$. In particular, for an element $p \in (\mathbb{C}\{\{t\}\}^*)^2$ and $-\mathrm{val}(p) = q$ we have $\mathrm{trop}(\psi_{\mathscr{A}}(p)) = (\mathrm{trop}(p^\alpha))_{\alpha \in \mathscr{A}} = (\langle q, \alpha \rangle)_{\alpha \in \mathscr{A}} = A^\top q$. In the real case we have to take the signs into account, i.e. for $p \in T^2_{\mathbb{R}}$ and $q = \mathrm{trop}_{\mathbb{R}}(p) = (s(p), -\mathrm{val}(p))$ we have

$$\mathrm{trop}_{\mathbb{R}}(\psi_{\mathscr{A}}(p)) = (s(p^\alpha), -\mathrm{val}(p^\alpha))_{\alpha \in \mathscr{A}} = (s(p_x)^{\alpha_x} s(p_y)^{\alpha_y}, \langle |q|, \alpha \rangle)_{\alpha \in \mathscr{A}}.$$

Hence, the real tropicalization of $a \cdot \psi_{\mathscr{A}}(p)$ is

$$\mathrm{trop}_{\mathbb{R}}(\psi_{\mathscr{A}}(p) \cdot a) = (s_\alpha s(p_x)^{\alpha_x} s(p_y)^{\alpha_y}, b_\alpha + \langle |q|, \alpha \rangle)_{\alpha \in \mathscr{A}} \in \mathbb{TR}^m,$$

where $\mathrm{trop}_{\mathbb{R}}(a) = (s(a_\alpha), -\mathrm{val}(a_\alpha))_{\alpha \in \mathscr{A}} = (s_\alpha, b_\alpha)_{\alpha \in \mathscr{A}} \in \mathbb{TR}^m$. Hence, we have

$$\mathrm{trop}_{\mathbb{R}}(a \cdot \psi_{\mathscr{A}}(p)) = \mathrm{trop}_{\mathbb{R}}(a) \odot_{\mathbb{R}} \mathrm{trop}_{\mathbb{R}}(\psi_{\mathscr{A}}(p)),$$

i.e. the modulus $b$ of $\mathrm{trop}_{\mathbb{R}}(a)$ is shifted by an element in the row space of $A$ (as in the complex case) and we perform a sign vector multiplication on the signs of $a$ that are defined by $\psi_{\mathscr{A}}$.

Remark 4.2.1.11 can be generalized to the following

**Lemma 4.2.1.12.** *Let $\mathscr{A} \subset \mathbb{Z}^n$ be a finite set of cardinality $m$ and let $\psi_{\mathscr{A}} : T^n_{\mathbb{R}} \to T^m_{\mathbb{R}}$ denote the monomial map according to $\mathscr{A}$. Moreover, let $\nabla_{\mathbb{R}}$ denote the family of real Laurent polynomials with support $\mathscr{A}$ that provide singular real hypersurfaces. Then*

$$\operatorname{trop}_{\mathbb{R}}(\nabla_{\mathbb{R}}) = \operatorname{trop}_{\mathbb{R}}(\nabla_{\mathbb{R},\mathbf{1}_n}) \odot_{\mathbb{R}} \operatorname{trop}_{\mathbb{R}}(\operatorname{Im}(\psi_{\mathscr{A}})).$$

*If we restrict the monomial map $\psi_{\mathscr{A}}$ to $\{\pm 1\}^n \cong \{\pm\}^n$ and define $G = \{\psi_{\mathscr{A}}(v) : v \in \{\pm\}^n\} \subset \{\pm\}^m$ then we have* $\operatorname{trop}_{\mathbb{R}}(\operatorname{Im}(\psi_{\mathscr{A}})) = G \times \operatorname{rowspace}(A')$.

**Definition 4.2.1.13** (Lineality group)**.** We call $G \times \operatorname{rowspace}(A')$ *lineality group*.

**Remark 4.2.1.14** (Lineality group)**.** Consider the equivalence class $s \times \sigma_T \subset \{\pm\}^m \times \operatorname{Sec}_{\mathscr{A}}$. The secondary fan $\operatorname{Sec}_{\mathscr{A}}$ contains the lineality space $\operatorname{rowspace}(A')$, generated by $\mathbf{1}_m$ and the two vectors containing the $x$-coordinates and $y$-coordinates of the points in $\mathscr{A}$. To see that $\sigma_T$ contains this lineality space note that shifting the heights defined by $u \in \operatorname{relint}(\sigma_T)$ with $\mathbf{1}_m$ does not change the regular subdivision. Moreover, inclining the heights $u$ by the $x$-coordinates (or $y$-coordinates respectively) does not affect the projection of upper faces. Consequently, $s \times \sigma_T$ is invariant under $(+)^m \times \operatorname{rowspace}(A')$ since it does not change the signs or subdivision inappropriately. As the sign of an equivalence class is unique up to sign equivalence we see that $s \times \sigma_T$ is invariant under the lineality group $G \times \operatorname{rowspace}(A')$ (cf. Remark 4.2.1.10).

### 4.2.2. Tropicalization of $\nabla_{\mathbb{R},\mathbf{1}_2}$

In this section we study the real tropicalization of $\nabla_{\mathbb{R},\mathbf{1}_2}$. We recall Notation 4.1 and parts of Section 3.1.2, in particular Section 3.1.2.1, since we deal with plane curves again.

Recall that $\nabla_{\mathbb{R},\mathbf{1}_2} = \mathscr{V}(\mathscr{I})$ where $\mathscr{I} = \langle F(\mathbf{1}_2), \frac{\partial F}{\partial x}(\mathbf{1}_2), \frac{\partial F}{\partial y}(\mathbf{1}_2)\rangle \subset \mathbb{K}_{\mathbb{R}}[y_1,\dots,y_m]$ (cf. Equation (78)). Thus $\mathscr{I}$ is a linear generated ideal. According to Equation (79) its coefficient matrix equals

$$A' = \begin{bmatrix} \mathbf{1}_m^\top \\ A \end{bmatrix} \in \mathbb{Z}^{3\times m}.$$

Here, $A \in \mathbb{Z}^{2\times m}$ denotes the matrix representation of $\mathscr{A}$ and $A'$ its shift. According to Lemma 3.1.2.3 we know the following facts: if (w.l.o.g.) $\{\alpha_1,\alpha_2,\alpha_3\}$ is a set of three affinely independent vectors of $\mathscr{A}$ we obtain an equivalent coefficient matrix of $\mathscr{I}$ of the form

$$A' = \begin{bmatrix} \mathbb{E}_3 & \bar{A}' \end{bmatrix}$$

such that $\bar{A}' \in \mathbb{Q}^{3\times m-3}$. Moreover, $\left(\bar{A}'\right)_j = \begin{bmatrix} a_j & b_j & c_j \end{bmatrix}^\top$ such that

$$\alpha_j = a_j\alpha_1 + b_j\alpha_2 + c_j\alpha_3. \tag{61}$$

Hence, column $(\bar{A}')_j$ with $j \in \{4,\dots,m\}$ of $\bar{A}'$ contains the coordinates of $\alpha_j$ with respect to the affine basis $\{\alpha_1,\alpha_2,\alpha_3\}$.

**Remark 4.2.2.15** (Gale duals III)**.** From the affine basis $\{\alpha_1,\alpha_2,\alpha_3\}$ we get the Gale dual

$$G = \begin{bmatrix} -(\bar{A}')^\top & \mathbb{E}_{m-3} \end{bmatrix}. \tag{62}$$

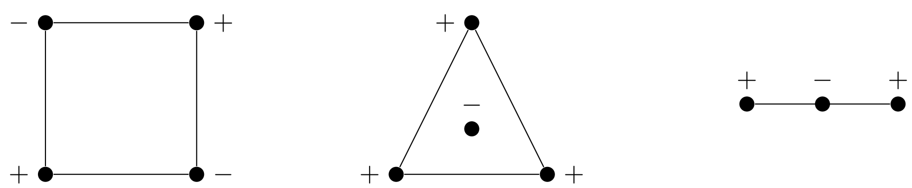
We refer to the $i$-th column of $G$ by $g_i$, i.e. $G = \begin{bmatrix} g_1 & \cdots & g_m \end{bmatrix}$. Note that $g_{i+3} = e_i$ for $1 \leq i \leq m-3$. The first three columns $g_1, g_2, g_3$ contain the $x$,$y$ and $z$ coordinates of the point configuration $\mathscr{A}$ with respect to the basis $\{\alpha_1, \alpha_2, \alpha_3\}$. For example, $(g_1)_j = -a_j$ (cf. Lemma 3.1.2.3). We refer to the first three columns as *special columns* of $G$.

We conclude that $\nabla_{\mathbb{R},\mathbf{1}_2} = \mathscr{V}(\mathscr{I}) = \ker(A')$ is a linear space. According to Remark 4.1.2.17 we can associate an oriented matroid $M$ to the linear ideal $\mathscr{I}$. The matroid $M$ describes the real tropicalization of $\ker(A')$ completely (cf. Theorem 4.1.2.18), i.e. we have $\operatorname{trop}_{\mathbb{R}}(\mathscr{V}(\mathscr{I})) = \bigcap_{C \in \mathscr{C}} \mathscr{T}_{\mathbb{R}}(l_C)$ (cf. Remark 4.1.2.17 for notations). Moreover, Theorem 4.1.2.19 shows that the $s$-chart of $\operatorname{trop}_{\mathbb{R}}(\mathscr{V}(\mathscr{I}))$ equals $\mathscr{B}^s(M)$. We studied signed Bergman fans in Section 4.1. We showed that

$$\mathscr{B}^s(M) = \bigcup_{\substack{\mathscr{F} \lhd \underline{M}: \\ \mathscr{F} \text{ is an } s\text{-flag}}} \sigma_{\mathscr{F}}.$$

Hence, we can study $\operatorname{trop}_{\mathbb{R}}(\mathscr{V}(\mathscr{I}))$ by studying its $s$-charts that are given by the signed Bergman fans $\mathscr{B}^s(M)$ where $M$ is the associated oriented matroid to $\mathscr{I}$. Due to [**BLVS$^+$99**, Proposition 4.1.13] the collection of zero sets of covectors of $M$ equals the collection of flats of $\underline{M}$. In [**MMS12a**] maximal flags of the unoriented matroid $\underline{M}$ were studied and classified. We enhance the classification to the oriented matroid $M$.

**Remark 4.2.2.16** (Affine dependencies)**.** Recall that $M[A']$ is the vector matroid arising from the linear dependencies among the columns of $A'$. As $A'$ is the matrix representation of $\mathscr{A}' = \{1\} \times \mathscr{A}$ it encodes the information about affine dependencies among the point configuration $\mathscr{A}$. The fixed point configuration $\mathscr{A}$ lives in the plane. In Figure 23 we list all planar circuits that may occur. We can




(A) Two dimensional circuit without interior point.

(B) Two dimensional circuit with an interior point.

(C) One dimensional circuit.

FIGURE 23. Planar circuits together with signs that can be realized.

recover the signs of a signed circuit $C$ of $M[A']$ from a single sign attached to a vertex of the unsigned circuit $\underline{C}$. If we equip a single arbitrary point of any of the sketched but unsigned circuits with a sign the remaining signs are uniquely determined by Radons Theorem (see also Remark 4.3.1.3). The distributions of signs correspond uniquely to signed circuits of $M[A']$ (cf. Definition 1.1.4.24).

**Remark 4.2.2.17** (Maximal flags in $\underline{M}$)**.** In [**MMS12a**, Lemma 3.7] all maximal flags of flats $\mathscr{F} \lhd \underline{M}$ were classified. With the notations introduced in Notation 4.1, Remark 4.2.2.15 and Definition 4.1.1.7, a flag of flats $\mathscr{F} = (F_1, \dots, F_{m-3}) \lhd \underline{M}$ satisfies either

(a) $|F_{m-3,m-4}| = 4$ and $|F_{j,j-1}| = 1$ for all $j \neq m-3$, or
(b) $|F_{m-3,m-4}| = 3$, $|F_{k,k-1}| = 2$ for some $k \neq m-3$ and $|F_{j,j-1}| = 1$ for all $j \neq k, m-3$.

In case (a) we have $F_{m-3,m-4} = \{i_1, i_2, i_3, i_4\} \subset [m]$ and any proper subset of $\{\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}, \alpha_{i_4}\}$ is affinely independent. Hence, $F_{m-4,m-3}$ is a (unsigned) circuit of type (A) or (B) (cf. Remark 4.2.2.16). In case (b) we have $F_{m-3,m-4} = \{i_1, i_2, i_3\}$ and $\{\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}\}$ is affinely dependent. Thus $F_{m-3,m-4}$ is a (unsigned) circuit of type (C). Moreover, all points $\alpha_r$ with $r \in F_{l,l-1}$ and $l > k$ are on the same line as $\{\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}\}$.

First, we like to know the set of covectors in $\mathscr{L}_M$ such that a given flag $\mathscr{F} \triangleleft \underline{M}$ of type (a) is a $s$-flag:

**Lemma 4.2.2.18** ($s$-flags of type (a))**.** *Let $A'$ denote the matrix representation of $\mathscr{A}' = \{1\} \times \mathscr{A}$ (cf. Notation 4.1), $G = \begin{bmatrix} g_1 & \cdots & g_m \end{bmatrix}$ a Gale dual to $A'$ (cf. Remark 4.2.2.15) and let $M = M[G]$ denote the oriented matroid associated to $G$. Moreover, let $M[A']$ be the oriented vector matroid associated to the shift of the point configuration $\mathscr{A}$ and let $\mathscr{F} \triangleleft \underline{M}$ be a maximal flag of type (a) (cf. Remark 4.2.2.17) and $s \in \{\pm\}^m$ a pure sign vector. Then $\mathscr{F}$ is an $s$-flag if and only if there exists a signed circuit $C \in \mathscr{C}(M[A'])$ such that $\underline{C} = F_{m-3,m-4}$ and $p_{\underline{C}}(s_C) = p_{\underline{C}}(s)$ (i.e. $F_{m-3,m-4} = C$) where $s_C$ denotes the sign vector associated to $C$ (cf. Remark 1.3.2.9).*

PROOF. Recall that $\mathscr{F}$ is a $s$-flag if there is a set of elements $y_1, \ldots, y_{m-3} \in \left(\mathbb{R}^{m-3}\right)^\vee$ such that $v_i = (\operatorname{sign}(\langle y_i, g_j \rangle))_{j=1,\ldots,m}$ is a covector satisfying $v_i \subseteq s$ and $v_i^0 = F_i$ (cf. Section 1.3.2). We like to work with a convenient Gale dual $G$ that allows an easy description of $\mathscr{F}$. Therefore, we rename the elements of $E = [m]$ such that $F_{m-3,m-4} = \{a, b, c, i_{m-3}\}$ and $F_{j,j-1} = \{i_j\}$ for all $j \neq m-3$. Thus $E = \{a, b, c, i_1, \ldots, i_{m-3}\}$ is the ground set, $B = \{i_1, \ldots, i_{m-3}\}$ is a basis of $\underline{M}$ and $\{\alpha_a, \alpha_b, \alpha_c\}$ is affinely independent since $F_{m-3,m-4}$ is a circuit. We consider $A'$ with resorted columns, i.e.

$$
\text{(63)} \qquad A' = \begin{array}{c} \begin{array}{cccccc} a & b & c & i_1 & \cdots & i_{m-3} \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ \alpha_a & \alpha_b & \alpha_c & \alpha_{i_1} & \cdots & \alpha_{i_{m-3}} \end{pmatrix} \end{array}.
$$

Using Remark 4.2.2.15 we work with a Gale dual of the form

$$
\text{(64)} \qquad G = \begin{array}{c} \\ i_1 \\ \vdots \\ i_{m-3} \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & i_1 & \cdots & i_{m-3} \end{array} \\ \begin{pmatrix} -a_{i_1} & -b_{i_1} & -c_{i_1} & 1 & & \\ \vdots & \vdots & \vdots & & \ddots & \\ -a_{i_{m-3}} & -b_{i_{m-3}} & -c_{i_{m-3}} & & & 1 \end{pmatrix} \end{array}.
$$

Recall that the entries of $g_a, g_b, g_c$ in row $k$ are the negatives of the coordinates of $\alpha_k$ in the affine basis $\alpha_a, \alpha_b, \alpha_c$, in particular $1 - a_{i_j} - b_{i_j} - c_{i_j} = 0$ (cf. Lemma 3.1.2.3).

First, we show "$\Rightarrow$": suppose $\mathscr{F}$ is an maximal $s$-flag. Thus all flats $F_i$ are $s$-flats, i.e. we have a covector $v_i \in \mathscr{L}_{M[G]}$ such that $v_i \subseteq s$ and $v_i^0 = F_i$ for all $i = 1, \ldots, m-3$. Let $y_i \in \left(\mathbb{R}^{m-3}\right)^\vee$ denote an element providing $v_i$, i.e. $v_i = (\operatorname{sign}(\langle g_j, y_i \rangle))_{j \in E}$. Due to the shape of $G$ we have $g_{i_j} = e_j$ for $j = 1, \ldots, m-3$. From this we can immediately conclude:

$$
\text{(65)} \qquad \forall l \in \{1, \ldots, m-3\}, j \in \{1, \ldots, m-3\} : (v_l)_{i_j} = 0 \quad \Leftrightarrow \quad (y_l)_j = 0.
$$

Now we focus on the covectors. To begin with, $v_{m-3} = \mathbf{0}_m \in \mathscr{L}_{M[G]}$ such that $v_{m-3}^0 = F_{m-3} = E$. We have $F_{m-3,m-4} = \{a, b, c, i_{m-3}\}$ and, therefore, the covector $v_{m-4}$ satisfies $(v_{m-4})_j = 0$ if and only if $j \neq a, b, c, i_{m-3}$ (cf. Remark 4.1.1.8). By definition, $v_{m-4} = (\operatorname{sign}(\langle y_{m-4}, g_j \rangle))_{j \in E}$ where

$y_{m-4} \in \left(\mathbb{R}^{m-3}\right)^\vee$. From Equation (65) we conclude that $(y_{m-4})_j = 0$ for all $j \in \{1,\dots,m-4\}$. Thus $y_{m-4} = te_{m-3}$ and, therefore, we get

$$v_{m-4} = (\operatorname{sign}(\langle y_{m-4}, g_j\rangle))_{j\in E} = (\operatorname{sign}(\langle te_{m-3}, g_j\rangle))_{j\in E} = (\operatorname{sign}(t(g_j)_{m-3}))_{j\in E}.$$

Hence, the signs of $v_{m-4}$ are determined by the $i_{m-3}$-th row of $G$.

$$e_{m-3}\cdot G = \begin{array}{cccccccc} & a & b & c & i_1 & \cdots & i_{m-4} & i_{m-3} \\ & \multicolumn{7}{c}{\begin{pmatrix} -a_{i_{m-3}} & -b_{i_{m-3}} & -c_{i_{m-3}} & 0 & \cdots & 0 & 1 \end{pmatrix}} \end{array} \tag{66}$$

As outlined in the beginning, we have the affine relation $\alpha_{i_{m-3}} = a_{i_{m-3}}\alpha_a + b_{i_{m-3}}\alpha_b + c_{i_{m-3}}\alpha_c$ with $a_{i_{m-3}} + b_{i_{m-3}} + c_{i_{m-3}} = 1$. Thus the non-zero entries of row $i_{m-3}$ of $G$ form the coefficients of a signed circuit $C \in \mathscr{C}(M[A'])$, i.e. $C^\pm = \{i \in \{a,b,c,i_{m-3}\} : (g_i)_{i_{m-3}} \gtrless 0\}$. However, all entries are non-zero as $a,b,c \notin F_j$ for $j \le m-4$. This means, for example,

$$(v_{m-4})_a = \operatorname{sign}(\langle y_{m-4}, g_a\rangle) = \operatorname{sign}(t(-a_{i_{m-3}})) = \operatorname{sign}(t)\operatorname{sign}(-a_{i_{m-3}}).$$

Hence, $(v_{m-4})_j = \pm\operatorname{sign}(j)$ for all $j \in C$. Thus the signs of the covector $v_{m-4}$ at $\{a,b,c,i_{m-3}\}$ coincide with the signs of a signed circuit. By assumption we have $v_{m-4} \subseteq s$ and, therefore, $t \gtrless 0$ such that $s_j = (v_{m-4})_j$ for all $j \in C$. Hence, $p_{\underline{C}}(s) = p_{\underline{C}}(s_C)$.

Vice versa, suppose there is a signed circuit $C \in \mathscr{C}(M[A'])$ such that $s$ coincides with $s_C$ at $\underline{C}$, i.e. for all $i \in \underline{C}$ holds: $s_i = \pm$ if and only if $i \in C^\pm$. Now, we construct elements $y_i \in \left(\mathbb{R}^{m-3}\right)^\vee$ providing covectors $v_i = (\operatorname{sign}(\langle y_i, g_j\rangle))_{j\in E} \in \mathscr{L}_{M[G]}$ such that $v_i \subseteq s$ and $v_i^0 = F_i$ for $i = 0,\dots,m-3$. We make use of the identical Gale dual $G$ constructed above. In detail, the $j$-th component of an element $y \in \left(\mathbb{R}^{m-3}\right)^\vee$ determines the sign of $v$ in component $i_j$ uniquely for $j \in \{1,\dots,m-3\}$ as

$$(v)_{i_j} = \operatorname{sign}(\langle y, g_{i_j}\rangle) = \operatorname{sign}(\langle y, e_j\rangle) = \operatorname{sign}(y_j),$$

cf. Equation (65). Consequently, by fixing the components individually, it is no problem to provide elements $y \in \left(\mathbb{R}^{m-3}\right)^\vee$ such that $v$ has desired signs in the components indexed by $\{i_1,\dots,i_{m-3}\}$. Thus, for any choice of $s$ we can define elements $y \in \left(\mathbb{R}^{m-3}\right)^\vee$ such that $v = (\operatorname{sign}(\langle y, g_j\rangle))_{j\in E}$ coincides with $s$ in the components $\{i_1,\dots,i_{m-3}\}$. The major task is to guarantee that we can do this such that the signs of the remaining components indexed by $\underline{C} = \{a,b,c,i_{m-3}\}$ coincide with those of $s$. To begin with, note that $y_{m-3} = \mathbf{0}_{m-3}$ such that $v_{m-3}^0 = \mathbf{0}_m$ in order to satisfy $v_{m-3}^0 = F_{m-3} = E$. As above, row $i_{m-3}$ of $G$ contains the coefficients that provide the affine dependency

$$\alpha_{i_{m-3}} + (-a_{i_{m-3}})\alpha_a + (-b_{i_{m-3}})\alpha_b + (-c_{i_{m-3}})\alpha_c = 0$$

where $1 - a_{i_{m-3}} - b_{i_{m-3}} - c_{i_{m-3}} = 0$. For the signs at $\underline{C}$ we have

$$s_a = \pm\operatorname{sign}(-a_{i_{m-3}}),\ s_b = \pm\operatorname{sign}(-b_{i_{m-3}}),\ s_c = \pm\operatorname{sign}(-c_{i_{m-3}}) \text{ and } s_{i_{m-3}} = \pm\operatorname{sign}(1)$$

as the circuit is unique up to a common factor $\pm 1$ and $p_{\underline{C}}(s) = p_{\underline{C}}(s_C)$. For $v_{m-4}$ it is required that $v_{m-4}^0 = E \setminus F_{m-3,m-4}$. Thus $(v_{m-4})_{i_j} = 0$ for all $j \in \{1,\dots,m-4\}$ and due to Equation (65) this implies that $(y_{m-4})_j = 0$ for all $j \in \{1,\dots,m-4\}$. Hence, $y_{m-4} = t_{m-3}e_{m-3}$. Let us consider the sign at component $a$, i.e.

$$(v_{m-4})_a = \operatorname{sign}(\langle g_a, y_{m-4}\rangle) = \operatorname{sign}(t_{m-3}(-a_{i_{m-3}})) = \pm s_a.$$

As $v_{m-4} \subseteq s$ we pick $t_{m-3} \gtrless 0$ such that $(v_{m-4})_a = s_a$. This way we can specify the entries of $v_{m-4}$ at $\underline{C}$ conform to $s$, i.e. $(v_{m-4})_j = s_j$ for $j \in \underline{C}$. The next step is to define $v_{m-5}, \dots, v_0$ iteratively, based on $v_{m-4}$ as it is necessary to satisfy $v_i \subseteq v_{i-1}$ for $i = 2, \dots, m-3$ (cf. Remark 4.1.1.8). Note that $F_{j,j-1} = \{i_j\}$ for $j \neq m-3$, i.e. $v_j$ differs from $v_{j-1}$ in component $i_j$. Hence, $y_j$ differs from $y_{j-1}$ in component $j$. For example, $F_{m-4,m-5} = \{i_{m-4}\}$ such that we want $(v_{m-5})_{i_{m-4}} = s_{i_{m-4}}$ so that we define $y_{m-5} = y_{m-4} + t_{m-4}e_{m-4}$ with $t_{m-4} \gtrless 0$ according to $s_{i_{m-4}}$. In general, we define

$$y_{m-k-1} = y_{m-k} + t_{m-k}e_{m-k}$$

for $k = 4, \dots, m-2$ with $t_{m-k} \gtrless 0$ according to $s_{i_{m-k}}$. However, the specifications of $y_{m-k}$ at components $j \in \{1, \dots, m-4\}$ may affect the signs of the components indexed by $\underline{C}$ of the corresponding covector. In detail, we have $(v_l)_{i_j} = \operatorname{sign}(t_j)$ for all $l \in \{1, \dots, m-5\}$ and $l < j \leq m-3$ whereas e.g. $(v_l)_a = \operatorname{sign}\left(\sum_{j=3}^{s+1} t_{m-j}e_{m-j}(-a_{i_{m-j}})\right)$. Therefore, we stick to the following convention: we choose a very large $t_{m-3} \in \mathbb{R}$ such that $|t_{m-3}a_{i_{m-3}}| > \sum_{j=4}^{m-1} |t_{m-j}a_{i_{m-j}}|$, i.e. the signs at $\underline{C}$ remain fixed by the choice for $t_{m-3}$ and we get $v_{m-3} = \mathbf{0}_m \subseteq v_{m-4} \subseteq \ldots \subseteq v_1 \subseteq s$ with $v_i^0 = F_i$ for all $i$. □

**Example 4.2.2.19.** Consider the point configuration $\mathscr{A} \subset \mathbb{Z}^2$ given by the columns of the matrix $A$, and the Gale dual $G$ of $A'$ as stated in Equation (67).

$$A = \begin{bmatrix} 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & -2 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 & 0 & 1 \end{bmatrix}. \tag{67}$$

We focus on the flag $\mathscr{F} \lhd \underline{M} = M[G]$ defined by

$$\emptyset \subsetneq F_1 = \operatorname{cl}(4) \subsetneq F_2 = \operatorname{cl}(4,6) \subsetneq F_3 = \operatorname{cl}(4,5,6) = E.$$

Then $F_{3,2} = \{1,2,3,5\}$, i.e. $\mathscr{F}$ is a flag of type (a). Fix the sign vector $s = (+,-,-,+,+,-) \in \{\pm\}^6$. Lemma 4.2.2.18 states that $\mathscr{F}$ is an $s$-flag if and only if there is a signed circuit $C \in \mathscr{C}(M[A'])$ such that the signs of $s$ and $s_C$ coincide at $\underline{C} = \{1,2,3,5\}$. The points $\alpha_1, \alpha_2, \alpha_3$ and $\alpha_5$ form a circuit $C$ of type (A) (cf. Remark 4.2.2.16) with $C^+ = \{1,5\}$ and $C^- = \{2,3\}$. Note that row 2 of $G$ (containing the negatives of the coordinates of $\alpha_5$ in the affine basis $\alpha_1, \alpha_2, \alpha_3$) contains the coefficients of the affine relation $\alpha_5 - a_5\alpha_1 - b_5\alpha_2 - c_5\alpha_3 = 0$ with $1 - a_5 - b_5 - c_5 = 0$. Thus $p_{\underline{C}}(s) = (+,-,-,+) = p_{\underline{C}}(s_C)$ and we conclude that $\mathscr{F}$ is an $s$-flag. Let us specify the set of covectors such that $\mathscr{F}$ becomes an $s$-flag. First, we define $v_3 = \mathbf{0}_6$ providing $F_3 = E = v_3^0$. The next step is $v_2$ providing $v_2^0 = F_2 = \{4,6\}$. We immediately see that any $y_2 \in \left(\mathbb{R}^3\right)^\vee$ providing $v_2$ satisfies $(y_2)_1 = (y_2)_3 = 0$, i.e. $y_2 = te_2$. In particular:

$$v_2 = (\operatorname{sign}(\langle y_2, g_j \rangle))_{j=1,\dots,6} = (\operatorname{sign}(\langle te_2, g_j \rangle))_{j=1,\dots,6} = (\operatorname{sign}(t), \operatorname{sign}(-t), \operatorname{sign}(-t), 0, \operatorname{sign}(t), 0).$$

For any $t > 0$ the signs of $v_2$ coincide with those of $s$ at $\underline{C}$. As $F_1 \subseteq F_2$ we have $v_2 \subseteq v_1$ (cf. Remark 4.1.1.8), i.e. $v_1$ has to inherit the signs of $v_2$. Since $F_{2,1} = \{6\}$ we want $(v_1)_6 \neq 0$ and, therefore, $y_1 = y_2 + t'e_3$ with $t' \neq 0$. In order to prevent that signs of $v_1$ differ from signs of $v_2$ at $E \setminus F_2$ due to $t'$ we pick $t$ as large as necessary so that $t$ determines the signs at $E \setminus F_2$. Thus $v_1$ differs from $v_2$ at only in the 6-th coordinate. Note that the sign of $(v_1)_6$ is determined by $t'$, i.e. whatever sign $s_6$ is we can guarantee that $(v_1)_6 = s_6$. We see that $\mathscr{F}$ is an $s$-flag for all $s \in \{\pm\}^6$ with $p_{\underline{C}}(s) = p_{\underline{C}}(s_C)$.

The situation for maximal flags $\mathscr{F} \lhd \underline{M}$ of type (b) is quite similar but not identical. The proof of the previous lemma reveals that the signs at $F_{m-3,m-4}$ are related whereas all signs at $F_{j,j-1}$ with $j \neq m-3$ are unrelated. For a flag $\mathscr{F}$ of type (b) we have $|F_{m-3,m-4}| = 3$, $|F_{k,k-1}| = 2$ for some $k \neq m-3$ and $|F_{j,j-1}| = 1$ for all $j \neq k, m-3$. As one might expected the signs in $F_{m-3,m-4}$ and $F_{k,k-1}$ are related:

**Lemma 4.2.2.20** ($s$-flags of type (b)). *Let $A'$ denote the matrix representation of $\mathscr{A}' = \{1\} \times \mathscr{A}$ (cf. Notation 4.1), $G = \begin{bmatrix} g_1 & \cdots & g_m \end{bmatrix}$ a Gale dual to $A'$ (cf. Remark 4.2.2.15) and let $M = M[G]$ denote the oriented matroid associated to $G$. Moreover, let $M[A']$ be the oriented vector matroid associated to the shift of the point configuration $\mathscr{A}$ and let $\mathscr{F} \lhd \underline{M}$ be a maximal flag of type (b) (cf. Remark 4.2.2.17), i.e. $|F_{m-3,m-4}| = 3$ and there is some $k \neq m-3$ such that $|F_{k,k-1}| = 2$ and $|F_{j,j-1}| = 1$ for all $j \neq k, m-3$ and let $s \in \{\pm\}^m$ be a pure sign vector. Recall that the affine span of the points in $\mathscr{A}$ corresponding to elements of $F_{m-3,m-4}$ forms a line $L$. Then $\mathscr{F}$ is an $s$-flag if and only if there exists a signed circuit $C \in \mathscr{C}(M[A'])$ such that $\underline{C} = F_{m-3,m-4}$ and $p_{\underline{C}}(s) = p_{\underline{C}}(s_C)$ and, if $F_{k,k-1} = \{a,b\}$, $s_a = \pm s_b$ depending on whether $\alpha_a$ and $\alpha_b$ are on different/the same side of $L$.*

PROOF. Similarly to the proof of Lemma 4.2.2.18 we begin with specifying a Gale dual $G$ that is convenient for $\mathscr{F}$. We reindex such that $F_{m-3,m-4} = \{i_{m-3}, a, b\}$, $F_{k,k-1} = \{i_k, c\}$ and $F_{j,j-1} = \{i_j\}$ for all $j \neq k, m-3$. Thus $E = \{a, b, c, i_1, \ldots, i_{m-3}\}$ and the three elements $a, b, c$ provide the affine independent set $\{\alpha_a, \alpha_b, \alpha_c\}$. This is true since $\alpha_a, \alpha_b$ are affinely independent as $F_{m-3,m-4}$ is a circuit and $\alpha_c$ is not contained in the line $L$ containing the circuit $F_{m-3,m-4}$. Moreover, $B = \{i_1, \ldots, i_{m-3}\}$ forms a basis of $\underline{M}$. We consider $A'$ with re-sorted columns (cf. Equation (63)). The first three special columns of the Gale dual $G$ obtained from $A'$ using the construction in Remark 4.2.2.15 contain the negative coordinates of $\alpha_{i_j}$ with respect to the basis $\{\alpha_a, \alpha_b, \alpha_c\}$ for $j \in [m-3]$ (cf. Equation (64)).
First, we show "$\Rightarrow$": suppose $\mathscr{F}$ is a maximal $s$-flag, i.e. all flats $F_i$ are $s$-flats and there is a covector $v_i \in \mathscr{L}_{M[G]}$ such that $v_i \subseteq s$ and $v_i^0 = F_i$ for all $i \in E = [m-3]$. In particular, $v_{m-3} = \mathbf{0}_m \in \mathscr{L}_{M[G]}$ as we require $v_{m-3}^0 = F_{m-3} = E$. We have $F_{m-3,m-4} = \{i_{m-3}, a, b\}$ and, therefore, the element $v_{m-4}$ satisfies $(v_{m-4})_j = 0$ if and only if $j \in E \setminus \{i_{m-3}, a, b\}$. Let $y_{m-4} \in \left(\mathbb{R}^{m-3}\right)^\vee$ be an element such that $v_{m-4} = (\mathrm{sign}(\langle y_{m-4}, g_j \rangle))_{j \in E}$. Recall that we have $g_{i_j} = e_j$ for $j \in \{1, \ldots, m-3\}$ such that

$$(v_{m-4})_{i_j} = \mathrm{sign}(\langle y_{m-4}, g_{i_j} \rangle) = \mathrm{sign}(\langle y_{m-4}, e_j \rangle) = \mathrm{sign}((y_{m-4})_j)$$

for $j \in \{1, \ldots, m-3\}$. Hence, $(y_{m-4})_j = (0, \ldots, 0, t) = te_{m-3} \in \left(\mathbb{R}^{m-3}\right)^\vee$ for some $t \neq 0$. Now, consider the row of $G$ indexed by $i_{m-3}$ (cf. Equation (66)). We have $(g_a)_{i_{m-3}} = -a_{i_{m-3}}$ and $(g_b)_{i_{m-3}} = -b_{i_{m-3}}$ whereas $(g_c)_{i_{m-3}} = -c_{i_{m-3}} = 0$ as $F_{m-3,m-4} = \{i_{m-3}, a, b\}$ is a circuit. According to Lemma 3.1.2.3, these coordinates provide the affine relation $\alpha_{i_{m-3}} = a_{i_{m-3}}\alpha_1 + b_{i_{m-3}}\alpha_2$ (or, equivalently, $\alpha_{i_{m-3}} + (-a_{i_{m-3}})\alpha_1 + (-b_{i_{m-3}})\alpha_2 = 0$) with $1 - a_{i_{m-3}} - b_{i_{m-3}} - c_{i_{m-3}} = 0$. Since we have $(g_{i_{m-3}})_{i_{m-3}} = 1$, the $i_{m-3}$-th row of $G$ contains the coefficients of a signed circuit $C$ such that $\underline{C} = F_{m-3,m-4}$. Recall that $(v_{m-4})_j \neq 0$ if and only if $j = a, b, i_{m-3}$. We know that $y_{m-4} = te_{i_{m-3}}$. Thus, for example,

$$(v_{m-4})_a = \mathrm{sign}(\langle y_{m-4}, g_a \rangle) = \mathrm{sign}(t(-a_{i_{m-3}})) = \mathrm{sign}(t)\mathrm{sign}(-a_4).$$

Hence, $(v_{m-4})_j = \pm\mathrm{sign}(j)$ for $j \in C$. Since $v_{m-4} \subseteq s$ we have $t \gtrless 0$ such that $s_j = (v_{m-4})_j$ for $j = a, b, i_{i_{m-3}}$, i.e. there is a signed circuit $C \in \mathscr{C}(M[A'])$ such that $\underline{C} = F_{m-3,m-4}$ and $p_{\underline{C}}(s) = p_{\underline{C}}(s_C)$.

For $m-4 \geq j \geq k+1$ we have $F_{j,j-1} = i_j$ and $\alpha_{i_j}$ is contained in the affine line spanned by $\{\alpha_a, \alpha_b\}$. Hence, we have $(g_c)_{i_j} = -c_{i_j} = 0$ for all $j$ satisfying $k+1 \leq j \leq m-3$ (cf. Equation (64)). Since $F_{k,k-1} = \{i_k, c\}$ we have $(g_c)_{i_k} = -c_{i_k} \neq 0$ — otherwise, $c \notin F_{k,k-1}$.

$$G = \begin{array}{c|c} & \begin{array}{cccccccccc} a & b & c & i_1 & \cdots & i_{k-1} & i_k & i_{k+1} & \cdots & i_{m-3} \end{array} \\ \begin{array}{c} i_1 \\ \vdots \\ i_{k-1} \\ i_k \\ i_{k+1} \\ \vdots \\ i_{m-3} \end{array} & \begin{pmatrix} -a_{i_1} & -b_{i_1} & -c_{i_1} & 1 & & & & & & \\ \vdots & \vdots & \vdots & & \ddots & & & & & \\ -a_{i_{k-1}} & -b_{i_{k-1}} & -c_{i_{k-1}} & & & 1 & & & & \\ -a_{i_k} & -b_{i_k} & -c_{i_k} \neq 0 & & & & 1 & & & \\ -a_{i_{k+1}} & -b_{i_{k+1}} & 0 & & & & & 1 & & \\ \vdots & \vdots & \vdots & & & & & & \ddots & \\ -a_{i_{m-3}} & -b_{i_{m-3}} & 0 & & & & & & & 1 \end{pmatrix} \end{array}. \tag{68}$$

Moreover, $(v_{k-1})_{i_k} \neq 0$ and consequently $(y_{k-1})_k \neq 0$. This implies $(v_{k-1})_c \neq 0$ and $(v_{k-1})_{i_k} \neq 0$. Recall that row $i_k$ of $G$ contains the coordinates of $\alpha_{i_k}$ with respect to the affine basis $\{\alpha_a, \alpha_b, \alpha_c\}$. Hence, the indices of the non-zero entries of the $i_k$-th row of $G$ form a circuit $C' \in \mathscr{C}(M[A'])$. Due to the reindexing we have $F_{k-1} = \{i_1, \ldots, i_{k-1}\}$, i.e. $(y_{k-1})_j = 0$ for all $j \in \{1, \ldots, k-1\}$. By assumption, $v_{k-1} \subseteq s$, i.e. we have $(y_{k-1})_k \gtrless 0$ such that

$$(v_{k-1})_{i_k} = \operatorname{sign}(\langle y_{k-1}, g_{i_k} \rangle) = \operatorname{sign}(\langle y_{k-1}, e_k \rangle) = \operatorname{sign}((y_{k-1})_k) = s_{i_k}.$$

However, this implies $(v_{k-1})_c = \operatorname{sign}(\langle y_{k-1}, g_c \rangle) = \operatorname{sign}((y_{k-1})_k(-c_{i_k})) = s_c$, i.e. we get the condition $s_c = \operatorname{sign}(-c_{i_k}) s_{i_k}$ on the signs at $i_k$ and $c$ in $s$. The sign of $(-c_{i_k})$ purely depends on the relative position of $\alpha_{i_k}$ to $\alpha_a, \alpha_b$ and $\alpha_c$. Let $L_{ab}$ denote the affine line through $\alpha_a$ and $\alpha_b$.

- Let $\alpha_c$ and $\alpha_{i_k}$ be separated by $L_{ab}$ (cf. Figure 25). Suppose $\alpha_a, \alpha_b, \alpha_c, \alpha_{i_k}$ form a circuit of type (A). Then $\operatorname{sign}(-c_{i_k}) = +$ since $\operatorname{sign}(1) = +$, as 1 is the coefficient of $\alpha_{i_k}$ in the affine relation. If $\alpha_a, \alpha_b, \alpha_c, \alpha_{i_k}$ form a circuit of type (B) then $\operatorname{sign}(-c_{i_k}) = +$ as well since $\operatorname{sign}(1) = +$ and $L_{ab}$ separates $\alpha_c$ and $\alpha_{i_k}$. If $\alpha_a, \alpha_b, \alpha_c, \alpha_{i_k}$ is a circuit of type (C) then (w.l.o.g.) $-a_{i_k} = 0$ and due to $\operatorname{sign}(1) = +$ we have $\operatorname{sign}(-c_{i_k}) = +$. We conclude: if $\alpha_c$ and $\alpha_{i_k}$ are separated by $L_{ab}$ then $s_{i_k} = s_c$.
- Assume that $\alpha_c$ and $\alpha_{i_k}$ are on the same side of $L_{ab}$ (cf. Figure 26). If $\alpha_a, \alpha_b, \alpha_c, \alpha_{i_k}$ form a circuit of type (A) then $\operatorname{sign}(-c_{i_k}) = -$ since the sign of the coefficient of $\alpha_{i_k}$, 1, is $+$. If $\alpha_a, \alpha_b, \alpha_c, \alpha_{i_k}$ form a circuit of type (B) then either $\alpha_c$ or $\alpha_{i_k}$ is the interior point. We conclude that their signs differ, and since $\operatorname{sign}(1) = +$ we have $\operatorname{sign}(-c_{i_k}) = -$. If $\alpha_a, \alpha_b, \alpha_c, \alpha_{i_k}$ form a circuit of type (C) we have (w.l.o.g.) $-a_{i_k} = 0$. Then $\alpha_c$ or $\alpha_{i_k}$ is the interior point — otherwise $L_{ab}$ would separate $\alpha_c$ and $\alpha_{i_k}$. However, this implies $\operatorname{sign}(-c_{i_k}) \neq \operatorname{sign}(1) = +$. We conclude: if $\alpha_c$ and $\alpha_{i_k}$ are on the same side of $L_{ab}$ then $s_{i_k} \neq s_c$.

Consequently, we have $s_c = s_{i_k}$ if and only if $L_{ab}$ separates $\alpha_c$ and $\alpha_{i_k}$.

For the other direction we use identical notations. Suppose we have a circuit $C \in \mathscr{C}(M[A'])$ such that $p_{F_{m-3,m-4}}(s) = s_C$ and $s_c = \pm s_{i_k}$, depending on the relative position of $\alpha_c$ and $\alpha_{i_k}$ to $L_{ab}$. The goal is to construct elements $y_i \in (\mathbb{R}^m)^\vee$ such that $v_i = (\operatorname{sign}(\langle y_i, g_j \rangle))_{j \in E}$ for $i = 1, \ldots, m-3$ satisfying $v_i^0 = F_i$ and $v_i \subseteq s$. We begin with $v_{m-3}$. We require $v_{m-3}^0 = F_{m-3} = E$, i.e. $v_{m-3} = \mathbf{0}_m$. Recall from the proof of Lemma 4.2.2.18 that, for $j \in \{1,\dots,m-3\}$, the $j$-th component of $y \in \left(\mathbb{R}^{m-3}\right)^{\vee}$ determines the sign of $v = (\operatorname{sign}(\langle g_j, y\rangle))_{j\in E}$ in component $i_j$ as $g_{i_j} = e_j$. Consequently, it is no problem to provide elements $y \in \left(\mathbb{R}^{m-3}\right)^{\vee}$ such that the corresponding covectors have desired signs at components indexed by $\{i_1,\dots,i_{m-3}\}$. It remains to show that the signs at $\{a,b,c\}$ remain unchanged, i.e. as initially required by the assumption.

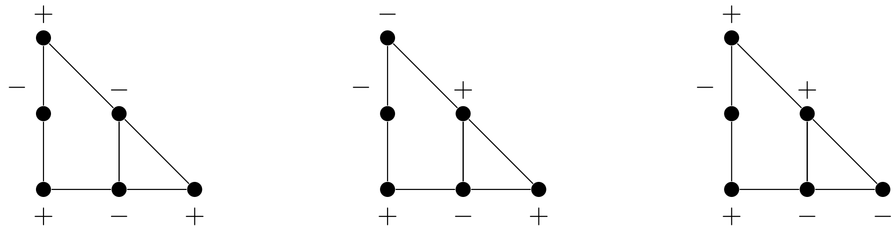
Due to Equation (65) we have $g_{i_j} = e_j$, i.e. $y_{m-3} = \mathbf{0}_{m-3}$ provides $v_{m-3}$. We look for $v_{m-4}$ satisfying $v^0_{m-4} = F_{m-4}$. Note that $F_{m-3,m-4} = \{a,b,i_{m-3}\}$, i.e. we want $(v_{m-4})_j \neq 0$ if and only if $j \in \{a,b,i_{m-3}\}$. Equivalently, $(v_{m-4})_j = 0$ if and only if $j \in \{i_1,\dots,i_{m-4},c\}$. As $(y_{m-4})_j$ determines $(v_{m-4})_{i_j}$ we conclude that $(y_{m-4})_j = 0$ for all $j \in \{1,\dots,m-4\}$. Hence, $y_{m-4} = t_{m-3}e_{m-3}$ with $t_{m-3} \neq 0$. Recall that row $i_{m-3}$ of $G$ contains the negative coefficients of the affine relation $a_{i_{m-3}}\alpha_a + b_{i_{m-3}}\alpha_b = \alpha_{i_{m-3}}$. The indices form the signed circuit $C$. As $p_{\underline{C}}(s) = p_{\underline{C}}(s_C)$ we have $s_a = \pm\operatorname{sign}(-a_{i_{m-3}}), s_b = \pm\operatorname{sign}(-b_{i_{m-3}})$ and $s_{i_{m-3}} = \pm\operatorname{sign}(1)$ as the circuit is unique up to a common factor $\pm 1$. We have seen that $y_{m-4} = t_{m-3}e_{m-3}$. We have $t_{m-3} \gtrless 0$ such that

$$(v_{m-4})_a = \operatorname{sign}(\langle g_a, y_{m-4}\rangle) = \operatorname{sign}(\langle g_a, t_{m-3}e_{m-3}\rangle) = \operatorname{sign}(t_{m-3}(g_a)_{m-3}) = s_a. \tag{69}$$

Then $(v_{m-4})_j = \operatorname{sign}(j)$ for all $j \in C$, i.e. $v_{m-4} \subseteq s$. We define $v_{m-5},\dots,v_0$ iteratively, based on $y_{m-4}$ (cf. Remark 4.1.1.8). Due to the reindexing we have $F_{j,j-1} = \{i_j\}$ for all $j \neq k, m-3$. We define $y_{m-j-1} = y_{m-j} + t_{m-j}e_{m-j}$ iteratively for $j = 4,\dots,m-1$ with $t_{m-j} \gtrless 0$ according to $s_{i_{m-j}}$. This way we can specify all signs according to those of $s$ at $\{i_1,\dots,i_{m-4}\}$. The signs at $a,b,i_{m-3}$ were fixed with $y_{m-4}$. It remains to show what happens to the sign at $c$. This is determined in $v_{k-1}$: $v_k$ and $v_{k-1}$ differ at two indices, $i_k$ and $c$. Recall that $y_{k-1} = y_k + t_k e_k$. We pick $t_k \gtrless 0$ such that $(v_{k-1})_{i_k} = s_{i_k}$. Also note that $(g_c)_j = 0$ for all $j \in \{i_{k+1},\dots,i_{m-3}\}$, i.e. $(v_l)_c = 0$ for all $l \in \{k,\dots,m-3\}$. Now, as $(y_{k-1})_k \neq 0$ we have $(v_{k-1})_c = \operatorname{sign}(\langle g_c, y_{k-1}\rangle) = \operatorname{sign}((g_c)_k t_k) \neq 0$ because $(g_c)_k \neq 0$. This is true since $\alpha_{i_k}$ is not contained in $L_{ab}$. However, we have $(g_c)_k = -c_{i_k}$. Moreover, row $i_k$ of $G$ contains the coefficients of the affine relation between $\alpha_{i_k}, \alpha_a, \alpha_b$ and $\alpha_c$. As $\operatorname{sign}(i_k) = +$ we can determine the sign of $c$:

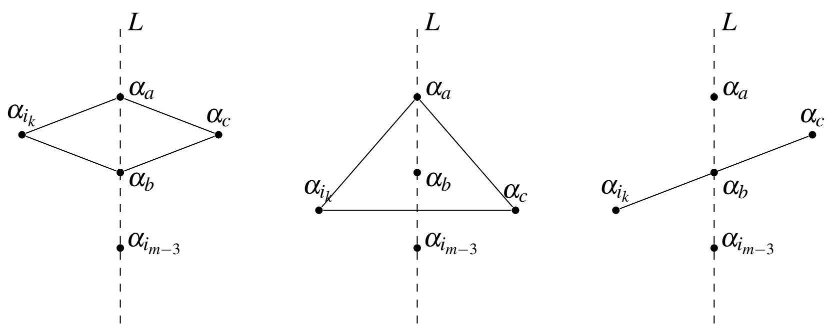
- Suppose $L_{ab}$ separates $\alpha_c$ and $\alpha_{i_k}$ (cf. Figure 25). If $\alpha_a, \alpha_b, \alpha_c$ and $\alpha_{i_k}$ form a circuit of type (A) then $\operatorname{sign}(c) = +$. If it is a circuit of type (B) then either $\alpha_a$ or $\alpha_b$ is the interior point. Hence, $\operatorname{sign}(c) = +$. If it is a circuit of type (C) we have (w.l.o.g.) $a_{i_k} = 0$, i.e. $\alpha_c$ is on a line with $\alpha_{i_k}$ and $\alpha_b$. Consequently, $\operatorname{sign}(c) = \operatorname{sign}(i_k) = +$.
- Suppose $L_{ab}$ does not separate $\alpha_c$ and $\alpha_{i_k}$ (cf. Figure 26). If $\alpha_a, \alpha_b, \alpha_c$ and $\alpha_{i_k}$ form a circuit of type (A) we have $\operatorname{sign}(c) = - \neq \operatorname{sign}(i_k) = +$. If it is a circuit of type (B) then either $\alpha_{i_k}$ or $\alpha_c$ is the interior point — their signs are not equal, i.e. $\operatorname{sign}(c) = -$. If it is a circuit of type (C) we have (w.l.o.g.) $a_{i_k} = 0$, i.e. $\alpha_c$, $\alpha_{i_k}$ and $\alpha_b$ are on a line. Thus (w.l.o.g.) $\alpha_c$ is the interior point and $\alpha_c$ and $\alpha_{i_k}$ are the boundary points such that $\operatorname{sign}(c) = -$ since $\operatorname{sign}(i_k) = +$.

We conclude that we get a sign at $c$ and $i_k$ in $v_{k-1}$, depending on the affine relations explained above. By assumption $s_c$ and $s_{i_k}$ satisfy the identical condition, i.e. we can pick $t_k \gtrless 0$ such that $s_{i_k} = (v_{k-1})_{i_k}$ and $s_c = (v_{k-1})_c$ in order to satisfy $v_{k-1} \subseteq s$. In order to guarantee that all subsequent covectors $v_j$ with $j \leq k-2$ stick to these signs at $c$ and $i_k$ we pick $t_k$ big enough (cf. discussions about $t_{m-3}$ before). $\square$



(A) $s = (+,-,-,+,-,+)$. (B) $s = (+,-,-,+,+,-)$ (C) $s = (+,-,-,-,+,+)$

FIGURE 24. Different sign distributions for the regular marked subdivision of $\Delta = \mathrm{conv}(0, 2e_1, 2e_2)$.




(A) Circuit of type (A). (B) Circuit of type (B). (C) Circuit of type (C).

FIGURE 25. $L$ separates $\alpha_c$ and $\alpha_{i_k}$.

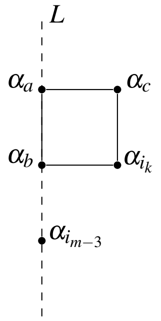
**Example 4.2.2.21.** We continue Example 4.2.2.19, i.e. we use the identical point configuration $\mathscr{A} \subset \mathbb{Z}^2$, matrix $A'$ and Gale dual $G$. We focus on the flag $\mathscr{F} \triangleleft \underline{M} = M[G]$ defined by

$$\emptyset \subsetneq F_1 = \mathrm{cl}(4) \subsetneq F_2 = \mathrm{cl}(4,5) \subsetneq F_3 = \mathrm{cl}(4,5,6) = E.$$

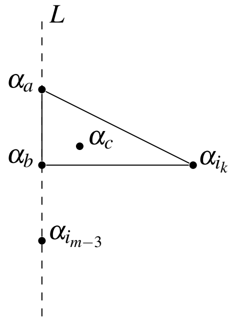
Then $F_{3,2} = \{1,3,6\}$ and $F_{2,1} = \{2,5\}$, i.e. $\mathscr{F}$ is a flag of type (b). Similarly to the case of flags of type (a) discussed in Example 4.2.2.19 we have to care about the signs at $F_{3,2}$ and $F_{2,1}$. The flag $\mathscr{F}$ is an $s$-flag if and only if the signs indexed by $F_{3,2}$ follow a sign distribution of a signed circuit in $M[A']$ whose support equals $F_{3,2}$ and the signs at $F_{2,1}$ need to be different. In Figure 24 we see three sign distributions on the regular marked subdivision of $\Delta = \mathrm{conv}(\mathscr{A})$ induced by $u \in \mathrm{relint}(\sigma_{\mathscr{F}})$. With respect to the first sign vector $s = (+,-,-,+,-,+)$ the flag $\mathscr{F}$ is no $s$-flag because the points $\alpha_2$ and $\alpha_5$ have equal signs. The second sign vector $s = (+,-,-,+,+,-)$ fails to provide an $s$-flag $\mathscr{F}$ since the signs at $\alpha_1, \alpha_3$ and $\alpha_6$ cannot be realized by a signed circuit $C \in M[A']$. However, the third sign vector $s = (+,-,-,-,+,+)$ meets all requirements in Lemma 4.2.2.20, i.e. $\mathscr{F}$ is an $(+,-,-,-,+,+)$-flag.

### 4.2.3. A First Classification of Real Plane Tropical Curves with a Singularity in $\mathbf{1}_2$

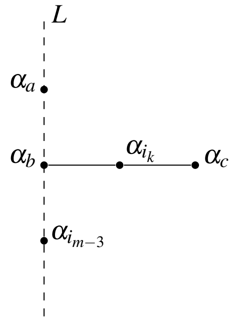
In this section we investigate the signed regular marked subdivisions arising from weight classes of $\mathrm{trop}_{\mathbb{R}}(\nabla_{\mathbb{R},\mathbf{1}_2})$ (cf. [**MMS12a**, Section 3.2]). We focus on the $(+,+)$-chart/subdivision as we fixed the singularity at $p = \mathbf{1}_2$. We stick to notations defined in Remark 4.2.2.16 and Remark 4.2.2.17. In




(A) Circuit of type (A).




(B) Circuit of type (B).




(C) Circuit of type (C).

FIGURE 26. $L$ does not separate $\alpha_c$ and $\alpha_{i_k}$.

particular, for a given real tropical Laurent polynomial $f \in \mathbb{TR}\left[w_1^{\pm}, w_2^{\pm}\right]$ such that $\mathbf{0}_2^+ \in \mathscr{T}_{\mathbb{R}}(f)$ is a singularity we classify the local picture around $\mathbf{0}_2^+ \in C_{f,(+,+)}$ (cf. Theorem 4.2.3.24).

**Remark 4.2.3.22** (Classification of signed regular marked subdivisions)**.** Let $\mathscr{F} \lhd \underline{M}$ be a maximal flag and let $s \in \{\pm\}^m$ be a sign vector.

(a) Suppose we have a maximal $s$-flag $\mathscr{F}$ of type (a) such that $\sigma_{\mathscr{F}} \subset \operatorname{trop}_{\mathbb{R}}(\nabla_{\mathbb{R},\mathbf{1}_2})$. Consequently, $F_{m-3,m-4} = \{a,b,c,d\}$ and its corresponding signed circuit is of type (A) or (B). We have $w_a = w_b = w_c = w_d \geq w_j$ for all $j \in F_k$ with $k \leq m-4$. Hence, the points $\alpha_a, \alpha_b, \alpha_c$ and $\alpha_d$ get the highest but equal height. Thus $\operatorname{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_d)$ is a triangle or quadrangle in the signed regular marked subdivision $T = T_{(+,+)}$ obtained from any $(s,u) \in \sigma_{\mathscr{F}}$. Let $f$ denote the real tropical polynomial obtained from $(s,u)$. Note that the coefficients at monomials corresponding to the vertices of the circuit $\operatorname{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_d)$ have equal modulus. In particular:

(i) If $F_{m-3,m-4} = \{a,b,c,d\}$ corresponds to a signed circuit of type (A) then the signed regular marked subdivision $T$ contains a quadrangle $\operatorname{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_d)$. The signs at vertices coincide with the signed circuit, i.e. they are alternating (cf. Figure 23a). Hence, any edge of the quadrangle is dual to an (maybe unbounded) edge in $C_{f,(+,+)}$. The quadrangle itself is dual to a vertex in $C_{f,(+,+)}$. There, $|f|$ attains the maximum at all four terms corresponding to the vertices. Since the coefficients of all terms have equal modulus the vertex is at $(0,0)$. As explained before the four edges of the quadrangle are dual to edges of $C_{f,(+,+)}$ attached to $(0,0)$, i.e. the quadrangle is dual to a 4-valent vertex at $(0,0)$ in $C_{f,(+,+)}$.

(ii) If $F_{m-3,m-4} = \{a,b,c,d\}$ is a signed circuit of type (B) then $\operatorname{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_d)$ is a triangle with an interior point in the signed regular marked subdivision $T$. The signs at vertices coincide with the signs of $C$, i.e. the interior point obtains (w.l.o.g.) "$-$" and the exterior points forming the triangle get "$+$" (cf. Figure 23b). The triangle is dual to a vertex in $C_{f,(+,+)}$ and we can solve for its coordinates. The maximum of $|f|$ obtained from the real tropical polynomial $f$ attains its maximum at all four terms and since the sign at the interior point differs from the signs of exterior points $(0,0) \in C_{f,(+,+)}$ corresponds to the triangle. However, the edges forming the boundary of conv$(\alpha_a, \alpha_b, \alpha_c, \alpha_d)$ are incident to vertices with equal signs. Hence, $(0,0) \in C_{f,(+,+)}$ is an isolated point as the edges of the triangle have no dual counterpart in $C_{f,(+,+)}$.

(b) Suppose we have a $s$-flag $\mathscr{F}$ of type (b) such that $F_{m-3,m-4} = \{a,b,c\}$ and $F_{k,k-1} = \{d,e\}$ for some $k \neq m-3$. Thus $w_a = w_b = w_c \geq w_j$ for all $j \in F_l$ with $l \leq m-4$ and the points $\alpha_a, \alpha_b$ and $\alpha_c$ get highest and equal height. Consequently, we see a signed circuit conv$(\alpha_a, \alpha_b, \alpha_c)$ of type (C) (cf. Figure 23c) in the signed regular marked subdivision $T = T_{(+,+)}$ formed by the points $\alpha_a, \alpha_b$ and $\alpha_c$. The points $\alpha_d$ and $\alpha_e$ also get equal height. Moreover, these points get highest height of all points that are not on the line $L_{abc}$ through $\alpha_a, \alpha_b$ and $\alpha_c$. As $\alpha_a, \alpha_b$ and $\alpha_c$ get the highest height we can be sure that the signed circuit conv$(\alpha_a, \alpha_b, \alpha_c)$ is part of the subdivision $T$ induced by an element $(s,u) \in \sigma_{\mathscr{F}}$. Hence, $C_{f,(+,+)}$ contains a dual edge of weight at least two. This edge passes through $(0,0)$. Consider the real tropical polynomial $f$. The terms of $f$ corresponding to $\alpha_a, \alpha_b$ and $\alpha_c$ have equal coefficients. Then $f$ attains its maximum for $(0,0)$ at the terms corresponding to $\alpha_a, \alpha_b$ and $\alpha_c$. There is at least one polygon having the signed circuit conv$(\alpha_a, \alpha_b, \alpha_c)$ as a face. Such a polygon contains at least one more vertex which has lower height. Then $C_{f,(+,+)}$ contains a verticex dual to each polygon as the signed circuit conv$(\alpha_a, \alpha_b, \alpha_c)$ that is of type (C) has alternating signs. Hence, there is a line through $(0,0)$.

**Remark 4.2.3.23** ([**MMS12a**, Remark 3.9])**.** Unfortunately, it is not possible to say much more about the subdivisions $T_{(+,+)}$ arising from weight classes of type (b). The reason for this is that we cannot say how the polygons adjacent to the signed circuit conv$(\alpha_a, \alpha_b, \alpha_c)$ look like (cf. notations in Remark 4.2.3.22 (b)). Even though the points $\alpha_d$ and $\alpha_e$ get highest height of all points not on the line $L_{abc}$ through $\alpha_a, \alpha_b$ and $\alpha_c$ we cannot predict whether $(\alpha_d, w_d)$ and $(\alpha_e, w_e)$ are contained in an upper face of the shifted polytope $\Delta_w$. Consider the point configuration (as shown in Remark 3.9 in [**MMS12a**]) in Figure 27. The weight class $\sigma_{\mathscr{F}}$ we consider is given by the flag

$$\mathscr{F}: \quad \emptyset \subsetneq F_1 = \{h\} \subsetneq F_2 = \{h,g\} \subsetneq F_3 = \{h,g,f\} \subsetneq F_4 = \{h,g,f,d,e\} \subsetneq F_5 = \{h,g,f,d,e,a,b,c\}.$$

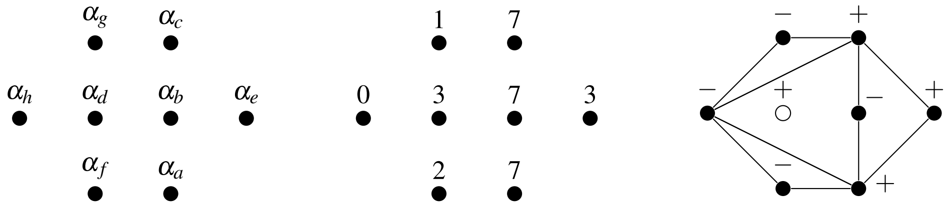
We see that $(\alpha_d, w_d)$ is not contained in an upper face of $\Delta_w$ and, therefore, not part of the subdivision. In particular, $\alpha_d$ is not marked.

However, this classification of signed regular marked subdivisions corresponding to flags of $M$ allows to state the following

THEOREM 4.2.3.24 (Singular real plane tropical curves with a singularity at $\mathbf{1}_2$). *Let $F \in \mathbb{K}_{\mathbb{R}}[x^{\pm}, y^{\pm}]$ be a real Laurent polynomial and $\mathbf{1}_2 \in \mathscr{V}(F)$ a singularity. Let $f = \mathrm{trop}_{\mathbb{R}}(F) \in \mathbb{TR}\left[w_1^{\pm}, w_2^{\pm}\right]$ be its corresponding real tropical Laurent polynomial dual to the signed regular marked subdivision $T$ and $\mathbf{0}_2^{+} = \mathrm{trop}_{\mathbb{R}}(\mathbf{1}_2)$. Then $\mathbf{0}_2^{+} \in C_{f,(+,+)}$ is a singularity in $\mathscr{T}_{\mathbb{R}}(f)$ of one of the following types:*

*(1) $\mathbf{0}_2^{+}$ is a 4-valent vertex dual to a signed circuit in $T_{(+,+)}$ of type (A),*
*(2) $\mathbf{0}_2^{+}$ is an isolated point dual to a signed circuit in $T_{(+,+)}$ of type (B), or*
*(3) $\mathbf{0}_2^{+}$ is contained in an edge of weight at least two dual to a signed circuit in $T_{(+,+)}$ of type (C).*

PROOF. Let $\mathscr{A} = \{\alpha_1, \ldots, \alpha_m\} \subset \mathbb{Z}^2$ be the support of $F$ such that we can write $F = \sum_i a_i x^{\alpha_i}$. Let $\Delta = \mathrm{Newt}(F) = \mathrm{conv}(\mathscr{A})$ denote its Newton polytope and $a = (a_1, \ldots, a_m) \in T_{\mathbb{R}}^m$ its coefficient




(A) Point configuration.

(B) Weights according to a flag of type (b).

(C) Regular subdivision with respect to weights coming from a flag of type (b) and signs.

FIGURE 27. A point configuration with a signed regular marked subdivision arising from a point of a weight class of type (b). From the picture we cannot say which two points get the second highest heights among all points not on the circuit of type (C).

vector. As $\mathbf{1}_2 \in \mathscr{V}(F)$ is a singularity we have $a \in \nabla_{\mathbb{R},\mathbf{1}_2}$. Let $b = \mathrm{trop}_{\mathbb{R}}(a)$ denote its real tropicalization. Then $b \in \mathrm{trop}_{\mathbb{R}}(\nabla_{\mathbb{R},\mathbf{1}_2}) = \mathrm{trop}_{\mathbb{R}}(\ker(A'))$ where $A'$ denotes the matrix representation of $\mathscr{A}'$, the shift of $\mathscr{A}$. For a fixed sign vector $s \in \{\pm\}^m$ the $s$-chart of $\mathrm{trop}_{\mathbb{R}}(\nabla_{\mathbb{R},\mathbf{1}_2})$ equals $\mathscr{B}^s(M)$ where $M$ denotes the oriented matroid associated to $\ker(A')$. From now on we denote the sign of $b$ by $s$ and its modulus by $u$ such that $b = (s,u)$. As $b \in \mathrm{trop}_{\mathbb{R}}(\nabla_{\mathbb{R},\mathbf{1}_2})$ there is a maximal $s$-flag $\mathscr{F} \lhd M$ such that $u \in \sigma_{\mathscr{F}} \subset \mathscr{B}^s(M)$. Note that the real tropical Laurent polynomial defined by $(s,u)$ equals $f$, i.e. $C_{f,(+,+)}$ is dual to $T_{(+,+)}$ where $T$ denotes the regular marked subdivision obtained from $u$. Now one of the following cases applies to $\mathscr{F}$:

- The flag $\mathscr{F}$ is an $s$-flag of type (a) and $F_{m-3,m-4} = \{a,b,c,d\}$ is a circuit of type (A) (cf. Lemma 4.2.2.18). Then $T_{(+,+)}$ contains a quadrangle $C = \mathrm{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_d)$ corresponding to $\mathbf{0}_2 \in C_{f,(+,+)}$ (cf. Remark 4.2.3.22). As the signs at its four vertices are alternating the singularity $\mathbf{0}_2 \in C_{f,(+,+)}$ is a 4-valent vertex.
- The flag $\mathscr{F}$ is an $s$-flag of type (a) and $F_{m-3,m-4} = \{a,b,c,d\}$ is a circuit of type (B) (cf. Lemma 4.2.2.18). Then $T_{(+,+)}$ contains a triangle (w.l.o.g.) $C = \mathrm{conv}(\alpha_a, \alpha_b, \alpha_c)$ corresponding to $\mathbf{0}_2 \in C_{f,(+,+)}$ (cf. Remark 4.2.3.22) having an interior point $\alpha_d$. As the sign of its interior point $\alpha_d$ differs to all three vertices of $C$ the singularity $\mathbf{0}_2 \in C_{f,(+,+)}$ is isolated.
- The flag $\mathscr{F}$ is an $s$-flag of type (b) and $F_{m-3,m-4} = \{a,b,c\}$ is a circuit of type (C) (cf. Lemma 4.2.2.20). Then $T_{(+,+)}$ contains an edge $C = \mathrm{conv}(\alpha_a, \alpha_b, \alpha_c)$ of length at least 2 with alternating signs. Hence, $C_{f,(+,+)}$ contains an edge of weight 2 passing through the singularity $\mathbf{0}_2 \in C_{f,(+,+)}$ (cf. Remark 4.2.3.22).

$\square$

**Remark 4.2.3.25.** Recall that $\mathrm{trop}_{\mathbb{R}}(\mathscr{V}(F)) \subseteq \mathscr{T}_{\mathbb{R}}(\mathrm{trop}_{\mathbb{R}}(F))$ for a given real Laurent polynomial $F \in \mathbb{K}_{\mathbb{R}}[x^{\pm}, y^{\pm}]$. Consequently, Theorem 4.2.3.24 does not tell us the exact local picture around $\mathbf{0}_2 \in \mathrm{trop}_{\mathbb{R}}(\mathscr{V}(F))$. Nevertheless, $\mathbf{0}_2^+ \in \mathrm{trop}_{\mathbb{R}}(\mathscr{V}(F))$ as $\mathbf{1}_2 \in \mathscr{V}(F)$ and $\mathrm{trop}_{\mathbb{R}}(\mathbf{1}_2) = \mathbf{0}_2^+$.

### 4.2.4. $\mathrm{trop}(\nabla_{\mathbb{R},\mathbf{1}_2})$ and the Signed Secondary Fan

In this section we investigate the fan structures of $\mathrm{trop}(\nabla_{\mathbb{R},\mathbf{1}_2})$ and $\{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$ and how they stick together. In particular, we like to understand the relationship of equivalence classes of $\{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$

and weight classes of $\mathrm{trop}_{\mathbb{R}}(\nabla_{\mathbb{R}})$. According to Lemma 4.2.1.12 $\mathrm{trop}_{\mathbb{R}}(\nabla_{\mathbb{R}})$ equals the lineality group $G \times \mathrm{rowspace}(A')$ acting on $\mathrm{trop}_{\mathbb{R}}(\nabla_{\mathbb{R},\mathbf{1}_2})$. In general, the two fan structures do not fit, e.g. we have to expect overlaps of cones. However, it appears that, in the complex case, we can compare the fan structures by merging certain sets of cones. Unfortunately, in contrast to the complex case, $\mathrm{trop}_{\mathbb{R}}(\nabla_{\mathbb{R},\mathbf{1}_2}) \odot_{\mathbb{R}} (G \times \mathrm{rowspace}(A'))$ is not a "proper subfan" of the signed secondary fan $\{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$, i.e. it cannot be described in terms of proper equivalence classes (cf. Example 4.2.4.32). If we restrict to the modulus and forget about signs the situation resembles the complex case and we use observations stated in [**MMS12a**, section 3.3].

Given an equivalence class $s \times \sigma_T \subset \{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$, we know from Lemma 1.1.4.23 that the codimension of $s \times \sigma_T$ equals the dimension of $\dim(L_T)$, the space describing the affine relations among the vertices of the regular subdivision $T$. In Section 4.2.3 (in particular, Remark 4.2.3.22) we have seen that all signed subdivisions obtained from weight classes of $\mathrm{trop}(\nabla_{\mathbb{R},\mathbf{1}_2})$ contain a circuit with a valid sign distribution in $T_{(+,+)}$ — either a 2-dimensional circuit, i.e. a quadrangle or triangle with an interior point, or a 1-dimensional line segment as a face of a signed marked polytope in $T_{(+,+)}$. Therefore, $\mathrm{trop}(\nabla_{\mathbb{R},\mathbf{1}_2})$ lives in the codimension one skeleton of $\{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$ if we forget about the sign equivalence. In particular, the $s$-flags described by Lemma 4.2.2.18 and Lemma 4.2.2.20 provide sets in $\{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$ with a fixed sign vector, i.e. they are not invariant under $G \times \mathbf{0}_m$. Moreover, the unsigned flags of type (a) and (b) of $\underline{M}$ do not contain the lineality space $\mathrm{rowspace}(A')$ of $\mathrm{Sec}_{\mathscr{A}}$, i.e. a weight class of $\mathscr{B}^s(M)$ is not invariant under $G \times \mathrm{rowspace}(A')$. However, for a weight class of $\mathrm{trop}(\nabla_{\mathbb{R},\mathbf{1}_2})$ arising from a $s$-flag of type (a) it is just the action of the lineality group $G \times \mathrm{rowspace}(A')$ that is missing to pass from the weight class to a codimension one equivalence class of $\{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$.

**Lemma 4.2.4.26.** *Let $\Delta = \mathrm{conv}(\mathscr{A})$ be the convex hull of $\mathscr{A} \subset \mathbb{Z}^2$, $A'$ the matrix representation of $\mathscr{A}'$ and $G$ a Gale dual to $A'$. Let $C \in M[A']$ be a signed circuit of type (A) or (B) (Remark 4.2.2.16) in $\mathscr{A}$, i.e. $\underline{C} = \{a,b,c,d\} \subset [m]$. Then the union over all $s$-flags of flats $\mathscr{F} \triangleleft M[G]$ ending with $F_{m-3,m-4} = \underline{C}$ and $p_{\underline{C}}(s) = p_{\underline{C}}(s_C)$ multiplied with the lineality group $G \times \mathrm{rowspace}(A')$ equals the union of equivalence classes $s \times \sigma_T$ such that $T_v$ contains $C$ for some $v \in \mathbb{Z}_2 \times \mathbb{Z}_2$, i.e.*

$$\bigcup_{\substack{s \in \{\pm\}^m,\\ \mathscr{F} \triangleleft M \text{ } s\text{-flag}:\\ p_{\underline{C}}(s) = p_{\underline{C}}(s_C).}} \sigma_{\mathscr{F}} \odot_{\mathbb{R}} (G \times \mathrm{rowspace}(A')) = \bigcup_{\substack{s,T:\\ \exists v \in \mathbb{Z}_2 \times \mathbb{Z}_2:\\ C \subset T_v}} s \times \sigma_T. \tag{70}$$

PROOF. We have seen that the signed regular marked subdivision $T_{(+,+)}$ of an element $(s,u)$ of any weight class corresponding to such a $s$-flag $\mathscr{F}$ contains the circuit $C$. Thus we have "$\subseteq$". Vice versa, pick any $(s,u) \in s \times \sigma_T$ such that there is $v \in \mathbb{Z}_2 \times \mathbb{Z}_2$ satisfying $C \subset T_v$. We can write $u$ as a sum of a vector in the lineality space and a vector such that the heights at $\alpha_a, \alpha_b, \alpha_c$ and $\alpha_d$ according to this vector are equal (cf. Remark 4.2.4.27 for details). Moreover, we define $s' = s \cdot \psi_{\mathscr{A}}(v)$ sign equivalent to $s$. Then $s \times \sigma_T = s' \times \sigma_{T'}$ where $T'$ is the signed regular marked subdivision according to $s'$ and $u$. Hence, $C \subset T' = T'_{(+,+)} = T_v$ completing the proof of "$\supseteq$". □

**Remark 4.2.4.27** (Weight classes of type (a) and the lineality space)**.** Consider $(s,u) \in s \times \sigma_T$ where $T$ is a regular marked subdivision containing a circuit $C$ of type (A) or (B). As the points of $C$ are marked they correspond to vertices of a face $C'$ of $\Delta_u^+$. W.l.o.g. we can assume that an edge of $C'$ is part of $\{x=0\}$. Using the $y$-coordinate-vector $l_y$ of the lineality space rowspace$(A')$ of $\mathrm{Sec}_{\mathscr{A}}$ we can incline the heights of $\Delta_u$ such that the heights of the points in $C' \cap \{x=0\}$ become equal. Let $u' = u + \mu l_y$ be this vector for some $\mu \in \mathbb{R}$. Then, using the $x$-coordinate-vector $l_x$, we can incline further such that the result is contained in a plane $\{z=c\}$ for some $c \in \mathbb{R}$, e.g. $u'' = u + \mu l_y + \lambda l_x$ for some $\lambda \in \mathbb{R}$. All these changes do not influence the projections of the upper parts, i.e. $p_{\{x,y\}}(\Delta^+_{u''}) = p_{\{x,y\}}(\Delta^+_u)$. Thus $\mathscr{F}(u'')$ is a flag of type (a) (cf. Remark 4.2.2.17) and the weight class $\sigma_{\mathscr{F}}$ does not contain rowspace$(A')/\mathbf{1}_3$, i.e. $\sigma_{\mathscr{F}} + \text{rowspace}(A')/\mathbf{1}_3$ is a direct sum. Moreover, the singularity at $(0,0)$ of $\mathscr{T}_{\mathbb{R}}(f')$ obtained from $(s,u'')$ corresponds to the singularity at $(\lambda,\mu)$ in $\mathscr{T}_{\mathbb{R}}(f)$ obtained from $(s,u)$.

We like to emphasize again that the sign condition of $s$-flags $\mathscr{F}$ of type (a) can be identified uniquely in a signed regular subdivision $T$, i.e. we actually see the signed circuit $C$ in $T$. The remaining part of this section deals with $s$-flags of type (b). In particular, we show that there is no statement analogue to Lemma 4.2.4.26 (cf. Example 4.2.4.32), in contrast to the complex case:

**Remark 4.2.4.28** (Subcases of flags of type (b))**.** Consider a $s$-flag $\mathscr{F} \lhd M$ of type (b) ending with $F_{m-3,m-4} = \{a,b,c\}$ and suppose $F_{k,k-1} = \{d,e\}$ for some $k \neq m-3$. Let $L_{abc}$ and $L_{de}$ denote the lines through $\alpha_a, \alpha_b, \alpha_c$ and $\alpha_d, \alpha_e$ respectively. We distinguish the two cases $L_{abc} \parallel L_{de}$ and $L_{abc} \nparallel L_{de}$. In the complex case (i.e. forgetting about signs) we get a full description of flags of type (b) in terms of cones in $\mathrm{Sec}_{\mathscr{A}}$ by restricting to flags of type (b) satisfying $L_{abc} \nparallel L_{de}$. According to [**MMS12a**, Lemma 3.12] the union of weight classes $\sigma_{\mathscr{F}}$ where $\mathscr{F} \lhd \underline{M}$ is a flag of type (b) ending with $F_{m-3,m-4} = \{a,b,c\}$, having $F_{k,k-1} = \{d,e\}$ and $L_{abc} \nparallel L_{de}$ plus the lineality space rowspace$(A')$ equals the union over all cones $\sigma_T$ of $\mathrm{Sec}_{\mathscr{A}}$ satisfying:

(1) if the circuit $C = \mathrm{conv}(\alpha_a, \alpha_b, \alpha_c)$ is contained in the boundary of $T$ then we take the union over all codimension one cones $\sigma_T$ corresponding to subdivisions $T$ containing $C$ such that the polygon $Q$ containing $C$ as a face has its third vertex at non-minimal distance to $C$,

(2) if the circuit $C = \mathrm{conv}(\alpha_a, \alpha_b, \alpha_c)$ is not contained in the boundary of $T$ then we take the union over all codimension one cones $\sigma_T$ corresponding to subdivisions $T$ containing $C$.

In Remark 4.2.4.29 (based on [**MMS12a**, Remark 3.11]) we briefly sketch why the case $L_{abc} \nparallel L_{de}$ is sufficient to get a full picture of all weight classes $\sigma_{\mathscr{F}}$ arising from flags of type (b) in terms of cones of $\mathrm{Sec}_{\mathscr{A}}$ and utilize this for the real case.

**Remark 4.2.4.29** ($L_{abc} \parallel L_{de}$)**.** Consider a $s$-flag $\mathscr{F}$ of $M$ of type (b) ending with $F_{m-3,m-4} = \{a,b,c\}$ and suppose $F_{k,k-1} = \{d,e\}$ for some $k \neq m-3$. Let $C \in \mathscr{C}(M[A'])$ denote the circuit such that $\underline{C} = F_{m-3,m-4}$. Assume that $L_{abc} \parallel L_{de}$ where $L_{abc}$ and $L_{de}$ denote the lines through $\alpha_a, \alpha_b, \alpha_c$ and $\alpha_d, \alpha_e$ respectively (cf. Remark 4.2.4.28). In particular, we then have $s_d \neq s_e$. Let $T = T_{(+,+)}$ denote the signed regular marked subdivision of $(s,u)$ with $u \in \mathrm{relint}(\sigma_{\mathscr{F}}) \subset \mathscr{B}^s(M)$. Let $Q \subset T$ be the polygon on the same side of $L_{abc}$ as $L_{de}$ containing the signed circuit $C = \mathrm{conv}(\alpha_a, \alpha_b, \alpha_c)$ (cf. the situation depicted in Figure 29). Then we face the following possibilities:

(a) Suppose $Q$ is spanned by $C$ and the points $\alpha_d$ and $\alpha_e$. This situation is sketched in Figure 29b. Then $\sigma_T$ is in the boundary of the cone $\sigma_S$ whose elements provide subdivisions where $Q$ is subdivided as exemplified shown in Figure 29b by the dashed line. Recall that $\mathrm{codim}(\sigma_T)$ equals the dimension of the space of affine relations among the marked polytopes of $T$. In this case we have $\dim(L_T) = 2$. We get a codimension one cone $\sigma_S$ from the subdivision $T$ by decreasing the height at a vertex of $C$ (e.g. $\alpha_a$ in Figure 29b). Moreover, we can pick a vertex such that the remaining elements of $\{\alpha_a,\dots,\alpha_e\}$ form a circuit of type (A). Thus, as shown in Figure 29b, $S$ contains a quadrangle $\mathrm{conv}(\alpha_b,\alpha_c,\alpha_d,\alpha_e)$ with a sign distribution corresponding to a signed circuit of $M[A']$ of type (A) (cf. Remark 4.2.2.16). These subdivisions were considered in Lemma 4.2.4.26, in particular the weight class $\sigma_{\mathscr{F}}$ and $\sigma_T$ are contained in the boundary of cones of the secondary fan belonging to weight classes of type (a).

(b) Suppose $Q$ contains a vertex whose distance to $L_{abc}$ is bigger than the distance of $L_{de}$ to $L_{abc}$. For example, consider the (unsigned) subdivision sketched in Figure 29a. In this situation the cones $\sigma_{\mathscr{F}}$ and $\sigma_T$ are contained in the boundary of cones of the secondary fan that correspond to weight classes of type (b) where $L_{abc}$ and $L_{de}$ are not parallel (for details see Remark 4.2.4.31). For example, we can achieve this for the subdivision shown in Figure 29a by adding a multiple of lineality vector containing the $x$-coordinates of all $\alpha_i$ such that $\alpha_f$ and $\alpha_e$ (and, therefore, $\alpha_d$) are on the same height. It follows that if $\mathscr{F}$ is of type (b) where $L_{abc}$ and $L_{de}$ are not parallel then $\sigma_{\mathscr{F}} + \mathrm{rowspace}(A')$ is a direct sum whereas in case of parallelism this is not true (cf. discussion above).

Parts of the proof of statement [**MMS12a**, Lemma 3.12] give evidence why an analogue statement for the real case is impossible. However, we can deduce a method that helps to decide whether a given signed regular marked subdivision containing a circuit of type (C) provides a singular real plane tropical curve or not.

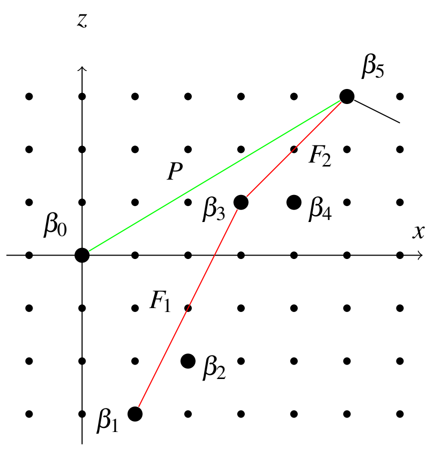
**Definition 4.2.4.30** (Height profile)**.** Let $\mathscr{A} \subset \mathbb{Z}^2$ be a finite set and $(s,u) \in \{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$ fixed. We define $\beta_i = (i, m_i)$ where $m_i = \max\{u_j : (\alpha_j)_x = i\}$ is the maximum of all heights of points in $\mathscr{A}$ on the line $\{x = i\}$ and we denote the boundaries of $\mathscr{A}$ along the $x$-axis by $b_1 = \min\{i : \{x = i\} \cap \mathscr{A} \neq \emptyset\}$ and $b_2 = \max\{i : \{x = i\} \cap \mathscr{A} \neq \emptyset\}$. Let $I = \{a_1,\dots,a_2\} \subset \{b_1,\dots,b_2\}$ be a subset of the supporting points of all $\beta_i$. The data consisting of the collection of points $\Xi = \{\beta_i : i \in I\}$ and the upper faces $\mathrm{conv}(\Xi)^+$ (cf. Definition 1.1.4.20) is called *height profile along $x$ with base points $I$*. A *signed height profile* is a height profile where the points $\beta_i$ come with sign sets, i.e. we have $(\beta_i, s_i)$ where

$$
s_i = \begin{cases} \{+\} & \text{if we have } s_\alpha = + \text{ for all } \alpha \in \mathscr{A} \text{ satisfying } \beta_i = ((\alpha)_x, u_\alpha), \\ \{-\} & \text{if we have } s_\alpha = - \text{ for all } \alpha \in \mathscr{A} \text{ satisfying } \beta_i = ((\alpha)_x, u_\alpha), \text{and} \\ \{\pm\} & \text{if there are } \alpha, \alpha' \in \mathscr{A} \text{ such that } \beta_i = ((\alpha)_x, u_\alpha) = ((\alpha')_x, u_{\alpha'}) \text{ and } s_\alpha \neq s_{\alpha'}. \end{cases} \tag{71}
$$

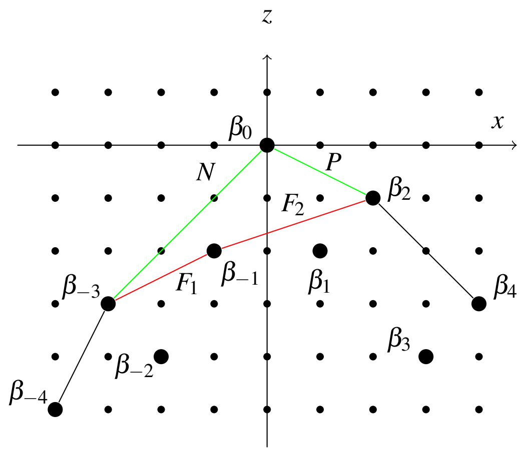
Note that $\{\pm\} = \{\mp\} = \{+,-\}$. If the subset $I = \{a_1,\dots,a_2\} \subset \{b_1,\dots,b_2\}$ contains all points between $a_1$ and $a_2$ we also say that $\Xi^+$ is the height profile from $a_1$ to $a_2$.

**Remark 4.2.4.31** (Weight classes of type (b) with $L_{abc} \not\parallel L_{de}$ and the lineality space)**.** The crucial part ("$\supseteq$") of the proof of [**MMS12a**, Lemma 3.12] (see Remark 4.2.4.28 for the statement) is to show that one can obtain a weight class of type (b) from a regular subdivision $T$ containing a circuit of type (C). We make use of the the basic idea of the proof and develop a quantitative description.

Let $T$ be a signed regular marked subdivision containing a signed circuit $C \in M[A']$ of type (C) satisfying the requirements of [**MMS12a**, Lemma 3.12]. Assume that $C$ is on the line $\{x = 0\}$ and consider any $(s,u) \in s \times \mathrm{relint}(\sigma_T)$. Using the $y$-coordinate-vector $l_y \in \mathrm{rowspace}(A')$ we can achieve




(A) $C \subset \{x=0\}$ and $C \subset \partial T$. In this situation we have $i_1 = 1$, $i_2 = 3$ and $i_3 = 5$.




(B) $C \subset \{x=0\}$ and $C \not\subset \partial T$. Here we have $i_1 = -3$, $i_2 = -1$ and $i_3 = 2$.

FIGURE 28. Examples for height profiles $D^+$ ($E^+$ respectively) in $x$-direction for the cases $C \subset \partial T$ and $C \not\subset \partial T$ with (without respectively) $x = 0$ as base point (cf. Remark 4.2.4.31). Here, $D^+$ ($E^+$ respectively) is given by the black and green (red respectively) edges.

that the heights at $\alpha_a, \alpha_b$ and $\alpha_c$ are equal (see also Remark 4.2.4.27), i.e. without restriction we have $u_a = u_b = u_c$. Using $\mathbf{1}_m \in \text{rowspace}(A')$ we can further assume that $u_a = u_b = u_c = 0$.
We seek for $\lambda \in \mathbb{R}$ such that $u' = u - \lambda l_x$ provides a ($s$-) flag $\mathscr{F}(u')$ of type (b). Therefore, $u'$ needs to satisfy $u'_a = u'_b = u'_c > u'_j$ for all $j \neq a, b, c$ and there need to be two elements $i, j \in [m]$ such that $u'_i = u'_j$ are the highest heights among all points not on the line $L_{abc}$. The addition of $l_x$ to $u$ can be described by a shear matrix $U = U(\lambda)$ that moves points parallel to the $z$-axis in the $\langle x, z\rangle$-plane), i.e. it suffices to consider the maximal points $(i, m_i)$. Let $D^+$ be the height profile of all $\beta_i = (i, m_i)$ along the $x$-axis (cf. Definition 4.2.4.30). Applying $U(\lambda)$ to $(i, m_i)$ gives the point $(i, m_i - \lambda i)$.

(a) Suppose $C$ is not contained in the boundary of $T$ (cf. Figure 28b). Let $Q_n$ and $Q_p$ be the polygons containing $C$ where $Q_n$ has a third vertex $\alpha$ with negative $x$-coordinate and $Q_p$ a third vertex $\alpha'$ with positive $x$-coordinate. Let $N$ ($P$ respectively) denote the upper face in $D^+$ corresponding to $Q_n$ ($Q_p$ respectively). Let $E^+$ be the height profile of all $\beta_i$ except $\beta_0$. Let $i_1, \ldots, i_k$ be the index set with $i_j < i_{j+1}$ for all $1 \leq j \leq k-1$ such that $\beta_{i_j} \in E^+$ is a proper vertex for all $j$ and $\beta_{i_1}$ and $\beta_{i_k}$ are the vertices of the upper faces $N$ and $P$ in $D^+$ besides $(0, u_a)$. We denote the face in $E^+$ formed by $\beta_{i_j}$ and $\beta_{i_{j+1}}$ by $F_j$. Let $\overline{\lambda} = \frac{m_{i_1}}{i_1}$ and $\underline{\lambda} = \frac{m_{i_k}}{i_k}$ denote the slopes of $N$ and $P$. We define $\lambda_j = \frac{m_{i_{j+1}} - m_{i_j}}{i_{j+1} - i_j}$ (the slope of $F_j$) and show that $u' = u - \lambda_j l_x$ is as desired for any $j \in [k-1]$. First, note that $\overline{\lambda}$ and $\underline{\lambda}$ form upper and lower bounds for all slopes $\lambda_i$, i.e. we have

$$\overline{\lambda} > \lambda_1 > \ldots > \lambda_{k-1} > \underline{\lambda}.$$

This is true since $N$ and $P$ are faces of $D^+$ and the vertices $\beta_{i_j}$ with $j \neq 1, k$ are below $N$ and $P$. Moreover, the height profile $E^+$ is convex. Fix any $j \in [k-1]$. The goal is to show that $m_l - \lambda_j l < m_{i_j} - \lambda_j i_j$ for all $l \neq i_j, i_{j+1}$. First, consider $l < i_j$. As $E^+$ is convex the slope of the line segment from $\beta_l$ to $\beta_{i_j}$ satisfies $\frac{m_{i_j}-m_l}{i_j-l} > \lambda_j$. This is equivalent to $m_l - m_{i_j} < \lambda_j(l-i_j)$ and, therefore, $m_l - \lambda_j l < m_{i_j} - \lambda_j i_j$. If $l > i_j$ then $\frac{m_l - m_{i_j}}{l-i_j} < \lambda_j$ as $E^+$ is convex and we conclude that $m_l - \lambda_j l < m_{i_j} - \lambda_j i_j$. Finally, by definition, we have $m_{i_j} - \lambda_j i_j = m_{i_{j+1}} - \lambda_j i_{j+1} > m_l - \lambda_j l$ for all $l \neq i_j, i_{j+1}$ as desired.

(b) Suppose $C$ is contained in the boundary of $T$ (cf. Figure 28a). We use identical notations: let $Q$ be the polygon containing $C$. By $\alpha$ we denote the third vertex of $Q$ besides the vertices of $C$. We only have $\beta_j = (j, m_j)$ with (w.l.o.g.) $j \geq 0$. As the third vertex of the polygon $Q$ containing $C$ is at non-minimal distance there are vertices between $L_{abc}$ and the parallel line $L_\alpha$ through $\alpha$. We define $k = (\alpha)_x$ and let $E^+$ denote the height profile of all $\beta_i$ except $\beta_0$. Let $i_1, \ldots, i_l$ be the index set of the vertices $\beta$ contained in $E^+$ satisfying $i_h < i_{h+1}$ for $1 \leq h \leq l-1$, and $i_1 = 1 > 0$ and $i_l = k$. As before we denote the face of $E^+$ given by the vertices $\beta_{i_j}$ and $\beta_{i_{j+1}}$ by $F_j$ and its slope by $\lambda_j$. Moreover, let $P$ denote the face of $D^+$ corresponding to $Q$, i.e. its vertices are $(0, u_a)$ and $(k, m_k)$. We denote the slope of $P$ by $\underline{\lambda}$. Then we have

$$\lambda_1 > \ldots > \lambda_{k-1} > \underline{\lambda}.$$

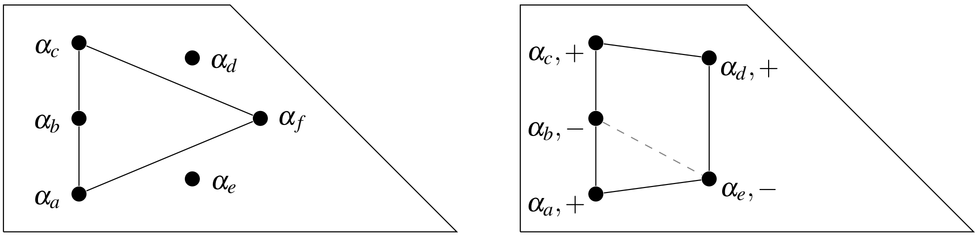
Now the proof that $u' = u - \lambda_j l_x$ is as desired for any $j \in [k-1]$ is completely analogous.

We can think of $\lambda_j$ as the parameter such that the shear matrix $U(\lambda_j)$ equalizes the heights of the points $\beta_{i_j}$ and $\beta_{i_{j+1}}$. Note that $\underline{\lambda}$ (and $\overline{\lambda}$ respectively) would shift the heights of the points corresponding to $\alpha$ (and $\alpha'$) to 0. As we require that $u_a = u_b = u_c > u_j$ for all $j \neq a, b, c$, and by assumption $u_a = u_b = u_c = 0$, the values $\underline{\lambda}$ and $\overline{\lambda}$ form bounds on the valid values of $\lambda \in \mathbb{R}$. The points $\lambda_j \in \mathbb{R}$ are precisely the values such that the shear maximizes, and at the same time equalizes, the heights of exactly two more points not on $L_{abc}$ to a height below zero. An element $\lambda \in (\lambda_i, \lambda_{i+1})$ has the effect that the point $\beta_{i+1}$ becomes the highest point.

Overall, we see that any $u \in \operatorname{relint}(\sigma_T)$ can be written as a sum of a lineality vector and an element $u'$ providing a flag of type (b). This representation is not unique as we have choices with regard to $\lambda_j$. Let us focus on the points $\alpha, \alpha' \in \mathscr{A}$ providing the vertices of the face $F_j$ if we shear with $U(\lambda_j)$. Essentially, we do not care about the signs of $\alpha$ and $\alpha'$ in the complex case. However, the real case highly depends on the signs at $\alpha$ and $\alpha'$. They determine whether the result (i.e. $u'$) provides a $s$-flag of type (b) (cf. Lemma 4.2.2.20) or not. It turns out that it heavily depend on the choice $u \in \operatorname{relint}(\sigma_T)$ as Example 4.2.4.32 shows.
It is worth to mention the following: if we keep in mind the slope $\mu$ of $(\alpha_a, u_a), (\alpha_b, u_b), (\alpha_c, u_c)$ along the $y$-axis in $\Delta_u$ and shear with $\lambda_j$ then $\mathscr{T}(|f|)$ obtained from $u$ has a singularity at $(-\lambda_j, -\mu)$.

However, in the real case, the union of weight classes arising from $s$-flags $\mathscr{F}$ of type (b) such that $F_{m-3,m-4} = \{a, b, c\}$, $F_{k,k-1} = \{d, e\}$ for some $k \in [m-4]$ and $L_{abc} \not\parallel L_{de}$ multiplied with the lineality group $G \times \operatorname{rowspace}(A')$ is not a union of equivalence classes $s \times \sigma_T$ for some $s \in \{\pm\}^m$ and subdivisions $T$ of $\Delta$ as the following example shows:

**Example 4.2.4.32.** Consider the equivalence class $s \times \sigma_T$ corresponding to the signed regular marked subdivision $T$ as shown in Figure 30. We use the techniques explained in Remark 4.2.4.31: by inclining with the $y$-coordinate-vector of $\operatorname{rowspace}(A')$ we can restrict on elements $(s, u)$ of $s \times \sigma_T$ that satisfy $u_a = u_b = u_c$. By adding a multiple of the $x$-coordinate-vector $l_x$ of $\operatorname{rowspace}(A')$ we can




(A) Polygon $Q$ formed by $C$ and a point $\alpha_f$ whose distance to $C$ is bigger than the distance of $L_{de}$ to $C$.

(B) Polygon $Q$ formed by $C$ and $\alpha_d, \alpha_e$.

FIGURE 29. Subdivisions corresponding to weight classes of type (b) with $L_{abc} \parallel L_{de}$.

ensure that the heights at $\alpha_a, \alpha_b$ and $\alpha_c$ get maximal. Additionally, two of the three points $\alpha_d, \alpha_e$ and $\alpha_f$ get the highest and equal height of the points not on $L_{abc}$. Let $u'$ denote the result of $u$ plus a multiple of $l_x$. The following situations may appear:

(1) Suppose $u'_d = u'_f > u'_e$. Then $\mathscr{F}(u')$ provides the flag of flats

$$\emptyset \subsetneq F_1 = \{e\} \subsetneq F_2 = \{d,e,f\} \subsetneq F_3 = \{a,b,c,d,e,f\}.$$

Note that $F_{2,1} = \{d, f\}$ and $s_d = s_f$. Moreover, there is no sign vector $s'$ sign equivalent to $s$ such that $s'_d \neq s'_e$. Recall $s'$ is sign equivalent to $s$ if there is $v \in \mathbb{Z}_2 \times \mathbb{Z}_2$ such that $s' = \psi_{\mathscr{A}}(v) \cdot s$. Equivalently, we can change signs of $s$ whose indices are contained in the sublattice corresponding to $v$. However, no sublattice (cf. Figure 20) changes signs only at one of the vertices $\alpha_d, \alpha_f$. Hence, $\mathscr{F}(u')$ is a flag of type (b) but it is not an $s'$-flag of type (b) for any sign equivalent sign vector $s'$. Thus $(s, u')$ is not contained in $\operatorname{trop}_{\mathbb{R}}(\nabla_{\mathbb{R},\mathbf{1}_2})$.

(2) Suppose $u'_e = u'_f > u'_d$. Then $\mathscr{F}(u')$ provides the flag of flats

$$\emptyset \subsetneq F_1 = \{d\} \subsetneq F_2 = \{d,e,f\} \subsetneq F_3 = \{a,b,c,d,e,f\}.$$

where $F_{2,1} = \{e, f\}$ and $s_e \neq s_f$. Hence, $\mathscr{F}(u')$ is an $s$-flag and $\sigma_{\mathscr{F}(u')} \subset \mathscr{B}^s(M)$ is part of $\operatorname{trop}_{\mathbb{R}}(\nabla_{\mathbb{R},\mathbf{1}_2})$.

Hence, only parts of $s \times \sigma_T$ belong to $\operatorname{trop}_{\mathbb{R}}(\nabla_{\mathbb{R}})$ and, therefore, we cannot expect a full description of $\operatorname{trop}_{\mathbb{R}}(\nabla_{\mathbb{R}})$ in terms of equivalence classes $s \times \sigma_T$ of $\{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$. In the complex case, there is no sign condition on the points not on $L_{abc}$ whose heights get adapted. The sign condition on this points in the real case causes the trouble.

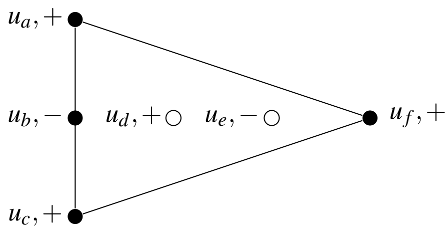



FIGURE 30. A regular marked subdivision $T$ with signs $s$. Only parts of $s \times \sigma_T$ corresponding to the shown signed regular marked subdivision belong to $\operatorname{trop}_{\mathbb{R}}(\nabla_{\mathbb{R}})$. For details see Example 4.2.4.32.

A priori, we cannot say if a signed regular marked subdivision $T$ (with signs $s$) containing a circuit $C$ of type (C) corresponds to a singular real plane tropical curve purely from the subdivision. However, for a given element $(s,u) \in s \times \sigma_T$ we can give an algorithmic answer to the posed question using the height profiles. First, note that we only consider signed regular marked subdivisions $T$ with circuits $C$ of type (C) that fulfill the conditions stated in Remark 4.2.4.29, i.e.

(1) if the circuit $C = \operatorname{conv}(\alpha_a, \alpha_b, \alpha_c)$ is contained in the boundary of $T$ then the polygon $Q$ containing $C$ as a face has its third vertex at non-minimal distance to $C$, or

(2) the circuit $C = \operatorname{conv}(\alpha_a, \alpha_b, \alpha_c)$ is not contained in the boundary of $T$.

Recall that $\partial T$ denotes the boundary of $T$ (cf. Remark 1.1.1.2).

**Proposition 4.2.4.33** ($C \subset \partial T$). *Let $s \times \sigma_T$ be an equivalence class of codimension one such that $T$ contains a circuit $C \subset \{x = 0\}$ of type (C) in its boundary and the third vertex $\alpha$ of the polygon $Q$ containing $C$ is at non-minimal distance. We define $k = (\alpha)_x$. Let $(s,u) \in s \times \sigma_T$ be a fixed element satisfying $u_a = u_b = u_c$ where $\underline{C} = \{a,b,c\} \subset [m]$. Let $f = \bigoplus_j (s_j, u_j) w^{\alpha_j} \in \mathbb{TR}\left[w_1^{\pm}, w_2^{\pm}\right]$ be the real tropical Laurent polynomial associated to $(s,u)$. Then $\mathscr{T}_{\mathbb{R}}(f)$ is a singular real plane tropical curve if and only if there is an element $v \in \mathbb{Z}_2 \times \mathbb{Z}_2$ such that the circuit $C \subset T_v$ is a signed circuit $C \in M[A']$ (cf. Notation 4.1) and the signed height profile $E^+$ from $1$ to $k$ (with respect to signs $s \cdot \psi_{\mathscr{A}}(v)$) satisfies one of the following conditions:*

- *$E^+$ contains a signed vertex $(\beta, s')$ with $s' = \pm$, or*
- *$E^+$ contains a face $F$ with signed vertices $(\beta, s')$ and $(\beta', s'')$ satisfying $s' \cup s'' = \{\pm\}$.*

Proof. For "$\Rightarrow$" suppose $q \in \mathscr{T}_{\mathbb{R}}(f)$ is a singularity. Without restriction, $s(q) = (+,+)$. The singularity is contained in an edge $e \in \mathscr{T}_{\mathbb{R}}(f)_{(+,+)}$ with $e \subset \{y = 0\}$ as we have $u_a = u_b = u_c$, i.e. $q = (\lambda, 0)$ for some $\lambda \in \mathbb{R}$. Then the real tropical hypersurface of $f'$ obtained from $(s,u')$ with $u' = u + A^\top q = u + \lambda l_x$ has a singularity at $\mathbf{0}_2$. Thus $\mathscr{F}(u')$ is a $s$-flag of type (b), i.e. we have $F_{m-3,m-4} = \{a,b,c\}$ and $F_{k,k-1} = \{d,e\}$ for some $k \in [m-4]$. In particular, $\alpha_d$ and $\alpha_e$ have highest height of all points not on $L_{abc}$. Note that $L_{abc} \parallel L_{de}$ or $L_{abc} \not\parallel L_{de}$. In both cases we have $s_d \neq s_e$ (cf. Lemma 4.2.2.20). In the first case we have $(\beta, s')$ with $s' = \pm$ corresponding to $\alpha_d, \alpha_e \in L_{de}$. As these points get the highest height among all points not on $L_{abc}$ we see $(\beta, s')$ in the signed height profile $E^+$ from 1 to $k$ according to $u'$. In the second case the points $(\alpha_d, u'_d)$ and $(\alpha_e, u'_e)$ correspond to vertices $(\beta, s')$ and $(\beta', s'')$ with the same height in the signed height profile $E^+$ from 1 to $k$ according to $u'$. As $s_d \neq s_e$ we have $s' \cup s'' = \{\pm\}$. Now, $u = u' - A^\top q = u - \lambda l_x$ is a shear of $u'$ where $\lambda \in \{\lambda_1, \ldots, \lambda_l\}$ is one of the slopes of the height profile $E^+$ with respect to $u$. To see this, recall that the shear with respect to $\lambda$ equalizes the heights at $\alpha_d$ and $\alpha_e$. Hence, $\lambda$ denotes the slope of the face $F$ in the signed height profile $E^+$ with base points $(\alpha_d)_x$ and $(\alpha_e)_x$.

Vice versa, suppose $v = (+,+)$. We use the facts and notations about the slopes of faces of $E^+$ explained in Remark 4.2.4.31. First, assume that the signed height profile $E^+$ contains a signed vertex $(\beta, s')$ with $s' = \pm$, i.e. there are two vertices $\alpha_1, \alpha_2 \in \mathscr{A} \setminus \{x = 0\}$ with maximal and equal heights $u_1 = u_2$, $s_1 \neq s_2$ and $(\alpha_1)_x = (\alpha_2)_x$. Hence, $(\beta, s')$ is a proper vertex of $E^+$, i.e. $\beta = (i_j, m_{i_j})$ for some $1 \leq i_j < k$ with $k \geq 2$ by assumption. Note that $i_j = k$ is not possible — otherwise $Q$ is not a triangle. For $1 < i_j < k$ we pick a slope $\lambda \in (\lambda_{i_{j-1}}, \lambda_{i_j})$. For $i_j = 1$ we pick $\lambda > \lambda_1$. In all cases the shear $U(\lambda)$ guarantees that $\beta$ becomes the vertex with the second highest height in $u' = U(\lambda)u$. Thus $\mathscr{F}(u')$ is an $s$-flag of type (b), i.e. $\mathscr{T}_{\mathbb{R}}(f)$ contains a singularity.

Now, we deal with the case that $E^+$ contains a face $F$ with signed vertices $(\beta, s')$ and $(\beta', s'')$ satisfying $s' \cup s'' = \{\pm\}$. If we have (w.l.o.g.) $s' = \pm$ for the vertex $\beta$ we are in the previous situation, i.e. we can assume that no vertex $\beta$ of $E^+$ comes with signs $\pm$. If we shear with the slope $\lambda$ of $F$ we equalize the heights of $\beta$ and $\beta'$ (cf. Remark 4.2.4.31). In particular, the heights are the highest heights among all points not contained in $\mathrm{aff}(C)$. Hence, we find $\lambda, \mu \in \mathbb{R}$ such that $\bar{u} = u + \lambda l_x + \mu l_y$ provides a flag of type (b) ending with $F_{m-3,m-4} = \{a,b,c\}$ and $F_{k,k-1} = \{d,e\}$ for some $k \in [m-4]$ and $d,e \in [m]$. In detail, let $\alpha_d$ correspond to $\beta$ and $\alpha_e$ to $\beta'$ and, moreover, we have $\bar{u}_d = \bar{u}_e$. By assumption, the two vertices $\beta, \beta'$ of $F$ have different signs, e.g. $s' \cup s'' = \{\pm\}$ and w.l.o.g. $s' = +$, $s'' = -$. Thus, we have $s_d \neq s_e$ for the corresponding vertices $\alpha_d, \alpha_e$. Then, $\mathscr{F}(\bar{u})$ is an $s$-flag of type (b), i.e. $\mathscr{T}_{\mathbb{R}}(f)$ contains a singularity (cf. Lemma 4.2.2.20). $\square$

We omit the proof of the analogue statement:

**Proposition 4.2.4.34** ($C \not\subset \partial T$)**.** *Let $s \times \sigma_T$ be an equivalence class of codimension one such that $T$ contains a circuit $C \subset \{x = 0\}$ of type (C) that is not contained in the boundary of $T$. Let $Q$ and $Q'$ denote the polygons in $T$ containing the circuit $C$. We denote the third vertices of $Q$ and $Q'$ by $\alpha$ and $\alpha'$. We define $a_1 = (\alpha)_x$ and $a_2 = (\alpha')_x$. Let $(s,u) \in s \times \sigma_T$ be a fixed element satisfying $u_a = u_b = u_c$ where $\underline{C} = \{a,b,c\} \subset [m]$. Let $f = \bigoplus_j (s_j, u_j) w^{\alpha_j} \in \mathbb{TR}\left[w_1^{\pm}, w_2^{\pm}\right]$ be the real tropical Laurent polynomial associated to $(s,u)$. Then $\mathscr{T}_{\mathbb{R}}(f)$ is a singular real plane tropical curve if and only if for some $v \in \mathbb{Z}_2 \times \mathbb{Z}_2$ the circuit $C \subset T_v$ is a signed circuit $C \in M[A']$ (cf. Notation 4.1) and the signed height profile $E^+$ of all $\beta_i$ with $a_1 \leq i \leq a_2$ and $i \neq 0$ (and with respect to signs $s \cdot \psi_{\mathscr{A}}(v)$) satisfies one of the following conditions:*

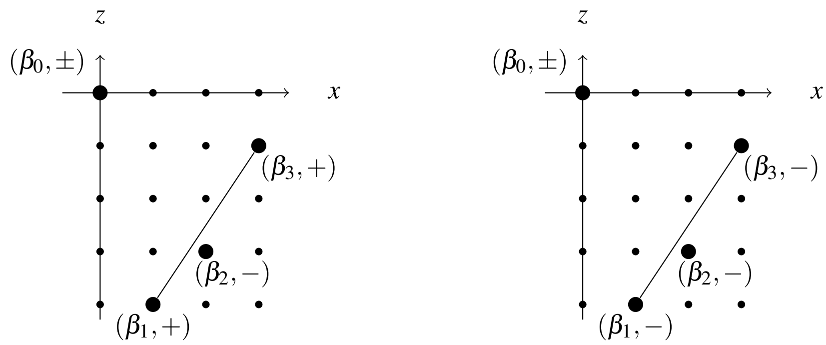
- *$E^+$ contains a signed vertex $(\beta, s')$ with $s' = \pm$, or*
- *$E^+$ contains a face $F$ with signed vertices $(\beta, s')$ and $(\beta', s'')$ satisfying $s' \cup s'' = \{\pm\}$.*

**Example 4.2.4.35.** First, consider the regular marked subdivision $T$ as shown in Figure 30. We pick $u = (0,0,0,-4,-3,-1) \in \sigma_T$ and the sign vector $s = (+,-,+,+,-,+)$. Figure 31 contains the height profiles $E^+$ with respect to the sign vectors $s \cdot \psi_{\mathscr{A}}(v)$ for $v \in \mathbb{Z}_2 \times \mathbb{Z}_2$ such that $T_v$ contains a signed circuit $C \in M[A']$ of type (C). Inclining with $\lambda_1 = \frac{3}{2}$ and $l_x \in \mathrm{rowspace}(A')$ provides an element $u'$ such that $u'_a = u'_b = u'_c > u'_d = u'_f > u'_e$. Note that the signs of the vertices of the face forming $E^+$ for $v = (+,+)$ and $v = (-,+)$ are positive. According to Proposition 4.2.4.33 this implies $\mathscr{T}_{\mathbb{R}}(f)$ is not singular.

The previous two statements are strongly connected to Euler derivatives.

### 4.2.5. Singular Real Plane Tropical Curves and Euler Derivatives

In this section we apply methods concerning Euler derivatives developed in Section 3.2 to real plane tropical curves, i.e. $k = 1$ and $n = 2$. Several results of Section 3.2 can be transferred to the real case (cf. [**Tab15**, Section 4] for $k = 1$). However, we focus on the posed questions concerning signed regular marked subdivisions containing a signed circuit of type (C) (cf. Remark 4.2.2.16 and Section 4.2.4). Basically, we introduce real (tropical) Euler derivatives (cf. Equation (73) and Equation (74)), the real tropical basis of $\nabla_{\mathbb{R},p} = \mathscr{V}(\mathscr{I}_p)$ in terms of real Euler derivatives (cf. Equation (76)) and state our main result (cf. Proposition 4.2.5.37).




(A) $v=(+,+)$. (B) $v=(-,+)$.

FIGURE 31. Height profiles for all $v \in \mathbb{Z}_2 \times \mathbb{Z}_2$, cf. Example 4.2.4.35.

Let $L = \sum_i \lambda_i x_i + v \in \mathbb{K}_{\mathbb{R}}[x^{\pm},y^{\pm}]$ be an affine linear form and $F = \sum_i a_i x^{\alpha_i} \in \mathbb{K}_{\mathbb{R}}[x^{\pm},y^{\pm}]$ a real Laurent polynomial with $\mathscr{A} = \{\alpha_1,\dots,\alpha_m\} \subset \mathbb{Z}^2$. The Euler derivative of $F$ with respect to $L$ is defined as in Definition 3.2.3.9, i.e.

$$\frac{\partial F}{\partial L} = \sum_i \lambda_i x_i \frac{\partial F}{\partial x_i} + vF. \tag{72}$$

Due to Lemma 3.2.3.10 we have

$$\frac{\partial F}{\partial L} = \sum_i L(\alpha_i) a_i x^{\alpha_i}. \tag{73}$$

We see that $L$ determines the terms of $F$ that appear in $\frac{\partial F}{\partial L}$ and, moreover, it may flip the signs of the coefficients. In particular, the sign of $a_i$ flips if and only if we have the $L(\alpha_i) < 0$. If $f \in \mathbb{TR}\left[w_1^{\pm},w_2^{\pm}\right]$ is a real tropical Laurent polynomial and $\mathrm{trop}_{\mathbb{R}}(F) = f$ then

$$\mathrm{trop}_{\mathbb{R}}\left(\frac{\partial F}{\partial L}\right) = \frac{\partial f}{\partial L} = \bigoplus_{i:L(\alpha_i)\neq 0} (s(a_i)s(L(\alpha_i)), -\mathrm{val}(a_i)) w^{\alpha_i}. \tag{74}$$

Thus the real version of Lemma 3.2.3.13 holds as well.

Now, we focus on the real tropical discriminant of singular real plane tropical curves. We stick to Notation 4.1, i.e. $F = \sum_i y_i x^{\alpha_i} \in \mathbb{K}_{\mathbb{R}}[y_1,\dots,y_m][x^{\pm},y^{\pm}]$ denotes a generic bivariate Laurent polynomial with fixed support $\mathscr{A}$. The vanishing locus of $F(\mathbf{1}_2) = \sum_i y_i$, $x\frac{\partial F}{\partial x}(\mathbf{1}_2) = \sum_i (\alpha_i)_x y_i$ and $y\frac{\partial F}{\partial y}(\mathbf{1}_2) = \sum_i (\alpha_i)_y y_i$ is $\nabla_{\mathbb{R},\mathbf{1}_2}$. If we pick a different point $\mathbf{1}_2 \neq p \in T_{\mathbb{R}}^2$ then $F(p) = \sum_i y_i p^{\alpha_i}$, $x\frac{\partial F}{\partial x}(p) = \sum_i (\alpha_i)_x y_i p^{\alpha_i}$ and $y\frac{\partial F}{\partial y}(\mathbf{1}_2) = \sum_i (\alpha_i)_y y_i p^{\alpha_i}$ are still linear but have non-constant coefficients. We define

$$\mathscr{I}_p = \langle F(p), x\frac{\partial F}{\partial x}(p), y\frac{\partial F}{\partial y}(p)\rangle \subset \mathbb{K}_{\mathbb{R}}\left[x^{\pm},y^{\pm}\right] \tag{75}$$

such that we have $\mathscr{V}(\mathscr{I}_p)) = \nabla_{\mathbb{R},p}$ (cf. Notation 4.1). However, the linear forms in $\mathscr{I}_p$ with minimal support form a tropical basis of $\mathscr{I}_p$ (Theorem 4.1.2.18). We know that an Euler derivative $\frac{\partial F}{\partial L}(p)$ with respect to the linear form $L \in \mathbb{K}_{\mathbb{R}}[x^{\pm},y^{\pm}]$ corresponds to a linear form of $\mathscr{I}_p$. Due to Lemma 3.2.3.10 the Euler derivative $\frac{\partial F}{\partial L}$ has minimal support if and only if $L$ vanishes on a maximal subset of $\mathscr{A}$ (see also Equation (73)). Let $\Lambda(\mathscr{A})$ be the set of affine integral linear forms $L \in \mathbb{Z}[x,y]$

that vanish on a maximal subset of $\mathscr{A}$ and whose coefficients have coefficients have gcd $= 1$. The set $\Lambda(\mathscr{A})$ is finite since $\mathscr{A}$ is finite. Then

$$P_p(\mathscr{A}) = \left\{ \frac{\partial F}{\partial L}(p) : L \in \Lambda(\mathscr{A}) \right\} \tag{76}$$

is a tropical basis of $\mathrm{trop}_{\mathbb{R}}(\nabla_{\mathbb{R},p})$ ([**Tab15**, Theorem 4.4], see also Equation (52) in Section 3.2). This allows to formulate an analogue statement of Proposition 3.2.4.16 for the real case (here we restrict ourselves to $k = 1$ and $n = 2$):

THEOREM 4.2.5.36 ([**Tab15**, Theorem 4.5]). *Let $f = \sum_i (p_i, s_i) w^{\alpha_i} \in \mathbb{TR}\left[w_1^{\pm}, w_2^{\pm}\right]$ be a real tropical Laurent polynomial with support $\mathscr{A}$. Then $q \in \bigcap_{L \in \Lambda(\mathscr{A})} \mathscr{T}_{\mathbb{R}}\left(\frac{\partial f}{\partial L}\right)$ if and only if $q \in \mathscr{T}_{\mathbb{R}}(f)$ is a singularity.*

Now, we focus on real tropical Laurent polynomials $f \in \mathbb{TR}\left[w_1^{\pm}, w_2^{\pm}\right]$ with support $\mathscr{A}$ arising from $(s,u) \in s \times \sigma_T$ such that $T$ is a regular marked subdivision that contains a circuit of type (C) (cf. Remark 4.2.2.16). Recall that if $\mathscr{T}_{\mathbb{R}}(f)$ contains a singularity then the singularity is contained in the edge dual to $C$ in some chart of $\mathscr{T}_{\mathbb{R}}(f)$ (cf. Remark 4.2.3.22 or Theorem 4.2.5.36).

**Proposition 4.2.5.37.** *Consider an equivalence class $s \times \sigma_T \subset \{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$ of codimension one such that $T$ contains a circuit $C = \mathrm{conv}(\alpha_a, \alpha_b, \alpha_c)$ of type (C). Let $L_C \in \mathbb{K}_{\mathbb{R}}[x^{\pm}, y^{\pm}]$ denote a affine linear form such that $C \subset \mathscr{V}(L_C)$. Let $f = \bigoplus_i (s_i, u_i) w^{\alpha_i} \in \mathbb{TR}\left[w_1^{\pm}, w_2^{\pm}\right]$ denote the real tropical Laurent polynomial associated to a fixed element $(s,u) \in s \times \sigma_T$. By $e \subset \mathscr{T}_{\mathbb{R}}(f)_{(+,+)}$ we denote the edge dual to the circuit $C$. Then $q \in e$ is a singularity if and only if $q \in \frac{\partial f}{\partial L_C}$.*

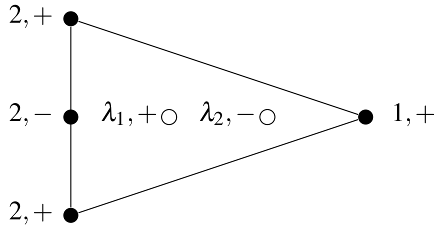
PROOF. First, we show "$\Rightarrow$". If $q \in \mathscr{T}_{\mathbb{R}}(f)$ is a singularity then $q \in \mathscr{T}_{\mathbb{R}}\left(\frac{\partial f}{\partial L}\right)$ for all affine linear forms $L \in \mathbb{K}_{\mathbb{R}}[x^{\pm}, y^{\pm}]$. In particular, $q \in \mathscr{T}_{\mathbb{R}}\left(\frac{\partial f}{\partial L_C}\right)$.
For "$\Leftarrow$", note that $q \in e \subset \mathscr{T}_{\mathbb{R}}(f)_{(+,+)}$ implies that $|f|(|q|)$ attains its maximum at monomials indexed by $\underline{C}$ and the signs differ since the signs of $C$ are alternating. Since $q \in \mathscr{T}_{\mathbb{R}}\left(\frac{\partial f}{\partial L_C}\right)$ and $\frac{\partial f}{\partial L_C} = \bigoplus_{i : L_C(\alpha_i) \neq 0} (s(L(\alpha_i)) s_i, u_i) w^{\alpha_i}$ we know that there are two monomials $\alpha, \alpha' \in \mathscr{A} \setminus \{L_C = 0\}$ such that $|\frac{\partial f}{\partial L_C}|(q)$ attains its maximum at $\alpha, \alpha'$ and the signs differ. Now, we show that $q \in \mathscr{T}_{\mathbb{R}}\left(\frac{\partial f}{\partial L}\right)$ for any affine linear form $L \in \mathbb{K}_{\mathbb{R}}[x^{\pm}, y^{\pm}]$. Therefore, let $L \in \mathbb{K}_{\mathbb{R}}[x^{\pm}, y^{\pm}]$ be any affine linear form.

(a) Suppose $|\mathscr{V}(L) \cap \{\alpha_a, \alpha_b, \alpha_c\}| = 0$. Then $\frac{\partial f}{\partial L}$ contains the terms of $f$ corresponding to $\alpha_a, \alpha_b$ and $\alpha_c$. Since $|f|(|q|)$ attains its maximum at $\alpha_a, \alpha_b$ and $\alpha_c$ we see that $|\frac{\partial f}{\partial L}|(|q|)$ attains its maximum at $\alpha_a, \alpha_b$ and $\alpha_c$ as well. If $\mathscr{V}(L) \cap \mathrm{conv}(\alpha_a, \alpha_b, \alpha_c) \neq \emptyset$ then (w.l.o.g.) $\alpha_a$ and $\alpha_b$ are contained in $L(x) > 0$. One of these points corresponds to the interior point of $C$. Hence, the signs of $\alpha_a$ and $\alpha_b$ in $\frac{\partial f}{\partial L}$ differ.

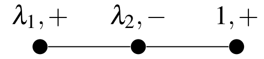
(b) Suppose $|\mathscr{V}(L) \cap \{\alpha_a, \alpha_b, \alpha_c\}| = 1$. W.l.o.g. this is $\alpha_a$. Moreover, $\alpha_b$ and $\alpha_c$ are contained in (w.l.o.g.) $L(x) > 0$ or $L$ separates $\alpha_b$ and $\alpha_c$. In the first case $\alpha_a$ is a vertex of $C$. Then the signs at $\alpha_b$ and $\alpha_c$ are alternating. If $\alpha_a$ is the interior point such that $L$ separates $\alpha_b$ and $\alpha_c$ then one sign flips as one point is contained in $L < 0$ and one is contained in $L > 0$.

(c) If $|\mathscr{V}(L) \cap \{\alpha_a, \alpha_b, \alpha_c\}| > 1$, all points of $C$ are contained in $L$ and we have $L \equiv L_C$. Then $q \in \mathscr{T}_{\mathbb{R}}\left(\frac{\partial f}{\partial L}\right)$ by assumption, i.e. $\frac{\partial f}{\partial L_C}(q)$ attains its maximum at the monomials $\alpha, \alpha'$.

Consequently, $q \in \mathscr{T}_{\mathbb{R}}\left(\frac{\partial f}{\partial L}\right)$ for all affine linear forms $L \in \mathbb{K}_{\mathbb{R}}[x^{\pm}, y^{\pm}]$, i.e. $q \in \mathscr{T}_{\mathbb{R}}(f)$ is a singularity (Theorem 4.2.5.36). $\square$




(A) Signed regular marked subdivision $T$ defined by $(s,u)$ corresponding to $f$.




(B) Signed regular marked subdivision $T^*$ defined by $(s^*,u^*)$ corresponding to $\frac{\partial f}{\partial L_C}$.

FIGURE 32. The signed regular marked subdivisions $T$ and $T^*$ dual to $f$ and $\frac{\partial f}{\partial L_C}$ (cf. Example 4.2.5.38).

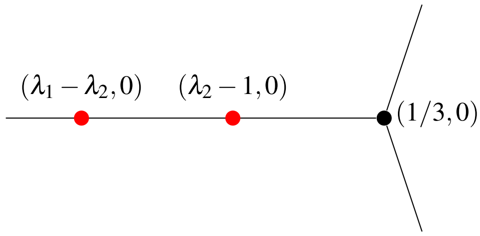



FIGURE 33. The $(+,+)$-chart of the real plane tropical curve $\mathscr{T}_{\mathbb{R}}(f)$ (cf. Example 4.2.5.38). The red dots illustrate the singularities.

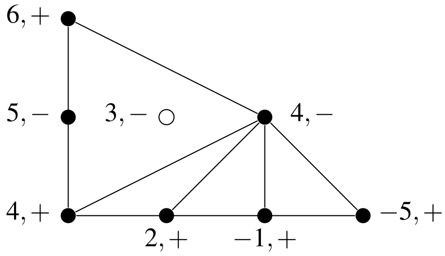
**Example 4.2.5.38.** We investigate the real plane tropical curve $\mathscr{T}_{\mathbb{R}}(f)$ defined in Example 4.2.4.35, i.e. we fix $u=(2,2,2,\lambda_1,\lambda_2,1)\in\sigma_T$ with $\lambda_1<\lambda_2<1$ and $2\lambda_2-\lambda_1>1$. We fix the sign vector $s=(+,-,+,+,-,+)$ such that $f=2^+\oplus 2^-w_2\oplus 2^+w_2^2\oplus\lambda_1^+w_1w_2\oplus\lambda_2^-w_1^2w_2\oplus 1^+w_1^3w_2$. The signed regular marked subdivision $T$ obtained from $(s,u)$ is shown in Figure 32a. We see a signed circuit $C=\operatorname{conv}(\alpha_a,\alpha_b,\alpha_c)$ and, therefore, we define $L_C(w)=w_1$. Then $\mathscr{A}^*=\mathscr{A}\setminus\{L_C=0\}$ is the support of $\frac{\partial f}{\partial L_C}$ and the corresponding signed regular marked subdivision $T^*$ defined by $(s^*,u^*)$ with $s^*=p_{\mathscr{A}^*}(s)$ and $u^*=p_{\mathscr{A}^*}(u)$ is shown in Figure 32b. The $(+,+)$-chart of $\mathscr{T}_{\mathbb{R}}(f^*)$ contains two infinite edges. Each edge intersects the edge $e\in\mathscr{T}_{\mathbb{R}}(f)_{(+,+)}$ dual to $C\subset T=T_{(+,+)}$ in a point, i.e. we have two singularities on $e\in\mathscr{T}_{\mathbb{R}}(f)_{(+,+)}$. This conforms with Proposition 4.2.4.33 as we see two valid slopes in the height profile $E^+$ of $\mathscr{A}^*$.

The set of singularities of a real plane tropical curve is not necessarily finite. The reason for this is that many algebraic curves tropicalize the tropical curve at hand. We exploit this phenomena explicitly in Section 4.3 where we consider singular real tropical surfaces. See also Theorem 4.3.2.8.

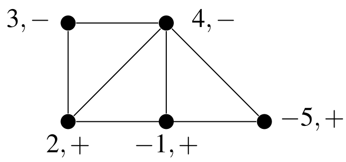
**Example 4.2.5.39.** Consider the real plane tropical curve $\mathscr{T}_{\mathbb{R}}(f)$ (cf. Figure 35) defined by

$$f=4^+\oplus 5^-w_2\oplus 6^+w_2^2\oplus 2^+w_1\oplus 3^-w_1w_2\oplus -1^+w_1^2\oplus 4^-w_1^2w_2\oplus -5^+w_1^3.$$

The signed regular marked subdivision $T$ dual to $f$ is shown in Figure 34a. We see a signed circuit $C=\operatorname{conv}(\alpha_a,\alpha_b,\alpha_c)$ contained in the affine line $L_C(w)=w_1$. Again $\mathscr{A}^*=\mathscr{A}\setminus\{L_C=0\}$ denotes the support of $\frac{\partial f}{\partial L_C}$ and the corresponding signed regular marked subdivision $T^*$ defined by $(s^*,u^*)$ with $s^*=p_{\mathscr{A}^*}(s)$ and $u^*=p_{\mathscr{A}^*}(u)$ is shown in Figure 34b. If we consider $u'=u-l_y$ we can see the distinctive feature of this curve. Then $u'_a=u'_b=u'_c$ become the highest heights and the points




(A) Signed regular marked subdivision $T$ defined by $(s,u)$ corresponding to $f$.




(B) Signed regular marked subdivision $T^*$ defined by $(s^*,u^*)$ corresponding to $\frac{\partial f}{\partial L_C}$.

FIGURE 34. The signed regular marked subdivisions $T$ and $T^*$ dual to $f$ and $\frac{\partial f}{\partial L_C}$ (cf. Example 4.2.5.39).

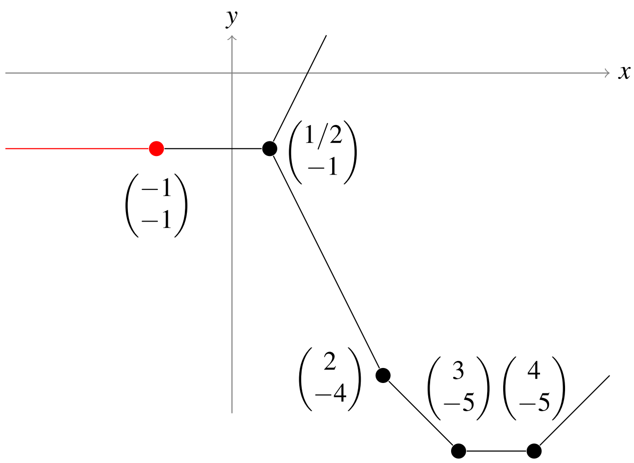



FIGURE 35. The $(+,+)$-chart of the real plane tropical curve $\mathscr{T}_{\mathbb{R}}(f)$ (cf. Example 4.2.5.39). The red colored parts illustrate the singularities of $\mathscr{T}_{\mathbb{R}}(f)$.

$\mathscr{A}\cap\{x=1\}$ have distinct signs and equal heights, i.e. we can change the heights with $\lambda l_x$ such that the heights at $\mathscr{A}\cap\{x=1\}$ get the highest heights among all points in $\mathscr{A}^*$. The value $\lambda$ is bounded from below by the slope $\lambda_1 = 1$ of the face formed by $\beta_1$ and $\beta_2$ in the height profile $E^+$ at $x=1$ of $\mathscr{A}^*$ with respect to $u'$. Hence, all $\lambda \geq \lambda_1$ are valid and provide real plane tropical curves (which are just translations of $\mathscr{T}_{\mathbb{R}}(f)$ along the $x$-axis) with a singularity at $\mathbf{0}_2$. Consequently, we get a infinite number of singularities on the edge dual to $C$ in $\mathscr{T}_{\mathbb{R}}(f)_{(+,+)}$.

### 4.2.6. Classification of Singular Real Plane Tropical Curves of Maximal Dimensional Type

As stated before the (signed) secondary fan is not the parameter space of (real) plane tropical curves since many (real) tropical polynomials can induce the same (real) plane tropical curve. In this section we restrict to equivalence classes of the signed secondary fan $\{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$ that do parametrize real plane tropical curves. We consider $\mathrm{trop}_{\mathbb{R}}(\nabla_{\mathbb{R}})$ within this subset of $\{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$. Then we can say a lot more about more about the subdivisions we get.

Consider a signed regular marked subdivision $T = \{(P_i, Q_i, s_{Q_i}) : i = 1,\dots,k\}$. Recall that the type of $T$ is $\mathscr{H} = \{P_i : i = 1,\dots,k\}$ (cf. Definition 4.2.1.1). In the complex case the type $\mathscr{H}$ is dual to a tropical curve $C'$ and we call $\mathscr{H}$ the type of $C'$. Let $C$ denote the real tropical curve dual to $T$. The chart $C_v$ of $C$ is dual to $T_v$ (cf. Remark 4.2.1.5) and, moreover, $|T_v| = |T|$, i.e. the type of $T_v$ equals the type of $T$. However, in contrast to the complex case, a chart $C_v$ of the real plane tropical curve $C$ dual to $T_v$ contains only parts of the curve $C'$ dual to $|T|$ owed to the sign conditions. The curves $C'$ dual to a given type $\mathscr{H}$ can be parametrized by an unbounded polyhedron in $\mathbb{R}^{b+2}$ where $b$ denotes the number of bounded edges in $C'$ and the two additional dimensions correspond to the spacial translation of the curve $C'$ in $\mathbb{R}^2$. Note that the lengths of the bounded edges cannot be changed individually if the tropical curve $C'$ is of genus $g \geq 1$. We get $2g$ not necessarily independent conditions corresponding to the closed circles in $C'$. The dimension of the parametrizing cone in $\mathbb{R}^{b+2}$ is called *type-dimension of* $\mathscr{H}$ and it is denoted by $\operatorname{tdim}(\mathscr{H})$ ([**MMS12a**, Subsection 2.2]). As the real tropical curve $C$ inherits its polyhedra from $C'$ this cone also parametrizes real tropical curves $C$ of type $\mathscr{H}$.

**Lemma 4.2.6.40** ([**MMS12a**, Lemma 2.5])**.** *Let $T$ be a signed marked subdivision of $\Delta = \operatorname{conv}(\mathscr{A})$ of type $\mathscr{H}$. Let $s \times \sigma_T \subset \{\pm\}^m \times \operatorname{Sec}_{\mathscr{A}} / \mathbf{1}_m$ denote the equivalence class of the secondary fan. Then,*

$$\operatorname{tdim}(\mathscr{H}) \leq \dim(s \times \sigma_T) \tag{77}$$

*and we have equality if and only if all lattice points of $\Delta$ are marked points in $T$.*

As a consequence we can study real plane tropical curves of maximal dimensional type by restricting to signed regular marked subdivisions without white points.

**Definition 4.2.6.41** (Maximal dimensional type)**.** An equivalence class $s \times \sigma_T \subset \{\pm\}^m \times \operatorname{Sec}_{\mathscr{A}}$ is of *maximal dimensional type* if all points of $T$ are marked, i.e. there are no white points in the subdivision $T$. By $\operatorname{trop}_{\mathbb{R}}(\nabla_{\mathbb{R}})_{\max}$ we denote the part of $\operatorname{trop}_{\mathbb{R}}(\nabla_{\mathbb{R}})$ that parametrizes singular real plane tropical curves of maximal dimensional type.

In the following we restrict ourselves to the parts of $\operatorname{trop}_{\mathbb{R}}(\nabla_{\mathbb{R}})_{\max}$ (and $\operatorname{trop}_{\mathbb{R}}(\nabla_{\mathbb{R},\mathbf{1}_2})_{\max}$) that have smallest codimension.

**Remark 4.2.6.42** ($\operatorname{trop}_{\mathbb{R}}(\nabla_{\mathbb{R}})_{\max}$ and $\{\pm\}^m \times \operatorname{Sec}_{\mathscr{A}}$)**.** Recall that top-dimensional weight classes $\sigma_{\mathscr{F}} \subset \mathscr{B}^s(M)$ arising from $s$-flags of type (a) live in equivalence classes $s \times \sigma_T \subset \{\pm\}^m \times \operatorname{Sec}_{\mathscr{A}}$ of codimension one (cf. Remark 4.2.4.27). If we restrict to parts that provide subdivisions of maximal dimensional type then Lemma 4.2.4.26 holds as well: the lineality group multiplied to the parts of all weight classes of type (a) providing subdivisions of maximal dimensional type corresponds to all equivalence classes of maximal dimensional type containing a circuit of type (A) or (B).
On the contrary, top-dimensional weight classes $\sigma_{\mathscr{F}} \subset \mathscr{B}^s(M)$ arising from $s$-flags of type (b) may partially live in equivalence classes of $\{\pm\}^m \times \operatorname{Sec}_{\mathscr{A}}$ of codimension two (cf. Remark 4.2.4.29). This is the case for $s$-flags $\mathscr{F}$ such that $F_{m-3,m-4} = \{a,b,c\} = \underline{C}$ for a signed circuit $C \in M[A']$ of type (C), $F_{k,k-1} = \{d,e\}$ for some $k \in [m-4]$ and $L_{abc} \parallel L_{de}$ (cf. notations in Remark 4.2.4.29). As we require that the points $\alpha_d$ and $\alpha_e$ are marked the regular marked subdivision belongs to a cone of $\operatorname{Sec}_{\mathscr{A}}$ of codimension two (cf. Remark 4.2.4.29 (a)). In particular, we see a quadrangle $\operatorname{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_d, \alpha_e)$ in the subdivision. To pass from this weight class to an equivalence class of $\{\pm\}^m \times \operatorname{Sec}_{\mathscr{A}}$ (and vice versa) we only have to add the $y$-coordinate-vector $l_y$ of rowspace$(A)$ — assumed that $C \subset \{x = c\}$ for some $c \in \mathbb{R}$.

At last we take a look at weight classes of type (b) as above but $L_{abc} \not\parallel L_{de}$. Then one point of $\mathscr{A}$ corresponding to an element of $F_{k,k-1} = \{d, e, \}$ is at non-minimal distance, e.g. $\alpha_d$. Let $Q$ be the polygon containing $C$ and $\alpha_d$. Assume that $Q$ has no more points in its boundary — otherwise the dimension drops for each additional point. Then, by Pick theorem, there is one interior point in $Q$ as its area is 2. This point must be marked as this is the requirement for all points in the subdivision. However, this point must have a higher height as $\alpha_d$ contradicting that we deal with heights of a flag of type (b). For more details see [**MMS12a**, Section 4].

In the remaining part of this section we go through the classification in Remark 4.2.3.22 and investigate what types of signed regular marked subdivisions we get from the parts of weight classes defined by flags of type (a) and (b) that provide subdivisions of maximal dimensional type. According to Remark 4.2.6.42 we have to study equivalence classes of codimension one of $\{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$ and those equivalence classes of codimension two that contain a quadrangle $\mathrm{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_d, \alpha_e)$ such that $\alpha_a, \alpha_b, \alpha_c \in L_{abc}$, $\alpha_d, \alpha_e \in L_{de}$ and $L_{abc} \parallel L_{de}$.

**Remark 4.2.6.43** (Classification of signed regular marked subdivisions of maximal dimensional type of $\mathrm{trop}_{\mathbb{R}}(\nabla_{\mathbb{R}})_{\max}$)**.** Let $s \in \{\pm\}^m$ be a sign vector and $\mathscr{F} \triangleleft \underline{M}$ a $s$-flag. Consider a signed regular marked subdivision $T = T_{(+,+)}$ induced by $(s,u) \in \mathrm{relint}(\sigma_{\mathscr{F}})$ without white points and minimal codimension. Here, we focus on the fact that there are no white points and extend the classification of Remark 4.2.3.22 for the maximal dimensional case.

(a) Suppose $\mathscr{F}$ is a $s$-flag of type (a), i.e. $F_{m-3,m-4} = \{a, b, c, d\}$ corresponds to a signed circuit of type (A) or (B). If the circuit is of type (A) we see a quadrangle $\mathrm{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_d)$ not covering any other points in the signed regular marked subdivision $T$. As the signs of the circuit are alternating the signed circuit is dual to a 4-valent vertex at $(0^+, 0^+)$. If the signed circuit is of type (B) then we see a triangle (w.l.o.g.) $\mathrm{conv}(\alpha_a, \alpha_b, \alpha_c)$ with one interior (and marked) point $\alpha_d$ in $T$. Then the signed circuit is dual to an isolated vertex at $(0^+, 0^+)$ of multiplicity 3. In order to get a maximal dimensional type of smallest codimension the remaining parts of $T$ are triangulated.

(b) Suppose $\mathscr{F}$ is a $s$-flag of type (b), i.e. $F_{m-3,m-4} = \{a, b, c\}$ and $F_{k,k-1} = \{d, e\}$ for some $k \in [m-4]$. As explained in Remark 4.2.3.22 we see a signed circuit $\mathrm{conv}(\alpha_a, \alpha_b, \alpha_c)$ of type (C) in $T$. We do not allow any white points in $T$ and, therefore, the points $\alpha_d$ and $\alpha_e$ have to be at minimal distance to the signed circuit $\mathrm{conv}(\alpha_a, \alpha_b, \alpha_c)$. Moreover, they have to be vertices of signed marked polytopes of $T$ in order to get a maximal dimensional type of smallest codimension. Let $L_{abc}$ denote the affine line through $\mathrm{conv}(\alpha_a, \alpha_b, \alpha_c)$. Recall that the heights at $\alpha_d$ and $\alpha_e$ are the highest heights among all points that are not on the line $L_{abc}$ (cf. Remark 4.2.2.17). Two subcases appear:

   (i) Suppose $L_{abc}$ separates $\alpha_d$ and $\alpha_e$. The vertices of $\mathrm{conv}(\alpha_a, \alpha_b, \alpha_c)$ have highest but equal height $\mu$ and $\alpha_d$ and $\alpha_e$ get highest height $\lambda$ of all points not on $L_{abc}$, i.e. $\mu \geq \lambda$. Hence, $\mathrm{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_d)$ and $\mathrm{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_e)$ form the triangles in $T$ that contain $C$ as we do not allow white points. Assume that $L_{abc} = \{x = 1\}$ and

$\alpha_d = \mathbf{0}_2$. We solve for the coordinates of the vertex dual to the triangle:

$$\begin{aligned} & \lambda + \langle \alpha_d, (x,y) \rangle = \mu + \langle \alpha_a, (x,y) \rangle = \mu + \langle \alpha_b, (x,y) \rangle \\ \Leftrightarrow \quad & \lambda = \mu + x + (\alpha_a)_2 y = \mu + x + (\alpha_b)_2 y. \end{aligned}$$

Without restriction, $\lambda = 0$ and, therefore, $\mu \geq 0$. Then $y = 0$ and $x = -\mu$, i.e. the vertex is at $(-\mu, 0)$. Using similar arguments and symmetry, and the fact that $\alpha_d$ and $\alpha_e$ have equal height, the second vertex is at $(\mu, 0)$. Moreover, the distance of $(0,0)$ to the vertices is equal. These vertices are part of $\mathscr{T}_{\mathbb{R}}(f)_{(+,+)}$ as the polygons $\mathrm{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_d)$ and $\mathrm{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_e)$ contain the signed circuit that has alternating signs. If the vertices dual to $\mathrm{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_d)$ and $\mathrm{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_e)$ are incident to other edges purely depends on their signs. Note that the vertex dual to $\mathrm{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_d)$ is incident to two more edges if and only if $s_a \neq s_d$ (without restriction we assume that $\alpha_a$ is a boundary vertex of the signed circuit $\mathrm{conv}(\alpha_a, \alpha_b, \alpha_c)$). As $\mathscr{F}$ is an $s$-flag we know that $s_d = s_e$, i.e. either both vertices dual to the polygons $\mathrm{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_d)$ and $\mathrm{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_e)$ are 3-valent or both vertices are 1-valent.

(ii) If $L_{abc}$ does not separate $\alpha_d$ and $\alpha_e$ then they are on the same side of $L_{abc}$. Recall that they need to be at minimal distance. Hence, they are on a line parallel to $L_{abc}$. Thus $\mathrm{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_d, \alpha_e)$ is a quadrangle not covering any other points. Let $\mu$ and $\lambda$ denote the heights of $\alpha_a, \alpha_b, \alpha_c$ and $\alpha_d, \alpha_e$ respectively. If $\mathrm{conv}(\alpha_a, \alpha_b, \alpha_c)$ is not contained in the boundary there is another vertex $\alpha_f$ with height $\nu$ at minimal distance on the other side of $L_{abc}$ that forms a triangle with $\alpha_a, \alpha_b, \alpha_c$. We solve for the coordinates of the vertex dual to the triangle as above, i.e. we suppose that $L_{abc} = \{x \in \mathbb{R}^2 : x_1 = 1\}$, $\alpha_f = \mathbf{0}_2$. Then:

$$\begin{aligned} & \nu + \langle \alpha_f, (x,y) \rangle = \mu + \langle \alpha_a, (x,y) \rangle = \mu + \langle \alpha_b, (x,y) \rangle \\ \Leftrightarrow \quad & \nu = \mu + x + (\alpha_a)_2 y = \mu + x + (\alpha_b)_2 y. \end{aligned}$$

By assumption we have $\nu < \lambda < \mu$ and we can assume that $\nu = 0$. Thus the vertex dual to the triangle $\mathrm{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_f)$ is at $(-\mu, 0)$. However, the vertex dual to the quadrangle $\mathrm{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_d, \alpha_e)$ is at $(\mu - \lambda, 0)$, due to its describing equations:

$$\begin{aligned} & \lambda + \langle \alpha_d, (x,y) \rangle = \lambda + \langle \alpha_e, (x,y) \rangle = \mu + \langle \alpha_a, (x,y) \rangle = \mu + \langle \alpha_b, (x,y) \rangle \\ \Leftrightarrow \quad & \lambda + 2x + (\alpha_d)_2 y = \lambda + 2x + (\alpha_e)_2 y = \mu + x + (\alpha_a)_2 y = \mu + x + (\alpha_b)_2 y. \end{aligned}$$

The distance from the vertex at $(-\mu, 0)$ dual to the triangle to the singularity at $\mathbf{0}_2 \in C_{(+,+)}$ is bigger than the distance from the distance of the vertex at $(\mu - \lambda, 0)$ dual to the quadrangle. In any case the vertex dual to $\mathrm{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_d, \alpha_e)$ is 3-valent as $\alpha_d$ and $\alpha_e$ are on the same side of $L_{abc}$, i.e. $s_d \neq s_e$. Thus one edge of $\mathrm{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_d, \alpha_e)$ has vertices with equal signs, i.e. the quadrangle is dual to a 3-valent vertex. Whether the vertex dual to the triangle $\mathrm{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_f)$ has higher valence purely depends on the sign $s_f$: if (w.l.o.g.) $\alpha_a$ is a vertex of $\mathrm{conv}(\alpha_a, \alpha_b, \alpha_c)$ then the vertex dual to the triangle is 3-valent if and only if $s_a \neq s_f$.

If $s_a = s_f$ then the vertex dual to the triangle is 1-valent. If $\mathrm{conv}(\alpha_a, \alpha_b, \alpha_c)$ is contained in the boundary we see the quadrangle $\mathrm{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_d, \alpha_e)$ in $T$ dual to a vertex at $(\mu - \lambda, 0) \in C_{(+,+)}$ and the singularity lies on the infinite edge dual to the circuit $\mathrm{conv}(\alpha_a, \alpha_b, \alpha_c)$.

THEOREM 4.2.6.44 (Classification of singular real plane tropical curves of maximal dimensional type). *Let $s \times \sigma_T \subset \{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$ be an equivalence class of maximal dimensional type. Consider the real tropical Laurent polynomial $f = \bigoplus_i (s_i, u_i) w^{\alpha_i} \in \mathbb{TR}\left[w_1^{\pm}, w_2^{\pm}\right]$ with support $\mathscr{A}$ defined by $(s,u) \in s \times \sigma_T$. Assume that $\mathscr{T}_{\mathbb{R}}(f)$ has a singularity at $\mathbf{0}_2^+$. Then the local picture of $\mathscr{T}_{\mathbb{R}}(f)_{(+,+)}$ around the singularity $\mathbf{0}_2^+$ matches with one of the following cases:*

*(a) $\mathbf{0}_2^+$ is a 4-valent vertex incident to 4 edges of weight one,*
*(b) $\mathbf{0}_2^+$ is an isolated vertex of multiplicity 3,*
*(c) $\mathbf{0}_2^+$ is the midpoint of an edge of weight 2 that is connecting*
- *either two 1-valent vertices, or*
- *two 3-valent vertices,*

*(d) $\mathbf{0}_2^+$ is contained in an interval of an edge of weight 2 connecting a 3-valent vertex and*
- *either a 1-valent vertex, or*
- *a 3-valent vertex,*

*and the boundaries of the interval are the 3-valent vertex and the midpoint, or*
*(e) $\mathbf{0}_2^+$ is contained in an infinite edge of weight 2 whose endpoint is a 3-valent vertex.*

### 4.3. Singular Real Tropical Surfaces

In this section we study singular real tropical surfaces. We adapt methods derived in Section 4.1 and Section 4.2. To keep things simple we use almost identical notations:

NOTATION 4.2. Let $\Delta \subset \mathbb{Z}^3$ be a convex lattice polytope and $\mathscr{A} = \Delta \cap \mathbb{Z}^3 = \{\alpha_1, \ldots, \alpha_m\}$ the set of lattice points of $\Delta$. For $\alpha \in \mathscr{A}$ we write $\alpha = (\alpha_x, \alpha_y, \alpha_z)$. Let $\mathbb{K}_{\mathbb{R}} = \mathbb{R}\{\{t\}\}$ be the ground field and $F = \sum_i y_i x^{\alpha_i} \in \mathbb{K}_{\mathbb{R}}[y_1, \ldots, y_m][x^{\pm}, y^{\pm}, z^{\pm}]$ a generic trivariate Laurent polynomial with fixed support $\mathscr{A}$ that is linear in the coefficients. We write $R = \mathbb{K}_{\mathbb{R}}[y_1, \ldots, y_m]$ for the polynomial ring forming the coefficients. By $F_a = \sum_i a_i x^{\alpha_i} \in \mathbb{K}_{\mathbb{R}}[x^{\pm}, y^{\pm}, z^{\pm}]$ we denote the Laurent polynomial obtained from $F$ with fixed coefficients $a = (a_1, \ldots, a_m) \in T_{\mathbb{R}}^m$ and $F(p) = \sum_i y_i p^{\alpha_i} \in \mathbb{K}_{\mathbb{R}}[y_1, \ldots, y_m]$ denotes the polynomial obtained from $F$ by evaluating at $p \in T_{\mathbb{R}}^3$. We study the family of real Laurent polynomials that provide a singular real surface:

$$\nabla_{\mathbb{R}} = \{a \in \mathbb{P}(T_{\mathbb{R}}^m :) \mathscr{V}(F_a) \text{ is singular.}\}.$$

In the following we write

$$\mathscr{I} = \langle F(\mathbf{1}_3), \frac{\partial F}{\partial x}(\mathbf{1}_3), \frac{\partial F}{\partial y}(\mathbf{1}_3), \frac{\partial F}{\partial z}(\mathbf{1}_3) \rangle \subset R \tag{78}$$

for the ideal generated by $F$ and its partial derivatives $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial z}$ evaluated at $\mathbf{1}_3$. Let $A \in \mathbb{Z}^{3 \times m}$ be the matrix representation of the point configuration $\mathscr{A}$. The coefficient matrix of $\mathscr{I}$ equals

$$A' = \begin{bmatrix} 1 & \cdots & 1 \\ \alpha_1 & \cdots & \alpha_m \end{bmatrix} = \begin{bmatrix} \mathbf{1}_m^{\top} \\ A \end{bmatrix} \in \mathbb{Z}^{4 \times m}, \tag{79}$$

It is the matrix representation of the shift $\mathscr{A}' = \{1\} \times \mathscr{A}$ of $\mathscr{A}$ into $\mathbb{R}^4$. By $\psi_{\mathscr{A}} : T^3_{\mathbb{R}} \to T^m_{\mathbb{R}}$ we denote the monomial map according to $\mathscr{A}$.

Most definitions and remarks of Section 4.2.1 can be transferred to surfaces if adapted for $n = 3$. In contrast to the curve case, we have $\{\pm\}^3 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and, therefore, a real tropical surface has $|\{\pm\}^3| = 8$ charts. Again, we write $G = \{\psi_{\mathscr{A}}(v) : v \in \{\pm\}^3\} \subset \{\pm\}^m$ such that we have

$$\mathrm{trop}_{\mathbb{R}}(\nabla_{\mathbb{R}}) = \mathrm{trop}_{\mathbb{R}}(\nabla_{\mathbb{R},\mathbf{1}_3}) \odot_{\mathbb{R}} G \times \mathrm{rowspace}(A)$$

according to Lemma 4.2.1.12. We start with the real tropicalization of $\nabla_{\mathbb{R},\mathbf{1}_3}$ analogue to Section 4.2.2.

### 4.3.1. Tropicalization of $\nabla_{\mathbb{R},\mathbf{1}_3}$

In this section we tropicalize $\nabla_{\mathbb{R},\mathbf{1}_3} = \mathscr{V}(\mathscr{I}) = \ker(A')$ (cf. Notation 4.2). Our point configuration $\mathscr{A}$ is a set in $\mathbb{R}^3$. Let $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be an affine independent set of $\mathscr{A}$. Then we can write

$$A' = \begin{bmatrix} \mathbb{E}_4 & \overline{A}' \end{bmatrix}$$

where $\overline{A}' \in \mathbb{Q}^{4 \times m-4}$ and $(\overline{A}')_j = \begin{bmatrix} a_j & b_j & c_j & d_j \end{bmatrix}^\top$ is the coefficient vector of $\alpha_j \in \mathscr{A}$ with respect to the affine basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ (cf. Lemma 3.1.2.3 (b)), i.e. we have $a_j + b_j + c_j + d_j = 1$ and $\alpha_j = a_j\alpha_1 + b_j\alpha_2 + c_j\alpha_3 + d_j\alpha_4$. Analogously to Remark 4.2.2.15 we obtain a Gale dual

$$G = \begin{bmatrix} -\left(\overline{A}'\right)^\top & \mathbb{E}_{m-4} \end{bmatrix} = \begin{bmatrix} g_1 & \cdots & g_m \end{bmatrix}.$$

We refer to $g_1, g_2, g_3$ and $g_4$ as the *special columns* of $G$.

In order to study $\mathrm{trop}_{\mathbb{R}}(\nabla_{\mathbb{R},\mathbf{1}_3})$ we can study $s$-flags of $M$, the oriented matroid associated to $\mathscr{I}$. These are heavily connected to affine dependencies among the points of $\mathscr{A}$:

**Remark 4.3.1.1** (Affine dependencies)**.** By $M[A']$ we denote the matroid encoding the affine dependencies among the points $\mathscr{A}$. Figure 36 shows a list of all signed circuits (up to a common change of signs) that may occur in $\mathbb{R}^3$. These can be categorized as follows: for a 3-dimensional circuit take a maximal affine independent set $B$ of points of $\mathscr{A}$ and consider its convex hull $P$. We obtain a 3-dimensional circuit $C$ from $P$ by adding a point to the point configuration $B$ that is either contained in the relative interior of $P$ (cf. Figure 36a) or it is not contained in $P$ (cf. Figure 36b). The same way we get all 2-dimensional circuits (cf. Figure 36c and Figure 36d). However, both cases coincide in one dimension (cf. Figure 36e). In this section we consider polytopes up to IUA-equivalence (cf. Remark 1.1.1.3). If nothing else is mentioned we consider all signed circuits in $\mathbb{R}^3$ with the signs shown in Figure 36.

**Lemma 4.3.1.2.** *Let $\mathscr{P} = \{\alpha_1, \dots, \alpha_n\} \in \mathbb{R}^n$ be an affine basis of an affine hyperplane $H \subset \mathbb{R}^n$ and $\beta_1, \beta_2 \in \mathbb{R}^n \backslash H$ two points. Then there is a circuit $C \in M[\mathscr{P} \cup \{\beta_1, \beta_2\}]$ with $\beta_1, \beta_2 \in C$ satisfying $(s_C)_{\beta_1} = \pm(s_C)_{\beta_2}$ if and only if $H$ separates/does not separate $\beta_1$ and $\beta_2$.*

Proof. Note that $\{\alpha_1, \dots, \alpha_n, \beta_i\}$ is affine independent in $\mathbb{R}^n$ for $i \in \{1,2\}$ whereas the set of points $\{\alpha_1, \dots, \alpha_n, \beta_1, \beta_2\}$ contains a circuit. Hence, there are $\lambda_1, \dots, \lambda_n, \mu_1, \mu_2 \in \mathbb{R}$ such that

$$\sum_{i,j} \lambda_i\alpha_i + \mu_j\beta_j = 0 \text{ and } \sum_{i,j} \lambda_i + \mu_j = 0.$$

In particular, we have $\mu_j \neq 0$ for $j = 1,2$. If this is not the case we have (w.l.o.g.) $\mu_1 = 0$ and, therefore, $\mu_2\beta_2 = -\sum_i \lambda_i\alpha_i$. Since $\mu_2 \neq 0$ we have $\beta_2 = -\sum_i \frac{\lambda_i}{\mu_2}\alpha_i$ with $-\sum_i \frac{\lambda_i}{\mu_2} = 1$, i.e. $\beta_2 \in H$ contradicting our assumptions. Therefore, we have $\mu_1, \mu_2 \neq 0$. Also note that at least one of the $\lambda_i$ is non-zero. Now, we show that $\mu_1, \mu_2 > 0$ if and only if $H$ separates $\beta_1$ and $\beta_2$.
First, suppose $\mu_1, \mu_2 > 0$. Hence, $\mu_1\beta_1 + \mu_2\beta_2 = -\sum_i \lambda_i\alpha_i$ and $-\sum_i \lambda_i = \mu_1 + \mu_2$. Then

$$\frac{1}{\mu_1+\mu_2}(\mu_1\beta_1 + \mu_2\beta_2) = \frac{1}{\mu_1+\mu_2}(-\sum_i \lambda_i\alpha_i) \tag{80}$$

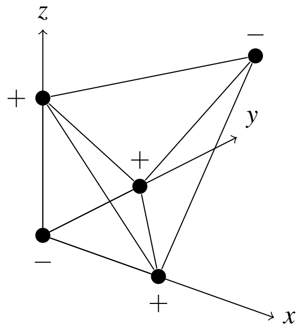
and $\frac{1}{\mu_1+\mu_2}(-\sum_i \lambda_i) = 1$. Hence, $\frac{1}{\mu_1+\mu_2}(\mu_1\beta_1 + \mu_2\beta_2) \in H$ as the right-hand side of Equation (80) is an affine combination of the $\alpha_i$. Moreover, the left-hand side of Equation (80) is a proper convex combination, i.e. $\frac{1}{\mu_1+\mu_2}(\mu_1\beta_1 + \mu_2\beta_2) \in \operatorname{conv}(\beta_1, \beta_2)$. Hence, $H$ separates $\beta_1$ and $\beta_2$.
Vice versa suppose $H$ separates $\beta_1$ and $\beta_2$. Let $L_{\beta_1,\beta_2} = \operatorname{conv}(\beta_1, \beta_2)$ denote the line segment connecting $\beta_1$ and $\beta_2$ and $v = L_{\beta_1,\beta_2} \cap H$ the intersection point. Then $\{\alpha_1, \ldots, \alpha_n, v\}$ contains a circuit, i.e. there are $\xi_i, \xi \in \mathbb{R}$ such that $\sum_i \xi_i\alpha_i + \xi v = 0$ and $\sum_i \xi_i + \xi = 0$. Moreover, the vertices $\beta_1, \beta_2, v$ form a circuit, i.e. there are $\nu_1, \nu_2, \nu \in \mathbb{R}$ such that $\sum_i \nu_i\beta_i + \nu v = 0$, $\sum_i \nu_i + \nu = 0$ and the signs satisfy $\operatorname{sign}(\nu_1) = \operatorname{sign}(\nu_2)$ (cf. Remark 4.2.2.16). Equivalently $v = \frac{1}{\nu}(-\sum_i \nu_i\beta_i)$. Substituting $v$ yields the equation $\sum_i \xi_i\alpha_i + \frac{\xi}{\nu}(-\sum \nu_i\beta_i) = 0$ where $\sum_i \xi_i + \frac{\xi}{\nu}(-\nu_1 - \nu_2) = 0$. Hence, $\operatorname{sign}(-\frac{\xi}{\nu}\nu_1) = \operatorname{sign}(-\frac{\xi}{\nu}\nu_2)$. □

**Remark 4.3.1.3** (Radon partitions)**.** Lemma 4.3.1.2 is a special case of the following well-known fact about *Radon partitions*: if $X$ is a subset of $d+2$ points in $\mathbb{R}^d$ and $Y \subset X$ a subset then $Y$ is contained in a *Radon half* if and only if $X \setminus Y$ is affinely dependent or $\operatorname{conv}(Y) \cap \operatorname{aff}(X \setminus Y) \neq \emptyset$ ([**Gru93**, Theorem 9.1]). A *Radon partition* of a subset $X$ of $\mathbb{R}^d$ is a pair $(A, B)$ of subsets such that $A \cup B = X$, $A \cap B = \emptyset$ and $\operatorname{conv}(A) \cap \operatorname{conv}(A) \neq \emptyset$. Each of $A$ and $B$ is called *Radon half*.

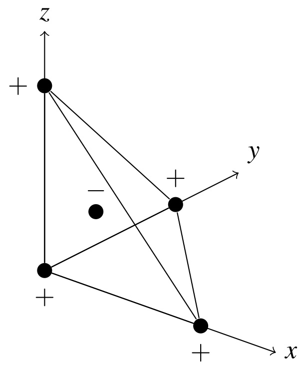
**Remark 4.3.1.4** (Maximal flags in $\underline{M}$)**.** [**MMS12b**, Lemma 10] classifies all maximal flags of flats $\mathscr{F} \lhd \underline{M}$. With the notations introduced in Definition 4.1.1.7 a maximal flag of flats $\mathscr{F} \lhd \underline{M}$ has one of the following forms:

(a) $|F_{m-3,m-4}| = 5$ and $|F_{j,j-1}| = 1$ for all $j \neq m-3$, or
(b) $|F_{m-3,m-4}| = 4$, $|F_{k,k-1}| = 2$ for some $k \neq m-3$ and $|F_{j,j-1}| = 1$ for all $j \neq k, m-3$, or
(c) $|F_{m-3,m-4}| = 3$, $|F_{k,k-1}| = 3$ for some $k \neq m-3$ and $|F_{j,j-1}| = 1$ for all $j \neq k, m-3$, or
(d) $|F_{m-3,m-4}| = 3$, $|F_{k,k-1}| = 2$, $|F_{l,l-1}| = 2$ for some $k < l \in [m-4]$ and $|F_{j,j-1}| = 1$ for all $j \neq l, k, m-3$.

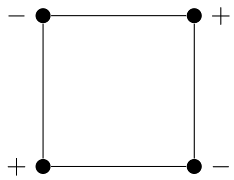
In case $(a)$ the set $F_{m-3,m-4}$ corresponds to a 3-dimensional circuit of type (A) (cf. Figure 36a) or (B) (cf. Figure 36b). In case $(b)$ the set $F_{m-3,m-4}$ corresponds to a 2-dimensional circuit $C$ of type (C) (cf. Figure 36c) or (D) (cf. Figure 36d), all points $\alpha_r$ with $r \in F_{m-3,k}$ are on the affine plane spanned by the circuit $C$ and no point of $F_{k,k-1}$ is on this plane. In case $(c)$ the set $F_{m-3,m-4}$ corresponds to a circuit $C$ of type (E) (cf. Figure 36e), all points $\alpha_r$ with $r \in F_{m-3,k}$ are contained in the affine line spanned by $C$ and each choice of two points indexed by $F_{k,k-1}$ spans the space together with the points indexed by $F_{m-3,m-4}$. In case $(d)$ the set $F_{m-3,m-4}$ corresponds to a circuit $C$ of type (E), all points $\alpha_r$ with $r \in F_{m-3,l}$ are contained in the affine line spanned by $C$, all points $\alpha_r$ with $r \in F_{l-1,k}$ are on the same plane as the three points of $F_{m-3,m-4}$ and the two points of $F_{l,l-1}$, and the two points indexed by $F_{k,k-1}$ are not contained in this plane.



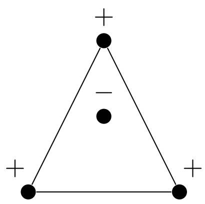

(A) Three dimensional circuit without interior point.



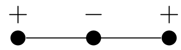

(B) Three dimensional circuit with an interior point.




(C) Two dimensional circuit without interior point.




(D) Two dimensional circuit with an interior point.




(E) One dimensional circuit.

FIGURE 36. Spacial circuits together with valid distributions of signs.

For flags $\mathscr{F} \lhd \underline{M}$ of type (a) and (b), analogous statements to Lemma 4.2.2.18 and Lemma 4.2.2.20 can be given. In contrast, cases (c) and (d) are considerably more complicated and require an intense study of all possible circuits with regard to the affine independent set $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Instead, we proceed as follows: we restrict to a coarsened and more general statement about $s$-flags of $M$ (cf. Lemma 4.3.1.5). As mentioned in the introduction we only deal with singular real tropical surfaces arising from *generic* points of $\mathrm{trop}_{\mathbb{R}}(\nabla_{\mathbb{R}})$ (introduced in Section 4.3.2 below). That is the locus where the structures of $\mathrm{trop}_{\mathbb{R}}(\nabla_{\mathbb{R}})$ and $\{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$ are best compatible ([**MMS12b**, section 3]). Thus we concentrate on those parts of $\mathrm{trop}_{\mathbb{R}}(\nabla_{\mathbb{R}})$ that guarantee *genericity*. The generic complex singular tropical surfaces are completely classified ([**MMS12b**, Theorem 2]). We can interpret an element of the (unsigned) secondary fan $\mathrm{Sec}_{\mathscr{A}}$, that is part of the complex tropical discriminant, as the modulus of an element in $\{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$. In order to become an element $\mathrm{trop}_{\mathbb{R}}(\nabla_{\mathbb{R}})$ these elements (the subdivisions respectively) have to satisfy certain sign conditions. This is where Lemma 4.3.1.5 comes into play. Hence, we bypass a direct classification of $s$-flags of $M$, go straight to the unsigned subdivisions arising from generic points of $\mathrm{trop}(\nabla)$ (classified in [**MMS12b**]) and study them considering Lemma 4.3.1.5:

**Lemma 4.3.1.5.** *Let $\mathscr{F} \lhd \underline{M}$ be a maximal flag of flats, $B \in M_{\mathscr{F}}$ a basis of $\underline{M}$ and $B^{\complement} = [m] \setminus B = \{b_1, b_2, b_3, b_4\}$ the index set of the affine basis $\{\alpha_{b_1}, \alpha_{b_2}, \alpha_{b_3}, \alpha_{b_4}\}$ of $\mathscr{A}$. Let $s \in \{\pm\}^m$ be a pure sign vector. Then $\mathscr{F}$ is an $s$-flag if and only if there is a signed circuit $C \in M[A']$ such that $F_{m-3,m-4} = \underline{C}$, $p_{F_{m-3,m-4}}(s) = p_{F_{m-3,m-4}}(s_C)$ and, moreover, for all $j \neq m-3$ such that $|F_{j,j-1}| > 1$ there is a signed circuit $C_j \in M[A']$ such that $\underline{C_j} \subset B^{\complement} \cup F_{j,j-1}$ and $p_{F_{j,j-1}}(s) = p_{F_{j,j-1}}(s_{C_j})$.*

### 4.3.2. Generic Singular Real Tropical Hypersurfaces

Remember that $\mathrm{trop}_{\mathbb{R}}(\nabla_{\mathbb{R}})$ is the union of all weight classes $\sigma_{\mathscr{F}} \subset \mathscr{B}^s(M)$ with $s \in \{\pm\}^m$ and $\mathscr{F} \triangleleft M$ multiplied with the lineality group $G \times \mathrm{rowspace}(A)$ (Lemma 4.2.1.12). For specific choices of $\mathscr{F}$ problems may arise, e.g. $\sigma_{\mathscr{F}} \odot_{\mathbb{R}} G \times \mathrm{rowspace}(A)$ is not top-dimensional. In this section we adapt the definition of a *generic* singular tropical hypersurface ([**MMS12b**, Definition 16]) for the real case. Recall that $\sigma(u)$ denotes the unique cone of the secondary fan $\mathrm{Sec}_{\mathscr{A}}$ corresponding to $u \in \mathbb{R}^m / \mathbf{1}_m$.

**Definition 4.3.2.6** (Defective weight class)**.** Let $M$ denote the oriented matroid associated to $\mathscr{I}$. A weight class $\sigma_{\mathscr{F}} \subset \mathscr{B}^s(M)$ is called *defective* if there exists a point $v \in \sigma_{\mathscr{F}} \odot_{\mathbb{R}} (G \times \mathrm{rowspace}(A))$ such that $\dim(|\sigma_{\mathscr{F}}| + \mathrm{rowspace}(A)) < \dim(\sigma(|v|))$.

Given a weight class $\sigma_{\mathscr{F}} \subset \mathrm{trop}_{\mathbb{R}}(\nabla_{\mathbb{R},\mathbf{1}_3})$ and $u \in \sigma_{\mathscr{F}}$ such that $\sigma(|u|)$ has codimension one, then the weight class $\sigma_{\mathscr{F}}$ is defective if and only if $\mathrm{aff}(\sigma_{\mathscr{F}}) \cap \mathrm{rowspace}(A) \neq \{0\}$ ([**MMS12b**, Remark 14]). The following definition is based on [**MMS12b**, Definition 16]:

**Definition 4.3.2.7** (Generic singular real tropical hypersurface.)**.** A point $v \in \mathrm{trop}_{\mathbb{R}}(\nabla_{\mathbb{R}})$ is called *generic* if the point is not contained in the union of all $\sigma_{\mathscr{F}} \odot_{\mathbb{R}} (G \times \mathrm{rowspace}(A))$ where the weight class $\sigma_{\mathscr{F}} \subset \mathrm{trop}_{\mathbb{R}}(\nabla_{\mathbb{R},\mathbf{1}_3})$ is either defective, or satisfies $\dim(|\sigma_{\mathscr{F}}| + \mathrm{rowspace}(A))) > \dim(\sigma(v))$, or is not top-dimensional. We call a singular real tropical hypersurface $\mathscr{T}_{\mathbb{R}}(f)$ generic if its defining real tropical Laurent polynomial $f$ comes from a generic element $(s,u) = (s_i,u_i)_{i\in[m]} \in \mathrm{trop}_{\mathbb{R}}(\nabla_{\mathbb{R}})$, i.e. $f = \bigoplus_i (s_i,u_i) w^{\alpha_i}$ with $(s,u)$ is generic.

The polyhedra of any $v$-chart of a real tropical surface $\mathscr{T}_{\mathbb{R}}(f)$ defined by $f \in \mathbb{TR}\left[w_1^{\pm}, w_2^{\pm}, w_3^{\pm}\right]$ come from polyhedra of $\mathscr{T}(|f|)$, i.e. $\mathscr{T}(|f|)$ can be seen as a bound on the variety of polyhedra in $\mathscr{T}_{\mathbb{R}}(f)$. The following theorem justifies why we spotlight on generic points of the real tropical discriminant:

THEOREM 4.3.2.8 ([**MMS12b**, Theorem 2]). *Let $\sigma_T \subset \mathrm{Sec}_{\mathscr{A}}$ be a cone of codimension $c$ and consider $f_u = \bigoplus_i u_i w^{\alpha_i} \in \mathbb{T}\left[w_1^{\pm}, w_2^{\pm}, w_3^{\pm}\right]$ for some $u \in \sigma_T$. If $u \in \sigma_T$ is generic then the set of singularities of $\mathscr{T}(f)$ is a finite union of finitely many polyhedra of dimension $c-1$.*

### 4.3.3. Classification of Generic Singular Real Tropical Surfaces of Maximal Dimensional Type

In order to study real tropical surfaces with a finite number of singularities we restrict to generic points in codimension one equivalence classes of $\mathrm{trop}_{\mathbb{R}}(\nabla_{\mathbb{R}})$ (Theorem 4.3.2.8). Lemma 4.2.6.40 holds for surfaces as well (cf. [**MMS12b**, Definition 2.2, Lemma 2.3] for details). Therefore, we additionally require that all lattice points are marked, i.e. we only deal with equivalence classes of maximal dimensional type (cf. Definition 4.2.6.41).

THEOREM 4.3.3.9 (Classification of generic singular real tropical surfaces of maximal dimensional type). *Let $f_{(s,u)} = \bigoplus_i (s_i,u_i) w^{\alpha_i} \in \mathbb{TR}\left[w_1^{\pm}, w_2^{\pm}, w_3^{\pm}\right]$ define a generic singular real tropical surface where $(s,u) \in s \times \sigma_T$ is of codimension one and $T$ is of maximal dimensional type. Then there is some $v \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ such that $T_v$ contains a signed circuit $C \in M[A']$ and $\mathscr{T}_{\mathbb{R}}(f)$ contains finitely many singularities. Their locations and dual polytopes, classified up to IUA-equivalence (cf. Remark 1.1.1.3) are as follows:*

(a) *If the circuit $C$ is 3-dimensional (cf. Figure 36a and Figure 36b) then the vertex $V$ dual to $C$ is the only singularity. Moreover:*

*(a.1)* *either V is adjacent to six edges and six polyhedra of dimension* 2 *dual to the faces of a signed pentatope C (cf. Figure 36a) given by the vertices (up to IUA-equivalence)* $v_1 = (0,0,0)$, $v_2 = (1,0,0)$, $v_3 = (0,1,0)$, $v_4 = (0,0,1)$, $v_5 = (1,p,q)$ *with* $p,q$ *coprime and the sign vector (unique up to a factor)* $(s)_{i=1,\dots,5} = (-,+,+,+,-)$ *(cf. Section 4.3.3.1),*

*(a.2)* *or V is an isolated vertex dual to a simplicial polytope with an interior point that (up to IUA-equivalence) has vertices* $v_1 = (0,0,0)$, $v_2 = (1,0,0)$, $v_3 = (0,1,0)$ *and* $v_4$ *which is one of the points* $(3,3,4)$, $(2,2,5)$, $(2,4,7)$, $(2,6,11)$, $(2,7,13)$, $(2,9,17)$, $(2,13,19)$ *or* $(3,7,20)$. *In any of these cases the proper simplex vertices have equal signs different to the sign of the interior point. The singularity V is distinct from other vertices of* $\mathscr{T}_{\mathbb{R}}(f)$ *by its multiplicity, i.e. the volume of the simplicial polytope (cf. Section 4.3.3.2).*

*(b)* *If the circuit C is* 2*-dimensional (cf. Figure 36c and Figure 36d) there is a dual edge E. The following cases may appear:*

*(b.1)* *The circuit C is IUA-equivalent to a triangle with vertices* $v_1 = (0,0,0)$, $v_2 = (0,1,2)$ *and* $v_3 = (0,2,1)$ *having an interior point* $v_4 = (0,1,1)$ *and signs (unique up to a factor)* $(s)_{i=1,2,3} = + \neq (s)_4$. *The dual edge E is not adjacent to 2-dimensional cells.*

*(b.1.1)* *E is bounded and both vertices of E are either adjacent to four edges and three 2-dimensional polyhedra or only adjacent to the edge E. There is a singularity at the midpoint of E or the singularity is at points that divide E with the ratio* $3:1$ *(cf. Section 4.3.3.3 part (1)).*

*(b.1.2)* *E is bounded or unbounded, the incident vertices may have different adjacencies and the singularity is contained in a virtual edge (cf. Section 4.3.3.4 part (2), for more explanations concerning the virtual edge see* [**MMS12b**, section 4.3.2]*).*

*(b.2)* *The circuit C is IUA-equivalent to a quadrangle whose vertices have alternating signs. The dual edge E is bounded and adjacent to four 2-dimensional polyhedra. Each vertex of E is adjacent to five edges. At one of these vertices, the four edges different to E split up in two pairs of edges that are adjacent to 2-dimensional polyhedra (cf. Section 4.3.3.4).*

*(c)* *If the circuit C is* 1*-dimensional (cf. Figure 36e) there is a dual* 2*-dimensional polyhedron S. The interior lattice point of C has a different sign as the boundary points.*

*(c.1)* *If S is a triangle there is a singularity at the weighted barycenter. The triangle is either isolated or there are two edges at each vertex and each edge is adjacent to two 2-dimensional polyhedra. These polyhedra split up in two disjoint polyhedral sets, each connected in codimension one (cf. Section 4.3.3.5).*

*(c.2)* *If S is a trapeze there is a singularity at its midpoint. The diametral vertices of S have equal adjacencies. Hence, the trapeze is either isolated, or each of the four vertices of the trapeze is adjacent to two more edges and a polyhedron, and two diametral edges of S are adjacent to two polyhedra, or each vertex of the trapeze S is adjacent to two edges and each edge of S is adjacent to two polyhedra and the set of adjacent edges*

*and polyhedra split up in two disjoint polyhedral sets, each connected in codimension one (cf. Section 4.3.3.6).*

*(c.3) S is an arbitrary polyhedron that admits finitely many extensions to a triangle or trapeze with a singularity as explained in (c.1) and (c.2). (cf. Section 4.3.3.5 and Section 4.3.3.6).*

Almost all parts in the following proof concerning the unsigned subdivisions and classification of polytopes were shown in [**MMS12b**]. However, in order to get a full picture, we summarize the basic steps of the proof of [**MMS12b**, Theorem 2] and then add the sign conditions.

PROOF. For simplicity we write $f = f_{(s,u)}$ in the following. Recall that $|T| = |T_v|$ for all $v \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathscr{T}_{\mathbb{R}}(f)$ inherits its polyhedra from $\mathscr{T}(|f|)$ where $|f|$ is dual to $|T|$. Without restriction we have $v = (+,+,+)$ so that $T$ contains a signed circuit $C \in M[A']$. We use the classification stated in [**MMS12b**, Section 4] dealing with the complex case. In particular, any $u \in \operatorname{trop}_{\mathbb{R}}(\nabla_{\mathbb{R}})$ can be written as $u = v + l$ for some $v \in \operatorname{trop}_{\mathbb{R}}(\nabla_{\mathbb{R},\mathbf{1}_3})$ and $l \in \operatorname{rowspace}(A)$. Therefore, we use the classification of all possible maximal flags of flats of $\underline{M}$ (cf. Remark 4.3.1.4) and take the sign conditions (cf. Lemma 4.3.1.5) into consideration. As the sum $u = v + l$ is not necessarily unique we also investigate the action of the lineality group in these cases.

#### 4.3.3.1. Weight Classes of Type (a), Circuit Type (A)

Let $u \in \operatorname{trop}_{\mathbb{R}}(\nabla_{\mathbb{R},\mathbf{1}_3})$ be contained in a weight class $\sigma_{\mathscr{F}}$ defined by a $s$-flag $\mathscr{F}$ of type (a) (cf. Remark 4.3.1.4) such that $C = F_{m-3,m-4} = \{a,b,c,d,e\}$ is a signed circuit of type (A). The points of $\mathscr{A}$ corresponding to $F_{m-3,m-4}$ get highest height, i.e. they form a pentatope that is part of the signed regular marked subdivision $T$ induced by $u$. We require that the pentatope does not contain any other lattice points beside those indexed by $F_{m-3,m-4}$ as $T$ must be of maximal dimensional type. Then, up to IUA-equivalence, the vertices are $\alpha_a = (0,0,0)$, $\alpha_b = (1,0,0)$, $\alpha_c = (0,1,0)$, $\alpha_d = (0,0,1)$ and $\alpha_e = (1,p,q)$ with $p$ and $q$ coprime (cf. [**MMS12b**, section 4.1]). Due to Lemma 4.3.1.5 the signs at $\alpha_b, \alpha_c$ and $\alpha_d$ are (w.l.o.g.) "+" and the remaining vertices $\alpha_a$ and $\alpha_e$ get "−". These vertices have equal heights, i.e. $u_{\alpha_i} = u_{\alpha_j}$ for $i,j \in \{a,b,c,d,e\}$ and all signs appear, i.e. the pentatope is dual to $\mathbf{0}_3^+ \in \mathscr{T}_{\mathbb{R}}(f)$. Note that $\alpha_b, \alpha_c, \alpha_d \in H_{\mathbf{1}_3,1}$ (cf. Example 1.1.1.4), i.e. they form a 2-dimensional simplex, and they have equal sign "+", i.e. there are no 2-dimensional polyhedra in $\mathscr{T}_{\mathbb{R}}(f)_{(+,+,+)}$ dual to the edges of the simplex formed by the vertices $\{\alpha_b, \alpha_c, \alpha_d\}$. As $\alpha_a$ and $\alpha_e$ have "−" the node $\mathbf{0}_3^+ \in \mathscr{T}_{\mathbb{R}}(f)_{(+,+,+)}$ is adjacent to six edges and six polyhedra of dimension 2. To see this consider $\alpha_a$ as the tip of the pyramid formed by the simplex in $H_{\mathbf{1}_3,1}$. The tip $\alpha_a$ has a different sign as the vertices of the simplex. Hence, each face provides a dual edge and each edge provides a dual 2-dimensional polyhedron. This shows (a.1).

#### 4.3.3.2. Weight Class of Type (a), Circuit Type (B)

Let $u \in \operatorname{trop}_{\mathbb{R}}(\nabla_{\mathbb{R},\mathbf{1}_3})$ be contained in a weight class $\sigma_{\mathscr{F}}$ defined by a $s$-flag $\mathscr{F}$ of type (a) (cf. Remark 4.3.1.4) such that $C = F_{m-3,m-4}$ is a signed circuit of type (B). The polytope formed by the five points of $\mathscr{A}$ indexed by $F_{m-3,m-4}$ is a simplex, i.e. there is an interior point. However, we require that this polytope contains no other points beside the lattice points corresponding to $F_{m-3,m-4}$, i.e. there is exactly one interior point. According to [**MMS12b**, Section 4.2] there are eight contemplable polytopes up to IUA-equivalence: $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and respectively $(3,3,4)$, $(2,2,5)$, $(2,4,7)$,

$(2,6,11)$, $(2,7,13)$, $(2,9,17)$, $(2,13,19)$ or $(3,7,20)$. In order to satisfy Lemma 4.3.1.5 the interior lattice point gets a "$-$" and the (proper) vertices get "$+$". As the heights are equal the polytope is dual to $\mathbf{0}_3^+$. Due to the sign distribution, $\mathbf{0}_3^+ \in \mathscr{T}_{\mathbb{R}}(f)_{(+,+,+)}$ is an isolated vertex. As the proper vertices have equal signs the faces of the polytope do not contribute to $\mathscr{T}_{\mathbb{R}}(f)_{(+,+,+)}$ by duality. The other simplicial polytopes of $T$ contain no other lattice points beside their proper vertices. Therefore, the singularity $\mathbf{0}_3^+$ has higher multiplicity. This shows (a.2).

#### 4.3.3.3. Weight Class Type (b), Circuit Type (C)

Consider a $s$-flag $\mathscr{F}$ of type (b) (cf. Remark 4.3.1.4), i.e. $C = F_{m-3,m-4} = \{a,b,c,d\}$ yields a signed circuit of type (C). Moreover, we have $F_{k,k-1} = \{e,f\}$ for some $k \in [m-4]$. Due to Remark 4.3.1.4 the points $\alpha_e$ and $\alpha_f$ are not contained in the affine plane $\operatorname{aff}(C)$. With regard to Lemma 4.3.1.2 we have $s_e = \pm s_f$ if and only if $\alpha_e$ and $\alpha_f$ are on different/the same side of $\operatorname{aff}(C)$. Keeping this in mind we go through the classification of subdivisions. We distinguish the two following situations:

(1) Both $\alpha_e$ and $\alpha_f$ form a pyramid with base $C = \operatorname{conv}(\alpha_a,\ldots,\alpha_d)$.

(2) At most one of the points $\alpha_e$ and $\alpha_f$ forms a pyramid with base $C = \operatorname{conv}(\alpha_a,\ldots,\alpha_d)$.

<u>(1):</u> If both $\alpha_e$ and $\alpha_f$ form pyramids with $C = \operatorname{conv}(\alpha_a,\ldots,\alpha_d)$ then $\operatorname{aff}(C)$ separates $\alpha_e$ and $\alpha_f$. As the subdivision needs to be of maximal dimensional type no further lattice points are allowed in these two pyramids. Thus, due to [**MMS12b**, Lemma 18], $\alpha_e$ and $\alpha_f$ must have integral distance 1 or 3 to the base $C = \operatorname{conv}(\alpha_a,\ldots,\alpha_d)$. Assume that $C \subset \{x = 0\}$ so that (w.l.o.g.) $(\alpha_e)_x < 0$ and $(\alpha_f)_x > 0$. Then either $|(\alpha_e)_x| = |(\alpha_f)_x|$ or $|(\alpha_e)_x| \neq |(\alpha_f)_x|$. We solve for the coordinates of the vertices dual to the pyramids. Therefore, let $\lambda$ be the height of $\alpha_a,\ldots,\alpha_d$ and $\mu$ the height of $\alpha_e,\alpha_f$. From $C$ we conclude that the dual edge satisfies $y = z = 0$ as the heights at $\alpha_a,\ldots,\alpha_d$ are equal and $C \subset \{x = 0\}$. Consequently, the vertex of the pyramid formed by $C$ and $\alpha_e$ is at $(\mu-\lambda,0)$ if $(\alpha_e)_x = -1$ and $\frac{1}{3}(\mu-\lambda,0)$ if $(\alpha_e)_x = -3$. We can solve for the coordinates of the vertex dual to the pyramid formed by $\alpha_f$ and $C$ analogously. Thus the singularity $\mathbf{0}_3^+$ is either exactly the middle of the edge dual to $C$ or it divides the edge into parts with ratio 1 : 3. As we have $s_e = s_f$ and the vertices on the boundary of $C$ have equal signs the edge dual to $C$ is either isolated or both vertices are adjacent to four edges and three 2-dimensional polyhedra each. In both cases the edge $E$ is not adjacent to 2-dimensional polyhedra. This shows $(b.1.1)$.

<u>(2):</u> Now, at least one of the points $\alpha_e$ and $\alpha_f$ does not form a pyramid with $C$. Without restriction let this be $\alpha_e$. We begin with a summary of some facts shown in [**MMS12b**, section 4.3.2] concerning the unsigned subdivisions. Any point $\alpha$ forming a pyramid with $C = \operatorname{conv}(\alpha_a,\alpha_b,\alpha_c,\alpha_d) \subset \{x = 0\}$ must have integral distance 1 or 3 to $C$ ([**MMS12b**, Lemma 18]). Any point $\alpha \neq \alpha_e$ that is one the same side of $\operatorname{aff}(C)$ as $\alpha_e$ and forms a pyramid with $C$ cannot have integral distance 1 — otherwise $\alpha_e$ would have higher height as $\alpha$ because $\alpha_e$ and $\alpha_f$ have highest height among all points not contained in $\operatorname{aff}(C)$. Hence, $\alpha$ has integral distance 3. From now on assume that $\alpha_e$ and $\alpha$ are on the positive side of $\operatorname{aff}(C)$. Then $\alpha_e$ has integral distance 1 to $C$: assume it is 2 then $\operatorname{conv}(\alpha_a,\alpha_b,\alpha_c,\alpha_d,\alpha_e)$ would contain other lattice points ([**MMS12b**, Lemma 18]) that cannot be marked. Hence, $(\alpha_e)_x = 1$. Let $u \in \operatorname{trop}_{\mathbb{R}}(\nabla_{\mathbb{R}})$ be an element that induces the unsigned subdivision. Now we take the action of the lineality group into consideration. First, we stay in the $v = (+,+,+)$-chart and manipulate the modulus only, i.e. we restrict to elements $(+^m,l) \in G \times \operatorname{rowspace}(A)$. There can be several possibilities for a decomposition $u = v \odot_{\mathbb{R}} (+^m,l)$ with $v \in \operatorname{trop}_{\mathbb{R}}(\nabla_{\mathbb{R},\mathbf{1}_3})$ and $l \in \operatorname{rowspace}(A)$ where

$v \in \sigma_{\mathscr{F}}$ is contained in a $s$-flag of type (b) satisfying our assumptions. This situation is analogous to Remark 4.2.4.31 in the curve case, see also Example 4.2.5.38 and Example 4.2.5.39. Recall that in the complex case any slope in the height profile in the valid $x$-range provided a feasible weight class. Now we examine the surface case: by adding a suitable element of rowspace$(A)$ we can assume that $|u|_a = |u|_b = |u|_c = |u|_d$ are the highest heights. The next step is to add a multiple of $l_x$ — the generator of rowspace$(A)$ containing all $x$-coordinates of all points of $\mathscr{A}$ — such that two points get equal heights and the highest heights of all points not on $\{x=0\}$. These are the points $\alpha_e$ and $\alpha_f$. From the assumption that we consider a generic point that induces this subdivision of maximal dimensional type we conclude that the height profile with respect to $|u|$ along the $x$-axis contains unique points at $x=\pm 1$ that are highest, and points at $x=\pm 3$ that form pyramids with $C$ as base. These are the candidates for $\alpha_e$ and $\alpha_f$, i.e. for a certain choice of two of the described points at $x=\pm 1$ and $x=\pm 3$ we can make their heights equal and the resulting element $|v| = |u| - l$ provides a valid (unsigned) weight class. Now we take the signs into consideration. Any choice has to satisfy that $s_e = \pm s_f$ if and only if $\alpha_e$ and $\alpha_f$ are separated/not separated by $\{x=0\}$. We examine the effects of changing signs with elements $g \in G$ (respectively $(g, \mathbf{0}_m) \in G \times \text{rowspace}(A)$). As $C \subset \{x=0\}$ and the $x$-coordinates of $\alpha_e$ and $\alpha_f$ differ by a multiple of 2, the element $\psi_{\mathscr{A}}((-,+,+)) \in G$ changes $s_e$ and $s_f$ simultaneously. Hence, we can restrict to the subgroup of $G$ generated by $\psi_{\mathscr{A}}((+,-,+))$ and $\psi_{\mathscr{A}}((+,+,-))$ that change signs in $\{x=0\}$. However, the signs at vertices of $C$ get changed by the elements in this subgroup such that they do not coincide with the signs of a signed circuit of type (C). Hence, if we see a signed circuit $C$ in $T$ then we have to deal with the signs of the points at $x=\pm 1$ and $x=\pm 3$. In total this leads to the following four possibilities for weight classes satisfying that $\alpha_e$ does not form a pyramid with $C$ and $(\alpha_e)_x = 1$:

(1) $(\alpha)_x = 3$, $(\alpha_f)_x = -1$, $s_e = s_f$ such that $\alpha$ and $\alpha_f$ form pyramids with $C$.
(2) $(\alpha)_x = 3$, $(\alpha_f)_x = -3$, $s_e = s_f$ such that $\alpha$ and $\alpha_f$ form pyramids with $C$.
(3) $(\alpha)_x = 3$, $(\alpha_f)_x = -1$, $s_e = s_f$, and there is some $\alpha'$ with $(\alpha')_x = -3$ such that $\alpha$ and $\alpha'$ form pyramids with $C$.
(4) $(\alpha_f)_x = 3$ and $\alpha_f$ forms a pyramid with $C$ and $s_e \neq s_f$.

In these situations the sum of rowspace$(A)$ and the corresponding (unsigned) weight classes is direct. As $\text{trop}_{\mathbb{R}}(\nabla_{\mathbb{R}}) = \text{trop}_{\mathbb{R}}(\nabla_{\mathbb{R},\mathbf{1}_3}) \odot_{\mathbb{R}} G \times \text{rowspace}(A)$ one of the four cases applies to our situation. Let us focus on the signs. Due to Lemma 4.3.1.5 the signs $s_e$ and $s_f$ are related by their positions relatively to the affine plane $\text{aff}(C)$. In the first three cases we have $s_e = s_f$ whereas in the last case we have $s_e = -s_f$. Note that the signs of $\alpha$ (and $\alpha'$ in the third case) can be arbitrary. Consequently, the vertices forming the tips of the pyramids can have identical or different signs.

Note that the location of the singularity only depends on the heights at $\alpha_a$, $\alpha_b$, $\alpha_c$, $\alpha_d$, $\alpha_e$ and $\alpha_f$, not on $\alpha$ and $\alpha'$. Therefore, *virtual vertices* and *virtual edges* were introduced (cf. [**MMS12b**, section 4.3.2]). The virtual vertex $v_e$ is the vertex dual to the pyramid formed by $\alpha_e$ and $C$, $v_f$ analogously. The virtual edge connects $v_e$ and $v_f$. It is shown in [**MMS12b**] that the virtual edge contains the actual edge with vertices corresponding to the actual pyramids containing $C$ and, moreover, the virtual vertex may coincide with the actual vertex. In the first and third case the singularity is the middle point of the virtual edge, in the second case it is a point that divides the virtual edge with ration 3:1 (cf. part (1.) :). This shows (b.1.2).

#### 4.3.3.4. Weight Class Type (b), Circuit Type (D)

Suppose $u \in \operatorname{trop}_{\mathbb{R}}(\nabla_{\mathbb{R},\mathbf{1}_3})$ is contained in a weight class $\sigma_{\mathscr{F}}$ defined by a $s$-flag $\mathscr{F}$ of type (b) (cf. Remark 4.3.1.4) and $C = F_{m-3,m-4} = \{a,b,c,d\}$ is a signed circuit of type (D). Moreover, we have $F_{k,k-1} = \{e,f\}$ for some $k \in [m-4]$. First, suppose $\alpha_e$ and $\alpha_f$ are separated by $\operatorname{aff}(C)$. Lemma 4.3.1.2 states that $s_e = s_f$ in this case. Moreover, these have highest height of all points not contained in $\operatorname{aff}(C)$, i.e. both form a pyramid with $C$. Due to [**MMS12b**, section 4.4] the tips, i.e. $\alpha_e$ and $\alpha_f$, have integral distance 1 to $C$. If we assume that $C \subset \{x = 0\}$ we can easily solve for the vertices dual to these two pyramids. Suppose $(\alpha_e)_x = -1$ and $(\alpha_f)_x = 1$ and let $\lambda$ be the height of $\alpha_a, \alpha_b, \alpha_c$ and $\mu$ the height of $\alpha_e$ and $\alpha_f$. Due to $C \subset \{x = 0\}$ and its quadrangle shape the dual edge satisfies $x = y = 0$. Analogously to the determination of the coordinates of the vertices dual to the pyramids in (b.1.1), the vertex dual to the pyramid with $\alpha_e$ is at $(\mu - \lambda, 0, 0)$ and the vertex dual to the pyramid with $\alpha_f$ is at $(\lambda - \mu, 0, 0)$. As the singularity is at $\mathbf{0}_3^+$ it is the middle point of the edge connecting $(\lambda - \mu, 0, 0)$ and $(\mu - \lambda, 0, 0)$. Recall that the vertices of $C$ have alternating signs. Two of the four edges of the pyramid connecting the tip with a base vertex have alternating signs. Since we have $s_e = s_f$ this is symmetric, i.e. the edges of the pyramids with alternating signs are adjacent. Consequently, each face containing the tip is dual to an edge and there are two 2-dimensional polyhedra adjacent dual to the edges connecting the tip with a base vertex of $C$. The edge dual to $C$ is adjacent to four 2-dimensional polyhedra.
Now suppose $\alpha_e$ and $\alpha_f$ are on the same side of $\operatorname{aff}(C)$. Due to [**MMS12b**, section 4.4] both points must have integral distance 1 to $C$. Then $\alpha_e$ and $\alpha_f$ form a “triangular roof” of $C$. However, the cone of $\operatorname{Sec}_{\mathscr{A}}$ corresponding to this (unsigned) regular marked subdivision does not have codimension one, i.e. the surface is not generic and we do not care for these cases. This ends the proof of (b.2).

#### 4.3.3.5. Weight Class Type (c), Circuit Type (E)

Consider a $s$-flag $\mathscr{F}$ of type (c) (cf. Remark 4.3.1.4) and $C = F_{m-3,m-4} = \{a,b,c\}$ is a signed circuit of type (E). Moreover, we have $F_{k,k-1} = \{d,e,f\}$ for some $k \in [m-4]$. We assume that $\alpha_a = (0,0,0)$, $\alpha_b = (0,0,1)$ and $\alpha_c = (0,0,2)$. We go through the classification of all polytopes spanned by $\alpha_a, \dots, \alpha_f$ stated in [**MMS12b**, section 4.5]. The following two cases were studied:

(1) There exists no plane $H \subset \mathbb{R}^3$ through the $z$-axis such that $\alpha_d, \alpha_e, \alpha_f \in H^+$.
(2) There exists a plane $H \subset \mathbb{R}^3$ through the $z$-axis such that $\alpha_d, \alpha_e, \alpha_f \in H^+$.

<u>(1):</u> In [**MMS12b**, Proposition 21] all polytopes $P$ matching our assumptions ($P = \operatorname{conv}(C, m, m', m'')$ where $C$ is a circuit of type (E) and $m$, $m'$, $m''$ lattice points such that any two of these span $\mathbb{R}^3$ together with the three points of $C$, $|P \cap \mathbb{Z}^3| = 6$ and there is no hyperplane $H$ through the $z$-axis such that $m$, $m'$ and $m''$ are all on one side of $H$) were classified:

(a) $m = (0,1,\gamma)$, $m' = (1,0,\gamma')$, $m'' = (-1,-1,\gamma'')$ with $\gamma, \gamma', \gamma'' \in \mathbb{Z}$ arbitrary.
(b) $m = (0,1,\gamma)$, $m' = (2,1,\gamma')$, $m'' = (-1,-1,\gamma'')$ with $\gamma, \gamma', \gamma'' \in \mathbb{Z}$ such that $\gamma \not\equiv \gamma' \mod 2$.
(c) $m = (0,1,\gamma)$, $m' = (3,1,\gamma')$, $m'' = (-1,-1,\gamma'')$ with $\gamma, \gamma', \gamma'' \in \mathbb{Z}$ and $\gamma \not\equiv \gamma' \mod 3$, $\gamma' \not\equiv \gamma'' \mod 2$.
(d) $m = (0,1,\gamma)$, $m' = (3,1,\gamma')$, $m'' = (-3,-2,\gamma'')$ with $\gamma, \gamma', \gamma'' \in \mathbb{Z}$ and $\gamma \not\equiv \gamma' \not\equiv \gamma'' \not\equiv \gamma \mod 3$.

We identify $m = \alpha_d$, $m' = \alpha_e$ and $m'' = \alpha_f$. Consider the affine basis $\mathscr{X} = \{\alpha_a, \alpha_b, \alpha_d, \alpha_e\}$ provided by $a, b \in F_{m-3,m-4}$ and $d, e \in F_{k,k-1}$. We have $p_{F_{m-3,m-4}}(s) = p_{F_{m-3,m-4}}(s_C)$ (Lemma 4.3.1.5) where $s_C$ is the sign vector of the signed circuit $C = \text{conv}(\alpha_a, \alpha_b, \alpha_c)$, i.e. w.l.o.g. $(s)_{a,c} = + \neq (s)_b$. Moreover, Lemma 4.3.1.5 determines the signs of $s$ at indices $d, e, f$. Let $C_k \in M\left[A'_{\{a,b,d,e,f\}}\right]$ be a signed circuit. Up to a common factor this signed circuit is unique as $\mathscr{X}$ is an affine basis. Let $H_{ab,i}$ denote the plane spanned by $\alpha_a$, $\alpha_b$ and $\alpha_i$ with $i \in \{d, e, f\}$. Then $H_{ab,i}$ separates $\alpha_x$ and $\alpha_y$ where $\{x, y\} = \{d, e, f\} \setminus \{i\}$. With Lemma 4.3.1.2 we conclude $(s)_d = (s)_e = (s)_f$.

Now, suppose the subdivision we consider contains one of the polytopes $P$ stated above, i.e. it is subdivided into the three polytopes $\Delta_{de} = \text{conv}(\alpha_a, \alpha_c, \alpha_d, \alpha_e)$, $\Delta_{df} = \text{conv}(\alpha_a, \alpha_c, \alpha_d, \alpha_f)$ and $\Delta_{ef} = \text{conv}(\alpha_a, \alpha_c, \alpha_e, \alpha_f)$. The dual of $C = \text{conv}(\alpha_a, \alpha_c)$ is a triangle in $\mathscr{T}_{\mathbb{R}}(f)_{(+,+,+)}$ as the signs of $C$ are alternating. Since $C \subset \Delta_{ij}$ for pairwise distinct $i, j \in \{d, e, f\}$ each polytope $\Delta_{ij}$ is dual to a vertex of this particular triangle. Moreover, we have $u_a = u_b = u_c > u_d = u_e = u_f$. From the circuit $C$ we conclude that the dual triangle is contained in $\{z = 0\}$. As explained in [**MMS12b**, section 4.5] the projection $p_{\{x,y\}}(P) = T_{de} \cup T_{df} \cup T_{ef}$ is a triangle that decomposes into the union of the triangles $T_{ij}$ corresponding to $\Delta_{ij}$, $i \neq j \in \{d, e, f\}$. Let $(p_i, q_i)$ and $(p_j, q_j)$ denote the vertices of $T_{ij}$. We obtain coordinates of the vertex $v_{ij}$ dual to $\Delta_{ij}$ if we assume that $f$ maximizes at all monomials corresponding to vertices in $\Delta_{ij}$. In particular, we get equations of the form

$$u_a = u_i + \langle (p_i, q_i), (x, y) \rangle = u_j + \langle (p_j, q_j), (x, y) \rangle.$$

Using $u = u_a - u_i$ for any $i \in \{d, e, f\}$ we get

$$(x, y) = \frac{1}{p_i q_j - p_j q_i} (q_j u - q_i u, p_i u - p_j u) = v_{ij}$$

as solution. If we define $p_i q_j - p_j q_i$ as weight (which is precisely the volume of the triangle $T_{ij}$) the singularity $\mathbf{0}_3^+$ equals the weighted barycenter of the triangle dual to $C$.
Let the endpoints of the circuit $C$ have positive signs, e.g. $(s)_a = (s)_c = +$. Since $(s)_d = (s)_e = (s)_f$ each vertex $v_{ij}$ is either purely adjacent to the triangle dual to $C$, i.e. the triangle is isolated, or each $v_{ij}$ is additionally adjacent to two more edges and each edge of the triangle is adjacent to two 2-dimensional polyhedra.

The subdivision around $C = \text{conv}(\alpha_a, \alpha_b, \alpha_c)$ can be more complicated, i.e. there may be more polytopes sharing the face $C$. However, the faces $\text{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_i)$ with $i \in \{d, e, f\}$ will still be part of the subdivision, i.e. $Q$ contains the edges dual to these faces. The intersections of these edges provide three points $A$, $B$, $C$ forming a *virtual triangle* ([**MMS12b**, section 4.5]). Then the singularity is the weighted barycenter according to this virtual triangle.

(2): Assume that there is a hyperplane through the $z$-axis such that $\alpha_d$, $\alpha_e$ and $\alpha_f$ are on one side of the hyperplane. We assume $\alpha_a = (0,0,0)$, $\alpha_b = (0,0,1)$ and $\alpha_c = (0,0,2)$ and $C = \text{conv}(\alpha_a, \alpha_b, \alpha_c)$. Let $P = \text{conv}(C, m, m', m'')$ be a polytope where $m$, $m'$ and $m''$ are three additional vectors such that $P$ contains only the six given lattice points, the three vectors $m$, $m'$ and $m''$ are on one side of a hyperplane through the $z$-axis. Then [**MMS12b**, Proposition 23] states that $P$ is $IUA$-equivalent to one of the following cases:

(a) $m = (-1, 0, \gamma)$, $m' = (0, 1, \gamma')$, $m'' = (\alpha'', 1, \gamma'')$ with $\alpha'' \geq 1$, $\gamma \in \mathbb{Z}$, $\gcd(\gamma'' - \gamma', \alpha'') = 1$.
(b) $m = (\alpha, 1, \gamma)$, $m' = (\alpha + l, 1, \gamma + k)$, $m'' = (\alpha + 2l, 1, \gamma + 2k)$ with $\alpha, \gamma \in \mathbb{Z}$ and $\gcd(l, k) = 1$.

(c) $m = (\alpha, 1, \gamma)$, $m' = (\alpha', 1, \gamma')$, $m'' = (\alpha'', 1, \gamma'')$ with

$$\det \begin{bmatrix} \alpha' - \alpha & \alpha'' - \alpha \\ \gamma' - \gamma & \gamma'' - \gamma \end{bmatrix} = \pm 1.$$

In particular, in all of these cases there is no hyperplane through the $z$-axis containing two of the three points $m$, $m'$, $m''$. As above we identify $\alpha_a = (0,0,0)$, $\alpha_b = (0,0,1)$, $\alpha_c = (0,0,2)$ whose convex hull is the signed circuit $C$ of type (E) and $m = \alpha_d$, $m' = \alpha_e$, $m'' = \alpha_f$. Consider the affine basis $\mathscr{Z} = \{\alpha_a, \alpha_b, \alpha_d, \alpha_e\}$ provided by $a, b \in F_{m-3,m-4}$ and $d, e \in F_{k,k-1}$. According to Lemma 4.3.1.5 we have $p_{F_{m-3,m-4}}(s) = p_{F_{m-3,m-4}}(s_C)$ where $s_C$ is the sign vector of the signed circuit $C = \operatorname{conv}(\alpha_a, \alpha_b, \alpha_c)$, i.e. w.l.o.g. $(s)_{a,c} = + \neq (s)_b$. Moreover, Lemma 4.3.1.5 determines the signs of $s$ at indices $d, e, f$. Let $C_k \in M\left[A'_{\{a,b,d,e,f\}}\right]$ be a signed circuit. Up to a common factor this signed circuit is unique as $\mathscr{Z}$ is an affine basis. Let $H_{ab,i}$ denote the plane spanned by $\alpha_a$, $\alpha_b$ and $\alpha_i$ with $i \in \{d, e, f\}$ and let $H$ denote the plane such that $\alpha_d$, $\alpha_e$ and $\alpha_f$ are on one side of the plane. If we project to the $\langle x, y \rangle$-plane we can order the points according to their angle with the line arising from $H$. Without restriction let $m' = \alpha_e$ be the central point. Hence, $H_{ab,e}$ separates $\alpha_d$ and $\alpha_f$ whereas $H_{ab,d}$ and $H_{ab,f}$ do not separate the remaining two points. Due to Lemma 4.3.1.2 this implies $(s)_d \neq (s)_e \neq (s)_f$.

Now, suppose that the subdivision purely consists of a polytope $P$ as defined in (a) (see above). Then $P$ contains two polytopes $\Delta_{de} = \operatorname{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_d, \alpha_e)$ and $\Delta_{ef} = \operatorname{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_e, \alpha_f)$ that contain $C$ — all other polytopes do not influence the singularity. As the sings of $C$ are alternating there is a dual 2-dimensional unbounded polyhedron $\sigma$. However, $\sigma$ has two vertices $A$ and $B$ dual to $\Delta_{de}$ and $\Delta_{ef}$. The faces $\operatorname{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_d)$ and $\operatorname{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_f)$ are dual to lines. Their intersection is dual to $\operatorname{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_d, \alpha_f)$ which is not part of the subdivision. Let $C$ denote the intersection point that is not contained in $\sigma$. Then their weighted sum (where we subtract the summand corresponding to $C$ as it is outside of $\sigma$) equals the singular point $\mathbf{0}_3^+$ and we call it *virtual weighted barycenter*.

We do not consider the cases (b) and (c) as these provide weight classes that have non-empty intersection with rowspace$(A)$.

If the subdivision is more complicated then we still get the points $A$, $B$ and $C$ and their weighted sum still provides the singular point. For details see [**MMS12b**, section 4.5.2].

#### 4.3.3.6. Weight Class Type (d), Circuit Type (E)

Consider a $s$-flag $\mathscr{F}$ of type (d) (cf. Remark 4.3.1.4) and $C = F_{m-3,m-4} = \{a, b, c\}$ is a signed circuit of type (E). Moreover, we have $F_{j,j-1} = \{d, e\}$ and $F_{i,i-1} = \{f, g\}$ for some $i < j \in [m-4]$. We assume that $\alpha_a = (0,0,1)$, $\alpha_b = (0,0,1)$ and $\alpha_c = (0,0,2)$. We know that $\alpha_d$ and $\alpha_e$ span a plane with $\alpha_a$, $\alpha_b$ and $\alpha_c$ (cf. Remark 4.3.1.4). We assume that this hyperplane is $\{y = 0\}$. First, assume that $(\alpha_d)_x < 0 < (\alpha_e)_x$. Then Lemma 4.3.1.2 implies $(s)_d = (s)_e$ and $\operatorname{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_d)$ and $\operatorname{conv}(\alpha_a, \alpha_b, \alpha_c, \alpha_e)$ are faces of polytopes in the subdivision. If $(\alpha_d) \neq -1$ then the face contains more lattice points, i.e. we have $|(\alpha_d)| = |(\alpha_e)| = 1$. Analogously $\alpha_f$ and $\alpha_g$ have integral distance one to $\{y = 0\}$. Suppose we have $(\alpha_f)_y = 1$ and $(\alpha_g)_y = -1$. Then $(s)_f = (s)_g$ according to Lemma 4.3.1.2. Suppose the subdivision contains only the polytopes $\operatorname{conv}(\alpha_a, \alpha_c, \alpha_d, \alpha_f)$, $\operatorname{conv}(\alpha_a, \alpha_c, \alpha_e, \alpha_f)$, $\operatorname{conv}(\alpha_a, \alpha_c, \alpha_d, \alpha_g)$, $\operatorname{conv}(\alpha_a, \alpha_c, \alpha_e, \alpha_g)$. Each polytope is dual to a vertex as $C$ has alternating signs. From $C$ we also conclude that the polygon dual to $C$ is contained in $\{z = 0\}$.

Each face of the four polytopes above is dual to an edge since $C$ is contained in each face and has alternating signs. Assuming that $\alpha_d = (1,0,\gamma)$, $\alpha_e = (-1,0,\gamma')$, $\alpha_f = (\xi,1,\gamma'')$ and $\alpha_g = (\xi',-1,\gamma''')$ and $u = u_a - u_d$, $w = u_d - u_f$ then the coordinates of the vertices are

$$A = (u, w+(1-\alpha)u),\ B = (-u, w+(1+\alpha)u),\ C = (u, -w+(\alpha'-1)u),\ D = (-u, -w-(1+\alpha)u)$$

Hence, the polygon dual to the signed circuit $C$ is a trapeze where $\mathbf{0}_3^+ = (+^3, \frac{A+B+C+D}{4})$, i.e. the singularity is its midpoint.

If the subdivision contains more polytopes around $C$ then the polygon has more edges. However, parts of the edges dual to the polytopes stated above are still part of the polygon. Thus we can introduce virtual vertices and virtual edges again. Nevertheless the singularity remains in the middle of the polygon.

All other cases (e.g. $\alpha_d$ and $\alpha_e$ on the same side etc.) correspond to defective weight classes and we do not consider these situations. $\square$

APPENDIX A

# SINGULAR Code

The following script completes the proof of Proposition 3.1.3.9:

```
ring r = (0,ax,ay,bx,by,cx,cy,dx,dy,ex,ey,fx,fy),(a4,a5,a6,b4,b5,b6,c4,c5,c6),dp;

poly c11 = ax*ay*(-ax-ay+1);
poly c12 = ax*by*((ax-1)*(by-1)-ay*bx)+bx*ay*((bx-1)*(ay-1)-by*ax);
poly c13 = ax*cy*((ax-1)*(cy-1)-ay*cx)+cx*ay*((cx-1)*(ay-1)-cy*ax);
poly c14 = ax*dy*((ax-1)*(dy-1)-ay*dx)+dx*ay*((dx-1)*(ay-1)-dy*ax);
poly c15 = ax*ey*((ax-1)*(ey-1)-ay*ex)+ex*ay*((ex-1)*(ay-1)-ey*ax);
poly c22 = bx*by*(-bx-by+1);
poly c23 = bx*cy*((bx-1)*(cy-1)-by*cx)+cx*by*((cx-1)*(by-1)-cy*bx);
poly c24 = bx*dy*((bx-1)*(dy-1)-by*dx)+dx*by*((dx-1)*(by-1)-dy*bx);
poly c25 = bx*ey*((bx-1)*(ey-1)-by*ex)+ex*by*((ex-1)*(by-1)-ey*bx);
poly c33 = cx*cy*(-cx-cy+1);
poly c34 = cx*dy*((cx-1)*(dy-1)-cy*dx)+dx*cy*((dx-1)*(cy-1)-dy*cx);
poly c35 = cx*ey*((cx-1)*(ey-1)-cy*ex)+ex*cy*((ex-1)*(cy-1)-ey*cx);
poly c44 = dx*dy*(-dx-dy+1);
poly c45 = dx*ey*((dx-1)*(ey-1)-dy*ex)+ex*dy*((ex-1)*(dy-1)-ey*dx);
poly c55 = ex*ey*(-ex-ey+1);

poly koef44 = c44 + c11*(a4)^2 +c12*(a4)*(b4) + c13*(a4)*(c4) - c14*(a4) + c22*(b4)^2 + c23*(b4)*(c4) - c24*(b4) + c33*(c4)^2 - c34*(c4);
poly koef55 = c55 + c11*(a5)^2 +c12*(a5)*(b5) + c13*(a5)*(c5) - c15*(a5) + c22*(b5)^2 + c23*(b5)*(c5) - c25*(b5) + c33*(c5)^2 - c35*(c5);
poly koef45 = c45 + 2*c11*(a4)*(a5) + c12*((a4)*(b5)+(a5)*(b4)) + c13*((a4)*(c5)+(a5)*(c4)) - c14*(a5) -c15*(a4) + 2*c22*(b4)*(b5) + c23*((b4)*(c5)+(b5)*(c4)) - c24*(b5) - c25*(b4) + 2*c33*(c4)*(c5) - c34*(c5) - c35*(c4);

poly koef_abc44 = subst(koef44,dx,a4*ax+b4*bx+c4*cx,dy,a4*ay+b4*by+c4*cy);
poly koef_abc45 = subst(koef45,dx,a4*ax+b4*bx+c4*cx,dy,a4*ay+b4*by+c4*cy,ex,a5*ax+b5*bx+c5*cx,ey,a5*ay+b5*by+c5*cy);
poly koef_abc55 = subst(koef55,ex,a5*ax+b5*bx+c5*cx,ey,a5*ay+b5*by+c5*cy);
```

APPENDIX B

# List of Symbols

In the following we list symbols that are frequently used all over the thesis.

| Symbol | Description | Page |
|---|---|---|
| $\mathbb{N}$ ($\mathbb{N}_0$ respectively) | Natural numbers without zero (with zero respectively). | 1 |
| $\mathbb{A}^n$, $\mathbb{P}^n$, $T^n$ | Affine space, projective space, Torus. | 1 |
| $S$, $R$, $L$ | Coordinate rings of $\mathbb{A}^n$, $\mathbb{P}^n$, $T^n$. | 1 |
| $p_X$ | Projection onto coordinates indexed by $X$. | 1 |
| $\mathscr{M}_{n,d}$ | The set of monomials in $n$ variables of degree $d$. | 1 |
| $N_n^d$ | The cardinality of $\mathscr{M}_{n,d}$. | 1 |
| $[m]$ | Set of cardinality $m$. | 1 |
| $\operatorname{aff}(X)$ | Smallest affine space containing $X$. | 2 |
| $H_A$ | Hyperplane specified by $A$. | 2 |
| $\operatorname{relint}(\sigma)/\partial\sigma$ | Relative interior/boundary of $\sigma$. | 2 |
| $\Delta_n$ | Standard simplex. | 2 |
| $\operatorname{vert}(X)$ | Set of 0-dimensional faces of a polytope $X$. | 2 |
| $\mathscr{N}_P$ | (Outer) normal fan of a polytope $P$. | 5 |
| $S_w$ | Induced subdivision by $w$. | 6 |
| $P_w^+$ | Upper faces of a shifted polytope. | 6 |
| $\mathrm{Sec}_{\mathscr{A}}$ | Secondary fan of $\mathscr{A}$. | 7 |
| $\sigma(w)$ | Cone of the secondary fan. | 7 |
| $\mathscr{B}$,$\mathscr{C}$ | Bases and circuits of a matroid. | 9 |
| $M\big\vert_X$ | Restricted matroid. | 9 |
| $M[A]$ | Vector matroid associated to a point configuration/matrix $A$. | 10 |
| $e_A$ | Incidence vector with respect to $A$ | 10 |
| $P_M$ | Matroid polytope. | 10 |
| $M_w$ | Initial matroid. | 11 |
| $B(M)$ | Bergman fan of a matroid $M$. | 11 |

| Symbol | Description | Page |
|---|---|---|
| $\mathscr{F} \triangleleft M$ | $\mathscr{F}$ is a flag of flats of a matroid $M$. | 12 |
| $\sigma_{\mathscr{F}}$ | Weight class with respect to $\mathscr{F}$. | 12 |
| $\sigma(M_w)$ | Cone of a Bergman fan with respect to its coarse subdivision. | 12 |
| $X^{\pm}$ | Partition of a signed set. | 14 |
| $\underline{M}$ | Underlying unoriented matroid arising from oriented matroid $M$. | 14 |
| ${}_{-A}M$ | Reorientation of a matroid $M$ according to $A$. | 14 |
| $s_C$ | Sign vector associated to signed circuit $C$. | 16 |
| $\mathscr{L}_M$ | The set of covectors of $M$. | 16 |
| $\mathbb{K}$, $\Bbbk$ | A valued field, its residue field. | 17 |
| $\mathbb{K}_{\mathbb{C}}$, $\mathbb{K}_{\mathbb{R}}$ | Complex Puiseux series, real Puiseux series. | 17 |
| $\mathbb{T}$ | Tropical semiring. | 18 |
| $\mathscr{T}(f)$ | Tropical hypersurface defined by $f$. | 19 |
| $\mathrm{Newt}(F)$ | Newton polytope of $F$. | 19 |
| $\mathrm{trop}(X)$ | Tropicalization of a variety $X$. | 21 |
| $\mathrm{in}_w(f)$ | Initial form of $f$ with respect to $w$. | 21 |
| $A(\mathscr{I})$ | Coefficient matrix of $\mathscr{I}$. | 24 |
| $M(\mathscr{I})$ | Matroid associated to a linear ideal $\mathscr{I}$. | 24 |
| $T^n_{\mathbb{R}}$ | $n$-dimensional torus over $\mathbb{K}_{\mathbb{R}}$. | 25 |
| $\mathbb{TR}$ | Real tropical group. | 26 |
| $\mathrm{trop}_{\mathbb{R}}$ | Real tropicalization. | 26 |
| $\mathscr{T}_{\mathbb{R}}(f)$ | Real tropical hypersurface. | 27 |
| $L_A$ | A line specified by $A$. | 28 |
| $T_p(V)$ | Tangent space of $V$ at $p$. | 28 |
| $\mathrm{Sing}_f$ | Singularities of $\mathscr{V}(f)$. | 29 |
| $\text{c-Sing}_f$ | Cusps of $\mathscr{V}(f)$. | 30 |
| $\mathrm{Sing}^k_f$ | $k$-fold singularities of $\mathscr{V}(f)$. | 31 |
| $\nabla$ | Laurent polynomials with support $\mathscr{A}$ providing singular hypersurfaces. | 31 |
| $\psi_{\mathscr{A}}$ | Monomial map according to $\mathscr{A} \subset \mathbb{Z}^n$ with $\lvert\mathscr{A}\rvert = m$. | 32 |
| $\mathrm{rowspace}(A)$ | Row space of a matrix $A$. | 32 |
| $\mathbb{R}_B$ | $\mathbb{R}^n$ restricted to the coordinates indexed by $B$. | 35 |
| $B(M)\big\rvert_B$ | Cones $\sigma(M_{\mathscr{F}})$ of $B(M)$ with $B \in M_{\mathscr{F}}$. | 38 |

| Symbol | Description | Page |
|---|---|---|
| $\bar{\mathscr{T}}_B$ | Image of codimension one skeleton of $B(M)$ under $p_B$. | 39 |
| $h_B$ | Lifting map inverse to $p_B$. | 40 |
| $\pi_B$ | Isomorphism between $R_{n+1}/\mathscr{I}$ and $R_B$. | 43 |
| $F_B$ | Image of $\bar{F} \in R_{n+1}/\mathscr{I}$ under $\pi_B$. | 43 |
| $\mathscr{N}_{\mathscr{B}}$ | Relative Newton polytope. | 43 |
| $f_*(\Sigma)$ | Push forward of a fan $\Sigma$ under a fan morphism $f$. | 44 |
| $\nu_d$ | Veronese map. | 59 |
| $V^{\top}$ | Tropical Veronese map. | 59 |
| $\mathscr{U}$ | Generating set of the Veronese ideal. | 59 |
| $\mathrm{h}_I$ | Hilbert function of ideal $I$. | 60 |
| $\mathbb{E}_{m\times n}$ | Unit matrix with $m$ rows and $n$ columns. | 62 |
| $\mathbb{U}_k$ | Upper right matrix of size $k$ with ones on diagonal. | 62 |
| $\mathscr{N}_P^T$ | Total fan of a polytope $P$. | 65 |
| $B^d$ | Subset of monomials of degree $d$ with indices in $B$. | 65 |
| $S(X)$ | The set of permutations/bijections on a finite set $X$. | 66 |
| $\sigma_\eta^T$ | Cone of a total fan of $B$ or $B^d$. | 66 |
| $\nabla^{\mathrm{cusp}}$ | Subset of $\nabla$ with cuspidal singularities. | 75 |
| $\nabla^{k+1}$ | Subset of $\nabla$ with $k+1$-fold singularities. | 75 |
| $\lvert\alpha,\alpha'\rvert$ | Determinant of $2\times 2$ matrix given by $\alpha,\alpha' \in \mathbb{R}^2$. | 76 |
| $L_A$ | Line specified by $A$. | 77 |
| $\mathrm{coef}_F(x^\alpha)$ | Coefficient of $x^\alpha$ in $F$. | 81 |
| $\mathscr{M}(k_1,k_2)$ | Monomials with length between $k_1$ and $k_2$. | 90 |
| $\mathscr{M}(k)$ | Monomials with length up to $k$. | 90 |
| $(x)_n$ | Falling factorial. | 91 |
| $\theta_\beta(\alpha)$ | The product of falling factorials of $\alpha$ with respect to $\beta$. | 92 |
| $\theta_k$ | Map whose components are defined by $\theta_\beta$. | 92 |
| $\frac{\partial F}{\partial L}$ | Euler derivative of $F$ with respect to $L$. | 93 |
| $\mathscr{B}^s(M)$ | Signed Bergman fan. | 103 |
| $\nabla_{\mathbb{R}}$ | $\nabla$ defined over $\mathbb{K}_{\mathbb{R}}$. | 106 |
| $C_v$ | Chart of a real tropical variety . | 108 |
| $\{\pm\}^m \times \mathrm{Sec}_{\mathscr{A}}$ | Signed secondary fan of $\mathscr{A}$. | 110 |

| Symbol | Description | Page |
|---|---|---|
| $G \times \text{rowspace}(A)$ | Lineality group. | 111 |
| $E^{+}$ | Height profile. | 125 |
| $U(\lambda)$ | A shear matrix along a coordinate axis. | 125 |
| $\text{tdim}(\mathscr{H})$ | Type-dimension of a subdivision of type $\mathscr{H}$. | 135 |

# Index

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