Structural properties of Cox rings of T-varieties

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CONTENTS

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INTRODUCTION

The main objects of the present thesis are $\mathbb{T}$-*varieties*, i.e. varieties X with an algebraic torus $\mathbb{T}$ acting effectively on them. The difference between the dimension of the variety and the dimension of the acting torus is called the *complexity* of a $\mathbb{T}$-variety. The best understood varieties of this kind are *toric varieties*, where the complexity equals zero. These varieties have been intensively studied [26, 27, 74, 75, 24, 40], and are of high interest due to their combinatorial description via fans. This thesis contributes to the case of higher complexity. These varieties occur naturally: For instance the surface quotient singularities are varieties of complexity one and the quadrics in projective space admit a natural action of the maximal torus of the orthogonal group. If X is a $\mathbb{T}$-variety of complexity c, then, as in the toric case, there is a combinatorial part reflecting the torus action, and a continuous part reflecting the geometry of a suitable variety representing the field $\mathbb{K}(X)^{\mathbb{T}}$ of rational invariants of X.

Our approach is based on Cox rings, which are a rich invariant of algebraic varieties. In fact the Cox ring of a variety fixes it up to small birational modifications. In the complete case the toric varieties are precisely those having a multigraded polynomial ring as Cox ring [23]. In general the *Cox ring* of a variety X is a graded algebra

$$\mathcal{R}(X) \ := \bigoplus_{[D]\in \mathrm{Cl}(X)} \Gamma(X, \mathcal{O}(D)),$$

and can be defined for any normal (irreducible) variety with only constant invertible functions and finitely generated divisor class group; see [6, 14, 52]. We are particularly interested in the case that $\mathcal{R}(X)$ is finitely generated. In that case, many of the geometric invariants can be read off. Moreover, it provides a canonical embedding of the variety into a toric one, which allows to deduce basic geometric properties from the toric embedding and thus describe them in combinatorial terms similar as in the toric case.

In the first part of this thesis our main focus is on $\mathbb{T}$-varieties of complexity one. For a complete, rational $\mathbb{T}$-variety of complexity one the Cox ring has been described in [49] via generators and specific trinomial relations. Applying the theory behind Cox rings to this special situation one obtains a combinatorial approach to these varieties. We extend the toolkit developed in [49] to the non-complete, e.g. affine, case. This includes i.a. a description of factorially graded affine algebras R of complexity one with only constant homogeneous invertible elements in terms of generators and relations. The first results in this direction treat the case $R_0 = \mathbb{K}$; see [66, 61, 44]. Our results complete the description of all factorially graded affine algebras of complexity one and thus the description of the Cox rings of rational $\mathbb{T}$-varieties of complexity one, compare Theorem 2.1.8, where the complete case treated in [49] is a subcase of Type 2:

Theorem. *Let X be a normal, rational $\mathbb{T}$-variety of complexity one with only constant invertible global functions and finitely generated divisor class group. Then, with $T_i^{l_i} = T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}}$, the Cox ring of X is described by trinomial relations of one of the following forms:*

$$\textit{Type 1: } T_1^{l_1} - T_2^{l_2} - \theta_1, \quad T_2^{l_2} - T_3^{l_3} - \theta_2, \quad \ldots, \quad T_{r-1}^{l_{r-1}} - T_r^{l_r} - \theta_{r-1},$$

$$\textit{Type 2: } T_0^{l_0} + T_1^{l_1} + T_2^{l_2}, \quad \theta_1 T_1^{l_1} + T_2^{l_2} + T_3^{l_3}, \quad \ldots, \quad \theta_{r-2} T_{r-2}^{l_{r-2}} + T_{r-1}^{l_{r-1}} + T_r^{l_r}.$$

As an immediate sample application we calculate the Cox rings of all affine $\mathbb{C}^*$-surfaces having at most log terminal singularities and their resolutions. It is well known that the log terminal surface singularities are exactly the surface quotient singularities, i.e., they arise as a quotient of $\mathbb{C}^2$ by a finite group $G \subseteq \mathrm{GL}_2$; see [4, 20, 30]. Moreover any log terminal surface singularity is in fact a $\mathbb{C}^*$-surface and with our explicit description of the Cox rings, one observes that the derived series of the group G reflects iteration of Cox rings.

We extend this picture to log terminal singularities in arbitrary dimension coming with a torus action of complexity one. As a first result, in Theorem 3.1.3 we obtain that the Cox rings of affine log terminal $\mathbb{T}$-varieties of complexity one are either of Type 1 or *platonic* of Type 2, i.e. $l_{ij} = 1$ holds for all $i \geq 3$, and all triples $(l_{0j_0}, l_{1j_1}, l_{2j_2})$ form a *platonic triple*, i.e., a triple of the form $(5,3,2), (4,3,2), (3,3,2), (x,2,2)$, or $(x,y,1)$, where $x, y \in \mathbb{Z}_{\geq 1}$. We obtain the following general statements on iteration of Cox rings, compare Theorems 3.1.5 and 3.1.6:

Theorem. *Let X_1 be a rational, normal, affine $\mathbb{T}$-variety of complexity one with Cox ring of Type 2 and at most log terminal singularities. Then the following assertions hold.*

(i) *There is a unique chain of quotients*

$$X_p \xrightarrow{/\!\!/ H_{p-1}} X_{p-1} \xrightarrow{/\!\!/ H_{p-2}} \quad \ldots \quad \xrightarrow{/\!\!/ H_3} X_3 \xrightarrow{/\!\!/ H_2} X_2 \xrightarrow{/\!\!/ H_1} X_1 \ ,$$

where $X_i = \mathrm{Spec}(R_i)$ holds, the ring R_p is factorial and each R_i is the Cox ring of X_{i-1}.

(ii) *X_1 is a quotient $X_1 = X_p /\!\!/ G$ by a solvable reductive group G.*

(iii) *The presentation of* (i) *is regained by $H_i := G^{(i-1)}/G^{(i)}$ and $X_i := X_p/G^{(i-1)}$, where $G^{(i)}$ is the i-th derived subgroup of G.*

Statement (ii) of the above Theorem shows that, in a large sense, the log terminal singularities with torus action of complexity one still can be regarded as quotient singularities:

the affine plane $\mathbb{C}^2$ and the finite group $G \subseteq \mathrm{GL}_2$ of the surface case have to be replaced with a factorial affine $\mathbb{T}$-variety of complexity one and a solvable reductive group.

Looking at this result about log terminal singularities the natural question arises, if there are any further varieties of complexity one admitting iteration of Cox rings despite the log terminal ones. Note that iteration of Cox rings requires in particular a finitely generated divisor class group of the spectrum of the Cox ring in each iteration step. As in case of complexity one, finite generation of the divisor class group turns out to be equivalent to rationality, the task is to give a criterion for the rationality of an affine variety X of complexity one. Having in mind that the generic quotient of a $\mathbb{T}$-variety of complexity one is the curve Y with function field $\mathbb{K}(X)^{\mathbb{T}}$ and rationality of this curve is equivalent to rationality of the variety X, we calculate a genus formula for Y leading to the following numerical criterion for the rationality of X. We call a ring R of Type 2 as above *hyperplatonic* if $\mathfrak{l}_i := \gcd(l_{i1}, \ldots, l_{in_i}) = 1$ holds for all $i \geq 3$ and $(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_2)$ is a platonic triple. We obtain the following criterion, see Theorem 4.1.1

Theorem. *Let X be a rational, normal $\mathbb{T}$-variety of complexity one with Cox ring of Type 2. Then the following statements are equivalent.*

(i) *The variety X admits iteration of Cox rings.*
(ii) *The variety X has a hyperplatonic or factorial Cox ring.*

For a variety with Cox ring of Type 1 the picture is much easier: it admits iteration of Cox rings if and only if the spectrum of its Cox ring is rational. Moreover, if the latter holds, the iteration of Cox rings stops after at most one step.

In the last part of this thesis we extend the Cox ring based combinatorial theory for rational varieties with torus action of complexity one to $\mathbb{T}$-varieties of arbitrary high complexity with finitely generated Cox ring. Recall that the situation can be described by a c-dimensional variety Y suitably realizing the field $\mathbb{K}(X)^{\mathbb{T}}$ of rational invariants of the $\mathbb{T}$-variety X and a combinatorial part reflecting the essential properties of the torus action. This approach has been taken in [1, 2], where the torus action is encoded via the combinatorial language of polyhedral divisors and Y is chosen as the Chow quotient.

We provide a more specific approach: We choose a rather minimal representative Y of the field $\mathbb{K}(X)^{\mathbb{T}}$ of rational invariants: the *maximal orbit quotient*; for a precise definition see Section 5.1. For our purposes, the crucial property is that Y has finitely generated Cox ring if and only if X has so, see [49, Thm. 1.1]. This allows us to make full use of the strongly combinatorial nature of varieties with finitely generated Cox ring [6, Chap. 3].

Given a variety Y with finitely generated Cox ring, we systematically construct varieties with torus action and maximal orbit quotient $\pi\colon X \dashrightarrow Y$. Our main tool is basic toric geometry: We start with a choice of Cox ring generators for Y, then fix a compatible fan Σ and finally deliver X as a closed subvariety of the toric variety Z associated with the fan Σ such that the torus action on X is inherited from a subtorus action on Z. As a byproduct of the construction, we obtain the Cox ring of X for free. Specializing to the case that Y is the projective or the affine line, we regain the Cox rings of Type 1 and 2 of the rational varieties with torus action of complexity one as described above.

As a sample class, we restrict to the special case of a maximal orbit quotient $\pi\colon X \dashrightarrow \mathbb{P}_c$ such that the critical values of π form a general hyperplane arrangement in $\mathbb{P}_c$. We call such a $\mathbb{T}$-variety X an *arrangement variety*. These varieties directly generalize the rational $\mathbb{T}$-varieties of complexity one and their Cox rings show indeed a very similar structure.

With our description we provide classification results on smooth *Fano varieties*, i.e., normal, projective varieties with ample anticanonical class. The interest in this class of varieties is due to their important role in the Minimal Model Program, an approach to classify projective varieties up to birational equivalence introduced by S. Mori [67, 68]. The classification of smooth Fano varieties was settled in dimension two by P. del Pezzo [77] and in dimension three by V. Iskovskikh [54, 55] and S. Mori/S. Mukai [69, 70]. Restricting to the toric case, the classification was done up to dimension nine by work of V. Batyrev, M. Kreuzer, B. Nill, M. Øbro and A. Paffenholz [10, 11, 60, 73, 78]. Extending recent classification work in complexity one [35], we take a closer look at smooth arrangement varieties of Picard numbers at most two. In Picard number one, we retrieve precisely the smooth projective quadrics. Similar to the case of complexity one, the situation in Picard number two is much more ample. For the case of complexity two, we obtain the following explicit descriptions, see Section 6.2.

Theorem. *Every non-toric smooth Fano arrangement variety of complexity two and Picard number two is isomorphic to precisely one of the following varieties X, specified by their Cox ring $\mathcal{R}(X)$ and the matrix $[w_1, \ldots, w_r]$ of generator degrees $w_i \in \mathrm{Cl}(X) = \mathbb{Z}^2$.*

No.	$\mathcal{R}(X)$	$[w_1, \ldots, w_r]$	$-\mathcal{K}_X$	$\dim(X)$
1	$\frac{\mathbb{K}[T_1,\ldots,T_9]}{\langle T_1T_2T_3^2+T_4T_5+T_6T_7+T_8T_9\rangle}$	$\begin{bmatrix} 0&0&1&1&1&1&1&1&1\\ 1&1&0&1&1&1&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 5\\ 6 \end{bmatrix}$	6
2	$\frac{\mathbb{K}[T_1,\ldots,T_9]}{\langle T_1T_2T_3+T_4T_5+T_6T_7+T_8T_9\rangle}$	$\begin{bmatrix} 0&0&1&1&0&1&0&1&0\\ 1&1&0&1&1&1&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 3\\ 6 \end{bmatrix}$	6
3	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2T_3^2+T_4T_5+T_6T_7+T_8^2\rangle}$	$\begin{bmatrix} 0&0&1&1&1&1&1&1\\ 1&1&0&1&1&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 4\\ 5 \end{bmatrix}$	5
4.A	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2^3+T_3T_4+T_5T_6+T_7T_8\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&2&1&2&1&2&1&2&\ldots&2\\ 1&0&1&0&1&0&1&0&1&\ldots&1 \end{array}\right]$	$\begin{bmatrix} 7+2m\\ 3+m \end{bmatrix}$	$m+5$
4.B	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2^3+T_3T_4^2+T_5T_6^2+T_7T_8^2\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&1&1&1&1&1&1&1&\ldots&1\\ 1&0&1&0&1&0&1&0&1&\ldots&1 \end{array}\right]$	$\begin{bmatrix} 4+m\\ 3+m \end{bmatrix}$	$m+5$
4.C	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6+T_7T_8\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&0&1&1&1&1&1&1&\ldots&1\\ 1&0&1&0&1&0&1&0&1&\ldots&1 \end{array}\right]$	$\begin{bmatrix} 4+m\\ 3+m \end{bmatrix}$	$m+5$
4.D	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2^2+T_3T_4+T_5T_6+T_7T_8\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccccc\|cccc} 0&1&1&1&1&1&1&1&d_1&1&\ldots&1\\ 1&0&1&0&1&0&1&0&1&1&\ldots&1 \end{array}\right]$ $d_1 \in \{0,1\}$	$\begin{bmatrix} 5+m-1+d_1\\ 3+m \end{bmatrix}$	$m+5$
4.E	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2^3+T_3T_4^3+T_5T_6^3+T_7T_8^3\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&0&1&0&1&0&1&0&\ldots&0\\ 1&0&1&0&1&0&1&0&1&\ldots&1 \end{array}\right]$	$\begin{bmatrix} 3\\ 3+m \end{bmatrix}$	$m+5$
4.F	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6^2+T_7T_8^2\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccccc\|cccc} 0&1&0&1&0&1&0&1&d_1&0&\ldots&0\\ 1&0&1&0&1&0&1&0&1&1&\ldots&1 \end{array}\right]$ $d_1 \in \{-1,0\}$	$\begin{bmatrix} 2+d_1\\ 3+m \end{bmatrix}$	$m+5$

<table>
<tr><td>4.G</td><td>$\dfrac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$
$m\geq 0$</td><td>$\left[\begin{array}{cccccccc|cccc}0&1&0&1&0&1&0&1&d_1&d_2&0\dots&0\\1&0&1&0&1&0&1&0&1&1&1\dots&1\end{array}\right]$
$d_1, d_2 \leq 0,\ d_1+d_2\geq -2$</td><td>$\begin{bmatrix}3+d_1+d_2\\3+m\end{bmatrix}$</td><td>$m+5$</td></tr>
<tr><td>5</td><td>$\dfrac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3^2T_4+T_5^2T_6+T_7^2T_8\rangle}$
$m\geq 1$</td><td>$\left[\begin{array}{cccccccc|ccc}0&2a+1&a&1&a&1&a&1&1&\dots&1\\1&1&1&0&1&0&1&0&0&\dots&0\end{array}\right]$
$a\geq 0, m>3a$</td><td>$\begin{bmatrix}3a+3+m\\3\end{bmatrix}$</td><td>$m+5$</td></tr>
<tr><td>6</td><td>$\dfrac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2T_6+T_7^2T_8\rangle}$
$m\geq 1$</td><td>$\left[\begin{array}{cccccccc|ccc}0&2a_3+1&a_1&a_2&a_3&1&a_3&1&1&\dots&1\\1&1&1&1&1&0&1&0&0&\dots&0\end{array}\right]$
$0\leq a_1\leq a_2,\ a_1+a_2=2a_3+1$
$m>4a_3+1$</td><td>$\begin{bmatrix}4a_3+3+m\\4\end{bmatrix}$</td><td>$m+5$</td></tr>
<tr><td>7</td><td>$\dfrac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2T_8\rangle}$
$m\geq 1$</td><td>$\left[\begin{array}{cccccccc|ccc}0&2a_5+1&a_1&a_2&a_3&a_4&a_5&1&1&\dots&1\\1&1&1&1&1&1&1&0&0&\dots&0\end{array}\right]$
$a_i\geq 0,$
$a_1+a_2=a_3+a_4=2a_5+1,$
$m>5a_5+2$</td><td>$\begin{bmatrix}5a_5+3+m\\5\end{bmatrix}$</td><td>$m+5$</td></tr>
<tr><td>8</td><td>$\dfrac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$
$1\leq m\leq 5$</td><td>$\left[\begin{array}{cccccccc|ccc}0&0&0&0&0&0&-1&1&1&\dots&1\\1&1&1&1&1&1&1&1&0&\dots&0\end{array}\right]$</td><td>$\begin{bmatrix}m\\6\end{bmatrix}$</td><td>$m+5$</td></tr>
<tr><td>9</td><td>$\dfrac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$
$m\geq 2$</td><td>$\left[\begin{array}{cccccccc|ccc}0&a_1&a_2&a_3&a_4&a_5&a_6&a_7&1&\dots&1\\1&1&1&1&1&1&1&1&0&\dots&0\end{array}\right]$
$a_i\geq 0,$
$a_1=a_2+a_3=a_4+a_5=a_6+a_7,$
$m>3a_1$</td><td>$\begin{bmatrix}3a_1+m\\6\end{bmatrix}$</td><td>$m+5$</td></tr>
<tr><td>10</td><td>$\dfrac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$
$m\geq 2$</td><td>$\left[\begin{array}{cccccccc|cccc}0&0&0&0&0&0&0&0&1&1&\dots&1\\1&1&1&1&1&1&1&1&0&d_2&\dots&d_m\end{array}\right]$
$0\leq d_2\leq\dots\leq d_m,\ 0<d_m\leq 5$
$m\cdot d_m<6+d_2+\ldots+d_m$</td><td>$\begin{bmatrix}m\\6+\sum d_k\end{bmatrix}$</td><td>$m+5$</td></tr>
<tr><td>11</td><td>$\dfrac{\mathbb{K}[T_1,\dots,T_7,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2\rangle}$
$1\leq m\leq 4$</td><td>$\left[\begin{array}{ccccccc|ccc}-1&1&0&0&0&0&0&1&\dots&1\\1&1&1&1&1&1&1&0&\dots&0\end{array}\right]$</td><td>$\begin{bmatrix}m\\5\end{bmatrix}$</td><td>$m+4$</td></tr>
<tr><td>12</td><td>$\dfrac{\mathbb{K}[T_1,\dots,T_7,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2\rangle}$
$m\geq 2$</td><td>$\left[\begin{array}{ccccccc|ccc}0&2a_5&a_1&a_2&a_3&a_4&a_5&1&\dots&1\\1&1&1&1&1&1&1&0&\dots&0\end{array}\right]$
$a_1+a_2=a_3+a_4=2a_5$
$a_i\geq 0$
$m>5a_5$</td><td>$\begin{bmatrix}m+5a_5\\5\end{bmatrix}$</td><td>$m+4$</td></tr>
<tr><td>13</td><td>$\dfrac{\mathbb{K}[T_1,\dots,T_7,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2\rangle}$
$m\geq 2$</td><td>$\left[\begin{array}{ccccccc|cccc}0&0&0&0&0&0&0&1&1&\dots&1\\1&1&1&1&1&1&1&0&d_2&\dots&d_m\end{array}\right]$
$0\leq d_2\leq\ldots\leq d_m$
$d_m>0$
$m\cdot d_m<5+d_2+\ldots+d_m$</td><td>$\begin{bmatrix}m\\5+\sum d_k\end{bmatrix}$</td><td>$m+4$</td></tr>
<tr><td>14</td><td>$\dfrac{\mathbb{K}[T_1,\dots,T_{10}]}{\left\langle\begin{array}{c}T_1T_2+T_3T_4+T_5T_6+T_7T_8,\\\lambda_1T_3T_4+\lambda_2T_5T_6+T_7T_8+T_9T_{10}\end{array}\right\rangle}$</td><td>$\begin{bmatrix}1&0&1&0&1&0&1&0&1&0\\0&1&0&1&0&1&0&1&0&1\end{bmatrix}$</td><td>$\begin{bmatrix}3\\3\end{bmatrix}$</td><td>$6$</td></tr>
</table>

Moreover, each of the listed data defines a smooth Fano arrangement variety of complexity two and Picard number two.

CHAPTER

ONE

BACKGROUND

In this chapter we introduce the fundamental definitions and concepts used throughout this thesis. Its content is well known and does not contain results by the author. Our main reference for this chapter is the book [6].

Throughout the whole thesis $\mathbb{K}$ is an algebraically closed field of characteristic zero. If not stated different a variety is always assumed to be irreducible.

1.1 Cox rings

At first we recall the basic definitions for divisors of normal varieties and divisorial sheaves. Let X be a normal variety over $\mathbb{K}$. A *prime divisor* D of X is an irreducible subvariety $D \subseteq X$ of codimension 1. We call the free abelian group generated by the prime divisors the *group of Weil divisors* of X and denote it by $\mathrm{WDiv}(X)$. To any nonzero rational function $f \in \mathbb{K}(X)^*$ we associate the Weil divisor

$$\mathrm{div}(f) := \sum_{D \text{ prime}} \mathrm{ord}_D(f) \cdot D,$$

where $ord_D(f)$ denotes the vanishing order of f along D. A Weil divisor E arising as $E = \mathrm{div}(f)$ for $f \in \mathbb{K}(X)^*$ is called *principal*. The set $\mathrm{PDiv}(X)$ of principal divisors of X and the set $\mathrm{CaDiv}(X)$ of locally principal divisors of X form subgroups of the group of Weil divisors. The *divisor class group* and the *Picard group* of X are given as

$$\mathrm{Cl}(X) := \mathrm{WDiv}(X)/\mathrm{PDiv}(X), \quad \mathrm{Pic}(X) := \mathrm{CaDiv}(X)/\mathrm{PDiv}(X).$$

The rank of the Picard group is called the *Picard number* of X and if X is $\mathbb{Q}$-factorial the Picard number equals the rank of the class group. With every divisor $D \in \mathrm{WDiv}(X)$ we associate the *divisorial sheaf* $\mathcal{O}_X(D)$ by defining its sections over an open $U \subseteq X$ to be

$$\Gamma(U, \mathcal{O}_X(D)) := \{f \in \mathbb{K}(X)^*;\ (\mathrm{div}(f) + D)|_U \geq 0\} \cup \{0\}.$$

Note that for any two functions $f_1 \in \Gamma(U, \mathcal{O}_X(D_1))$ and $f_2 \in \Gamma(U, \mathcal{O}_X(D_2))$ we have $f_1 f_2 \in \Gamma(U, \mathcal{O}_X(D_1 + D_2))$. Thus we obtain a sheaf of K-graded $\mathcal{O}_X$-algebras called the *sheaf of divisorial algebras* associated with a subgroup $K \subseteq \mathrm{WDiv}(X)$ by setting

$$\mathcal{S} := \bigoplus_{D \in K} \mathcal{S}_D, \text{ where } \mathcal{S}_D := \mathcal{O}_X(D).$$

Now we turn to the definition of the Cox sheaf and the Cox ring, which is a generalization of the homogeneous coordinate ring for toric varieties.

For this let X be a normal prevariety with $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ and finitely generated divisor class group $\mathrm{Cl}(X)$. Let $K \leq \mathrm{WDiv}(X)$ be a subgroup such that the map $c\colon K \to \mathrm{Cl}(X)$, $D \mapsto [D]$ is surjective. Denote by $K^0 \subseteq K$ the kernel of c and fix a group homomorphism $\chi\colon K^0 \to \mathbb{K}(X)^*$ with

$$\mathrm{div}(\chi(E)) = E \text{ for all } E \in K^0.$$

Denote by $\mathcal{I}$ the sheaf of ideals of $\mathcal{S}$ generated by the sections $1 - \chi(E)$, where E runs through K^0.

Definition 1.1.1. The *Cox sheaf* associated with K and χ is the quotient sheaf $\mathcal{R} := \mathcal{S}/\mathcal{I}$ together with the $\mathrm{Cl}(X)$-grading

$$\mathcal{R} := \bigoplus_{[D] \in \mathrm{Cl}(X)} \mathcal{R}_{[D]}, \qquad \text{where} \qquad \mathcal{R}_{[D]} := \pi\left(\bigoplus_{D' \in C^{-1}([D])} \mathcal{S}'_D\right),$$

and $\pi\colon \mathcal{S} \to \mathcal{R}$ is the projection. The *Cox ring of* X is the algebra of global sections $\mathcal{R}(X)$.

The Cox ring is up to isomorphy independent of the choices of K and χ. In case of a torsion free class group we can choose a subgroup $K \leq \mathrm{WDiv}(X)$ such that $c\colon K \to \mathrm{Cl}(X)$ as above is an isomorphism. In this case we have $\mathcal{R} = \mathcal{S}$ where $\mathcal{S}$ is the sheaf of divisorial algebras associated with K.

The Cox ring of a variety is in general not a unique factorization domain but has a similar property:

Definition 1.1.2. Let K be an abelian group and $R = \oplus_{w \in K} R_w$ an integral K-graded $\mathbb{K}$-algebra.

(i) A homogeneous element $0 \neq f \in R \setminus R^*$ is called *K-prime* if whenever $f|gh$ holds for homogeneous elements $g, h \in R$ we have $f|g$ or $f|h$.

(ii) We call R *factorially K-graded* if every homogeneous $0 \neq f \in R \setminus R^*$ is a product of K-prime elements.

Theorem 1.1.3. *Let X be a normal variety with $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ and finitely generated divisor class group $\mathrm{Cl}(X)$. Then the Cox ring $\mathcal{R}(X)$ is $\mathrm{Cl}(X)$-factorially graded. Moreover if $\mathrm{Cl}(X)$ is torsion free then $\mathcal{R}(X)$ is a unique factorization domain.*

Observe that the Cox ring of a variety is in general not finitely generated. This motivates the following definition.

Definition 1.1.4. Let X be a normal variety with $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ and finitely generated divisor class group. X is called a *Mori Dream Space (MDS)* if its Cox ring is finitely generated.

Let X be an MDS with Cox sheaf $\mathcal{R}$. Then finite generation of $\mathcal{R}(X)$ implies that $\mathcal{R}$ is locally of finite type, i.e., any $x \in X$ has an affine open neighbourhood U such that $\mathcal{R}(U)$ is finitely generated. Choosing an affine open cover of X we can define the relative spectrum $\widehat{X} := \mathrm{Spec}_X(\mathcal{R})$ by gluing the affine pieces $\mathrm{Spec}\,\mathcal{R}(U)$. We call $\widehat{X}$ the *characteristic space* and $\overline{X} := \mathrm{Spec}\,\mathcal{R}(X)$ the *total coordinate space of* X. Note that $\widehat{X}$ has a canonical open embedding into $\overline{X}$ and is big, i.e., the codimension of $\overline{X} \setminus \widehat{X}$ is at least two.

1.2 Graded algebras, quasitorus actions and good quotients

In this section we give the necessary background to show how to regain a variety back from its characteristic space. At first we introduce the correspondence between graded affine algebras and quasitorus actions on affine varieties. Moreover we provide the necessary background on good quotients.

An *(affine) algebraic group* is an (affine) variety G and a group such that the group operations

$$G \times G \to G,\ (g,h) \mapsto g \cdot h \quad \text{and} \quad G \to G,\ g \mapsto g^{-1}$$

are morphisms of algebraic varieties. A *homomorphism* of algebraic groups G and G' is a group homomorphism $G \to G'$, which is a morphism of the underlying varieties. A *character* of an algebraic group G is a homomorphism of algebraic groups $\chi\colon G \to \mathbb{K}^*$, where $\mathbb{K}^*$ is the multiplicative group of the underlying field $\mathbb{K}$. The *character group* of an algebraic group G is the set $\mathbb{X}(G)$ of all characters of G, which forms a group with respect to pointwise multiplication. A *quasitorus* is an affine algebraic group H such that the algebra of regular functions $\Gamma(H, \mathcal{O})$ is generated as a $\mathbb{K}$-vector space by the characters $\chi \in \mathbb{X}(H)$. A *torus* is a connected quasitorus. The *standard n-torus* is the algebraic torus $\mathbb{T}^n := (\mathbb{K}^*)^n$.

Let G be an affine algebraic group. A *G-variety* is a variety X together with an action $G \times X \to X,\ (g,x) \mapsto g \cdot x$, which is a morphism of the underlying varieties. If G is even a quasitorus, then any choice of characters $\chi_1, \ldots, \chi_r$ defines a *diagonal G-action* on $\mathbb{K}^r$ by setting

$$g \cdot z := (\chi_1(g)z_1, \ldots, \chi_r(g)z_r).$$

We briefly recall the equivalence of categories between the category of affine algebras graded by a finitely generated abelian group and the category of affine varieties with quasitorus action.

Construction 1.2.1. Let K be a finitely generated abelian group and $R = \oplus_{w\in K} R_w$ a K-graded affine $\mathbb{K}$-algebra. Set $X := \operatorname{Spec} R$. Fixing homogeneous generators $f_1, \dots f_r$ with $f_i \in R_{w_i}$ we obtain a closed embedding

$$X \to \mathbb{K}^r, \quad x \mapsto (f_1(x), \dots, f_r(x)).$$

We equip $\mathbb{K}^r$ with the diagonal action of the quasitorus $H := \operatorname{Spec} \mathbb{K}[K]$ given by the characters $\chi^{w_1}, \dots, \chi^{w_r}$. Then X is invariant under this action and thus an H-variety. This construction does not depend on the choice of the embedding $X \subseteq \mathbb{K}^r$ up to isomorphism.

We show that vice versa any affine variety with quasitorus action gives rise to an affine algebra graded by a finitely generated abelian group: Let a quasitorus H act on a variety X. Then the algebra of regular functions $\Gamma(X, \mathcal{O})$ on X becomes a rational H-module by setting

$$(h \cdot f)(x) := f(h \cdot x).$$

and it becomes a $\mathbb{X}(H)$-graded algebra by considering its decomposition into one-dimensional subrepresentations

$$\Gamma(X, \mathcal{O}) = \bigoplus_{\chi\in\mathbb{X}(H)} \Gamma(X, \mathcal{O})_\chi, \quad \text{where} \quad \Gamma(X, \mathcal{O})_\chi := \{f \in \Gamma(X, \mathcal{O});\ f(h \cdot x) = \chi(h) f(x)\}\,.$$

Definition 1.2.2. Let G be a reductive algebraic group acting on a prevariety X. A *good quotient* for this action is a morphism of prevarieties $p\colon X \to Y$ such that the following holds:

(i) $p\colon X \to Y$ is affine and G-invariant.
(ii) The pullback $p^*\colon \mathcal{O}_Y \to (p_*\mathcal{O}_X)^G$ is an isomorphism.

A good quotient is called *geometric* if it moreover separates the orbits, i.e., any fiber is a G-orbit.

Let $X := \operatorname{Spec} R$ be an affine G-variety, where G is a reductive algebraic group. Then the *algebra of invariants*

$$R^G := \{f \in R;\ f(g \cdot x) = f(x) \text{ for all } x \in X \text{ and } g \in G\}$$

is finitely generated. Thus we obtain a good quotient $p\colon X \to Y$, where $Y = \operatorname{Spec} R^G$. For non affine X good quotients are locally modeled on this concept.

Construction 1.2.3. Assume X is a Mori Dream Space. Then $\mathcal{R}(X)$ is a $\operatorname{Cl}(X)$-graded affine $\mathbb{K}$-algebra. This gives rise to an action of a quasitorus $H := \operatorname{Spec} \mathbb{K}[\operatorname{Cl}(X)]$ on its total coordinate space $\overline{X}$. The characteristic space $\widehat{X} \subseteq \overline{X}$ is an H-invariant open subset admitting a good quotient for this action. In particular we obtain the following diagramm:

$$\begin{array}{ccccccc} \operatorname{Spec}_X(\mathcal{R}) & = & \widehat{X} & \subseteq & \overline{X} & = & \operatorname{Spec} \mathcal{R}(X)\,. \\ & & \downarrow & & & & \\ & & X & & & & \end{array}$$

1.3 Bunched rings

In this section we summarize at first basic facts about toric varieties. Afterwards we give a short summary of the theory of bunched rings. Fixing a graded ring R the theory of bunched rings provides a construction for example for all quasi-projective varieties having R as their Cox ring and fulfilling some maximality property, see below. Moreover the theory provides an embedding of a variety X with Cox ring R into a toric variety and thus leads to combinatorial methods.

A *toric variety* is a normal variety Z with an effective torus action $T \times Z \to Z$ and a *base-point* $z_0 \in Z$ such that the orbit map $T \to Z,\ t \mapsto t \cdot z_0$ is an open embedding.

We recall the basic notions of the combinatorial description of toric varieties via fans.

Let N and M be mutually dual lattices and $N_{\mathbb{Q}}$ resp. $M_{\mathbb{Q}}$ the corresponding rational vector spaces. A *cone* is a convex polyhedral subset $\sigma \subseteq N_{\mathbb{Q}}$ such that for any $u \in \sigma$ and $t \in \mathbb{Q}_{\geq 0}$ we have $t \cdot u \in \sigma$. The *dimension* of a cone σ is the dimension of its linear hull. A *lattice cone* is a pair (σ, N) where N is a lattice and $\sigma \subseteq N_{\mathbb{Q}}$ is *pointed*, i.e., it contains no lines. The *dual cone of* σ is the polyhedral cone $\sigma^\vee := \{u \in M_{\mathbb{Q}};\ u|_\sigma \geq 0\}$. A *face* of a cone σ is a cone $\tau \subseteq \sigma$ such that there exists an $u \in \sigma^\vee$ with $u|_\tau = 0$. A 1-dimensional face is called *ray* and a face of codimension one is called *facet*.

A finite collection Σ of convex polyhedral cones in $N_{\mathbb{Q}}$ is called a *quasifan* if the following holds:

(i) Let $\sigma \in \Sigma$. Then $\sigma_0 \in \Sigma$ holds for any face $\sigma_0 \preccurlyeq \sigma$.

(ii) For any two cones $\sigma_1, \sigma_2 \in \Sigma$ the intersection $\sigma_1 \cap \sigma_2$ is a face of both cones.

A quasifan Σ is called a *fan* if all of its cones are pointed. A tuple (Σ, N) is called a *lattice fan*. The *support* $|\Sigma|$ of a fan Σ is the union of its cones.

Any lattice cone (σ, N) defines an affine toric variety $Z(\sigma) := \operatorname{Spec} \mathbb{K}[\sigma^\vee \cap M]$ with dense open torus $T_Z := \operatorname{Spec} \mathbb{K}[M]$. We can extend this construction to lattice fans (Σ, N) and toric varieties: For any two cones $\sigma_1, \sigma_2 \in \Sigma$ one glues the corresponding affine toric varieties $Z_{\sigma_1}, Z_{\sigma_2}$ along the affine toric variety $Z_{\sigma_1 \cap \sigma_2}$, which defines a common open subset. This gluing process provides a toric variety Z_Σ. Note that Z_Σ is a complete variety if and only if Σ is *complete*, i.e., $|\Sigma| = N_{\mathbb{Q}}$.

Let K be a finitely generated abelian group and consider a finitely generated K-factorially graded affine $\mathbb{K}$-algebra

$$R := \bigoplus_{w \in K} R_w.$$

Fix a set of pairwise non-associated K-prime homogeneous generators $\mathfrak{F} := (f_1, \ldots, f_r)$ and denote by $Q\colon \mathbb{Z}^r \to K$ the homomorphism of abelian groups, that maps the canonical basis vector $e_i \in \mathbb{Z}^r$ to the weights $w_i := \deg(f_i) \in K$.

(i) An $\mathfrak{F}$-*face* is a face $\gamma_0 \preccurlyeq \gamma := \mathbb{Q}^r_{\geq 0}$ of the positive orthant, such that there is a point $x \in \overline{X}$ with $x_i \neq 0$ if and only if $e_i \in \gamma_0$.

(ii) The K-grading of R is *almost free* if for every facet $\gamma_0 \preccurlyeq \gamma$ the image $Q(\gamma_0 \cap E)$ generates K as a group.

(iii) Let $\Omega_{\mathfrak{F}} = \{Q(\gamma_0);\ \gamma_0 \preccurlyeq \gamma \text{ is an } \mathfrak{F}\text{-face }\}$ denote the set of projected $\mathfrak{F}$-faces. An $\mathfrak{F}$-*bunch* is a nonempty subset $\Phi \subseteq \Omega_{\mathfrak{F}}$ satisfiying the following conditions:
   (a) For any two $\tau_1, \tau_2 \in \Phi$, we have $\tau_1^\circ \cap \tau_2^\circ \neq \emptyset$.
   (b) Let $\tau_1, \tau_2 \in \Omega_{\mathfrak{F}}$ with $\tau_1^\circ \subseteq \tau_2^\circ$. Then $\tau_1 \in \Phi$ implies $\tau_2 \in \Phi$.

(iv) An $\mathfrak{F}$-bunch Φ is called *true* if for every facet $\gamma_0 \preccurlyeq \gamma$ we have $Q(\gamma_0) \in \Phi$.

Definition 1.3.1. A *bunched ring* is a triple $(R, \mathfrak{F}, \Phi)$ consisting of an integral, normal, almost freely factorially K-graded affine $\mathbb{K}$-algebra R with only constant homogeneous invertible elements, a system $\mathfrak{F}$ of pairwise nonassociated K-prime generators for R and a true $\mathfrak{F}$-bunch Φ.

The following construction associates to a bunched ring $(R, \mathfrak{F}, \Phi)$ a variety X having R as its Cox ring.

Construction 1.3.2. Let $(R, \mathfrak{F}, \Phi)$ be a bunched ring with $Q\colon E \to K$ as above and $\mathfrak{F} = (f_1, \ldots, f_r)$. Then we obtain an action of the quasitorus $H := \operatorname{Spec} \mathbb{K}[K]$ on the affine variety $\overline{X} := \operatorname{Spec} R$. We define the localization of $\overline{X}$ with respect to an $\mathfrak{F}$-face γ_0 to be

$$\overline{X}_{\gamma_0} \ := \ \overline{X}_{f_1^{u_1} \cdots f_r^{u_r}} \qquad \text{for some} \ \ u = (u_1, \ldots, u_r) \in \gamma_0^\circ.$$

This set is independent of the choice of u and with the *collection of relevant faces* $\operatorname{rlv}(\Phi) := \{\gamma_0 \preccurlyeq \gamma;\ \gamma_0 \text{ is an } \mathfrak{F}\text{-face with } Q(\gamma_0) \in \Phi\}$ associated to Φ, we obtain an H-invariant open subset of $\overline{X}$:

$$\widehat{X} \ := \ \widehat{X}(R, \mathfrak{F}, \Phi) \ := \ \bigcup_{\gamma_0 \in \operatorname{rlv}(\Phi)} \overline{X}_{\gamma_0}.$$

The H-action on $\widehat{X}$ admits a good quotient [6, Prop. 3.1.3.8] and we set

$$X \ := \ X(R, \mathfrak{F}, \Phi) \ := \ \widehat{X} /\!\!/ H$$

and denote the quotient map by $p\colon \widehat{X} \to X$. The pieces $X_{\gamma_0} := p(\overline{X}_{\gamma_0}) \subseteq X$ form an affine cover of X. Moreover, every element f_i of $\mathfrak{F}$ defines a prime divisor $D_X^i := p(V_{\widehat{X}}(f_i))$ on X.

Recall that an A_2-*variety* is a variety X with the property that any two points of X admit a common affine open neighbourhood.

Theorem 1.3.3. *Let $\overline{X}$, $\widehat{X}$ and X arise from a bunched ring $(R, \mathfrak{F}, \Phi)$ as above. Then X is a normal A_2-variety with*

$$\dim(X) = \dim(\overline{X}) - \dim(K_{\mathbb{Q}}), \quad \Gamma(X, \mathcal{O}^*) = \mathbb{K}^*.$$

Moreover there is an isomorphism $\operatorname{Cl}(X) \cong K$, the Cox ring $\mathcal{R}(X)$ is isomorphic to R and the characteristic space of X equals $\widehat{X}$. In particular X is a Mori dream space.

We say that a variety is A_2-*maximal* if it has the A_2-property and cannot be realized as an open subset with nonempty complement of codimension at least two in another A_2-variety. Moreover, we call an $\mathfrak{F}$-bunch *maximal* if it cannot be enlarged by adding further projected $\mathfrak{F}$-faces. In the situation of Theorem 1.3.3 X is A_2-maximal if and only if Φ is maximal. Moreover, we obtain the following:

Theorem 1.3.4. *Let X be an A_2-maximal Mori Dream Space with Cox ring $R := \mathcal{R}(X)$ and let $\mathfrak{F}$ be any finite system of pairwise nonassociated* $\mathrm{Cl}(X)$*-prime generators for R. Then $X \cong X(R, \mathfrak{F}, \Phi)$ holds for some maximal $\mathfrak{F}$-bunch Φ.*

Every variety X defined by a bunched ring comes with a closed embedding into a toric variety Z.

Construction 1.3.5. Let $(R, \mathfrak{F}, \Phi)$ be a bunched ring and $Q\colon E \to K$ be as above. Setting $F := E^*$ and $M := \ker(Q)$, we obtain mutually dual exact sequences

$$0 \longrightarrow L \longrightarrow F \xrightarrow{\;P\;} N$$

$$0 \longleftarrow K \xleftarrow{\;Q\;} E \xleftarrow{\;P^*\;} M \longleftarrow 0$$

We define the *envelope* of the collection of relevant faces of Φ

$$\mathrm{Env}(\Phi) \;:=\; \{\gamma_0 \preceq \gamma;\ \exists\, \gamma_1 \in \mathrm{rlv}(\Phi) \text{ with } \gamma_1 \preceq \gamma_0 \text{ and } Q(\gamma_1)^\circ \subseteq Q(\gamma_0)^\circ\}.$$

Set $\delta := \gamma^\vee \subset F_{\mathbb{Q}}$ and for each $\gamma_0 \preceq \gamma$, let $\gamma_0^* := \gamma_0^\perp \cap \delta \preceq \delta$ be the corresponding face. We define the following fans in $F_{\mathbb{Q}}$ and $N_{\mathbb{Q}}$

$$\begin{aligned} \widehat{\Sigma} &:= \{\delta_0 \preceq \delta;\ \exists\, \gamma_0 \in \mathrm{Env}(\Phi) \text{ with } \delta_0 \preceq \gamma_0^*\}, \\ \Sigma &:= \{P(\gamma_0^*);\ \gamma_0 \in \mathrm{Env}(\Phi)\}. \end{aligned}$$

Let $\overline{\Sigma}$ be the fan consisting of all faces of δ, which is the fan corresponding to $\mathbb{K}^r$ with its natural toric structure. Since $\widehat{\Sigma}$ is a subfan of $\overline{\Sigma}$, there is an open embedding of the corresponding varieties $\widehat{Z} \subseteq \mathbb{K}^r$. Moreover, there is a map of fans $\widehat{\Sigma} \to \Sigma$ arising from $P\colon F \to N$. Denoting by Z the toric variety associated to Σ, we obtain a toric morphism $p\colon \widehat{Z} \to Z$. We obtain the following commutative diagram:

$$\begin{tikzcd} \overline{X} \arrow[r, "\bar{\imath}"] & \mathbb{K}^r \\ \widehat{X} \arrow[u] \arrow[r, "\hat{\imath}"] \arrow[d, "/\!\!/ H"'] & \widehat{Z} \arrow[u] \arrow[d, "/\!\!/ H"] \\ X \arrow[r, "\imath"] & Z \end{tikzcd}$$

where $\overline{X}$, $\widehat{X}$ and X are the varieties associated to the bunched ring $(R, \mathfrak{F}, \Phi)$ as in Construction 1.3.2. We call the closed embedding $i\colon X \to Z$ the *canonical toric embedding*.

Definition 1.3.6. Let $(R, \mathfrak{F}, \Phi)$ be a bunched ring with $\mathfrak{F} = (f_1, \dots, f_r)$. We say that it is a *complete intersection* if the kernel of the epimorphism $\mathbb{K}[T_1, \dots, T_r] \to R$, mapping T_i to f_i, is generated by K-homogeneous polynomials $g_1, \dots, g_s$, where $s = r - \dim(R)$. In this case we define the *degree vectors* of $(R, \mathfrak{F}, \Phi)$ as $(w_1, \dots, w_r)$ and $(u_1, \dots, u_s)$, where $w_i := \deg(f_i) \in K$ and $u_j := \deg(g_j) \in K$.

Proposition 1.3.7. *Let the bunched ring $(R, \mathfrak{F}, \Phi)$ be a complete intersection with degree vectors $(w_1, \dots, w_r)$ and $(u_1, \dots, u_s)$. Then the anticanonical divisor class of $X = X(R, \mathfrak{F}, \Phi)$ is given in $\mathrm{Cl}(X) = K$ as*

$$-\mathcal{K}_X \ = \ \sum_{i=1}^{r} w_i - \sum_{j=1}^{s} u_j \ .$$

The last part of this section is dedicated to projective varieties arising from a bunched ring. In particular we give a link between bunched rings providing projective varieties and the GIT-chamber decomposition of the weight cone of a graded ring.

Let K be a finitely generated abelian group and consider an affine K-graded $\mathbb{K}$-algebra R. Set $X := \operatorname{Spec} R$. The *weight cone* of X is the cone

$$\omega_X := \mathrm{cone}(w \in K;\ A_w \neq \{0\}).$$

The *orbit cone* of a point $x \in X$ is the convex cone $\omega_x \subseteq K_{\mathbb{Q}}$ generated by the *weight monoid*

$$S_x = \{w \in K;\ f(x) \neq 0 \text{ for some } f \in A_w\} \subseteq K.$$

The *GIT-cone* of an element $w \in \omega_X$ is the intersetion of all orbits containing it:

$$\lambda(w) := \bigcap_{x \in X, w \in \omega_x} \omega_x.$$

The set of all GIT-cones is a quasifan in $K_{\mathbb{Q}}$ having the weight cone ω_X as its support.

For any weight $w \in \omega_X$ we define the *set of semistable points* to be the H-invariant open subset

$$X^{ss}(w) \ := \ \{x \in X;\ f(x) \neq 0 \text{ for some } f \in A_{nw}, n > 0\} \subseteq X,$$

allowing a good quotient $X^{ss}(w) \to X^{ss}(w) /\!\!/ H$ onto a projective variety. Note that $X^{ss}(w_1) = X^{ss}(w_2)$ for any $w_2 \in \lambda(w_1)^{\circ}$. In particular every GIT-cone defines a good quotient as above.

We link this situation to the theory of bunched rings: Let $(R, \mathfrak{F}, \Phi)$ be a bunched ring and suppose the $\mathfrak{F}$-bunch Φ arises from a GIT-cone $\lambda(w)$, i.e.,

$$\Phi := \Phi(w) := \{Q(\gamma_0);\ \gamma_0 \preccurlyeq \gamma \ \mathfrak{F}\text{-face with } w \in Q(\gamma_0)^{\circ}\} .$$

Then $X(R, \mathfrak{F}, \Phi) = \overline{X}^{ss}(w) /\!\!/ H$ holds with $\overline{X} := \operatorname{Spec} R$. Note that any projective Mori dream space arises this way.

CHAPTER

TWO

NON COMPLETE RATIONAL T-VARIETIES OF COMPLEXITY ONE

In this chapter we consider rational varieties with a torus action of complexity one and extend the combinatorial approach via the Cox ring developed for the complete case in [49, 45, 44] to the non-complete, e.g. affine, case. This includes in particular a description of all factorially graded affine algebras of complexity one with only constant homogeneous invertible elements in terms of canonical generators and relations. The results of this chapter have been published in [50].

2.1 Factorially graded rings of complexity one

The basic task is to describe all Cox rings of normal rational varieties X with an effective torus action $T \times X \to X$ of complexity one even in the non-complete case. As it is needed for the uniqueness of the Cox ring, we require $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$. From the algebraic point of view, a Cox ring is firstly a finitely generated integral $\mathbb{K}$-algebra R, graded by a finitely generated abelian group K such that there are only constant invertible homogeneous elements. The most important characterizing property of a Cox ring is then *K-factoriality*, which means that we have unique factorization in the multiplicative monoid of non-zero homogeneous elements. Observe that for a torsion-free grading group, K-factoriality is equivalent to the usual unique factorization property but in general it is weaker.

In a first step, we describe all finitely generated integral $\mathbb{K}$-algebras R that admit an effective factorial K-grading of complexity one, where effective means that the weights $w \in K$ with $R_w \neq 0$ generate K as a group and complexity one means that the rational vector space $K \otimes \mathbb{Q}$ is of dimension one less than R. The first results in this direction concern the case $R_0 = \mathbb{K}$ in dimension two, see [66, 61]. The case $R_0 = \mathbb{K}$ in arbitrary dimension was settled in [44] and occurs as part of *Type 2* in our subsequent considerations. A simple example of *Type 1* presented below is the coordinate algebra of the special linear group SL(2):

$$R \ = \ \mathbb{K}[T_1,T_2,T_3,T_4]/\langle T_1T_2 - T_3T_4 - 1\rangle, \qquad Q \ = \ \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix},$$

where the matrix Q specifies a $\mathbb{Z}^2$-grading of R by assigning to the variable T_i the i-th column of Q as its degree; if $\mathbb{T} \subseteq \mathrm{SL}(2)$ denotes the diagonal torus, this grading reflects the action of $\mathbb{T} \times \mathbb{T}$ on SL(2) given by $(s,t)\cdot A = sAt^{-1}$. Here comes the general construction.

Construction 2.1.1. Fix integers $r, n > 0$, $m \geq 0$ and a partition $n = n_\iota + \ldots + n_r$ starting at $\iota \in \{0,1\}$. For each $\iota \leq i \leq r$, fix a tuple $l_i \in \mathbb{Z}_{>0}^{n_i}$ and define a monomial

$$T_i^{l_i} \ := \ T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}} \ \in \ \mathbb{K}[T_{ij}, S_k;\ \iota \leq i \leq r,\ 1 \leq j \leq n_i,\ 1 \leq k \leq m].$$

We will also write $\mathbb{K}[T_{ij}, S_k]$ for the above polynomial ring. We distinguish two settings for the input data A and P_0 of the graded $\mathbb{K}$-algebra $R(A,P_0)$.

Type 1. Take $\iota = 1$. Let $A := (a_1, \ldots, a_r)$ be a list of pairwise different elements of $\mathbb{K}$. Set $I := \{1, \ldots, r-1\}$ and define for every $i \in I$ a polynomial

$$g_i \ := \ T_i^{l_i} - T_{i+1}^{l_{i+1}} - (a_{i+1} - a_i) \ \in \ \mathbb{K}[T_{ij}, S_k].$$

We build up an $r \times (n+m)$ matrix from the exponent vectors $l_1, \ldots, l_r$ of these polynomials:

$$P_0 \ := \ \begin{bmatrix} l_1 & & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & & l_r & 0 & \ldots & 0 \end{bmatrix}.$$

Type 2. Take $\iota = 0$. Let $A := (a_0, \ldots, a_r)$ be a $2 \times (r+1)$-matrix with pairwise linearly independent columns $a_i \in \mathbb{K}^2$. Set $I := \{0, \ldots, r-2\}$ and for every $i \in I$ define

$$g_i \ := \ \det \begin{bmatrix} T_i^{l_i} & T_{i+1}^{l_{i+1}} & T_{i+2}^{l_{i+2}} \\ a_i & a_{i+1} & a_{i+2} \end{bmatrix} \ \in \ \mathbb{K}[T_{ij}, S_k].$$

We build up an $r \times (n+m)$ matrix from the exponent vectors $l_0, \ldots, l_r$ of these polynomials:

$$P_0 \ := \ \begin{bmatrix} -l_0 & l_1 & & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ -l_0 & 0 & & l_r & 0 & \ldots & 0 \end{bmatrix}.$$

We now define the ring $R(A,P_0)$ simultaneously for both types in terms of the data A and P_0. Denote by P_0^* the transpose of P_0 and consider the projection

$$Q\colon \mathbb{Z}^{n+m} \ \to \ K_0 \ := \ \mathbb{Z}^{n+m}/\mathrm{im}(P_0^*).$$

Denote by $e_{ij}, e_k \in \mathbb{Z}^{n+m}$ the canonical basis vectors corresponding to the variables T_{ij}, S_k. Define a K_0-grading on $\mathbb{K}[T_{ij}, S_k]$ by setting

$$\deg(T_{ij}) \ := \ Q(e_{ij}) \ \in \ K_0, \qquad \deg(S_k) \ := \ Q(e_k) \ \in \ K_0.$$

This is the coarsest possible grading of $\mathbb{K}[T_{ij}, S_k]$ leaving the variables and the g_i homogeneous. In particular, we have a K_0-graded factor algebra

$$R(A, P_0) \ := \ \mathbb{K}[T_{ij}, S_k]/\langle g_i; \ i \in I\rangle.$$

We gather the basic properties of the graded algebras just constructed; the corresponding proofs are given in Section 2.2. Below, we mean by a K_0-*prime* element a homogeneous non-zero non-unit which, whenever it divides a product of homogeneous elements, it also divides one of the factors.

Theorem 2.1.2. *Let $R(A, P_0)$ be a K_0-graded ring as provided by Construction 2.1.1.*

(i) *The ring $R(A, P_0)$ is an integral, normal complete intersection ring of dimension $n + m - r + 1$.*
(ii) *The K_0-grading on $R(A, P_0)$ is effective, factorial of complexity one and $R(A, P_0)$ has only constant invertible homogeneous elements.*
(iii) *The variables T_{ij} and S_k define pairwise nonassociated K_0-prime generators for $R(A, P_0)$.*
(iv) *In case of Type 1, suppose $r \geq 2$ and $n_i l_{ij} > 1$ for all i, j. Then $R(A, P_0)$ is factorial if and only if one of the following statements holds:*
  - *(a) One has $\gcd(l_{i1}, \ldots, l_{in_i}) = 1$ for $i = 1, \ldots, r$ or, equivalently, K_0 is torsion free.*
  - *(b) $P_0 = [2E_2, 0]$ holds.*

(v) *In case of Type 2, suppose $r \geq 2$ and $n_i l_{ij} > 1$ for all i, j. Then the following statements are equivalent:*
  - *(a) $R(A, P_0)$ is factorial.*
  - *(b) Any two of the $\mathfrak{l}_i := \gcd(l_{i1}, \ldots, l_{in_i})$, where $i = 0, \ldots, r$, are coprime.*
  - *(c) K_0 is torsion free.*

(vi) *In case of Type 1, the degree zero part $R(A, P_0)_0$ is isomorphic to a polynomial ring in one variable over $\mathbb{K}$, and in case of Type 2, one has $R(A, P_0)_0 = \mathbb{K}$.*

Observe that the situation of (iv) can always be achieved by eliminating the variables that occur in a linear term of some relation. The following result shows that Construction 2.1.1 yields in fact all affine algebras with property (ii) of the above theorem; see Section 2.2 for the proof.

Theorem 2.1.3. *Every finitely generated, integral, normal $\mathbb{K}$-algebra with an effective, factorial grading of complexity one by a finitely generated abelian group and only constant invertible homogeneous elements is isomorphic to a K_0-graded $\mathbb{K}$-algebra $R(A, P_0)$ as provided by Construction 2.1.1.*

We turn to Cox rings of rational varieties with a torus action of complexity one. They will be obtained as suitable downgradings of the algebras $R(A, P_0)$ of Construction 2.1.1. Here comes the precise recipe.

Construction 2.1.4. Let integers r, $n = n_\iota + \ldots + n_r$, m and data A and P_0 of Type 1 or Type 2 as in Construction 2.1.1. Fix $1 \le s \le n+m-r$, choose an integral $s \times (n+m)$ matrix d and build the $(r+s) \times (n+m)$ stack matrix

$$P \;:=\; \begin{bmatrix} P_0 \\ d \end{bmatrix}.$$

We require the columns of P to be pairwise different primitive vectors generating $\mathbb{Q}^{r+s}$ as a vector space. Let P^* denote the transpose of P and consider the projection

$$Q\colon \mathbb{Z}^{n+m} \;\to\; K \;:=\; \mathbb{Z}^{n+m}/\mathrm{im}(P^*).$$

Denoting as before by $e_{ij}, e_k \in \mathbb{Z}^{n+m}$ the canonical basis vectors corresponding to the variables T_{ij} and S_k, we obtain a K-grading on $\mathbb{K}[T_{ij}, S_k]$ by setting

$$\deg(T_{ij}) \;:=\; Q(e_{ij}) \;\in\; K, \qquad\qquad \deg(S_k) \;:=\; Q(e_k) \;\in\; K.$$

This K-grading coarsens the K_0-grading of $\mathbb{K}[T_{ij}, S_k]$ given in Construction 2.1.1. In particular, we have the K-graded factor algebra

$$R(A, P) \;:=\; \mathbb{K}[T_{ij}, S_k]/\langle g_i;\; i \in I\rangle.$$

We present the basic properties of this construction; see Section 2.2 for the proof. Recall from [6] that the K-grading of $R(A, P)$ is *almost free* if for any choice of $n + m - 1$ out of the $n + m$ variables T_{ij}, S_k, the respective degrees generate the grading group K; geometrically this means that the quasitorus $\mathrm{Spec}\,\mathbb{K}[K]$ acts freely on an open subset of $\mathrm{Spec}\,R(A, P)$ having complement of codimension at least two.

Theorem 2.1.5. *Let $R(A, P)$ be a K-graded ring as provided by Construction 2.1.4.*

(i) *The K-grading on $R(A, P)$ is almost free, factorial, and $R(A, P)$ has only constant invertible homogeneous elements.*
(ii) *The variables T_{ij} and S_k define pairwise different nonassociated K-prime generators for $R(A, P)$.*

Knowledge of the Cox ring allows to (re)construct the underlying varieties. As in the complete case, we will obtain A_2-varieties, i.e. varieties admitting an embedding into a toric variety. This comprises in particular the affine and, more generally, the quasiprojective case. We make use of the language of bunched rings, see Section 1.3 for an introduction.

Construction 2.1.6. Let $R(A, P)$ be a K-graded ring as provided by Construction 2.1.4 and $\mathfrak{F} = (T_{ij}, S_k)$ the canonical system of generators. Consider

$$H \;:=\; \mathrm{Spec}\,\mathbb{K}[K], \qquad\qquad \overline{X}(A, P) \;:=\; \mathrm{Spec}\,R(A, P),$$

Then H is a quasitorus and the K-grading of $R(A,P)$ defines an action of H on $\overline{X}$. Any true $\mathfrak{F}$-bunch Φ defines an H-invariant open set and a good quotient

$$\widehat{X}(A,P,\Phi) \subseteq \overline{X}(A,P), \qquad X(A,P,\Phi) := \widehat{X}(A,P,\Phi)/\!\!/H.$$

The action of $H_0 = \operatorname{Spec} \mathbb{K}[K_0]$ leaves $\widehat{X}(A,P,\Phi)$ invariant and induces an action of the torus $T = \operatorname{Spec} \mathbb{K}[\mathbb{Z}^s]$ on $X(A,P,\Phi)$.

From [6, Thm. 3.2.1.4] we infer the following properties of the varieties arising via this construction; the proof of rationality is given in Section 2.2.

Theorem 2.1.7. *Consider a T-variety $X = X(A,P,\Phi)$ as provided by Construction 2.1.6. Then X is a normal, rational A_2-variety with only constant invertible functions and the action of T on X is of complexity one. Dimension, divisor class group and Cox ring of X are given by*

$$\dim(X) = s+1, \qquad \operatorname{Cl}(X) \cong K, \qquad \mathcal{R}(X) \cong R(A,P).$$

Note that, according to Theorem 2.1.2 (v), for a given $X = X(A,P,\Phi)$, its Cox ring $R(A,P)$ arises from data of Type 2 if and only if X has only constant T-invariant functions. In case of Type 1, the algebra of T-invariant functions is a polynomial ring in one variable over $\mathbb{K}$.

The following converse for Theorem 2.1.7 is proven in Section 2.2 and concerns A_2-*maximal* varieties that means A_2-varieties that cannot be realized as an open subset with nonempty complement of codimension at least two in another A_2-variety; this setting includes in particular the affine and, more generally, the semiprojective case, i.e., varieties being projective over an affine one.

Theorem 2.1.8. *Let X be an irreducible, normal, rational, A_2-maximal variety with only constant invertible functions, finitely generated divisor class group and a torus action of complexity one. Then X is equivariantly isomorphic to a variety $X(A,P,\Phi)$ provided by Construction 2.1.6.*

In the case of *affine*, normal, rational varieties with a torus action of complexity one, the whole machinery boils down to the following statement.

Corollary 2.1.9. *Let X be an irreducible, normal, rational affine variety with only constant invertible functions, finitely generated divisor class group and an effective algebraic torus action of complexity one. Then X is equivariantly isomorphic to a variety* $\operatorname{Spec} R(A,P)_0$ *acted on by the torus H_0/H, where $R(A,P)$ is as in Construction 2.1.4 and the columns of P generate the extremal rays of a pointed cone in $\mathbb{Q}^{r+s}$.*

In Section 2.3 we turn towards the geometry of the varieties of complexity one constructed in this section, presenting i.a. methods for the resolution of singularities.

Finally, in Section 2.4, we illustrate our methods by discussing the well-known case of normal affine $\mathbb{K}^*$-surfaces [36, 38]. We take a closer look at du Val singularities and show how their Cox rings and resolutions are obtained using our framework; see [33, 62, 29] for earlier treatments based on other methods.

2.2 Proofs of the results of Section 2.1

We first show that the algebras provided by Constructions 2.1.1 and 2.1.4 have indeed the desired properties: the assertions of Theorem 2.1.2 are verified in Propositions 2.2.1 to 2.2.8, where we restrict to Type 1 and refer to [6, Section 3.4.2] for the corresponding statements on Type 2. Then Theorems 2.1.5, 2.1.7, 2.1.8 and Corollary 2.1.9 are proven. We work in the notation of Constructions 2.1.1 and 2.1.4.

Proposition 2.2.1. *Let $R(A, P_0)$ be a $\mathbb{K}$-algebra of Type 1 as in Construction 2.1.1. Then every K_0-homogeneous invertible element of $R(A, P_0)$ is constant.*

Lemma 2.2.2. *Notation as for Type 1 in Construction 2.1.1. For any two indices $1 \le i, j \le r$, set*

$$g_{ij} \ := \ T_i^{l_i} - T_j^{l_j} + a_i - a_j.$$

For any three $1 \le i, j, k \le r$, we have $g_{ij} = g_{ik} - g_{jk}$ and $G := \{g_{ir};\ 1 \le i \le r-1\}$ is a reduced Gröbner basis with respect to the lexicographical ordering for $\langle g_1, \ldots, g_{r-1}\rangle$.

Proof. The identities among the g_{ij} are obvious. Since $g_i = g_{i\,i+1}$ holds, we see that G generates $\langle g_1, \ldots, g_{r-1}\rangle$. With $\alpha_{ij} := a_j - a_i$, the S-polynomials of G are of the form

$$T_i^{l_i}T_r^{l_r} - T_j^{l_j}T_r^{l_r} + T_i^{l_i}\alpha_{jr} - T_j^{l_j}\alpha_{ir} \ = \ g_{ir}(T_r^{l_r} + \alpha_{jr}) - g_{jr}(T_r^{l_r} + \alpha_{ir}).$$

In particular, they all reduce to zero with respect to G and thus G is the desired Gröbner basis for $\langle g_1, \ldots, g_{r-1}\rangle$. Obviously G is reduced. □

Proof of Proposition 2.2.1. Let $f \in \mathbb{K}[T_{ij}, S_k]$ define a K_0-homogeneous unit in $R(A, P_0)$ with inverse defined by $g \in \mathbb{K}[T_{ij}, S_k]$. We first show that f and hence g is of K_0-degree zero. We have a presentation

$$fg - 1 \ = \ \sum_{i=1}^{r-1} h_i g_i, \qquad h_i \in \mathbb{K}[T_{ij}, S_k].$$

Suppose that f is of nonzero K_0-degree. Then g is so and the constant term of $fg - 1$ equals -1. Thus, at least one of the h_i must have a nonzero constant term and we may rewrite the presentation as

$$fg - 1 \ = \ \sum_{i=1}^{r-1} \tilde{h}_i g_i + \beta_i g_i \ = \ \sum_{i=1}^{r-1} \tilde{h}_i g_i + \beta_i(T_i^{l_i} - T_{i+1}^{l_{i+1}} - \alpha_{i\,i+1}),$$

where the $\tilde{h}_i \in \mathbb{K}[T_{ij}, S_k]$ have constant term zero and at least one β_i is nonzero. Adding 1 to the left and the right hand side gives

$$fg \ = \ \sum_{i=1}^{r-1} \tilde{h}_i g_i + \beta_i(T_i^{l_i} - T_{i+1}^{l_{i+1}}).$$

By Lemma 2.2.2, at least two different $T_j^{l_j}, T_k^{l_k}$ are not cancelled on the right hand side. Consider monomials f_j, f_k of f dividing $T_j^{l_j}, T_k^{l_k}$ respectively. Since f is K_0-homogeneous, the exponents of f_j and f_k differ by an element of the row lattice of P_0. This works only for $f_j = 1$ or $f_j = T_j^{l_j}$. We conclude that f and hence g is of K_0-degree zero; a contradiction.

Having seen that f and g are of K_0-degree zero, we conclude that they are polynomials in the $T_i^{l_i}$. Using the structure of the g_{ir} we may bring the representatives f and g in the form

$$f \;=\; \sum_i \lambda_i (T_r^{l_r})^i, \qquad\qquad g \;=\; \sum_j \kappa_j (T_r^{l_r})^j.$$

Then also $fg - 1$ is a polynomial in $T_r^{l_r}$. Since $fg - 1$ belongs to $\langle g_{1\,r}, \ldots, g_{r-1\,r}\rangle$, its reduction by the set G of Lemma 2.2.2 equals zero. This means $fg - 1 = 0$ and thus $f, g \in \mathbb{K}^*$. $\square$

Proposition 2.2.3. *Let $R(A, P_0)$ be a $\mathbb{K}$-algebra of Type 1 as in Construction 2.1.1. Then $R(A, P_0)$ is an integral, regular complete intersection of dimension $n + m - r + 1$. The K_0-grading of $R(A, P_0)$ is effective and of complexity one. Moreover, the degree zero part $R(A, P_0)_0$ is a polynomial ring in one variable over $\mathbb{K}$.*

Lemma 2.2.4. *Let G be a quasitorus and X a (normal) affine G-variety with only constant invertible homogeneous functions. Then X is connected (irreducible).*

Proof. Consider the induced action of G on the set $Y = \{X_1, \ldots, X_r\}$ of connected components. Then Y is a single G-orbit, because otherwise we can write X as a union of disjoint open G-invariant sets which in turn yields nonconstant invertible functions on X. The stabilizer $G_1 \subseteq G$ of $X_1 \in Y$ is a closed subgroup and we have the homomorphism $\pi\colon G \to G/G_1$. Write $X_i = g_i \cdot X_1$ with suitable $g_1, \ldots, g_r \in G$. Then, for every character $\mathbb{X}$ on G/G_1, we obtain an invertible regular function $f_{\mathbb{X}}$ on X sending $x \in g_i \cdot X_1$ to $\mathbb{X}(\pi(g_i))$. By construction, $f_{\mathbb{X}}$ is homogeneous with respect to $\mathbb{X}$. Thus every $f_{\mathbb{X}}$ is constant, which means $G = G_1$ and thus $X = X_1$. $\square$

Proof of Proposition 2.2.3. Consider $\overline{X} := V(g_1, \ldots, g_{r-1}) \subseteq \mathbb{K}^{m+n}$. We first show that for every $z \in \overline{X}$ the Jacobian of $g_1, \ldots, g_{r-1}$ is of full rank. The Jacobian is of the form $(J_g, 0)$ with

$$J_g \;:=\; \begin{bmatrix} \delta_{1,1} & \delta_{1,2} & 0 & & \cdots & & & 0 \\ 0 & \delta_{2,2} & \delta_{1,3} & 0 & & & & \\ \vdots & & & & \vdots & & & \vdots \\ & & & & 0 & \delta_{2,r-2} & \delta_{1,r-1} & 0 \\ 0 & & \cdots & & & 0 & \delta_{2,r-1} & \delta_{1,r} \end{bmatrix}$$

where each $\delta_{t,i}$ is a nonzero multiple of $\delta_i := \operatorname{grad} T_i^{l_i}$. Let $z \in \mathbb{K}^{m+n}$ be any point with $J_g(z)$ not of full rank. Then $\delta_i(z) = \delta_j(z) = 0$ for some $i \neq j$. This implies $z_{ik} = 0 = z_{jl}$

for some $0 \le k \le n_i$, $0 \le l \le n_j$. It follows $T_i^{l_i}(z) = T_j^{l_j}(z) = 0$ and thus $z \notin \overline{X}$. So the Jacobian is of full rank for any $z \in \overline{X}$.

We conclude that $g_1, \ldots, g_{r-1}$ generate the vanishing ideal of $\overline{X}$ and that $\overline{X} = \operatorname{Spec} R(A, P_0)$ is smooth. Lemma 2.2.4 yields that $\overline{X}$ is connected and, by smoothness, irreducible. Thus $R(A, P_0)$ is integral. Moreover the dimension of $\overline{X}$ and hence $R(A, P_0)$ is $n + m - (r-1)$ and thus $R(A, P_0)$ is a complete intersection.

Effectivity of the K_0-grading means that the degrees of the generators T_{ij} and S_k generate K_0 as a group and is given by construction. As well by construction, the monomials $T_1^{l_1}, \ldots, T_r^{l_r}$ generate the degree zero part of the K_0-grading of $\mathbb{K}[T_{ij}, S_k]$. Lemma 2.2.2 yields the relations $T_1^{l_1} = T_i^{l_i} - a_1 + a_i$, where $2 \le i \le r$, in $R(A, P_0)$ and we arrive at $R(A, P_0)_0 = \mathbb{K}[T_1^{l_1}] \cong \mathbb{K}[T]$. □

Proposition 2.2.5. *Let $R(A, P_0)$ be of Type 1. Then the variables T_{ij}, S_k define pairwise nonassociated K_0-prime elements in $R(A, P_0)$. If furthermore the ring $R(A, P_0)$ is factorial, $P_0 \neq [2E_2, 0]$ and $n_i l_{ij} > 1$ holds, then T_{ij} is even prime.*

Proof. First observe that, by the nature of relations, any two different variables define a zero set of codimension at least two in $\overline{X}$. Thus the variables are pairwise non-associated. Since $R(A, P_0)$ is integral and $R(A, P_0) \cong R(A, P_0')[S_1, \ldots, S_m]$ holds with P_0' obtained from P_0 by deleting the zero columns, the S_k are even prime.

We now turn to the T_{ij} and exemplarily treat T_{11}. The task is to show that the divisor of T_{11} in $\overline{X}$ is H_0-prime that means that its prime components have multiplicity one and are transitively permuted by H_0. First we claim

$$V(\overline{X}; T_{11}) \ := \ V(T_{11}) \cap \overline{X} \ = \ \overline{H_0 \cdot z} \ \subseteq \ \mathbb{K}^{n+m}.$$

Indeed, the zero set of T_{11} in $\overline{X}$ is given by the equations

$$T_{11} \ = \ 0, \qquad T_s^{l_s} \ = \ a_s - a_1, \qquad 2 \le s \le r.$$

Set $h := T_{12} \cdots T_{rn_r} \cdot S_1 \cdots S_m$ and let $z \in \mathbb{K}_h^{n+m}$ be a point satisfying the above equations. Then z is of the form $(0, z_{12}, \ldots, z_{rn_r}, z_1, \ldots, z_m)$ with nonzero z_{ij} and z_k and any other such point $z' \in \mathbb{K}_h^{n+m}$ is given as

$$z' = t \cdot z = (0, t_{12}z_{12}, \ldots, t_{rn_r}z_{rn_r}, t_1 z_1, \ldots, t_m z_m), \quad t \in (\mathbb{K}^*)^{n+m}, \quad t_s^{l_s} = 1.$$

This means $t \in H_0$ and $V(\overline{X}_h; T_{11}) = H_0 \cdot z$. Since the common zero set of any two different variables is of codimension at least two in $\overline{X}$, our claim follows. In particular, H_0 permutes transitively the components of the divisor defined by T_{11} on $\overline{X}$. To obtain only multiplicities one, observe that the Jacobian of the above equations is of full rank at any point of $H_0 \cdot z \subseteq V(\overline{X}; T_{11})$.

The supplement is shown in 4.4.11. □

Proposition 2.2.6. *Let $R(A, P_0)$ be of Type 1. Then $R(A, P_0)$ is K_0-factorial.*

Proof. First observe that the quasitorus $H_0 \cong \operatorname{Spec} \mathbb{K}[K_0]$ equals the kernel of the homomorphism of tori

$$\varphi\colon \mathbb{T}^{n+m} \ \to\ \mathbb{T}^r, \qquad (t_{ij}, t_k) \ \mapsto\ (t_1^{l_1}, \ldots, t_r^{l_r}).$$

Denote the coordinates of $\mathbb{T}^r$ by $U_1, \ldots, U_r$. Then the relations g_i are pullbacks of the affine linear forms

$$g_i \ = \ \varphi^*(h_i), \qquad h_i \ := \ U_i - U_{i+1} - (a_{i+1} - a_i) \ \in \ \mathbb{K}[U_1^{\pm}, \ldots, U_r^{\pm 1}].$$

The h_i generate the vanishing ideal of an r times punctured affine line in $\mathbb{T}^r$ and thus

$$(R(A, P_0)_t)_0 \ = \ \mathbb{K}[U_1^{\pm}, \ldots, U_r^{\pm 1}] / \langle h_1, \ldots, h_{r-1} \rangle$$

is a factorial ring, where t is the product over all the variables T_{ij} and S_k. Now Proposition 2.2.5 and [6, Cor. 3.4.1.6] tell us that $R(A, P_0)$ is K_0-factorial. □

Proposition 2.2.7. *Let $R(A, P_0)$ be of Type 1. Then the variable T_{ij} is prime in $R(A, P_0)$ if and only if $1 = \gcd(l_{k1}, \ldots, l_{kn_k})$ holds for all $k \neq i$.*

Proof. We treat exemplarily T_{11}. By Lemma 2.2.2, the ideal of relations of $R(A, P_0)$ is generated by $g_{12}, \ldots, g_{1r}$. Thus T_{11} generates a prime ideal if and only if the following ideal is prime

$$\langle T_j^{l_j} + a_j - a_1;\ j \neq 1 \rangle \ \subseteq\ \mathbb{K}[T_{ij};\ (i,j) \neq (1,1)].$$

This is equivalent to the statement that $(l_2, 0, \ldots, 0), \ldots, (0, \ldots, 0, l_r)$ generate a primitive sublattice of $\mathbb{Z}^{n-n_1}$. This in turn holds if and only if $l_{k1}, \ldots, l_{kn_k}$ have greatest common divisor one for all $k \neq 1$. □

Proposition 2.2.8. *Let $R(A, P_0)$ be of Type 1 with $P_0 \neq [2E_2, 0]$ and suppose that $r \geq 2$ and $n_i l_{ij} > 1$ hold for all i, j. Then the following statements are equivalent.*

(i) *The ring $R(A, P_0)$ is factorial.*
(ii) *The group K_0 is torsion free.*
(iii) *We have $\gcd(l_{i1}, \ldots, l_{in_i}) = 1$ for $i = 1, \ldots, r$.*
(iv) *The variables T_{ij} are prime for all i, j.*

Proof. Let K_0 be torsion free. Then K_0-factoriality implies factoriality of $R(A, P_0)$, see [6, Thm. 3.4.1.11]. If $R(A, P_0)$ is factorial, then Proposition 2.2.5 says that the generators T_{ij} are prime. This implies $\gcd(l_{k1}, \ldots, l_{kn_k}) = 1$ for all k, see Proposition 2.2.7. If the latter holds, then the rows of P_0 generate a primitive sublattice of $\mathbb{Z}^{n+m}$ and thus K_0 is torsion free. □

Proof of Theorem 2.1.5. We first show that every K-homogeneous unit $f \in R(A, P)_w$ is constant. For this, it suffices to show that f is K_0-homogeneous, see Proposition 2.2.1. From [6, Rem. 3.4.3.2] we infer that the downgrading map $K_0 \to K$ has kernel $\mathbb{Z}^s$.

Consider the inverse $g \in R(A,P)_{-w}$ of $f \in R(A,P)_w$ and the decompositions into K_0-homogeneous parts

$$f \;=\; \sum f_i, \quad f_i \in R(A,P_0)_{u_i}, \qquad\qquad g \;=\; \sum g_j, \quad g_j \in R(A,P_0)_{v_j},$$

where $u_i = w_0 + u_i'$ with $u_i' \in \mathbb{Z}^s$ and $v_j = -w_0 + v_j'$ with $v_j' \in \mathbb{Z}^s$ for some fixed $w_0 \in K_0$ projecting to $w \in K$; we identify the kernel of $K_0 \to K$ with $\mathbb{Z}^s$. Let $f_{i_0}, f_{i_1}, g_{j_0}, g_{j_1}$ denote the terms, where u_{i_0}', v_{j_0}' are minimal and u_{i_1}', v_{j_1}' are maximal with respect to the lexicographical ordering on $\mathbb{Z}^s$. As 1 is of K_0-degree zero, we obtain

$$0 \;=\; \deg(f_{i_0} g_{j_0}) \;=\; w_0 + u_{i_0}' - w_0 + v_{j_0}' \;=\; u_{i_0}' + v_{j_0}'.$$

and analogously $u_{i_1}' + v_{j_1}' = 0$. We conclude $u_{i_0}' = u_{i_1}'$ and $v_{j_0}' = v_{j_1}'$. Consequently, f is homogeneous with respect to the K_0-grading.

Using [6, Lemma 2.1.4.1], we see that the K-grading of $R(A,P)$ is almost free. By [6, Lemma 3.4.3.5], the variables T_{ij} and S_k define pairwise nonassociated K-primes in $R(A,P)$. Finally, [6, Thms. 3.4.1.5, 3.4.1.11 and Cor. 3.4.1.6] show that the K-grading of $R(A,P)$ is factorial. $\square$

Now we turn to the converse statements. For this, we adapt the ideas of [49] to our more general setting.

Proof of Theorem 2.1.3. Consider $X = \operatorname{Spec} R$ with the action of $H := \operatorname{Spec} \mathbb{K}[K]$ defined by the grading. We follow the lines of [6, Sec. 4.4.2]. Denote by $E_1, \ldots, E_m$ the prime divisors on X such that for any $x \in E_k$ the isotropy group H_x is infinite and consider the H-invariant open subset

$$X_0 \;:=\; \{x \in X;\; H_x \text{ is finite}\} \;\subseteq\; X.$$

Then there is a geometric quotient $X_0 \to X_0/H$ with a possibly non-separated smooth curve X_0/H. Consider the separation $X_0/H \to Y$ and let $a_\iota, \ldots, a_r \in Y$ be points such that every fiber of $X_0/H \to Y$ comprising more than one point lies over some a_i and every prime divisor of X with non-trivial general H-isotropy lies over some a_i; we denote these prime divisors by D_{ij}, where the i indicates that D_{ij} lies over a_i.

According to [6, Thm. 4.4.2.1], this quotient is the characteristic space over the possibly non-separated curve X_0/H and we have a canonical well-defined pullback isomorphism of K-graded algebras

$$\mathcal{R}(X_0/H)[T_{ij}, S_k]/\langle T_{ij}^{l_{ij}} - 1_{z_{ij}}\rangle \;\to\; \Gamma(X, \mathcal{O}),$$

where S_k and T_{ij} are sent to functions with divisor E_k and D_{ij} respectively and $1_{z_{ij}}$ is the pullback of the canonical section of a point $z_{ij} \in X_0/H$ lying over $y_i \in Y$. As X_0/H is smooth, has only constant invertible global functions and finitely generated divisor class group, we end up with Y being either the affine or the projective line. The Cox ring of X_0/H is given as

$$\mathcal{R}(X_0/H) \;=\; \mathcal{R}(Y)[U_{ij}]/\langle U_{i1} \cdots U_{in_i} - 1_{a_i}\rangle,$$

where the U_{ij} represent the canonical sections of the points $z_{i1}, \ldots, z_{in_i} \in X_0/H$ lying over $a_i \in Y$ and 1_{a_i} is the pullback of the canonical section of a_i with respect to $X_0/H \to Y$, see [6, Prop. 4.4.3.4]. Now, if $Y = \mathbb{K}$ holds, we set $\imath := 1$ and represent $\mathcal{R}(Y)$ as

$$\mathcal{R}(Y) \ = \ \mathbb{K}[V_1, \ldots, V_r]/\langle V_i - V_{i+1} - (a_{i+1} - a_i)\rangle.$$

Plugging this into the above descriptions of $\mathcal{R}(X_0/H)$ and $\Gamma(X, \mathcal{O})$ gives us Type 1 of Construction 2.1.1. If $Y = \mathbb{P}_1$ holds, then we set $\imath := 0$, replace the $a_i \in Y$ with representatives $a_i \in \mathbb{K}^2 \setminus \{0\}$ and obtain

$$\mathcal{R}(Y) \ = \ \mathbb{K}[V_0, \ldots, V_r]/\langle g_0, \ldots, g_{r-2}\rangle, \qquad h_i \ := \ \det \begin{bmatrix} V_i & V_{i+1} & V_{i+2} \\ a_i & a_{i+1} & a_{i+2} \end{bmatrix}.$$

Combining this description with the above presentations of $\mathcal{R}(X_0/H)$ and $\Gamma(X, \mathcal{O})$ leads to Type 2 of Construction 2.1.1.

So far, we verified that the algebra $R = \Gamma(X, \mathcal{O})$ has the desired generators and relations. The generators are homogeneous with respect to $K = \mathbb{X}(H)$. As the K_0-grading of $R(A, P_0)$ is the finest possible with this property, we obtain a downgrading map $K_0 \to K$. Using the arguments of the proof of [6, Thm. 4.4.2.2], we see that $K_0 \to K$ is an isomorphism. □

Proof of Theorem 2.1.7. From [6, Thm. 3.2.1.4] we infer all listed properties except rationality. For the latter, let $U \subseteq X(A, P)$ be the open subset obtained by removing the prime divisors corresponding to the T_{ij} and S_k. Then U is affine, T acts freely on U with a geometric quotient $p\colon U \to C$ onto a smooth affine curve C. Suitably shrinking U, we find invertible T-homogeneous functions $f_1, \ldots, f_s$ on U, the weights of which form a $\mathbb{Z}$-basis of the character group of T. Then $(p, f_1, \ldots, f_s)$ defines an isomorphism $U \cong C \times (\mathbb{K}^*)^s$. Since $\mathrm{Cl}(X)$ is finitely generated, also $\mathrm{Cl}(C)$ is so. Consequently, C is rational and thus X is rational. □

Proof of Theorem 2.1.8. One follows exactly the proof of [6, Thm. 4.4.1.6], but uses our more general Theorem 2.1.5 instead of [6, Thm. 4.4.2.2]. □

Proof of Corollary 2.1.9. Theorem 2.1.8 tells us $X \cong X(A, P, \Phi)$ as in Construction 2.1.6. Since X is affine, the open subset $\widehat{X}(A, P, \Phi)$ equals the total coordinate space $\overline{X}(A, P)$. The latter means that Φ contains the trivial cone $\{0\}$. This is equivalent to saying that the columns of P generate the extremal rays of a pointed cone in $\mathbb{Q}^{r+s}$. □

2.3 Geometry of complexity one T-varieties

In this section we adapt the construction of a canonical toric ambient variety from [6, Sec. 3.2.5]. As a first application we show that varieties $X(A, P, \Phi)$ with Cox ring $R(A, P)$ of Type 1 such that $l_{i1} + \ldots + l_{in_i} > 1$ holds for all $i = \iota, \ldots, r$ are non-toric. Moreover we extend the resolution of singularities [6, Thm. 3.4.4.9] to our setting.

Remark 2.3.1. Consider the defining matrix P of a K-graded ring $R(A, P)$ as in Construction 2.1.4. Write $v_{ij} = P(e_{ij})$ and $v_k = P(e_k)$ for the columns of P. The i-th column block of P is $(v_{i1}, \ldots, v_{in_i})$ and by the data of this block we mean l_i and the $s \times n_i$ block d_i of d. We introduce *admissible operations* on P:

(i) swap two columns inside a block $v_{i1}, \ldots, v_{in_i}$,
(ii) exchange the data l_{i_1}, d_{i_1} and l_{i_2}, d_{i_2} of two column blocks,
(iii) add multiples of the upper r rows to one of the last s rows,
(iv) any elementary row operation among the last s rows,
(v) swapping among the last m columns.

The operations of type (iii) and (iv) do not change the associated ring $R(A, P)$, whereas the types (i), (ii), (v) correspond to certain renumberings of the variables of $R(A, P)$ keeping the (graded) isomorphy type.

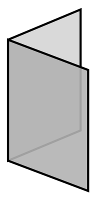
Remark 2.3.2. If $R(A, P)$ is not a polynomial ring, then we can always assume that P is *irredundant* in the sense that $l_{i1} + \ldots + l_{in_i} > 1$ holds for $i = \iota, \ldots, r$. Indeed, if P is redundant, then we have $n_i = 1$ and $l_{i1} = 1$ for some i. After an admissible operation of type (ii), we may assume $i = r$. Now, erasing v_{r1} and the r-th row of P and the last column from A produces new data defining a ring $R(A, P)$ isomorphic to the previous one. Iterating this procedure leads to an $R(A, P)$ isomorphic to the initial one but with irredundant P.

Toric embeddability is important in our subsequent considerations. More specifically, there is even a canonical embedding $X \to Z$ into a toric variety such that X inherits many geometric properties from Z. The construction makes use of the tropical variety of X.

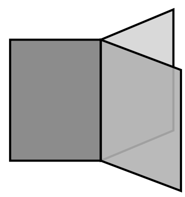
Construction 2.3.3. Let $X = X(A, P, \Phi)$ be obtained from Construction 2.1.6. The *tropical variety* of X is the fan $\mathrm{trop}(X)$ in $\mathbb{Q}^{r+s}$ consisting of the cones

$$\lambda_i \ := \ \mathrm{cone}(v_{i1}) + \mathrm{lin}(e_{r+1}, \ldots, e_{r+s}) \text{ for } i \ = \ \iota, \ldots, r, \qquad \lambda \ := \ \lambda_\iota \cap \ldots \cap \lambda_r,$$

where $v_{ij} \in \mathbb{Z}^{r+s}$ denote the first n columns of P and $e_k \in \mathbb{Z}^{r+s}$ the k-th canonical basis vector; we call λ_i a *leaf* and λ the *lineality part* of $\mathrm{trop}(X)$.



Type 1



Type 2

Construction 2.3.4. Let $X = X(A, P, \Phi)$ be obtained from Construction 2.1.6. For a face $\delta_0 \preceq \delta$ of the orthant $\delta \subseteq \mathbb{Q}^{n+m}$, let $\delta_0^* \preceq \delta$ denote the complementary face and call δ_0 *relevant* if

- the relative interior of $P(\delta_0)$ intersects $\mathrm{trop}(X)$,
- the image $Q(\delta_0^*)$ comprises a cone of Φ,

where $Q\colon \mathbb{Z}^{n+m} \to K = \mathbb{Z}^{n+m}/P^*(\mathbb{Z}^{r+s})$ is the projection. Then we obtain fans $\widehat{\Sigma}$ in $\mathbb{Z}^{n+m}$ and Σ in $\mathbb{Z}^{r+s}$ of pointed cones by setting

$$\widehat{\Sigma} \ := \ \{\delta_1 \preceq \delta_0;\ \delta_0 \preceq \delta \text{ relevant}\}, \qquad \Sigma \ := \ \{\sigma \preceq P(\delta_0);\ \delta_0 \preceq \delta \text{ relevant}\}.$$

The toric varieties $\widehat{Z}$ and Z associated with $\widehat{\Sigma}$ and Σ, respectively, and $\overline{Z} = \mathbb{K}^{n+m}$ fit into a commutative diagramm of characteristic spaces and total coordinate spaces

$$\begin{array}{ccc}
\overline{X}(A,P) & \subseteq & \overline{Z} \\
\cup\mid & & \cup\mid \\
\widehat{X}(A,P,\Phi) & \subseteq & \widehat{Z} \\
{\scriptstyle /\!\!/ H}\downarrow & & \downarrow{\scriptstyle /\!\!/ H} \\
X(A,P,\Phi) & \subseteq & Z
\end{array}$$

The horizontal inclusions are T-equivariant closed embeddings, where T acts on Z as the subtorus of the $(r+s)$-torus corresponding to $0 \times \mathbb{Z}^s \subseteq \mathbb{Z}^{r+s}$. Moreover, $X(A,P,\Phi)$ intersects every closed toric orbit of Z.

We call Z from Construction 2.3.4 the *minimal toric ambient variety* of $X = X(A,P,\Phi)$. Observe that the rays of the fan Σ of Z have precisely the columns of the matrix P as its primitive generators. In particular, every ray of Σ lies on the tropical variety $\mathrm{trop}(X)$.

Theorem 2.3.5. *Let $X := X(A,P,\Phi)$ be as in Construction 2.1.6 with $R(A,P)$ irredundant of Type 1. Then X is not a toric variety.*

Proof. Assume X is a toric variety. Then its Cox ring $\mathcal{R}(X) = R(A,P)$ is a polynomial ring and $\overline{X} := \mathrm{Spec}\, R(A,P) \cong \mathbb{K}^t$ holds. By construction $\overline{X}$ is endowed with an $H_0 := \mathrm{Spec}\, \mathbb{K}[K_0]$-action of complexity one. As $\overline{X}$ is factorial H_0 is indeed a torus and the H_0-action on $\overline{X}$ induces an action of a torus $\mathbb{T}^{t-1}$ on $\mathbb{K}^t$. By [15, 16] this torus action arises as a subtorus action of the maximal torus. Consider the H-invariant isomorphic open subsets

$$(\mathbb{K}^t)_0 := \left\{x \in \mathbb{K}^t;\ \mathbb{T}^{t-1}_x \text{ is finite}\right\} \subseteq \mathbb{K}^t, \qquad \overline{X}_0 := \left\{x \in \overline{X};\ (H_0)_x \text{ is finite}\right\} \subseteq \overline{X}.$$

Then there are geometric quotients

$$(\mathbb{K}^t)_0 \to (\mathbb{K}^t)_0 /\!\!/ \mathbb{T}^{t-1} \text{ and } \overline{X}_0 \to \overline{X}_0 /\!\!/ H_0$$

with possibly non-separated smooth curves $(\mathbb{K}^t)_0 /\!\!/ \mathbb{T}^{t-1}$ and $\overline{X}_0 /\!\!/ H_0$. Consider the separation $\overline{X}_0 /\!\!/ H_0 \to Y$ and call a points $a \in Y$, where a fiber of $\overline{X}_0 /\!\!/ H_0 \to Y$ comprising more than one point lies over a or a prime divisor of $\overline{X}_0$ with non-trivial general H_0-isotropy lies over a, a doubling point of Y. Analogously, consider the separation $(\mathbb{K}^t)_0 /\!\!/ \mathbb{T}^{t-1} \to Y'$ and call the points $a' \in Y'$, where a fiber of $(\mathbb{K}^t)_0 /\!\!/ \mathbb{T}^{t-1} \to Y'$ comprising more than one point lies over a' or a prime divisor of $(\mathbb{K}^t)_0$ with non-trivial general $\mathbb{T}^{t-1}$-isotropy lies over a', a doubling points of Y'. Note that as $\overline{X} \cong \mathbb{K}^t$ holds as T-varieties, the number of doubling points of Y' and Y coincide. Thus we compare the number of doubling points of Y' with the ones of Y.

As $\mathbb{T}^{t-1}$ acts as a subtorus of the maximal torus the quotient $(\mathbb{K}^t)_0 /\!\!/ \mathbb{T}^{t-1}$ has at most one doubling point at zero.

For counting the doubling points of Y we consider the embedding $\overline{X} \subseteq \mathbb{K}^{n+m}$. We obtain the following commutative diagramm:

$$\begin{array}{ccc}
\overline{X}_0 & \subseteq & \mathbb{K}_0^{n+m} \\
\downarrow & & \downarrow \\
\overline{X}_0/H_0 & \subseteq & \mathbb{K}_0^{n+m}/H_0 \\
\downarrow & & \downarrow \\
Y & \subseteq & \mathbb{K}^r,
\end{array}$$

where $\mathbb{K}_0^{n+m} \subseteq \mathbb{K}^{n+m}$ is the subset of all points with finite H_0-isotropy and $\overline{X}_0 = \mathbb{K}_0^{n+m} \cap \overline{X}$ holds. First we determine the orders of isotropy groups. Every point in $\mathbb{T}^{n+m}$ has trivial H_0-isotropy. Thus, we only have to look what happens on the sets $V(T_{ij}) \cap \mathbb{K}_0^{n+m}$. Applying [6, Prop. 2.1.4.2] we obtain that the order of isotropy group of H_0 at any point $x \in V(T_{ij}) \cap \mathbb{K}_0^{n+m}$ equals l_{ij}. Moreover the H_0-invariant divisors $V(\overline{X}, T_{ij})$ are prime and two divisors $V(\overline{X}, T_{ij})$ and $V(\overline{X}, T_{ij'})$ are identified isomorphically under the separation map $\overline{X}_0/H_0 \to Y$ In particular any term of the defining relations of $\overline{X}$ with $l_{i1} + \ldots + l_{in_i} > 1$ gives rise to exactly one doubling point of Y. As $R(A, P)$ has at least one relation and P is irredundant we obtain more than two doubling points on Y. This contradicts $\overline{X} \cong \mathbb{K}^t$. □

We turn towards resolution of singularities. The minimal toric ambient variety is crucial for the resolution of singularities. The following recipe for resolving singularities directly generalizes [6, Thm. 3.4.4.9]; a related approach using polyhedral divisors is presented in [62].

Construction 2.3.6. Let $X = X(A, P, \Phi)$ be obtained from Construction 2.1.6 and consider the canonical toric embedding $X \subseteq Z$ and the defining fan Σ of Z.

- Let $\Sigma' = \Sigma \sqcap \operatorname{trop}(X)$ be the coarsest common refinement.
- Let Σ'' be any regular subdivision of the fan Σ'.

Then $\Sigma'' \to \Sigma$ defines a proper toric morphism $Z'' \to Z$ and with the proper transform $X'' \subseteq Z''$ of $X \subseteq Z$, the morphism $X'' \to X$ is a resolution of singularities.

Remark 2.3.7. In the setting of Construction 2.3.6, the variety X'' has again a torus action of complexity one and thus is of the form $X'' = X(A'', P'', \Phi'')$. We have $A'' = A$

and P'' is obtained from P by inserting the primitive generators of Σ'' as new columns. Moreover, Φ'' is the Gale dual of Σ'', that means that with the corresponding projection Q'' and orthant δ'' we have

$$\Phi'' \;=\; \{Q''(\delta_0^*);\ \delta_0 \preceq \delta'';\ P''(\delta_0) \in \Sigma''\}.$$

Proposition 2.3.8. *Consider a variety $X = X(A, P, \Phi)$ of Type 2 as provided by Construction 2.1.6. Then the following statements are equivalent.*

(i) *One has $\widehat{X} = \overline{X}$*
(ii) *The variety X is affine.*
(iii) *The minimal toric ambient variety Z of X is affine.*
(iv) *One has $\widehat{Z} = \overline{Z} = \mathbb{K}^{n+m}$.*

If one of these statements holds, then the columns of P generate the extremal rays of a full-dimensional cone $\sigma \subseteq \mathbb{Q}^{r+s}$ and we have $Z = \operatorname{Spec} \mathbb{K}[\sigma^\vee \cap \mathbb{Z}^{r+s}]$.

Proof. Only for the implication "(ii)⇒(iii)" there is something to show. As X is of Type 2, we have $0 \in \overline{X} \subseteq \overline{Z} = \mathbb{K}^{n+m}$. Since X is affine, we have $\overline{X} = \widehat{X}$ and thus $0 \in \widehat{Z}$. We conclude $\widehat{Z} = \overline{Z}$ and thus $Z = \overline{Z} /\!\!/ H$ is affine. □

The characterization 2.3.8 (i) allows us to omit the bunch of cones Φ in the affine case: we may just speak of the affine variety $X = X(A, P) := \overline{X} /\!\!/ H$.

Corollary 2.3.9. *Let $X = X(A, P)$ be affine of Type 2. Then the following statements are equivalent.*

(i) *The variety X is $\mathbb{Q}$-factorial.*
(ii) *The variety Z is $\mathbb{Q}$-factorial.*
(iii) *The columns of P are linearly independent.*

Proof. The equivalence of (i) and (ii) is [6, Cor. 3.3.1.7], The equivalence of (ii) and (iii) is [24, Thm. 3.1.19 (b)]. □

Corollary 2.3.10. *Let $X = X(A, P)$ be affine of Type 2. Then the Picard group of X is trivial.*

Proof. Proposition 2.3.8 says that the minimal toric ambient variety Z is affine. Thus, Z has trivial Picard group; see [24, Prop. 4.2.2]. According to [6, Cor. 3.3.1.12], the Picard group of X equals that of Z. □

More generally one can show that in fact every normal affine variety admitting a torus action with an attractive orbit has trivial Picard group: every bundle can be linearized and the non-vanishing loci of its homogeneous sections form an invariant trivializing open cover. As one of these covering sets contains the attractive orbit, the bundle is trivial.

2.4 Application: affine $\mathbb{K}^*$-surfaces

To illustrate our methods, we consider the well-known case of normal affine $\mathbb{K}^*$-surfaces [36, 38] and take a closer look at those with at most du Val singularities. By Corollary 2.1.9, any *affine* rational normal variety X with only constant invertible functions and a torus action of complexity one is of the form

$$X \;=\; X(A,P) \;:=\; \operatorname{Spec} R(A,P)_0.$$

Moreover, using the fact that the columns of P generate the extremal rays of a pointed cone in $\mathbb{Q}^{r+s}$ we directly obtain the following.

Remark 2.4.1. Consider a rational normal affine $\mathbb{K}^*$-surface $X = X(A,P)$. Then we have $s = 1$ and there are three possible cases for the defining matrix P:

(i) The *elliptic* case: we are in Type 2 and we have $n_0 = \ldots = n_r = 1$ and $m = 0$.
(ii) The *parabolic* case: we are in Type 1 and we have $n_1 = \ldots = n_r = 1$ and $m = 1$.
(iii) The *hyperbolic* case: we are in Type 1 and we have $n_1, \ldots, n_r \leq 2$ and $m = 0$.

Example 2.4.2. We consider the unique normal affine $\mathbb{K}^*$-surface X of parabolic type with Cox ring

$$R(A,P) := \mathbb{K}[T_{11}, T_{21}, S_1]/\langle T_{11}^2 + T_{21}^2 + 1\rangle,$$

where

$$P \;:=\; \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

In this case the total coordinate space $\overline{X}$ is isomorphic to $\mathbb{K}^* \times \mathbb{K}$. The divisor class group of X is $\operatorname{Cl}(X) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and the $\operatorname{Cl}(X)$-grading of the Cox ring $\mathcal{R}(X) = R(A,P)$ is given by

$$\deg(T_{11}) = (\bar{0}, \bar{1}), \quad \deg(T_{21}) = (\bar{0}, \bar{1}), \quad \deg(S_1) = (\bar{1}, \bar{1}).$$

We have

$$X \cong \operatorname{Spec} R(A,P)^{\operatorname{Cl}(X)} \cong \mathrm{V}(T_1^2 T_2 + T_1 T_2 + T_3^2) \subseteq \mathbb{K}^3$$

and X is endowed with a $\mathbb{K}^*$-action given by $t \cdot x := (x_1, t^2 x_2, t x_3)$ for $x \in X$ and $t \in \mathbb{K}^*$. We obtain a fixed point curve consisting of the points $(x_1, 0, 0)$ mapping isomorphically onto the quotient $X /\!\!/ \mathbb{K}^* \cong \mathbb{K}$ and thus X is of parabolic type as claimed.

We sketch how to regain the Cox ring out of the geometric data of X: The fixed point curve $(x_1, 0, 0)$ corresponds to the free variable S_1 in the Cox ring of X. Moreover, the orbits $\mathbb{K}^* \cdot (0,1,0)$ and $\mathbb{K}^* \cdot (-1,1,0)$ are of generic isotropy 2 and give rise to the relation $T_1^2 + T_2^2 + 1$ in the Cox ring of X.

We take a closer look at the surfaces $X = X(A,P)$ with at most du Val singularities. Recall that these are exactly the singularities with a resolution graph of type $\boldsymbol{A}$, $\boldsymbol{D}$ or $\boldsymbol{E}$. The singularities of type $\boldsymbol{A}$ are precisely the toric du Val surface singularities; we refer to [24] for an exhaustive treatment of this case.

Proposition 2.4.3. *Let X be a parabolic or hyperbolic normal affine $\mathbb{K}^*$-surface. If $x_0 \in X$ is a du Val singularity, then it is of type $\boldsymbol{A}$.*

Proof. In the parabolic and hyperbolic cases, the fan Σ of the canonical ambient toric variety is supported on the tropical variety $\mathrm{trop}(X)$ and we have $\Sigma' = \Sigma$ in the first step of the resolution of singularities according to Construction 2.3.6. The second step means regular subdivision of the purely two-dimensional fan $\Sigma' = \Sigma$ and, in the du Val case, we end up with resolution graphs of type $\boldsymbol{A}$; use [6, Sec. 5.4.2] for computing intersection numbers. $\square$

We turn to the elliptic case. In case of a singularity of type $\boldsymbol{D}$ or $\boldsymbol{E}$, we determine the possible X and present the defining data and the Cox ring for X as well as for the minimal resolution $\tilde{X}$; see [33, 62, 29] for other approaches.

Proposition 2.4.4. *Let X be an elliptic normal affine $\mathbb{K}^*$-surface with a du Val singularity $x_0 \in X$. If x_0 is of type $\boldsymbol{A}$, then X is an affine toric surface. If x_0 is of type $\boldsymbol{D}$ or $\boldsymbol{E}$, then $X \cong X(A,P)$, where*

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix},$$

the defining matrix P depends on the type of x_0 as shown in the table below; we additionally present a defining equation for $X \subseteq \mathbb{K}^3$ from [64] and the relation g of the Cox ring $\mathcal{R}(X) = \mathbb{K}[T_1, T_2, T_3]/\langle g \rangle$.

x_0	*equation in* $\mathbb{K}^3$	*matrix* P	*relation* g
$\boldsymbol{D}_q$	$T_1^2 + T_2T_3^2 + T_2^{q-1}$	$\begin{bmatrix} -2 & q-2 & 0 \\ -2 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}$	$T_1^2 + T_2^{q-2} + T_3^2$
$\boldsymbol{E}_6$	$T_1^2 + T_2^3 + T_3^4$	$\begin{bmatrix} -3 & 3 & 0 \\ -3 & 0 & 2 \\ -2 & 1 & 1 \end{bmatrix}$	$T_1^3 + T_2^3 + T_3^2$
$\boldsymbol{E}_7$	$T_1^2 + T_2^3 + T_2T_3^3$	$\begin{bmatrix} -4 & 3 & 0 \\ -4 & 0 & 2 \\ -3 & 1 & 1 \end{bmatrix}$	$T_1^4 + T_2^3 + T_3^2$
$\boldsymbol{E}_8$	$T_1^2 + T_2^3 + T_3^5$	$\begin{bmatrix} -5 & 3 & 0 \\ -5 & 0 & 2 \\ -4 & 1 & 1 \end{bmatrix}$	$T_1^5 + T_2^3 + T_3^2$

Moreover, Construction 2.3.6 provides a minimal resolution of singularities $\tilde{X} \to X$ with $\tilde{X} = X(A, \tilde{P}, \tilde{\Phi})$, where $\tilde{P}$ depends on the type of x_0 as shown below; we list the relation

$\tilde{g}$ of the Cox ring $\mathcal{R}(\tilde{X}) = \mathbb{K}[T_{ij}, S_1]/\langle \tilde{g} \rangle$.

x_0	*matrix* $\tilde{P}$	*relation* $\tilde{g}$
$\boldsymbol{D}_q$	$\begin{bmatrix} -2 & -1 & q-2 & q-3 & \dots & 1 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 & \dots & 0 & 2 & 1 & 0 \\ -1 & 0 & 1 & 1 & \dots & 1 & 1 & 1 & 1 \end{bmatrix}$	$T_{11}^2 T_{12} + T_{21}^{q-2} \cdots T_{2,q-2} + T_{31}^2 T_{32}$
$\boldsymbol{E}_6$	$\begin{bmatrix} -3 & -2 & -1 & 3 & 2 & 1 & 0 & 0 & 0 \\ -3 & -2 & -1 & 0 & 0 & 0 & 2 & 1 & 0 \\ -2 & -1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$T_{11}^3 T_{12}^2 T_{13} + T_{21}^3 T_{22}^2 T_{23} + T_{31}^2 T_{32}$
$\boldsymbol{E}_7$	$\begin{bmatrix} -4 & -3 & -2 & -1 & 3 & 2 & 1 & 0 & 0 & 0 \\ -4 & -3 & -2 & -1 & 0 & 0 & 0 & 2 & 1 & 0 \\ -3 & -2 & -1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$T_{11}^4 \cdots T_{14} + T_{21}^3 T_{22}^2 T_{23} + T_{31}^2 T_{32}$
$\boldsymbol{E}_8$	$\begin{bmatrix} -5 & -4 & -3 & -2 & -1 & 3 & 2 & 1 & 0 & 0 & 0 \\ -5 & -4 & -3 & -2 & -1 & 0 & 0 & 0 & 2 & 1 & 0 \\ -4 & -3 & -2 & -1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$T_{11}^5 \cdots T_{15} + T_{21}^3 T_{22}^2 T_{23} + T_{31}^2 T_{32}$

The fan $\tilde{\Sigma}$ of the canonical ambient toric variety $\tilde{Z}$ of $\tilde{X}$ is the unique fan with only two-dimensional maximal cones, all of them lying on $\operatorname{trop}(X)$*, and having as one-dimensional cones precisely the rays through the columns of $\tilde{P}$. The corresponding bunch of cones $\tilde{\Phi}$ is the Gale dual of $\tilde{\Sigma}$.*

Proof. We only have to consider the case that $X = X(A, P)$ is not a toric surface and thus can assume $r \geq 2$ and $l_i := l_{i1} > 1$ for all $i = 0, \dots, r$. We resolve the singularity $x_0 \in X$ according to Construction 2.3.6. The first step gives us a fan with $r+1$ maximal cones, each of dimension two:

$$\operatorname{cone}(v_0, e_{r+1}), \dots, \operatorname{cone}(v_r, e_{r+1}),$$

where $v_i \in \mathbb{Q}^{r+1}$ denotes the i-th column of P and we may assume that e_{r+1} is the $(r+1)$-th canonical basis vector. In the second step, we perform the minimal regular subdivision of these cones. This gives indeed a minimal resolution $\tilde{X} \to X$ of x_0 and the resulting picture reflects the resolution graph. We see that x_0 cannot be of type $\boldsymbol{A}$ und thus is of type $\boldsymbol{D}$ or $\boldsymbol{E}$. We end up with $r = 2$ and defining data

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \qquad P = \begin{bmatrix} -l_0 & l_1 & 0 \\ -l_0 & 0 & l_2 \\ d_0 & d_1 & d_2 \end{bmatrix}.$$

Moreover, because all exceptional curves are of self intersection -2, we must have $d_i \equiv 1 \mod l_i$ for $i = 1, 2, 3$. That means, that we inserted $l_i - 1$ new rays to obtain the minimal regular subdivision of the i-th cone.

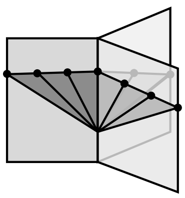
As a sample, we continue the case of an $\boldsymbol{E}_6$-singularity. By the shape of the corresponding resolution graph, we have $l_0 = l_1 = 3$ and $l_2 = 2$ after renumbering the columns suitably. This establishes the P_0-block. By suitable row operations we can achieve

$$P \;=\; P_c \;:=\; \begin{bmatrix} -3 & 3 & 0 \\ -3 & 0 & 2 \\ 1+3c & 1 & 1 \end{bmatrix}.$$

Every $c \in \mathbb{Z}$ yields a matrix P_c admissible for Construction 2.1.4. The minimal resolution $\tilde{X}_c$ of X_c is the $\mathbb{K}^*$-surface defined by A and the matrix

$$\tilde{P}_c \;=\; \begin{bmatrix} -3 & -2 & -1 & 3 & 2 & 1 & 0 & 0 & 0 \\ -3 & -2 & -1 & 0 & 0 & 0 & 2 & 1 & 0 \\ 1+3c & 1+2c & 1+c & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Only for $c = -1$, we obtain self intersection -2 for all exceptional curves; in fact, the one corresponding to $(0,0,1)$ is important here. The fan $\tilde{\Sigma}$ of the canonical ambient toric variety $\tilde{Z}$ of $\tilde{X} = \tilde{X}_{-1}$ sits on the tropical variety $\operatorname{trop}(X)$ and looks as follows:



Resolution of the $\boldsymbol{E}_6$-Singularity

$\square$

Remark 2.4.5. Note that the approach via the defining data A and P establishes *a posteriori* that every du Val surface singularity can be realized as the fixed point of an elliptic $\mathbb{K}^*$-surface. Similarly, the defining equation for $X \subseteq \mathbb{K}^3$ is easily seen to be the defining relation of the Veronese subalgebra $\Gamma(X, \mathcal{O}) = R(A,P)_0$ of the Cox ring $\mathcal{R}(X) = R(A,P)$.

CHAPTER

THREE

LOG TERMINAL VARIETIES AS QUOTIENTS

Looking at the well understood case of log terminal surface singularities, one observes that each of them is the quotient of a factorial one by a finite solvable group. The derived series of this group reflects an iteration of Cox rings of surface singularities. We extend this picture to log terminal singularities in any dimension coming with a torus action of complexity one. In this setting, the previously finite groups become solvable finite torus extensions. The results of this chapter have been published in the joint publication [5].

3.1 Platonic tuples and iteration of Cox rings

We begin with a brief discussion of the well known surface case [4, 20, 30]. The two-dimensional log terminal singularities are exactly the quotient singularities $\mathbb{C}^2/G$, where G is a finite subgroup of the general linear group GL(2). The particular case that G is a subgroup of SL(2) leads to the du Val singularities A_n, D_n, E_6, E_7 and E_8, named according to their resolution graphs. They are precisely the rational double points, and are also characterized by being the canonical surface singularities. The du Val singularities fill the middle row of the following commutative diagram involving *all* two-dimensional log terminal singularities:

$$
\begin{tikzcd}
 & \mathbb{C}^2 \arrow[dl] \arrow[d, "\mathrm{CR}"] \arrow[drr, bend left] \arrow[drrrr, bend left] \arrow[drrrrr, bend left] & & & & & \\
E_8 \arrow[d] & A_n \arrow[rr, "\mathrm{CR}"] \arrow[d] \arrow[dr, "n \text{ odd}"] & & D_{n+3} \arrow[rr, "\mathrm{CR}", "n=1"'] \arrow[d] \arrow[dr, "n=1"] & & E_6 \arrow[r, "\mathrm{CR}"] \arrow[d] & E_7 \arrow[d] \\
E_8^{\imath} & A_{n,k}^{\imath} & D_{(n+3)/2}^{2,\imath} & D_{n+3}^{\imath} & E_6^{3,\imath} & E_6^{\imath} & E_7^{\imath}
\end{tikzcd}
$$

Here, all arrows indicate quotients by finite groups. The label "CR" tells us that this quotient represents a Cox ring; recall that the Cox rings of (the resolutions of) the du Val singularities $\mathbb{C}^2/G$ have been computed in [29, 33], see also the example given below. So, E_6 is the spectrum of the Cox ring of E_7 etc.. In fact, the chain of Cox rings reflects the derived series of the binary octahedral group $\widetilde{S}_4 \subseteq \mathrm{SL}(2)$, producing the E_7 singularity:

$$\widetilde{S}_4 \supseteq \widetilde{A}_4 \supseteq \widetilde{D}_4 \supseteq \{\pm I_2\} \supseteq \{I_2\},$$

where $\widetilde{A}_4$ is the binary tetrahedral group, $\widetilde{D}_4$ the binary dihedral group, and I_2 stands for the 2×2 unit matrix. The respective CR labelled arrows stand for quotients by the factors of this derived series. The arrows passing from the middle to the lower row indicate index-one covers: the upper surface is Gorenstein, one divides by a cyclic group of order $\imath$ and the lower surface is of Gorenstein index $\imath$. Finally, the superscripts 2 in $D^{2,\imath}_{(n+3)/2}$ and 3 in $E_6^{3,\imath}$ denote the "canonical multiplicity" of the singularity, generalizing the "exponent" discussed in [28, 31]; see 3.3.2. For a discussion of the surface case based on the methods provided in this chapter, see Example 3.3.8.

Another feature of the log terminal surface singularities is that, as quotients $\mathbb{C}^2/G$ by a finite subgroup $G \subseteq \mathrm{GL}(2)$, they all come with a non-trivial $\mathbb{C}^*$-action, induced by scalar multiplication on $\mathbb{C}^2$. The higher dimensional analogue of $\mathbb{C}^*$-surfaces are T-varieties X of complexity one, that means varieties X with an effective action of an algebraic torus T which is of dimension one less than X. The notion of log terminality is defined in general via discrepancies in the ramification formula; see Section 3.2 for a brief reminder. In higher dimensions, log terminal singularities form a larger class than the quotient singularities $\mathbb{C}^n/G$ with G a finite subgroup of $\mathrm{GL}(n)$. Our aim is, however, to extend the picture drawn at the beginning for the surface case to log terminal singularities with a torus action of complexity one in any dimension.

If X comes with a torus action of complexity one, then the Cox ring $\mathcal{R}(X)$ admits an explicit description in terms of generators and very specific trinomial relations. Vice versa, one can abstractly write down all rings that arise as the Cox ring of some T-variety X of complexity one. Let us briefly summarize the procedure; see Section 2.1 and [44] for the details.

Construction 3.1.1. Fix integers $m \geq 0$, $\iota \in \{0,1\}$ and $r, n > 0$ and a partition $n = n_\iota + \cdots + n_r$. For every $i = \iota, \ldots, r$ let $l_i := (l_{i1}, \ldots, l_{in_i}) \in \mathbb{Z}^{n_i}_{>0}$ with $l_{i1} \geq \ldots \geq l_{in_i}$ and $l_{\iota 1} \geq \ldots \geq l_{r1}$ and define a monomial

$$T_i^{l_i} \; := \; T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}}.$$

Denote the polynomial ring $\mathbb{C}[T_{ij}, S_k;\, i = \iota, \ldots, r,\, j = 1, \ldots, n_i,\, k = 1, \ldots, m]$ for short by $\mathbb{C}[T_{ij}, S_k]$. We distinguish two types of rings:

Type 1. Take $\iota = 1$ and pairwise different scalars $\theta_1 = 1, \theta_2, \ldots, \theta_{r-1} \in \mathbb{C}^*$ and define for each $i = 1, \ldots, r-1$ a trinomial

$$g_i \; := \; T_i^{l_i} - T_{i+1}^{l_{i+1}} - \theta_i.$$

Then we obtain a factor ring

$$R \;=\; \mathbb{C}[T_{ij}, S_k]/\langle g_1, \ldots, g_{r-1}\rangle.$$

Type 2. Take $\iota = 0$ and pairwise different scalars $\theta_0 = 1, \theta_1, \ldots, \theta_{r-2} \in \mathbb{C}^*$ and define for each $i = 0, \ldots, r-2$ a trinomial

$$g_i \;:=\; \theta_i T_i^{l_i} + T_{i+1}^{l_{i+1}} + T_{i+2}^{l_{i+2}}.$$

Then we obtain a factor ring

$$R \;=\; \mathbb{C}[T_{ij}, S_k]/\langle g_0, \ldots, g_{r-2}\rangle.$$

As we explain later, the rings R come with a natural grading by a finitely generated abelian group K_0 and suitable downgradings $K_0 \to K$ give us Cox rings of rational, normal, varieties X with $\mathrm{Cl}(X) = K$ that come with a torus action of complexity one. More geometrically, X arises as a quotient of an open set $\widehat{X} \subseteq \overline{X}$ of the total coordinate space $\overline{X} = \operatorname{Spec} R$ by the quasitorus H having K as its character group. Conversely, basically every rational, normal variety X with a torus action of complexity one can be presented this way.

Geometrically speaking, Type 1 leads to the T-varieties of complexity one that admit non-constant global invariant functions and Type 2 to those having only constant global invariant functions. The varieties of Type 1 turn out to be locally isomorphic to toric varieties. In particular, they are all log terminal and the study of their singularities is essentially toric geometry, see Corollary 3.2.7 for a precise formulation. We therefore mainly concentrate on Type 2. There, the true non-toric phenomena occur, as for instance the singularities D_n, E_6, E_7 and E_8 in the surface case.

Characterizing log terminality for a T-variety of complexity one of Type 2 involves *platonic triples*, that means, triples of the form

$$(5,3,2), \quad (4,3,2), \quad (3,3,2), \quad (x,2,2), \quad (x,y,1),$$

where $x \geq y \in \mathbb{Z}_{\geq 0}$. We say that positive integers $a_0, \ldots, a_r$ form a *platonic tuple* if, after reordering decreasingly, the first three numbers are a platonic triple and all others equal one. Moreover, in the setting of Construction 3.1.1, we say that a ring R of Type 2 is *platonic* if every $(l_{0j_0}, \ldots, l_{rj_r})$ is a platonic tuple.

Example 3.1.2. The platonic rings of Type 2 in dimension two are the polynomial ring $\mathbb{C}[T_1, T_2]$ and the factor rings $\mathbb{C}[T_1, T_2, T_3]/\langle f\rangle$, where f is one of

$$T_1^y + T_2^2 + T_3^2,\ y \in \mathbb{Z}_{>1}, \quad T_1^3 + T_2^3 + T_3^2, \quad T_1^4 + T_2^3 + T_3^2, \quad T_1^5 + T_2^3 + T_3^2.$$

Endowed with a suitable grading, $\mathbb{C}[T_1, T_2]$ is the Cox ring of A_n, and the other rings, according to the above order of listing, are the Cox rings of D_{y-2}, E_6, E_7 and E_8.

Our first result says that a rational, normal variety X with a torus action of complexity one of Type 2 has at most log terminal singularities if and only if there occur enough platonic tuples $(l_{0j_0}, \ldots, l_{rn_r})$ in the Cox ring R; see Theorem 3.2.15 for the precise meaning of "enough". In the affine case, the result specializes to the following; compare also [38, Ex. 2.20] for an earlier result in a particular case and [62, Cor. 5.8] for a related characterization.

Theorem 3.1.3. *An affine, normal, $\mathbb{Q}$-Gorenstein, rational variety X with torus action of complexity one of Type 2 has at most log terminal singularities if and only if its Cox ring R is a platonic ring.*

Set for the moment $\mathfrak{l}_i := \gcd(l_{i1}, \ldots, l_{in_i})$. Then, by Proposition 2.2.8, a ring R of Type 1 is factorial if and only if $\mathfrak{l}_i = 1$ holds for all $i = 1, \ldots, r$. Moreover, a ring R of Type 2 is factorial if and only if the $\mathfrak{l}_i$ are pairwise coprime for $i = 0, \ldots, r$, see Theorem 2.1.2 or [44, Thm. 1.1].

Example 3.1.4. In dimension two, the factorial platonic rings R of Type 2 are the polynomial ring $\mathbb{C}[T_1, T_2]$ and the ring $\mathbb{C}[T_1, T_2, T_3]/\langle T_1^5 + T_2^3 + T_3^2 \rangle$.

To extend the iteration of Cox rings $\mathbb{C}^2 \to A_1 \to D_4 \to E_6 \to E_7$ observed in the surface case to higher dimensions, we have to allow instead of only finite abelian groups also non-finite abelian groups in the respective quotients.

Theorem 3.1.5. *Let X_1 be a rational, normal, affine variety with a torus action of complexity one of Type 2 and at most log terminal singularities. Then there is a unique chain of quotients*

$$X_p \xrightarrow{/\!\!/ H_{p-1}} X_{p-1} \xrightarrow{/\!\!/ H_{p-2}} \quad \ldots \quad \xrightarrow{/\!\!/ H_3} X_3 \xrightarrow{/\!\!/ H_2} X_2 \xrightarrow{/\!\!/ H_1} X_1 \ ,$$

where $X_i = \operatorname{Spec}(R_i)$ holds with a platonic ring R_i for $i \geq 2$, the ring R_p is factorial and each $X_i \to X_{i-1}$ is the total coordinate space.

Note that iteration of Cox rings requires in each step finite generation of the divisor class group $\operatorname{Cl}(\overline{X})$ of the total coordinate space of X. The latter merely means that the curve Y with function field $\mathbb{C}(\overline{X})^{H_0^0}$ is of genus zero, where $H_0^0 \subseteq H_0$ is the unit component of the quasitorus H_0 with character group $\operatorname{Cl}(X)$. In Theorem 3.4.3, we establish a formula for the genus of Y in terms of the entries l_{ij} of the defining matrix P of $R = \mathcal{R}(X)$, generalizing the case of $\mathbb{C}^*$-surfaces settled in [76, Prop. 3, p. 64]. This allows us to conclude that for log terminal affine X, the total coordinate space is always rational. Together with the fact that the total coordinate space of a log terminal affine X is canonical, see Proposition 3.4.1, we obtain that Cox ring iteration is possible in the log terminal case; see Remark 3.4.12 for a discussion of a non log terminal example with rational Cox ring. The final step in proving Theorem 3.1.5 is to show that the Cox ring iteration even stops after finitely many steps. For this, we compute explicitly in Proposition 3.5.6 the equations of the iterated Cox ring. It seems to be interesting to study Cox ring iteration also more generally; note that a $\mathbb{Q}$-factorial variety has a log terminal Cox ring if and only if it is log Fano [21, 41].

The next result shows that, in a large sense, the log terminal singularities with torus action of complexity one still can be regarded as quotient singularities: the affine plane $\mathbb{C}^2$ and the finite group $G \subseteq \mathrm{GL}(2)$ of the surface case have to be replaced with a factorial affine T-variety of complexity one and a solvable reductive group.

Theorem 3.1.6. *Let X be a rational, normal, affine variety of Type 2 with a torus action of complexity one and at most log terminal singularities.*

(i) *X is a quotient $X = X' /\!\!/ G$ of a factorial affine variety $X' := \mathrm{Spec}(R')$ by a solvable reductive group G, where R' is a factorial platonic ring.*
(ii) *The presentation of Theorem 3.1.5 is regained by $H_i := G^{(i-1)}/G^{(i)}$ and $X_i := X'/G^{(i-1)}$, where $G^{(i)}$ is the i-th derived subgroup of G.*

Example 3.1.7. Every log terminal affine $\mathbb{C}^*$-surface is a quotient of $\mathbb{C}^2$ or the E_8-singular surface $V(T_1^5 + T_2^3 + T_3^2) \subseteq \mathbb{C}^3$ by a finite solvable group.

3.2 The anticanonical complex and singularities

First recall the basic singularity types arising in the minimal model programme. Let X be a $\mathbb{Q}$-Gorenstein variety, i.e., some non-zero multiple of a canonical divisor D_X on X is an integral Cartier divisor. Then, for any resolution of singularities $\varphi\colon X' \to X$, one has the ramification formula

$$D_{X'} - \varphi^*(D_X) \;=\; \sum a_i E_i,$$

where the E_i are the prime components of the exceptional divisors and the coefficients $a_i \in \mathbb{Q}$ are the discrepancies of the resolution. The variety X is said to have at most *log terminal (canonical, terminal)* singularities, if for every resolution of singularities the discrepancies a_i satisfy $a_i > -1$ ($a_i \geq 0$, $a_i > 0$).

Remark 3.2.1. In our subsequent considerations we will use the description of the Cox rings of varieties of complexity one as introduced in Constructions 2.1.1 and 2.1.4. This allows more flexibility than the simpler version presented in 3.1.1. However, given any $R(A, P)$ as in Construction 2.1.4, we can achieve $l_{i1} \geq \ldots \geq l_{in_i}$ for all i and $l_{\iota 1} \geq \ldots \geq l_{r1}$ by means of admissible operations of type (i) and (ii), see Remark 2.3.1. Moreover, via suitable scalings of the variables T_{ij}, we can turn the coefficients of the relations g_i into those presented in Section 3.1.

In [13], the "anticanonical complex" has been introduced for Fano varieties $X(A, P, \Phi)$ and served as a tool to study singularities of the above type. The purpose of this section is to extend this approach and to generalize results from [13] to the non-complete and non-$\mathbb{Q}$-factorial cases. As an application, we characterize log terminality in Theorem 3.2.15

via platonic triples occuring in the Cox ring. For the affine case, the result specializes to Theorem 3.1.3.

Now, let $X = X(A, P, \Phi)$ be a rational T-variety of complexity one arising from Construction 2.1.6. Consider the embedding $X \subseteq Z$ into the minimal toric ambient variety. Then X and Z share the same divisor class group

$$K \;=\; \mathrm{Cl}(X) \;=\; \mathrm{Cl}(Z)$$

and the same degree map $Q\colon \mathbb{Z}^{n+m} \to K$ for their Cox rings. Let $e_Z \in \mathbb{Z}^{n+m}$ denote the sum over the canonical basis vectors e_{ij} and e_k of $\mathbb{Z}^{n+m}$. Then, with the defining relations $g_\iota, \ldots, g_{r-2+\iota}$ of the Cox ring $R(A, P)$, the canonical divisor classes of Z and X are given as

$$\mathcal{K}_Z \;=\; -Q(e_Z) \;\in\; K, \qquad\qquad \mathcal{K}_X \;=\; \sum_{i=\iota}^{r-2+\iota} \deg(g_i) + \mathcal{K}_Z \;\in\; K.$$

Observe that if X is of Type 1, then its canonical divisor class equals that of the minimal toric ambient variety Z. Define a (rational) polyhedron

$$B(-\mathcal{K}_X) \;:=\; Q^{-1}(-\mathcal{K}_X) \cap \mathbb{Q}^{n+m}_{\geq 0} \;\subseteq\; \mathbb{Q}^{n+m}$$

and let $B := B(g_\iota) + \ldots + B(g_{r-2+\iota}) \subseteq \mathbb{Q}^{n+m}$ denote the Minkowski sum of the Newton polytopes $B(g_i)$ of the relations $g_\iota, \ldots, g_{r-2+\iota}$ of $R(A, P)$.

Definition 3.2.2. Let $X = X(A, P, \Phi)$ such that $-\mathcal{K}_X$ is ample and denote by Σ the fan of the minimal toric ambient variety Z of X.

(i) The *anticanonical polyhedron* of X is the dual polyhedron $A_X \subseteq \mathbb{Q}^{r+s}$ of the polyhedron
$$B_X \;:=\; (P^*)^{-1}(B(-\mathcal{K}_X) + B - e_\Sigma) \;\subseteq\; \mathbb{Q}^{r+s}.$$

(ii) The *anticanonical complex* of X is the coarsest common refinement of polyhedral complexes
$$A_X^c \;:=\; \mathrm{faces}(A_X) \sqcap \Sigma \sqcap \mathrm{trop}(X).$$

(iii) The *relative interior* of A_X^c is the interior of its support with respect to the intersection $\mathrm{Supp}(\Sigma) \cap \mathrm{trop}(X)$.

(iv) The *relative boundary* ∂A_X^c is the complement of the relative interior of A_X^c in A_X^c.

Remark 3.2.3. Consider a subdivision $\Sigma' \to \Sigma$ of fans in $\mathbb{Q}^n$ and the associated toric morphism $Z_{\Sigma'} \to Z_\Sigma$. Then the toric Cox constructions $P\colon \mathbb{Z}^{R'} \to \mathbb{Z}^n$ and $P'\colon \mathbb{Z}^{R'} \to \mathbb{Z}^n$, where $R = \Sigma(1)$ and $R' = \Sigma'(1)$ define homomorphisms of tori

$$\mathbb{T}^{R'} \to \mathbb{T}^n \leftarrow \mathbb{T}^R.$$

For a polynomial $g \in \mathbb{C}[T_\varrho;\ \varrho \in R]$ without monomial factors the *push-down of* g is the unique polynomial $p_*(g) \in \mathbb{C}[T_1, \ldots, T_n]$ without monomial factors such that $T^\mu p^*(p_*(g)) = g$ holds for some Laurent monomial $T^\mu \in \mathbb{C}[T_\varrho^{\pm 1};\ \varrho \in R]$. We define the *shift* of g to be the unique $g' \in \mathbb{C}[T_{\varrho'};\ \varrho' \in R']$ without monomial factors such that $p'_*(g') = p_*(g)$.

Let $X' \to X$ be a resolution of singularities as in Construction 2.3.6. Then the Cox ring of X' is given as

$$\mathcal{R}(X') = \mathbb{C}[T_{\varrho'};\ \varrho' \in R']/\langle g'_\iota, \dots, g'_{r-2+\iota}\rangle,$$

with the shift g'_i of g_i. Due to [13, Lemma 2.4] the exponents of g_i correspond to the exponents of g'_i and for any exponent ν of g_i the corresponding exponent ν' of g'_i satisfies $\nu'_\varrho = \nu_\varrho$.

A first statement expresses the discrepancies of a given resolution of singularities via the anticanonical complex; the proof is a straightforward generalization of the one given in [13] for the Fano case but for the sake of completeness we give a proof here.

Proposition 3.2.4. *Let $X = X(A, P, \Phi)$ such that $-\mathcal{K}_X$ is ample and $\varphi\colon X' \to X$ a resolution of singularities as in Construction 2.3.6. For any ray $\varrho \in \Sigma''$, let v_ϱ be its primitive generator, v'_ϱ its leaving point of A_X^c provided $\varrho \not\subseteq A_X^c$ and D_ϱ the corresponding prime divisor on X''. Then the discrepancy a_ϱ along D_ϱ satisfies*

$$a_\varrho \;=\; -1 + \frac{||v_\varrho||}{||v'_\varrho||} \quad \text{if } \varrho \not\subseteq A_X^c, \qquad\qquad a_\varrho \le -1 \quad \text{if } \varrho \subseteq A_X^c.$$

Proof. We use the notation of Remark 3.2.3. Note that the exceptional divisors of φ are exactly the divisors $D_{X'}^{\varrho'}$ obtained as pullbacks of the toric divisors in Z_Σ given by the rays $\varrho' \in \Sigma'(1) \setminus \Sigma(1)$. We fix such a ray ϱ' and compute the discrepancy of φ along $D_{X'}^{\varrho'}$. Let $B := B(g_\iota) + \ldots + B(g_{r-2+\iota})$ and $B' := B(g'_\iota) + \ldots + B(g'_{r-2+\iota})$ be the Minkowski sums of the Newton polytopes $B(g_i)$ and $B(g'_i)$. The inverse image $P^{-1}(\varrho')$ is contained in a maximal cone $\tau \in \mathcal{N}(B(-\mathcal{K}_X) + B)$. Let $\eta \in B(-\mathcal{K}_X) + B$ be the vertex corresponding to τ. Then $\eta = \nu_{-\mathcal{K}_X} + \nu$ with vertices $\nu_{-\mathcal{K}_X} \in B(-\mathcal{K}_X)$ and $\nu \in B$. Write $\nu' \in B'$ for the vertex corresponding to $\nu \in B$ in the sense of Remark 3.2.3. We fix the following representatives of the anticanonical classes of X and X':

$$D_X^c := \sum_{\varrho \in R} (-1 + \nu_\varrho) D_X^\varrho, \qquad D_{X'}^c := \sum_{\varrho \in R'} (-1 + \nu'_\rho) D_{X'}^\varrho.$$

Note that $D_{X'}^c - \varphi^* D_X^c$ is supported on the exceptional locus as $\nu'_\varrho = \nu_\varrho$ holds for all $\varrho \in \Sigma(1)$. We use this representatives to compute the discrepancy of ϱ along $D_{X'}^{\varrho'}$.

Let $\sigma \in \Sigma$ be the cone with $\mathrm{relint}(\varrho') \subseteq \mathrm{relint}(\sigma)$. Then, on the corresponding chart $X_\sigma = X \cap Z_\sigma$, the divisor D_X^c is rationally principal and we claim that on X_σ the divisor has a presentation

$$D_X^c = \frac{1}{m}\mathrm{div}(\chi^{mu}) \quad \text{with } u := (P^*)^{-1}(\nu_{-\mathcal{K}_X} + \nu - e_\Sigma),$$

where $m \in \mathbb{Z}_{>0}$ such that mu is integral and χ^{mu} denotes the pullback of the toric character function on Z_Σ associated to mu. We obtain

$$\frac{1}{m}\mathrm{div}(\chi^{mu}) = \sum_{\varrho \in \sigma(1)} \langle u, v_\varrho\rangle D_X^\varrho = \sum_{\varrho \in \sigma(1)} \langle P^* u, e_\varrho\rangle D_X^\varrho = \sum_{\varrho \in \sigma(1)} \langle \nu_{-\mathcal{K}_X} + \nu - e_\Sigma, e_\varrho\rangle D_X^\varrho$$

Thus in order to verify the claim, we have to show that $\langle \nu_{-\mathcal{K}_X}, e_\varrho \rangle = 0$ holds for all rays ϱ of σ. Due to amplesness of the anticanonical class we obtain $B(-\mathcal{K}_X) \cap \mathrm{relint}(\widehat{\sigma}^\perp \cap \gamma_R)$ is non-empty and thus contains some element e. As $\mathrm{relint}(\varrho') \subseteq \mathrm{relint}(\sigma)$ holds, we can choose a vector $\mu = \sum_{\varrho \in \sigma(1)} b_\varrho \varrho$ with positive b_ϱ in the preimage $P^{-1}(\varrho')$. By the choice of e we have $\langle e, \mu \rangle = 0$ and as $\nu_{-\mathcal{K}_X} \in B(-\mathcal{K}_X)$ is a minimizing vertex of μ, we conclude $\langle \nu_{-\mathcal{K}_X}, \mu \rangle = 0$ and thus $\langle \nu_{-\mathcal{K}_X}, e_\varrho \rangle = 0$ for all rays ϱ of σ.

We obtain that the discrepancy $a_{\varrho'}$ of $\varphi\colon X' \to X$ along $D_{X'}^{\varrho'}$ is the multiplicity of $D_{X'}^{\mathrm{c}} - \mathrm{div}(\chi^u)$ along $D_{X'}^{\varrho'}$ and thus

$$a_{\varrho'} = -1 + \nu'_{\varrho'} - \langle u, v_{\varrho'} \rangle.$$

In a last step we show that $\nu'_{\varrho'}$ equals zero. Then evaluating $\langle u, v_{\varrho'} \rangle$ gives the assertion. First note that we can find a decomposition $\nu = \nu_\iota + \ldots + \nu_{r-2+\iota}$, where $\nu_i \in B(g_i)$. Let ν'_i be the corresponding exponent vector of the shift g'_i. Then we have a decomposition $\nu' = \nu'_\iota + \ldots + \nu'_{r-2+\iota}$. We claim that $\nu'_{i\varrho'} = 0$ for all $i = \iota, \ldots, r-2+\iota$. By definition, ν'_i lies in the face of $B(g'_i)$ which is cut out by $P'^{-1}(\varrho')$. Consequently, the corresponding exponent vector of the pushed down equation $p_*(g_i)$ lies in the face of $B(p_*(g_i))$ that is cut out by ϱ'. Then [13, Lemma 2.5] gives the assertion. □

The next result characterizes the existence of at most log terminal (canonical, terminal) singularities in terms of the anticanonical complex; again, this generalizes a result from [13].

Theorem 3.2.5. *Let $X = X(A, P, \Phi)$ be such that $-\mathcal{K}_X$ is ample. Then the following statements hold.*

(i) *A_X^{c} contains the origin in its relative interior and all primitive generators of the fan Σ are vertices of A_X^{c}.*
(ii) *X has at most log terminal singularities if and only if the anticanonical complex A_X^{c} is bounded.*
(iii) *X has at most canonical singularities if and only if 0 is the only lattice point in the relative interior of A_X^{c}.*
(iv) *X has at most terminal singularities if and only if 0 and the primitive generators v_ϱ for $\varrho \in \Sigma^{(1)}$ are the only lattice points of A_X^{c}.*

Proof. By construction of the desingularization 2.3.6 X is strongly tropically resolvable. Thus following the lines of [13, Theorem 1.4] but replacing [13, Proposition 2.3] with our more general Proposition 3.2.4 gives the assertions. □

We describe the structure of the anticanonical complex in more detail, which generalizes in particular statements on the $\mathbb{Q}$-factorial Fano case obtained in [13]. For Type 1, the situation turns out to be simple, whereas Type 2 is more ample.

Proposition 3.2.6. *Let $X = X(A, P, \Phi)$ be of Type 1 such that $-\mathcal{K}_X$ is ample. Let Σ be the fan of the minimal toric ambient variety of X and denote by $\lambda_0, \ldots, \lambda_r$ the leaves of* $\mathrm{trop}(X)$.

(i) *Every cone $\sigma \in \Sigma$ is contained in a leaf $\lambda_i \subseteq \mathrm{trop}(X)$. In particular, $\Sigma \sqcap \mathrm{trop}(X)$ equals Σ.*
(ii) *The boundary of A_X^c is the union of all faces of A_X that are contained in* $\mathrm{Supp}(\Sigma)$*.*
(iii) *The non-zero vertices of A_X^c are the primitive generators of Σ, i.e. the columns of P.*

Corollary 3.2.7. *Let $X = X(A, P, \Phi)$ be a T-variety of Type 1. Then X has at most log-terminal singularities. Moreover, it has at most canonical (terminal) singularities if and only if its minimal toric ambient variety Z does so.*

Construction 3.2.8. Let $X = X(A, P, \Phi)$ be of Type 2 and Σ the fan of the minimal toric ambient variety of Z. Write $v_{ij} := P(e_{ij})$ and $v_k := P(e_k)$ for the columns of P. Consider a pointed cone of the form

$$\tau \;=\; \mathrm{cone}(v_{0j_0}, \ldots, v_{rj_r}) \;\subseteq\; \mathbb{Q}^{r+s},$$

that means that τ contains exactly one v_{ij} for every $i = 0, \ldots, r$. We call such τ a *P-elementary cone* and associate the following numbers with τ:

$$\ell_{\tau,i} \;:=\; \frac{l_{0j_0} \cdots l_{rj_r}}{l_{ij_i}} \text{ for } i = 0, \ldots, r, \qquad \ell_\tau \;:=\; (1-r) l_{0j_0} \cdots l_{rj_r} + \sum_{i=0}^{r} l_{\tau,i}.$$

Moreover, we set

$$v(\tau) \;:=\; \ell_{\tau,0} v_{0j_0} + \ldots + \ell_{\tau,r} v_{rj_r} \;\in\; \mathbb{Z}^{r+s}, \qquad \varrho(\tau) \;:=\; \mathbb{Q}_{\ge 0} \cdot v(\tau) \;\in\; \mathbb{Q}^{r+s}.$$

We denote by $\mathrm{T}(A, P, \Phi)$ the set of all P-elementary cones $\tau \in \Sigma$. For a given $\sigma \in \Sigma$, we denote by $\mathrm{T}(\sigma)$ the set of all P-elementary faces of σ.

Remark 3.2.9. Let $X = X(A, P, \Phi)$ be of Type 2. Let Σ be the fan of the minimal toric ambient variety of X and $\lambda_0, \ldots, \lambda_r \subseteq \mathrm{trop}(X)$ the leaves of the tropical variety of X. As in [13, Def. 4.1], we say that

(i) a cone $\sigma \in \Sigma$ is a *leaf cone* if $\sigma \subseteq \lambda_i$ holds for some $i = 0, \ldots, r$,
(ii) a cone $\sigma \in \Sigma$ is called *big* if $\sigma \cap \lambda_i^\circ \neq \emptyset$ holds for all $i = 0, \ldots, r$.

Observe that a given cone $\sigma \in \Sigma$ is big if and only if σ contains some P-elementary cone as a subset.

Proposition 3.2.10. *Let $X = X(A, P, \Phi)$ be of Type 2 such that $-\mathcal{K}_X$ is ample. Let Σ be the fan of the minimal toric ambient variety of X, denote by $\lambda_0, \ldots, \lambda_r$ the leaves of* $\mathrm{trop}(X)$ *and by $\lambda = \lambda_0 \cap \ldots \cap \lambda_r$ its lineality part.*

(i) *The fan $\Sigma \sqcap \mathrm{trop}(X)$ consists of the cones $\sigma \cap \lambda$ and $\sigma \cap \lambda_i$, where $\sigma \in \Sigma$ and $i = 0, \ldots, r$. Here, one always has $\sigma \cap \lambda \preceq \sigma \cap \lambda_i$.*
(ii) *The fan $\Sigma \sqcap \mathrm{trop}(X)$ is a subfan of the normal fan of the polyhedron B_X. In particular, for every cone $\sigma \cap \lambda_i$, there is a vertex $u_{\sigma,i} \in B_X$ with*

$$\partial A_X^c \cap \sigma \cap \lambda_i \;=\; \{v \in \sigma \cap \lambda_i;\; \langle u_{\sigma,i}, v \rangle = -1\}.$$

(iii) *If a P-elementary cone τ is contained in some $\sigma \in \Sigma$, then τ is simplicial, $v(\tau) \in \tau^\circ$ holds, $\varrho(\tau)$ is a ray, $\varrho(\tau) = \tau \cap \lambda$ holds as well as $\mathbb{Q}\varrho(\tau) = \mathbb{Q}\tau \cap \lambda$.*

(iv) *Let $\sigma \in \Sigma$ be any cone. Then, for every $i = 0, \ldots, r$, the set of extremal rays of $\sigma \cap \lambda_i \in \Sigma \sqcap \operatorname{trop}(X)$ is given by*

$$(\sigma \cap \lambda_i)^{(1)} \;=\; \{\varrho(\sigma_0);\ \sigma_0 \in \mathrm{T}(\sigma)\} \cup \{\varrho \in \sigma^{(1)};\ \varrho \subseteq \lambda_i\}.$$

(v) *The set of rays of $\Sigma \sqcap \operatorname{trop}(X)$ consists of the rays of Σ and the rays $\varrho(\sigma_0)$, where $\sigma_0 \in \mathrm{T}(A, P, \Phi)$.*

(vi) *If a P-elementary cone τ is contained in some $\sigma \in \Sigma$, then the minimum value among all $\langle u, v(\tau)\rangle$, where $u \in B_X$, equals $-\ell_\tau$.*

(vii) *Let the P-elementary cone τ be contained in $\sigma \in \Sigma$. Then $\varrho(\tau) \not\subseteq A_X^c$ holds if and only if $\ell_\tau > 0$ holds; in this case, $\varrho(\tau)$ leaves A_X^c at $v(\tau)' = \ell_\tau^{-1} v(\tau)$.*

(viii) *The vertices of A_X^c are the primitive generators of Σ, i.e. the columns of P, and the points $v(\sigma_0)' = \ell_{\sigma_0}^{-1} v(\sigma_0)$, where $\sigma_0 \in \mathrm{T}(A, P, \Phi)$ and $\ell_{\sigma_0} > 0$.*

Proof. Assertion (i) holds more generally. Indeed, the coarsest common refinement $\Sigma_1 \sqcap \Sigma_2$ of any two quasifans Σ_i in a common vector space consists of the intersections $\sigma_1 \cap \sigma_2$, where $\sigma_i \in \Sigma_i$. Moreover, the faces of a given cone $\sigma_1 \cap \sigma_2$ of $\Sigma_1 \sqcap \Sigma_2$ are precisely the cones $\sigma_1' \cap \sigma_2'$, where $\sigma_i' \preceq \sigma_i$.

We show (ii). Let Σ' be the complete fan in $\mathbb{Q}^{r+s}$ defined by the class $-\mathcal{K}_X \in K$. Since $-\mathcal{K}_X$ is ample, the fan Σ is a subfan of Σ'. The preimage $P^{-1}(\Sigma')$ consists of the cones $P^{-1}(\sigma')$, where $\sigma' \in \Sigma'$, and is the normal fan of $B(-\mathcal{K}_X) \subseteq \mathbb{Q}^{n+m}$. Moreover, $P^{-1}(\operatorname{trop}(X))$ turns out to be a subfan of the normal fan of $B \subseteq \mathbb{Q}^{n+m}$. It follows that $P^{-1}(\Sigma') \sqcap P^{-1}(\operatorname{trop}(X))$ is a subfan of the normal fan of $B(-\mathcal{K}_X) + B$. Projecting the involved fans via P to $\mathbb{Q}^{r+s}$ gives the assertion.

To obtain (iii), consider first any P-elementary $\tau = \operatorname{cone}(v_{0j_0}, \ldots, v_{rj_r})$. Then $v_{0j_0}, \ldots, v_{rj_r}$ is linearly dependent if and only if $v(\tau) = 0$ holds. The latter is equivalent to 0 being an inner point of τ. Thus, if τ is contained in some $\sigma \in \Sigma$, then τ is pointed an thus must be simplicial. The remaining part is then obvious; recall that the lineality part of $\operatorname{trop}(X)$ equals the vector subspace $0 \times \mathbb{Q}^s \subseteq \mathbb{Q}^{r+s}$.

We turn to (iv). First, we claim that if $\sigma_0 \in \Sigma$ is big and $\varrho(\tau) = \varrho(\tau')$ holds for any two P-elementary cones $\tau, \tau' \subseteq \sigma$, then σ_0 is P-elementary. Assume that σ_0 is not P-elementary. Then we find some $1 \le t \le r$ and cones

$$\begin{aligned} \tau &= \operatorname{cone}(v_{0j_0}, \ldots, v_{tj_{t-1}}, v_{tj_t}, v_{tj_{t+1}}, \ldots, v_{rj_r}) \subseteq \sigma_0, \\ \tau' &= \operatorname{cone}(v_{0j_0}, \ldots, v_{tj_{t-1}}, v_{tj_t'}, v_{tj_{t+1}}, \ldots, v_{rj_r}) \subseteq \sigma_0 \end{aligned}$$

with $j_t \neq j_t'$ and thus $\tau \neq \tau'$. Here, we may assume that $c_\tau^{-1} l_{tj_t} \ge c_{\tau'}^{-1} l_{tj_t'}$ holds with the greatest common divisors c_τ and $c_{\tau'}$ of the entries of $v(\tau)$ and $v(\tau')$ respectively. Then even $c_\tau^{-1}\ell_{\tau,i} \ge c_{\tau'}^{-1}\ell_{\tau',i}$ must hold for all $1 \le i \le r$. Since, the rays $\varrho(\tau)$ and $\varrho(\tau')$ coincide, also their primitive generators $c_{\tau'}^{-1} v(\tau')$ and $c_\tau^{-1} v(\tau)$ coincide. By the definition of $v(\tau)$ and $v(\tau')$, this implies

$$c_{\tau'}^{-1}\ell_{\tau',t} v_{tj_t'} \;=\; c_\tau^{-1}\ell_{\tau,k} v_{tj_t} + \sum_{i \neq t} (c_\tau^{-1}\ell_{\tau,i} - c_{\tau'}^{-1}\ell_{\tau',i}) v_{ij_i}.$$

We conclude $v_{tj'_t} \in \tau$. Since $v_{tj'_t}$ is an extremal ray of σ_0 and $\tau' \subseteq \sigma_0$ holds, $v_{tj'_t}$ generates an extremal ray of τ. This is a contradiction to the choice of j'_t and the claim is verified.

Now, consider the equation of (iv). To verify "$\subseteq$", let ϱ be an extremal ray of $\sigma \cap \lambda_i$. We have to show that $\varrho = \varrho(\sigma_0)$ holds for some $\sigma_0 \in \mathrm{T}(\sigma)$ or that ϱ is a ray of σ with $\varrho \subseteq \lambda_i$. According to (ii), there is a face $\sigma_\varrho \preceq \sigma$ such that $\varrho = \sigma_\varrho \cap \lambda$ or $\varrho = \sigma_\varrho \cap \lambda_i$ holds. We choose σ_ϱ minimal with respect to this property, that means that we have $\varrho^\circ \subseteq \sigma_\varrho^\circ$. We distinguish the following cases.

Case 1. We have $\varrho = \sigma_\varrho \cap \lambda$. If $\sigma_\varrho \subseteq \lambda$ holds, then we obtain $\varrho = \sigma_\varrho$ and thus $\varrho \subseteq \lambda_i$ is an extremal ray of σ. So, assume that σ_ϱ is not contained in λ. Then, because of $\sigma_\varrho^\circ \cap \lambda \neq \emptyset$, there is a P-elementary cone $\tau \subseteq \sigma_\varrho$. Using (i), we obtain

$$\varrho(\tau) \;=\; \tau \cap \lambda \;\subseteq\; \sigma_\varrho \cap \lambda \;=\; \varrho$$

and thus $\varrho = \varrho(\tau)$. As this does not depend on the particular choice of the P-elementary cone $\tau \subseteq \sigma_\varrho$, the above claim yields $\sigma_0 := \sigma_\varrho \in \mathrm{T}(\sigma)$ and $\varrho = \varrho(\sigma_0)$.

Case 2. We don't have $\varrho = \sigma_\varrho \cap \lambda$. Then $\varrho = \sigma_\varrho \cap \lambda_i$ and $\varrho^\circ \subseteq \lambda_i^\circ$ hold. If $\sigma_\varrho \subseteq \lambda_i$ holds, then we obtain $\varrho = \sigma_\varrho$ and thus $\varrho \subseteq \lambda_i$ is an extremal ray of σ. So, assume that σ_ϱ is not contained in λ_i. Then $\sigma_\varrho \cap \lambda_j^\circ$ is non-empty for all $j = 0, \ldots, r$. Thus, there is a P-elementary cone $\tau \subseteq \sigma_\varrho$. Using (i), we obtain

$$\varrho(\tau) \;=\; \tau \cap \lambda \;\subseteq\; \sigma_\varrho \cap \lambda \;=\; \varrho$$

and thus $\varrho = \varrho(\tau)$. As this does not depend on the particular choice of the P-elementary cone $\tau \subseteq \sigma_\varrho$, the above claim yields $\sigma_0 := \sigma_\varrho \in \mathrm{T}(\sigma)$ and $\varrho = \varrho(\sigma_0)$.

We verify the inclusion "$\supseteq$". Consider a face $\sigma_0 \in \mathrm{T}(\sigma)$. As seen just before, the extremal rays of $\sigma_0 \cap \lambda_i$ are $\varrho(\sigma_0)$ and the rays of σ_0 that lie in λ_i. Since $\sigma_0 \cap \lambda_i$ is a face of $\sigma \cap \lambda_i$, the ray $\varrho(\sigma_0)$ is an extremal ray of $\sigma \cap \lambda_i$. Finally, consider an extremal ray $\varrho \preceq \sigma$ with $\varrho \subseteq \lambda_i$. Then $\varrho = \varrho \cap \lambda_i$ is a face of $\sigma \cap \lambda_i$.

The proof of Assertion (iv) is complete now. Assertion (v) is a direct consequence of (iv).

We turn to Assertions (vi), (vii) and (viii). Let $\widehat{\tau} \preceq \widehat{\sigma} \preceq \mathbb{Q}^{n+m}_{\ge 0}$ be the faces with $P(\widehat{\tau}) = \tau$ and $P(\widehat{\sigma}) = \sigma$. Moreover, let $e_\tau \in \widehat{\tau}$ be the (unique) point with $P(e_\tau) = v(\tau)$. The minimum value $\langle u, v(\tau)\rangle$ is attained at some vertex $u \in B_X$. For this u, we find vertices $e_\sigma \in B(-\mathcal{K}_X)$ and $e_B \in B$ with

$$u \;=\; (P^*)^{-1}(e_\sigma + e_B - e_Z).$$

Here, e_σ is any vertex of $B(-\mathcal{K}_X)$ such that $\widehat{\sigma}$ is contained in the cone of the normal fan of $B(-\mathcal{K}_X)$ associated with e_σ; such e_σ exists due to ampleness of $-\mathcal{K}_X$ and e_σ vanishes along $\widehat{\sigma}$. Together we have

$$e_\tau \;=\; \sum_{i=0}^{r} l_{ij_i} e_{ij_i}, \qquad\qquad \langle u, v(\tau)\rangle \;=\; \langle e_\sigma + e_B - e_Z, e_\tau\rangle.$$

As mentioned, $\langle e_\sigma, e_\tau\rangle = 0$ holds. Moreover, $\langle e, e_\tau\rangle = (r-1) l_{0j_0} \cdots l_{rj_r}$ holds for every $e \in B$. We conclude $\langle u, v(\tau)\rangle = -\ell_\tau$ and Assertion (vi). Moreover, Assertions (vii) and (viii) are direct consequences of (vi) and (ii). $\square$

Example 3.2.11. Consider the E_6-singular affine surface $X = V(z_1^4 + z_2^3 + z_3^2) \subseteq \mathbb{C}^3$. It inherits a $\mathbb{C}^*$-action from the action

$$t \cdot (z_1, z_2, z_3) \;=\; (t^3 z_1, t^4 z_2, t^6 z_3)$$

on $\mathbb{C}^3$. The divisor class group and the Cox ring of the surface X are explicitly given by

$$\mathrm{Cl}(X) \;=\; \mathbb{Z}/3\mathbb{Z}, \qquad \mathcal{R}(X) \;=\; \mathbb{C}[T_1, T_2, T_3]/\langle T_1^3 + T_2^3 + T_3^2 \rangle,$$

where the $\mathrm{Cl}(X)$-degrees of T_1, T_2, and T_3 are $\bar{1}$, $\bar{2}$ and $\bar{0}$. The minimal toric ambient variety is affine and corresponds to the cone

$$\sigma \;=\; \mathrm{cone}((-3,-3,-2), (3,0,1), (0,2,1)).$$

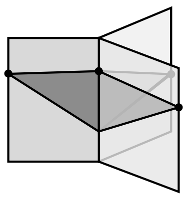
Denoting by $e_i \in \mathbb{Q}^3$ the i-th canonical basis vector, the tropical variety $\mathrm{trop}(X)$ in $\mathbb{Q}^3$ is given as

$$\mathrm{trop}(X) \;=\; \mathrm{cone}(e_1, \pm e_3) \cup \mathrm{cone}(e_2, \pm e_3) \cup \mathrm{cone}(-e_1 - e_2, \pm e_3).$$

The anticanonical polyhedron $A_X \subseteq \mathbb{Q}^3$ is not bounded with recession cone generated by $(-1,-1,-1)$, $(1,0,0)$, $(0,1,0)$. The vertices of A_X are

$$(-3,-3,-2), \;\; (3,0,1), \;\; (0,2,1), \;\; (0,0,1).$$

The anticanonical complex $A_X^c = A_X \sqcap \Sigma \sqcap \mathrm{trop}(X)$ lives inside $\mathrm{trop}(X)$ and looks as follows.



Corollary 3.2.12. *Let $X = X(A, P, \Phi)$ be of Type 2 such that $-\mathcal{K}_X$ is ample. Let τ be a P-elementary cone contained in some $\sigma \in \Sigma$. Assume $\varrho(\tau) \not\subseteq A_X^c$ and denote by c_τ the greatest common divisor of the entries of $v(\tau)$. Then, for any resolution of singularities $\varphi\colon X'' \to X$ provided by 2.3.6, the discrepancy along the prime divisor of X'' corresponding to $\varrho(\tau)$ equals $c_\tau^{-1}\ell_\tau - 1$.*

Corollary 3.2.13. *Let $X = X(A, P, \Phi)$ be of Type 2 such that $-\mathcal{K}_X$ is ample and let $\tau = \mathrm{cone}(v_{0j_0}, \ldots, v_{rj_r})$ be contained in some $\sigma \in \Sigma$.*

(i) *If X has at most log terminal singularities, then $l_{0j_0}^{-1} + \ldots + l_{rj_r}^{-1} > r - 1$ holds.*
(ii) *If X has at most canonical singularities, then $l_{0j_0}^{-1} + \ldots + l_{rj_r}^{-1} \geq r - 1 + c_\tau l_{0j_0}^{-1} \cdots l_{rj_r}^{-1}$ holds.*
(iii) *If X has at most terminal singularities, then $l_{0j_0}^{-1} + \ldots + l_{rj_r}^{-1} > r - 1 + c_\tau l_{0j_0}^{-1} \cdots l_{rj_r}^{-1}$ holds.*

Remark 3.2.14. Let $a_0, \ldots, a_r$ be positive integers. Then $a_0^{-1} + \ldots + a_r^{-1} > r - 1$ holds if and only if $(a_0, \ldots, a_r)$ is a platonic tuple.

Theorem 3.2.15. *Let $X = X(A, P, \Phi)$ be of Type 2 such that $-\mathcal{K}_X$ is ample and let Σ be the fan of the minimal toric ambient variety of X. Then the following statements are equivalent.*

(i) *The variety X has at most log terminal singularities.*
(ii) *For every P-elementary $\tau = \mathrm{cone}(v_{0j_0}, \ldots, v_{rj_r})$ contained in a cone of Σ, the exponents $l_{0j_0}, \ldots, l_{rj_r}$ form a platonic tuple.*

Proof. Assume that $X = X(A, P, \Phi)$ is log terminal. Then Corollary 3.2.13 (i) tells us that for every P-elementary $\tau = \mathrm{cone}(v_{0j_0}, \ldots, v_{rj_r})$ contained in a cone of Σ, the corresponding exponents $l_{0j_0}, \ldots, l_{rj_r}$ form a platonic tuple.

Now assume that (ii) holds. Then every $(l_{0j_0}, \ldots, l_{rj_r})$ is a platonic tuple. Consequently, we have $\ell_\tau > 0$ for every P-elementary cone τ. Proposition 3.2.10 shows that A_X^c is bounded for $X = X(A, P, \Phi)$. Theorem 3.2.5 (ii) tells us that X is log terminal. □

Remark 3.2.16. Let $X = X(A, P, \Phi)$ be affine of Type 2 such that $\mathcal{K}_X$ is $\mathbb{Q}$-Cartier. Then $-\mathcal{K}_X$ is ample. The fan Σ of the minimal toric ambient variety Z of X consists of all the faces of the cone σ generated by the columns of P. In particular, every P-elementary cone is contained in σ. Thus, Theorem 3.1.3 follows from Theorem 3.2.15. Moreover, the rays $\varrho(\sigma_0)$, where $\sigma_0 \in \mathrm{T}(A, P, \Phi)$, are precisely the extremal rays of the intersection of σ and the lineality part of $\mathrm{trop}(X)$.

3.3 Gorenstein index and canonical multiplicity

If a normal variety X is $\mathbb{Q}$-Gorenstein, then, by definition, some multiple of its canonical class $\mathcal{K}_X$ is Cartier. The *Gorenstein index* of X is the smallest positive integer $\imath_X$ such that $\imath_X \mathcal{K}_X$ is Cartier. We attach another invariant to the canonical divisor of X.

Remark 3.3.1. Let $X = X(A, P)$ be a $\mathbb{Q}$-Gorenstein, affine T-variety of Type 2. We consider canonical divisors D_X on X that are of the following form, cf. [6, Prop. 3.3.3.2]:

$$-\sum_{i,j} D_{ij} - \sum_k E_k + \sum_{\alpha=1}^{r-1} \sum_{j=0}^{n_{i_\alpha}} l_{i_\alpha j} D_{i_\alpha j}, \qquad 0 \le i_\alpha \le r. \tag{3.1}$$

Corollary 2.3.10 says that $\imath_X D_X$ is the divisor of a T-homogeneous rational function. Any two $\imath_X D_X$ with D_X of shape (3.1) differ by the divisor of a T-invariant rational function, and thus, all the functions with divsors $\imath_X D_X$, where D_X as in (3.1), are homogeneous with respect to the same weight $\eta_X \in \mathbb{X}(T)$.

Definition 3.3.2. Let $X = X(A, P)$ be a $\mathbb{Q}$-Gorenstein, affine T-variety of Type 2. We call $\eta_X \in \mathbf{X}(T)$ of Remark 3.3.1 the *canonical weight* of X. The *canonical multiplicity* of X is the minimal non-negative integer ζ_X such that $\eta_X = \zeta_X \cdot \eta'_X$ holds with a primitive element $\eta'_X \in \mathbb{X}(T)$.

Proposition 3.3.3. *Let $X = X(A,P)$ be a $\mathbb{Q}$-Gorenstein, affine T-variety of Type 2 with at most log terminal singularities. Then $\zeta_X > 0$ holds. Moreover, for any positive integer $\imath$, the following statements are equivalent.*

(i) *The variety X is of Gorenstein index $\imath$.*

(ii) *There exist integers $\mu_1, \ldots, \mu_r$ with $\gcd(\mu_1, \ldots, \mu_r, \zeta_X, \imath) = 1$ such that with $\mu_0 := \imath(r-1) - \mu_1 - \ldots - \mu_r$ we obtain integral vectors*

$$\nu_i := (\nu_{i1}, \ldots, \nu_{in_i}) \text{ with } \nu_{ij} := \frac{\imath - \mu_i l_{ij}}{\zeta_X},$$

$$\nu' := (\nu'_1, \ldots, \nu'_m) \text{ with } \nu'_k := \frac{\imath}{\zeta_X}$$

and by suitable elementary row operations on the (d,d')-block, the matrix P gains $(\nu_0, \ldots, \nu_r, \nu')$ as its last row, i.e., turns into the shape

$$\tilde{P} = \begin{pmatrix} -l_0 & l_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -l_0 & 0 & \ldots & l_r & 0 \\ * & * & \ldots & * & * \\ \nu_0 & \nu_1 & \ldots & \nu_r & \nu' \end{pmatrix}.$$

Proof. We work with an anticanonical divisor D_X on X such that $-D_X$ is of the form (3.1):

$$D_X := \sum_{i,j} D_{ij} + \sum_k E_k - (r-1)\sum_{j=1}^{n_0} l_{0j} D_{0j}.$$

According to Corollary 2.3.10, the Picard group of X is trivial. Thus, $\imath_X D_X$ is the divisor of some toric character χ^u, where

$$u = (\mu_1, \ldots, \mu_r, \eta_1, \ldots, \eta_s) \in \mathbb{Z}^{r+s}.$$

Note that $-(\eta_1, \ldots, \eta_s) \in \mathbb{Z}^s = \mathbb{X}(T)$ is the canonical weight η_X of X. Moreover, the divisor $\imath_X D_X = \operatorname{div}(\chi^u)$ corresponds to the vector $P^* \cdot u \in \mathbb{Z}^{m+n}$ under the identification of toric divisors with lattice points via $D_{ij} \mapsto e_{ij}$ and $E_k \mapsto e_k$.

We claim that η_X is non-trivial. Otherwise, $\eta_1 = \ldots = \eta_s = 0$ holds. As noted, the ij-th and k-th components of the vector $P^* \cdot u$ are the multiplicities of D_{ij} and D_k in $\imath_X D_X$, respectively. More explicitly, this leads to the conditions

$$m = 0, \qquad \imath_X((r-1)l_{0j} - 1) = (\mu_1 + \ldots + \mu_r) l_{0j}, \qquad \imath_X = \mu_i l_{ij}$$

for all i and j. Plugging the third into the second one, we obtain that $l_{0j_0}^{-1} + \ldots + l_{rj_r}^{-1}$ equals $r-1$ for any choice of $1 \le j_i \le n_i$. According to Corollary 3.2.13 (i), this contradicts to log terminality of X. Knowing that η_X is non-zero, we obtain that ζ_X is non-zero.

Now, assume that (i) holds, i.e., we have $\imath = \imath_X$. Let $u \in \mathbb{Z}^{r+s}$ as above. Then we have $\zeta_X = \gcd(\eta_1, \ldots, \eta_s)$ and $\operatorname{div}(\chi^u) = \imath D_X$ implies $\gcd(\mu_1, \ldots, \mu_r, \zeta_X, \imath) = 1$. Next, choose a unimodular $s \times s$ matrix $\mathcal{B}$ with $\mathcal{B}^{-1} \cdot (\eta_1, \ldots, \eta_s) = (0, \ldots, 0, \zeta_X)$. Consider $\tilde{P} := \operatorname{diag}(E_r, \mathcal{B}^*) \cdot P$ and

$$\tilde{u} \;=\; (\mu_1, \ldots, \mu_r, 0, \ldots, 0, \zeta_X) \;\in\; \mathbb{Z}^{r+s}.$$

Observe that we have $P^* \cdot u = \tilde{P}^* \cdot \tilde{u}$. Comparing the entries of $\tilde{P}^* \cdot \tilde{u}$ with the multiplicities of the prime divisors D_{ij} and D_k in $\imath D_X$ shows that the last row of $\tilde{P}$ is as claimed.

Conversely, if (ii) holds, consider $u := (\mu_1, \ldots, \mu_r, 0, \ldots, 0, \zeta_X)$. Then we obtain $\imath D_X = \operatorname{div}(\chi^u)$. Using $\gcd(\mu_1, \ldots, \mu_r, \zeta_X, \imath) = 1$, we conclude that $\imath$ is the Gorenstein index of X. □

Remark 3.3.4. Let $X = X(A, P)$ be a $\mathbb{Q}$-Gorenstein, affine T-variety of Type 2 and D_X a canonical divisor on X as in (3.1). Then $\imath_X D_X$ is the divisor of some toric character χ^u, where

$$u \;=\; (\mu_1, \ldots, \mu_r, \eta_1, \ldots, \eta_s) \;\in\; \mathbb{Z}^{r+s}.$$

In this situation, we have $\eta_X = (\eta_1, \ldots, \eta_s) \in \mathbb{X}(T)$ for the canonical weight of X and the canonical multiplicity of X is given by $\zeta_X = \gcd(\eta_1, \ldots, \eta_s)$. If P is in the shape of Proposition 3.3.3, then $\eta_X = (0, \ldots, 0, \zeta_X)$ holds and $-\mu_1, \ldots, -\mu_r$ satisfy the conditions of 3.3.3 (ii).

Remark 3.3.5. The defining matrix P of a given $\mathbb{Q}$-Gorenstein, affine T-variety $X = X(A, P)$ is in the shape of Proposition 3.3.3 if and only if for every $i = 0, \ldots, r$, the numbers $\mu_i := (\imath_X - \zeta_X \nu_{i1}) l_{i1}^{-1}$ satisfy

(i) $\zeta_X \nu_{ij} + \mu_i l_{ij} = \imath_X$ for $i = 1, \ldots, r$ and $j = 1, \ldots, n_i$,
(ii) $\zeta_X \nu_{0j} + \mu_0 l_{0j} = \imath_X$, for $\mu_0 := \imath_X(r-1) - \mu_1 - \ldots - \mu_r$ and $j = 1, \ldots, n_0$,
(iii) $\gcd(\mu_1, \ldots, \mu_r, \zeta_X, \imath_X) = 1$,
(iv) $\zeta_X \nu'_k = \imath_X$ for $k = 1, \ldots, m$.

Corollary 3.3.6. *Let $X = X(A, P)$ be a $\mathbb{Q}$-Gorenstein, affine T-variety of Type 2 with at most log terminal singularities. Then, for every $\imath \in \mathbb{Z}_{\geq 1}$, the following statements are equivalent.*

(i) *The variety X is of Gorenstein index $\imath$ and of canonical multiplicity one.*
(ii) *One can choose the defining matrix P to be of the shape*

$$\begin{pmatrix} -l_0 & l_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -l_0 & 0 & \ldots & l_r & 0 \\ * & * & \ldots & * & * \\ \boldsymbol{\imath} - \imath(r-1) l_0 & \boldsymbol{\imath} & \ldots & \boldsymbol{\imath} & \boldsymbol{\imath} \end{pmatrix},$$

where $\boldsymbol{\imath}$ stands for a vector $(\imath, \ldots, \imath)$ of suitable length.

Proof. If (i) holds, then we may assume P to be as $\tilde{P}$ in Proposition 3.3.3. Adding the μ_i-fold of the i-th row to the last row brings P into the desired form. If (ii) holds, take $u = (0, \ldots, 0, -1) \in \mathbb{Z}^{r+s}$. Then $P^* \cdot u \in \mathbb{Z}^{n+m}$ defines a divisor $\imath D_X$ with D_X a canonical divisor of shape (3.1) and we see $\zeta_X = 1$. □

Proposition 3.3.7. *Let $X = X(A, P)$ be a $\mathbb{Q}$-Gorenstein affine T-variety of Type 2 with at most log terminal singularities and canonical multiplicity $\zeta_X > 1$. Then we can choose P of shape 3.3.3 (ii) such that $l_{ij} = 1$ and $\nu_{ij} = 0$ holds for $i = 3, \ldots, r$ and $j = 1, \ldots, n_i$ and, moreover, P satisfies one of the following cases:*

Case	(l_{01}, l_{11}, l_{21})	(ν_0, ν_1, ν_2)	ζ_X	$\imath_X$
(i)	$(4, 3, 2)$	$\frac{1}{2}(\boldsymbol{\imath_X} + l_0,\ \boldsymbol{\imath_X} - \imath_X l_1,\ \boldsymbol{\imath_X} - l_2)$	2	0 mod 2
(ii)	$(3, 3, 2)$	$\frac{1}{3}(\boldsymbol{\imath_X} - l_0,\ \boldsymbol{\imath_X} + l_1,\ \boldsymbol{\imath_X} - \imath_X l_2)$	3	0 mod 3
(iii)	$(2k+1, 2, 2)$	$\frac{1}{4}(\boldsymbol{\imath_X} - \imath_X l_0,\ \boldsymbol{\imath_X} - l_1,\ \boldsymbol{\imath_X} + l_2)$	4	2 mod 4
(iv)	$(2k, 2, 2)$	$\frac{1}{2}(\boldsymbol{\imath_X} - l_0,\ \boldsymbol{\imath_X} + l_1,\ \boldsymbol{\imath_X} - \imath_X l_2)$	2	0 mod 2
(v)	$(k, 2, 2)$	$\frac{1}{2}(\boldsymbol{\imath_X} - \imath_X l_0,\ \boldsymbol{\imath_X} - l_1,\ \boldsymbol{\imath_X} + l_2)$	2	0 mod 2
(vi)	$(k_0, k_1, 1)$	$(\nu_0,\ \nu_1,\ \zeta_X^{-1}(\boldsymbol{\imath_X} - \imath_X l_2))$		

where $\boldsymbol{\imath_X}$ stands for a vector $(\imath_X, \ldots, \imath_X)$ of suitable length, and in Case (vi), all the numbers $(\imath_X - \nu_{0j_0}\zeta_X)/l_{0j_0}$ and $(\nu_{1j_1}\zeta_X - \imath_X)/l_{1j_1}$ are integral and coincide.

Proof. Since $X = X(A, P)$ has at most log terminal singularities, Theorem 3.1.3 guarantees that the Cox ring $\mathcal{R}(X) = R(A, P)$ is platonic. Thus, suitably exchanging data column blocks, we achieve $l_{ij} = 1$ for all $i \geq 3$. Next, we bring P in to the form of Proposition 3.3.3 (ii). Finally, subtracting the ν_{ij}-fold of the i-th row from the last one, we achieve $\nu_{ij} = 0$ for $i = 3, \ldots, r$.

Observe that our new matrix P still satisfies the conditions of Remark 3.3.5. For the integers μ_i defined there, we have

$$\mu_0 + \mu_1 + \mu_2 \ = \ \mu_3 \ = \ \ldots \ = \ \mu_r \ = \ \imath_X. \tag{3.2}$$

Moreover, for $i = 0, 1, 2$ set $\ell_i := l_{01}l_{11}l_{21}/l_{i1}$. Then, because of $\imath_X + \mu_i l_{ij} = \nu_{ij}\zeta_X$, we obtain

$$\gcd(\ell_0, \ell_1, \ell_2)^{-1} \sum_{i=0}^{2} \ell_i(\imath_X - \mu_i l_{ij}) \ = \ \alpha\zeta_X \quad \text{for some } \alpha \in \mathbb{Z}. \tag{3.3}$$

Finally, Remark 3.3.5 ensures

$$1 \ = \ \gcd(\mu_1, \ldots, \mu_r, \zeta_X, \imath_X) \ = \ \gcd(\mu_1, \mu_2, \zeta_X, \imath_X). \tag{3.4}$$

We will now apply these conditions to establish the table of the assertion. Since (l_{01}, l_{11}, l_{21}) is a platonic triple, we have to discuss the following cases.

Case 1: (l_{01}, l_{11}, l_{21}) equals $(5, 3, 2)$. Our task is to rule out this case. Using (3.2) and (3.3), we see that ζ_X divides

$$\imath_X \;=\; 31\imath_X - 30(\mu_0 + \mu_1 + \mu_2) \;=\; 6(\imath_X - 5\mu_0) + 10(\imath_X - 3\mu_1) + 15(\imath_X - 2\mu_2).$$

Consequently, (3.4) becomes $\gcd(\mu_1, \mu_2, \zeta_X) = 1$ and from $\imath_X - \mu_i l_{ij} = \nu_{ij}\zeta_X$ we infer that ζ_X divides $5\mu_0$, $3\mu_1$ and $2\mu_2$. This leaves us with the three possibilities $\zeta_X = 2, 3, 6$.

If $\zeta_X = 2$ holds, then ζ_X divides μ_0 and μ_1 but not μ_2; if $\zeta_X = 3$ holds, then ζ_X divides μ_0 and μ_2 but not μ_1. Both contradicts to the fact that ζ_X divides $\imath_X = \mu_0 + \mu_1 + \mu_2$. Thus, only $\zeta_X = 6$ is left. In that case, ζ_X must divide μ_0. Since ζ_X divides $\imath_X = \mu_0 + \mu_1 + \mu_2$, we see that ζ_X divides $\mu_1 + \mu_2$. Moreover, $\zeta_X \mid 3\mu_1$ gives $\mu_1 = 2\mu_1'$ and $\zeta_X \mid 2\mu_2$ gives $\mu_2 = 3\mu_2'$ with integers μ_1', μ_2'. Now, as $\zeta_X = 6$ divides $2\mu_1' + 3\mu_2'$, we obtain that μ_2' and hence μ_2 are even. This contradicts $\gcd(\mu_1, \mu_2, \zeta_X) = 1$.

Case 2: (l_{01}, l_{11}, l_{21}) equals $(4, 3, 2)$. Similarly as in the preceding case, we apply (3.2) and (3.3) to see that ζ_X divides

$$\imath_X \;=\; 13\imath_X - 12(\mu_0 + \mu_1 + \mu_2) \;=\; \frac{1}{2}(6(\imath_X - 4\mu_0) + 8(\imath_X - 3\mu_1) + 12(\imath_X - 2\mu_2)).$$

As before, we conclude $\gcd(\mu_1, \mu_2, \zeta_X) = 1$ and obtain that ζ_X divides $4\mu_0$, $3\mu_1$ and $2\mu_2$. This reduces to $\zeta_X = 2, 3, 6$.

If $\zeta_X = 3$ holds, then ζ_X divides μ_0 and μ_2 but not μ_1, contradicting the fact that ζ_X divides $\imath_X = \mu_0 + \mu_1 + \mu_2$. If $\zeta_X = 6$ holds, then we obtain $\mu_0 = 3\mu_0'$, $\mu_1 = 2\mu_1'$ and $\mu_2 = 3\mu_2'$ with suitable integers μ_i'. Since ζ_X divides $\imath_X = \mu_0 + \mu_1 + \mu_2$, we obtain that μ_2 is divisible by 3, contradicting $\gcd(\mu_1, \mu_2, \zeta_X) = 1$.

Thus, the only possibility left is $\zeta_X = 2$. We show that this leads to Case (i) of the assertion. Observe that μ_1 is even, μ_2 is odd because of $\gcd(\mu_1, \mu_2, \zeta_X) = 1$ and μ_2 is odd because $\imath_X = \mu_0 + \mu_1 + \mu_2$ is even. Recall that the vectors ν_i in the last row of P are given as

$$\nu_i \;=\; \frac{1}{\zeta_X}(\boldsymbol{\imath_X} - \mu_i l_i) \;=\; \frac{1}{2}\boldsymbol{\imath_X} - \frac{\mu_i}{2} l_i.$$

Thus, adding the $(-\mu_0 - \mu_2)/2$-fold of the first row and the $(\mu_2 - 1)/2$-fold of the second row to the last row brings P into the shape of Case (i).

Case 3: (l_{01}, l_{11}, l_{21}) equals $(3, 3, 2)$. As in the two preceding cases, we infer from (3.2) and (3.3) that ζ_X divides

$$\imath_X \;=\; 7\imath_X - 6(\mu_0 + \mu_1 + \mu_2) \;=\; \frac{1}{3}(6(\imath_X - 3\mu_0) + 6(\imath_X - 3\mu_1) + 9(\imath_X - 2\mu_2)).$$

Since $\gcd(\mu_1, \mu_2, \zeta_X) = 1$ and ζ_X divides $3\mu_0$, $3\mu_1$, $2\mu_2$, we are left with $\zeta_X = 2, 3, 6$. If $\zeta_X = 2$ or $\zeta_X = 6$ holds, then μ_0, μ_1 and $\imath_X = \mu_0 + \mu_1 + \mu_2$ must be even. Thus also μ_2 must be even, contradicting $\gcd(\mu_1, \mu_2, \zeta_X) = 1$.

Let $\zeta_X = 3$. We show that this leads to Case (ii) of the assertion. First, 3 divides μ_2 and $\imath_X = \mu_0 + \mu_1 + \mu_2$, hence also $\mu_0 + \mu_1$. Moreover, 3 divides neither μ_0 nor μ_1 because of $\gcd(\mu_1, \mu_2, \zeta_X) = 1$. Interchanging, if necessary, the data of the column blocks no. 0 and 1, we achieve that 3 divides $\mu_0 - 1$ and $\mu_1 + 1$. So, at the moment, the ν_i in the last row of P are of the form

$$\nu_i \;=\; \frac{1}{\zeta_X}(\boldsymbol{\imath_X} - \mu_i l_i) \;=\; \frac{1}{3}\boldsymbol{\imath_X} - \frac{\mu_i}{3} l_i.$$

Adding the $(\mu_1 + 1)/3$-fold of the first and the $(-\mu_0 - \mu_1)/3$-fold of the second to the last row of P, we arrive at Case (ii).

Case 4: (l_{01}, l_{11}, l_{21}) equals $(k, 2, 2)$ with $k \geq 3$ odd. Then (3.2) and (3.3) show that ζ_X divides

$$2\imath_X \;=\; (2+2k)\imath_X - 2k(\mu_0 + \mu_1 + \mu_2) \;=\; \frac{1}{2}(4(\imath_X - k\mu_0) + 2k(\imath_X - 2\mu_1) + 2k(\imath_X - 2\mu_2)).$$

Case 4.1: ζ_X doesn't divide $\imath_X$. Then we have $2\imath_X = \alpha\zeta_X$ with $\alpha \in \mathbb{Z}$ odd. Thus, ζ_X is even and $2\mu_i = \imath_X - \nu_{ij}\zeta_X$ implies that $4\mu_i$ is an odd multiple of ζ_X for $i = 1, 2$. In particular, 4 divides ζ_X. Moreover, (3.4) implies $\gcd(\mu_1, \mu_2, \zeta_X/2) = 1$ and we obtain $\zeta_X = 4$. That means $\imath_X \equiv 2 \mod 4$. Since $\zeta_X = 4$ divides $\imath_X - k\mu_0$ and k is odd, we conclude $\mu_0 \equiv 2 \mod 4$. Then $\mu_0 + \mu_1 + \mu_2 = \imath_X \equiv 2 \mod 4$ implies that 4 divides $\mu_1 + \mu_2$. Interchanging, if necessary, the data of the column blocks no. 1 and 2, we can assume $\mu_1 \equiv -\mu_2 \equiv 1 \mod 4$. Then, adding the $(\mu_1 - 1)/4$-fold of the first and the $(\mu_2 + 1)/4$-fold of the second to the last row of P, we arrive at Case (iii) of the assertion.

Case 4.2: ζ_X divides $\imath_X$. Then (3.4) becomes $\gcd(\mu_1, \mu_2, \zeta_X) = 1$. Since ζ_X divides $2\mu_1$ and $2\mu_2$, we see that $\zeta = 2$ holds and μ_1, μ_2 are odd. Adding the $(\mu_1 - 1)/2$-fold of the first and the $(\mu_2 + 1)/2$-fold of the second to the last row of P leads to Case (v) of the assertion.

Case 5: (l_{01}, l_{11}, l_{21}) equals $(k, 2, 2)$ with $k \geq 2$ even. Then (3.2) and (3.3) show that ζ_X divides

$$\imath_X \;=\; (k+1)\imath_X - k(\mu_0 + \mu_1 + \mu_2) \;=\; \frac{1}{4}(4(\imath_X - k\mu_0) + 2k(\imath_X - 2\mu_1) + 2k(\imath_X - 2\mu_2)).$$

As earlier, we conclude that $\zeta_X | 2\mu_i$ for $i = 1, 2$ and $\zeta_X = 2$. Since $\gcd(\mu_1, \mu_2, 2) = 1$ holds and $\mu_0 + \mu_1 + \mu_2 = \imath_X$ is even, two of the μ_i are be odd and one is even. If μ_1 and μ_2 are odd, then adding the $(\mu_1 - 1)/2$-fold of the first and the $(\mu_2 + 1)/2$-fold of the second to the last row of P leads to Case (v). Now, let μ_0 be odd. Interchanging, if necessary, the data of the column blocks no. 1 and 2, we achieve that μ_1 is odd. Then we add the $(\mu_1 + 1)/2$-fold of the first and the $(-\mu_0 - \mu_1)/2$-fold of the second to the last row of P and arrive at Case (iv) of the assertion.

Case 6. (l_{01}, l_{11}, l_{21}) equals $(k_0, k_1, 1)$, where $k_0, k_1 \in \mathbb{Z}_{>0}$. We subtract the ν_{21}-fold of the second row of P from the last one. Since $\nu_{21} = (\imath_X - \mu_2)/\zeta_X$ holds, we obtain

$\nu_2 = \zeta_X^{-1}(\boldsymbol{\imath_X} - \imath_X l_2)$. Moreover, (3.2) becomes $\mu_0 + \mu_1 = 0$. We arrive at Case (vi) of the assertion by observing

$$(\imath_X - \nu_{0j_0}\zeta_X)/l_{0j_0} \;=\; \mu_0 \;=\; -\mu_1 \;=\; (\nu_{1j_1}\zeta_X - \imath_X)/l_{1j_1}.$$

□

Example 3.3.8. We discuss the rational affine $\mathbb{C}^*$-surfaces X with at most log terminal singularities. First, the affine toric surfaces $X = \mathbb{C}^2/C_k$ show up here, where C_k is the cyclic group of order k acting diagonally. In terms of toric geometry, these surfaces are given as

$$X \;=\; \operatorname{Spec}\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^2], \qquad \sigma = \operatorname{cone}((k,\imath),\,(\imath,k+m)),$$

where $k, m \in \mathbb{Z}_{>0}$ with $\gcd(k,\imath) = \gcd(k+m,\imath) = 1$ and $\imath$ is the Gorenstein index of X; see [24, Chap. 10] for more background. Now consider a non-toric $\mathbb{C}^*$-surface $X = X(A,P)$ of Type 2. As a quotient of $\mathbb{C}^2$ by a finite group, X has finite divisor class group and thus P is a 3×3 matrix of the shape

$$P \;=\; \begin{bmatrix} -l_{01} & l_{11} & 0 \\ -l_{01} & 0 & l_{21} \\ d_{01} & d_{11} & d_{21} \end{bmatrix}.$$

Theorem 3.1.3 says that (l_{01}, l_{11}, l_{21}) is a platonic triple. Moreover, Corollary 3.3.6 and Proposition 3.3.7 provide us with constraints on the d_{i1}. Having in mind that P is of rank three with primitive columns, one directly arrives at the following possibilities, where $\zeta = \zeta_X$ is the canonical multiplicity and $\imath = \imath_X$ the Gorenstein index:

Type	P	ζ	$\imath$
$D_n^{1,\imath}$	$\begin{bmatrix} -n+2 & 2 & 0 \\ -n+2 & 0 & 2 \\ -n\imath+3\imath & \imath & \imath \end{bmatrix}$	1	$\gcd(\imath, 2n) = 1$
$D_{2n+1}^{2,\imath}$	$\begin{bmatrix} -2n+1 & 2 & 0 \\ -2n+1 & 0 & 2 \\ (1-n)\imath & \imath/2+1 & \imath/2-1 \end{bmatrix}$	2	$\gcd(\imath, 8n-4) = 4$
$E_6^{1,\imath}$	$\begin{bmatrix} -3 & 3 & 0 \\ -3 & 0 & 2 \\ -2\imath & \imath & \imath \end{bmatrix}$	1	$\gcd(\imath, 6) = 1$
$E_6^{3,\imath}$	$\begin{bmatrix} -3 & 3 & 0 \\ -3 & 0 & 2 \\ \imath/3-1 & \imath/3+1 & -\imath/3 \end{bmatrix}$	3	$\gcd(\imath, 18) = 9$
$E_7^{1,\imath}$	$\begin{bmatrix} -4 & 3 & 0 \\ -4 & 0 & 2 \\ -3\imath & \imath & \imath \end{bmatrix}$	1	$\gcd(\imath, 6) = 1$
$E_8^{1,\imath}$	$\begin{bmatrix} -5 & 3 & 0 \\ -5 & 0 & 2 \\ -4\imath & \imath & \imath \end{bmatrix}$	1	$\gcd(\imath, 30) = 1$

For geometric details on these surfaces, we refer to the work of Brieskorn [20], and, in the context of the McKay Correspondence, Wunram [88] and Wemyss [86].

3.4 Geometry of the total coordinate space

We take a closer look at the geometry of the total coordinate space $\overline{X}$ of a T-variety X of complexity one. The first result says in particular that $\overline{X}$ is Gorenstein and canonical provided that X is log terminal and affine.

Proposition 3.4.1. *Let $R(A, P_0)$ be a platonic ring of Type 2. Then the affine variety $\overline{X} = \operatorname{Spec} R(A, P_0)$ is Gorenstein and has at most canonical singularities.*

Proof. Adding suitable rows, we complement the matrix P_0 to a square matrix P of full rank with last row $(\mathbf{1} - (r-1)l_0, \mathbf{1}, \dots, \mathbf{1})$, where $\mathbf{1}$ indicates vectors of length n_i with all entries equal to one; this is possible, because the last row is not in the row space of P_0. Then $X = X(A, P)$ is a $\mathbb{Q}$-factorial affine T-variety. Theorem 3.1.3 tells us that X has at most log terminal singularities and Corollary 3.3.6 ensures that X is Gorenstein. Thus, X has at most canonical singularities. Since $\overline{X} \to X$ is finite with ramification locus of codimension at least two, we can use [53, Thm. 6.2.9] to see that $\overline{X}$ is Gorenstein with at most canonical singularities. $\square$

Now we investigate the generic quotient Y of $\overline{X}$ by the action of the unit component $H_0^0 \subseteq H_0$, in other words, the smooth projective curve Y with function field $\mathbb{C}(Y) = \mathbb{C}(\overline{X})^{H_0^0}$. Note that the curve Y occurs also in [3], where it carries the polyhedral divisor of the Cox ring.

Definition 3.4.2. Consider the defining matrix P_0 of a ring $R(A, P_0)$ of Type 2 and the vectors $l_i = (l_{i1}, \dots, l_{in_i})$ occuring in the rows of P_0. Set

$$\mathfrak{l}_i := \gcd(l_{i1}, \dots, l_{in_i}), \qquad \mathfrak{l} := \gcd(\mathfrak{l}_0, \dots, \mathfrak{l}_r), \qquad \mathfrak{l}_{ij} := \gcd(\mathfrak{l}^{-1}\mathfrak{l}_i, \mathfrak{l}^{-1}\mathfrak{l}_j),$$

$$\bar{\mathfrak{l}} := \operatorname{lcm}(\mathfrak{l}_0, \dots, \mathfrak{l}_r), \qquad b_i := \mathfrak{l}_i^{-1}\bar{\mathfrak{l}}, \qquad b(i) := \gcd(b_j;\ j \neq i).$$

Theorem 3.4.3. *Let $R(A, P_0)$ be of Type 2 and consider the action of the unit component $H_0^0 \subseteq H_0$ of the quasitorus $H_0 = \operatorname{Spec} \mathbb{C}[K_0]$ on $\overline{X} = \operatorname{Spec} R(A, P_0)$. Then the smooth projective curve Y with function field $\mathbb{C}(Y) = \mathbb{C}(\overline{X})^{H_0^0}$ is of genus*

$$g(Y) \;=\; \frac{\mathfrak{l}_0 \cdots \mathfrak{l}_r}{2\bar{\mathfrak{l}}} \left((r-1) - \sum_{i=0}^{r} \frac{b(i)}{\mathfrak{l}_i} \right) + 1.$$

Lemma 3.4.4. *Let $R(A, P_0)$ be of Type 2, consider the degree $u := \deg(g_0) \in K_0$ of the defining relations and the subgroup*

$$K_0(u) \;:=\; \{w \in K_0;\ \alpha w \in \mathbb{Z}u \text{ for some } \alpha \in \mathbb{Z}_{>0}\} \;\subseteq\; K_0.$$

Then the Veronese subalgebra $R(A, P_0)(u)$ of $R(A, P_0)$ associated with $K_0(u)$ of K_0 is generated by the monomials $T_0^{l_0/\mathfrak{l}_0}, \dots, T_r^{l_r/\mathfrak{l}_r}$.

Proof. First, observe that every element of $R(A, P_0)(u)$ is a polynomial in the variables T_{ij}. Now consider a monomial T^l in the T_{ij} of degree $w \in K_0(u)$, where $l \in \mathbb{Z}^{n+m}$. Then $\alpha w \in \beta_0 u$ holds for some $\alpha \in \mathbb{Z}_{>0}$ and $\beta_0 \in \mathbb{Z}$. Moreover, there are $\beta_1, \ldots, \beta_r \in \mathbb{Z}$ with

$$\alpha l = \beta_0 l'_0 + \beta_1(l'_0 - l'_1) + \ldots + \beta_r(l'_0 - l'_r), \text{ where } l'_i := l_{i1}e_{i1} + \ldots + l_{in_i}e_{in_i},$$

reflecting the fact that $\alpha l - \beta_0 l'_0$ lies in the row space of P_0. Consequently, we obtain $l = \beta'_0 l'_0 + \ldots + \beta'_r l'_r$ for suitable $\beta'_i \in \mathbb{Q}$. Since l has only non-negative integer entries, we conclude that every β'_i is a non-negative integral multiple of $\mathfrak{l}_i^{-1}$. Thus, T^l is a monomial in the $T_i^{l_i/\mathfrak{l}_i}$. The assertion follows. □

Proof of Theorem 3.4.3. The curve Y occurs as a GIT-quotient: $Y = \overline{X}^{ss}(u^0)/H_0^0$, where $u^0 \in \mathbb{X}(H_0^0)$ represents the character induced by $u = \deg(g_0) \in K_0 = \mathbb{X}(H_0)$. In other words, we have $Y = \operatorname{Proj} R(A, P_0)(u^0)$ with the Veronese subalgebra defined by u^0. We may replace u^0 with

$$w^0 \;:=\; \frac{1}{\bar{\mathfrak{l}}} u^0 \;\in\; \mathbb{X}(H_0^0).$$

Then $R(A, P_0)(u^0)$ is replaced with $R(A, P_0)(w^0)$ which in turn equals the Veronese subalgebra treated in Lemma 3.4.4. Moreover, the generators $T_i^{l_i/\mathfrak{l}_i} \in R(A, P_0)(w^0)$ are of degree $b_i w^0 \in \mathbb{X}(H_0^0)$, respectively. We obtain a closed embedding into a weighted projective space

$$Y \;=\; V(h_0, \ldots, h_{r-2}) \;\subseteq\; \mathbb{P}(b_0, \ldots, b_r), \qquad h_i \;:=\; \det \begin{bmatrix} T_i^{\mathfrak{l}_i} & T_{i+1}^{\mathfrak{l}_{i+1}} & T_{i+2}^{\mathfrak{l}_{i+2}} \\ a_i & a_{i+1} & a_{i+2} \end{bmatrix},$$

where the h_i generate the ideal of relations among the generators of the Veronese subalgebra $R(A, P_0)(w^0)$. The idea is now to construct a ramified covering $Y' \to Y$ with a suitable curve Y' and then to compute the genus of Y via the Hurwitz formula. Consider

$$Y' \;=\; V(h'_0, \ldots, h'_{r-2}) \;\subseteq\; \mathbb{P}_r, \qquad h'_i \;:=\; \det \begin{bmatrix} T_i^{\bar{\mathfrak{l}}} & T_{i+1}^{\bar{\mathfrak{l}}} & T_{i+2}^{\bar{\mathfrak{l}}} \\ a_i & a_{i+1} & a_{i+2} \end{bmatrix}.$$

The $Y' \subseteq \mathbb{P}_r$ is a smooth complete intersection curve. Computing the genus of Y' according to [39], we obtain

$$g(Y') \;=\; \frac{1}{2}((r-1)\bar{\mathfrak{l}}^r - (r+1)\bar{\mathfrak{l}}^{r-1}) + 1.$$

The morphism $\mathbb{P}_r \to \mathbb{P}(b_0, \ldots, b_r)$ sending $[z_0, \ldots, z_r]$ to $[z_0^{b_0}, \ldots, z_r^{b_r}]$ restricts to a morphism $Y' \to Y$ of degree $b_0 \cdots b_r$. The intersection $Y \cap U_i$ with the i-th coordinate hyperplane $U_i \subseteq \mathbb{P}_r$ contains precisely $\bar{\mathfrak{l}}^{r-1}$ points and each of these points has ramification order $b_i \cdot b(i) - 1$. Outside the U_i, the morphism $Y' \to Y$ is unramified. The Hurwitz formula then gives $g(Y)$. □

We now use Theorem 3.4.3 to characterize rationality of $\overline{X} = \operatorname{Spec} R(A, P_0)$. For the special case of Pham-Brieskorn surfaces, the following statement has been obtained in [9].

Proposition 3.4.5. *Let $R(A, P_0)$ be of Type 2 with $r = 2$, that means that $\overline{X} = \operatorname{Spec} R(A, P_0)$ is given as*

$$\overline{X} \;\cong\; V(T_{01}^{l_{01}} \cdots T_{0n_0}^{l_{0n_0}} + T_{11}^{l_{11}} \cdots T_{1n_1}^{l_{1n_1}} + T_{21}^{l_{21}} \cdots T_{2n_2}^{l_{2n_2}}) \;\subseteq\; \mathbb{C}^n.$$

Then the hypersurface $\overline{X}$ is rational if and only if one of the following conditions holds:

(i) *there are pairwise coprime positive integers c_0, c_1, c_2 and a positive integer s such that, after suitable renumbering, one has*

$$\gcd(c_2, s) = 1, \quad \mathfrak{l}_0 = sc_0, \quad \mathfrak{l}_1 = sc_1, \quad \mathfrak{l}_2 = c_2;$$

(ii) *there are pairwise coprime positive integers c_0, c_1, c_2 such that*

$$\mathfrak{l}_0 = 2c_0, \quad \mathfrak{l}_1 = 2c_1, \quad \mathfrak{l}_2 = 2c_2.$$

Lemma 3.4.6. *For $i = 0, 1, 2$, let $l_i = (l_{i1}, \ldots, l_{in_i})$ be tuples of positive integers. Define $\mathfrak{l}$, $\mathfrak{l}_i$ and $\mathfrak{l}_{ij}$ as in Definition 3.4.2 for $r = 2$. Then the following statements are equivalent.*

(i) *We have $\mathfrak{l}(\mathfrak{l}\mathfrak{l}_{01}\mathfrak{l}_{02}\mathfrak{l}_{12} - (\mathfrak{l}_{01} + \mathfrak{l}_{02} + \mathfrak{l}_{12})) = -2$.*

(ii) *One of the following two conditions holds:*

  (a) there are pairwise coprime positive integers c_0, c_1, c_2 and a positive integer s such that, after suitable renumbering, one has

$$\gcd(c_2, s) = 1, \quad \mathfrak{l}_0 = sc_0, \quad \mathfrak{l}_1 = sc_1, \quad \mathfrak{l}_2 = c_2;$$

  (b) there are pairwise coprime positive integers c_0, c_1, c_2 such that

$$\mathfrak{l}_0 = 2c_0, \quad \mathfrak{l}_1 = 2c_1, \quad \mathfrak{l}_2 = 2c_2.$$

Proof. If (ii) holds, then a simple computation shows that (i) is valid. Now, assume that (i) holds. Then the following cases have to be considered.

Case 1. We have $\mathfrak{l} = 1$. Then $\mathfrak{l}_{01}(\mathfrak{l}_{02}\mathfrak{l}_{12} - 1) = \mathfrak{l}_{02} + \mathfrak{l}_{12} - 2$ holds. From this we deduce

$$\begin{aligned} \mathfrak{l}_{01}(\mathfrak{l}_{02}\mathfrak{l}_{12} - 1) &= (\mathfrak{l}_{01} - 1)(\mathfrak{l}_{02}\mathfrak{l}_{12} - 1) + (\mathfrak{l}_{02} - 1)(\mathfrak{l}_{12} - 1) + \mathfrak{l}_{02} + \mathfrak{l}_{12} - 2 \\ &\geq \mathfrak{l}_{02} + \mathfrak{l}_{12} - 2, \end{aligned}$$

where equality holds if and only if at least two of $\mathfrak{l}_{01}, \mathfrak{l}_{02}, \mathfrak{l}_{12}$ equal one. So, we arrive at Condition (a).

Case 2. We have $\mathfrak{l} = 2$. Then we have $\mathfrak{l}_{01}(2\mathfrak{l}_{02}\mathfrak{l}_{12} - 1) + 1 = \mathfrak{l}_{02} + \mathfrak{l}_{12}$. In this situation, we conclude

$$\begin{aligned} \mathfrak{l}_{01}(2\mathfrak{l}_{02}\mathfrak{l}_{12} - 1) + 1 &= (\mathfrak{l}_{01} - 1)(2\mathfrak{l}_{02}\mathfrak{l}_{12} - 1) + \mathfrak{l}_{02}\mathfrak{l}_{12} \\ &\qquad + (\mathfrak{l}_{02} - 1)(\mathfrak{l}_{12} - 1) + \mathfrak{l}_{02} + \mathfrak{l}_{12} - 1 \\ &\geq \mathfrak{l}_{02} + \mathfrak{l}_{12}, \end{aligned}$$

where equality holds if and only if we have $\mathfrak{l}_{01} = \mathfrak{l}_{02} = \mathfrak{l}_{12} = 1$. Thus, we arrive at Condition (b). $\square$

Proof of Proposition 3.4.5. First, observe that $\overline{X}$ is rational if and only if Y is rational or, in other words, of genus zero. For $r = 2$, Theorem 3.4.3 gives

$$g(Y) \;=\; \frac{\mathfrak{l}}{2}(\mathfrak{l}\mathfrak{l}_{01}\mathfrak{l}_{02}\mathfrak{l}_{12} - \mathfrak{l}_{01} - \mathfrak{l}_{02} - \mathfrak{l}_{12}) + 1.$$

Thus, according to Lemma 3.4.6, condition $g(Y) = 0$ holds if and only if (i) or (ii) of the proposition holds. □

Remark 3.4.7. If the defining polynomial in Proposition 3.4.5 is classically homogeneous, then it defines a projective hypersurface $X' \subseteq \mathbb{P}^{n-1}$ and the following statements are equivalent.

(i) X' is rational.
(ii) $\mathrm{Cl}(X')$ is finitely generated.
(iii) Condition 3.4.5 (i) or (ii) holds.

Corollary 3.4.8. *Let $R(A, P_0)$ be of Type 2. Then $\overline{X} = \operatorname{Spec} R(A, P_0)$ is rational if and only if one of the following conditions holds:*

(i) *We have $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = 1$ for all $0 \le i < j \le r$, in other words, $R(A, P_0)$ is factorial.*
(ii) *There are $0 \le i < j \le r$ with $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) > 1$ and $\gcd(\mathfrak{l}_u, \mathfrak{l}_v) = 1$ whenever $v \notin \{i, j\}$.*
(iii) *There are $0 \le i < j < k \le r$ with $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = \gcd(\mathfrak{l}_i, \mathfrak{l}_k) = \gcd(\mathfrak{l}_j, \mathfrak{l}_k) = 2$ and $\gcd(\mathfrak{l}_u, \mathfrak{l}_v) = 1$ whenever $v \notin \{i, j, k\}$.*

Lemma 3.4.9. *Let A, P_0 be defining data of Type 2, enhance A to A' by attaching a further column and P_0 to P_0' by attaching l_{r+1} to $l_0, \ldots, l_r$. If $\gcd(\mathfrak{l}_i, \mathfrak{l}_{r+1}) = 1$ holds for $i = 0, \ldots, r$, then we have $g(Y) = g(Y')$ for the curves associated with $R(A, P)$ and $R(A', P_0')$ respectively.*

Proof. Denote the numbers arising from P' in the sense of Definition 3.4.2 by $\mathfrak{l}_i'$, $\mathfrak{l}'$ etc. Then we have

$$\bar{\mathfrak{l}}' \;=\; \bar{\mathfrak{l}}\mathfrak{l}_{r+1}, \quad b'(i) \;=\; \gcd(\bar{\mathfrak{l}}, \bar{\mathfrak{l}}'/\bar{\mathfrak{l}}_j;\ j \ne i) \;=\; b(i), \quad i = 0, \ldots, r,$$

$$b(r+1) \;=\; \gcd(\bar{\mathfrak{l}}'/\bar{\mathfrak{l}}_0, \ldots, \bar{\mathfrak{l}}'/\bar{\mathfrak{l}}_r) \;=\; \mathfrak{l}_{r+1}.$$

Plugging these identities into the genus formula of Theorem 3.4.3, we directly obtain $g(Y') = g(Y)$. □

Lemma 3.4.10. *Let $R(A, P_0)$ be of Type 2 and assume that the curve Y associated with $R(A, P)$ is of genus zero. Then there are $0 \le i \le j \le k \le r$ with $\gcd(\mathfrak{l}_u, \mathfrak{l}_v) = 1$ whenever $v \notin \{i, j, k\}$.*

Proof. According to Theorem 3.4.3, the fact that the curve Y associated with $R(A, P)$ is of genus zero implies

$$\sum_{i=0}^{r} \frac{b(i)}{\mathfrak{l}_i} \;=\; (r-1) + \frac{2\bar{\mathfrak{l}}}{\mathfrak{l}_0 \cdots \mathfrak{l}_r} \;>\; r - 1.$$

As $b(i)$ divides $\mathfrak{l}_i$, we see that $b(i) \neq \mathfrak{l}_i$ can happen at most three times. Moreover, $b(i) = \mathfrak{l}_i$ is equivalent to $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = 1$ for all $j \neq i$. □

Proof of Corollary 3.4.8. We may assume that the indices i, j and k of Lemma 3.4.10 are $0, 1$ and 2. Then Lemma 3.4.9 says that $\overline{X}$ is rational if and only if the trinomial hypersurface defined by the exponent vectors l_0, l_1, l_2 is rational. Thus, Proposition 3.4.5 gives the assertion. □

Corollary 3.4.11. *Let $R(A, P_0)$ be a platonic ring of Type 2. Then $\overline{X} = \operatorname{Spec} R(A, P_0)$ is rational.*

Remark 3.4.12. It may happen that for a rational T-variety X of complexity one, the total coordinate space $\overline{X}$ is rational, but the total coordinate space of $\overline{X}$ is not rational any more. For instance consider

$$X_3 \ := \ V(T_1^4 + T_2^4 + T_3^4) \ \subseteq \ \mathbb{C}^3.$$

Then, according to Proposition 3.4.5, the surface X_3 is not rational. Moreover, X_3 is the total coordinate space of an affine rational $\mathbb{C}^*$-surface X_2 with defining matrix

$$P_2 \ = \ \begin{bmatrix} -4 & 4 & 0 \\ -4 & 0 & 4 \\ -3 & 1 & 1 \end{bmatrix}.$$

The divisor class group of X_2 is $\operatorname{Cl}(X_2) = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ and the $\operatorname{Cl}(X_2)$-grading of the Cox ring $\mathcal{R}(X_2) = \mathbb{C}[T_1, T_2, T_3]/\langle T_1^4 + T_2^4 + T_3^4 \rangle$ is given by

$$\deg(T_1) \ = \ (\overline{1}, \overline{1}), \quad \deg(T_2) \ = \ (\overline{1}, \overline{2}), \quad \deg(T_3) \ = \ (\overline{2}, \overline{1}).$$

For an equation for X_2, compute the degree zero subalgebra of $\mathcal{R}(X_2)$: it has three generators S_1, S_2, S_3 and $S_1^3 + S_2^3 + S_3^4$ as defining relation. Thus,

$$X_2 \ \cong \ V(S_1^3 + S_2^3 + S_3^4) \ \subseteq \ \mathbb{C}^3.$$

To obtain a rational affine $\mathbb{C}^*$-surface having X_2 as its total coordinate space, we take X_1, defined by

$$P_1 \ := \ \begin{bmatrix} -3 & 3 & 0 \\ -3 & 0 & 4 \\ -2 & 1 & 1 \end{bmatrix}.$$

The divisor class group of X_1 is $\operatorname{Cl}(X_1) = \mathbb{Z}/3\mathbb{Z}$ and the $\operatorname{Cl}(X_1)$-grading of the Cox ring $\mathcal{R}(X_1) = \mathbb{C}[S_1, S_2, S_3]/\langle S_1^3 + S_2^3 + S_3^4 \rangle$ is given by

$$\deg(T_1) \ = \ \overline{1}, \quad \deg(T_2) \ = \ \overline{2}, \quad \deg(T_3) \ = \ \overline{0}.$$

We have constructed a chain of total coordinate spaces $X_3 \to X_2 \to X_1$, where X_1 is a rational affine $\mathbb{C}^*$-surface, X_2 is rational and X_3 not.

Finally, we determine the factor group of the maximal quasitorus by its unit component acting on a given trinomial hypersurface; the proof is a direct consequence of the subsequent lemma.

Proposition 3.4.13. *Let $R(A,P)$ be any ring of Type 2, where $r = 2$. Then, for the quasitorus H_0 acting on the corresponding trinomial hypersurface*

$$\overline{X} \;\cong\; V(T_{01}^{l_{01}} \cdots T_{0n_0}^{l_{0n_0}} + T_{11}^{l_{11}} \cdots T_{1n_1}^{l_{1n_1}} + T_{21}^{l_{21}} \cdots T_{2n_2}^{l_{2n_2}}) \subseteq \mathbb{C}^n,$$

the factor group H_0/H_0^0 by the unit component $H_0^0 \subseteq H_0$ is isomorphic to the product of cyclic groups $C(\mathfrak{l}) \times C(\mathfrak{l}\mathfrak{l}_{01}\mathfrak{l}_{02}\mathfrak{l}_{12})$.

Lemma 3.4.14. *Consider a matrix P_0 with $m = 0$ and $r = 2$ as in Type 2 of Construction 2.1.4:*

$$P_0 \;=\; \begin{bmatrix} -l_0 & l_1 & 0 \\ -l_0 & 0 & l_2 \end{bmatrix}.$$

As earlier, set $\mathfrak{l}_i = \gcd(l_{i1}, \ldots, l_{in_i})$. Then, with $\mathfrak{l}_{ij} = \gcd(\mathfrak{l}_i, \mathfrak{l}_j)$ and $\mathfrak{l} = \gcd(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_2)$, we obtain

$$K_0^{\mathrm{tors}} \;=\; (\mathbb{Z}^n/\mathrm{im}(P_0^*))^{\mathrm{tors}} \;\cong\; C(\mathfrak{l}) \times C(\mathfrak{l}\mathfrak{l}_{01}\mathfrak{l}_{02}\mathfrak{l}_{12}).$$

Proof. Suitable elementary column operations to P_0 transform the entries l_i to $(\mathfrak{l}_i, 0, \ldots, 0)$. Thus, $K_0^{\mathrm{tors}} \cong (\mathbb{Z}^3/\mathrm{im}(P_1^*))^{\mathrm{tors}}$ holds with the 2×3 matrix

$$P_1 \;:=\; \begin{bmatrix} -\mathfrak{l}_0 & \mathfrak{l}_1 & 0 \\ -\mathfrak{l}_0 & 0 & \mathfrak{l}_2 \end{bmatrix}.$$

The determinantal divisors of P_0 are $\gcd(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_2)$ and $\gcd(\mathfrak{l}_0\mathfrak{l}_1, \mathfrak{l}_0\mathfrak{l}_2, \mathfrak{l}_1\mathfrak{l}_2)$. Thus, the invariant factors of P_0 are $\mathfrak{l}$ and $\mathfrak{l}\mathfrak{l}_{01}\mathfrak{l}_{02}\mathfrak{l}_{12}$; see [72]. □

3.5 Proof of Theorems 3.1.5 and 3.1.6

We are ready to prove the main results of this chapter. The proof of Theorem 3.1.5 will be in fact constructive in the sense that it allows to compute the defining equations of the Cox ring in every iteration step; see Proposition 3.5.6.

Remark 3.5.1. Let $R(A,P)$ resp. $R(A,P_0)$ be a ring of Type 2. Applying suitable admissible operations, one achieves that P resp. P_0 (is *ordered* in the sense that $l_{i1} \geq \ldots \geq l_{in_i}$ for all $i = 0, \ldots, r$ and $l_{01} \geq \ldots \geq l_{r1}$ hold. For an ordered P resp. P_0, the ring $R(A,P)$ resp. $R(A,P_0)$ is platonic if and only if (l_{01}, l_{11}, l_{21}) is a platonic triple and $l_{i1} = 1$ holds for $i \geq 3$.

Definition 3.5.2. The *leading platonic triple* of a ring $R(A,P)$ resp. $R(A,P_0)$ of Type 2 is the triple (l_{01}, l_{11}, l_{21}) obtained after ordering P resp. P_0.

Lemma 3.5.3. *Let $R(A, P_0)$ be of Type 2 and platonic such that $l_{i1} \geq \ldots \geq l_{in_i}$ holds for all i and $l_{i1} = 1$ for $i \geq 3$. Moreover, assume $\gcd(\mathfrak{l}_1, \mathfrak{l}_2) = \mathfrak{l}$. Then, with $K_0 = \mathbb{Z}^{n+m}/\mathrm{im}(P_0^*)$, the kernel of $\mathbb{Z}^{n+m} \to K_0/K_0^{\mathrm{tors}}$ is generated by the rows of the matrix*

$$P_1 \; := \; \begin{bmatrix} \frac{-1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)} l_0 & \frac{1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)} l_1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \frac{-1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_2)} l_0 & 0 & \frac{1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_2)} l_2 & 0 & 0 & & & \\ -l_0 & 0 & & \mathbf{1} & 0 & \vdots & & \vdots \\ \vdots & & & \vdots & \ddots & \vdots & & \\ -l_0 & 0 & \ldots & 0 & \mathbf{1} & 0 & \ldots & 0 \end{bmatrix},$$

where, as before, the symbols $\mathbf{1}$ indicate vectors of length n_i with all entries equal to one.

Proof. Observe that the rows of P_0 generate a sublattice of finite index in the row lattice P_1. Thus, we have a commutative diagram

$$\begin{tikzcd} K_0 \arrow[rr] \arrow[dr] & & K_0/K_0^{\mathrm{tors}} \\ & \mathbb{Z}^{n+m}/\mathrm{im}(P_1^*) \arrow[ur] & \end{tikzcd}$$

It suffices to show, that $\mathbb{Z}^{n+m}/\mathrm{im}(P_1^*)$ is torsion free. Applying suitable elementary column operations to P_1, reduces the problem to showing that for the 2×3 matrix

$$\begin{bmatrix} \frac{\mathfrak{l}_0}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)} & \frac{\mathfrak{l}_1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)} & 0 \\ \frac{\mathfrak{l}_0}{\gcd(\mathfrak{l}_0,\mathfrak{l}_2)} & 0 & \frac{\mathfrak{l}_2}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)} \end{bmatrix},$$

all determinantal divisors equal one. The entries of the above matrix are coprime and its 2×2 minors are

$$\frac{\mathfrak{l}_0\mathfrak{l}_2}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)\gcd(\mathfrak{l}_0,\mathfrak{l}_2)}, \qquad \frac{\mathfrak{l}_1\mathfrak{l}_2}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)\gcd(\mathfrak{l}_0,\mathfrak{l}_2)}, \qquad \frac{\mathfrak{l}_0\mathfrak{l}_1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)\gcd(\mathfrak{l}_0,\mathfrak{l}_2)}.$$

up to sign. By assumption, we have $\gcd(\mathfrak{l}_1, \mathfrak{l}_2) = \mathfrak{l}$. Consequently, we obtain

$$\gcd(\mathfrak{l}_0\mathfrak{l}_2, \mathfrak{l}_0\mathfrak{l}_1, \mathfrak{l}_1\mathfrak{l}_2) \; = \; \gcd(\mathfrak{l}_0\mathfrak{l}, \mathfrak{l}_1\mathfrak{l}_2) \; = \; \gcd(\mathfrak{l}_0,\mathfrak{l}_1)\gcd(\mathfrak{l}_0,\mathfrak{l}_2)$$

and therefore the second determinantal divisor equals one. As remarked, the first one equals one as well and the assertion follows. $\square$

Lemma 3.5.4. *Let $R(A, P_0)$ be of Type 2 and $\overline{X} = \operatorname{Spec} R(A, P_0)$. Then, for the generator T_{01} of $R(A, P_0)$, we have*

$$V(\overline{X}, T_{01}) \; \cong \; V(T_{01}) \cap V(T_1^{l_1} - T_i^{l_i};\ i = 2, \ldots, r) \; \subseteq \; \mathbb{C}^{n+m}.$$

In particular, the number of irreducible components of $V(\overline{X}, T_{01})$ *equals the product of the invariant factors of the matrix*

$$\begin{bmatrix} -\mathfrak{l}_1 & \mathfrak{l}_2 & & 0 \\ \vdots & & \ddots & \\ -\mathfrak{l}_1 & 0 & & \mathfrak{l}_r \end{bmatrix}.$$

Proof. First observe that the ideal $\langle T_{01}, g_0, \ldots, g_{r-2} \rangle \subseteq \mathbb{C}[T_{ij}, S_k]$ is generated by binomials which can be brought into the above form by scaling the variables appropriately. Now consider the homomorphism of tori

$$\pi\colon \mathbb{T}^{n_1+\ldots+n_r} \ \to \ \mathbb{T}^{r-1}, \qquad (t_1, \ldots, t_r) \ \mapsto \ \left(\frac{t_2^{l_2}}{t_1^{l_1}}, \ldots, \frac{t_r^{l_r}}{t_1^{l_1}} \right).$$

Then the number of connected components of $\ker(\pi)$ equals the product of the invariant factors of the above matrix. Moreover, $\mathbb{T}^{n_0-1} \times \ker(\pi) \times \mathbb{T}^m$ is isomorphic to $V(\overline{X}, T_{01}) \cap \mathbb{T}^{n+m}$. Finally, one directly checks that $V(\overline{X}, T_{01})$ has no further irreducible components outside $\mathbb{T}^{n+m}$. $\square$

Lemma 3.5.5. *Let* $R(A, P_0)$ *be of Type 2 and platonic. Assume that* P_0 *is ordered. Then the number* $c(i)$ *of irreducible components of* $V(\overline{X}, T_{ij})$ *is given as*

i	0	1	2	≥ 3
$c(i)$	$\gcd(\mathfrak{l}_1, \mathfrak{l}_2)$	$\gcd(\mathfrak{l}_0, \mathfrak{l}_2)$	$\gcd(\mathfrak{l}_0, \mathfrak{l}_1)$	$\mathfrak{l}^2 \mathfrak{l}_{01} \mathfrak{l}_{02} \mathfrak{l}_{12}$

Proof. Suitable admissible operations turn T_{ij} to T_{01}. Then the number of components is computed via Lemma 3.5.4. $\square$

Proposition 3.5.6. *Let* $R(A, P_0)$ *be of Type 2, platonic and non-factorial. Assume that* P_0 *is ordered and let* P_1 *be as in Lemma 3.5.3. Set*

$$n_{i,1}, \ldots, n_{i,c(i)} \ := \ n_i, \qquad l_{ij,1}, \ldots, l_{ij,c(i)} \ := \ \gcd((P_1)_{1,ij}, \ldots, (P_1)_{r,ij}).$$

The $l_{i,\alpha} := (l_{i1,\alpha}, \ldots, l_{in_i,\alpha}) \in \mathbb{Z}^{n_{i,\alpha}}$ *build up an* $r' \times (n' + m)$ *matrix* P_0'*, where* $n' := c(0)n_0 + \ldots + c(r)n_r$*. With a suitable matrix* A'*, the following holds.*

(i) *The affine variety* $\operatorname{Spec} R(A', P_0')$ *is the total coordinate space of the affine variety* $\operatorname{Spec} R(A, P_0)$*,*

(ii) *The leading platonic triple (l.p.t.) of* $R(A', P_0')$ *can be expressed in terms of that of* $R(A, P_0)$ *as*

l.p.t. of $R(A, P_0)$	*l.p.t. of* $R(A', P_0')$
$(4, 3, 2)$	$(3, 3, 2)$
$(3, 3, 2)$	$(2, 2, 2)$
$(y, 2, 2)$	$(z, z, 1)$ *or* $(\frac{y}{2}, 2, 2)$
$(x, y, 1)$	$\left(\frac{x}{\gcd(\mathfrak{l}_0, \mathfrak{l}_1)}, \frac{y}{\gcd(\mathfrak{l}_0, \mathfrak{l}_1)}, 1\right)$

Proof. We compute the Cox ring of $\overline{X} = \operatorname{Spec} R(A, P_0)$ according to [6, Thm. 4.4.1.6]; use Corollary 1.9 [50] to obtain the statement given there also in the affine case. That means that we have to figure out which invariant divisors are identified under the rational map onto the curve Y with function field $\mathbb{C}(\overline{X})^{H_0^0}$ and we have to determine the orders of isotropy groups of invariant divisors.

Let P_1 be as in Lemma 3.5.3. Then the torus H_0^0 acts diagonally on $\mathbb{C}^{n+m}$ with weights provided by the projection $Q_1\colon \mathbb{Z}^{n+m} \to K_0^0$, where $K_0^0 = \mathbb{Z}^{n+m}/\mathrm{im}(P_1^*)$ equals the character group of H_0^0. Consider the commutative diagram

$$\begin{tikzcd}
\overline{X}_0 \arrow[d] & \subseteq & \mathbb{C}_0^{n+m} \arrow[d] \\
\overline{X}_0/H_0^0 \arrow[d] & \subseteq & \mathbb{C}_0^{n+m}/H_0^0 \arrow[d] \\
Y & \subseteq & \mathbb{P}
\end{tikzcd}$$

where $\overline{X}_0 \subseteq \overline{X}$ and $\mathbb{C}_0^{n+m} \subseteq \mathbb{C}^{n+m}$ denote the open H_0^0-invariant subsets obtained by removing all coordinate hyperplanes $V(S_k)$ and all intersections $V(T_{i_1j_1}, T_{i_2j_2})$ with $(i_1, j_1) \neq (i_2, j_2)$ from $\mathbb{C}^{n+m}$. Moreover, the geometric quotient spaces in the middle row are possibly non-separated and the maps to the lower row are separation morphisms.

We determine the orders of isotropy groups. Every point in $\mathbb{T}^{n+m}$ has trivial H_0^0-isotropy. Thus, we only have to look what happens on the sets $V(T_{ij}) \cap \mathbb{C}_0^{n+m}$. According to [6, Prop. 2.1.4.2], the order of isotropy group of H_0^0 at any point $x \in V(T_{ij}) \cap \mathbb{C}_0^{n+m}$ equals the greatest common divisor of the entries of the ij-th column of P_1:

$$|H_{0,x}^0| \;=\; l'_{ij} \;:=\; \gcd((P_1)_{1,ij}, \ldots, (P_1)_{r,ij}) \quad \text{for all } x \in V(T_{ij}) \cap \mathbb{C}_0^{n+m}.$$

Now we figure out which H_0^0-invariant divisors of $\overline{X}_0$ are identified under the map $\overline{X}_0 \to Y$. Lemma 3.5.5 provides us explicit numbers $c(0), \ldots, c(r)$ such that for fixed i and $j = 1, \ldots, n_i$, we have the decomposition into prime divisors

$$V(\overline{X}, T_{ij}) \;=\; D_{ij,1} \cup \ldots \cup D_{ij,c(i)},$$

in particular, the number $c(i)$ does not depend on the choice of j. The components $D_{ij,1}, \ldots, D_{ij,c(i)}$ lie in the common affine chart $W_0 \subseteq \overline{X}_0$ obtained by localizing at all $T_{i'j'}$ different from T_{ij}. Their images thus lie in the affine chart $W_0/H_0^0 \subseteq \overline{X}_0/H_0^0$. Consequently, the $D_{ij,1}, \ldots, D_{ij,c(i)}$ have pairwise disjoint images under the composition $\overline{X}_0 \to \overline{X}_0/H_0^0 \to Y$.

On the other hand, $V(\overline{X}, T_{ij})$ and $V(\overline{X}, T_{ij'})$ are identified isomorphically under the separation map $\overline{X}_0/H_0^0 \to Y$ Thus, suitably numbering, we obtain for every i, and $\alpha = 1, \ldots, c(i)$ a chain

$$D_{i1,\alpha}, \ldots, D_{in_i,\alpha},$$

of divisors identified under the morphism $\overline{X}_0/H_0^0 \to Y$. The order of isotropy for any $x \in D_{ij,\alpha}$ equals l'_{ij}. Now, using [6, Thm. 4.4.1.6], we can compute the defining relations of the Cox ring of $\overline{X}$, which establishes the two assertions. □

Remark 3.5.7. Let $R(A, P_0)$ be a non factorial platonic ring with ordered P_0 and leading platonic triple (l_{01}, l_{11}, l_{21}). Denote by $R(A', P'_0)$ the Cox ring of $\operatorname{Spec} R(A, P_0)$. Then the exponents of the defining relations of $R(A', P'_0)$ are listed in the following table, where $\mathbf{1}_{n_1}$ denotes the vector of length n_i with all entries equal to one.

leading plat. triple	exponents in $R(A', P'_0)$
$(4, 3, 2)$	l_1, l_1, $l_0/2$, $\mathbf{1}_{n_2}$ and $2 \times \mathbf{1}_{n_i}$ for $i \geq 3$.
$(3, 3, 2)$	l_2, l_2, l_2, $\mathbf{1}_{n_0}$, $\mathbf{1}_{n_1}$ and $3 \times \mathbf{1}_{n_i}$ for $i \geq 3$.
$(x, 2, 2)$ and $\mathfrak{l} = 2$	$l_0/2$, $l_0/2$, $2 \times \mathbf{1}_{n_1}$, $2 \times \mathbf{1}_{n_2}$, and $4 \times \mathbf{1}_{n_i}$ for $i \geq 3$.
$(x, 2, 2)$ and $2 \nmid \mathfrak{l}_0$	l_0, l_0, $\mathbf{1}_{n_1}$, $\mathbf{1}_{n_2}$ and $2 \times \mathbf{1}_{n_i}$ for $i \geq 3$.
$(x, 2, 2)$ and $\mathfrak{l}_2 = 1$	$l_0/2$, l_2, l_2, $\mathbf{1}_{n_1}$ and $2 \times \mathbf{1}_{n_i}$ for $i \geq 3$.
$(x, y, 1)$	$\frac{l_0}{\gcd(\mathfrak{l}_0, \mathfrak{l}_1)}$, $\frac{l_1}{\gcd(\mathfrak{l}_0, \mathfrak{l}_1)}$, $\gcd(\mathfrak{l}_0, \mathfrak{l}_1) \times \mathbf{1}_{n_i}$ for $i \geq 2$.

Proof of Theorem 3.1.5. We start with a rational, normal, affine, log terminal X_1 of complexity one. According to Theorem 3.1.3, the Cox ring R_2 of X_1 is a platonic ring. If the greatest common divisors of pairs $\mathfrak{l}_i, \mathfrak{l}_j$ of R_2 all equal one, then R_2 is factorial by [44, Thm. 1.1] and we are done. If not, then we pass to the Cox ring R_3 of $X_2 := \operatorname{Spec} R_2$ and so on. Proposition 3.5.6 ensures that this procedure terminates with a factorial platonic ring R_p. □

Proof of Theorem 3.1.6. Let X_1 be any rational, normal, affine variety with a torus action of complexity one of Type 2 and at most log terminal singularities. Then Theorem 3.1.5 provides us with a chain of quotients

$$X_p \xrightarrow{/\!\!/ H_{p-1}} X_{p-1} \xrightarrow{/\!\!/ H_{p-2}} \quad \ldots \quad \xrightarrow{/\!\!/ H_3} X_3 \xrightarrow{/\!\!/ H_2} X_2 \xrightarrow{/\!\!/ H_1} X_1 \ ,$$

such that $X_i = \operatorname{Spec}(R_i)$ holds with a platonic ring R_i when $i \geq 2$, the ring R_p is factorial and each $X_{i+1} \to X_i$ is the total coordinate space. The idea is to construct stepwise solvable linear algebraic groups $G_i \subseteq \operatorname{Aut}(X_{i+1})$ acting algebraically on X_{i+1} such that the unit component $G_i^0 \subseteq G_i$ is a torus, G_i contains H_i as a normal subgroup, $G_{i-1} = G_i/H_i$ holds and we have $G_1 = H_1$.

Start with $G_1 := H_1$, acting on X_2. According to [6, Thm. 2.4.3.2], there exists an (effective) action of a torus $\mathcal{G}_1$ on X_3 lifting the action of G_1^0 on X_2 and commuting with the action of H_2 on X_3. Moreover, [7, Thm. 5.1] provides us with an exact sequence of groups

$$1 \longrightarrow H_2 \longrightarrow \operatorname{Aut}(X_3, H_2) \xrightarrow{\pi} \operatorname{Aut}(X_2) \longrightarrow 1 \ ,$$

where $\mathrm{Aut}(X_3, H_2)$ denotes the group of automorphisms of X_3 normalizing the quasitorus H_2. Set $G_2 := \pi^{-1}(G_1)$. Then $H_2^0 \mathcal{G}_1$, as a factor group of the torus $H_2^0 \times \mathcal{G}_1$ by a closed subgroup, is an algebraic torus and it is of finite index in G_2. Thus, G_2 is an affine algebraic group with $G_2^0 = H_2^0 \mathcal{G}_1$ being a torus. By construction, $H_2 \subseteq G_2$ is the kernel of $\alpha_1 := \pi|_{G_2}$ and hence a normal subgroup. Moreover, G_2 is solvable and acts algebraically on X_3. Iterating this procedure gives a sequence

$$G_{p-1} \xrightarrow{\alpha_{p-2}} G_{p-2} \xrightarrow{\alpha_{p-3}} \quad \ldots \quad \xrightarrow{\alpha_2} G_2 \xrightarrow{\alpha_1} G_1 \xrightarrow{\alpha_0} 1$$

of group epimorphisms, where, as wanted, G_i is a solvable reductive group acting algebraically on X_{i+1} such that $H_i = \ker(\alpha_{i-1})$ is the characteristic quasitorus of X_i. In particular, the group $G := G_{p-1} \subseteq \mathrm{Aut}(X_p)$ satisfies the first assertion of the theorem.

We turn to the second assertion. From [6, Prop. 1.6.1.6], we infer that $G_1 = H_1$ acts freely on the preimage $U_2 \subseteq X_2$ of the set of smooth points $U_1 \subseteq X_1$ and moreover, the complement $X_2 \setminus U_2$ is of codimension at least two in X_2. Let $U_3 \subseteq X_3$ be the preimage of $U_2 \subseteq X_2$. Again, the complement of U_3 is of codimension at least two in X_3 and, as U_2 consists of smooth points of X_2, the quasitorus H_2 acts freely on U_3. Because of $G_2/H_2 = G_1$, we conclude that U_3 is G_2-invariant and G_2 acts freely on U_2. Repeating this procedure, we end up with an open set $U_p \subseteq X_p$ having complement of codimension at least two such that G acts freely on U_p. Thus, G acts strongly stably on X_p. Now consider

$$G = \mathcal{D}_0 \supseteq \mathcal{D}_1 \supseteq \ldots \supseteq \mathcal{D}_{p-2} \supseteq \mathcal{D}_{p-1} = 1, \qquad \mathcal{D}_i := \ker(\alpha_i \circ \ldots \circ \alpha_{p-2}).$$

Then we have $X_i = X_p /\!\!/ \mathcal{D}_{i-1}$ and $H_i = \mathcal{D}_{i-1}/\mathcal{D}_i$. Moreover for each $\mathcal{D}_i$, its action on X_p is strongly stable, as remarked before, and X_p is G-factorial because it is factorial. Using [7, Prop. 3.5], we obtain a commutative diagram

$$\begin{tikzcd}
X_p /\!\!/ [\mathcal{D}_i, \mathcal{D}_i] \arrow[rr, "\beta"] \arrow[dr, "/\!\!/ \mathcal{D}_i/[\mathcal{D}_i,\mathcal{D}_i]"'] & & X_p /\!\!/ \mathcal{D}_{i+1} \arrow[dl, "/\!\!/ \mathcal{D}_i/\mathcal{D}_{i+1}"] \\
& X_p /\!\!/ \mathcal{D}_i &
\end{tikzcd}$$

where the left downward map is a total coordinate space. As $\mathcal{D}_i/\mathcal{D}_{i+1} = H_{i+1}$ is abelian, $[\mathcal{D}_i, \mathcal{D}_i]$ is contained in $\mathcal{D}_{i+1}$ and we have the horizontal morphism β. Since the right hand side is a total coordinate space as well, we infer from [6, Sec. 1.6.4] that β is an isomorphism. This implies $\mathcal{D}_{i+1} = [\mathcal{D}_i, \mathcal{D}_i]$, proving the second assertion. □

CHAPTER

FOUR

CHARACTERIZATION OF ITERABILITY OF COX RINGS

We consider normal algebraic varieties X defined over the field $\mathbb{C}$ of complex numbers with finitely generated divisor class group K and only constant invertible global regular functions. If the K-graded Cox ring R_1 of X is a finitely generated $\mathbb{C}$-algebra, then one has the total coordinate space $X_1 := \operatorname{Spec} R_1$. We say that X admits *iteration of Cox rings* if there is a chain

$$X_p \xrightarrow{/\!\!/ H_{p-1}} X_{p-1} \xrightarrow{/\!\!/ H_{p-2}} \quad \dots \quad \xrightarrow{/\!\!/ H_2} X_2 \xrightarrow{/\!\!/ H_1} X_1$$

dominated by a factorial variety X_p where in each step, X_{i+1} is the total coordinate space of X_i and $H_i = \operatorname{Spec} \mathbb{C}[K_i]$ the characteristic quasitorus of X_i, having the divisor class group K_i of X_i as its character group. Note that if the divisor class group K of X is torsion free, then R_1 is a unique factorization domain and iteration of Cox rings is trivially possible. As soon as K has torsion, it may happen that during the iteration process a total coordinate space with non-finitely generated divisor class group pops up and thus there is no chain of total coordinate spaces as above, see Remark 3.4.12.

In Chapter 3 we showed that for affine X with $\Gamma(X, \mathcal{O})^{\mathbb{T}} = \mathbb{C}$ and at most log terminal singularities, the iteration of Cox rings is possible. In this chapter, we characterize all varieties X with a torus action of complexity one that admit iteration of Cox rings. The results of the Sections 4.1 to 4.3 of this chapter have been published in [51].

4.1 Iterability of Cox rings

First consider the case $\Gamma(X, \mathcal{O})^{\mathbb{T}} = \mathbb{C}$. In order to have finitely generated divisor class group, X must be rational and then the Cox ring of X is of Type 2 as introduced in Construction 2.1.1 and thus of the form $R = \mathbb{C}[T_{ij}, S_k]/I$, with a polynomial ring $\mathbb{C}[T_{ij}, S_k]$ in variables T_{ij} and S_k modulo the ideal I generated by the trinomial relations

$$T_0^{l_0} + T_1^{l_1} + T_2^{l_2}, \quad \theta_1 T_1^{l_1} + T_2^{l_2} + T_3^{l_3}, \quad \dots, \quad \theta_{r-2} T_{r-2}^{l_{r-2}} + T_{r-1}^{l_{r-1}} + T_r^{l_r},$$

with $T_i^{l_i} = T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}}$. For each exponent vector l_i set $\mathfrak{l}_i := \gcd(l_{i1}, \ldots, l_{in_i})$. The Cox ring R is factorial if and only if the $\mathfrak{l}_i$ are pairwise coprime; see [44, Thm. 1.1]. We say that R is *hyperplatonic* if $\mathfrak{l}_0^{-1} + \ldots + \mathfrak{l}_r^{-1} > r - 1$ holds. After reordering $\mathfrak{l}_0, \ldots, \mathfrak{l}_r$ decreasingly, the latter condition precisely means that $\mathfrak{l}_i = 1$ holds for all $i \geq 3$ and $(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_2)$ is a *platonic triple*, i.e., a triple of the form

$$(5,3,2), \quad (4,3,2), \quad (3,3,2), \quad (x,2,2), \quad (x,y,1), \quad x,y \in \mathbb{Z}_{\geq 1}.$$

Theorem 4.1.1. *Let X be a normal $\mathbb{T}$-variety of complexity one with $\Gamma(X, \mathcal{O})^{\mathbb{T}} = \mathbb{C}$. Then the following statements are equivalent.*

(i) *The variety X admits iteration of Cox rings.*
(ii) *The variety X is rational with hyperplatonic or factorial Cox ring.*

Note that we have to check rationality in each iteration step. See Remark 3.4.12 for an example, where X and its total coordinate space $\overline{X}$ are rational but the total coordinate space of $\overline{X}$ is not any more.

We turn to the case $\Gamma(X, \mathcal{O})^{\mathbb{T}} \neq \mathbb{C}$. Here, $\mathcal{O}(X)^* = \mathbb{C}^*$ holds and finite generation of the divisor class group of X force $\Gamma(X, \mathcal{O})^{\mathbb{T}} = \mathbb{C}[T]$. We end up with a Cox ring of Type 1 and obtain the following simple characterization.

Theorem 4.1.2. *Let X be a normal $\mathbb{T}$-variety of complexity one with $\Gamma(X, \mathcal{O})^{\mathbb{T}} \neq \mathbb{C}$. Then X admits iteration of Cox rings if and only if X and its total coordinate space are rational with only constant globally invertible functions. Moreover, if the latter holds, then the iteration of Cox rings stops after at most one step.*

Note that there exist indeed rational T-varieties of complexity one with only constant globally invertible functions, having a rational non-factorial total coordinate space $\overline{X}$ of Type 1 with non-constant globally invertible functions; see Remark 4.4.12 for an example.

As a consequence of the two theorems above, we obtain the following structural result, generalizing Theorem 3.1.6 but using analogous ideas for the proof.

Corollary 4.1.3. *Let X be a normal, rational, affine variety with a torus action of complexity one admitting iteration of Cox rings. Then X is a quotient $X = X' /\!\!/ G$ of a factorial affine variety $X' := \operatorname{Spec}(R')$, where R' is a factorial ring and G is a solvable reductive group.*

On our way of proving Theorem 4.1.1, we give in Proposition 4.2.5 an explicit description of the Cox ring of a variety $\operatorname{Spec} R$ for a hyperplatonic ring R. This allows us to describe the possible Cox ring iteration chains in more detail. After reordering the numbers $\mathfrak{l}_0, \ldots, \mathfrak{l}_r$ associated with R decreasingly, we call $(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_2)$ the *basic platonic triple* of R.

Corollary 4.1.4. *The possible sequences of basic platonic triples arising from iteration of Cox rings of normal, rational varieties with a torus action of complexity one and hyperplatonic Cox ring are the following:*

(i) $(1,1,1) \to (2,2,2) \to (3,3,2) \to (4,3,2)$,
(ii) $(1,1,1) \to (x,x,1) \to (2x,2,2)$,
(iii) $(1,1,1) \to (x,x,1) \to (x,2,2)$,
(iv) $(\mathfrak{l}_{01}^{-1}\mathfrak{l}_0, \mathfrak{l}_{01}^{-1}\mathfrak{l}_1, 1) \to (\mathfrak{l}_0, \mathfrak{l}_1, 1)$, *where* $\mathfrak{l}_{01} := \gcd(\mathfrak{l}_0, \mathfrak{l}_1) > 1$.

4.2 Proof of Theorem 4.1.1

In order to iterate a Cox ring $R(A, P_0)$, it is necessary that $\operatorname{Spec} R(A, P_0)$ has finitely generated divisor class group. The latter turns out to be equivalent to rationality of $\operatorname{Spec} R(A, P_0)$. From Corollary 3.4.8, we infer the following rationality criterion.

Remark 4.2.1. Let $R(A, P_0)$ be a ring of Type 2 as in Construction 2.1.1 and set $\mathfrak{l}_i := \gcd(l_{i1}, \ldots, l_{in_i})$. Then $\operatorname{Spec} R(A, P_0)$ is rational if and only if one of the following conditions holds:

(i) We have $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = 1$ for all $0 \le i < j \le r$, in other words, $R(A, P_0)$ is factorial.
(ii) There are $0 \le i < j \le r$ with $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) > 1$ and $\gcd(\mathfrak{l}_u, \mathfrak{l}_v) = 1$ whenever $v \notin \{i, j\}$.
(iii) There are $0 \le i < j < k \le r$ with $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = \gcd(\mathfrak{l}_i, \mathfrak{l}_k) = \gcd(\mathfrak{l}_j, \mathfrak{l}_k) = 2$ and $\gcd(\mathfrak{l}_u, \mathfrak{l}_v) = 1$ whenever $v \notin \{i, j, k\}$.

Definition 4.2.2. Let $R(A, P_0)$ be a ring of Type 2 such that $\operatorname{Spec} R(A, P_0)$ is rational. We say that P_0 is gcd-*ordered* if it satisfies the following two properties

(i) $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = 1$ for all $i = 0, \ldots, r$ and $j = 3, \ldots, r$,
(ii) $\gcd(\mathfrak{l}_1, \mathfrak{l}_2) = \gcd(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_2)$.

Observe that if $\operatorname{Spec} R(A, P_0)$ is rational, then one can always achieve that P_0 is gcd-ordered by suitably reordering $l_0, \ldots, l_r$. This does not affect the K_0-graded algebra $R(A, P_0)$ up to isomorphy.

Lemma 4.2.3. *Let* $R(A, P_0)$ *be a ring of Type 2 such that* $\operatorname{Spec} R(A, P_0)$ *is rational and* P_0 *is* gcd-*ordered. Then, with* $K_0 = \mathbb{Z}^{n+m}/\operatorname{im}(P_0^*)$*, the kernel of* $\mathbb{Z}^{n+m} \to K_0/K_0^{\mathrm{tors}}$ *is generated by the rows of*

$$P_1 := \begin{bmatrix} \frac{-1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)} l_0 & \frac{1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)} l_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \frac{-1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_2)} l_0 & 0 & \frac{1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_2)} l_2 & 0 & 0 & & & \\ -l_0 & 0 & & l_3 & 0 & \vdots & & \vdots \\ \vdots & & & \vdots & \ddots \ \vdots & & & \\ -l_0 & 0 & \cdots & 0 & l_r & 0 & \cdots & 0 \end{bmatrix}.$$

Proof. The arguments are similar as for Lemma 3.5.3. The row lattice of P_0 is a sublattice of finite index of that of P_1 and thus there is a commutative diagram

$$\begin{tikzcd}
K_0 \arrow[rr] \arrow[dr] & & K_0/K_0^{\mathrm{tors}} \\
& \mathbb{Z}^{n+m}/\mathrm{im}(P_1^*) \arrow[ur] &
\end{tikzcd}$$

We have to show, that $\mathbb{Z}^{n+m}/\mathrm{im}(P_1^*)$ is torsion free. Suitable elementary column operations on P_1 reduce the problem to showing that for the $r \times (r+1)$ matrix

$$\begin{bmatrix}
\frac{-1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)}\mathfrak{l}_0 & \frac{1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)}\mathfrak{l}_1 & 0 & \dots & 0 \\
\frac{-1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_2)}\mathfrak{l}_0 & 0 & \frac{1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_2)}\mathfrak{l}_2 & 0 & 0 \\
-\mathfrak{l}_0 & 0 & & \mathfrak{l}_3 & 0 \\
\vdots & & & \vdots & \ddots & \vdots \\
-\mathfrak{l}_0 & 0 & \dots & 0 & \mathfrak{l}_r
\end{bmatrix}$$

the r-th determinantal divisor and therefore the product of the invariant factors equals one. Up to sign, the $r \times r$ minors of the above matrix are

$$\frac{1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)\gcd(\mathfrak{l}_0,\mathfrak{l}_2)}\mathfrak{l}_0 \cdots \mathfrak{l}_{i-1} \cdot \mathfrak{l}_{i+1} \cdots \mathfrak{l}_r, \qquad \text{where } i = 0, \dots, r.$$

Suppose that some prime p divides all these minors. Then $p \nmid \mathfrak{l}_j$ holds for all $j \geq 3$, because otherwise we find an $i \neq j$ with $p \mid \mathfrak{l}_i$, contradicting gcd-orderedness of P_0. Thus, p divides each of the numbers

$$\frac{\mathfrak{l}_0\mathfrak{l}_2}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)\gcd(\mathfrak{l}_0,\mathfrak{l}_2)}, \qquad \frac{\mathfrak{l}_1\mathfrak{l}_2}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)\gcd(\mathfrak{l}_0,\mathfrak{l}_2)}, \qquad \frac{\mathfrak{l}_0\mathfrak{l}_1}{\gcd(\mathfrak{l}_0,\mathfrak{l}_1)\gcd(\mathfrak{l}_0,\mathfrak{l}_2)}.$$

By the assumption of the lemma, $\mathfrak{l} := \gcd(\mathfrak{l}_1,\mathfrak{l}_2)$ equals $\gcd(\mathfrak{l}_0,\mathfrak{l}_1,\mathfrak{l}_2)$. Consequently, we obtain

$$\gcd(\mathfrak{l}_0\mathfrak{l}_2,\mathfrak{l}_0\mathfrak{l}_1,\mathfrak{l}_1\mathfrak{l}_2) \;=\; \gcd(\mathfrak{l}_0\mathfrak{l},\mathfrak{l}_1\mathfrak{l}_2) \;=\; \gcd(\mathfrak{l}_0,\mathfrak{l}_1)\gcd(\mathfrak{l}_0,\mathfrak{l}_2).$$

We conclude $p = 1$; a contradiction. Being the greatest common divisor of the above minors, the r-th determinantal divisor equals one. □

Lemma 4.2.4. *Let $R(A,P_0)$ be a ring of Type 2 and $X := \operatorname{Spec} R(A,P_0)$ be rational. Assume that P_0 is* gcd*-ordered. Then, with $\mathfrak{l} := \gcd(\mathfrak{l}_1,\mathfrak{l}_2)$, the number $c(i)$ of irreducible components of $V(X,T_{ij})$, where $j = 1,\dots,n_i$, is given by*

i	0	1	2	≥ 3
$c(i)$	$\gcd(\mathfrak{l}_1,\mathfrak{l}_2)$	$\gcd(\mathfrak{l}_0,\mathfrak{l}_2)$	$\gcd(\mathfrak{l}_0,\mathfrak{l}_1)$	$\frac{1}{\mathfrak{l}}\gcd(\mathfrak{l}_1,\mathfrak{l}_2)\gcd(\mathfrak{l}_0,\mathfrak{l}_2)\gcd(\mathfrak{l}_0,\mathfrak{l}_1)$

Proof. The assertion is a direct consequence of Lemma 3.5.5 □

We are ready for the main ingredient of the proof of Theorem 4.1.1, the explicit description of the iterated Cox ring.

Proposition 4.2.5. *Let $R(A, P_0)$ be of Type 2 non-factorial with* $\operatorname{Spec} R(A, P_0)$ *rational. Assume that P_0 is* gcd*-ordered and let P_1 be as in Lemma 4.2.3. Define numbers $n' := c(0)n_0 + \ldots + c(r)n_r$ and*

$$n_{i,1}, \ldots, n_{i,c(i)} \;:=\; n_i, \qquad l_{ij,1}, \ldots, l_{ij,c(i)} \;:=\; \gcd((P_1)_{1,ij}, \ldots, (P_1)_{r,ij}).$$

Then the vectors $l_{i,\alpha} := (l_{i1,\alpha}, \ldots, l_{in_i,\alpha}) \in \mathbb{Z}^{n_{i,\alpha}}$ build up an $r' \times (n'+m)$ matrix P_0' with $r' = c(0) + \ldots + c(r) - 1$. With a suitable matrix A', the affine variety $\operatorname{Spec} R(A', P_0')$ *is the total coordinate space of the affine variety* $\operatorname{Spec} R(A, P_0)$.

Proof. The idea is to work with the action of the torus $H_0^0 := \operatorname{Spec} \mathbb{C}[K_0/K_0^{\mathrm{tors}}]$ on $X := \operatorname{Spec} R(A, P_0)$ and to use the description of the Cox ring of a variety with torus action provided in [49]. For this, one has to look at the exceptional fibers of the map $\pi\colon X_0 \to Y$, where $X_0 \subseteq X$ is the set of points with at most finite H_0^0-isotropy and the curve Y is the separation of X_0/H_0^0. Following the lines of the proof of Proposition 3.5.6, one uses Lemma 4.2.4 to determine the number of components for each fiber of π and Lemma 4.2.3 to determine the order of the general (finite) H_0^0-isotropy groups on each component. The rest is application of [49]. $\square$

The defining property of a hyperplatonic ring $R(A, P_0)$ is $\mathfrak{l}_0^{-1} + \ldots + \mathfrak{l}_r^{-1} \geq r - 1$. Thus, for any such ring we find a (unique) platonic triple $(\mathfrak{l}_i, \mathfrak{l}_j, \mathfrak{l}_k)$ with i, j, k pairwise different and all $\mathfrak{l}_u$ with u different from i, j, k equal one. We call $(\mathfrak{l}_i, \mathfrak{l}_j, \mathfrak{l}_k)$ the *basic platonic triple (bpt)* of $R(A, P_0)$.

Remark 4.2.6. Let $R(A, P_0)$ be non-factorial and hyperplatonic with basic platonic triple $(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_2)$. Then Remark 4.2.1 ensures that $X := \operatorname{Spec} R(A, P_0)$ is rational. Moreover, Lemma 4.2.4 and Proposition 4.2.5 yield that the exponent vectors of the defining relations of the Cox ring $R(A', P_0')$ of X are computed in terms of the exponent vectors $l_0, \ldots, l_r$ of $R(A, P_0)$ according to the table below, where "$a \times l_i$" means that the vector l_i shows up a times.

bpt of $R(A, P_0)$	exponent vectors in $R(A', P')$
$(4, 3, 2)$	$2 \times l_1$, $\frac{1}{2}l_0$, $\frac{1}{2}l_2$ and $2 \times l_i$ for $i \geq 3$
$(3, 3, 2)$	$3 \times l_2$, $\frac{1}{3}l_0$, $\frac{1}{3}l_1$ and $3 \times l_i$ for $i \geq 3$
$(x, 2, 2)$ and $2 \mid x$	$2 \times \frac{1}{2}l_0$, $2 \times \frac{1}{2}l_1$, $2 \times \frac{1}{2}l_2$ and $4 \times l_i$ for $i \geq 3$
$(x, 2, 2)$ and $2 \nmid x$	$2 \times l_0$, $\frac{1}{2}l_1$, $\frac{1}{2}l_2$ and $2 \times l_i$ for $i \geq 3$
$(x, y, 1)$	$\frac{1}{\gcd(x,y)}l_0$, $\frac{1}{\gcd(x,y)}l_1$ and $\gcd(x, y) \times l_i$ for $i \geq 2$

Lemma 4.2.7. *Let $R(A, P_0)$ be a ring of Type 2, non-factorial and assume that $X := \operatorname{Spec} R(A, P_0)$ is rational. If the total coordinate space of X is rational as well, then $\mathfrak{l}_i > 1$ holds for at most three $0 \le i \le r$.*

Proof. We may assume that P_0 is gcd-ordered. Then Proposition 4.2.5 provides us with the exponent vectors of the Cox ring $R(A', P_0')$ of X. As $R(A, P_0)$ is rational and non-factorial, Remark 4.2.1 leaves us with the following two cases.

Case 1. We have $\gcd(\mathfrak{l}_0, \mathfrak{l}_1) > 1$ and $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = 1$ whenever $j \ge 2$. This means in particular $\mathfrak{l}_0, \mathfrak{l}_1 > 1$. Assume that there are $2 \le i < j \le r$ with $\mathfrak{l}_i, \mathfrak{l}_j > 1$. According to Proposition 4.2.5, we find $c(i)$ times the exponent vector l_i and $c(j)$ times the exponent vector l_j in P_0'. Lemma 4.2.4 tells us $c(j) = c(i) = \gcd(\mathfrak{l}_0, \mathfrak{l}_1) > 1$. Thus, for the first two copies of l_i and l_j, we obtain $\gcd(l_{i,1}, l_{i,2}) = \mathfrak{l}_i > 1$ and $\gcd(l_{j,1}, l_{j,2}) = \mathfrak{l}_j > 1$ respectively. Remark 4.2.1 shows that $\operatorname{Spec} R(A', P_0')$ is not rational; a contradiction.

Case 2. We have $\gcd(\mathfrak{l}_0, \mathfrak{l}_1) = \gcd(\mathfrak{l}_0, \mathfrak{l}_2) = \gcd(\mathfrak{l}_1, \mathfrak{l}_2) = 2$. Assume that there is an index $3 \le i \le r$ with $\mathfrak{l}_i > 1$. Proposition 4.2.5 and Lemma 4.2.4 yield that the exponent vector l_i occurs $c(k) = 4$ times in the matrix P_0'. As in the previous case we conclude via Remark 4.2.1 that the total coordinate space $\operatorname{Spec} R(A', P_0')$ is not rational; a contradiction. □

Proof of Theorem 4.1.1. We prove "(ii)⇒(i)". Then X is rational and has a ring $R(A, P_0)$ of Type 2 as provided by Construction 2.1.1 as its Cox ring. If $R(A, P_0)$ is factorial, then there is nothing to show. So, let $R(A, P_0)$ be non-factorial and hyperplatonic. Then, after reordering the $\mathfrak{l}_i$ decreasingly, $(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_2)$ is the basic platonic triple of $R(A, P_0)$. From Remark 4.2.6 we infer that $X_1 := \operatorname{Spec} R(A, P_0)$ is rational with hyperplatonic Cox ring $R(A', P_0')$. So, we can pass to $X_2 := R(A', P_0')$ and so forth. The table of possible basic platonic triples given in Remark 4.2.6 shows that the iteration process terminates at a factorial ring.

We prove "(i)⇒(ii)". Since X has a Cox ring, X must have finitely generated divisor class group. As for any $\mathbb{T}$-variety of complexity one, the latter is equivalent to X being rational. The Cox ring of X is a ring $R(A, P_0)$ of Type 2. If $R(A, P_0)$ is factorial, then we are done. So, let $R(A, P_0)$ be non-factorial. Then we may assume that P_0 is gcd-ordered and, moreover, $\mathfrak{l}_{01} \neq 1$. Since $X_1 = \operatorname{Spec} R(A, P_0)$ has a Cox ring $R(A', P_0')$, it must be rational. Preserving the gcd-orderedness, due to Lemma 4.2.7 we may assume $\mathfrak{l}_j = 1$ whenever $j \ge 3$ holds. Remark 4.2.1 leaves us with the following cases.

Case 1. We have $\mathfrak{l}_{01} := \gcd(\mathfrak{l}_0, \mathfrak{l}_1) > 1$ and $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = 1$ whenever $j \ge 2$ holds. Then we may assume $\mathfrak{l}_0 \ge \mathfrak{l}_1$.

1.1. Consider the case $\mathfrak{l}_{01} > 3$. By Lemma 4.2.4, the exponent vector l_2 occurs $\mathfrak{l}_{01}$ times in the defining relations of the Cox ring $R(A', P_0')$ of X_1. Since $\operatorname{Spec} R(A', P_0')$ is rational, Remark 4.2.1 yields $\mathfrak{l}_2 = 1$. We conclude that $(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_2)$ is platonic.

1.2. Assume $\mathfrak{l}_{01} = 3$. Then l_2 occurs 3 times as exponent vector in the defining relations of $R(A', P_0')$. Remark 4.2.1 shows $\mathfrak{l}_2 \le 2$. Thus, $(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_2)$ is platonic.

1.3. Let $\mathfrak{l}_{01} = 2$. If $\mathfrak{l}_0 = \mathfrak{l}_1 = 2$ holds, then $(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_2)$ is a not necessarily ordered platonic triple for any $\mathfrak{l}_2$. So, assume $\mathfrak{l}_0 > \mathfrak{l}_1 \geq 2$. As we are in Case 1, the number $\mathfrak{l}_2$ must be odd. If $\mathfrak{l}_2 = 1$ holds, then $(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_2)$ is a platonic triple and we are done. So, assume $\mathfrak{l}_2 \neq 1$. By Proposition 4.2.5 and Lemma 4.2.4, we find the exponent vectors $1/2\, l_0$ and $1/2\, l_1$ as well as twice l_2 in P_0'. Since $X_1 = \operatorname{Spec} R(A', P_0')$ is rational and $\mathfrak{l}_0 > \mathfrak{l}_1$ holds, Lemma 4.2.7 shows $\mathfrak{l}_1 = 2$ and the triple of non-trivial gcd's of exponent vectors of P_0' is $(\mathfrak{l}_0/2, \mathfrak{l}_2, \mathfrak{l}_2)$. After gcd-ordering P_0', we can apply Case 1.1 and with $\mathfrak{l}_0/2 > 1$ we obtain $\mathfrak{l}_0 = 4$ and $\mathfrak{l}_2 = 3$. In particular, $(\mathfrak{l}_0, \mathfrak{l}_2, \mathfrak{l}_1)$ is platonic.

Case 2: We have $\gcd(\mathfrak{l}_0, \mathfrak{l}_1) = \gcd(\mathfrak{l}_0, \mathfrak{l}_2) = \gcd(\mathfrak{l}_1, \mathfrak{l}_2) = 2$. Then we may assume $\mathfrak{l}_0 \geq \mathfrak{l}_1 \geq \mathfrak{l}_2$. Proposition 4.2.5 and Lemma 4.2.4 tell us that each of the exponent vectors $1/2\, l_0$, $1/2\, l_1$ and $1/2\, l_2$ occurs twice in P_0'. Since $\operatorname{Spec} R(A', P_0')$ is rational, Lemma 4.2.7 yields $\mathfrak{l}_1 = \mathfrak{l}_2 = 2$. Thus, $(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_2)$ is platonic. □

4.3 Proof of Theorem 4.1.2

As a first step we relate the total coordinate space of a rational variety with torus action of complexity one admitting non-constant invariant functions to the total coordinate space of one with only constant invariant functions; see Corollary 4.3.3. This allows us to characterize rationality of the total coordinate space using previous results; see Corollary 4.3.4. Then we determine in a similar manner as before, the iterated Cox ring; see Proposition 4.3.6. This finally allows us to prove Theorem 4.1.2.

Recall from Chapter 2 that the suitable downgradings of the rings $R(A, P_0)$ of Type 1 as provided by Construction 2.1.1 yield precisely the Cox rings of the normal rational $\mathbb{T}$-varieties X of complexity one with $\Gamma(X, \mathcal{O})^{\mathbb{T}} = \mathbb{C}[T]$. Suitable downgradings of the rings $R(A, P_0)$ of Type 2 yield precisely the Cox rings of the normal rational $\mathbb{T}$-varieties X of complexity one with $\Gamma(X, \mathcal{O})^{\mathbb{T}} = \mathbb{C}$.

Construction 4.3.1. Consider a ring $R(A, P_0)$ of Type 1 as in Construction 2.1.1 with $A = (a_1, \ldots, a_r)$. Set $\mathfrak{l}_i := \gcd(l_{i1}, \ldots, l_{in_i})$ and $\ell := \operatorname{lcm}(\mathfrak{l}_1, \ldots, \mathfrak{l}_r)$. Then, writing L_0 for the column vector $-(\ell, \ldots, \ell) \in \mathbb{Z}^r$, we obtain a ring $R(\tilde{A}, \tilde{P}_0)$ of Type 2 with defining matrices

$$\tilde{A} \;:=\; \begin{bmatrix} -1 & a_1 & \ldots & a_r \\ 0 & 1 & \ldots & 1 \end{bmatrix}, \qquad \tilde{P}_0 \;:=\; [L_0, P_0]\,.$$

Proposition 4.3.2. *Let $R(A, P_0)$ be a ring of Type 1 and $R(\tilde{A}, \tilde{P}_0)$ the associated ring of Type 2 obtained via Construction 4.3.1. Fix $\alpha_{ij} \in \mathbb{Z}$ with $\mathfrak{l}_i = \alpha_{i1} l_{i1} + \ldots + \alpha_{in_i} l_{in_i}$. Then one obtains an isomorphism of graded $\mathbb{C}$-algebras*

$$R(\tilde{A}, \tilde{P}_0)_{\tilde{T}_{01}} \;\to\; R(A, P_0)[T_{01}, T_{01}^{-1}], \qquad \tilde{T}_{01} \;\mapsto\; T_{01}, \quad \tilde{T}_{ij} \;\mapsto\; T_{ij} T_{01}^{\frac{\ell}{\mathfrak{l}_i}\alpha_{ij}}.$$

Proof. By construction, $R(\tilde{A}, \tilde{P}_0)$ is a factor algebra of $\mathbb{C}[\tilde{T}_{ij}, \tilde{S}_k]$ and $R(A, P_0)$ of $\mathbb{C}[T_{ij}, S_k]$. We have an isomorphism of $\mathbb{C}$-algebras

$$\psi\colon \mathbb{C}[\tilde{T}_{ij}, \tilde{S}_k]_{\tilde{T}_{01}} \to \mathbb{C}[T_{ij}, S_k][T_{01}, T_{01}^{-1}], \quad \tilde{T}_{01} \mapsto T_{01},\ \tilde{T}_{ij} \mapsto T_{ij} T_{01}^{\frac{\ell}{\mathfrak{l}_i}\alpha_{ij}},\ \tilde{S}_k \mapsto S_k.$$

Observe $\psi(\tilde{T}_i^{l_i}) = T_{01}^{\ell} T_i^{l_i}$. We claim that ψ is compatible with the gradings by $\tilde{K}_0$ on the l.h.s. and by $\mathbb{Z} \times K_0$ on the r.h.s., where the latter grading is given by

$$\deg(T_{01}) \ = \ (1,0) \ \in \ \mathbb{Z} \times K_0, \qquad \deg(T_{ij}) \ = \ (0, e_{ij} + \operatorname{im}(P_0^*)) \ \in \ \mathbb{Z} \times K_0.$$

Indeed, because of $\psi(\tilde{T}_{01}^{-\ell} \tilde{T}_i^{l_i}) = T_i^{l_i}$, the kernels of the respective downgrading maps

$$\mathbb{Z}^{n+1+m} \ \to \ \tilde{K}_0, \qquad \qquad \mathbb{Z}^{n+1+m} \ \to \ \mathbb{Z} \times K_0,$$

generated by the rows $\tilde{P}_0$ and P_0, correspond to each other under ψ. The defining ideal of $R(\tilde{A}, \tilde{P}_0)$ is generated by the polynomials $\tilde{g}_1, \ldots, \tilde{g}_{r-1}$, where

$$\tilde{g}_i \ := \ \det \begin{bmatrix} \tilde{T}_0^{\ell} & T_i^{l_i} & T_{i+1}^{l_{i+1}} \\ -1 & a_i & a_{i+1} \\ 0 & 1 & 1 \end{bmatrix}.$$

The above isomorphism sends $\tilde{g}_i$ to $T_0^{\ell} g_i$, where the g_i are the generators of the defining ideal of $R(A, P_0)$, and thus induces the desired isomorphism. $\square$

Corollary 4.3.3. *Let $X := \operatorname{Spec} R(A, P_0)$ be the affine variety arising from a ring of Type 1 and $\tilde{X} := \operatorname{Spec} R(\tilde{A}, \tilde{P}_0)$ the one arising from the associated ring of Type 2. Then $X \times \mathbb{C}^*$ is isomorphic to the principal open subset $\tilde{X}_{\tilde{T}_{01}} \subseteq \tilde{X}$. In particular, X is rational if and only if $\tilde{X}$ is so.*

Proof. Only for the supplement, there is something to show. If X is rational, then obviously $\tilde{X}$ is so. Now, let $\tilde{X}$ be rational. Then the divisor class group $\operatorname{Cl}(\tilde{X})$ is finitely generated. Thus, also $\operatorname{Cl}(\tilde{X}_{\tilde{T}_{01}}) = \operatorname{Cl}(X \times \mathbb{C}^*) = \operatorname{Cl}(X)$ is finitely generated and, as it carries a torus action of complexity one, X must be rational; see [6, Rem. 4.4.1.5]. $\square$

Corollary 4.3.4. *Let $R(A, P_0)$ be a ring of Type 1. Then $X := \operatorname{Spec} R(A, P_0)$ is rational if and only if one of the following conditions holds:*

- (i) *One has $\mathfrak{l}_i = 1$ for all $1 \leq i \leq r$, in other words, $R(A, P_0)$ is factorial.*
- (ii) *There is exactly one $1 \leq i \leq r$ with $\mathfrak{l}_i > 1$.*
- (iii) *There are $1 \leq i < j \leq r$ with $\mathfrak{l}_i = \mathfrak{l}_j = 2$ and $\mathfrak{l}_u = 1$ whenever $u \notin \{i, j\}$*

Proof. Combine Corollary 4.3.3 with the rationality criterion of Remark 4.2.1. $\square$

Lemma 4.3.5. *Let $R(A, P_0)$ be of Type 1 with $X := \operatorname{Spec} R(A, P_0)$ rational and assume that $(\mathfrak{l}_1, \ldots, \mathfrak{l}_r)$ is decreasingly ordered. Then the number $c(i)$ of irreducible components of $V(X, T_{ij})$ is given as*

$$\begin{array}{c|c|c|c} i & 1 & 2 & \geq 3 \\ \hline c(i) & \mathfrak{l}_1 & \mathfrak{l}_2 & \mathfrak{l}_1 \mathfrak{l}_2 \end{array}.$$

Proof. Due to Corollary 4.3.3, we can realize $X \times \mathbb{C}^*$ as a principal open subset of the associated variety $\tilde{X}$ of Type 2. Then the irreducible components of $V(X, T_{ij}) \times \mathbb{C}^*$ are in one-to-one correspondence with the irreducible components $X \cap V(\tilde{X}, \tilde{T}_{ij})$. The assertions follows. $\square$

Proposition 4.3.6. *Let* $R(A, P_0)$ *be non-factorial of Type 1 with* $\operatorname{Spec} R(A, P_0)$ *rational with only constant globally invertible functions and* $(\mathfrak{l}_1, \ldots, \mathfrak{l}_r)$ *decreasingly ordered. Define numbers* $n' := c(1)n_1 + \ldots + c(r)n_r$ *and*

$$n_{i,1}, \ldots, n_{i,c(i)} \ := \ n_i, \qquad\qquad l_{i,1}, \ldots, l_{i,c(i)} \ := \ \frac{1}{\mathfrak{l}_i} l_i.$$

Then the vectors $l_{i,\alpha} \in \mathbb{Z}^{n_{i,\alpha}}$ *build up an* $r' \times (n' + m)$ *matrix* P_0'. *With a suitable matrix* A' *the affine variety* $\operatorname{Spec} R(A', P_0')$ *is the total coordinate space of the affine variety* $\operatorname{Spec} R(A, P_0)$.

Proof. First observe that the kernel of $\mathbb{Z}^{n+m} \to K_0/K_0^{\mathrm{tors}}$ is generated by the rows of the following $r \times (n+m)$ matrix:

$$\begin{bmatrix} \frac{1}{\mathfrak{l}_1} l_1 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \ldots & \frac{1}{\mathfrak{l}_r} l_r & 0 & \ldots & 0 \end{bmatrix}.$$

Now one determines the Cox ring of $X = \operatorname{Spec} R(A, P_0)$ in the same manner as in the proof of Proposition 4.2.5 by exchanging the matrix P_1 used there by the matrix above and applying Lemma 4.3.5. $\square$

Proof of Theorem 4.1.2. If the Cox ring of X is a factorial ring $R(A, P_0)$ of Type 1, then we are done. So, let $R(A, P_0)$ be non-factorial and rational of Type 1. Then Proposition 4.3.6 shows that the Cox ring of $\operatorname{Spec} R(A, P_0)$ is factorial. Thus, iteration of Cox rings is possible for X if and only if the total coordinate space of X is rational. Moreover, if the latter holds then the iteration of Cox rings ends after at most one step. $\square$

4.4 Divisor class groups of total coordinate spaces

To complete the picture drawn in 4.1, we calculate explicitely the divisor class groups of all affine rational $\mathbb{T}$-varieties $X := \operatorname{Spec} R(A, P_0)$ arising from a ring of Type 2. We obtain the following result:

Theorem 4.4.1. *Let* $X := \operatorname{Spec} R(A, P_0)$ *be an affine, rational, non-factorial variety arising from a ring of Type 2 and set* $\tilde{n} := \sum_{i=0}^{r} ((c(i) - 1)n_i - c(i) + 1)$.

(i) *If* $c := \gcd(\mathfrak{l}_0, \mathfrak{l}_1) > 1$ *and* $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = 1$ *holds whenever* $j \notin \{0, 1\}$, *then the class group* $\operatorname{Cl}(X)$ *is isomorphic to*

$$(\mathbb{Z}/\mathbb{Z}\mathfrak{l}_2)^{c-1} \times \ldots \times (\mathbb{Z}/\mathbb{Z}\mathfrak{l}_r)^{c-1} \times \mathbb{Z}^{\tilde{n}}.$$

(ii) *If* $\gcd(\mathfrak{l}_0, \mathfrak{l}_1) = \gcd(\mathfrak{l}_1, \mathfrak{l}_2) = \gcd(\mathfrak{l}_0, \mathfrak{l}_2) = 2$ *and* $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = 1$ *holds whenever* $j \notin \{0, 1, 2\}$, *then the class group* $\mathrm{Cl}(X)$ *is isomorphic to*

$$\mathbb{Z}/(\mathfrak{l}_0\mathfrak{l}_1\mathfrak{l}_2/4)\mathbb{Z} \times (\mathbb{Z}/\mathfrak{l}_3\mathbb{Z})^3 \times \ldots \times (\mathbb{Z}/\mathfrak{l}_r\mathbb{Z})^3 \times \mathbb{Z}^{\tilde{n}}.$$

The rationality criterion of Corollary 3.4.8 shows that indeed all rational varieties are treated in the above Theorem.

Remark 4.4.2. As a direct consequence of the above theorem, we can compute the divisor class groups of all affine varieties arising from a hyperplatonic Cox ring. We list the basic platonic tuple (bpt) of $R(A, P_0)$ and the divisor class group of $X := \mathrm{Spec}\, R(A, P_0)$ in a table:

Case	bpt of $R(A, P_0)$	divisor class group
(i)	$(4, 3, 2)$	$\mathbb{Z}^{n_1+n_3+\cdots+n_r-(r-1)} \times \mathbb{Z}/3\mathbb{Z}$
(ii)	$(3, 3, 2)$	$\mathbb{Z}^{2\cdot(n_2+\cdots+n_r-(r-1))} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
(iii)	$(x, y, 1)$	$\mathbb{Z}^{(\gcd(x,y)-1)\cdot(n_2+\cdots+n_r-(r-1))}$
(iv)	$(x, 2, 2)$ and $2 \nmid x$	$\mathbb{Z}^{n_0+n_3+\cdots+n_r-(r-1)} \times \mathbb{Z}/x\mathbb{Z}$
(v)	$(x, 2, 2)$ and $2 \mid x$	$\mathbb{Z}^{n_0+n_1+n_2+3\cdot(n_3+\cdots+n_r-(r-1))} \times \mathbb{Z}/x\mathbb{Z}$

.

Recall that the exponents of the relations of the Cox ring $R(A, P_0)$ of a variety of complexity one give rise to the matrix P_0, see Construction 2.1.1. This matrix defines the maximal grading keeping the relations and the variables of the Cox ring homogeneous and any other such grading coarsens this maximal one. Moreover, if we endow $R(A, P_0)$ with a grading, such that it arises as the Cox ring of a variety of complexity one, we find a description of this grading via a stack matrix

$$P := \begin{bmatrix} P_0 \\ d \end{bmatrix}$$

as defined in Construction 2.1.4. In particular its transpose P^* defines an injective map. Thus, let P_0 be as above and define $K_0 := \mathbb{Z}^{n+m}/\mathrm{im}(P_0^*)$. Then

$$K_0^{\mathrm{tors}} \subseteq K := \mathbb{Z}^{n+m}/\mathrm{im}(P^*)$$

holds for any matrix P as above and we call $\mathrm{Cl}(X)^{\mathrm{ctors}} := K_0^{\mathrm{tors}}$ the *compulsory torsion* of the class group $\mathrm{Cl}(X)$ of any X having $R(A, P_0)$ as its Cox ring.

We use the notation introduced in Proposition 4.2.5.

Lemma 4.4.3. *Let* $R(A, P_0)$ *be non factorial of Type 2 such that* $X := \mathrm{Spec}\, R(A, P_0)$ *is rational.*

(i) *If* $c := \gcd(\mathfrak{l}_0, \mathfrak{l}_1) > 1$ *and* $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = 1$ *holds whenever* $j \notin \{0, 1\}$*, then the compulsory torsion of the class group of* X *is*

$$(\mathbb{Z}/\mathfrak{l}_2\mathbb{Z})^{c-1} \times \cdots \times (\mathbb{Z}/\mathfrak{l}_r\mathbb{Z})^{c-1}.$$

(ii) *If* $\gcd(\mathfrak{l}_0, \mathfrak{l}_1) = \gcd(\mathfrak{l}_1, \mathfrak{l}_2) = \gcd(\mathfrak{l}_0, \mathfrak{l}_2) = 2$ *and* $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = 1$ *holds whenever* $j \notin \{0, 1, 2\}$*, then the compulsory torsion of the class group of* X *is*

$$\mathbb{Z}/(\mathfrak{l}_0/2)\mathbb{Z} \times \mathbb{Z}/(\mathfrak{l}_1/2)\mathbb{Z} \times \mathbb{Z}/(\mathfrak{l}_2/2)\mathbb{Z} \times (\mathbb{Z}/\mathfrak{l}_3\mathbb{Z})^3 \times \cdots \times (\mathbb{Z}/\mathfrak{l}_r\mathbb{Z})^3.$$

Proof. We prove (i). With our subsequent considerations we obtain that the class group of X is given as $\mathbb{Z}^{n'+m}/\mathrm{im}(P')$, where P' is some $(r'+s') \times (n'+m)$ stack matrix

$$\begin{bmatrix} P_0' \\ d' \end{bmatrix},$$

of full row rank, and with Proposition 4.2.5 we get that P_0' is the $r' \times (n'+m)$ matrix build up by the exponent vectors $c^{-1}l_0$, $c^{-1}l_1$ and c copies $l_{i,1}, \dots, l_{i,c}$ of l_i for $i \geq 2$. Thus, to obtain the assertion, we compute the Smith Normal Form of P_0'. Suitable elementary column operations transform P_0' into

$$\begin{bmatrix} c^{-1}\mathfrak{l}_0 & c^{-1}\mathfrak{l}_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ c^{-1}\mathfrak{l}_0 & 0 & \mathfrak{l}_{2,1} & & 0 & & & \\ \vdots & & & \ddots & \vdots & & & \\ c^{-1}\mathfrak{l}_0 & 0 & \dots & & \mathfrak{l}_{r,c} & 0 & \dots & 0 \end{bmatrix}.$$

As $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = 1$ holds for $i, j \notin \{0, 1\}$ we obtain for $1 \leq t \leq c$ that the $(r'-t+1)$-th determinantal divisor of P_0' equals $\mathfrak{l}_2^{c-t} \cdots \mathfrak{l}_r^{c-t}$. The assertion follows.

For the proof of (ii) we note that in this case P_0' is built up by 2 copies of $1/2l_0, 1/2l_1$ and $1/2l_2$ and 4 copies of each term l_i for $i \geq 3$. Then, applying the same arguments as above, we obtain the assertion. ☐

Construction 4.4.4. Let X be an irreducible, normal variety with $\mathcal{O}(X)^* = \mathbb{C}^*$ and finitely generated divisor class group. Denote by $\mathrm{WDiv}(X)$ the group of Weil-divisors of X and fix a finitely generated subgroup $\mathbb{Z}^n \cong \langle D_1, \dots, D_n \rangle \leq \mathrm{WDiv}(X)$ such that the map $\pi\colon \mathbb{Z}^n \to \mathrm{Cl}(X)$ sending each Weil divisor D to its class $[D] \in \mathrm{Cl}(X)$ is surjective. Let $f_1, \dots, f_r$ be any linear relations such that

$$f_j([D_1], \dots, [D_n]) = \sum_{i=1}^{n} \alpha_{ij}[D_i] = [0] \in \mathrm{Cl}(X)$$

and set

$$P := \begin{bmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{r1} & \dots & \alpha_{rn} \end{bmatrix}.$$

Then there is a commutative diagramm:

$$\begin{tikzcd} \mathbb{Z}^n \arrow[rr, "\pi"] \arrow[dr] & & \mathrm{Cl}(X) \\ & \mathbb{Z}^n/\mathrm{im}(P^*) \arrow[ur] & \end{tikzcd}$$

In particular $\mathrm{Cl}(X)$ is a factor group of $\mathbb{Z}^n/\mathrm{im}(P^*)$.

Lemma 4.4.5. *Let $l_i \in \mathbb{Z}^{n_i}_{>0}$ be any tuple, $k \in \mathbb{Z}_{\geq 1}$ and consider the matrix*

$$A(k, l_i) := \begin{bmatrix} l_i & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & l_i \\ E_{n_i} & \dots & E_{n_i} \end{bmatrix} \in \mathrm{Mat}(k + n_i, k \cdot n_i, \mathbb{Z}),$$

where E_{n_i} denotes the identity matrix of size n_i. Then $A(k, l_i)$ has rank $n_i - 1 + k$ and the $(n_i - 1 + k)$-th determinantal divisor divides $\mathfrak{l}_i^{k-1}$ with $\mathfrak{l}_i = \gcd(l_{i1}, \dots, l_{in_i})$.

Proof. Choose for any $2 \leq t \leq k$ an integer $1 \leq j_t \leq n_i$ and denote by e_{j_t} the column vector having 1 as j_t-th entry and all other entries equal zero. Consider the following $(n_i - 1 + k) \times (n_i - 1 + k)$ square matrix obtained by deleting the first row and several of the last $(k-1) \cdot n_i$ columns of $A(k, l_i)$

$$\begin{bmatrix} 0 & \dots & 0 & l_{ij_2} & \dots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & l_{ij_k} \\ & E_{n_i} & & e_{j_2} & \dots & e_{j_k} \end{bmatrix}.$$

The determinant of this matrix equals up to sign $l_{ij_2} \cdots l_{ij_k}$. With $\mathfrak{l}_i = \gcd(l_{i1}, \dots, l_{in_i})$ we obtain

$$\gcd(\prod_{t=2}^{k} l_{ij_t};\ j_t \in \{1, \dots, n_i\}) = \mathfrak{l}_i^{k-1}.$$

This shows that the $(n_i - 1 + k)$-th determinantal divisor divides $\mathfrak{l}_i^{k-1}$. Moreover as $A(k, l_i)$ is obviously not of full rank this proves the assertions. □

Remark 4.4.6. Let $R(A, P_0)$ be a ring of Type 2 defining a rational variety $X := \mathrm{Spec}\, R(A, P_0)$. Then the prime divisors $D_{ij,1}, \dots D_{ij,c(i)}$ inside $\mathrm{V}(X; T_{ij})$, where $1 \leq j \leq n_i$, correspond to the variables $T_{ij,1}, \dots, T_{ij,c(i)}$ in the Cox ring $R(A', P_0')$ of X as described in Proposition 4.2.5. In particular we have

$$\deg(T_{ij,t}) = [D_{ij,t}] \in \mathrm{Cl}(X).$$

Moreover each free variable S_k gives rise to a prime divisor $\mathrm{V}(X; S_k) = E_k$ with infinite H_0^0-isotropy. This leads to a free variable S_k' in $R(A', P_0')$ with

$$\deg(S_k') = [E_k] = [0] \in \mathrm{Cl}(X).$$

Note that all free variables of $R(A', P_0')$ arise this way.

Lemma 4.4.7. *Let $R(A, P_0)$ be a ring of Type 2 defining a rational variety $X := \operatorname{Spec} R(A, P_0)$ such that $\gcd(\mathfrak{l}_0, \mathfrak{l}_1) > 1$ and $\gcd(\mathfrak{l}_i, \mathfrak{l}_j) = 1$ holds whenever $j \notin \{0, 1\}$. Then the defining relations of the Cox ring $R(A', P_0')$ of X have $\operatorname{Cl}(X)$-degree zero.*

Proof. Note that for a ring $R(A, P_0)$ as above there is at least one integer $i \in \{0, 1, 2\}$ such that $V(X, T_{ij}) = D_{ij,1}$ is irreducible for $j = 1, \ldots, n_i$. Thus K_0-primeness of the variable T_{ij} implies that $D_{ij,1}$ is a principal divisors for $j = 1, \ldots, n_i$. We conclude

$$\deg(T_{i,1}^{l_{i,1}}) = \sum_{j=1}^{n_0} l_{ij,1}[D_{ij,1}] = 0 \in \operatorname{Cl}(X).$$

As $T_{i,1}^{l_{i,1}}$ occurs as a term in at least one relation of $R(A', P_0')$ the assertion follows. □

Proof of Theorem 4.4.1, Case (i). Set $H_0^0 := H_0/H_0^{\mathrm{tors}}$. We recall that the H_0^0-invariant prime divisors with finite isotropy generate the class group of $X = \operatorname{Spec} R(A, P_0)$ and those are exactly the irreducible components of $V(X, T_{ij})$, where $i = 0, \ldots, r$ and $1 \leq j \leq n_i$. Our aim is to determine some relations between the $\operatorname{Cl}(X)$-degrees of the divisors arising this way. Using Construction 4.4.4 this gives rise to an abelian group having $\operatorname{Cl}(X)$ as a factor group.

Let $D_{ij,1} \cup \cdots \cup D_{ij,c(i)}$ be the decomposition of $V(X, T_{ij})$ into prime divisors. As $R(A, P_0)$ is K_0-factorial and T_{ij} is K_0-prime (see [6, Theo. 3.4.2.3]) we get

$$\sum_{t=1}^{c(i)} [D_{ij,t}] = 0 \in \operatorname{Cl}(X). \tag{4.1}$$

Moreover we observe that in the Cases (i)-(iv) $\gcd(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_2) = 1$ holds and so due to Lemma 4.4.7 the defining relations of $R(A', P_0')$ have degree zero. In particular, due to Proposition 4.2.5 we obtain a term $T_{i,t}^{l_{i,t}} = T_{i1,t}^{l_{ij,t}} \cdots T_{ij,t}^{l_{in_i,t}}$ of degree zero for fixed i and t. This gives rise to relations

$$\sum_{j=1}^{n_i} l_{ij,t}[D_{ij,t}] = 0 \in \operatorname{Cl}(X), \tag{4.2}$$

where $i = 0, \ldots, r$ and $t = 1, \ldots, c(i)$. As $l_{i,1} = \cdots = l_{i,c(i)}$ holds for any $i = 0, \ldots, r$, the relations (4.1) and (4.2) give rise to block matrices $A(c(i), l_{i,1})$ in a matrix P as in Construction 4.4.4. In particular we get an $m' \times n'$ matrix with $m' := \sum_{i=0}^{r}(n_i + c(i))$ and $n' := \sum_{i=0}^{r} c(i) \cdot n_i$ of the following form

$$P := \begin{bmatrix} A(c(0), l_{0,1}) & 0 & \cdots & 0 \\ 0 & A(c(1), l_{1,1}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A(c(r), l_{r,1}) \end{bmatrix}. \tag{4.3}$$

Note that P is of rank $\sum_{i=0}^{r}(n_i - 1 + c(i))$ and the $\mathrm{rk}(P)$-th determinantal divisor of P equals the product of the $(n_i - 1 + c(i))$-th determinantal divisors of the block matrices $A(c(i), l_{i,1})$. With Lemma 4.4.5 we conclude that the class group of X is isomorphic to a factor group of the group

$$\mathbb{Z}^{n'}/\mathrm{im}(P^*) \cong \mathbb{Z}^{n'-\mathrm{rk}(P)} \times G \tag{4.4}$$

with some finite abelian group G of order k with $k | \mathfrak{l}_{0,1}^{c(0)-1} \cdots \mathfrak{l}_{r,1}^{c(r)-1}$.

We show that even $\mathbb{Z}^{n'}/\mathrm{im}(P^*) \leq \mathrm{Cl}(X)$ and therefore equality holds. For this purpose we compare the dimensions of $X = \mathrm{Spec}\, R(A, P_0)$ and $\overline{X} = \mathrm{Spec}\, R(A', P_0')$:

$$\begin{aligned}\dim(\overline{X}) - \dim(X) &= n' - (r' - 1) - (n - (r - 1)) \\ &= n' - \sum_{i=0}^{r} c(i) + 2 - \sum_{i=0}^{r} n_i + (r - 1) = n' - \mathrm{rk}(P).\end{aligned}$$

With $X = \overline{X} /\!\!/ \mathrm{Spec}\, \mathbb{C}[\mathrm{Cl}(X)]$ we conclude $\mathbb{Z}^{n'-\mathrm{rk}(P)} \leq \mathrm{Cl}(X)$. The assertion follows with

$$|\,\mathrm{Cl}(X)^{\mathrm{ctors}}| \leq |G| \leq |\,\mathrm{Cl}(X)^{\mathrm{ctors}}|.$$

□

We turn towards the proof of the second assertion of Theorem 4.4.1.

Definition 4.4.8. Let X be an irreducible normal variety and $Y \subseteq X$ a prime divisor. Let furthermore $\mathfrak{A} := \langle f_1, \ldots, f_r \rangle \leq \mathcal{O}(X)$ be any ideal. Then we define the *order of* $\mathfrak{A}$ *along* Y to be $\min(\mathrm{ord}_Y(f_i);\ i = 1, \ldots, r) =: \mathrm{ord}_Y(\mathfrak{A})$.

Lemma 4.4.9. *Let X be an irreducible normal variety, $\mathfrak{A} := \langle f_1, \ldots, f_r \rangle \leq \mathcal{O}(X)$ any ideal and $f \in \mathcal{O}(X)$. Then the following statements are equivalent:*

(i) $\mathrm{ord}_Y(\mathfrak{A}) = \mathrm{ord}_Y(f)$ *holds for all prime divisors* $Y \subseteq X$.
(ii) $\langle f \rangle = \mathfrak{A}$ *holds, i.e.* $\mathfrak{A}$ *is a principal ideal.*

In particular the Weil-divisor $D := \sum \mathrm{ord}_Y(\mathfrak{A})$, where the sum runs over all prime divisors $Y \subseteq X$, is principal if and only if $\mathfrak{A}$ is a principal ideal.

Proof. We prove (i) ⇒ (ii). Observe that $f \mid f_i$ holds for $i = 1, \ldots, r$ as $\mathrm{div}(f) \leq \mathrm{div}(f_i)$ by construction. In particular $\langle f \rangle \supseteq \mathfrak{A}$. We prove the other inclusion. Consider the covering $\cup_{i=1}^{r} U_i$ of X where

$$U_i := X \setminus (Y_{i_1} \cup \cdots \cup Y_{i_{k_i}}),$$

where all prime divisors Y with $\mathrm{ord}_Y(f_i) \neq \mathrm{ord}_Y(\mathfrak{A})$ occur among the Y_{i_t}. Then inside U_i we have $f_i \mid f$. We obtain $c_i \cdot f_i = f$ with $c_i \in \mathcal{O}(U)^*$. Considering the associated sheaf $\tilde{\mathfrak{A}}$ of $\mathfrak{A}$ we obtain $f \in \tilde{\mathfrak{A}}(X) = \mathfrak{A}$. The other implication is clear. □

Lemma 4.4.10. *Let $R(A, P_0) = \mathbb{C}[T_{ij}, S_k]/I$ be a hyperplatonic ring with g_0 of the form $T_0^{l_0} + T_1^{l_1} + T_2^{l_2}$ and basic platonic triple $(x, 2, 2)$. Fix an integer $y \in \mathbb{Z}_{\geq 0}$ with $y \mid x$ and set*

$$\mathfrak{A}_y := \langle T_1^{1/2l_1} + i \cdot T_2^{1/2l_2}, T_0^{1/y \cdot l_0} \rangle \leq R(A, P_0).$$

Then $\mathfrak{A}$ is a principal ideal if and only if $y = 1$ holds.

Proof. Note that $\mathfrak{A}_1 = \langle T_1^{1/2l_1} + iT_2^{1/2l_2} \rangle$ holds in $R(A, P_0)$. So let $y \neq 1$ and assume there is an $f \in \mathfrak{A}_y$ with $\langle f \rangle = \mathfrak{A}$. Then there exist $g_1, g_2, h_1, h_2 \in \mathbb{K}[T_{ij}, S_k]$ with $g_1 \cdot f + I = T_0^{1/y \cdot l_0} + I$ and $g_2 \cdot f + I = T_1^{1/2l_1} + iT_2^{1/2l_2} + I$ and

$$h_1 \cdot T_0^{1/y \cdot l_0} + h_2 \cdot (T_1^{1/2l_1} + iT_2^{1/2l_2}) + I = f + I.$$

Inserting the third formula into the first one we obtain

$$T_0^{1/y \cdot l_0} + I = g_1 \cdot h_1 \cdot T_0^{1/y \cdot l_0} + g_1 \cdot h_2 \cdot (T_1^{1/2l_1} + iT_2^{1/2l_2}) + I$$

and so in particular

$$h := (g_1 \cdot h_1 - 1) \cdot T_0^{1/y \cdot l_0} + g_1 \cdot h_2 \cdot (T_1^{1/2l_1} + iT_2^{1/2l_2}) \in I. \tag{4.5}$$

As there can not occur any term $T_0^{1/y \cdot l_0}$ in I for $y \neq 1$, we conclude that g_1 and h_1 each have a constant term. Inserting the third formula above into the second we obtain a constant term in g_2 and h_2 with similar arguments. But this leads to a term $\lambda \cdot (T_1^{1/2l_1} + i \cdot T_2^{1/2l_2})$ with $\lambda \neq 0$ in (4.5), which contradicts $h \in I$. □

Proof of Theorem 4.4.1, Case (ii). With the same arguments as in the Case (i) we get relations of the form (4.1). Moreover since the degrees of the relations and thus all terms occuring in the Cox ring $R(A', P_0')$ of $X = \operatorname{Spec} R(A, P_0)$ coincide, we obtain

$$\sum_{j=1}^{n_0} l_{0j,1}[D_{0j,1}] = \sum_{j=1}^{n_i} l_{ij,t(i)}[D_{ij,t(i)}] \in \operatorname{Cl}(X), \tag{4.6}$$

where $i = 0, \ldots, r$ and $1 \leq t(i) \leq c(i)$. Those replace the relations (4.2). Suitably ordered this gives rise to a matrix

$$\left[\begin{array}{cc|ccc} -1/2l_0 & 1/2l_0 & 0 & \cdots & 0 \\ E_{n_0} & E_{n_0} & 0 & \cdots & 0 \\ \hline * & 0 & A(c(1), l_{1,1}) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & 0 & 0 & \ldots & A(c(r), l_{r,1}) \end{array}\right],$$

where we use $c(0) = 2$, $l_{0,1} = l_{0,2} = 1/2 \cdot l_0$. and the $*$ indicates that there might be some entries not equal to zero. By suitably swapping columns and applying elementary row operations one achieves a matrix

$$P' := \left[\begin{array}{c|cccc|c} -l_0 & 0 & & \cdots & & 0 \\ \hline * & A(c(1), l_{1,1}) & \cdots & 0 & & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ * & 0 & \ldots & A(c(r), l_{r,1}) & & 0 \\ \hline E_{n_0} & 0 & \ldots & 0 & & E_{n_0} \end{array}\right].$$

The rank of P' equals $\sum_{i=0}^{r}(n_i - 1 + c(i))$. Using $l_{i,1} = l_{i,2} = l_i/2$ for $i = 1, 2$, we obtain with Lemma 4.4.5 that the $(n_i - 1 + c(i))$-th determinantal divisors of $A(c(i), l_{i,1})$ divides $\mathfrak{l}_i/2$ for $i = 1, 2$. Using $l_{i,1} = \ldots = l_{i,4} = l_i$ for $i \geq 3$ we obtain that the $(n_i - 1 + c(i))$-th determinantal divisors of $A(c(i), l_{i,1})$ divides $\mathfrak{l}_i^3$ for $i \geq 3$. Thus considering the maximal square submatrices just including one of the first n_0 columns, Laplace expansion with respect to the first row shows that the $\mathrm{rk}(P')$-th determinantal divisor of P' divides $\mathfrak{l}_0$. If we delete all of the first n_0 columns we observe that the $(\mathrm{rk}(P') - 1)$-th determinantal divisor of P' divides 1, i.e., it equals 1. Thus $\mathrm{Cl}(X)$ is a factor group of

$$\mathbb{Z}^{n'}/\mathrm{im}(\mathrm{P}^*) \cong \mathbb{Z}^{n' - \mathrm{rk}(P')} \times G,$$

where G is a finite group of order k with $k | \mathfrak{l}_0(\mathfrak{l}_1/2)(\mathfrak{l}_2/2)\mathfrak{l}_3^3 \ldots \mathfrak{l}_r^3$.
We show equality of these groups. Observe that we may assume the relation g_0 of $R(A, P_0)$ to be of the form $T_0^{l_0} + T_1^{l_1} + T_2^{l_2}$. In particular the irreducible components $D_{0j,1}$ and $D_{0j,2}$ of $\mathrm{V}(X; T_{0j})$ are of the form

$$D_{0j,1} = \mathrm{V}(T_{0j}, T_1^{1/2l_1} + i \cdot T_2^{1/2l_2}) \quad \text{and} \quad D_{0j,2} = \mathrm{V}(T_{0j}, T_1^{1/2l_1} - i \cdot T_2^{1/2l_2}).$$

We conclude that for $y \in \mathbb{Z}_{\geq 0}$ with $y \mid \mathfrak{l}_0$

$$D := \sum_{j=1}^{n_0} \frac{1}{y} l_{0j} D_{0j,1} = \sum_{Y} \mathrm{ord}_Y(\mathfrak{A}_y)$$

holds with $\mathfrak{A}_y$ as in Lemma 4.4.10. As $\mathfrak{A}_y$ is principal if and only if $y = 1$ holds, we obtain $\mathbb{Z}/\mathfrak{l}_0\mathbb{Z}$ and thus in particular $\mathbb{Z}/2\mathbb{Z}$ as a factor of the class group of X. Calculating the difference between the dimensions of $\mathrm{Spec}\, R(A, P_0)$ and $\mathrm{Spec}\, R(A', P_0')$ as in the proof of the case (i) we conclude $\mathbb{Z}^{n' - \mathrm{rk}(P')} \leq \mathrm{Cl}(X)$ and the assertion follows with

$$2 \cdot |\,\mathrm{Cl}(X)^{\mathrm{ctors}}| \leq |G| \leq 2 \cdot |\,\mathrm{Cl}(X)^{\mathrm{ctors}}|.$$

$\square$

To conclude this chapter we give a necessary criterion for the factoriality of a ring $R(A, P_0)$ of Type 1 using the methods developed so far.

Proposition 4.4.11. *Let $R(A, P_0)$ be a factorial ring of Type 1 with $l_{i1}n_i > 1$ for all $i = 1, \ldots, r$. Then one of the following statements hold:*

(i) We have $\mathfrak{l}_i = 1$ for all $i = 1, \ldots, r$.
(ii) We have $P_0 = [2E_2, 0]$.

Moreover, if (i) holds, the variables T_{ij} are even prime.

Proof. Factoriality of $R(A, P_0)$ implies rationality of $\operatorname{Spec} R(A, P_0)$. Thus it suffices to go through the cases of Corollary 4.3.4. In Case 4.3.4(i) we have $\mathfrak{l}_i = 1$ for all i and it is nothing to show.

Now let $R(A, P_0)$ be as in Case 4.3.4(ii). Then, after suitable renumbering, $\mathfrak{l}_1 > 1$ and $\mathfrak{l}_i = 1$ holds for all $i \geq 2$. The associated ring $R(\tilde{A}, \tilde{P}_0)$ of Type 2 has the terms $T_{01}^{\mathfrak{l}_1}, T_1^{l_1}, \ldots, T_r^{l_r}$ in the defining relations. In particular due to [6, Theorem 3.4.2.3] it is not factorial. As T_{01} is prime due to [6, Lemma 3.4.2.7] we conclude that the localization $R(\tilde{A}, \tilde{P}_0)_{T_{01}} \cong R(A, P_0)[T_{01}, T_{01}^{-1}]$ is not factorial and thus $R(A, P_0)$ is not factorial, see [65, Theorem 20.2]; a contradiction.

We turn to Case 4.3.4(iii). After suitable renumbering $\mathfrak{l}_1 = \mathfrak{l}_2 = 2$ and $\mathfrak{l}_i = 1$ holds for all $i \geq 3$. In particular $l_{i1} n_i > 1$ implies $n_i \geq 2$ for all $i = 3, \ldots, r$. We are left with the following two cases:

We have $r = 2$ and $n_1 = n_2 = 1$: In this case we have

$$R(A, P_0) \cong \mathbb{C}[T_{11}, T_{21}, S_k]/\langle T_{11}^2 + T_{21}^2 + 1 \rangle \cong \mathbb{C}[T, T^{-1}, S_k]$$

and $P_0 = [2E_2, 0]$ holds.

We have $r = 2$ and $n_1 \leq n_2 \geq 2$ or $r \geq 3$: We show that these assumptions contradict factoriality of $R(A, P_0)$. Consider the corresponding ring $R(\tilde{A}, \tilde{P}_0)$ of Type 2, which is hyperplatonic with basic platonic tuple $(2, 2, 2)$ and has the terms $T_{01}^2, T_1^{l_1}, \ldots, T_r^{l_r}$ in the defining relations. Set $X := \operatorname{Spec} R(A, P_0)$ and $\tilde{X} := \operatorname{Spec} R(\tilde{A}, \tilde{P}_0)$ and note that $\operatorname{div}(T_{01}) = D_{01,1} + D_{01,2}$ holds with prime divisors $D_{01,1}, D_{01,2} \subseteq \tilde{X}$. In particular $[D_{01,1}] = -[D_{01,2}] \in \operatorname{Cl}(\tilde{X})$ holds. Using this and factoriality of $X \times \mathbb{C}^* \cong \tilde{X} \setminus V(T_{01})$, we obtain an exact sequence

$$\mathbb{Z}^2 \xrightarrow{\pi} \operatorname{Cl}(\tilde{X}) \longrightarrow \operatorname{Cl}(X \times \mathbb{C}^*) = 0,$$

where $\pi(e_1) = [D_{01,1}]$ and $\pi(e_2) = [D_{01,2}] = -[D_{01,1}]$ and thus $\pi(\mathbb{Z}^2)$ is a cyclic group. Due to Remark 4.4.2 we have

$$\operatorname{Cl}(\tilde{X}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^{n_0 + n_1 + n_2 + 3(n_3 + \ldots + n_r - (r-1))}$$

and $n_0 + n_1 + n_2 + 3(n_3 + \ldots + n_r - (r - 1)) \geq 1$ holds by assumption. As the image of π is a cyclic group this contradicts surjectivity of π and thus factoriality of $R(A, P_0)$.

The supplement is a direct consequence of Proposition 2.2.7. $\square$

Remark 4.4.12. It may happen that for a rational T-variety of complexity one with only constant globally invertible functions, the total coordinate space $\overline{X}$ is rational and non-factorial of Type 1 but has non-constant globally invertible functions. For instance consider

$$X_2 := V(T_{11}^2 T_{12}^2 - T_{21}^2 - 1) \subseteq \mathbb{C}^3,$$

where $T_{11}T_{12} + T_{21} \in \mathcal{O}(X)^*$. According to Corollary 4.3.4 and Proposition 4.4.11 the surface X_2 is rational and non-factorial. Moreover X_2 is the total coordinate space of an affine rational $\mathbb{C}^*$-surface X_1 with defining matrix

$$P_0 := \begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}.$$

X_1 has only constant globally invertible functions, see [6, Theorem 3.2.1.4], the divisor class group of X_1 is $\mathrm{Cl}(X_1) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ and the $\mathrm{Cl}(X_1)$-grading of the Cox ring $\mathcal{R}(X_1) = \mathbb{C}[T_{11}, T_{12}, T_{21}]/\langle T_{11}^2 T_{12}^2 - T_{21}^2 - 1\rangle$ is given by

$$\deg(T_{11}) = (\overline{0}, \overline{1}), \quad \deg(T_{12}) = (\overline{1}, \overline{3}), \quad \deg(T_{21}) = (\overline{1}, \overline{2}).$$

CHAPTER

FIVE

VARIETIES WITH TORUS ACTION OF HIGHER COMPLEXITY

In this chapter we extend the Cox ring based combinatorial theory for rational varieties with torus action of complexity one to Mori dream spaces with torus action of arbitrary high complexity. The key idea is to work over the maximal orbit quotient, which keeps finite generation of the Cox ring. As a sample class we investigate Mori dream spaces with a projective space as maximal orbit quotient having a general hyperplane arrangement as critical locus. Here we obtain simply structured resulting Cox rings directly generalizing the case of rational $\mathbb{T}$-varieties of complexity one. The results of this and the following chapter have been published in the joint publication [46].

5.1 Mori dream spaces with torus action

In this section, we introduce a general framework to construct Mori dream spaces with torus action. The key input is the main result of [49]. There, the Cox ring

$$\mathcal{R}(X) = \bigoplus_{\mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D))$$

of a normal variety X with an effective torus action $\mathbb{T} \times X \to X$, only constant invertible global functions and finitely generated divisor class group $\mathrm{Cl}(X)$ was described in terms of the open subset of points with finite $\mathbb{T}$-isotropy and a certain quotient:

$$X_0 \ = \ \{x \in X;\ \mathbb{T}_x \text{ is finite}\} \ \subseteq \ X, \qquad\qquad \pi\colon X \ \dashrightarrow \ Y.$$

More precisely, [83, Cor. 3] yields a quotient $\kappa\colon X_0 \to X_0/\mathbb{T}$, where the orbit space $X_0/\mathbb{T}$ is a normal, possibly non-separated prevariety. Using [49, Prop. 3.5], we obtain a normal variety Y and a commutative diagram of rational maps

$$\begin{tikzcd} X_0 \arrow[rr, "\kappa"] \arrow[dr, dashed, "\pi"'] & & X_0/\mathbb{T} \arrow[dl, dashed, "\sigma"] \\ & Y & \end{tikzcd}$$

such that there are an open set $W \subseteq X_0$ with complement $X_0 \setminus W$ of codimension at least two and prime divisors $C_0, \ldots, C_r$ on Y with the following properties:

(i) the map π is defined on W, the image $V := \pi(W) \subseteq Y$ is open with complement of codimension at least two,

(ii) the image $\kappa(W) \subseteq X_0/\mathbb{T}$ is open, $\sigma\colon \kappa(W) \to V$ is a surjective local isomorphism and it is an isomorphism over $V \setminus (C_0 \cup \ldots \cup C_r)$,

(iii) for every $i = 0, \ldots, r$, the inverse image $\pi^{-1}(C_i) \subseteq W$ is a union of prime divisors $D_{i1}, \ldots, D_{in_i} \subseteq W$ and all prime divisors of X_0 with nontrivial generic $\mathbb{T}$-isotropy occur among the D_{ij}.

We call the rational map $\pi\colon X \dashrightarrow Y$ the *maximal orbit quotient*, the morphism $\pi\colon W \to V$ a *big representative* and $C_0, \ldots, C_r$ the *doubling divisors* of π. Keeping their notation, we extend the D_{ij} to X by passing to their closures. Moreover, we denote by $E_1, \ldots, E_m$ the prime divisors in the complement $X \setminus X_0$. Then the main result of [49] says that the Cox ring $\mathcal{R}(X)$ of X is given as

$$\mathcal{R}(X) \;\cong\; \mathcal{R}(Y)[T_{ij}, S_k]/\langle T_i^{l_i} - 1_{C_i}\rangle, \qquad\qquad T_i^{l_i} := T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}},$$

where $\mathcal{R}(Y)$ is the Cox ring of Y, by $1_{C_i} \in \mathcal{R}(Y)$ we denote the canonical section of C_i, the variables T_{ij}, S_k represent the canonical sections of D_{ij}, E_k and l_{ij} is the order of the isotropy group $\mathbb{T}_x$ for a general $x \in D_{ij}$. Moreover, the $\mathrm{Cl}(X)$-grading on the r.h.s. assigns to T_{ij}, S_k the classes of D_{ij}, E_k and turns the $\mathrm{Cl}(Y)$-grading of $\mathcal{R}(Y)$ into a $\mathrm{Cl}(X)$-grading via the pullback homomorphism π^*.

The idea of this section is to go in the reverse direction. That means that we start with a normal variety Y having only constant invertible global functions, finitely generated divisor class group $\mathrm{Cl}(Y)$ and finitely generated Cox ring $\mathcal{R}(Y)$; for example, Y might be any Mori dream space. The aim is to construct from Y in a systematic way basically all varieties X with finitely generated Cox ring coming with a torus action that have maximal orbit quotient $X \dashrightarrow Y$.

Our construction will link to toric geometry by using suitable toric varieties as ambient spaces. This means in particular, that we have to deal with varieties Y admitting toric embeddings. According to [87], the latter just means that Y is an A_2*-variety*, that means that any two points of Y admit a common affine open neighborhood; this is the case, for instance, if Y is affine or projective.

We are ready to enter the construction. The reader preferring to see a concrete example before may jump directly to Example 5.2.10.

Construction 5.1.1. Let Y be a normal A_2-variety with only constant invertible global functions, finitely generated divisor class group $\mathrm{Cl}(Y)$ and finitely generated Cox ring

$\mathcal{R}(Y)$. Fix a choice $\alpha = (f_0, \dots, f_r)$ of pairwise non-associated $\mathrm{Cl}(Y)$-prime generators of $\mathcal{R}(Y)$ and an associated toric embedding $Y \subseteq Z_\Delta$, where Z_Δ arises from the fan Δ in the lattice $\mathbb{Z}^t$. Denote by $u_0, \dots, u_r$ the primitive generators of the rays of Δ and write them as the columns in a $t \times (r+1)$ matrix

$$B \;=\; \begin{bmatrix} u_0 & \dots & u_r \end{bmatrix}.$$

Note that $\mathrm{Cl}(Y) = \mathrm{Cl}(Z_\Delta)$ equals $K_B := \mathbb{Z}^{r+1}/\mathrm{im}(B^*)$. We build a larger matrix P from B as follows. Fix positive integers $n_0, \dots, n_r$ and set $n := n_0 + \dots + n_r$. Let m, s be nonnegative integers such that $t + s \leq n + m$. For every pair i, j, where $i = 0, \dots, r$ and $j = 1, \dots, n_i$, fix a positive integer l_{ij} and an intergral vector $d_{ij} \in \mathbb{Z}^s$. Moreover, fix integral vectors $d'_1, \dots, d'_m \in \mathbb{Z}^s$. Set $u_{ij} := l_{ij} u_i$ and consider the $(t+s) \times (n+m)$ matrix

$$P \;=\; \begin{bmatrix} u_{01} & \dots & u_{0n_0} & \dots & u_{r1} & \dots & u_{rn_r} & 0 & \dots & 0 \\ d_{01} & \dots & d_{0n_0} & \dots & d_{r1} & \dots & d_{rn_r} & d'_1 & \dots & d'_m \end{bmatrix},$$

where we require that the columns $v_{ij} = (u_{ij}, d_{ij})$ and $v_k = (0, d'_k)$ are pairwise different and primitive and generate $\mathbb{Q}^{t+s}$ as a vector space. Now choose any fan Σ in $\mathbb{Z}^{t+s}$ having the columns v_{ij}, v_k as the primitive generators of its rays and denote by Z_Σ the associated toric variety. We obtain a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, dashed] & Z_\Sigma \arrow[d, dashed] \\ Y \arrow[r] & Z_\Delta \end{tikzcd}$$

where the downwards rational map from Z_Σ to Z_Δ is given by the projection of tori $\mathbb{T}^{t+s} \to \mathbb{T}^t$ and we define

$$X \;=\; X(\alpha, P, \Sigma) \;:=\; \overline{(Y \cap \mathbb{T}^t) \times \mathbb{T}^s} \;\subseteq\; Z_\Sigma$$

to be the closure of the inverse image of Y under $\mathbb{T}^{t+s} \to \mathbb{T}^t$. Then X is invariant under the action of $\mathbb{T}^s$. We have

$$K_P \;:=\; \mathbb{Z}^{n+m}/\mathrm{im}(P^*) \;=\; \mathrm{Cl}(Z_\Sigma).$$

Now, consider the monomials $T_i^{l_i} := T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}} \in \mathbb{K}[T_{ij}, S_k]$ and let $h_1, \dots, h_q$ be generators for the ideal of relations between $f_0, \dots, f_r$. Then the factor ring

$$R(\alpha, P) \;:=\; \mathbb{K}[T_{ij}, S_k] / \langle h_1(T_0^{l_0}, \dots, T_r^{l_r}), \dots, h_q(T_0^{l_0}, \dots, T_r^{l_r}) \rangle$$

becomes K_P-graded by assigning to the generators T_{ij}, S_k the classes of the canonical basis vectors e_{ij}, e_k in K_P as their degrees. Moreover, we have a unique homomorphism of graded rings $\mathcal{R}(Y) \to R(\alpha, P)$ sending f_i to $T_i^{l_i}$.

Remark 5.1.2. If, in Construction 5.1.1, the toric ambient variety Z_Σ is affine (complete, projective), then the resulting X is affine (complete, projective).

Proposition 5.1.3. *Let $X = X(\alpha, P, \Sigma)$ arise from Construction 5.1.1. Suppose that $R(\alpha, P)$ is integral, normal with only constant homogeneous units and the variables T_{ij} define pairwise nonassociated K_P-primes in $R(\alpha, P)$. Then the following statements hold.*

(i) *The $\mathbb{T}^s$-variety X is normal with only constant invertible global functions, is of dimension $s + \dim(Y)$, has divisor class group $\mathrm{Cl}(X) = K_X$, Cox ring $\mathcal{R}(X) = R(\alpha, P)$ and it comes with a $\mathbb{T}^s$-equivariant toric embedding $X \subseteq Z_\Sigma$.*

(ii) *Let $Z_\Sigma^1 \subseteq Z_\Sigma$ be the union of the open toric orbit and all those corresponding to variables T_{ij} and $Z_\Delta^1 \subseteq Z_\Delta$ the union of all toric orbits of codimension at most one. Then $X_1 := X \cap Z_\Sigma^1 \subseteq X_0$ maps onto $Y_1 := Y \cap Z_\Delta^1$ and $X_1 \to Y_1$ is a big representative of of the maximal orbit quotient $\pi\colon X \dashrightarrow Y$.*

Proof of Construction 5.1.1 and Proposition 5.1.3. Consider the toric Cox constructions of the fan Δ living in $N_\Delta := \mathbb{Z}^t$ and the fan Σ living in $N_\Sigma := \mathbb{Z}^{t+s}$; see for example [24, Sec. 5.1]. They fit into a commutative ladder of lattices with exact rows

$$\begin{tikzcd} 0 \arrow[r] & L_\Sigma \arrow[r] \arrow[d] & F_\Sigma \arrow[r, "P"] \arrow[d, "A"'] & N_\Sigma \arrow[d] \\ 0 \arrow[r] & L_\Delta \arrow[r] & F_\Delta \arrow[r, "B"'] & N_\Delta \end{tikzcd}$$

where the lifting $A\colon F_\Sigma \to F_\Delta$ of the projection $N_\Sigma \to N_\Delta$ sends the canonical basis vectors $e_{ij} \in F_\Sigma = \mathbb{Z}^{n+m}$ to $l_{ij}e_i \in F_\Delta = \mathbb{Z}^{r+1}$ and $e_k \in F_\Sigma = \mathbb{Z}^{n+m}$ to $0 \in F_\Delta = \mathbb{Z}^{r+1}$. Dualizing gives a commutative ladder of abelian groups with exact rows

$$\begin{tikzcd} 0 & K_P \arrow[l] & E_\Sigma \arrow[l, "Q"'] & M_\Sigma \arrow[l, "P^*"'] & 0 \arrow[l] \\ 0 & K_B \arrow[l] \arrow[u, "\imath"] & E_\Delta \arrow[l, "C"] \arrow[u, "A^*"] & M_\Delta \arrow[l, "B^*"] \arrow[u] & 0 \arrow[l] \end{tikzcd}$$

By construction, $e_i \in E_\Delta$ is sent by C to $\deg(f_i) \in K_B = \mathrm{Cl}(Y)$. Consequently, the induced map $\imath\colon K_B \to K_P$ sending $\deg(f_i) \in K_B$ to the class of $l_{i1}e_{i1} + \ldots + l_{in_i}e_{in_i}$ in K_P. The fact that we have a homomorphism of graded rings $\mathcal{R}(Y) \to R(\alpha, P)$ sending f_i to $T_i^{l_i}$ is then obvious. This proves all statements made in Construction 5.1.1.

Let $\overline{Y} \subseteq \mathbb{K}^{r+1}$ and $\overline{X} \subseteq \mathbb{K}^{n+m}$ denote the closures of the inverse images of $Y \cap \mathbb{T}^t$ and $X \cap \mathbb{T}^{t+s}$ under the homomorphisms of tori $b\colon \mathbb{T}^{r+1} \to \mathbb{T}^t$ and $p\colon \mathbb{T}^{n+m} \to \mathbb{T}^{t+s}$ defined by B and P respectively. Then $\overline{Y}$ is the total coordinate space of Y and has $\mathcal{R}(Y)$ as its algebra of functions. Observe that with the quasitori $H_Y := \operatorname{Spec} \mathbb{K}[K_B]$ and $H_X := \operatorname{Spec} \mathbb{K}[K_P]$ and the homomorphism of tori $a\colon \mathbb{T}^{n+m} \to \mathbb{T}^{r+1}$ defined by A, we have a commutative diagram

$$\begin{tikzcd} \overline{X} \cap \mathbb{T}^{n+m} \arrow[r, "/H_X", "p"'] \arrow[d, "a"'] & X \cap \mathbb{T}^{t+s} \arrow[d, "/\mathbb{T}^s"] \\ \overline{Y} \cap \mathbb{T}^{r+1} \arrow[r, "/H_Y", "b"'] & Y \cap \mathbb{T}^t, \end{tikzcd}$$

Consider the product $f \in \mathcal{R}(Y)$ over all the generators f_i of $\mathcal{R}(Y)$ and the product $g \in R(\alpha, P)$ over all the generators T_{ij} and S_k of $R(\alpha, P)$. Then, using the above diagram, we see

$$(\mathcal{R}(Y)_f)^{H_Y} \;\cong\; a^*\left(\mathcal{R}(Y)_f\right)^{H_Y} \;=\; \left((R(\alpha,P)_g)^{H_X}\right)^{\mathbb{T}^s}.$$

Since the l.h.s. ring is factorial, also the r.h.s. ring is so. By assumption, $R(\alpha, P)$ is integral, normal and the generators T_{ij} are K_P-prime. Thus, we can apply [6, Cor. 3.4.1.6] and see that $R(\alpha, P)$ is factorially K_P-graded. Consequently, we are in the setting of [6, Constr. 3.2.1.3] which establishes Proposition 5.1.3 (i).

For the second assertion of the Proposition, observe that $Z_\Sigma^1 \to Z_\Delta^1$ defines the maximal orbit quotient of the $\mathbb{T}^s$-action on Z_Σ. As toric prime divisors of Z_Σ^1 and Z_Δ^1 cut down to prime divisors of X_1 and Y_1 respectively, we can conclude that $X_1 \to Y_1$ bigly represents the maximal orbit quotient of the $\mathbb{T}^s$-variety X. $\square$

Theorem 5.1.4. *Let X be an irreducible, normal, A_2-maximal variety with torus action having only constant invertible global functions, finitely generated divisor class group and finitely generated Cox ring. Then X is equivariantly isomorphic to a variety $X(\alpha, P, \Sigma)$ arising from Construction 5.1.1.*

Proof. Consider the maximal orbit quotient $X_0 \dashrightarrow Y$. As outlined at the beginning of the section, the main result of [49] yields a presentation of the Cox ring of X via $\mathrm{Cl}(X)$-homogeneous generators and relations:

$$\mathcal{R}(X) \;\cong\; \mathcal{R}(Y)[T_{ij}, S_k]/\langle T_i^{l_i} - 1_{C_i}\rangle, \qquad\qquad T_i^{l_i} \;:=\; T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}},$$

where we ensure that the canonical sections $1_{C_0}, \ldots, 1_{C_r}$ generate the Cox ring of Y. The $\mathrm{Cl}(X)$-grading of $\mathcal{R}(X)$ reflects the action of the characteristic quasitorus $H := \operatorname{Spec} \mathbb{K}[\mathrm{Cl}(X)]$ on the total coordinate space $\overline{X} := \operatorname{Spec} \mathcal{R}(X)$. Moreover, there is an H-invariant open subset $\widehat{X} \subseteq \overline{X}$ with complement of codimension at least two in $\overline{X}$ such that we have a good quotient $p \colon \widehat{X} \to X = \widehat{X} /\!\!/ H$.

Let $\mathbb{T} \times X \to X$ be the torus action on X. According to [6, Thm. 4.2.3.2], there is an action of $\mathbb{T}$ on $\widehat{X}$ such that we have $t \cdot h \cdot x = h \cdot t \cdot x$ and $p(t \cdot x) = t^b \cdot p(x)$ with a fixed positive integer b for all $t \in \mathbb{T}$, $h \in H$ and $x \in \widehat{X}$. Since $\widehat{X} \subseteq \overline{X}$ has a small complement and $\overline{X}$ is normal, we can extend the $\mathbb{T}$-action to $\overline{X}$.

The Cox ring generators T_{ij} and S_k are H-homogeneous. We show that they are also $\mathbb{T}$-homogeneous. Consider $f := T_{ij} \in \mathcal{R}(X)$. Since $\mathrm{div}(T_i^{l_i}) = p^*(D_{ij})$ is $\mathbb{T}$-invariant, also the component $\operatorname{div}(f)$ of this divisor is $\mathbb{T}$-invariant. For each $t \in \mathbb{T}$, we define a rational function on $\overline{X}$ by

$$g_t \colon x \mapsto \frac{f(t \cdot x)}{f(x)}.$$

Numerator and denominator have the same divisor and both are H-homogeneous. Thus, g_t is an invertible H-homogeneous element of $\mathcal{R}(X)$ and hence constant; see [6]. We conclude that there is a character $\chi \in \mathbb{X}(\mathbb{T})$ with $f(t \cdot x) = \chi(t) f(x)$. The same arguing works in the case $f = S_k$.

The toric embedding $X \subseteq Z$ defined by the $(\mathbb{T} \times H)$-homogeneous Cox ring generators T_{ij} and S_k is $\mathbb{T}$-equivariant, where $\mathbb{T}$ acts as a subtorus of the acting torus $\mathbb{T}_Z$ of the ambient toric variety Z. The inclusion $\mathbb{T} \subseteq \mathbb{T}_Z$ is reflected by a splitting $\mathbb{Z}^t \times \mathbb{Z}^s$ of the lattice of one parameter subgroups of $\mathbb{T}_Z$, where $\mathbb{Z}^s$ represents the factor $\mathbb{T} = \mathbb{T}^s$. The toric variety Z is defined by a fan Σ in $\mathbb{Z}^t \times \mathbb{Z}^s$ and the projection $\mathbb{Z}^t \times \mathbb{Z}^s \to \mathbb{Z}^s$ gives rise to a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, dashed] & Z_\Sigma \arrow[d, dashed] \\ Y \arrow[r] & Z_\Delta, \end{tikzcd}$$

where Δ in $\mathbb{Z}^s$ is the fan having the projected rays corresponding to the generators T_{ij} as its maximal cones. The r.h.s. downwards arrow defines the maximal orbit quotient for the $\mathbb{T}$-action on $Z = Z_\Sigma$ and as the toric divisors of Z_Σ cut out the prime divisors of X, the l.h.s. downwards arrow defines the maximal orbit quotient for the $\mathbb{T}$-action on X.

Now consider the toric Cox construction of Z_Σ. It is given by a homomorphism $\mathbb{Z}^{n+m} \to \mathbb{Z}^{t+s}$. Let P denote the corresponding $(n+m) \times (t+s)$ matrix and write v_{ij}, v_k for the columns indexed according to the Cox ring generators T_{ij} and S_k. Computing the $\mathbb{T}$-isotropy along the toric divisors of Z_Σ according to [6, Prop. 2.1.4.2], we obtain $v_1, \ldots, v_m \in \{0\} \times \mathbb{Q}^s$ and see that the v_{ij} have a non-trivial $\mathbb{Z}^t$-part being the l_{ij}-fold multiple of the primitive generator $w_i \in \mathbb{Z}^t$ of the ray through the image of v_{ij}. Thus, P looks as in Construction 5.1.1.

To conclude the proof, we still have to show that $Y \subseteq Z_\Delta$ is the toric embedding arising from the Cox ring generators $1_{C_0}, \ldots, 1_{C_r}$. By construction, the pullbacks to X of the divisors on $Y \subseteq Z_\Delta$ cut out by the toric prime divisors equal the pullbacks to X of the divisors $C_0, \ldots, C_r$. Thus, $C_0, \ldots, C_r$ are in fact the divisors cut out by the toric prime divisors of Z_Δ. The toric Cox construction of Z_Δ is given by the lattice homomorphism $\mathbb{Z}^{r+1} \to \mathbb{Z}^t$ sending the i-th canonical basis vector to w_i. The monomial map

$$\mu \colon \mathbb{K}^{n+m} \to \mathbb{K}^{r+1}, \qquad (z, w) \mapsto (z_0^{l_0}, \ldots, z_r^{l_r})$$

is the categorical quotient by the action of the quasitorus $\ker(\mu)$ on $\mathbb{K}^{n+m}$. The total coordinate space $\overline{X} \subseteq \mathbb{K}^{n+m}$ is invariant and thus maps onto a closed set $\overline{Y} \subseteq \mathbb{K}^{r+1}$. By construction, $\overline{Y}$ lies over $Y \subseteq Z_\Delta$. Moreover, we have

$$\mathcal{O}(\overline{Y}) \cong \mathcal{O}(\overline{X})^{\ker(\mu)} = \mathcal{R}(Y).$$

Thus, $\overline{Y}$ is a total coordinate space for Y, showing that $Y \subseteq Z_\Delta$ is the toric embedding arising from the Cox ring generators $1_{C_0}, \ldots, 1_{C_r}$. $\square$

Corollary 5.1.5. *Let X be a Mori dream space with effective torus action $\mathbb{T} \times X \to X$. Then the $\mathbb{T}$-variety X arises from Construction 5.1.1.*

Remark 5.1.6. If, in Construction 5.1.1, we fix α and P, then the possible choices of polytopal fans Σ having the columns of P as their primitive generators give us all projective Mori dream spaces sharing the K_P-graded ring $R(\alpha, P)$ as Cox ring.

Remark 5.1.7. In order to describe a projective Mori dream space with torus action via polyhedral divisors [1, 2], it happens that one has to start with a non Mori dream space. For example, the maximal torus action on the Grassmannian $G(2,n)$ has the moduli space $\overline{M}_{0,n}$ as its Chow quotient and for $n \geq 10$, it is known that $\overline{M}_{0,n}$ and hence all its blow ups have a non-finitely generated Cox ring [22, 42, 48].

5.2 First properties and examples

We begin this section with adapting concepts and statements from [6, Chap. 3] to the setting of Construction 5.1.1. This allows us to describe basic geometric properties of the resulting varieties. Then we turn to more specific properties around the torus action. Finally, we elaborate an explicit example, showing how Construction 5.1.1 works in practice and we indicate how an existing description of rational $\mathbb{T}$-varieties of complexity one fits into the framework of Construction 5.1.1.

Remark 5.2.1. Let $X = X(\alpha, P, \Sigma)$ and the toric ambient variety $Z = Z_\Sigma$ be as in 5.1.1 and 5.1.3. The *total coordinate spaces* $\overline{X}$ and $\overline{Z}$, that means the spectra of the Cox rings $\mathcal{R}(X)$ and $\mathcal{R}(Z)$, are explicitly given as

$$\overline{X} \ := \ \overline{X}(\alpha, P) \ := \ V(h_1(T_0^{l_0}, \ldots, T_r^{l_r}), \ldots, h_q(T_0^{l_0}, \ldots, T_r^{l_r})) \ \subseteq \ \mathbb{K}^{n+m} \ =: \ \overline{Z}.$$

The grading of $\mathcal{R}(X)$ and $\mathcal{R}(Z)$ by $K_p = \mathrm{Cl}(X) = \mathrm{Cl}(Z)$ defines the actions of the *characteristic quasitorus* $H = \operatorname{Spec} \mathbb{K}[K_P]$ on $\overline{X}$ and $\overline{Z}$, which respect the embedding $\overline{X} \subseteq \overline{Z}$. Moreover, we have a commutative diagram

$$\begin{tikzcd} \widehat{X} \arrow[d, "/\!\!/ H"'] & \subseteq & \widehat{Z} \arrow[d, "/\!\!/ H"] \\ X & \subseteq & Z \end{tikzcd}$$

where $\widehat{Z} \to Z$ is the toric Cox construction [24, Sec. 5.1] and $\widehat{X} = \overline{X} \cap \widehat{Z}$ holds. The induced good quotient $\widehat{X} \to X$ is the *characteristic space* over X.

We take a closer look at the decomposition of $X = X(\alpha, P, \Sigma)$ obtained by cutting down the orbit decomposition of the ambient toric variety $Z = Z_\Sigma$. Recall that, for $\sigma \in \Sigma$, the associated distinguished point $z_\sigma \in Z$ is the common limit point of all one-parameter groups given by vectors from the relative interior of σ.

Definition 5.2.2. Let $X = X(\alpha, P, \Sigma)$ be as in 5.1.1 and 5.1.3. Set $\gamma := \mathbb{Q}^{n+m}_{\geq 0}$. An $\overline{X}$*-face* is a face $\gamma_0 \preccurlyeq \mathbb{Q}^{n+m}$ such that the complementary face $\gamma_0^* \preccurlyeq \gamma$ satisfies

$$\mathbb{K}^{n+m} \supseteq \overline{X}(\gamma_0) := \overline{X} \cap \mathbb{T}^{n+m} \cdot z_{\gamma_0^*} \neq \emptyset.$$

For a cone $\sigma \in \Sigma$ and the face $\gamma_0 \preccurlyeq \gamma$ with $P(\gamma_0^*) = \sigma$, consider the intersection of the corresponding toric orbit of $Z = Z_\Sigma$ with X:

$$X(\gamma_0) := X(\sigma) := X \cap \mathbb{T}^{t+s} \cdot z_\sigma \subseteq Z.$$

We say that $\sigma \in \Sigma$ and $\gamma_0 \preccurlyeq \gamma$ are X*-relevant* if $X(\gamma_0) = X(\sigma)$ is non-empty. Moreover, we denote

$$\operatorname{rlv}(X) := \{\gamma_0 \preccurlyeq \gamma;\ \gamma \text{ is } X\text{-relevant}\}.$$

Note that each $X(\gamma_0) \subseteq X$ is locally closed and X is the disjoint union of the $X(\gamma_0)$, where $\gamma_0 \preccurlyeq \gamma$ runs through the X-relevant faces. Moreover, if $\gamma_0 \preccurlyeq \gamma$ is X-relevant, then we have $\overline{X}(\gamma_0) \subseteq \widehat{X}$ and $\overline{X}(\gamma_0)$ maps onto $X(\gamma_0)$. In terms of the pieces $X(\gamma_0) \subseteq X$, we can characterize the following local properties; for the proofs see [6, 3.3.1.8 to 3.3.1.12], the notation is as in Construction 5.1.1.

Proposition 5.2.3. *Let $X = X(\alpha, P, \Sigma)$ be as in 5.1.1 and 5.1.3. Let $\gamma_0 \preccurlyeq \gamma$ and thus $\sigma = P(\gamma_0^*) \in \Sigma$ be X-relevant. Then the following statements are equivalent.*

(i) *The piece $X(\sigma)$ consists of $\mathbb{Q}$-factorial points of X.*
(ii) *The cone σ is simplicial.*
(iii) *The cone $Q(\gamma_0) \subseteq K_{\mathbb{Q}}$ is of full dimension.*

Proposition 5.2.4. *Let $X = X(\alpha, P, \Sigma)$ be as in 5.1.1 and 5.1.3. Let $\gamma_0 \preccurlyeq \gamma$ and thus $\sigma = P(\gamma_0^*) \in \Sigma$ be X-relevant. Then the following statements are equivalent.*

(i) *The piece $X(\sigma)$ consists of locally factorial points of X.*
(ii) *The cone σ is regular.*
(iii) *The set $Q(\gamma_0 \cap \mathbb{Z}^{n+m})$ generates K as a group.*

Moreover, $X(\sigma)$ consists of smooth points of X if and only if one of the above statements holds and $\overline{X}(\gamma_0)$ consists of smooth points of $\overline{X}$.

As well, we can use the X-relevant faces to describe global data as the Picard group and the various cones of divisor classes; compare [6, Cor. 3.3.1.6 and Prop. 3.3.2.9].

Proposition 5.2.5. *Let $X = X(\alpha, P, \Sigma)$ be as in 5.1.1 and 5.1.3. Then, in $K_P = \mathrm{Cl}(X)$, the Picard group of X is given by*

$$\operatorname{Pic}(X) = \bigcap_{\gamma_0 \in \operatorname{rlv}(X)} Q(\operatorname{lin}_{\mathbb{Q}}(\gamma_0) \cap \mathbb{Z}^{n+m}).$$

Moreover, in $(K_P)_{\mathbb{Q}} = \mathrm{Cl}_{\mathbb{Q}}(X)$, the cones of effective, movable, semiample and ample divisor classes are given by

$$\operatorname{Eff}(X) = Q(\gamma), \qquad \operatorname{Mov}(X) = \bigcap_{\gamma_0 \preccurlyeq \gamma \text{ facet}} Q(\gamma_0),$$

$$\mathrm{SAmple}(X) \;=\; \bigcap_{\gamma_0 \in \mathrm{rlv}(X)} Q(\gamma_0), \qquad\qquad \mathrm{Ample}(X) \;=\; \bigcap_{\gamma_0 \in \mathrm{rlv}(X)} Q(\gamma_0)^\circ.$$

Remark 5.2.6. Let $X = X(\alpha, P, \Sigma)$ be as in 5.1.1 and 5.1.3. Assume that X is projective, and take any $u \in (K_P)_{\mathbb{Q}}$ from the relative interior of the ample cone $\mathrm{Ample}(X)$. Then Σ can be chosen as the normal fan $\Sigma(u)$ of the polytope

$$(P^*)^{-1}(Q^{-1}(u) \cap \gamma) - e) \;\subseteq\; \mathbb{Q}^{t+s},$$

where $\gamma = \mathbb{Q}^{n+m}_{\ge 0}$ and $e \in \mathbb{Q}^{n+m}$ is any element with $Q(e) = u$; note that in terms of the faces $\gamma_0 \preccurlyeq \gamma$, the normal fan is given as

$$\Sigma(u) \;=\; \{P(\gamma_0^*);\ \gamma_0 \preccurlyeq \gamma \text{ with } u \in Q(\gamma_0)^\circ\}.$$

Conversely, for any $u' \in \mathrm{Mov}(X)^\circ$, the normal fan $\Sigma(u')$ defines a projective variety $X' = X(\alpha, P, \Sigma(u'))$ and there is a small quasimodification $X \dashrightarrow X'$, which is an isomorphism if and only if u and u' belong to the same Mori chamber.

We turn to more specific properties of the varieties produced by Construction 5.1.1, involving in particular the torus action.

Proposition 5.2.7. *Let $X = X(\alpha, P, \Sigma)$ be as in 5.1.1 and 5.1.3. Suppose that the Cox ring presentation $\mathcal{R}(Y) = \mathbb{K}[f_1, \ldots, f_r]/\langle h_1, \ldots, h_q\rangle$ is a complete intersection. Then, with $h'_u := h_u(T_0^{l_0}, \ldots, T_r^{l_r})$, also the Cox ring presentation*

$$\mathcal{R}(X) \;=\; \mathbb{K}[T_{ij}, S_k]/\langle h'_1, \ldots, h'_q\rangle$$

is a complete intersection. Moreover, in the latter case, the canonical divisor class of X is given by

$$\mathcal{K}_X \;=\; -\sum_{i=0}^{r}\sum_{j=1}^{n_i} \deg(T_{ij}) - \sum_{k=1}^{m} \deg(S_k) + \sum_{u=1}^{q} \deg(h'_u) \;\in\; K_P \;=\; \mathrm{Cl}(X).$$

In particular, with the canonical divisor class $\mathcal{K}_Y \in K_B = \mathrm{Cl}(Y)$ and the maximal orbit quotient $\pi\colon X \dashrightarrow Y$, we have

$$\mathcal{K}_X - \pi^*(\mathcal{K}_Y) \;=\; \sum_{i=0}^{r}\sum_{j=1}^{n_i} (l_{ij} - 1)\deg(T_{ij}) - \sum_{k=1}^{m} \deg(S_k).$$

Proof. The second and third statement follow from [6, Prop. 3.3.3.2]. The first one is seen via a simple dimension computation:

$$\begin{aligned}
\dim(\overline{X}) &= \dim(X) + \mathrm{rk}\,(\mathrm{Cl}(X)) \\
&= s + \dim(Y) + \mathrm{rk}\,(\mathrm{Cl}(X)) \\
&= s + \dim(\overline{Y}) - \mathrm{rk}\,(\mathrm{Cl}(Y)) + \mathrm{rk}\,(\mathrm{Cl}(X)) \\
&= s + (r + 1 - q) - (r + 1 - t) + (n + m - t - s) \\
&= n + m - q.
\end{aligned}$$

$\square$

For the next observation, note that in Construction 5.1.1, we may remove successively maximal cones that are not X-relevant from the fan Σ. The result is a minimal fan Σ defining still the initial X. We call Z_Σ in this case the *minimal ambient toric variety* of X.

Proposition 5.2.8. *Let $X = X(\alpha, P, \Sigma)$ be as in 5.1.1 and 5.1.3. Consider the sublattice $L := \{0\} \times \mathbb{Z}^s \subseteq \mathbb{Z}^{t+s}$ corresponding to the inclusion $\mathbb{T}^s \subseteq \mathbb{T}^{t+s}$ of tori and assume that Z_Σ is the minimal toric ambient variety of X.*

(i) *The normalization of the general $\mathbb{T}^s$-orbit closure of X is the toric variety defined by the fan Σ_L in L, where*

$$\Sigma_L \ := \ \{\tau;\ \tau \preccurlyeq (\sigma \cap L_{\mathbb{Q}}),\ \sigma \in \Sigma\}.$$

(ii) *If the maximal orbit quotient $\pi\colon X \dashrightarrow Y$ is a morphism, then Σ_L is a subfan of Σ.*

Proof. As Z_Σ is the minimal toric embedding, the general $\mathbb{T}^s$-orbit closure of X equals the general $\mathbb{T}^s$-orbit closure of Z_Σ. This reduces the problem to standard toric geometry. □

Corollary 5.2.9. *Let $X = X(\alpha, P, \Sigma)$ be as in 5.1.1 and 5.1.3. Assume that X is complete and Σ_L is a subfan of Σ. Then we have*

$$\mathrm{rk}\,(\mathrm{Cl}(X)) - \mathrm{rk}\,(\mathrm{Cl}(Y)) \ > \ n - r - 1.$$

Proof. According to Proposition 5.2.8, the general $\mathbb{T}^s$-orbit closure of X has divisor class group of rank $m - s > 0$. Thus, the assertion follows from

$$\mathrm{rk}\,(\mathrm{Cl}(X)) \ = \ n + m - t - s, \qquad\qquad \mathrm{rk}\,(\mathrm{Cl}(Y)) \ = \ r + 1 - t.$$

□

We conclude the section by producing an explicit example of a Mori dream space with torus action via Construction 5.1.1.

Example 5.2.10. Consider the surface $Y := \mathbb{P}_1 \times \mathbb{P}_1$. Then we have $\mathrm{Cl}(Y) = \mathbb{Z}^2$ and the Cox ring of Y is the polynomial ring $\mathbb{K}[T_0, T_1, T_2, T_3]$, where the $\mathbb{Z}^2$-grading is given by

$$\deg(T_0) = \deg(T_1) = (1, 0), \qquad \deg(T_2) = \deg(T_3) = (0, 1).$$

Consider the redundant system $\alpha = (f_0, \ldots, f_5)$ of generators for $\mathcal{R}(Y)$ consisting of $f_i := T_i$ for $i = 0, \ldots, 3$ and the canonical sections of the diagonals

$$f_4 \ := \ T_0T_3 - T_1T_2, \qquad f_5 \ := \ T_0T_2 - T_1T_3,$$

both being of degree $(1, 1)$. A matrix B of relations between the degrees of generators $f_0, \ldots, f_5$ is given by

$$B \ := \ \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}.$$

Then Y is embedded into the toric variety Z_Δ, the fan Δ of which lives in $\mathbb{Z}^4$ and has the following four cones as its maximal ones

$$\mathrm{cone}(v_i, v_j, v_k, v_4, v_5), \qquad 0 \le i \le j \le k \le 3,$$

where v_i denotes the i-th column of B. Note that Y is given in Cox coordinates by the equation $f_4 = f_0 f_3 - f_1 f_2$ and $f_5 = f_0 f_2 - f_1 f_3$. To build the variety X, consider the matrix

$$P \ := \ \left[\begin{array}{cccccccc} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 2 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 & 2 & 0 \\ \hline -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 2 & -1 \end{array}\right]$$

obtained from B by firstly doubling the last column, then multiplying its last and third last columns with 2, adding a zero column and, after that, adding two new rows as d, d' part. We gain polynomials by modifying the variables of the describing relations of $Y \subseteq Z_\Delta$ accordingly to the column modifications:

$$g_1 := T_{41}^2 - T_{01}T_{31} + T_{11}T_{21}, \qquad\qquad g_2 := T_{51}T_{52}^2 - T_{01}T_{21} + T_{11}T_{31},$$

By construction, the polynomials g_i are homogeneous with respect to the grading of $\mathbb{K}[T_{ij}, S_1]$ given by

$$\deg(T_{ij}) \ := \ Q(e_{ij}) \ \in \ K, \qquad\qquad \deg(S_1) \ := \ Q(e_1) \ \in \ K,$$

where $Q\colon \mathbb{Z}^8 \to K := \mathbb{Z}^8/\mathrm{im}(P^*) \cong \mathbb{Z}^2$, is the projection and $e_{ij}, e_1 \in \mathbb{Z}^8$ are the canonical basis vectors, numbered according to the variables T_{ij} and S_1. Let $\Sigma = \Sigma(u)$ in $\mathbb{Z}^6$ be the normal fan of the polytope

$$(P^*)^{-1}(Q^{-1}(u) \cap \gamma) - e) \ \subseteq \ \mathbb{Q}^6,$$

where $u := (8, -4) \in K$ and $e \in \mathbb{Z}^8$ is any point with $Q(e) = u$. Then Σ has the columns of P as its primitive generators. Moreover, the projection $\mathbb{Z}^8 \to \mathbb{Z}^6$ onto the first six coordinates sends the rays of Σ into the rays of Δ. This gives a rational toric map $\pi\colon Z_\Sigma \dashrightarrow Z_\Delta$. Now, define a variety

$$X \ = \ X(\alpha, P, \Sigma) \ := \ \overline{\pi^{-1}(Y \cap \mathbb{T}^4)} \ \subseteq \ Z_\Sigma.$$

Then X is invariant under the action of the subtors $\mathbb{T} := (1, 1, 1, 1, \mathbb{K}^*, \mathbb{K}^*)$ of the acting torus $\mathbb{T}^6$ of $Z_{\Sigma(u)}$. The $\mathbb{T}$-variety X is normal, of dimension four with divisor class group and Cox ring given by

$$\mathrm{Cl}(X) \ = \ \mathbb{Z}^2, \qquad\qquad \mathcal{R}(X) \ = \ \mathbb{K}[T_{ij}, S_1]/\langle g_1, g_2\rangle,$$

where the grading of the Cox ring is the one given above. This involves application of Proposition 5.1.3; the necessary assumptions are directly verified. Now, applying for instance Propositions 5.2.3, 5.2.5 and 5.2.7, we obtain that X is a $\mathbb{Q}$-factorial Fano variety of Gorenstein index 30.

Example 5.2.11. We show how to retrieve the description of rational T-varieties of complexity one provided in Chapter 2 via Construction 5.1.1.

Type 1. We have $Y = \mathbb{K}$. Then $\mathrm{Cl}(Y) = \{0\}$ and $\mathcal{R}(Y) = \mathbb{K}[T]$ hold. As a system of Cox ring generators, take $\alpha = (f_0, \ldots, f_r)$, where $f_i = T - a_i$ with $a_i \in \mathbb{K}$. Then Construction 5.1.1 succeeds with the unit matrix $B = E_{r+1}$ and the relations

$$h_i \;=\; S_i - S_{i+1} - (a_i - a_{i+1}) \;\in\; \mathbb{K}[S_0, \ldots, S_r], \quad i = 0, \ldots, r-1.$$

Type 2. We have $Y = \mathbb{P}_1$. Then $\mathrm{Cl}(Y) = \mathbb{Z}$ holds and the Cox ring is $\mathcal{R}(Y) = \mathbb{K}[T_1, T_2]$ with the classical grading. As a system of Cox ring generators, take $\alpha = (f_0, \ldots, f_r)$, where $f_i := a_{i1}T_1 + a_{i2}T_2$ and $[a_{i1} : a_{i2}] \in \mathbb{P}_1$ are pairwise different points for $i = 0, \ldots, r$. The matrix

$$B = [e_0, e_1, \ldots, e_r], \qquad e_0 := -e_1 - \ldots - e_r$$

defines the fan Δ of the projective space $Z_\Delta = \mathbb{P}_r$ and Construction 5.1.1 succeeds with the relations

$$h_i \;:=\; \det \begin{bmatrix} a_{i,1} & a_{i+1,1} & a_{i+2,1} \\ a_{i,2} & a_{i+1,2} & a_{i+2,2} \\ S_i & S_{i+1} & S_{i+2} \end{bmatrix}.$$

Then one has to verify the assumptions of Proposition 5.1.3 for both types. Together with Theorem 5.1.4, this basically gives the desired results.

5.3 Arrangement varieties

We use the results of Section 5.1 to produce all $\mathbb{T}$-varieties X with maximal orbit quotient $X \dashrightarrow \mathbb{P}_c$ such that the doubling divisors form a general hyperplane arrangement in the projective space $\mathbb{P}_c$. This leads to a natural and direct extension of the Cox ring based approach to complete rational $\mathbb{T}$-varieties of complexity one developed in [45, 49, 44, 6]. The resulting Cox rings $\mathcal{R}(X)$ allow a direct description. We proceed by presenting and discussing the Cox rings first and then see how the varieties X arise via Construction 5.1.1.

Construction 5.3.1. Fix integers $r \geq c > 0$ and $n_0, \ldots, n_r > 0$ as well as $m \geq 0$. Set $n := n_0 + \ldots + n_r$. The input data is a pair (A, P_0), where

- A is a $(c+1) \times (r+1)$ matrix over $\mathbb{K}$ such that any $c+1$ of its columns $a_0, \ldots, a_r$ are linearly independent,
- P_0 is an integral $r \times (n+m)$ matrix built from tuples of positive integers $l_i = (l_{i1}, \ldots, l_{in_i})$, where $i = 0, \ldots, r$, as follows

$$P_0 \;:=\; \begin{bmatrix} -l_0 & l_1 & & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ -l_0 & 0 & & l_r & 0 & \ldots & 0 \end{bmatrix}.$$

Write $\mathbb{K}[T_{ij}, S_k]$ for the polynomial ring in the variables T_{ij}, where $i = 0, \ldots, r$, $j = 1, \ldots, n_i$, and S_k, where $k = 1, \ldots, m$. Every l_i defines a monomial

$$T_i^{l_i} \ := \ T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}} \ \in \ \mathbb{K}[T_{ij}, S_k].$$

Moreover, for every $t = 1, \ldots, r-c$, we obtain a polynomial g_t by computing the following $(c+2) \times (c+2)$ determinant

$$g_t \ := \ \det \begin{bmatrix} a_0 & \ldots & a_c & a_{c+t} \\ T_0^{l_0} & \ldots & T_c^{l_c} & T_{c+t}^{l_{c+t}} \end{bmatrix} \ \in \ \mathbb{K}[T_{ij}, S_k].$$

Now, let $e_{ij} \in \mathbb{Z}^n$ and $e_k \in \mathbb{Z}^m$ denote the canonical basis vectors and consider the projection

$$Q_0 \colon \mathbb{Z}^{n+m} \ \to \ K_0 := \mathbb{Z}^{n+m}/\mathrm{im}(P_0^*)$$

onto the factor group by the row lattice of P_0. Then the *K_0-graded $\mathbb{K}$-algebra associated with* (A, P_0) is defined by

$$R(A, P_0) \ := \ \mathbb{K}[T_{ij}, S_k]/\langle g_1, \ldots, g_{r-c} \rangle,$$

$$\deg(T_{ij}) := Q_0(e_{ij}), \qquad \deg(S_k) := Q_0(e_k).$$

We list the basic properties of the resulting graded algebra. Recall that a grading of a $\mathbb{K}$-algebra $R = \oplus_K R_w$ by a finitely generated abelian group is *effective* if the weights $w \in K$ with $R_w \neq \{0\}$ generate K as a group and *pointed*, if $R_0 = \mathbb{K}$ holds and $R_w \neq \{0\} \neq R_{-w}$ is only possible for torsion elements $w \in \mathbb{K}$. Finally, we say that the grading is of *complexity* c if $\dim(R) - \mathrm{rk}\,(K) = c$ holds.

Theorem 5.3.2. *Let $R(A, P_0)$ be a K_0-graded $\mathbb{K}$-algebra arising from Construction 5.3.1. Then $R(A, P_0)$ is an integral, normal, complete intersection ring satisfying*

$$\dim(R(A, P_0)) \ = \ n + m - r + c, \qquad R(A, P_0)^* \ = \ \mathbb{K}^*.$$

The K_0-grading of $R(A, P_0)$ is effective, pointed, factorial and of complexity c. The variables T_{ij}, S_k define pairwise nonassociated K_0-primes in $R(A, P_0)$, and for $c \geq 2$, they define even primes.

The following auxiliary statements for the proof of this theorem are also used later. We begin with discussing the specific nature of the matrix A and its impact on the ideal of relations of $R(A, P)$.

Remark 5.3.3. Situation as in Construction 5.3.1. For any tuple $I = (i_1, \ldots, i_{c+2})$ of stricly increasing integers from $[0, r]$, consider the matrix

$$A(I) \ := \ \left[a_{i_1}, \ldots, a_{i_{c+2}}\right],$$

Let $w(I) \in \mathbb{K}^{c+2}$ denote the cross product of the rows of $A(I)$ and define a vector $v(I) \in \mathbb{K}^{r+1}$ by putting the entries of $w(I)$ at the right places:

$$v(I)_i \ := \ \begin{cases} w(I)_j, & i = i_j \text{ occurs in } I = (i_1, \ldots, i_{c+2}), \\ 0, & \text{else.} \end{cases}$$

Then any linearly independent choice of vectors $v(I_1), \ldots, v(I_{r-c})$ is a basis for $\ker(A)$. Note that any non-zero $v \in \ker(A)$ has at least $c+2$ non-zero coordinates.

Remark 5.3.4. Situation as in Construction 5.3.1. Every vector $v \in \ker(A) \subseteq \mathbb{K}^{r+1}$ defines a polynomial

$$g_v \ := \ v_0 T_0^{l_0} + \ldots + v_r T_r^{l_r} \ \in \ \langle g_1, \ldots, g_{r-c} \rangle.$$

Moreover, if a subset $B \subseteq \ker(A)$ generates $\ker(A)$ as a vector space, then the polynomials g_v, $v \in B$, generate the ideal $\langle g_1, \ldots, g_{r-c} \rangle$. In particular, we have

$$\langle g_1, \ldots, g_{r-c} \rangle \ = \ \langle g_{v(I)};\ I = (i_1, \ldots, i_{c+2}),\ 0 \le i_1 < \ldots < i_{c+2} \le r \rangle,$$

with the tuples I from Remark 5.3.3. Observe that each g_v, $0 \neq v \in \ker(A)$, has at least $c+2$ of the monomials $T_i^{l_i}$ and all the g_v share the same K_0-degree.

Lemma 5.3.5. *Let $R(A, P_0)$ be a graded algebra arising from Construction 5.3.1.*

(i) *If we have $l_{i1} + \ldots + l_{in_i} = 1$ for some i, then $R(A, P_0)$ is isomorphic to a ring $R(A', P_0')$ with data $r' = r - 1$ and $c' = c$.*

(ii) *For any generator T_{ij}, the factor ring $R(A, P_0)/\langle T_{ij} \rangle$ is isomorphic to a ring $R(A', P_0')$ with data $r' = r - 1$ and $c' = c - 1$.*

Proof. To obtain (i), let A' be the matrix obtained by deleting the i-th column from A. Then the respective ideals defined by A and A' produce isomorphic rings. Adapting the matrix P_0 accordingly, gives the desired P_0'.

We show (ii). As elementay row operations on A neither change the required properties of A nor the defining ideal of $R(A, P)$, we may assume that $a_{i1} \neq 0$ holds and all other entries of the i-th column of A equal zero. Then the matrix A' obtained by deleting the first row and the i-th column from A satisfies the assumptions of Construction 5.3.1 with $r' = r - 1$ and $c' = c - 1$. Using Remarks 5.3.3 and 5.3.4, we see that the ideal defined by A' corresponds to the defining ideal of $R(A, P_0)/\langle T_{ij} \rangle$. Again, adapting the matrix P_0 accordingly, gives the desired P_0'. □

Lemma 5.3.6. *Situation as in Construction 5.3.1. Let us say that a point $z \in \mathbb{K}^{n+m}$ with coordinates z_{ij}, z_k is of*

- *big type, if for every $i = 0, \ldots, r$, there is an index $1 \le j_i \le n_i$ such that $z_{ij_i} = 0$ holds,*
- *leaf type, if there is a set $I_z = \{i_1, \ldots, i_c\}$ of indices $0 \le i_1 < \ldots < i_c \le r$, such that for all i, j, we have $z_{ij} = 0 \Rightarrow i \in I_z$.*

If $z \in \mathbb{K}^{n+m}$ is of one of these types, then also all translates $t \cdot z$, where $t \in \mathbb{T}^{n+m}$, are so. Moreover, for $\overline{X} = V(g_1, \ldots, g_{r-c}) \subseteq \mathbb{K}^{n+m}$ we have the following statements.

(i) *Every point $z \in \overline{X}$ is of big type or of leaf type.*
(ii) *Every $z \in \mathbb{K}^{n+m}$ of big type is contained in $\overline{X}$.*
(iii) *For every $z \in \mathbb{K}^{n+m}$ of leaf type, there is a $t \in \mathbb{T}^{n+m}$ with $t \cdot z \in \overline{X}$.*

Proof. To obtain (i), we have to show that any $z \in \overline{X}$ which is not of big type must be of leaf type. Otherwise, there are indices $i_1 < \ldots < i_{c+1}$ and associated j_q with $z_{i_q j_q} = 0$. As z is not of big type, there is at least one index i_0 with $z_{i_0 j} \neq 0$ for all $j = 1, \ldots, n_{i_0}$. Remarks 5.3.3 and 5.3.4 provide us with a relation $g \in \langle g_1, \ldots, g_{r-c} \rangle$ involving precisely the monomials $T_i^{l_i}$ for $i = i_0, i_1, \ldots, i_{c+1}$. Then $g(z) = 0$ implies $z_{i_0 j} = 0$ for some $j = 1, \ldots, n_{i_0}$; a contradiction.

We verify (ii) and (iii). Let $z \in \mathbb{K}^{n+m}$. If z is of big type, then we obviously have $g_i(z) = 0$ for $i = 1, \ldots, r-c$. Thus, $z \in \overline{X}$. Now, assume that z is of leaf type. First consider the case $I_z = \{1, \ldots, c\}$. Then, suitably scaling $z_{c+1,1}$, we achieve $g_1(z) = 0$. Next we scale $z_{c+2,1}$ to ensure $g_2(z) = 0$, and so on, until we have also $g_{r-c}(z) = 0$. Then we have found our $t \in \mathbb{T}^{n+m}$ with $t \cdot z \in \overline{X}$. Given an arbitrary I_z, Remarks 5.3.3 and 5.3.4 yield a suitable system $g_1', \ldots, g_{r-c}'$ of ideal generators that allows us to argue analogously. ☐

Lemma 5.3.7. *Situation as in Construction 5.3.1. Let $\overline{X} = V(g_1, \ldots, g_{r-c}) \subseteq \mathbb{K}^{n+m}$ and denote by J the Jacobian of $g_1, \ldots, g_{r-c}$. Then, for any $z \in \overline{X}$, the following statements are equivalent:*

(i) *The Jacobian $J(z)$ is not of full rank, i.e., we have $\operatorname{rk}(J(z)) < r-c$.*
(ii) *The point $z \in \overline{X}$ is of big type and there are $i_1 < \ldots < i_{c+2}$ such that each of these i_q fulfills one of the subsequent two conditions:*
   - *$z_{i_q j_q} = 0$ and $l_{i_q j_q} \geq 2$ hold for at least one $1 \leq j_q \leq n_{i_q}$,*
   - *$z_{i_q j} = 0$ and $l_{i_q j} = 1$ hold for at least two $1 \leq j \leq n_{i_q}$.*

In particular, the set of points $z \in \overline{X}$ with $J(z)$ not of full rank is of codimension at least $c+1$ in $\overline{X}$.

Proof. Assertion (ii) directly implies the supplement and, by a simple computation, also (i). We are left with proving "(i)⇒(ii)". So, let $z \in \overline{X}$ be a point such that $J(z)$ is not of full rank. Then there is a non-trivial linear combination annulating the lines of $J(z)$:

$$\eta_1 \operatorname{grad}(g_1)(z) + \ldots + \eta_{r-c} \operatorname{grad}(g_{r-c})(z) \ = \ 0.$$

The corresponding $g := \eta_1 g_1 + \ldots + \eta_{r-c} g_{r-c}$ satisfies $\operatorname{grad}(g)(z) = 0$ and is of the form $g = g_v$ with a non-zero $v \in \ker(A)$ as in Remark 5.3.4. The condition $\operatorname{grad}(g)(z) = 0$ implies $z_{i j_i} = 0$ for some $1 \leq j_i \leq n_i$ whenever the monomial $T_i^{l_i}$ shows up in g. As observed in Remark 5.3.4, the polynomial g has at least $c+2$ monomials. Thus, we have $z_{i j_i} = 0$ for at least $c+2$ different i. By Lemma 5.3.6, the point $z \in \overline{X}$ is of big type. Moreover, the two conditions of (ii) reflect the fact $\operatorname{grad}(g)(z) = 0$. ☐

Proof of Theorem 5.3.2. For $c = 1$, the statement is proven in [44, Thm. 1.1 and Prop. 2.2]. So, assume $c \geq 2$. First we show that $\overline{X} = V(g_1, \ldots, g_{r-c}) \subseteq \mathbb{K}^{n+m}$ is connected. By construction, the quasitorus $H_0 \subseteq \mathbb{T}^{n+m}$ is the kernel of the homomorphism $\mathbb{T}^{n+m} \to \mathbb{T}^r$ defined by P_0. Consider the multiplicative one-parameter subgroup $\mathbb{K}^* \to H_0$, $t \mapsto (t^\zeta, t^\xi)$, where

$$\zeta \;=\; \left(\frac{n_0 \cdots n_r l_{01} \cdots l_{rn_r}}{n_0 l_{01}}, \ldots, \frac{n_0 \cdots n_r l_{01} \cdots l_{rn_r}}{n_r l_{rn_r}}\right) \;\in\; \mathbb{T}^n, \quad \xi \;=\; (1, \ldots, 1) \;\in\; \mathbb{T}^m.$$

This gives rise to a $\mathbb{K}^*$-action on $\overline{X}$ having the origin as an attractive fixed point. Consequently, $\overline{X}$ is connected. Moreover, we can conclude that all invertible functions as well as all H_0-invariant functions are constant on $\overline{X}$.

Now, Lemma 5.3.7 allows us to apply Serre's criterion and thus we obtain that $R(A, P_0)$ is an integral, normal, complete intersection. By construction, the K_0-grading is effective and as seen above, it is pointed. To obtain factoriality of the K_0-grading, localize $R(A, P_0)$ by the product over all generators T_{ij}, S_k, observe that the degree zero part of the resulting ring is a polynomial ring and apply [12, Thm. 1.1]. Finally, primeness of the generators T_{ij} follows from Lemma 5.3.5 (i). $\square$

Construction 5.3.8. Let (A, P_0) be input data as in Construction 5.3.1. Moreover, fix $1 \leq s \leq n + m - r$ and let d be an integral $s \times (n + m)$ matrix such that the columns v_{ij}, v_k of the $(r + s) \times (n + m)$ stack matrix

$$P \;:=\; \begin{bmatrix} P_0 \\ d \end{bmatrix}$$

are pairwise different, primitive and generate $\mathbb{Q}^{r+s}$ as a vector space. Consider the factor group $K := \mathbb{Z}^{n+m}/\mathrm{im}(P^*)$. Then the projection $Q \colon \mathbb{Z}^{n+m} \to K$ factors through Q_0 and we obtain the *K-graded $\mathbb{K}$-algebra associated with* (A, P):

$$R(A, P) \;:=\; \mathbb{K}[T_{ij}, S_k]/\langle g_1, \ldots, g_{r-c}\rangle,$$

$$\deg(T_{ij}) := w_{ij} := Q(e_{ij}), \qquad \deg(S_k) := w_k \;:=\; Q(e_k).$$

Now, let Σ be any fan in $\mathbb{Z}^{r+s}$ having precisely the rays through the columns of P as its one-dimensional cones and let Z be the associated toric variety. Then we have a commutative diagram

$$\begin{tikzcd}
V(g_1, \ldots, g_{r-c}) \arrow[r, phantom, "="] & \overline{X} \arrow[r, phantom, "\subseteq"] & \overline{Z} \arrow[r, phantom, "="] & \mathbb{Z}^{n+m} \\
& \widehat{X} \arrow[u, phantom, "\cup" rotate=90] \arrow[r, phantom, "\subseteq"] \arrow[d, "/\!\!/H"'] & \widehat{Z} \arrow[u, phantom, "\cup" rotate=90] \arrow[d, "/\!\!/H"] & \\
& X \arrow[r, phantom, "\subseteq"] \arrow[d, dashed] & Z \arrow[d, dashed] & \\
& \mathbb{P}_c \arrow[r] & \mathbb{P}_r &
\end{tikzcd}$$

with the quasitorus $H = \operatorname{Spec}\mathbb{K}[K]$, the toric Cox construction $\widehat{Z} \to Z$ and the induced quotient $\widehat{X} \to X$, where $\widehat{Z} := \overline{X} \cap \widehat{Z}$. The resulting variety $X = X(A, P, \Sigma)$ is normal with dimension, invertible functions, divisor class group and Cox ring given by

$$\dim(X) = s + c, \qquad \Gamma(X, \mathcal{O}^*) = \mathbb{K}^*, \qquad \mathrm{Cl}(X) = K, \qquad \mathcal{R}(X) = R(A, P).$$

Moreover, the inclusion $\mathbb{Z}^c \subseteq \mathbb{Z}^{s+c}$ defines a subtorus $T \subseteq T_Z$ of the acting torus of Z leaving $X \subseteq Z$ invariant and the induced T-action on X is effective and of complexity c. Finally, the dashed arrows indicate the maximal orbit quotients for the T-actions and $\mathbb{P}_c \subseteq \mathbb{P}_r$ is the linear subspace given by

$$\mathbb{P}_c \ = \ V(h_1, \ldots, h_{r-c}), \qquad h_t \ := \ \det \begin{bmatrix} a_0 & \ldots & a_c & a_{c+t} \\ U_0 & \ldots & U_c & U_{c+t} \end{bmatrix} \ \in \ \mathbb{K}[U_0, \ldots, U_r].$$

The doubling divisors of $X_0 \to \mathbb{P}_c$ are precisely the intersection of $\mathbb{P}_c$ with the coordinate hyperplanes of $\mathbb{P}_r$ and thus form the general hyperplane arrangement

$$H_0, \ldots, H_c \ \subseteq \ \mathbb{P}_c, \qquad H_i \ := \ \{z \in \mathbb{P}_c;\ a_{i0}z_0 + \ldots + a_{ic}z_c = 0\}.$$

Remark 5.3.9. Situation as in Construction 5.3.8. Then the Cox ring $\mathcal{R}(Y)$ of $Y := \mathbb{P}_c$ is generated by the canonical sections $f_i \in \mathcal{R}(Y)$ of the hyperplanes $H_i \subseteq \mathbb{P}_c$, where $i = 0, \ldots, r$. Enter Construction 5.1.1 with $Y = \mathbb{P}_c$ and $\alpha = (f_0, \ldots, f_r)$. Set $t := c$ and let $\Delta \in \mathbb{Z}^t$ the standard fan of $Y = \mathbb{P}_c$, that means

$$B \ = \ \begin{bmatrix} -1 & 1 & & 0 \\ \vdots & & \ddots & \\ -1 & 0 & & 1 \end{bmatrix}.$$

Then running Construction 5.1.1 leads to a variety $X(\alpha, P, \Sigma) = X(A, P, \Sigma)$, where the $(c+1) \times (r+1)$ matrix A has the normal vectors $a_i \in \mathbb{K}^{c+1}$ of the hyperplanes $H_i \subseteq \mathbb{P}_c$ as its columns. This verifies in particular all claims made in Construction 5.3.8.

Remark 5.3.10. According to Lemma 5.3.5 (i), we may always assume that the defining data P of Construction 5.3.8 is *irredundant* in the sense that $l_{i0} + \ldots + l_{in_i} \geq 2$ holds for every $i = 0, \ldots, r$. In this case, we also say that $X(A, P, \Sigma)$ is *irredundant*.

Definition 5.3.11. By an *arrangement variety* we mean a normal projective $\mathbb{T}$-variety X with only constant invertible global functions and maximal orbit quotient $\pi\colon X \dashrightarrow \mathbb{P}_c$ such that the doubling divisors $C_0, \ldots, C_r \subseteq \mathbb{P}_c$ form a general hyperplane arrangement.

Theorem 5.3.12. *Let X be an A_2-maximal arrangement variety. Then X is $\mathbb{T}$-equivariantly isomorphic to some $X(A, P, \Sigma)$ arising from Construction 5.3.8.*

Proof. Take the canonical sections of the doubling divisors on the maximal orbit quotient $Y = \mathbb{P}_c$ as generators of the Cox ring $\mathcal{R}(Y)$ and enter Construction 5.1.1. As outlined in the proof of Theorem 5.1.4, this reproduces $X = X(\alpha, P)$. Thus, Remark 5.3.9 gives the assertion. $\square$

5.4 Examples and first properties

We begin with two example classes. First, in Example 5.4.1, we show how intrinsic quadrics arise as arrangement varieties. Second, in Examples 5.4.2 and 5.4.14, we exhibit a series of arrangement varieties producing many smooth Fano examples. Then, we provide basic structural properties of arrangement varieties, also needed in the subsequent sections. Finally, as a first application, we show that the smooth projective arrangement varieties of Picard number one are just the classical smooth projective quadrics; see Proposition 5.4.15.

Example 5.4.1. An *intrinsic quadric* is a normal projective variety with a Cox ring defined by a single quadratic relation; see [18, 34]. From [34, Prop. 2.1], we infer that every intrinsic quadric admits a representation $X = X(A, P, \Sigma)$ in the sense of Construction 5.3.8 with a matrix P having left upper block

$$\begin{bmatrix} -l_0 & l_1 & & 0 \\ \vdots & & \ddots & \\ -l_0 & 0 & & l_r \end{bmatrix}, \qquad l_0 = \ldots = l_q = (1,1), \quad l_{q+1} = \ldots = l_r = (2),$$

where $-1 \leq q \leq r$ and the variables T_{i1} with $i = q+1, \ldots, r$ have pairwise distinct K-degrees. In particular, we obtain that intrinsic quadrics are arrangement varieties. Moreover, for the dimension of X, the rank of the divisor class group and the complexity of the torus action, we have

$$\dim(X) \ = \ r - 1 + s, \qquad \mathrm{rk}\,(\mathrm{Cl}(X)) \ = \ m + q + 2 - s, \qquad c \ = \ r - 1.$$

Example 5.4.2. Fix integers $r > c \geq 1$. Consider the product $Z = \mathbb{P}_r \times \mathbb{P}_r$ and the intersection $X = V(g_1) \cap \ldots \cap V(g_{r-c}) \subseteq Z$ of the $r - c$ divisors of bidegree (a, b) in Z given by

$$\begin{aligned} g_1 &= \lambda_{1,0} T_{01}^a T_{02}^b + \lambda_{1,1} T_{11}^a T_{12}^b + \ldots + \lambda_{1,c} T_{c1}^a T_{c2}^b + T_{c+1,1}^a T_{c+1,2}^b, \\ &\ \ \vdots \\ g_{r-c} &= \lambda_{r-c,0} T_{01}^a T_{02}^b + \lambda_{r-c,1} T_{11}^a T_{12}^b + \ldots + \lambda_{r-c,c} T_{c1}^a T_{c2}^b + T_{r1}^a T_{r2}^b, \end{aligned}$$

where $a, b > 0$ are coprime integers and any $c+1$ of the vectors $\lambda_i = (\lambda_{i,0}, \ldots, \lambda_{i,c})$ are linearly independent. Observe that for $r > c+1$, the divisors $V(g_i) \subseteq Z$ are singular. We have $X = X(A, P, \Sigma)$ in the sense of Construction 5.3.8, where the stack matrix P has upper and lower blocks

$$P_0 \ = \ \begin{bmatrix} -l_0 & l_1 & & 0 \\ \vdots & & \ddots & \\ -l_0 & 0 & & l_r \end{bmatrix}, \qquad l_0 = \ldots = l_r = (a,b),$$

$$d \ = \ \begin{bmatrix} -d_0 & d_1 & & 0 \\ \vdots & & \ddots & \\ -d_0 & 0 & & d_r \end{bmatrix}, \qquad d_0 = \ldots = d_r = (v,u),$$

where u and v are integers with $ua - vb = 1$. Observe that the toric ambient variety $Z = Z_\Sigma$ is indeed the product $\mathbb{P}_r \times \mathbb{P}_r$. To see this, apply the following unimodular matrix to P from the left:

$$\begin{bmatrix} u \cdot E_r & -b \cdot E_r \\ -v \cdot E_r & a \cdot E_r \end{bmatrix}.$$

Moreover, X is of dimension $r + c$ and comes with an effective r-torus action. The anticanonical class of X is given by

$$-\mathcal{K}_X \;=\; ((a-1)r - ac - 1,\, (b-1)r - bc - 1) \;\in\; \mathrm{Cl}(X) \;=\; \mathbb{Z}^2,$$

see Proposition 5.2.7. In particular, X is a Fano variety if and only if $(a-1)r - ac > 1$ and $(b-1)r - bc > 1$ hold.

We begin with our collection of structural properties of arrangement varieties. The first one shows that there may occur inavoidable torsion in the divisor class group.

Proposition 5.4.3. *Let $X = X(A, P, \Sigma)$ arise from Construction 5.3.8. Then the finite group $\mathbb{Z}^r/\mathrm{im}(P_0)$ is a subgroup of the divisor class group $\mathrm{Cl}(X)$.*

Proof. The divisor class group of X equals $K = \mathbb{Z}^{n+m}/\mathrm{im}(P^*)$. Moreover, $\mathbb{Z}^r/\mathrm{im}(P_0)$ is the torsion part K_0^{tors} of the factor group $K_0 = \mathbb{Z}^{n+m}/\mathrm{im}(P_0^*)$. Applying the snake Lemma to the exact sequences arising from P_0^* and P^* yields that the kernel of $K_0 \to K$ injects into $\mathbb{Z}^s$. Consequently, the torsion part K_0^{tors} maps injectively into K. $\square$

Definition 5.4.4. Consider the setting of Construction 5.3.8 and let $\sigma \in \Sigma$. We say that the cone σ is

(i) *big (elementary big)* if σ contains at least (precisely) one column v_{ij} of P for every $i = 0, \ldots, r$,
(ii) a *leaf cone* if there is a set $I_\sigma = \{i_1, \ldots, i_c\}$ of indices $0 \le i_1 < \ldots < i_c \le r$ such that for any i, we have $v_{ij} \in \sigma \Rightarrow i \in I_\sigma$.

Proposition 5.4.5. *Let $X = X(A, P, \Sigma)$ arise from Construction 5.3.8. Then, for every $\sigma \in \Sigma$, the following statements are equivalent.*

(i) *The cone σ is X-relevant.*
(ii) *The cone σ is big or a leaf cone.*

Proof. Consider the face $\gamma_0 \preceq \gamma$ with $P(\gamma_0^*) = \sigma$. Then the points $x \in \overline{X}(\gamma_0)$ are precisely those $x \in \overline{X}$ satisfying $x_{ij} = 0$ if and only if $v_{ij} \in \sigma$. The assertion thus follows from Lemma 5.3.6. $\square$

Remark 5.4.6. Consider the setting of Construction 5.3.8. Set $L := \{0\} \times \mathbb{Z}^s$. Then, for any $\sigma \in \Sigma$, the following statements are equivalent.

(i) The cone σ is big,
(ii) The projection $\mathbb{Q}^{r+s} \to \mathbb{Q}^r$ maps σ onto $\mathbb{Q}^r$,
(iii) We have $\sigma \not\subseteq L_\mathbb{Q}$ and $\sigma^\circ \cap L_\mathbb{Q} \neq \emptyset$.

Proposition 5.4.7. *Consider the setting of Construction 5.3.8. Assume $r > c$. Set $L := \{0\} \times \mathbb{Z}^s$ and let Σ_L be the fan in $\mathbb{Z}^{r+s}$ consisting of all the faces of the cones $\sigma \cap L_{\mathbb{Q}}$, where $\sigma \in \Sigma$. Then the following statements are equivalent.*

(i) *Σ_L is a subfan of Σ.*
(ii) *Σ contains no big cone.*
(iii) *Σ consists of leaf cones.*

Proof. The equivalence of (ii) and (iii) is clear by $r > c$. We prove "(i)$\Rightarrow$(ii)". Assume that there is a big cone $\sigma \in \Sigma$. Then $\sigma \cap L_{\mathbb{Q}}$ belongs to Σ_L but not to Σ according to 5.4.6 (iii); a contradiction. We turn to "(ii)$\Rightarrow$(i)". The task is to show that for every cone $\sigma \in \Sigma$, the intersection $\sigma \cap L_{\mathbb{Q}}$ is a face of σ. Let $\tau \preceq \sigma$ be the minimal face containing $\sigma \cap L_{\mathbb{Q}}$. Then $\tau^\circ \cap L_{\mathbb{Q}}$ is non-empty. Since $\tau \in \Sigma$ is not big, we can use 5.4.6 (iii) to conclude $\tau \subseteq L_{\mathbb{Q}}$. This means $\sigma \cap L_{\mathbb{Q}} = \tau$. □

Proposition 5.4.8. *Let $X = X(A, P, \Sigma)$ arise from Construction 5.3.8. Assume that P is irredundant, X is locally factorial, Σ consists of leaf cones and each of the sets cone$(v_{i1}) + L_{\mathbb{Q}}$ is covered by cones of Σ. Then $n_i \geq 2$ holds for all $i = 0, \ldots, r$.*

Proof. Assume that $n_i = 1$ holds for some i. Let ϱ denote the ray through v_{i1} and consider the cone $\tau := \varrho + L_{\mathbb{Q}}$. We claim that for every $\sigma \in \Sigma$, the intersection $\tau \cap \sigma$ is a face of σ. Indeed, as Σ consists of leaf cones, the image of $\mathrm{pr}(\sigma)$ under the projection $\mathrm{pr}\colon \mathbb{Q}^{r+s} \to \mathbb{Q}^r$ is a pointed cone, having $\mathrm{pr}(\varrho)$ as an extremal ray. Thus, $\tau = \mathrm{pr}^{-1}(\mathrm{pr}(\varrho))$ cuts out a face from σ.

By our assumptions, the above claim implies that $\tau = \varrho + L_{\mathbb{Q}}$ is a union of cones of Σ. Any cone of $\Sigma \setminus \Sigma_L$ contained in τ is necessarily of the form $\varrho + \sigma_L \in \Sigma$ with $\sigma_L \in \Sigma_L$. We conclude that in particular all the cones $\sigma = \varrho + \sigma_L$, where $\dim(\sigma_L) = s$, must belong to Σ. As σ and σ_L are leaf cones, they are X-relevant by Proposition 5.4.10. Thus, Proposition 5.2.4 yields that σ and σ_L are regular. This implies $l_{i1} = 1$; a contradiction to the assumption that P is irredundant. □

Corollary 5.4.9. *Let $X = X(A, P, \Sigma)$ arise from Construction 5.3.8. Assume that X is non-toric, projective, locally factorial and that Σ consists of leaf cones. Then the Picard number of X satisfies*

$$\rho(X) \;\geq\; r + 3 \;\geq\; c + 4.$$

Proof. Since X is non-toric, we may assume that P is irredundant with $r > c$. Moreover, as X is projective, we may assume that Σ is complete. Thus, Proposition 5.4.8 applies and we obtain $n \geq 2r + 2$. Then Corollary 5.2.9 yields the desired estimate. □

Proposition 5.4.10. *Let $X = X(A, P, \Sigma)$ arise from Construction 5.3.8. Assume that X is $\mathbb{Q}$-factorial. If Σ admits a big cone, then it admits an elementary big cone.*

Proof. Let $\sigma \in \Sigma$ be a big cone. Then σ is X-relevant according to Proposition 5.4.10. Proposition 5.2.3 tells us that σ is simplicial. Now, any elementary big face of σ is as wanted. □

Corollary 5.4.11. *Let $X = X(A, P, \Sigma)$ arise from Construction 5.3.8. Assume that X is non-toric, projective and locally factorial. If X is of Picard number $\rho(X) \leq c + 3$, then Σ admits an elementary big cone.*

Definition 5.4.12. Let $X = X(A, P, \Sigma)$ arise from Construction 5.3.8. We say that X is *quasismooth* if for every X-relevant face $\gamma_0 \preccurlyeq \gamma$, the set $\overline{X}(\gamma_0) \subseteq \overline{X}$ consists of smooth points of $\overline{X}$.

Proposition 5.4.13. *Let $X = X(A, P, \Sigma)$ arise from Construction 5.3.8. Assume that P is irredundant, X is quasismooth and $\sigma = \operatorname{cone}(v_{0j_0} + \ldots + v_{rj_r})$ is an elementary big cone of Σ.*

(i) *We have $l_{ij_i} \geq 2$ for at most $c + 1$ different $i = 0, \ldots, r$.*
(ii) *We have $n_i = 1$ for at most $c + 1$ different $i = 0, \ldots, r$.*

Proof. We have $\sigma = P(\gamma_0^*)$ with an X-relevant face $\gamma_0 \preccurlyeq \gamma$. Since X is quasismooth, every $z \in \overline{X}(\gamma_0)$ is a smooth point of $\overline{X}$ and thus the Jacobian $J(z)$ is of full rank. Because of $z_{ij_i} = 0$ for every $i = 0, \ldots, r$, Lemma 5.3.7 implies that $l_{ij_i} \geq 2$ or $n_i \geq 2$ can hold for at most $c + 1$ different i. □

Example 5.4.14. We continue Example 5.4.2. Note that suitably renumbering the variables we achieve $a \geq b$. Then X is smooth if and only if one of the following conditions is satisfied.

(i) We have $r = c + 1$, $a \geq 1$ and $b = 1$.
(ii) We have $r = c + 2$ and $a = b = 1$ holds.

Indeed, one first checks that $\operatorname{cone}(v_{0j_0}, \ldots, v_{rj_r})$, where $\{j_0, \ldots, j_r\}$ equals $\{1, 2\}$, are precisely the elementary big cones of Σ. Then Lemma 5.3.7 (ii) and Proposition 5.4.13 verify the claim.

Proposition 5.4.15. *Let X be a non-toric, smooth, projective arrangement variety of Picard number one. Then X is a quadric $V(T_0^2 + \ldots + T_r^2) \subseteq \mathbb{P}_r$.*

Proof. According to Theorem 5.3.12, we may assume $X = X(A, P, \Sigma)$ is as in Construction 5.3.8. Moreover, we may assume that P is irredundant and $n_0 \geq \ldots \geq n_r$ holds. Finally, we have $K_{\mathbb{Q}} = \mathbb{Q}$ and may assume that the effective cone of X is $\mathbb{Q}_{\geq 0}$.

First we show that $m = 0$ holds. Otherwise, consider the X-relevant face $\gamma_1 = \operatorname{cone}(e_1) \preceq \gamma$. Smoothness of X implies that the Jacobian of $g_1, \ldots, g_{r-c}$ does not vanish at the point $x_1 \in \overline{X}(\gamma_1)$ having $x_1 = 1$ as its only non-zero coordinate; see Proposition 5.2.4. This implies $l_{i1} + \ldots + l_{in_i} = 1$ for some i, contradicting irredundance of P.

According to Corollary 5.4.11, the fan Σ admits an elementary big cone. Proposition 5.4.13 tells us $n_0 \geq 2$. Thus $\gamma_{0j} = \operatorname{cone}(e_{0j}) \preceq \gamma$ is an X-relevant face. Proposition 5.2.4 yields that $\deg(T_{0j})$ generates K. We conclude $K = \mathbb{Z}$ and $\deg(T_{0j}) = 1$. Additionally, smoothness of $X(\gamma_{01})$ implies that $\operatorname{grad}(g_1)(x) \neq 0$ holds for every point $x \in \overline{X}(\gamma_{01})$. We conclude $n_0 = 2$ and $\deg(g_1) = 2$. This implies $\deg(T_{ij}) = 1$ and for all i, j, we obtain $l_{ij} = 1$ or $l_{ij} = 2$ according to $n_i = 2$ or $n_i = 1$.

Finally, observe that $c = r-1$ holds, i.e., that there is only one defining relation. Indeed, otherwise, we find generators $g'_1, \ldots, g'_{r-c}$, each involving precisely $c+2$ monomials and g'_{r-c} all different from $T_0^{l_0}$. Then the corresponding Jacobian vanishes at any $x \in \overline{X}(\gamma_{01})$, showing that $X(\gamma_{01})$ is singular. A contradiction. $\square$

Remark 5.4.16. Consider $X = X(A, P, \Sigma)$ as in Construction 5.3.8 such that X is smooth, projective, of Picard number one and P is irredundant. By Proposition 5.4.15, the divisor class group $\mathrm{Cl}(X)$ is torsion free. Thus, Proposition 5.4.3 yields

$$P_0 \;=\; \begin{bmatrix} -l_0 & l_1 & & 0 \\ \vdots & & \ddots & \\ -l_0 & 0 & & l_r \end{bmatrix}, \qquad l_0 = \ldots = l_{r-1} = (1,1), \quad l_r \;=\; \begin{cases} (1,1), & n \text{ even}, \\ (2), & n \text{ odd}. \end{cases}$$

Moreover, the torus action on X is the action of the maximal torus of $\mathrm{Aut}(X) = \mathrm{O}(n)$. In particular, the torus action on X is of complexity

$$c \;=\; \begin{cases} \frac{n}{2} - 2, & n \text{ even}, \\ \frac{n-1}{2} - 1, & n \text{ odd}. \end{cases}$$

CHAPTER

SIX

CLASSIFICATION RESULTS FOR SMOOTH ARRANGEMENT VARIETIES WITH $\rho(X) = 2$

In this chapter we contribute to the classification of smooth Fano varieties. In the toric case classification was done up to dimension nine by work of V. Batyrev, M. Kreuzer, B. Nill, M. Øbro and A. Paffenholz [10, 11, 60, 73, 78]. Extending recent classification work in complexity one [35], we consider smooth arrangement varieties of Picard number at most two. For Picard number one, we obtained in Proposition 5.4.15 precisely the projective quadrics. The situation in Picard number two is much more ample: In Section 6.1 we derive constraints on the defining data and prove our classification results, which are listed in Section 6.2.

6.1 Towards the classification

According to Theorem 5.3.12, we may assume that X arises from Construction 5.3.8. Here are first bounds on the defining data.

Proposition 6.1.1. *Let $X = X(A, P, \Sigma)$ arise from Construction 5.3.8, where P is irredundant and we have $n_0 \geq \ldots \geq n_r$. Assume that X is smooth, projective of Picard number two and that the torus action is of complexity two. Then we have $\mathrm{Cl}(X) = \mathbb{Z}^2$ and one of the following statements holds.*

(I) We have $r = 3$ and the tuple (n_0, n_1, n_2, n_3) together with the number m fits into one of the cases below, where $n_0 \geq n_1 \geq 3$:

- *(a) $m \geq 0$ and $(n_0, n_1, 2, 2)$,*
- *(b) $m \geq 0$ and $(n_0, 2, 2, 2)$,*
- *(c) $m \geq 0$ and $(n_0, 2, 2, 1)$,*
- *(d) $m = 0$ and $(3, 2, 1, 1)$,*
- *(e) $m = 0$ and $(3, 1, 1, 1)$,*
- *(f) $m \geq 0$ and $(2, 2, 2, 2)$,*
- *(g) $m \geq 0$ and $(2, 2, 2, 1)$,*
- *(h) $m \geq 0$ and $(2, 2, 1, 1)$,*
- *(i) $m > 0$ and $(2, 1, 1, 1)$.*

(II) We have $r = 4$ and $m = 0$ and the tuple $(n_0, n_1, n_2, n_3, n_4)$ is one of

$$(2,2,2,2,2),\ (2,2,2,2,1),\ (2,2,2,1,1),\ (2,2,1,1,1).$$

The proposition is a direct consequence of the more general statements 6.1.5, 6.1.6 and 6.1.7, presented and proven below. As in the corresponding case of complexity one, elaborated, in [35], the idea is to extract bounding conditions on the defining data of X from smoothness of suitable small strata $X(\gamma_0) \subseteq X$. The following applies to arbitrary $X(A, P, \Sigma)$ and generalizes [35, Lemma 3.9].

Lemma 6.1.2. *Situation as in Construction 5.3.8. Consider the orthant $\gamma = \mathbb{Q}_{\ge 0}^{n+m}$, its extremal rays $\gamma_{ij} := \mathrm{cone}(e_{ij})$ and $\gamma_k := \mathrm{cone}(e_k)$ and the two-dimensional faces*

$$\gamma_{k_1,k_2} := \gamma_{k_1} + \gamma_{k_2}, \quad \gamma_{ij,k} := \gamma_{ij} + \gamma_k, \quad \gamma_{i_1j_1,i_2j_2} := \gamma_{i_1j_1} + \gamma_{i_2j_2}.$$

(i) *All γ_k, resp. γ_{k_1,k_2}, are $\overline{X}$-faces and each $\overline{X}(\gamma_k)$, resp. $\overline{X}(\gamma_{k_1,k_2})$, consists of singular points of $\overline{X}$.*
(ii) *A given γ_{ij}, resp. $\gamma_{ij,k}$, is an $\overline{X}$-face if and only if $n_i \ge 2$ holds. In that case, $\overline{X}(\gamma_{ij})$, resp. $\overline{X}(\gamma_{ij,k})$, consists of smooth points of $\overline{X}$ if and only if $r = c+1$, $n_i = 2$ and $l_{i,3-j} = 1$ hold.*
(iii) *A given γ_{ij_1,ij_2} with $j_1 \ne j_2$ is an $\overline{X}$-face if and only if $n_i \ge 3$ holds. In that case, $\overline{X}(\gamma_{ij_1,ij_2})$ consists of smooth points of $\overline{X}$ if and only if $r = c+1$, $n_i = 3$ and $l_{ij} = 1$ for the $j \ne j_1, j_2$ hold.*
(iv) *A given $\gamma_{i_1j_1,i_2j_2}$ with $i_1 \ne i_2$ is an $\overline{X}$-face if and only if we have either $n_{i_1}, n_{i_2} \ge 2$ or $n_{i_1} = n_{i_2} = 1$ and $r = c+1$. In the former case $\overline{X}(\gamma_{i_1j_1,i_2j_2})$ consists of smooth points of $\overline{X}$ if and only if one of the following holds:*
   - *$r = c+1$, $n_{i_t} = 2$ and $l_{i_t,3-j_t} = 1$ for a $t \in \{1,2\}$,*
   - *$r = c+2$, $n_{i_1} = n_{i_2} = 2$, $l_{i_1,3-j_1} = l_{i_2,3-j_2} = 1$.*

Proof. Lemmas 5.3.6 and 5.3.7 directly yield the assertions. □

Observe that the above statements (iii), (iv) and (v) depend on the complexity c. To proceed, we have to figure out the X-relevant ones from the above $\overline{X}$-faces in our concrete situation. Propositions 5.2.3 and 5.2.5 lead to the following description.

Remark 6.1.3. Let $X = X(A, P, \Sigma)$ arise from Construction 5.3.8. Assume that X is projective and has divisor class group $\mathrm{Cl}(X)$ of rank two. Then the effective cone of X is of dimension two and decomposes as

$$\mathrm{Eff}(X) \ = \ \tau^+ \cup \tau_X \cup \tau^-,$$

where $\tau_X \subseteq \mathrm{Eff}(X)$ is the ample cone, τ^+, τ^- are closed cones not intersecting τ_X and $\tau^+ \cap \tau^-$ consists of the origin. Due to $\tau_X \subseteq \mathrm{Mov}(X)$, each of the cones τ^+ and τ^- contains at least two of the weights

$$w_{ij} \ = \ \deg(T_{ij}) \ = \ Q(e_{ij}), \qquad\qquad w_k \ = \ \deg(S_k) \ = \ Q(e_k).$$

Moreover, for every $\overline{X}$-face $\{0\} \neq \gamma_0 \preccurlyeq \gamma$ precisely one of the following inclusions holds:

$$Q(\gamma_0) \ \subseteq \ \tau^+, \qquad \tau_X \ \subseteq \ Q(\gamma_0)^\circ, \qquad Q(\gamma_0) \ \subseteq \ \tau^-.$$

The X-relevant faces are exactly the $\overline{X}$-faces $\gamma_0 \preccurlyeq \gamma$ with $\tau_X \subseteq Q(\gamma_0)^\circ$. Note that the ample cone τ_X is of dimension two if and only if X is $\mathbb{Q}$-factorial.

Lemma 6.1.4. *Let $X = X(A, P, \Sigma)$ arise from Construction 5.3.8. Assume that X is projective and has divisor class group $\mathrm{Cl}(X)$ of rank two.*

(i) *Suppose that X is $\mathbb{Q}$-factorial. Then $w_k \notin \tau_X$ holds for all $1 \leq k \leq m$ and for all $0 \leq i \leq r$ with $n_i \geq 2$ we have $w_{ij} \notin \tau_X$, where $1 \leq j \leq n_i$.*
(ii) *Suppose that X is quasismooth, $m > 0$ holds and there is $0 \leq i_1 \leq r$ with $n_{i_1} \geq 3$. Then the w_{ij}, w_k with $n_i \geq 3$, $j = 1, \ldots, n_i$ and $k = 1, \ldots, m$ lie either all in τ^+ or all in τ^-.*
(iii) *Suppose that X is quasismooth and there is $0 \leq i_1 \leq r$ with $n_{i_1} \geq 4$. Then the w_{ij} with $n_i \geq 4$ and $j = 1, \ldots, n_i$ lie either all in τ^+ or all in τ^-.*
(iv) *Suppose that X is quasismooth and there exist $0 \leq i_1 < i_2 \leq r$ with $n_{i_1}, n_{i_2} \geq 3$. Then the w_{ij} with $n_i \geq 3$, $j = 1, \ldots, n_i$ lie either all in τ^+ or all in τ^-.*
(v) *Suppose that X is quasismooth. Then $w_1, \ldots, w_m$ lie either all in τ^+ or all in τ^-.*

Proof. Follow the lines of the proof of [35, Lemma 3.11], replacing [35, Lemma 3.9] with the more general Lemma 6.1.2. □

Proposition 6.1.5. *Let $X = X(A, P, \Sigma)$ arise from Construction 5.3.8, where P is irredundant and $n_0 \geq \ldots \geq n_r$ holds. Let X be non-toric, projective, quasismooth with divisor class group of rank two. Assume that $m > 0$ holds and that Σ admits an elementary big cone.*

(i) *We have $r = c + 1$ and are in one of the following situations:*
   (a) *We have $n_0 = 2$ and there exist indices i and j such that $n_i = 2$ holds and $\gamma_{ij,k}$ is X-relevant for all k.*
   (b) *We have $n_0 \geq 3$ and there exist indices $i_1 \neq i_2$ and j_1, j_2 such that $n_{i_1} = n_{i_2} = 2$ holds and $\gamma_{i_1 j_1,k}, \gamma_{i_2 j_2,k}$ are X-relevant for all k.*
(ii) *Assume $c = 2$. Then we have $r = 3$ and the constellation of the n_i is $(n_0, n_1, 2, 2)$, $(n_0, 2, 2, 2)$, $(n_0, 2, 2, 1)$ $(2, 2, 2, 2)$, $(2, 2, 2, 1)$, $(2, 2, 1, 1)$ or $(2, 1, 1, 1)$, where $n_0 \geq n_1 \geq 3$.*

Proof. Due to Lemma 6.1.4 (v), we may assume $w_1, \ldots, w_m \in \tau^+$. As X is non-toric we have at least one relation g_1. Thus, $r \geq c + 1$ holds and Proposition 5.4.13 (ii) yields $n_0 \geq 2$. Lemma 6.1.4 (i) says that none of the w_{ij} with $n_i \geq 2$ lies in τ_X. Moreover, at least one of the w_{ij} with $n_i \geq 2$ lies in τ^-; otherwise, since all relations g_i share the same degree, we had $w_{i1} \in \tau^+$ for all i with $n_i = 1$, meaning that τ^- contains no weights at all; a contradiction. In particular, if $n_0 = 2$ holds, then there exists a $w_{ij} \in \tau^-$ with $n_i = 2$ and all $\gamma_{ij,k}$ are X-relevant. Assume $n_0 \geq 3$. Then Lemma 6.1.4 (ii) yields $w_{ij} \in \tau^+$ whenever $n_i \geq 3$. Moreover, because all relations g_i have the same degree, $w_{ij} \in \tau^+$ holds for all i with $n_i = 1$. Since τ^- contains at least two weights, we find

i_1, i_2 and j_1, j_2 with $n_{i_1} = n_{i_2} = 2$ and $w_{i_1j_1}, w_{i_2j_2} \in \tau^-$. Note that all $\gamma_{i_1j_1,k}, \gamma_{i_2j_2,k}$ are X-relevant. Now, Lemma 6.1.2 (ii) yields $r = c + 1$. Thus, Assertion (i) is proven. Assertion (ii) is a direct consequence. $\square$

Proposition 6.1.6. *Let $X = X(A, P, \Sigma)$ arise from Construction 5.3.8, where P is irredundant and $n_0 \geq \ldots \geq n_r$ holds. Let X be non-toric, projective, quasismooth with divisor class group of rank two. Assume $m = 0$ holds and that Σ admits an elementary big cone.*

(i) *We are in one of the following situations:*
  - *(a) We have $r = c + 1$, $n_0 = 3 > n_1$ and there exists an index j such that $\gamma_{01,0j}$ is X-relevant.*
  - *(b) We have $r = c + 1$ and there exist indices $0 \leq i_1 < i_2$ with $n_{i_1} = n_{i_2} = 2$ and indices j_0, j_2 such that γ_{0j_0,i_2j_2} is X-relevant.*
  - *(c) We have $r = c + 2$ and $n_0 = n_1 = 2$ and there exist indices $0 < i_1$ and j_0, j_1 such that γ_{0j_0,i_1j_1} is X-relevant.*

(ii) *Assume $c = 2$. Then the constellation of the n_i is one of the following, where $n_0 \geq n_1 \geq 3$ holds:*

$$
\begin{aligned}
r = 3: \quad & (n_0, n_1, 2, 2), (n_0, 2, 2, 2), (n_0, 2, 2, 1), (3, 2, 1, 1), (3, 1, 1, 1), \\
& (2, 2, 2, 2), (2, 2, 2, 1), (2, 2, 1, 1). \\
r = 4: \quad & (2, 2, 2, 2, 2), (2, 2, 2, 2, 1), (2, 2, 2, 1, 1), (2, 2, 1, 1, 1).
\end{aligned}
$$

Proof. Only for the first assertion, there is something to show. As X is non-toric we have at least one relation g_1 and conclude $r \geq c + 1$. Moreover, Proposition 5.4.13 (ii) yields $n_0 \geq 2$. Finally, Lemma 6.1.4 (i) shows that none of the w_{ij} with $n_i \geq 2$ lies in τ_X. We distinguish the following cases.

First, let $n_0 \geq 4$ or $n_0 = n_1 = 3$. By Lemma 6.1.4 (iii) and (iv), we may assume $w_{ij} \in \tau^+$ for all i with $n_i \geq 3$. Then $w_{ij} \in \tau^+$ holds as well for all i with $n_i = 1$. Since τ^- contains at least two weights, there are $i_1 < i_2$ and j_1, j_2 with $n_{i_1} = n_{i_2} = 2$ and $w_{i_1j_1}, w_{i_2j_2} \in \tau^-$. Observe that γ_{01,i_2j_2} is X-relevant. Moreover, Lemma 6.1.2 (iv) shows $r = c + 1$. We arrive at Case (b) of (i).

Next, let $n_0 = 3 > n_1$. If all weights w_{0j} lie either in τ^+ or in τ^-, then we can argue as above and end up in Case (b) of (i). Otherwise, w_{01} and some w_{0j} for $j = 2, 3$ lie on different sides of τ_X. Then $\gamma_{01,0j}$ is X-relevant. Lemma 6.1.2 (iii) yields $r = c + 1$ and we are in Case (a) of (i).

Finally, let $n_0 = 2$. The common degree of $g_1, \ldots, g_{r-c}$ and hence all w_{ij} with $n_i = 1$ lie in precisely one of the cones τ^+, τ^- or τ_X, where we may assume that this is not τ^-. Then no pair w_{i1}, w_{i2} lies in τ^-. As there must be at least two weights in τ^-, we conclude $n_1 = 2$ and find the desired $\gamma_{0j_0,1j_1}$. Lemma 6.1.2 (iv) yields $r \leq c + 2$. Thus, we are in one of the Cases (b) or (c) of (i). $\square$

Corollary 6.1.7. *Let X be a smooth projective arrangement variety of Picard number two. Then we have* $\mathrm{Cl}(X) = \mathrm{Pic}(X) = \mathbb{Z}^2$.

Proof. Corollary 5.4.11 tells us that Σ admits an elementary big cone. Thus Propositions 6.1.5 and 6.1.6 provide an X-relevant face $\gamma_0 \preccurlyeq \gamma$. Then the two weights stemming from γ_0 generate K as a group. This implies $\mathrm{Cl}(X) \cong K \cong \mathbb{Z}^2$. □

The further proof of Theorems 6.2.1, 6.2.2 and 6.2.4 goes through the list of cases established in Proposition 6.1.1.

Remark 6.1.8. Let $X = X(A, P, \Sigma)$ as in Construction 5.3.8. be smooth, projective and of Picard number two. Corollary 6.1.7 ensures $\mathrm{Cl}(X) = \mathbb{Z}^2$ and we will write

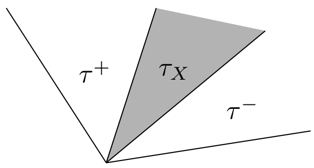
$$\deg(T_{ij}) \;=\; Q(e_{ij}) \;=\; w_{ij} \;=\; (x_{ij}, y_{ij}) \;\in\; \mathbb{Z}^2,$$

$$\deg(T_k) \;=\; Q(e_k) \;=\; w_k \;=\; (x_k, y_k) \;\in\; \mathbb{Z}^2$$

for the weights. Moreover, the (common) degree of the relations $g_1, \ldots, g_{r-c}$ will be denoted as $\deg(g_i) = \mu = (\mu_1, \mu_2) \in \mathbb{Z}^2$. Recall that for each $i = 0, \ldots, r$ we have

$$\mu_1 \;=\; \sum_{j=1}^{n_i} l_{ij} x_{ij}, \qquad\qquad \mu_2 \;=\; \sum_{j=1}^{n_i} l_{ij} y_{ij}.$$

Consider the decomposition of the effective cone $\mathrm{Eff}(X) = \tau^- \cup \tau_X \cup \tau^+$ from Remark 6.1.3. Choosing names suitably, we can fix the following orientation:



If a pair $w, w' \in \mathbb{Q}^2$ is positively oriented, for instance $w \in \tau^-$ and $w' \in \tau^+$, then $\det(w, w')$ is positive. Moreover, if w, w' are the weights stemming from a two-dimensional X-relevant face $\gamma_0 \preccurlyeq \gamma$, then we have $\det(w, w') = 1$ by Proposition 5.2.4. In that case, we can achieve

$$w \;=\; (1, 0), \qquad\qquad w' \;=\; (0, 1)$$

by a suitable unimodular coordinate change on $\mathbb{Z}^2$. Then $w'' = (x'', 1)$ holds whenever w, w'' stems from a two-dimensional X-relevant face and, similary, $w'' = (1, y'')$ holds whenever w'', w' stems from a two-dimensional X-relevant face.

Lemma 6.1.9. *In the situation of Proposition 6.1.1, consider the case $r = 3$, $m \geq 0$ and $n_0 \geq 3 > n_1 = n_2 = 2 \geq n_3$. Then the following constellation of weights can't occur:*

$$w_{01}, \ldots, w_{0n_0}, w_{12}, w_{22} \;\in\; \tau^+, \qquad\qquad w_{11}, w_{21} \;\in\; \tau^-.$$

Proof. We may assume $w_{02}, \ldots, w_{0n_0}, w_{21} \in \operatorname{cone}(w_{01}, w_{11})$. Applying Remark 6.1.8 at first to $\gamma_{01,11} \in \operatorname{rlv}(X)$ and then to all $\gamma_{01,21}, \gamma_{22,11}, \gamma_{0j,11}, \gamma_{i,11} \in \operatorname{rlv}(X)$, where $j = 1, \ldots, n_0$ and $i = 1, \ldots, m$, turns the degree matrix Q into the shape

$$Q = \left[\begin{array}{cccc|cc|cc|ccc||ccc} 0 & x_{02} & \ldots & x_{0n_0} & 1 & x_{12} & 1 & x_{22} & x_{31} & \ldots & x_{3n_1} & x_1 & \ldots & x_m \\ 1 & 1 & \ldots & 1 & 0 & y_{12} & y_{21} & 1 & y_{31} & \ldots & y_{3n_1} & 1 & \ldots & 1 \end{array}\right],$$

where $x_{0j}, y_{21} \geq 0$ holds. Moreover, $\gamma_{01,11}, \gamma_{01,21} \in \operatorname{rlv}(X)$ implies $l_{12} = l_{22} = 1$ due to Lemma 6.1.2 (iv). With $\gamma_{21,12} \in \operatorname{rlv}(X)$ we infer $y_{12} = 1 + y_{21}x_{12}$ from $\det(w_{21}, w_{12}) = 1$ and, by the shape of Q, obtain

$$3 \leq l_{01} + \cdots + l_{0n_0} = \mu_2 = y_{12} = 1 + y_{21}x_{12}.$$

We conclude $x_{12} > 0$. Using $\gamma_{0j,21} \in \operatorname{rlv}(X)$ gives $\det(w_{21}, w_{0j}) = 1$ and thus $x_{0j}y_{21} = 0$. As the effective cone of X is pointed, $w_{21} \in \tau^-$ implies $y_{21} > 0$. We arrive at $x_{0j} = 0$ and thus $\mu_1 = 0 = l_{11} + x_{12}$. A contradiction to $l_{11}, x_{12} > 0$. □

Case 6.1.1 (I)(a). *We have $r = 3$, $m \geq 0$ and $n_0 \geq n_1 \geq 3 > n_2 = n_3 = 2$. This setting allows no examples satisfying the assumptions of Theorem 6.2.1.*

Proof. By Lemma 6.1.4 (iv) and (ii), we may assume that the weights $w_{01}, \ldots, w_{0n_0}$, $w_{11}, \ldots, w_{1n_1}$ and $w_1, \ldots, w_m$ all lie in τ^+. At least two other weights lie in τ^-. Renumbering suitably, we arrive at $w_{21}, w_{31} \in \tau^-$ and $w_{22}, w_{32} \in \tau^+$ because of $\mu \in \tau^+$. Thus, Lemma 6.1.9 gives the assertion. □

Case 6.1.1 (I)(b). *We have $r = 3$, $m \geq 0$ and $n_0 \geq 3 > n_1 = n_2 = n_3 = 2$. This gives the varieties Nos. 1 and 2 of Theorem 6.2.1.*

Proof. We claim that each of τ^+ and τ^- contains weights from $w_{01}, \ldots, w_{0n_0}$. Otherwise, due to Lemma 6.1.4 (i), we may assume that all w_{0j} lie in τ^+. If $m > 0$ holds, Lemma 6.1.4 (ii) yields $w_1, \ldots, w_m \in \tau^+$. As τ^- contains at least two weights, we can achieve $w_{11}, w_{21} \in \tau^-$ and $w_{12}, w_{22} \in \tau^+$ by suitable renumbering; note that $w_{i1}, w_{i2} \in \tau^-$ is not possible for $i = 1, 2, 3$ because of $\mu \in \tau^+$. Lemma 6.1.9 then verifies the claim.

By the claim, we may assume $w_{01}, w_{02} \in \tau^+$ and $w_{03} \in \tau^-$. Lemma 6.1.4 (ii) shows $m = 0$ and Lemma 6.1.4 (iii) gives $n_0 = 3$. There must be at least one more weight in τ^-, say w_{11}. Applying Lemma 6.1.2 (iii) to $\gamma_{0j,03} \in \operatorname{rlv}(X)$ we obtain $l_{01} = l_{02} = 1$. Applying Lemma 6.1.2 (iv) to suitable $\gamma_{0j,i_2j_2} \in \operatorname{rlv}(X)$, we obtain

$$l_{11} = l_{12} = l_{21} = l_{22} = l_{31} = l_{32} = 1.$$

We may assume $w_{02} \in \operatorname{cone}(w_{01}, w_{03})$. Then, applying Remark 6.1.8 to $\gamma_{01,03} \in \operatorname{rlv}(X)$ and afterwards to $\gamma_{01,11}, \gamma_{02,03} \in \operatorname{rlv}(X)$ turns the degree matrix Q into the following shape

$$Q := \left[\begin{array}{ccc|cc|cc|cc} 0 & x_{02} & 1 & 1 & x_{12} & x_{21} & x_{22} & x_{31} & x_{32} \\ 1 & 1 & 0 & y_{11} & y_{12} & y_{21} & y_{22} & y_{31} & y_{32} \end{array}\right].$$

Note that we have $x_{02} \geq 0$ because of $w_{02} \in \operatorname{cone}(w_{01}, w_{03})$. We distinguish the following three cases according to the possible positions of the weights w_{21} and w_{22}.

We have $w_{21}, w_{22} \in \tau^-$. Then $\mu \in \tau^-$ holds and we may assume $w_{31} \in \tau^-$. Moreover, we have $\gamma_{01,21}, \gamma_{01,22}, \gamma_{01,31} \in \operatorname{rlv}(X)$ and conclude

$$x_{21} = x_{22} = x_{31} = 1, \qquad \mu = (2,2), \qquad x_{12} = x_{22} = x_{32} = 1.$$

The determinants corresponding to $\gamma_{02,21}, \gamma_{02,22} \in \operatorname{rlv}(X)$ both equal one, which implies $y_{21}x_{02} = 0$ and $y_{22}x_{02} = 0$. Because of $y_{21} + y_{22} = \mu_2 = 2$, we obtain

$$x_{02} = 0, \qquad\qquad l_{03} = \mu_1 = 2.$$

The considerations performed so far show that the defining relation g_1 and the degree matrix Q are of the following shape:

$$g_1 = T_{01}T_{02}T_{03}^2 + T_{11}T_{12} + T_{21}T_{22} + T_{31}T_{32},$$

$$Q = \left[\begin{array}{ccc|cc|cc|cc} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & a_1 & 2-a_1 & a_2 & 2-a_2 & a_3 & 2-a_3 \end{array}\right].$$

We claim that all w_{ij}, where $i = 1,2,3$, lie in τ^-. That means that we have to show $w_{12}, w_{32} \in \tau^-$. Otherwise, if $w_{12} \in \tau^+$ holds, then $\gamma_{03,12} \in \operatorname{rlv}(X)$ leads to

$$1 = \det(w_{03}, w_{12}) = a_1.$$

This implies $w_{11} = w_{21} \in \tau^- \cap \tau^+$, which is impossible. Analogously, one excludes $w_{32} \in \tau^+$. Thus, we may assume $a_1 \leq a_2 \leq a_3$ and $a_i \geq 2 - a_i$. The latter implies $a_i \geq 1$ and

$$\operatorname{SAmple}(X) = \overline{\tau_X} = \operatorname{cone}((1, a_3), (0,1)).$$

We have $w_{21}, w_{22} \in \tau^+$. Then we have $\mu \in \tau^+$ and thus $w_{12} \in \tau^+$. Consequently $\gamma_{03,12}, \gamma_{03,21}, \gamma_{21,22} \in \operatorname{rlv}(X)$ holds and we conclude

$$y_{12} = y_{21} = y_{22} = 1, \qquad \mu_2 = 2, \qquad y_{11} = 1.$$

Looking at the determinants associated with $\gamma_{02,11}, \gamma_{11,21}, \gamma_{11,22} \in \operatorname{rlv}(X)$ we see $x_{02} = x_{21} = x_{22} = 0$. This gives $l_{03} = \mu_1 = x_{21} + x_{22} = 0$. A contradiction.

We have $w_{21} \in \tau^-$ *and* $w_{22} \in \tau^+$. Then we may assume $w_{31} \in \tau^-$ and $w_{32} \in \tau^+$, as otherwise, up to renumbering, we are in one of the preceding cases. Applying Remark 6.1.8 to $\gamma_{01,21}, \gamma_{01,31}, \gamma_{03,22}, \gamma_{03,32} \in \operatorname{rlv}(X)$ and using $\mu_2 = 2$, one obtains

$$x_{21} = x_{31} = y_{22} = y_{32} = 1, \qquad\qquad y_{21} = y_{31} = 1.$$

We claim $y_{11} \neq 0$. Otherwise, $y_{12} = \mu_2 = 2$ holds. This implies $\det(w_{03}, w_{12}) = 2$, hence $\gamma_{03,12} \notin \operatorname{rlv}(X)$ and thus $w_{12} \in \tau^-$. Then $\gamma_{01,12} \in \operatorname{rlv}(X)$ leads to $x_{12} = 1$ and $\mu_1 = 2$. Thus $w_{22} = (1,1) = w_{21} \in \tau^-$. A contradiction. Now, $y_{11} \neq 0$ yields

$$x_{02} = x_{22} = x_{32} = 0, \qquad \mu = (1,2), \qquad l_{03} = 1, \qquad x_{12} = 0.$$

due to $\gamma_{11,02}, \gamma_{11,22}, \gamma_{11,32} \in \mathrm{rlv}(X)$ and homogeneity of the relation g_1. We conclude $w_{12} = (0, y_{12}) \in \tau^+$ and $\gamma_{03,12} \in \mathrm{rlv}(X)$ shows $y_{12} = 1$. Finally, $y_{11} = \mu_2 - y_{12} = 1$ holds. For the relation, the degree matrix and the ample cone this means

$$\begin{aligned} g_1 &= T_{01}T_{02}T_{03} + T_{11}T_{12} + T_{21}T_{22} + T_{31}T_{32}, \\ Q &= \left[\begin{array}{ccc|cc|cc|cc} 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right], \\ \mathrm{SAmple}(X) &= \overline{\tau_X} = \mathrm{cone}((0,1),(1,1)). \end{aligned}$$

□

Case 6.1.1 (I)(c). *We have $r = 3$, $m \geq 0$ and $n_0 \geq 3 > n_1 = n_2 = 2 > n_1 = 1$. This gives the variety No. 3 in the Theorems 6.2.1 and 6.2.2.*

Proof. We claim that each of τ^+ and τ^- contains weights from $w_{01}, \dots, w_{0n_0}$. Otherwise, due to Lemma 6.1.4 (i), we may assume that all w_{0j} lie in τ^+. Then we have $\mu \in \tau^+$ and thus $w_{31} \in \tau^+$. If $m > 0$ holds, Lemma 6.1.4 (ii) yields $w_1, \dots, w_m \in \tau^+$. As τ^- contains at least two weights, we can achieve $w_{11}, w_{21} \in \tau^-$ and $w_{12}, w_{22} \in \tau^+$ by suitable renumbering; note that $w_{i1}, w_{i2} \in \tau^-$ is not possible for $i = 1, 2$ because of $\mu \in \tau^+$. Lemma 6.1.9 then verifies the claim.

By the claim, we may assume $w_{01}, w_{02} \in \tau^+$ and $w_{03} \in \tau^-$. Lemma 6.1.4 (ii) shows $m = 0$ and Lemma 6.1.4 (iii) gives $n_0 = 3$. We claim that $w_{i1}, w_{i2} \in \tau^+$ is not possible for $i = 1, 2$. Otherwise $\mu \in \tau^+$ implies $w_{31} \in \tau^+$ and there is no weight left to lie in τ^-. Thus we may assume $w_{11} \in \tau^-$. Applying Lemma 6.1.2 (iii) to $\gamma_{0j,03} \in \mathrm{rlv}(X)$ we obtain $l_{01} = l_{02} = 1$. Applying Lemma 6.1.2 (iv) to suitable $\gamma_{0j,i_2j_2} \in \mathrm{rlv}(X)$, where $n_{i_2} = 2$, we obtain

$$l_{11} = l_{12} = l_{21} = l_{22} = 1.$$

We may assume $w_{02} \in \mathrm{cone}(w_{01}, w_{03})$. Then applying Remark 6.1.8 to $\gamma_{01,03}$ and afterwards to $\gamma_{01,11}, \gamma_{02,03} \in \mathrm{rlv}(X)$ turns the degree matrix Q into the following shape:

$$Q = \left[\begin{array}{ccc|cc|cc|c} 0 & x_{02} & 1 & 1 & x_{12} & x_{21} & x_{22} & x_{31} \\ 1 & 1 & 0 & y_{11} & y_{12} & y_{21} & y_{22} & y_{31} \end{array}\right],$$

Note that because of $w_{02} \in \mathrm{cone}(w_{01}, w_{03})$ we have $x_{02} \geq 0$. We distinguish the following three cases according to the possible positions of the weights w_{21}, w_{22}.

We have $w_{21}, w_{22} \in \tau^-$. Then $\mu \in \tau^-$ and thus $w_{31} \in \tau^-$. Moreover we have $\gamma_{01,21}, \gamma_{01,22} \in \mathrm{rlv}(X)$ and conclude

$$x_{21} = x_{22} = 1, \quad \mu = (2, 2), \quad x_{12} = 1.$$

Irredundancy of (A, P) implies $l_{31} = 2$ and $x_{31} = y_{31} = 1$. The determinants corresponding to $\gamma_{02,21}, \gamma_{02,22} \in \mathrm{rlv}(X)$ both equal one, which implies $x_{02}y_{21} = 0$ and $x_{02}y_{22} = 0$. Because of $y_{21} + y_{22} = \mu_2 = 2$, we obtain

$$x_{02} = 0, \qquad l_{03} = \mu_1 = 2.$$

The considerations performed so far show that the defining relation g_1 and the degree matrix Q are of the following shape:

$$Q \;=\; g_1 := T_{01}T_{02}T_{03}^2 + T_{11}T_{12} + T_{21}T_{22} + T_{31}^2$$

$$\left[\begin{array}{ccc|cc|cc|c} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & a_1 & 2-a_1 & a_2 & 2-a_2 & 1 \end{array}\right].$$

We claim that $w_{12} \in \tau^-$ and thus all weights w_{ij} for $i = 1, 2, 3$ lie in τ^-. Otherwise $\gamma_{03,12} \in \mathrm{rlv}(X)$ leads to

$$1 = \det(w_{03}, w_{12}) = a_1.$$

This implies $w_{11} = w_{21} \in \tau^+ \cap \tau^-$, which is impossible. Thus we may assume $a_1 \leq a_2$ and $a_i \geq 2 - a_i$. The latter implies $a_i \geq 1$ and

$$\mathrm{SAmple}(X) = \overline{\tau_X} = \mathrm{cone}((0,1),(1,a_2)).$$

The anticanonical class is $-\mathcal{K}_X = (4,5)$. In particular the variety is Fano if and only if $a_1 = a_2 = 1$ holds and there exists no truly almost Fano variety in this case.

We have $w_{21}, w_{22} \in \tau^+$: This case does not provide any smooth projective variety. The proof is exactly the same as in Case 6.1.1 (I)(b) with $w_{21}, w_{22} \in \tau^+$.

We have $w_{21} \in \tau^-, w_{22} \in \tau^+$: Applying Remark 6.1.8 to $\gamma_{01,21}, \gamma_{03,22} \in \mathrm{rlv}(X)$ and using $\mu_2 = 2$ we obtain

$$x_{21} = y_{22} = 1, \qquad y_{21} = 1.$$

We claim $y_{11} \neq 0$. Otherwise $y_{12} = \mu_2 = 2$. This implies $\det(w_{03}, w_{12}) = 2$, hence $\gamma_{03,12} \notin \mathrm{rlv}(X)$ and thus $w_{12} \in \tau^-$. Then $\gamma_{01,12} \in \mathrm{rlv}(X)$ leads to $x_{12} = 1$ and $\mu_1 = 2$. Thus $w_{22} = (1,1) = w_{21} \in \tau^-$. A contradiction. Now $y_{11} \neq 0$ yields

$$x_{22} = 0, \qquad \mu = (1,2), \qquad l_{31} = 1$$

due to $\gamma_{11,22} \in \mathrm{rlv}(X)$. This contradicts irredundancy of (A, P). □

Case 6.1.1 (I) (d) and (e). *We have $r = 3$, $m = 0$ and $n_0 = 3 > n_1 \geq n_2 = n_3 = 1$. This setting allows no examples satisfying the assumptions of Theorem 6.2.1.*

Proof. We claim that each of τ^+ and τ^- contains at least one weight w_{0j}. Otherwise, due to Lemma 6.1.4 (i), we may assume that all w_{0j} lie in τ^+. We conclude $\mu \in \tau^+$ and thus we may assume $w_{i1} \in \tau^+$ for all $i = 1, 2, 3$. As $n_1 \leq 2$ holds there is at most one weight w_{12} left to lie in τ^-. As each of τ^+ and τ^- has to contain at least two weights this is impossible.

By the claim, we may assume $w_{01} \in \tau^+$ and $w_{02}, w_{03} \in \tau^-$. Applying Lemma 6.1.2 (iii) to $\gamma_{01,03} \in \mathrm{rlv}(X)$ and afterwards to $\gamma_{01,02} \in \mathrm{rlv}(X)$ and using Remark 6.1.8 we obtain

$$l_{01} = l_{02} = 1, \qquad w_{01} = (0,1), w_{03} = (1,0), \qquad y_{02} = 1, \qquad \mu_2 = 2.$$

Thus irredundancy of P implies $l_{21} = l_{31} = 2$ and we obtain torsion in $\mathrm{Cl}(X)$. Corollary 6.1.7 gives the assertion. □

Case 6.1.1 (I)(f). *We have $r = 3$, $m \geq 0$ and $n_0 = n_1 = n_2 = n_3 = 2$. This gives the varieties Nos. 4 to 10 in the Theorems 6.2.1, 6.2.2 and 6.2.4,*

Proof. Note that due to Lemma 6.1.4 (i) any weight w_{ij} lies either in τ^+ or in τ^-. Moreover, if there is an integer i_1 such that $w_{i_1 1}, w_{i_1 2} \in \tau^+$ holds, we obtain $\mu \in \tau^+$ and thus we may assume $w_{i1} \in \tau^+$ for each $i = 0, \ldots, 3$. As each of the cones τ^+ and τ^- contains at least two weights we are left with the following five cases according to the possible position of weights.

We have $w_{i1} \in \tau^+$ and $w_{i2} \in \tau^-$ for each $i = 0, \ldots 3$. Applying Lemma 6.1.4 (v) we may assume $w_k \in \tau^+$ for all k. We may further assume $w_{ij} \in \operatorname{cone}(w_{01}, w_{12})$ for all $i = 1, 2, 3$. Then, applying Remark 6.1.8 to $\gamma_{01,12} \in \operatorname{rlv}(X)$ and afterwards to $\gamma_{01,22}, \gamma_{01,32}, \gamma_{12,21}, \gamma_{12,31}, \gamma_{12,k} \in \operatorname{rlv}(X)$, where $k = 1, \ldots, m$, turns the degree matrix into the following shape:

$$Q = \left[\begin{array}{cc|cc|cc|cc||ccc} 0 & x_{02} & x_{11} & 1 & x_{21} & 1 & x_{31} & 1 & x_1 & \ldots & x_m \\ 1 & y_{02} & y_{11} & 0 & 1 & y_{22} & 1 & y_{32} & 1 & \ldots & 1 \end{array}\right].$$

Note that $w_{ij} \in \operatorname{cone}(w_{01}, w_{12})$ implies $x_{ij}, y_{ij} \geq 0$ for all $i = 1, 2, 3$ and thus $\mu_1, \mu_2 > 0$. For any integers $0 \leq i_1, i_2 \leq 3$ with $i_1 \neq i_2$ we have $\gamma_{i_1 1, i_2 2} \in \operatorname{rlv}(X)$. Thus applying Lemma 6.1.2 (iv) yields $l_{i_1 2} = 1$ or $l_{i_2 1} = 1$.

We claim that we may assume $l_{01} = l_{11} = l_{21} = l_{31} = 1$. Otherwise after renumbering we have $l_{31} \geq 2$ and $l_{32} = 1$. Applying Lemma 6.1.2 (iv) to $\gamma_{01,32}, \gamma_{11,32}, \gamma_{21,32} \in \operatorname{rlv}(X)$ we obtain

$$l_{02} = l_{12} = l_{22} = 1.$$

Thus after suitably renumbering we always have $l_{01} = l_{11} = l_{21} = l_{31} = 1$.

Note that due to Remark 6.1.8 the tuples $(w_{02}, w_{01}), (w_{22}, w_{21}), (w_{32}, w_{31})$ are positively oriented and we obtain

$$x_{02} \geq 0, \qquad x_{21} y_{22} = 0, \qquad x_{31} y_{32} = 0.$$

With $0 < \mu_2 = 1 + y_{02} l_{02}$ we conclude $y_{02} \geq 0$ and thus all entries x_{ij}, y_{ij} of Q are non negative.

Considering the determinants corresponding to $\gamma_{02,11}, \gamma_{02,21} \in \operatorname{rlv}(X)$ we obtain

$$x_{02} y_{11} = 1 + y_{02} x_{11}, \qquad x_{02} - y_{02} x_{21} = 1.$$

We claim $y_{22} = 0$. Otherwise $0 = y_{22} x_{21}$ implies $x_{21} = 0$ and we obtain $x_{02} = 1$. We conclude

$$1 + y_{02} x_{11} = y_{11} = \mu_2 = 1 + y_{02} l_{02}.$$

Thus $y_{02} = 0$ or $x_{11} = l_{02}$ holds. Assume $y_{02} = 0$ holds. Then $\mu_2 = 1$ and thus $y_{22} = 0$. A contradiction. If $x_{11} = l_{02}$ holds then

$$l_{02} = l_{02} x_{02} = \mu_1 = x_{11} + l_{12} = l_{02} + l_{12}.$$

This is impossible.

By the claim, we obtain

$$\mu_2 = 1, \qquad y_{02} = y_{32} = 0, \qquad y_{11} = 1.$$

For the relation g_1, the degree matrix Q this means

$$g_1 = T_{01}T_{02}^{l_{02}} + T_{11}T_{12}^{l_{12}} + T_{21}T_{22}^{l_{22}} + T_{31}T_{32}^{l_{32}},$$

$$Q = \left[\begin{array}{cc|cc|cc|cc||ccc} 0 & 1 & a_1 & 1 & a_2 & 1 & a_3 & 1 & d_1 & \dots & d_m \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots & 1 \end{array}\right],$$

where we have $a_i \geq 0$, $l_{02} = a_1 + l_{12} = a_2 + l_{22} = a_3 + l_{33}$ and we may assume $0 \leq a_1 \leq a_2 \leq a_3$ and $d_1 \leq \ldots \leq d_m$. The semiample cone and the anticanonical class are given as

$$\mathrm{SAmple}(X) = \overline{\tau_X} = \mathrm{cone}((1,0),(d,1)), \quad -\mathcal{K}_X = (4 + a_2 + a_3 + \sum d_k - l_{12}, 3 + m),$$

where $d := \max(a_3, d_m)$. In particular X is Fano if and only if the following inequality holds

$$(3+m)\cdot d < 4 + a_2 + a_3 + \sum d_k - l_{12}.$$

With $l_{12} \geq 1$ we conclude $d \leq 2$. We list all possibilities for the entries of Q and the exponents of g_1 in a table.

		$(a_1, a_2, a_3, l_{02}, l_{12}, l_{22}, l_{32})$	restrictions on d_k
$d = 2$	$l_{12} = 1$	$(2,2,2,3,1,1,1)$	$d_k = 2$ for all k.
$d = 1$	$l_{12} = 2$	$(1,1,1,3,2,2,2)$, $(0,1,1,2,2,1,1)$	$d_k = 1$ for all k.
	$l_{12} = 1$	$(1,1,1,2,1,1,1)$	$0 \leq d_1 \leq d_2 = 1$.
$d = 0$	$l_{12} = 3$	$(0,0,0,3,3,3,3)$	$d_k = 0$ for all k
	$l_{12} = 2$	$(0,0,0,2,2,2,2)$	$-1 \leq d_1 \leq d_2 = 0$
	$l_{12} = 1$	$(0,0,0,1,1,1,1)$	$-2 \leq d_1 + d_2 \leq 0$

Moreover X is truly almost Fano if and only if the following equality holds:

$$(3+m)\cdot d = 4 + a_2 + a_3 + \sum d_k - l_{12}.$$

We list all possibilities for the entries of Q and the exponents of g_1 in a table.

		$(a_1, a_2, a_3, l_{02}, l_{12}, l_{22}, l_{32})$	restrictions on d_k
$d = 3$	$l_{12} = 1$	$(3, 3, 3, 4, 1, 1, 1)$	$d_k = 3$ for all k
$d = 2$	$l_{12} = 1$	$(2, 2, 2, 3, 1, 1, 1)$	$d_1 = 1$, $d_k = 2$ for $k \neq 1$
	$l_{12} = 2$	$(1, 2, 2, 3, 2, 1, 1)$	$d_k = 2$ for all k
		$(2, 2, 2, 4, 2, 2, 2)$	$d_k = 2$ for all k
$d = 1$	$l_{12} = 1$	$(0, 0, 0, 1, 1, 1, 1)$	$d_k = 1$ for all k
		$(1, 1, 1, 2, 1, 1, 1)$	$d_1 = d_2 = 0$, $d_k = 1$ for all $k \geq 3$
		$(1, 1, 1, 2, 1, 1, 1)$	$d_1 = -1$, $d_k = 1$ for all $k \neq 1$
	$l_{12} = 2$	$(0, 0, 1, 2, 2, 2, 1)$	$d_k = 1$ for all k
		$(1, 1, 1, 3, 2, 2, 2)$	$d_1 = 0$, $d_k = 1$ for all $k \neq 1$
		$(0, 1, 1, 2, 2, 1, 1)$	$d_1 = 0$, $d_k = 1$ for all $k \neq 1$
	$l_{12} = 3$	$(1, 1, 1, 4, 3, 3, 3)$	$d_k = 1$ for all k
		$(0, 1, 1, 3, 2, 2, 2)$	$d_k = 1$ for all k
$d = 0$	$l_{12} = 1$	$(0, 0, 0, 1, 1, 1, 1)$	$\sum d_k = -3$
	$l_{12} = 2$	$(0, 0, 0, 2, 2, 2, 2)$	$\sum d_k = -2$
	$l_{12} = 3$	$(0, 0, 0, 3, 3, 3, 3)$	$d_1 = -1$, $d_k = 0$ for all $k \neq 1$
	$l_{12} = 4$	$(0, 0, 0, 4, 4, 4, 4)$	$d_k = 0$ for all k.

We have $w_{i1} \in \tau^+$ and $w_{i2} \in \tau^-$ for $i = 1, 2, 3$ and $w_{01}, w_{02} \in \tau^+$. We may assume $w_{02}, w_{12}, w_{22} \in \operatorname{cone}(w_{01}, w_{32})$. Then, applying Remark 6.1.8 to $\gamma_{01,32} \in \operatorname{rlv}(X)$ and afterwards to $\gamma_{02,32}, \gamma_{11,32}, \gamma_{21,32}, \gamma_{12,01}, \gamma_{22,01} \in \operatorname{rlv}(X)$ turns the degree matrix Q into the following shape

$$Q := \left[\begin{array}{cc|cc|cc|cc||ccc} 0 & x_{02} & x_{11} & 1 & x_{21} & 1 & x_{31} & 1 & x_1 & \dots & x_m \\ 1 & 1 & 1 & y_{12} & 1 & y_{22} & y_{31} & 0 & y_1 & \dots & y_m \end{array}\right].$$

Note that we have $x_{02}, y_{12}, y_{22} \geq 0$ because of $w_{02}, w_{12}, w_{22} \in \operatorname{cone}(w_{01}, w_{32})$. Moreover $2 \leq \mu_2$ holds and thus $y_{31} > 0$.

We claim $l_{11}, l_{21} \geq 2$. Otherwise after renumbering we may assume $l_{11} = 1$. With $2 \leq \mu_2 = 1 + l_{12}y_{12}$ we conclude $y_{12} > 0$. The determinant corresponding to $\gamma_{02,12} \in \operatorname{rlv}(X)$ equals one, which implies $x_{02}y_{12} = 0$. This gives

$$x_{02} = 0, \qquad l_{31}x_{31} + l_{32} = 0, \qquad x_{31} < 0.$$

This contradicts $y_{31} - y_{12}x_{31} = \det(w_{12}, w_{31}) = 1$.

Thus applying Lemma 6.1.2 (iv) to $\gamma_{02,12}, \gamma_{01,12}, \gamma_{21,12}, \gamma_{31,12} \in \operatorname{rlv}(X)$ we obtain

$$l_{01} = l_{02} = l_{32} = l_{22} = 1, \qquad \mu_2 = 2, \qquad l_{11} = l_{21} = 2, \qquad y_{22} = y_{12} = 0$$

As the determinant corresponding to $\gamma_{12,31} \in \operatorname{rlv}(X)$ equals one we obtain $y_{31} = 1$ and $l_{31} = \mu_2 = 2$. Thus applying Lemma 6.1.2 (iv) to $\gamma_{11,32}, \gamma_{21,32} \in \operatorname{rlv}(X)$ implies $l_{12} = 1 = l_{22}$. With

$$0 \leq \mu_1 = x_{02} = 1 + 2 \cdot x_{11} = 1 + 2 \cdot x_{21} = 1 + 2 \cdot x_{31}$$

we obtain $x_{11} = x_{21} = x_{31} \geq 0$ and $x_{02} > 0$. With $w_{11}, w_{21} \in \tau^+$ and Lemma 6.1.2 (ii) we conclude that the possible weights of type w_k lie in τ^-. Thus applying Remark 6.1.8 to $\gamma_{01,k} \in \mathrm{rlv}(X)$ we obtain $x_k = 1$. Finally as the determinant corresponding to $\gamma_{02,k} \in \mathrm{rlv}(X)$ equals one and $x_{02} > 0$ holds we obtain $y_k = 0$. For the relation g_1, the degree matrix Q and the ample cone this means

$$g_1 = T_{01}T_{02} + T_{11}^2T_{12} + T_{21}^2T_{22} + T_{31}^2T_{32}$$

$$Q \;=\; \left[\begin{array}{cc|cc|cc|cc||ccc} 0 & 2a+1 & a & 1 & a & 1 & a & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & \dots & 0 \end{array}\right],$$

$$\mathrm{SAmple}(X) = \overline{\tau_X} = \mathrm{cone}((1,0),(2a+1,1)),$$

where $a \geq 0$. The anticanonical class is $-\mathcal{K}_X = (3a+3+m, 3)$. In particular X is Fano if and only if $3a+3+m > 6a+3$ holds. This is equivalent to $m > 3a$. Moreover X is truly almost Fano if and only if $m = 3a$ holds.

We have $w_{01}, w_{02}, w_{11}, w_{12}, w_{21}, w_{31} \in \tau^+$ *and* $w_{22}, w_{32} \in \tau^-$. We may assume $w_{02}, w_{11}, w_{12}, w_{22} \in \mathrm{cone}(w_{01}, w_{32})$. Then, applying Remark 6.1.8 to $\gamma_{01,32} \in \mathrm{rlv}(X)$ and afterwards to $\gamma_{02,32}, \gamma_{11,32}, \gamma_{12,32}, \gamma_{21,32}, \gamma_{01,22} \in \mathrm{rlv}(X)$ turns the degree matrix X into the following shape

$$Q := \left[\begin{array}{cc|cc|cc|cc||ccc} 0 & x_{02} & x_{11} & x_{12} & x_{21} & 1 & x_{31} & 1 & x_1 & \dots & x_m \\ 1 & 1 & 1 & 1 & 1 & y_{22} & y_{31} & 0 & y_1 & \dots & y_m \end{array}\right].$$

Note that we have $x_{02}, x_{11}, x_{12}, y_{22} \geq 0$ because of $w_{02}, w_{11}, w_{12}, w_{22} \in \mathrm{cone}(w_{01}, w_{32})$. Moreover $\mu_2 \geq 2$ holds and thus $y_{31} > 0$.

We claim $l_{21} \geq 2$. Otherwise $l_{21} = 1$ holds. With $2 \leq \mu_2 = 1 + y_{22}l_{22}$ we conclude $y_{22} > 0$. The determinant corresponding to $\gamma_{02,22} \in \mathrm{rlv}(X)$ equals one, which implies $x_{02}y_{22} = 0$. This gives

$$x_{02} = 0, \qquad \mu_1 = 0, \qquad l_{31}x_{31} + l_{32} = 0, \qquad x_{31} < 0.$$

This contradicts $y_{31} - y_{22}x_{31} = \det(w_{22}, w_{31}) = 1$.

Thus applying Lemma 6.1.2 (iv) to $\gamma_{02,22}, \gamma_{01,22} \in \mathrm{rlv}(X)$ implies

$$l_{01} = l_{02} = 1, \qquad \mu_2 = 2, \qquad l_{11} = l_{12} = 1, \qquad l_{21} = 2, \qquad y_{22} = 0.$$

As the determinant corresponding to $\gamma_{22,31} \in \mathrm{rlv}(X)$ equals one we obtain $y_{31} = 1$ and $2 = \mu_2 = l_{31}$. Thus applying Lemma 6.1.2 (iv) to $\gamma_{22,31}, \gamma_{21,32} \in \mathrm{rlv}(X)$ we obtain $l_{32} = l_{22} = 1$. With $w_{21} \in \tau^+$ and Lemma 6.1.2 (ii) we conclude that possible weights of type w_k lie in τ^-. Thus applying Remark 6.1.8 to $\gamma_{01,k} \in \mathrm{rlv}(X)$ we obtain $x_k = 1$. Finally as the determinant corresponding to $\gamma_{02,k} \in \mathrm{rlv}(X)$ equals one and $x_{02} = 2x_{21} + 1 \geq 0$ we conclude $x_{02} > 0$ and thus $y_k = 0$. For the relation g_1 and the degree matrix Q this means

$$g_1 = T_{01}T_{02} + T_{11}T_{12} + T_{21}^2T_{22} + T_{31}^2T_{32}$$

$$Q = \left[\begin{array}{cc|cc|cc|cc||ccc} 0 & 2a_3+1 & a_1 & a_2 & a_3 & 1 & a_3 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots & 0 \end{array}\right],$$

where $2a_3 + 1 = a_1 + a_2$, $a_i \geq 0$ and we may assume $a_1 \leq a_2$. The semiample cone and the anticanonical class are given as

$$\mathrm{SAmple}(X) = \overline{\tau_X} = \mathrm{cone}((1,0),(2a_3+1,1)), \qquad -\mathcal{K}_X = (4a_3+3+m, 4).$$

In particular X is Fano if and only if $4a_3 + 3 + m > 8a_3 + 4$ holds. This is equivalent to $m > 4a_3 + 1$. Moreover X is truly almost Fano if and only if $m = 4a_3 + 1$.

We have $w_{01}, \dots, w_{31} \in \tau^+$ *and* $w_{32} \in \tau^-$. As each of τ^- and τ^+ contain at least two weights we have $m \geq 1$ and $w_k \in \tau^-$. We may assume $w_i, w_{02}, \dots, w_{22} \in \mathrm{cone}(w_1, w_{01})$. Then, applying Remark 6.1.8 at to $\gamma_{1,01} \in \mathrm{rlv}(X)$ and afterwards to $\gamma_{1,02}, \dots, \gamma_{1,31}, \gamma_{01,32} \in \mathrm{rlv}(X)$ turns the degree matrix Q into the following shape

$$Q := \left[\begin{array}{cc|cc|cc|cc||cccc} 0 & x_{02} & x_{11} & x_{12} & x_{21} & x_{22} & x_{31} & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & y_{32} & 0 & y_2 & \dots & y_m \end{array}\right],$$

Note that because of $w_i, w_{02}, \dots, w_{22} \in \mathrm{cone}(w_1, w_{01})$ all entries of Q except x_{31} and y_{32} non-negative.

Applying Lemma 6.1.2 (ii) to $\gamma_{1,01}, \dots, \gamma_{1,31} \in \mathrm{rlv}(X)$, we obtain

$$l_{01} = l_{02} = l_{11} = l_{12} = l_{21} = l_{22} = l_{32} = 1, \qquad \mu_2 = 2$$

As the determinants corresponding to $\gamma_{02,32}, \gamma_{11,32}, \gamma_{12,32}, \gamma_{21,32}, \gamma_{22,32} \in \mathrm{rlv}(X)$ all equal one we conlude $y_{32} = 0$ or $x_{02} = \ldots = x_{22} = 0$.

Assume $y_{32} = 0$. Then $2 = \mu_2 = l_{31}$ holds. Moreover $\mu_1 = x_{02} = 2x_{31} + 1$ and $x_{02} \geq 0$ implies $x_{31} \geq 0$ and $x_{02} > 0$. As the determinants corresponding to $\gamma_{02,k} \in \mathrm{rlv}(X)$ for $k = 1, \dots, m$ we obtain $y_2 = \ldots = y_m = 0$. For the relation g_1 and the degree matrix Q this means

$$g_1 = T_{01}T_{02} + T_{11}T_{12} + T_{21}T_{22} + T_{31}^2 T_{32}$$

$$Q = \left[\begin{array}{cc|cc|cc|cc||ccc} 0 & 2a_5+1 & a_1 & a_2 & a_3 & a_4 & a_5 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & \dots & 0 \end{array}\right],$$

where $2a_5 + 1 = a_1 + a_2 = a_3 + a_4$ and $a_i \geq 0$. The semiample cone and the anticanonical class are given as

$$\mathrm{SAmple}(X) = \overline{\tau_X} = \mathrm{cone}((1,0),(2a_5+1,1)), \qquad -\mathcal{K}_X = (5a_5+3+m, 5).$$

In particular X is Fano if and only if $10a_5 + 5 < 5a_5 + 3 + m$ holds. This is equivalent to $5a_5 + 2 < m$. Moreover X is truly almost Fano if and only if $5a_5 + 2 = m$ holds.

Assume $x_{02} = \ldots = x_{22} = 0$. We have $\mu_1 = 0$ and thus $l_{31}x_{31} = -1$. This implies $l_{31} = 1$ and $x_{31} = -1$. Thus $\mu_2 = 2 = 1 + y_{32}$ and we conclude $y_{32} = 1$. Finally as the determinants corresponding to $\gamma_{31,k} \in \mathrm{rlv}(X)$ all equal one, we conclude $y_i = 0$. For the relation g_1, the degree matrix Q and the ample cone this means

$$g_1 = T_{01}T_{02} + T_{11}T_{12} + T_{21}T_{22} + T_{31}T_{32}$$

$$Q \;=\; \left[\begin{array}{cc|cc|cc|cc||ccc} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \dots & 0 \end{array}\right].$$

$$\mathrm{SAmple}(X) = \overline{\tau_X} = \mathrm{cone}((1,1),(0,1)).$$

The anticanonical class is $-\mathcal{K}_X = (m, 6)$. In particular X is Fano if and only if $m < 6$ holds and truly almost Fano in the case $m = 6$.

We have $w_{ij} \in \tau^+$ for all i, j. As each of τ^- and τ^+ contain at least two weights applying we conclude with Lemma 6.1.4 (v) $m \geq 2$ and $w_k \in \tau^-$ for all $k = 1, \dots, m$. Moreover $\gamma_{ij,1} \in \mathrm{rlv}(X)$ implies $l_{ij} = 1$ for all i, j. We may assume $w_{ij}, w_k \in \mathrm{cone}(w_{01}, w_1)$ for all i, j, k. Then, applying Remark 6.1.8 at to $\gamma_{01,1}$ and afterwards to all other $\gamma_{ij,1}, \gamma_{01,k} \in \mathrm{rlv}(X)$ turns the grading matrix Q into th following shape

$$Q := \left[\begin{array}{cc|cc|cc|cc||cccc} 0 & x_{02} & x_{11} & x_{12} & x_{21} & x_{22} & x_{31} & x_{32} & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & y_2 & \dots & y_m \end{array}\right].$$

Note that all entries of Q are non negative because of $w_{ij}, w_k \in \mathrm{cone}(w_{01}, w_1)$. We distinguish between the case that all entries $y_2, \dots, y_m$ equal zero and the case they do not.

We have $y_k = 0$ holds for all k. Then we have

$$x_{02} = x_{11} + x_{12} = x_{21} + x_{22} = x_{31} + x_{32}$$

and the relation g_1 and the grading matrix Q have the following shape.

$$g_1 = T_{01}T_{02} + T_{11}T_{12} + T_{21}T_{22} + T_{31}T_{32}$$

$$Q \;=\; \left[\begin{array}{cc|cc|cc|cc||ccc} 0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \dots & 0 \end{array}\right],$$

where $a_1 = a_2 + a_3 = a_4 + a_5 = a_6 + a_7$ and $a_i \geq 0$. The semiample cone and the anticanonical class are

$$\mathrm{SAmple}(X) = \overline{\tau_X} = \mathrm{cone}((1,0),(a_1,1)), \qquad -\mathcal{K}_X = (3a_1 + m, 6).$$

In particular X is Fano if and only if $3a_1 + m > 6a_1$ and this is equivalent to $m > 3a_1$. Moreover X is truly almost Fano if and only if $m = 3a_1$ holds.

We have $y_k > 0$ for at least one k. We may assume $0 \leq y_2 \leq \ldots \leq y_m$ and $y_m > 0$. As the determinants correspoding to $\gamma_{ij,k} \in \mathrm{rlv}(X)$ all equal one we conclude $x_{ij} = 0$. For the relation g_1 and the grading matrix Q this means

$$g_1 = T_{01}T_{02} + T_{11}T_{12} + T_{21}T_{22} + T_{31}T_{32}$$

$$\left[\begin{array}{cc|cc|cc|cc||cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & a_2 & \dots & a_m \end{array}\right],$$

where $0 \leq a_2 \leq \ldots \leq a_m$ and $a_m > 0$. The semiample cone and the anticanonical class are

$$\mathrm{SAmple}(X) = \overline{\tau_X} = ((0,1),(1,a_m)), \qquad -\mathcal{K}_X = (m, 6 + a_2 + \ldots + a_m).$$

In particular X is Fano if and only if $m \cdot a_m < 6 + a_2 + \ldots + a_m$. This implies $a_m \leq 5$. Furthermore X is truly almost Fano if and only if $m \cdot a_m = 6 + a_2 + \ldots + a_m$, which implies $m \leq 6$. □

Lemma 6.1.10. *In the situation of Proposition 6.1.1, consider the case $r = 3$, $m \geq 0$ and $n_0 = n_1 = 2 \geq n_2 \geq n_3 = 1$. Then the following constellation of weights can't occur:*

$$w_{02}, w_{12} \in \tau^+ \qquad w_{01}, w_{11} \in \tau^-.$$

Proof. We may assume $w_{11} \in \mathrm{cone}(w_{01}, w_{12})$. Applying Remark 6.1.8 to $\gamma_{01,12} \in \mathrm{rlv}(X)$ we obtain

$$w_{01} = (1,0), \qquad w_{12} = (0,1), \qquad x_{11}, y_{11} \geq 0.$$

Moreover the position of weights implies $\det(w_{11}, w_{12}) > 0$ and $\det(w_{01}, w_{02}) > 0$ and thus $x_{11} > 0$ and $y_{02} > 0$. Applying Lemma 6.1.2 (iv) to $\gamma_{01,12} \in \mathrm{rlv}(X)$ we obtain $l_{02} = 1$ or $l_{11} = 1$. With $\gamma_{02,11} \in \mathrm{rlv}(X)$ we obtain

$$\begin{aligned} l_{02} = 1 \quad &\Longrightarrow \quad 1 = \det(w_{11}, w_{02}) = \det(w_{11}, l_{12}w_{12} - l_{01}w_{01}) = l_{12}x_{11} + l_{01}y_{11}, \\ l_{11} = 1 \quad &\Longrightarrow \quad 1 = \det(w_{11}, w_{02}) = \det(l_{01}w_{01} - l_{12}w_{12}, w_{02}) = l_{01}y_{02} + l_{12}x_{02}, \end{aligned}$$

where the second equality on the r.h.s. holds due to homogeneity of the relation.

We show $l_{02} > 1$. Otherwise the above considerations show

$$y_{11} = 0, \qquad l_{12} = x_{11} = 1, \qquad \mu_2 = 1, \quad l_{31} = y_{31} = 1.$$

This contradicts irredundancy of P.

Thus we have $l_{11} = 1$. Note that $y_{02} > 0$ implies $x_{02} \leq 0$. The corresponding determinant of $\gamma_{02,11} \in \mathrm{rlv}(X)$ equals one and we obtain $1 = x_{11}y_{02} - y_{11}x_{02}$. This implies

$$y_{11}x_{02} = 0, \qquad x_{11} = y_{02} = 1, \qquad \mu_1 = 1, \qquad l_{31} = x_{31} = 1.$$

This again contradicts irredundancy of P. □

Case 6.1.1 (I)(g). *We have $r = 3$, $m \geq 0$ and $n_0 = n_1 = n_2 = 2 > n_3 = 1$. This gives the varieties Nos. 11, 12 and 13 in the Theorems 6.2.1, 6.2.2 and 6.2.4.*

Proof. With Lemma 6.1.10 and Lemma 6.1.4 (i) we may assume $w_{21}, w_{22} \in \tau^+$. We conclude $\mu \in \tau^+$ and thus $w_{31} \in \tau^+$. We distinguish between the following two cases according to the possible positions of weights.

We have $w_{02}, w_k \in \tau^-$ and all other weights in τ^+. As each of τ^+ and τ^- contains at least two weights we conclude $m \geq 1$ and Lemma 6.1.4 (v) yields $w_k \in \tau^-$ for all k.

Applying Lemma 6.1.2 (iv) to $\gamma_{01,1}, \gamma_{11,1}, \gamma_{12,1}, \gamma_{21,1}\gamma_{22,1} \in \mathrm{rlv}(X)$ we obtain

$$l_{02} = l_{11} = l_{12} = l_{21} = l_{22} = 1.$$

Then applying Remark 6.1.8 to $\gamma_{11,1} \in \mathrm{rlv}(X)$ and afterwards to $\gamma_{01,1}, \gamma_{12,1}, \gamma_{21,1}\gamma_{22,1}\gamma_{02,11}, \gamma_{11,k} \in \mathrm{rlv}(X)$ turns the degree matrix Q into the following shape:

$$Q := \left[\begin{array}{cc|cc|cc|c||cccc} x_{01} & 1 & 0 & x_{12} & x_{21} & x_{22} & x_{31} & 1 & 1 & \dots & 1 \\ 1 & y_{02} & 1 & 1 & 1 & 1 & y_{31} & 0 & y_2 & \dots & y_m \end{array}\right].$$

We conclude $\mu_2 = 2$ and irredundancy of P gives $l_{31} = 2$ and $y_{31} = 1$. We obtain

$$\mu_1 = l_{01}x_{01} + 1 = x_{12} = l_{31}x_{31} = 2x_{31}$$

Thus μ_1 is even and we conclude that l_{01} and x_{01} are odd. Thus $y_{02} = \mu_2 - l_{01}$ is odd and thus nonzero. The determinants corresponding to $\gamma_{02,12}, \gamma_{02,22} \in \mathrm{rlv}(X)$ both equal one, which implies $y_{02}x_{12} = 0$ and $y_{02}x_{22} = 0$. We obtain

$$x_{12} = x_{22} = 0 = \mu_1, \qquad x_{21} = x_{31} = 0, \qquad x_{01} = -1, \qquad l_{01} = 1, \qquad y_{02} = 1.$$

Finally $\gamma_{01,k} \in \mathrm{rlv}(X)$ leads to $\det(w_k, w_{01}) = 1$ and thus $y_k = 0$. For the relation, the degree matrix and the ample cone this means

$$g_1 := T_{01}T_{02} + T_{11}T_{12} + T_{21}T_{22} + T_{31}^2$$

$$Q = \left[\begin{array}{cc|cc|cc|c||ccc} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \dots & 0 \end{array}\right]$$

$$\mathrm{SAmple}(X) = \overline{\tau_X} = \mathrm{cone}((0,1),(1,1)).$$

Moreover the anticanonical class is given as $-\mathcal{K}_X = (m, 5)$. Thus X is Fano if and only if $m < 5$ and truly almost Fano if $m = 5$ holds.

We have $w_{ij} \in \tau^+$ for all i. Consider the fact that each of τ^+ and τ^- contains at least two weights and applying Lemma 6.1.4 (v) we conclude $m \geq 2$ and $w_k \in \tau^-$ for all k.

We may assume $w_{ij}, w_k \in \mathrm{cone}(w_{01}, w_1)$ for all i, j, k. Applying Remark 6.1.8 to $\gamma_{01,1} \in \mathrm{rlv}(X)$ and afterwards to $\gamma_{02,1}, \dots, \gamma_{22,1}, \gamma_{01,k} \in \mathrm{rlv}(X)$ turns the degree matrix Q into the following shape

$$Q := \left[\begin{array}{cc|cc|cc|c||cccc} 0 & x_{02} & x_{11} & x_{12} & x_{21} & x_{22} & x_{31} & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & y_{31} & 0 & y_2 & \dots & y_m \end{array}\right].$$

Note that we have $x_{ij}, x_k \geq 0$ for all i, j, k because of $w_{ij}, w_k \in \mathrm{cone}(w_{01}, w_1)$ for all i, j, k. Applying Lemma 6.1.2 (ii) to $\gamma_{ij,k} \in \mathrm{rlv}(X)$ for $i = 0, 1, 2$ we obtain

$$l_{01} = \dots = l_{22} = 1, \qquad \mu_2 = 2$$

and thus irredundancy of P implies $l_{31} = 2$ and $y_{31} = 1$.

We distinguish between the case that all entries y_k equal zero and the case that there exists at least one k with $y_k > 0$.

We have $y_k = 0$ for all k. Then the defining relation g_1, the degree matrix Q and the ample cone are of the following shape:

$$g_1 = T_{01}T_{02} + T_{11}T_{12} + T_{21}T_{22} + T_{31}^2$$

$$Q = \left[\begin{array}{cc|cc|cc|c||ccc} 0 & 2a_5 & a_1 & a_2 & a_3 & a_4 & a_5 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \dots & 0 \end{array}\right],$$

$$\mathrm{SAmple}(X) = \overline{\tau_X} = \mathrm{cone}((1,0),(2a_5,1)).$$

where $a_1 + a_2 = a_3 + a_4 = 2a_5$ holds and $a_i \geq 0$ for all i. The anticanonical class is given as $-\mathcal{K}_X = (m + 5a_5, 5)$. In particular X is Fano if and only if $m > 5a_5$ and truly almost Fano if equality holds.

We have $y_k > 0$ for at least one k. We may assume $0 = y_1 \leq \ldots \leq y_m$ and $y_m > 0$. Then $\gamma_{02,m} \in \mathrm{rlv}(X)$ leads to $\det(w_m, w_{02}) = 1$ and with $x_{ij} \geq 0$ for all i this implies

$$x_{02} = 0, \qquad \mu_1 = 0, \qquad x_{12} = x_{21} = x_{22} = x_{31} = 0$$

For the relation g_1, the degree matrix Q and the ample cone this means

$$g_1 = T_{01}T_{02} + T_{11}T_{12} + T_{21}T_{22} + T_{31}^2$$

$$Q = \left[\begin{array}{cc|cc|cc|c||cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & d_2 & \dots & d_m \end{array}\right],$$

$$\mathrm{SAmple}(X) = \overline{\tau_X} = \mathrm{cone}((0,1),(1,d_m))$$

where $d_2 \leq \ldots \leq d_m$. The anticanonical class is given as

$$-\mathcal{K}_X = (m, 5 + \sum d_i).$$

In particular X is Fano if and only if $m \cdot d_m < 5 + \sum d_i$ and we obtain furthermore $m < 5$. X is truly almost Fano if $m \cdot d_m = 5 + \sum d_i$ holds. □

Case 6.1.1 (I)(h). *We have $r = 3$, $m \geq 0$ and $n_0 = n_1 = 2 > n_2 = n_3 = 1$. This setting allows no examples satisfying the assumptions of Theorem 6.2.1.*

Proof. For $m > 0$ we may assume due to Lemma 6.1.4 (v) that $w_1, \ldots, w_m \in \tau^-$ holds. Applying Lemma 6.1.10 we may always assume $w_{11}, w_{12} \in \tau^+$ and thus $\mu \in \tau^+$. This implies $w_{21} = w_{31} \in \tau^+$. As each of the cones τ^- and τ^+ contains at least two weights we are left with the following two possible position of weights.

We have $w_{02} \in \tau^-$ and $w_{01} \in \tau^+$. Applying Lemma 6.1.2 (iv) to $\gamma_{01,1}, \gamma_{12,1}, \gamma_{11,1} \in \mathrm{rlv}(X)$ we obtain

$$l_{02} = l_{11} = l_{12} = 1.$$

Applying Remark 6.1.8 to $\gamma_{11,1} \in \mathrm{rlv}(X)$ and afterwards to $\gamma_{01,1}, \gamma_{12,1}, \gamma_{02,11}, \gamma_{11,k} \in \mathrm{rlv}(X)$ turns the degree matrix Q into the following shape

$$Q := \left[\begin{array}{cc|cc|c|c||cccc} x_{01} & 1 & 0 & x_{12} & x_{21} & x_{31} & 1 & 1 & \dots & 1 \\ 1 & y_{02} & 1 & 1 & y_{21} & y_{31} & 0 & y_2 & \dots & y_m \end{array}\right].$$

We obtain $\mu_2 = 2$. Thus either $l_{21} = l_{31} = 2$ holds, which contradicts torsion freeness of the divisor class group of X or at least one of l_{21} or l_{31} equals one. This contradicts irredundancy of P.

We have $w_{01}, w_{02} \in \tau^+$. Applying Remark 6.1.8 and Lemma 6.1.2 (iv) to $\gamma_{01,1}, \gamma_{02,1}, \gamma_{11,1}, \gamma_{12,1} \in \mathrm{rlv}(X)$ we obtain

$$x_{01} = x_{02} = x_{11} = x_{12} = 1, \qquad l_{01} = l_{02} = l_{11} = l_{12} = 1, \qquad \mu_1 = 2$$

Irredundancy of P thus implies $l_{21} = l_{31} = 2$, which leads to torsion in the divisor class group of X; a contradiction. □

Case 6.1.1 (I)(i). *We have $r = 3$, $m > 0$ and $n_0 = 2 > n_1 = n_2 = n_3 = 1$. This setting allows no examples satisfying the assumptions of Theorem 6.2.1.*

Proof. Applying Lemma 6.1.4 (v) we may assume $w_1, \dots, w_m \in \tau^-$. Moreover we have $\mu = w_{11} = w_{21} = w_{31}$.

We claim that one of the weights w_{0j} lies in τ^+. Otherwise Lemma 6.1.4 (i) implies $w_{0j} \in \tau^-$ for $j = 1, 2$. This implies $\mu \in \tau^-$ and there are no weights left to lie in τ^+, which is impossible. Thus we may assume $w_{01} \in \tau^+$.

Applying Lemma 6.1.2 (ii) and Remark 6.1.8 to $\gamma_{01,1} \in \mathrm{rlv}(X)$ we obtain

$$l_{02} = 1, \qquad w_{01} = (0,1), \qquad w_1 = (1,0).$$

Moreover as the class group of X is torsion free and P is irredundant we have pairwise coprime $l_{11}, l_{21}, l_{31} > 1$. We distinguish the following two cases according to the possible position of weights.

We have $w_{02} \in \tau^-$. As τ^+ contains at least two weights we may assume $w_{11} \in \tau^+$. This implies $\mu \in \tau^+$ and thus $w_{21} = w_{31} \in \tau^+$. With $w_{02} \in \tau^-$ we obtain $\mathrm{cone}(w_{02}, w_{01}, w_{11}) \in \mathrm{rlv}(X)$. As X is locally factorial this implies $(0,1), (x_{02}, y_{02}), (x_{11}, y_{11})$ generate $\mathbb{Z}^2$ as a group. Thus $\gcd(x_{02}, x_{11}) = 1$ holds and in particular not both equal zero. We obtain

$$0 < \mu_1 = x_{02} = l_{11}x_{11} = l_{21}x_{21}.$$

With $\gcd(l_{11}, l_{21}) = 1$ we conclude $l_{21}|x_{02}$ and $l_{21}|x_{11}$; This contradicts $\gcd(x_{02}, x_{11}) = 1$.

We have $w_{02} \in \tau^+$. We obtain $w_{11} = w_{21} = w_{31} = \mu \in \tau^+$. As τ^- contains at least two weights we obtain $m \geq 2$. Applying Lemma 6.1.2 (ii) and Remark 6.1.8 to $\gamma_{02,1} \in \mathrm{rlv}(X)$ we obtain

$$l_{01} = 1, \qquad y_{02} = 1, \qquad \mu_2 = 2.$$

Irredundanccy of P implies $l_{11} = l_{21} = l_{31} = 2$ contradicting coprimeness. □

Case 6.1.1 (II). *We have $r = 4$, $m = 0$ and $n_0 = n_1 = 2 \geq n_2 \geq n_3 \geq n_4$. This leads to No. 14 in Theorems 6.2.1 and 6.2.2.*

We treat the cases (a) to (d) at once. Observe that if two weights $w_{i_1 1}, w_{i_1 2}$ lie in one cone τ^- or τ^+ homogeneity of the relations implies that all w_{ij} with $n_i = 1$ lie in this cone as well. As each of τ^+ and τ^- contains at least two weights and $m = 0$ holds we may thus assume $w_{01}, w_{11} \in \tau^-$ and $w_{02}, w_{12} \in \tau^+$. In particular for each $w_{i_1 j_1}$ with $n_{i_1} = 2$ there exist at least one $w_{i_2 j_2}$ with $\gamma_{i_1 j_1, i_2 j_2} \in \operatorname{rlv}(X)$. Thus considering $r = 4$ and applying Lemma 6.1.2 (iv) we obtain $l_{ij} = 1$ for all i with $n_i = 2$. We may assume $w_{11} \in \operatorname{cone}(w_{01}, w_{12})$. Applying Remark 6.1.8 to $\gamma_{01,12} \in \operatorname{rlv}(X)$ we obtain

$$w_{01} = (1, 0), \qquad w_{12} = (0, 1), \qquad x_{11}, y_{11} \geq 0.$$

Applying Remark 6.1.8 to $\gamma_{02,11} \in \operatorname{rlv}(X)$ we obtain

$$1 = \det(w_{11}, w_{02}) = x_{11} y_{02} - x_{02} y_{11} = x_{11} + y_{11},$$

where the last equality follows with $\mu_2 = y_{02} = y_{11} + 1$. As by assumption $w_{11} \notin \tau^-$ we have $x_{11} > 0$ and conclude

$$x_{11} = 1, \qquad y_{11} = 0, \qquad \mu = (1, 1), \qquad x_{02} = 0, \qquad y_{02} = 1.$$

This implies $l_{ij} = 1$ for all i with $n_i = 1$. As P is irredundant the only possible constellation is

$$n_0 = n_1 = n_2 = n_3 = n_4 = 2, \qquad l_{01} = \ldots = l_{42} = 1.$$

With Lemma 6.1.4 (i) may assume $w_{21}, w_{31}, w_{41} \in \tau^-$ and applying Remark 6.1.8 to $\gamma_{21,02}, \gamma_{31,02}, \gamma_{41,02} \in \operatorname{rlv}(X)$ we obtain

$$x_{21} = x_{31} = x_{41} = 1, \qquad x_{22} = x_{32} = x_{42} = 0.$$

We conclude $w_{22}, w_{32}, w_{42} \in \tau^+$. This in turn implies $\gamma_{01,22}, \gamma_{01,32}, \gamma_{01,42} \in \operatorname{rlv}(X)$ and applying Remark 6.1.8 once more we obtain

$$1 = y_{22} = y_{32} = y_{42}, \qquad y_{21} = y_{31} = y_{41} = 0.$$

For the defining relations g_1, g_2, the grading matrix Q and the ample cone we obtain

$$g_0 := T_{01}T_{02} + T_{11}T_{12} + T_{21}T_{22} + T_{31}T_{32}$$

$$g_1 := \lambda_1 T_{01}T_{02} + \lambda_2 T_{11}T_{12} + T_{21}T_{22} + T_{41}T_{42}$$

$$\left[\begin{array}{cc|cc|cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right],$$

$$\operatorname{SAmple}(X) = \overline{\tau_X} = (\mathbb{Q}_{\geq 0})^2.$$

The anticanonical class is $-\mathcal{K}_X = (3, 3)$. In particular the variety is Fano.

6.2 Classification Results

Here we give a complete list of all non-toric smooth arrangement varieties of complexity two and Picard number two and classify in every dimension the smooth (almost) Fano varieties of complexity two and Picard number two.

Theorem 6.2.1. *Every non-toric smooth projective arrangement variety of complexity two and Picard number two is isomorphic to precisely one of the following varieties X, specified by their Cox ring $\mathcal{R}(X)$, the matrix $[w_1, \ldots, w_r]$ of generator degrees and an ample class $u \in \mathrm{Cl}(X) = \mathbb{Z}^2$.*

No.	$\mathcal{R}(X)$	$[w_1, \ldots, w_r]$	u	$\dim(X)$
1	$\frac{\mathbb{K}[T_1,\ldots,T_9]}{\langle T_1T_2T_3^2+T_4T_5+T_6T_7+T_8T_9\rangle}$	$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & a_1 & 2-a_1 & a_2 & 2-a_2 & a_3 & 2-a_3 \end{bmatrix}$ $1 \le a_1 \le a_2 \le a_3$	$\begin{bmatrix} 1 \\ a_3+1 \end{bmatrix}$	6
2	$\frac{\mathbb{K}[T_1,\ldots,T_9]}{\langle T_1T_2T_3+T_4T_5+T_6T_7+T_8T_9\rangle}$	$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	6
3	$\frac{\mathbb{K}[T_1,\ldots,T_8]}{\langle T_1T_2T_3^2+T_4T_5+T_6T_7+T_8^2\rangle}$	$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & a_1 & 2-a_1 & a_2 & 2-a_2 & 1 \end{bmatrix}$ $1 \le a_1 \le a_2$	$\begin{bmatrix} 1 \\ a_2+1 \end{bmatrix}$	5
4	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2^{l_2}+T_3T_4^{l_4}+T_5T_6^{l_6}+T_7T_8^{l_8}\rangle}$ $m \ge 0$	$\left[\begin{array}{cccccccc\|ccc} 0 & 1 & a_1 & 1 & a_2 & 1 & a_3 & 1 & d_1 & \ldots & d_m \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \ldots & 1 \end{array}\right]$ $0 \le a_1 \le a_2 \le a_3$ $l_2 = a_1 + l_4 = a_2 + l_6 = a_3 + l_8$ $d_1 \le \ldots \le d_m$	$\begin{bmatrix} d+1 \\ 1 \end{bmatrix}$ $d := \max(a_3, d_m)$	$m+5$
5	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2+T_3^2T_4+T_5^2T_6+T_7^2T_8\rangle}$ $m \ge 0$	$\left[\begin{array}{cccccccc\|ccc} 0 & 2a+1 & a & 1 & a & 1 & a & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & \ldots & 0 \end{array}\right]$ $a \ge 0$	$\begin{bmatrix} 2a+2 \\ 1 \end{bmatrix}$	$m+5$
6	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2T_6+T_7^2T_8\rangle}$ $m \ge 0$	$\left[\begin{array}{cccccccc\|ccc} 0 & 2a_3+1 & a_1 & a_2 & a_3 & 1 & a_3 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \ldots & 0 \end{array}\right]$ $2a_3+1 = a_1+a_2,\ 0 \le a_1 \le a_2$	$\begin{bmatrix} 2a_3+2 \\ 1 \end{bmatrix}$	$m+5$
7	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2T_8\rangle}$ $m \ge 1$	$\left[\begin{array}{cccccccc\|ccc} 0 & 2a_5+1 & a_1 & a_2 & a_3 & a_4 & a_5 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & \ldots & 0 \end{array}\right]$ $2a_5+1 = a_1+a_2 = a_3+a_4$ $a_i \ge 0$	$\begin{bmatrix} 2a_5+2 \\ 1 \end{bmatrix}$	$m+5$
8	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$ $m \ge 1$	$\left[\begin{array}{cccccccc\|ccc} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \ldots & 0 \end{array}\right]$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$m+5$
9	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$ $m \ge 2$	$\left[\begin{array}{cccccccc\|ccc} 0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \ldots & 0 \end{array}\right]$ $a_1 = a_2+a_3 = a_4+a_5 = a_6+a_7$ $a_i \ge 0$	$\begin{bmatrix} a_1+1 \\ 1 \end{bmatrix}$	$m+5$
10	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$ $m \ge 2$	$\left[\begin{array}{cccccccc\|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & d_2 & \ldots & d_m \end{array}\right]$ $0 \le d_2 \le \cdots \le d_m, d_m > 0$	$\begin{bmatrix} 1 \\ d_m+1 \end{bmatrix}$	$m+5$
11	$\frac{\mathbb{K}[T_1,\ldots,T_7,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2\rangle}$ $m \ge 1$	$\left[\begin{array}{ccccccc\|ccc} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \ldots & 0 \end{array}\right]$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$m+4$
12	$\frac{\mathbb{K}[T_1,\ldots,T_7,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2\rangle}$ $m \ge 2$	$\left[\begin{array}{ccccccc\|ccc} 0 & 2a_5 & a_1 & a_2 & a_3 & a_4 & a_5 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \ldots & 0 \end{array}\right]$ $a_1+a_2 = a_3+a_4 = 2a_5$ $a_i \ge 0$	$\begin{bmatrix} 2a_5+1 \\ 1 \end{bmatrix}$	$m+4$

13	$\frac{\mathbb{K}[T_1,\dots,T_7,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2\rangle}$ $m\geq 2$	$\left[\begin{array}{ccccccc\|cccc} 0&0&0&0&0&0&0&1&1&\dots&1\\ 1&1&1&1&1&1&1&0&d_2&\dots&d_m \end{array}\right]$ $0\leq d_2\leq\dots\leq d_m$ $d_m>0$	$\begin{bmatrix}1\\ d_m+1\end{bmatrix}$	$m+4$
14	$\frac{\mathbb{K}[T_1,\dots,T_{10}]}{\left\langle\begin{array}{c}T_1T_2+T_3T_4+T_5T_6+T_7T_8,\\ \lambda_1T_3T_4+\lambda_2T_5T_6+T_7T_8+T_9T_{10}\end{array}\right\rangle}$	$\begin{bmatrix}1&0&1&0&1&0&1&0&1&0\\ 0&1&0&1&0&1&0&1&0&1\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$	6

Moreover, each of the listed data defines a smooth projective arrangement variety of complexity two and Picard number two.

As direct applications, we can classify in Theorem 6.2.2 in every dimension the (finitely many) smooth Fano arrangement varieties of complexity two and Picard number two and in Theorem 6.2.4 the smooth truly almost Fano arrangement varieties of complexity two and Picard number two, where truly almost Fano means that the anticanonical divisor is semiample but not ample.

Theorem 6.2.2. *Every non-toric smooth Fano arrangement variety of complexity two and Picard number two is isomorphic to precisely one of the following varieties X, specified by their Cox ring $\mathcal{R}(X)$ and the matrix $[w_1, \dots, w_r]$ of generator degrees $w_i \in \mathrm{Cl}(X) = \mathbb{Z}^2$.*

No.	$\mathcal{R}(X)$	$[w_1,\dots,w_r]$	$-\mathcal{K}_X$	$\dim(X)$
1	$\frac{\mathbb{K}[T_1,\dots,T_9]}{\langle T_1T_2T_3^2+T_4T_5+T_6T_7+T_8T_9\rangle}$	$\begin{bmatrix}0&0&1&1&1&1&1&1&1\\ 1&1&0&1&1&1&1&1&1\end{bmatrix}$	$\begin{bmatrix}5\\6\end{bmatrix}$	6
2	$\frac{\mathbb{K}[T_1,\dots,T_9]}{\langle T_1T_2T_3+T_4T_5+T_6T_7+T_8T_9\rangle}$	$\begin{bmatrix}0&0&1&1&0&1&0&1&0\\ 1&1&0&1&1&1&1&1&1\end{bmatrix}$	$\begin{bmatrix}3\\6\end{bmatrix}$	6
3	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\langle T_1T_2T_3^2+T_4T_5+T_6T_7+T_8^2\rangle}$	$\begin{bmatrix}0&0&1&1&1&1&1&1\\ 1&1&0&1&1&1&1&1\end{bmatrix}$	$\begin{bmatrix}4\\5\end{bmatrix}$	5
4.A	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^3+T_3T_4+T_5T_6+T_7T_8\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&2&1&2&1&2&1&2&\dots&2\\ 1&0&1&0&1&0&1&0&1&\dots&1 \end{array}\right]$	$\begin{bmatrix}7+2m\\ 3+m\end{bmatrix}$	$m+5$
4.B	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^3+T_3T_4^2+T_5T_6^2+T_7T_8^2\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&1&1&1&1&1&1&1&\dots&1\\ 1&0&1&0&1&0&1&0&1&\dots&1 \end{array}\right]$	$\begin{bmatrix}4+m\\ 3+m\end{bmatrix}$	$m+5$
4.C	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6+T_7T_8\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&0&1&1&1&1&1&1&\dots&1\\ 1&0&1&0&1&0&1&0&1&\dots&1 \end{array}\right]$	$\begin{bmatrix}4+m\\ 3+m\end{bmatrix}$	$m+5$
4.D	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^2+T_3T_4+T_5T_6+T_7T_8\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccccc\|cccc} 0&1&1&1&1&1&1&1&d_1&1&\dots&1\\ 1&0&1&0&1&0&1&0&1&1&\dots&1 \end{array}\right]$ $d_1\in\{0,1\}$	$\begin{bmatrix}5+m-1+d_1\\ 3+m\end{bmatrix}$	$m+5$
4.E	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^3+T_3T_4^3+T_5T_6^3+T_7T_8^3\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&0&1&0&1&0&1&0&\dots&0\\ 1&0&1&0&1&0&1&0&1&\dots&1 \end{array}\right]$	$\begin{bmatrix}3\\ 3+m\end{bmatrix}$	$m+5$
4.F	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6^2+T_7T_8^2\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccccc\|cccc} 0&1&0&1&0&1&0&1&d_1&0&\dots&0\\ 1&0&1&0&1&0&1&0&1&1&\dots&1 \end{array}\right]$ $d_1\in\{-1,0\}$	$\begin{bmatrix}2+d_1\\ 3+m\end{bmatrix}$	$m+5$
4.G	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccccc\|ccccc} 0&1&0&1&0&1&0&1&d_1&d_2&0&\dots&0\\ 1&0&1&0&1&0&1&0&1&1&1&\dots&1 \end{array}\right]$ $d_1,d_2\leq 0,\ d_1+d_2\geq -2$	$\begin{bmatrix}3+d_1+d_2\\ 3+m\end{bmatrix}$	$m+5$

5	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2+T_3^2T_4+T_5^2T_6+T_7^2T_8\rangle}$ $m\geq 1$	$\left[\begin{array}{cccccccc\|ccc} 0 & 2a+1 & a & 1 & a & 1 & a & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & \ldots & 0 \end{array}\right]$ $a\geq 0, m>3a$	$\begin{bmatrix} 3a+3+m \\ 3 \end{bmatrix}$	$m+5$
6	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2T_6+T_7^2T_8\rangle}$ $m\geq 1$	$\left[\begin{array}{cccccccc\|ccc} 0 & 2a_3+1 & a_1 & a_2 & a_3 & 1 & a_3 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \ldots & 0 \end{array}\right]$ $0\leq a_1\leq a_2,\ a_1+a_2=2a_3+1$ $m>4a_3+1$	$\begin{bmatrix} 4a_3+3+m \\ 4 \end{bmatrix}$	$m+5$
7	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2T_8\rangle}$ $m\geq 1$	$\left[\begin{array}{cccccccc\|ccc} 0 & 2a_5+1 & a_1 & a_2 & a_3 & a_4 & a_5 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & \ldots & 0 \end{array}\right]$ $a_i\geq 0,$ $a_1+a_2=a_3+a_4=2a_5+1,$ $m>5a_5+2$	$\begin{bmatrix} 5a_5+3+m \\ 5 \end{bmatrix}$	$m+5$
8	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$ $1\leq m\leq 5$	$\left[\begin{array}{cccccccc\|ccc} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \ldots & 0 \end{array}\right]$	$\begin{bmatrix} m \\ 6 \end{bmatrix}$	$m+5$
9	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$ $m\geq 2$	$\left[\begin{array}{cccccccc\|ccc} 0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \ldots & 0 \end{array}\right]$ $a_i\geq 0,$ $a_1=a_2+a_3=a_4+a_5=a_6+a_7,$ $m>3a_1$	$\begin{bmatrix} 3a_1+m \\ 6 \end{bmatrix}$	$m+5$
10	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$ $m\geq 2$	$\left[\begin{array}{cccccccc\|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & d_2 & \ldots & d_m \end{array}\right]$ $0\leq d_2\leq\cdots\leq d_m,\ 0<d_m\leq 5$ $m\cdot d_m<6+d_2+\ldots+d_m$	$\begin{bmatrix} m \\ 6+\sum d_k \end{bmatrix}$	$m+5$
11	$\frac{\mathbb{K}[T_1,\ldots,T_7,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2\rangle}$ $1\leq m\leq 4$	$\left[\begin{array}{ccccccc\|ccc} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \ldots & 0 \end{array}\right]$	$\begin{bmatrix} m \\ 5 \end{bmatrix}$	$m+4$
12	$\frac{\mathbb{K}[T_1,\ldots,T_7,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2\rangle}$ $m\geq 2$	$\left[\begin{array}{ccccccc\|ccc} 0 & 2a_5 & a_1 & a_2 & a_3 & a_4 & a_5 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \ldots & 0 \end{array}\right]$ $a_1+a_2=a_3+a_4=2a_5$ $a_i\geq 0$ $m>5a_5$	$\begin{bmatrix} m+5a_5 \\ 5 \end{bmatrix}$	$m+4$
13	$\frac{\mathbb{K}[T_1,\ldots,T_7,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2\rangle}$ $m\geq 2$	$\left[\begin{array}{ccccccc\|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & d_2 & \ldots & d_m \end{array}\right]$ $0\leq d_2\leq\ldots\leq d_m$ $d_m>0$ $m\cdot d_m<5+d_2+\ldots+d_m$	$\begin{bmatrix} m \\ 5+\sum d_k \end{bmatrix}$	$m+4$
14	$\frac{\mathbb{K}[T_1,\ldots,T_{10}]}{\left\langle \begin{array}{c} T_1T_2+T_3T_4+T_5T_6+T_7T_8, \\ \lambda_1T_3T_4+\lambda_2T_5T_6+T_7T_8+T_9T_{10} \end{array} \right\rangle}$	$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 3 \end{bmatrix}$	6

Moreover, each of the listed data defines a smooth Fano arrangement variety of complexity two and Picard number two.

Remark 6.2.3. Some of the above Fano varieties are intrinsic quadrics. Here is the overlap with [34, Cor. 1.2]:

(i) Cases 10 and 13 are intrinsic quadrics of Type 1,
(ii) Cases 9 and 12 are intrinsic quadrics of Type 2,
(iii) Cases 8 and 11 are intrinsic quadricsof Type 3,
(iv) Case 4.G is an intrinsic quadric of Type 4.

Theorem 6.2.4. *Every non-toric smooth projective truly almost Fano arrangement variety of complexity two and Picard number two is isomorphic to precisely one of the following varieties X, specified by their Cox ring $\mathcal{R}(X)$, the matrix $[w_1,\ldots,w_r]$ of generator degrees and an ample class $u\in\mathrm{Cl}(X)=\mathbb{Z}^2$.*

No.	$\mathcal{R}(X)$	$[w_1, \dots, w_r]$	u	$\dim(X)$
4.A	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^4+T_3T_4+T_5T_6+T_7T_8\rangle}$, $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&3&1&3&1&3&1&3&\dots&3 \\ 1&0&1&0&1&0&1&0&1&\dots&1 \end{array}\right]$	$\begin{bmatrix} 4 \\ 1 \end{bmatrix}$	$m+5$
4.B	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^3+T_3T_4+T_5T_6+T_7T_8\rangle}$, $m\geq 0$	$\left[\begin{array}{cccccccc\|cccc} 0&1&2&1&2&1&2&1&1&2&\dots&2 \\ 1&0&1&0&1&0&1&0&1&1&\dots&1 \end{array}\right]$	$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$	$m+5$
4.C	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^3+T_3T_4^2+T_5T_6+T_7T_8\rangle}$, $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&1&1&2&1&2&1&2&\dots&2 \\ 1&0&1&0&1&0&1&0&1&\dots&1 \end{array}\right]$	$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$	$m+5$
4.D	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^4+T_3T_4^2+T_5T_6^2+T_7T_8^2\rangle}$, $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&2&1&2&1&2&1&2&\dots&2 \\ 1&0&1&0&1&0&1&0&1&\dots&1 \end{array}\right]$	$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$	$m+5$
4.E	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$, $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&0&1&0&1&0&1&1&\dots&1 \\ 1&0&1&0&1&0&1&0&1&\dots&1 \end{array}\right]$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	$m+5$
4.F	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^2+T_3T_4+T_5T_6+T_7T_8\rangle}$, $m\geq 0$	$\left[\begin{array}{cccccccc\|ccccc} 0&1&1&1&1&1&1&1&0&0&1&\dots&1 \\ 1&0&1&0&1&0&1&0&1&1&1&\dots&1 \end{array}\right]$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	$m+5$
4.G	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^2+T_3T_4+T_5T_6+T_7T_8\rangle}$, $m\geq 0$	$\left[\begin{array}{cccccccc\|cccc} 0&1&1&1&1&1&1&1&-1&1&\dots&1 \\ 1&0&1&0&1&0&1&0&1&1&\dots&1 \end{array}\right]$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	$m+5$
4.H	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6^2+T_7T_8\rangle}$, $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&0&1&0&1&1&1&1&\dots&1 \\ 1&0&1&0&1&0&1&0&1&\dots&1 \end{array}\right]$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	$m+5$
4.I	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^3+T_3T_4^2+T_5T_6^2+T_7T_8^2\rangle}$, $m\geq 0$	$\left[\begin{array}{cccccccc\|cccc} 0&1&1&1&1&1&1&1&0&1&\dots&1 \\ 1&0&1&0&1&0&1&0&1&1&\dots&1 \end{array}\right]$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	$m+5$
4.J	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6+T_7T_8\rangle}$, $m\geq 0$	$\left[\begin{array}{cccccccc\|cccc} 0&1&0&1&1&1&1&1&0&1&\dots&1 \\ 1&0&1&0&1&0&1&0&1&1&\dots&1 \end{array}\right]$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	$m+5$
4.K	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^4+T_3T_4^3+T_5T_6^3+T_7T_8^3\rangle}$, $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&1&1&1&1&1&1&1&\dots&1 \\ 1&0&1&0&1&0&1&0&1&\dots&1 \end{array}\right]$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	$m+5$
4.L	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^3+T_3T_4^2+T_5T_6^2+T_7T_8^2\rangle}$, $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&0&1&1&1&1&1&1&\dots&1 \\ 1&0&1&0&1&0&1&0&1&\dots&1 \end{array}\right]$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	$m+5$
4.M	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$, $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&0&1&0&1&0&1&d_1&\dots&d_m \\ 1&0&1&0&1&0&1&0&1&\dots&1 \end{array}\right]$ $d_k \leq 0,\ \sum d_k = -3$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$m+5$
4.N	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6^2+T_7T_8^2\rangle}$, $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&0&1&0&1&0&1&d_1&\dots&d_m \\ 1&0&1&0&1&0&1&0&1&\dots&1 \end{array}\right]$ $d_k \leq 0,\ \sum d_k = -2$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$m+5$
4.O	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^3+T_3T_4^3+T_5T_6^3+T_7T_8^3\rangle}$, $m\geq 0$	$\left[\begin{array}{cccccccc\|cccc} 0&1&0&1&0&1&0&1&-1&0&\dots&0 \\ 1&0&1&0&1&0&1&0&1&1&\dots&1 \end{array}\right]$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$m+5$
4.P	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2^4+T_3T_4^4+T_5T_6^4+T_7T_8^4\rangle}$, $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&1&0&1&0&1&0&1&0&\dots&0 \\ 1&0&1&0&1&0&1&0&1&\dots&1 \end{array}\right]$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$m+5$
5	$\frac{\mathbb{K}[T_1,\dots,T_8,S_1,\dots,S_m]}{\langle T_1T_2+T_3^2T_4+T_5^2T_6+T_7^2T_8\rangle}$, $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0&2a+1&a&1&a&1&a&1&1&\dots&1 \\ 1&1&1&0&1&0&1&0&0&\dots&0 \end{array}\right]$ $a \geq 0,\ m = 3a$	$\begin{bmatrix} 2a+2 \\ 1 \end{bmatrix}$	$m+5$

6	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2T_6+T_7^2T_8\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccccc\|ccc} 0 & 2a_3+1 & a_1 & a_2 & a_3 & 1 & a_3 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \ldots & 0 \end{array}\right]$ $0 \leq a_1 \leq a_2, 0 \leq a_3,$ $m = 4a_3 + 1$	$\begin{bmatrix} 2a_3+2 \\ 1 \end{bmatrix}$	$m+5$
7	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2T_8\rangle}$ $m\geq 1$	$\left[\begin{array}{cccccccc\|ccc} 0 & 2a_5+1 & a_1 & a_2 & a_3 & a_4 & a_5 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & \ldots & 0 \end{array}\right]$ $a_i \geq 0,$ $a_1 + a_2 = a_3 + a_4 = 2a_5 + 1,$ $m = 5a_5 + 2$	$\begin{bmatrix} 2a_5+2 \\ 1 \end{bmatrix}$	$m+5$
8	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$ $m=6$	$\left[\begin{array}{cccccccc\|ccc} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \ldots & 0 \end{array}\right]$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$m+5$
9	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$ $m\geq 2$	$\left[\begin{array}{cccccccc\|ccc} 0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \ldots & 0 \end{array}\right]$ $a_i \geq 0,$ $a_1 = a_2 + a_3 = a_4 + a_5 = a_6 + a_7,$ $m = 3a_1$	$\begin{bmatrix} a_1+1 \\ 1 \end{bmatrix}$	$m+5$
10	$\frac{\mathbb{K}[T_1,\ldots,T_8,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7T_8\rangle}$ $m\geq 2$	$\left[\begin{array}{cccccccc\|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & d_2 & \ldots & d_m \end{array}\right]$ $0 \leq d_2 \leq \cdots \leq d_m, 0 < d_m \leq 6$ $m \cdot d_m = 6 + d_2 + \ldots + d_m$	$\begin{bmatrix} 1 \\ d_m+1 \end{bmatrix}$	$m+5$
11	$\frac{\mathbb{K}[T_1,\ldots,T_7,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2\rangle}$ $m=5$	$\left[\begin{array}{ccccccc\|ccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \ldots & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & \ldots & 1 \end{array}\right]$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$m+4$
12	$\frac{\mathbb{K}[T_1,\ldots,T_7,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2\rangle}$ $m\geq 2$	$\left[\begin{array}{ccccccc\|ccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \ldots & 0 \\ 0 & 2a_5 & a_1 & a_2 & a_3 & a_4 & a_5 & 1 & \ldots & 1 \end{array}\right]$ $2a_5 = a_1 + a_2 = a_3 + a_4,$ $a_i \geq 0$ $m = 5a_5$	$\begin{bmatrix} 2a_5+1 \\ 1 \end{bmatrix}$	$m+4$
13	$\frac{\mathbb{K}[T_1,\ldots,T_7,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6+T_7^2\rangle}$ $m\geq 2$	$\left[\begin{array}{ccccccc\|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & d_2 & \ldots & d_m \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \ldots & 1 \end{array}\right]$ $0 \leq d_2 \leq \ldots \leq d_m$ $m \cdot d_m = 5 + d_2 + \ldots + d_m$	$\begin{bmatrix} 1 \\ d_m+1 \end{bmatrix}$	$m+4$

Moreover, each of the listed data defines a smooth truly almost Fano arrangement variety of complexity two and Picard number two.

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