Analyse von Ausfallrisiken

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Geleitwort

Zu den großen Herausforderungen unserer Zeit gehört in der Finanzwirtschaft zweifelsohne das Ausfallrisiko, auch Credit Risk genannt. Dieses Risiko betrifft alle Arten von Finanzgeschäften, sofern die Möglichkeit besteht, dass vereinbarte Zahlungen womöglich nicht vollumfänglich geleistet werden und somit zumindest teilweise ausfallen. Deshalb stellt Credit Risk eine wichtige Risikoquelle für die folgenden Finanzmarktakteure dar:

- Risikomanager, die die bestehenden Risiken identifizieren und im gewünschten Umfang adjustieren müssen.
- Investoren, die genau den Umfang und die Art dieses Risikos abschätzen und geeignet in Bezug zu entsprechenden Risikoprämien setzen wollen, um von solchen Positionen zu profitieren.
- Aufsichtsbehörden, die bei Übernahme von Kreditrisiken über den Umfang des bei Banken zu unterlegenden Kapitals zu entscheiden haben.

Mit dem vorliegenden Werk ist es Thomas Schön sowohl gelungen die Struktur des Credit Risk im Kern zu identifizieren als auch Ausfallrisiken mit modernen Finanzprodukten wie Conditional Credit Default Swaps (CCDS) zu steuern und in Form von aufsichtsrechtlichen Kennzahlen wie dem Credit Value Adjustment (CVA) geeignet zu erfassen.

Durch eine konsequente empirische sowie modelltheoretische Analyse gelingt es Herrn Schön die entscheidenden ökonomischen Effekte zu identifizieren und empirisch nachzuweisen. Insbesondere kann er so das umfassende Ausfallrisiko in drei verschiedene Subausfallarten für Einzelausfälle, Multiausfälle und systemische Ausfälle unterteilen und markante Veränderungen dieser in der Wahrnehmung des Kapitalmarktes durch die Finanzkrise aufzeigen. Die Betrachtung dieser Subkategorien ist auch für Credit Default Swaps (CDSs) auf Einzeladressen von höchster Relevanz, da diese je nach Bonität unterschiedlich, aber in vorhersehbarer Weise auf CDS-Preise einwirken.

Ferner meistert Herr Schön die Bewertungsherausforderungen von CCDS. Mittels eines Strukturmodells mit Zins- und Bonitätsrisiko erfasst er die spezifischen Unterschiede im Vergleich zu herkömmlichen CDS und kann dennoch eine einfache, geeignete Näherungsformel, die lediglich auf Marktpreisen von gehandelten Instrumenten basiert, für CCDS angeben.

Schließlich zeigt er die Modellabhängigkeit der CVA sowohl bei Aktien- als auch Zinsderivaten auf.

Mit dieser Arbeit liefert Herr Schön für alle drei genannten Interessensgruppen relevante Beiträge, die ein neues Verständnis von Credit Risk bei der Risikoerfassung, -bewertung und aufsichtsrechtlichen Behandlung ermöglichen.

Ich kann sowohl allen Forschenden wie auch Risikomanagern, Tradern, Portfolio Managern sowie Supervisorn in Aufsichtsbehörden, die mit Kreditrisiken zu tun haben, die Lektüre der äußerst gelungenen Dissertationsschrift von Thomas Schön nur empfehlen. Ich wünsche dieser Arbeit eine gute Aufnahme in diesen Leserkreisen.

Prof. Dr. Christian Koziol

Vorwort

Die vorliegende Arbeit entstand in den Jahren 2012 – 2015 während meiner Zeit als wissenschaftlicher Angestellter am Lehrstuhl für Finance der Eberhard Karls Universität Tübingen von Herrn Prof. Dr. Christian Koziol. Sie wurde dort im Wintersemester 2015/2016 von der Wirtschafts- und Sozialwissenschaftlichen Fakultät als Dissertation angenommen.

Mein ganz herzlicher Dank gilt meinem Doktorvater, Herrn Prof. Dr. Christian Koziol, für die wissenschaftliche Betreuung der Arbeit und die uneingeschränkte Unterstützung. Er hat mir während des Entstehungsprozesses stets den notwendigen Freiraum eingeräumt und mit sehr wertvollen Diskussionen und Anregungen maßgeblich zum Gelingen der Dissertation beigetragen. Herrn Prof. Dr. Joachim Grammig danke ich für die rasche Erstellung des Zweitgutachtens sowie Herrn Prof. Dr. Werner Neus für die Übernahme des Prüfungsvorsitzes im Rahmen der Disputation. Ebenfalls möchte ich mich bei Herrn Dr. Philipp Koziol und weiteren Mitarbeitern der Deutschen Bundesbank, Frankfurt am Main, für wertvolle Diskussionen und die Unterstützung im Rahmen des Forschungsprojektes "Are Credit Risks Consistently Priced? An Empirical Analysis of CDS and CDO Markets during Pre-Crisis, Crisis, and Post-Crisis" bedanken, dessen Erkenntnisse in meine Dissertation eingeflossen sind.

Neben den akademischen Wegbegleitern trägt jedoch insbesondere auch das private Umfeld zum Gelingen einer Dissertation bei. Daher möchte ich mich zunächst bei meinen Freunden bedanken, die mich während des Entstehungsprozesses begleitet haben und stets für die nötige Abwechslung zu Sorgen wussten. Für den langjährigen Austausch zu quantitativen Themen und wegen des gemeinsamen Interesses an wissenschaftlichen Fragestellungen verdient Herr Florian Häußermann an dieser Stelle besondere Erwähnung.

Meinen Eltern Gisela und Siegfried Schön danke ich aus tiefstem Herzen für die unbeschwerte, wunderbare Kindheit, die sie mir und meinem Bruder geschenkt haben. Dadurch, dass sie meine Ausbildung stets gefördert und mich all die Jahre unterstützt haben, legten sie den Grundstein für die vorliegende Arbeit.
Meinem Bruder Dr. Stephan Schön und seiner Frau Amela danke ich für den steten Rückhalt und die schöne gemeinsame Zeit, die ich nicht missen möchte. Manches lässt sich schwer in Worte fassen, jedoch möchte ich mich ganz besonders bei meinem Onkel Harald Schön bedanken. Und zwar nicht nur dafür, dass er als erster mein Interesse an der Wissenschaft geweckt hat, sondern auch für alles andere. Ihm ist diese Arbeit gewidmet.

Frankfurt am Main, im November 2015 Thomas Schön

Für Harry

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Chapter 1

Introduction

This thesis comprises three essays on the pricing of default risk. It analyzes latest developments in this field empirically and theoretically delivering deeper insights into the questions of how firms default and how default risk is priced in interest rate and equity instruments.

The origin of default risk lies in the possibility that a particular firm may not be able to follow its financial obligations, only making fractions of promised payments to its financial counterparties. This might lead to losses at a counterparty that in turn can cause its default, too. Very common but still highly important financial instruments in this context are debt securities such as bonds or loans. In the course of such a bond or loan, an investor lends a predefined amount of capital to the issuing firm which is then able to make investments that, for example, aim at expanding its business. Ideally, the firm pays back the capital at the maturity of the security and compensates the investor for lending his capital by paying interest on a regular basis. But since the outcome of an investment is uncertain in advance, in a worst case scenario, the firm can go bankrupt and impose a severe loss on the investor.

Thus, an investor has a deep interest in managing the default risk of his exposures. An important financial innovation that was developed in the 1990s and that gained considerable trading volumes in the 2000s are credit default swaps (CDS). A CDS insures an investor against the default of a predefined debt security. In case the counterparty of the investor defaults, he will receive a payment amount from the CDS contract that restores the amount of capital that he lost because of the default which can be considered as a perfect hedge of default risk.

Very important characteristics of CDS contracts are their very high liquidity and their very high sensitivity towards default risk which make them ideal instruments for default risk studies. For example, CDS can be pooled in a portfolio which again can be regarded

as an underlying for other securities such as index credit default swaps or tranches of collateralized debt obligations (CDOs). With the help of these instruments, an investor can insure himself against losses of a portfolio of debt securities. Thus, their prices reveal whether the default risks of single firms do only depend on their creditworthiness or whether they might also be subject to defaults of other firms. The default of the major parts supplier Delphi in 2005 and its consequences for the creditworthiness of General Motors showed that a single default can cause a chain reaction on financial markets. *Longstaff and Rajan (2008)* find that such correlated defaults are clearly priced in CDO tranches. But their paper raises the question which firms exactly are subject to the correlated default factors that they find for CDO tranche products. In the second chapter of this thesis, we investigate this question empirically and find that especially firms with a high creditworthiness are subject to correlated default factors.
Despite the fact that CDS contracts have gained high practical importance in the last years, one of their major drawbacks lies in their restriction to plain vanilla debt securities such as bonds or loans. There are, however, many other and more complex financial instruments traded actively on the markets whose value can be clearly affected by the default of a counterparty. Interest rate swaps (IR swaps) are one of the most important representatives of non-vanilla instruments that are subject to default. Their importance is attributed to the fact that IR swaps are used by many financial institutions for managing interest rate risk. In contrast to plain vanilla debt securities whose future default loss can be roughly estimated at valuation date, the future payments of IR swaps are stochastic because of uncertain interest rate payments. In this context, the management of possible default losses is highly complex and the valuation of defaultable IR swaps is non-trivial. Contingent credit default swaps (CCDS) provide protection against the default of IR swaps in a similarly convenient way as CDS do for plain vanilla debt securities. But the crucial question is whether CCDS exhibit a similar pricing behavior as CDS do or whether they are basically different instruments with distinct properties. In the third chapter, we investigate this question in a semi-analytical setting and find that CCDS exhibit fundamentally different pricing properties as CDS contracts.
One practical reason why CCDS have not gained high trading volumes lies in their complex payoff profile. No mentionable market has emerged since dealers and buyers avoid highly complex default products that are difficult to evaluate. Another major reason lies in the way the Basel II/III accords stipulate banks to mitigate default risks. On the one hand, a bank could offset its default risk from complex transactions by buying a CCDS. But it can also mitigate its default risks by correcting the value of

their portfolios for all default losses that might occur which is commonly known as credit valuation adjustment (CVA). The CVA is computed by subtracting the value of a defaultable financial instrument from its non-defaultable value. Therefore, it equals the default risk premium of a specific security. The major problem related to CVA lies in its computation since highly sophisticated models commonly need to be deployed. However, there are no established standard models for CVA computations which enables financial institutions to choose from a variety of available market models. As a consequence, the CVA can differ across models to a certain extent. In the fourth chapter of this thesis, we analyze the model dependence of the CVA for standard instruments in equity and interest rate markets. The main finding is that the CVA can be highly model dependent with obvious important implications for market participants and regulators. The last chapter concludes this thesis.

Chapter 2

Do Correlated Defaults Matter for CDS?*

Abstract

Correlated defaults and systemic risk are clearly priced in credit portfolio securities such as CDOs or index CDSs. In this paper we study an extensive CDX data set for evidence whether correlated defaults are also present in the underlying CDS market. We develop a cash flow based top-down approach for modeling CDSs from which we can derive the following major contributions: (I) Correlated defaults did not matter for CDS prices prior to the financial crisis in 2008. During and after the crisis, however, their importance has increased strongly. (II) In line with a plausible default order, we observe that correlated defaults primarily impact the CDS prices of firms with an overall low CDS level. (III) Idiosyncratic risk factors for each single CDS play a major (minor) role when the CDS premia are high (low).

* This chapter is based on the working paper *"Do Correlated Defaults Matter for CDS Premia? An Empirical Analysis"* by Koziol, Koziol and Schön (2014).

2.1 Introduction

Credit default swaps (CDSs) provide protection against the default of single name borrowers. Their prices are therefore highly dependent on the future default probability of the respective single name and they should only react to changes in their associated creditworthiness. However, if CDSs on different single names are pooled together in a portfolio, prices of securities on that portfolio, e.g. collateralized debt obligation (CDO) tranches, reveal that not only single defaults are priced in the CDS pool but correlated defaults are as well *see e.g. Longstaff and Rajan (2008)*. As a consequence, we have a paradoxical situation for the pricing of CDSs on single entities. On the one hand, a default event and therefore a payoff from the CDS only depends on the solvency of the particular firm. On the other hand, the market price for the CDS is also impacted by other characteristics outside the firm such as correlation effects. This property suggests that not only the individual default risks of a single name are relevant for pricing CDS but also systemic factors that can lead to the default of many single names simultaneously due to default correlation.

The aim of this paper is to analyze risk factors for correlated defaults that drive CDS quotes. In particular, we strive for answers to the following research questions: (1) Has the financial crisis changed the relevance of correlated default factors? (2) Which CDSs are primarily impacted by correlated default factors? (3) Which CDSs require a further idiosyncratic factor beyond the common default factors of the portfolio to be reasonably explained?

To analyze these research questions, we use CDX data for CDSs and CDO tranches retrieved from Markit for the time period from September 2005 until September 2012. In the first step, we follow the approach proposed by *Longstaff and Rajan (2008)* to calibrating default risk factors that explain CDO prices. We also find that three factors — single defaults, industry defaults and systemic defaults — represent market prices reasonably well. In the second step, we derive a cash flow based top-down approach that translates CDO prices into CDS quotes. The notion behind this model is that an observed change of a CDS translates with different sensitivities to the various CDO tranches in which this entity is included. Furthermore, our top-down approach allows for idiosyncratic risk factors that can perfectly explain empirically observed CDS premia.

These estimations provide us with the following conclusions: before the crisis, correlated default factors, i.e. industry and systemic defaults, played a minor role in the pricing of CDSs. More than 80% of the observed default risk was caused by the single default factor. During and subsequent to the crisis, correlated default factors strongly enhanced their importance. During the crisis, correlated default factors accounted for about 80% of the default risk, and even after the crisis, their fraction is still above 50%.

Furthermore, we can observe that the CDO tranche sensitivities to the various CDSs contained therein exhibit a reasonable default order. In other words, the low CDS premia are primarily relevant for the senior tranche in a CDO, while the high CDS quotes drive the equity tranche of a CDO. As a consequence of the further observation that the equity tranche can be primarily characterized by the single default factor, whereas the correlated default factors explain the more senior tranches, we can confirm the following important relationship for the pricing of CDSs: high CDS premia are primarily driven by the single default factor. For low CDS quotes, the correlated default factors are a relevant issue.

The methods applied in this paper are related to other important studies. *Giesecke et al. (2011)* introduce a top-down approach based on a default matrix implied by CDO and CDS prices. We modify their approach by adapting the default matrix to observable cash flows. That way, we achieve a low parameterization and a high analytical and empirical tractability. To put our approach to work, we have to use a model for the CDO portfolio for which a wide variety of literature exists. First of all, models that belong to the category of top models can be employed within our framework. Top models are used to directly model the portfolio loss distribution without considering the single names of the underlying portfolio. Examples can be found in *Longstaff and Rajan (2008)*, *Schönbucher (2006)*, *Arnsdorf and Halperin (2007)*, *Brigo et al. (2007)*, *Ding et al. (2009)* and many others. Since we do not impose certain restrictions on the CDO model but only assume the ability to model CDO cash flows, we could also use bottom-up models that capture the portfolio loss distribution from the underlying single-name portfolio. *Li (2000)*, *Hull and White (2004)* and *Lopatin (2011)* belong to this category. *Ascheberg et al. (2013)* investigate how these models perform empirically in hedging situations. *Junge and Trolle (2013)* construct a liquidity risk measure for CDS markets in comparison to index CDSs. Our paper contributes to their discussion through the introduction of idiosyncratic risk factors that explain the spread of index-to-theoretical bases.

The remainder of the paper is organized as follows: in Section 2.2, we introduce our cash flow based top-down approach. Section 2.3 presents a step-wise calibration procedure for the top-down model that we apply in Section 2.4 to the CDX North America Investment Grade index in order to analyze the question whether correlated defaults are priced in CDS markets. Section 2.5 concludes.

2.2 The Model

At first glance, single-name CDSs are only subject to individual default risk. However, if they are pooled in a portfolio such as for CDOs, the prices of tranches reveal that they contain not only individual, but also correlated default risks. In order to understand which portfolio risk factors are priced in single-name CDSs we propose a *top-down* approach that splits the cash flows of CDO tranches (*top* level) to the single name CDSs of the underlying portfolio (*down* level). The cash flow allocation is based on sensitivities that specify to what extent the cash flow of a specific CDS can be attributed to a given CDO tranche. This approach is based on the characteristic that a portfolio of CDSs provides equivalent cash flows to its corresponding CDO. Moreover, to capture potential deviations, we extend our top-down approach by including an idiosyncratic risk factor for each single-name CDS. This factor accounts for individual default risk that is not priced in the CDO market and should therefore facilitate the interpretation of any deviations between CDO-induced model premia and observed CDS premia.
In the following, we present how CDSs and CDO tranches are priced in our cash flow based top-down approach in general and how the idiosyncratic risk factor is embedded in our model for every single-name CDS. Furthermore, we outline the top model of *Longstaff and Rajan (2008)* that we use to price CDOs and a pointwise-homogeneous Poisson process for the idiosyncratic risk factors of single name CDS.

2.2.1 CDO Valuation

Let L_τ denote the accumulated loss of a CDO portfolio for any time τ with $t \leq \tau \leq T$, where t denotes the valuation date and T the maturity of the CDO with notional 1. Then, for the possible portfolio loss outcomes during the lifetime of the CDO $0 \leq L_\tau < 1$ holds. Furthermore, let a_p and d_p denote the attachment and the detachment point of the tranche p. The accumulated loss process L_τ^p of tranche p is then expressed by

$$L_\tau^p = \frac{1}{d_p - a_p} \left(\max[0, L_\tau - a_p] - \max[0, L_\tau - d_p] \right). \tag{2.1}$$

The equation shows that the notional $1 - L_\tau^p$ of the tranche p is not affected by portfolio losses that occur below its attachment point, $L_\tau < a_p$. For higher losses $L_\tau \geq a_p$, the notional $1 - L_\tau^p$ of p linearly decreases for increasing L_τ until L_τ hits the detachment point of p leading to $L_\tau^p = 1$.

The payment obligations of a CDO tranche become effective at payment dates t_n for which $t < t_n \leq T$ holds. We denote the time period between t_{n-1} and t_n as Δ_{t_n} where usually Δ_{t_n} takes values that are close to the quarter of a year depending on the day count convention of the CDO.
The protection leg of a CDO tranche compensates for losses in the underlying portfolio interval $[a_p, d_p)$ that occur between two payment dates t_{n-1} and t_n. Thus, the value of the protection leg at t under the risk-neutral measure $\mathbb{Q}$ is equal to

$$PF_t^{p,\text{prot}} = \sum_{n=1}^{N} b_{t,t_n} \cdot E_t^{\mathbb{Q}} \left(L_{t_n}^p - L_{t_{n-1}}^p \right) \tag{2.2}$$

where b_{t,t_n} denotes the discount factor at t with time horizion t_n. In exchange for the loss compensation, a CDO investor has to pay a premium c_t^p at every date t_n on the intact capital of the portfolio interval. The value of the premium leg referring to a premium amount of 1 is then expressed by

$$PF_t^{p,\text{prem}} = \sum_{n=1}^{N} b_{t,t_n} \cdot \Delta_{t_n} \cdot E_t^{\mathbb{Q}} \left(1 - L_{t_{n-1}}^p \right). \tag{2.3}$$

There are two conventions governing how CDO tranche premia are quoted in the markets. The approach that was mainly used before the financial crisis of 2008 is known as the running spread convention. At the trading date t, the two counterparties of a CDO tranche trade agree that the protection buyer will pay a premium $c_t^p \Delta_{t_n}$ on the remaining intact capital of the tranche to the protection seller at each payment date t_n. The premium c_t^p may change for every other CDO tranche trade and consequently, it is subject to market risk. Thus, as the market risky premium c_t^p is paid on a recurring basis its quoting convention is known as the running spread convention. Its value is derived from the assumption that under the risk-neutral measure $\mathbb{Q}$ the value of the protection leg has to equal the value of the premium leg leading to:

$$c_t^p = \frac{PF_t^{p,\text{prot}}}{PF_t^{p,\text{prem}}}. \tag{2.4}$$

The other approach to quoting CDO tranches is known as the upfront payment convention. It was already used for junior tranches before the financial crisis but it has since become the market standard. Its major benefit lies in the simplification of trade processing and the higher flexibility and efficiency in trade settlements. Unlike for c_t^p, the running spread $c^{p,\text{fix}}$ is set for a standard amount and is not subject to market risk.

In order to account for the value of the CDO portfolio, the protection buyer pays an upfront payment $c_t^{p,\text{up}}$ to the protection seller at t that is subject to market risk. When the value of the protection leg is equal to that of the premium leg, $c_t^{p,\text{up}}$ is equal to zero. Otherwise,

$$c_t^{p,\text{up}} = PF_t^{p,\text{prot}} - c^{p,\text{fix}} PF_t^{p,\text{prem}} \tag{2.5}$$

holds. Equation (2.5) implies that $c_t^{p,\text{up}}$ can take negative values which might seem unrealistic at first glance. But there are certain cases in which the value of the fixed premium leg might be too high with respect to the quality of an underlying portfolio interval and therefore the protection seller compensates the buyer for overpayments that occur during the course of the CDO tranche.
Typically, the intervals $[a_p, d_p)$ are parameterized in such a way that their union yields the interval $[0, 1)$. Furthermore, they are disjoint sets, making them adjacent intervals. In the following, we will assume that $p = 1$ marks the most junior tranche of a CDO which is commonly known as the equity tranche. As long as no default happens in the CDO portfolio, the capital of the equity tranche remains unaffected. But as soon as losses occur, the capital of the equity tranche will be reduced first until it is completely exhausted. Further portfolio losses will then affect the next most junior tranche after the equity tranche, $p = 2$, also known as the junior mezzanine tranche. Increasing portfolio losses will consume the capital of $p = 2$ until it is completely exhausted, too. In this way, at least in theory, the capital of the CDO is consumed tranche after tranche until the capital of the entire portfolio is consumed.

2.2.2 CDS Valuation

In our top-down approach, the value of a CDS k is represented by its sensitivities q_k^p towards the CDO tranches $p = 1, \ldots, P$. In other words, the sensitivities q_k^p split the cash flows of all CDO tranches to all the single-name CDSs of the underlying portfolio. Therefore, in a perfect world, the cash flow of a CDO tranche p is completely allocated to the underlying CDS portfolio leading to equation (2.6). Additionally, as we only split a cash flow into positive amounts, inequation (2.7) has to hold:

$$\sum_{k=1}^{K} q_k^p = 1, \forall p, \tag{2.6}$$

$$q_k^p \geq 0, \forall p, \forall k. \tag{2.7}$$

(2.6) and (2.7) represent restrictions that CDS model premia have to adhere to and that are especially important during calibration.
The cash flows of a CDO tranche are allocated to a single-name CDS as follows: as can be seen from equation (2.1) the cash flows of a CDO tranche refer to its notional with amount 1. However, the notional of the CDO portfolio adds up to 1 as well. Consequently, CDO tranche cash flows need to be rescaled to the original portfolio notional which can be achieved by multiplying equation (2.2) by $(d_p - a_p)$. In the next step, the rescaled CDO tranche cash flows are split to the single-name CDSs of the portfolio by multiplying them by the tranche sensitivities q_k^p. For the protection leg of a CDS k, the value of allocated tranche cash flows is expressed by:

$$PF_t^{k,\text{prot}} = \sum_{n=1}^{N} b_{t,t_n} \sum_{p=1}^{P} q_k^p \cdot (d_p - a_p) \cdot E_t^{\mathbb{Q}} \left(L_{t_n}^p - L_{t_{n-1}}^p \right) \tag{2.8}$$

If both markets, the CDO as well as the CDS market, valued risks equivalently, equation (2.8) would be sufficient for valuing the protection leg cash flow of any CDS. However, this may not always be the case, and in order to account for CDS premia that are not in line with CDO cash flows, we extend the CDS valuation by including idiosyncratic risk factors. For this purpose, we introduce a stochastic default time η_k for each CDS k where the distribution of η_k is driven by a k-specific idiosyncratic risk factor. Then the value of the protection leg induced by the idiosyncratic risk factor is

$$I_t^{k,\text{prot}} = (1 - \varphi) \cdot \sum_{n=1}^{N} b_{t,t_n} \cdot P\left[t_{n-1} < \eta_k \leq t_n\right], \tag{2.9}$$

where φ denotes the recovery rate. As in the protection leg, the portfolio-related part of the premium leg of k is computed as

$$PF_t^{k,\text{prem}} = \sum_{n=1}^{N} b_{t,t_n} \cdot \Delta_{t_n} \cdot \sum_{p=1}^{P} q_k^p \cdot (d_p - a_p) \cdot E_t^{\mathbb{Q}} \left(1 - L_{t_{n-1}}^p \right) \tag{2.10}$$

and the part of the idiosyncratic risk factor as

$$I_t^{k,\text{prem}} = \sum_{n=1}^{N} b_{t,t_n} \cdot \Delta_{t_n} \cdot P\left[t_n < \eta_k\right]. \tag{2.11}$$

Finally, the model premium of a CDS k in our top-down approach is expressed by

$$f_t^k = \frac{PF_t^{k,\text{prot}} + I_t^{k,\text{prot}}}{PF_t^{k,\text{prem}} + I_t^{k,\text{prem}}}. \tag{2.12}$$

If $q_k^p = 0$ for all p of a given k, then equation (2.12) reduces to the valuation formula that is commonly used in the literature, e.g. as in *Longstaff et al. (2005)*. It would also mean that a CDS is not correlated with a given CDO portfolio at all.

Default Order Equations (2.8) to (2.12) show that in our approach mainly four types of input are necessary for pricing CDS premia: the discount factors b_{t,t_n}, the expected tranche loss $E_t^{\mathbb{Q}}\left(L_{t_n}^p\right)$, the sensitivities q_k^p that determine the impact of the expected tranche loss on the CDS premium, and finally the survival probabilities $P\left[t_n < \eta_k\right]$ deduced from the idiosyncratic risk factor. While the expected tranche loss is computed from CDO data, the sensitivities q_k^p and the idiosyncratic risk factor are retrieved from information priced in the CDS market. The levels of their values have a strong economic impact as they reveal which kind of portfolio risk is priced in a given CDS. For example, let us consider the sensitivity q_k^1 that is associated with the equity tranche of a CDO. If q_1^1 is noticeably higher than all other q_k^1 then the bulk of the expected losses of the equity tranche are priced in CDS $k = 1$. Another example deals with the important question of whether systemic risk is priced in only a few single names or in all names of a portfolio. Given the nature of systemic risk, one might expect that the effects of a catastrophic event, e.g. a severe economic crisis, would lead to a substantial number of defaults in the portfolio. Therefore, if CDSs and CDO markets are priced consistently it is plausible to suggest that most of the single names are exposed to systemic risk, implying equally high q_k^p for all k with respect to the senior tranche $p = 4$. In other words, the risk of junior tranches should be mapped to only a few single names that have a high probability of default. By contrast, the risk inherent in senior tranches should be priced in single names with small CDS premia because they will most likely only be affected by a catastrophic event. Consequently, the values q_k^p shed light on the implicit default order that can be deduced from CDS and CDO premia. If the default order prevails, then a high (low) CDS quote is supposed to have a high sensitivity to the equity tranche $p = 1$ (senior tranche $p = 4$). A major question for our empirical study in Section 2.4 will be, whether the default order can be verified empirically.

2.2.3 Portfolio Model

There are several possibilities for modeling the loss distribution of a portfolio. One common way is to model the default of every single name first and then to aggregate the resulting single-name loss distributions to a portfolio loss distribution. This approach is known as the bottom-up approach and is used e.g. in the base correlation model (*O'Kane and Livesey (2004)*). Although we are free to employ such a model in our valuation approach, it seems to be more purposeful to model the portfolio loss distribution *directly* without considering the risks inherent in single names. In this way, one does not need to account for the dependence structure between all single names which results in a much lower parameterization and simpler calibration of the model. The model of *Longstaff and Rajan (2008)* combines these desirable properties and is the portfolio model that we use throughout this paper.
First, we assume that the loss dynamic of a portfolio is driven by three independent Cox processes $i = 1, 2, 3$. The intensity dynamic of each process is given in the form of a *Cox et al. (1985)* process without drift as in

$$d\lambda_\tau^i = \sigma_i \sqrt{\lambda_\tau^i} dY_\tau^i, \tag{2.13}$$

where dY_τ^i marks the independent increment of a Wiener process related to process i and σ_i its volatility. λ_τ^i denotes the jump intensity. Let $P_t[j = N_T^i]$ denote the probability that conditional on time t process i has jumped j times at time T. It can then be shown that the following equation holds for the jump probabilities of each process:

$$P_t[j = N_T^i] = \exp(-A^i(T-t) \cdot \lambda_t^i) \cdot \sum_{k=0}^{j} B_{j,k}^i(T-t) \cdot (\lambda_t^i)^k, \tag{2.14}$$

$$A^i(T-t) = \frac{4\sigma_i^2}{\sqrt{2}\sigma_i^3 \cdot \left[1 + \exp(-\sqrt{2}\sigma_i \cdot (T-t))\right]} - \frac{\sqrt{2}}{\sigma_i} \tag{2.15}$$

where $B_{0,0}^i(T-t) = 1$, $B_{j,0}^i(T-t) = 0$ for $j > 0$, $B_{j,k}^i(0) = 0$ for $j > 0$, $k > 0$. The remaining functions $B_{j,k}^i(T-t)$, $1 \leq k \leq j-1$ are computed numerically from the following system of ODEs:

$$dB_{j,j}^i(\tau) = j \cdot (B_{j-1,j-1}^i(\tau) - \sigma_i^2 \cdot A^i(\tau) \cdot B_{j,j}^i(\tau))d\tau, \tag{2.16}$$

$$dB_{j,k}^i(\tau) = \left(jB_{j-1,k-1}^i(\tau) - k\sigma_i^2 A^i(\tau) B_{j,k}^i(\tau) + \frac{(k+1)k\sigma_i^2}{2} B_{j,k+1}^i(\tau) \right) d\tau. \tag{2.17}$$

As a result of the distributions for the number of jumps N_τ^i and the jump size γ_i, we obtain the following possible outcomes for the portfolio losses:

$$L_\tau = 1 - \exp\left(-\sum_{i=1}^{3} \gamma_i N_\tau^i\right). \tag{2.18}$$

Obviously, $0 \leq L_\tau < 1$ holds for $N_\tau^i \in \mathbb{N}_0$. If $N_\tau^i = 0, \forall i$ then the exponential function is equal to one and the portfolio loss takes the value $L_\tau = 0$. For increasing N_τ^i the portfolio loss will increase as well until it takes values close to one.

2.2.4 Single-Name Model

In general, there are two model classes that are suitable for computing the default probabilities of the idiosyncratic risk factors in equations (2.11) and (2.9): structural and reduced form models. While structural models are particularly useful for an economic explanation of the sources leading to the default of a company, reduced form models allow for a higher flexibility and do not need any assumptions with regard to the liability structure. For these two reasons, we employ a pointwise-homogeneous reduced form model in the context of *Lando (1998)* in order to compute default probabilities from the idiosyncratic risk factor. Let θ_τ^k mark the default intensity of the idiosyncratic risk factor of single name k at time τ. The associated solution for the survival probabilities $P_t[T < \eta_k]$ can be derived as

$$P_t[T < \eta_k] = \exp(-(\theta_t^k + \omega_t^k) \cdot (T - t)) \tag{2.19}$$

where ω_τ^k marks a technical intensity that does not exhibit a specific economic meaning. The need for ω_τ^k arises because of the structure of equation (2.12) and the calibration pattern introduced in Section 2.3. The first step in the pattern calibrates the ratio $PF_t^{k,\text{prot}}/PF_t^{k,\text{prem}}$ to observed CDS premia. Afterwards, for calibrating the idiosyncratic risk factor, it first needs to be adapted to the protection and the default leg induced by the tranche cash flows and the calibrated q_k^p as in (2.12). That means that

$$\frac{PF_t^{k,\text{prot}} + I_t^{k,\text{prot}}}{PF_t^{k,\text{prem}} + I_t^{k,\text{prem}}} = \frac{PF_t^{k,\text{prot}}}{PF_t^{k,\text{prem}}} \tag{2.20}$$

for calibrated ω_τ^k and $\theta_\tau^k = 0$. To match observed CDS premia, we allow θ_τ^k to take values that are greater than $-\omega_\tau^k$. This way, the calibrated θ_τ^k reflect the true level of idiosyncratic risk.

2.3 Model Calibration

For an accurate calibration, we propose a stepwise calibration pattern that successively calibrates the models involved from the *top* level of the CDO portfolio to the *down* level of each single-name CDS.

2.3.1 Portfolio Level

Let c_t^{*p} denote the observable market quote of the CDO tranche p at time t. For a given data set, we formulate the calibration problem as follows:

$$\begin{aligned} \min_{\lambda_t,\sigma,\gamma} \quad & \sum_t \sum_p \left| \frac{c_t^p - c_t^{*p}}{c_t^{*p}} \right| \\ \text{s.t.} \quad & \lambda_t^i \geq 0, \forall i, \forall t, \\ & \sigma_i \geq 0, \forall i, \\ & \gamma_i \geq 0, \forall i. \end{aligned} \tag{2.21}$$

Calibration errors that are composed of absolute values of relative differences offer several advantages in CDO calibration. First, the information of all tranches is incorporated into the calibration problem to the same extent. This effect avoids an overemphasis of junior tranches with relatively high premia and ensures that the information contained in senior tranches is taken into account in a balanced way. The approach is therefore superior for calibrating correlated default factors that drive prices of senior tranches. Moreover, in comparison with a quadratic calibration error, the absolute error forces the optimization algorithm to further minimize errors that are below one, whereas quadratic errors tend to overemphasize very large deviations and to neglect very small ones. The problem is solved with a gradient-based method.

2.3.2 Single-Name Level

In our approach, there are two types of risks that are priced in CDS premia: CDO-induced risk and idiosyncratic risk. While the latter is only related to the single-name k and therefore does not require any information about the other single names of a portfolio, the former is split from the portfolio to all single names. This is why restrictions (2.6) and (2.7) have to be adhered to during the calibration of q_k^p. We formulate the general calibration problem as

$$\begin{aligned}
\min_{q,\theta} \quad & \sum_t \sum_{k=1}^{K} (f_t^k - f_t^{*k})^2 \\
\text{s.t.} \quad & \sum_{k=1}^{K} q_k^p = 1, \forall p, \\
& q_k^p \geq 0, \forall p, \forall k, \\
& \omega_t^k \geq 0, \forall k, \\
& \theta_t^k \geq -\omega_t^k, \forall k,
\end{aligned} \tag{2.22}$$

where f_t^{*k} denotes the values of observed CDS market premia. We do not choose an absolute error as in (2.21) because we want to make use of a quadratic optimization algorithm that facilitates very fast and accurate calibration[1].

Since θ_t^k is not restricted to positive values the idiosyncratic risk factor can lead to higher *or* lower CDS model premia compared to the case in which only portfolio risks are priced. Let us assume that the model premium which is only induced by portfolio risk is lower than the observed market premium. Then $\theta_t^k > 0$ has to hold as additional default mass needs to be induced by the idiosyncratic risk factor. For the other case that the portfolio model premium is too high, $\theta_t^k < 0$ leads to a reduction of the priced default mass in the model premium. This is why one can directly infer from the value of the intensities θ_t^k whether a single name is subject to high or low idiosyncratic risk. The restriction $\theta_t^k \geq -\omega_t^k$ needs to be imposed because otherwise negative intensities $\theta_t^k + \omega_t^k$ would be possible in (2.19), and the related default probabilities $P_t[T < \eta_k]$ would not meet the usual requirements for a probability measure.

Although the idiosyncratic risk factors provide easy and useful explanations they complicate the calibration of CDS premia. One possibility for addressing this problem would be to calibrate all the parameters related to CDSs at once: $q_k^p, \omega_t^k, \theta_t^k$. However, it is non-trivial to solve such a high-dimensional problem. For this reason, we propose a step-wise calibration approach that we outline in the following.

We split the problem (2.22) into two problems that are easier to solve. First, we calibrate the sensitivities q_k^p only and afterwards the idiosyncratic risk factor where we take the calibrated q_k^p from the first step as fixed values. This approach is motivated by the assumption that all single names — and only them — are part of the CDO portfolio

[1] As problem (2.21) would still be highly non-linear with a quadratic error, the absolute error there is the better choice for an accurate calibration.

and that all losses that are priced in the CDO should map, overall , to the single names. Any further deviations that cannot be explained by the portfolio dynamics are then captured by the idiosyncratic risk factor.
In accordance with the approach of *Giesecke et al. (2011)*, let $q \in \mathbb{R}^{(k \cdot p) \times 1}$ denote the stapled vectors q_k each of which contains the sensitivities q_k^p of a given k. With e denoting a vector of ones for which $e \in \mathbb{R}^{p \times 1}$ holds, we solve the following quadratic problem for q:

$$\begin{aligned} \min_q \quad & \frac{1}{2} q^T \cdot Q \cdot q \\ \text{s.t.} \quad & A \cdot q = e \\ & q \geq 0. \end{aligned} \tag{2.23}$$

Q is a diagonal matrix with matrices Q_k on its diagonal and zeros otherwise with

$$Q_k = \sum_\tau \operatorname{diag}(z_\tau^k) \cdot e \cdot e' \cdot \operatorname{diag}(z_\tau^k), \tag{2.24}$$

where the elements of the vector $z_t^k \in \mathbb{R}^{p \times 1}$ are defined as

$$z_t^{k,p} = \sum_{n=1}^N b_{t,t_n} \cdot (d_p - a_p) \cdot \left(\Delta_{t_n} \cdot E_t^{\mathbb{Q}} \left(1 - L_{t_n}^p \right) - \frac{E_\tau^{\mathbb{Q}} \left(L_{t_n}^p - L_{t_{n-1}}^p \right)}{f_t^{*k}} \right). \tag{2.25}$$

Equation (2.25) is obtained by subtracting (2.10) from (2.8) and rearranging. The error is squared in equation (2.24) for a given k and stapled in Q for all single names. With the help of a quadratic optimizer, the calibration of q turns out to be very fast, accurate and unambiguous.
In the second step, we solve problem (2.22) for every k independently from the rest of the portfolio but with fixed q_k^p from the previous calibration. This way, the intensities θ_t^k are obtained easily by applying a gradient-based method.

2.4 Empirical Analysis

After describing the formal foundations of our analysis, we can turn to our overall question of which default factors are priced in CDS markets empirically. We outline the specifics of the CDX data set, provide corresponding results of the calibration routine presented in Section (2.3) and finally show the test results which reveal the role of correlated defaults in CDS markets in the years 2005 - 2012 which include the subprime credit crisis.

2.4.1 Data Set

Our data set comprises daily CDO, index CDS and single-name CDS quotes of the CDX North America Investment Grade Index from September 2005 until September 2012 with a maturity of five years which to our knowledge, is the most extensive CDX data set used so far in the literature. The data set was completely retrieved from Markit, which is the provider of the CDX index. The CDX comprises the cross-industry single names that exhibit the highest liquidity in the credit derivatives market. Therefore, our analysis focuses on the overall credit risk perception among representative single names during that period.

There are some important characteristics of the CDX data set that we outline in the following. Before the beginning of the financial crisis in 2007, the CDX index was reconstituted every six months in March and September, a process which is commonly known as index roll. Immediately before an index roll occurs, Markit conducts a poll in which licensed CDS dealers agree on the most liquid 125 single names that will constitute the next CDX index. The first index of that kind was the CDX 1 that began trading in September 2003 and was labeled as an on-the-run index, which means that it was constituted of the most liquid single names at that time. Afterwards, the CDX 2 began trading in March 2004 as an on-the-run index and so forth. From this point, trades in the CDX 1 were still possible as long as the maturity of the underlying products was not reached. However, the CDX 1 was entitled to continue as an off-the-run index as it did not represent the most liquid single names at that time. However, no CDO data are available for the first four CDX indices and for this reason, we exclude them from our analysis.

We include the four subsequent indices CDX 5 through 8 that exhibit workable time series on daily CDO as well as CDS quotes and the same CDO tranche borders: $0-0.03$, $0.03-0.07$, $0.07-0.10$, $0.10-0.15$ and $0.15-0.30$. For these indices, we also include index CDS[2] data to supplement the missing tranche that would cover the last CDO interval ending with detachment point 1. The only tranche of these indices that was quoted according to the upfront payment convention (2.5) was the equity tranche with $c^{1,\text{fix}} = 500$ bp, $a_1 = 0$ and $d_1 = 0.03$. The quotes of all other tranches correspond to the running spread convention (2.4).

For the following three years, the CDX data exhibit a peculiar feature: although index rolls were conducted every half-year, CDX 9 is considered to be the most liquid reference index during the crisis. Because of this and the fact that CDX 9 has the most workable time series in these three years, we exclude the indices CDX 10 through 13. Furthermore, we do not supplement CDX 9 data with index CDS because the time series of the $0.3 - 1$ tranche is available.

After CDX 9, three major changes occured for liquidity reasons. First, index rolls were only conducted every full year. Thus, even index numbers are not available any more because the odd numbers started to trade in the September of each year and traded for the entire following year. Second, the tranche borders of the CDO were restructured to four tranches $0 - 0.03$, $0.03 - 0.07$, $0.07 - 0.15$ and $0.15 - 1$. Third, the quotation convention was changed to the upfront convention for all tranches which — in that order — exhibit the fixed running premia 500 bp, 100 bp, 100 bp and 50 bp. The last two indices that we include in our data sample are CDX 15 and CDX 17, of which the former was on-the-run from September 2010 until September 2011 and the latter the subsequent year.

In the whole observation period, four credit events occured that led to payouts of CDO tranches and CDSs. The first two are related to Fannie Mae and Freddie Mac, which were placed into conservatorship on September 7, 2008. Washington Mutual filed for Chapter 11 bankruptcy on September 27, 2008, followed by CIT Group about one year later on November 1, 2009. Until the respective credit event, we include the CDS time series of all four entities. The absolute CDO tranche premia were reduced after each credit event, but unlike the CDSs they continued trading on the markets as the capital of the underlying portfolio intervals was not exhausted. After each credit event, the version of the corresponding index was increased by one. So before September 7, 2008, the CDX 9 data referred to its first version. Afterwards, the second version of CDX 9 began trading and so forth.

In the CDS space, major changes occured on April 8, 2009 and they are known as the CDS Big Bang. The thrust of these changes was to improve the efficiency of central clearing and trade processing in CDS markets. Of all the changes in the context of

[2] The index CDS can be considered as a tranche on the whole portfolio interval $0 - 1$.

the CDS Big Bang, the only one that may be important to us is the change of the quotation convention. Before that date, CDS premia were quoted according to the running spread convention. Afterwards, CDS dealers quoted them in terms of the upfront payment convention. Another characteristic of the CDX data set is that all CDS quotes after the CDS Big Bang were still quoted in terms of the running spread convention. The conversion from upfront payments to running spreads is facilitated by a model converter that Markit offers on its website. It is common market practice to use models because otherwise — as can be seen from equations (2.4) and (2.5)[3] — conversion would not be possible.

The characteristics of the data set show that a uniform presentation of all indices is not possible because of changing tranche borders and quotation conventions. Therefore, we have two possible ways to describe the data set: first, we could present the descriptive statistics for each single CDX index with its fixed characteristics in terms of quotation convention and tranche borders. This would truly reflect all observed data and provide full transparency but would involve costs regarding the associated scope and readability. For this reason, we choose the second option which entails the use of model premia computed from the calibrated top model. The advantage of this approach lies in the fact that it facilitates the presentation of uniform time series for CDO tranche premia which can be compared among indices. Clearly, this implies the drawback of imposing model risk on the descriptive statistics. Because of the good model fit, which we will present later, we consider this handicap to be negligible.

All CDX indices that we include in our study exhibit different tranche borders and quotation conventions. As we seek to unify tranches across all indices we need to fix the tranche borders and conventions. This leads us to the problem that we need to reduce and convert observed data to premia that can be computed from any index. For example, from the premia of the $0.07-0.10$ and the $0.10-0.15$ tranche we can compute a model-implied premium for a non-existent $0.07 - 0.15$ tranche, but not vice versa because of missing information. Consequently, we have to fix our uniform tranche borders to the borders of the CDX15 and CDX17 indices: $0.00 - 0.03$, $0.03 - 0.07$, $0.07 - 0.15$ and $0.15 - 1.00$. The model gives us the flexibility to use the quotation convention for tranches that best fits our purposes. As outlined above, CDS premia are quoted in accordance with the running spread convention throughout our data set. Furthermore, according to the running spread convention, quotations can never take negative values, which attributes a higher expressiveness to statistics. For these

[3] The facts also hold for CDS model premia.

two reasons, we use the running spread convention to represent tranche data that are, provided that no workable observed time series exists, computed from the top model.

2.4.2 Descriptive Statistics

In the course of this paper, we investigate how the market perception of default factors has changed over time. In order to draw meaningful conclusions from our data set, we divide it into three parts that are in line with *Kahle and Stulz (2010)*: the pre-crisis period ranging from September 2005 until September 2007, which comprises the CDX 5 through 8 indices. The crisis period lasting from September 2007 until September 2010, which consists solely of CDX 9 data. And finally, the post-crisis period from September 2010 until September 2012, which contains the CDX 15 and 17 indices. Thus, we are capable of comparing the role of default factors in the CDS market before, during and after the financial crisis.

Figure 2.1 plots CDO premia that were retrieved from the data set and that were supplemented by model premia where necessary. The upper graph shows that until spring 2007, the CDO market was in a comparably smooth state with equity tranche premia just short of 2000 basis points. The premia of the $0.03 - 0.07$, $0.07 - 0.15$ and $0.15 - 1.00$ tranches were negligibly small during that period. The outbreak of the crisis in summer 2007 is reflected in increased premia for all tranches. The roll to the CDX 9 index saw a sharp drop in the equity tranche premia followed by very high market uncertainty in the overall credit derivatives market. The credit events of Fannie Mae, Freddie Mac and Washington Mutual in September 2008 caused the CDX 9 tranches to peak at the beginning of 2009. Afterwards, the situation relaxed until summer 2010 when premia widened again, although no credit event occured in the index. This rise may have been driven by the Greek sovereign debt crisis which started at that time and may have channelled through to North American credit derivatives markets. Another explanation might be the drop in CDX 9 liqudity in anticipation of the roll to the CDX 15 index which started trading at considerably lower premia. The low premium levels of CDX 15 persisted throughout CDX 17 and were — at least for the equity tranche — at levels similar to those immediately before the crisis. However, for the more senior tranches the premium levels after the crisis are considerably higher than before the crisis. As can be seen from the lower graph of Figure 2.1, the more senior tranches in particular gained in relative premium levels during the crisis. This observation indicates that correlated defaults may have played an increasingly important role during the crisis because they mainly affect the premium levels of tranches with high seniority.

Table 2.1 presents the descriptive statistics for the four tranche premia throughout the complete data set[4]. The correlation between the time series decreases with the seniority of the tranche. For example, the equity tranche is highly correlated with the junior mezzanine tranche but exhibits almost no correlation with the senior tranche with borders $0.15 - 1.00$. This property suggests that different risk factors are driving the premia of different tranches and therefore justify the use of the three-factor model. Furthermore, premium levels decrease with the seniority of a tranche because the capital of junior tranches is consumed first when defaults occur. Since all time series exhibit a very high serial correlation, the descriptive statistics for their first differences are also reported in Table 2.1. The values show that the main findings of the original time series hold for the differentiated time series as well and that the former are not due to the high serial correlation.

[4] Descriptive statistics for the pre-crisis, crisis and post-crisis periods are reported in the appendix.

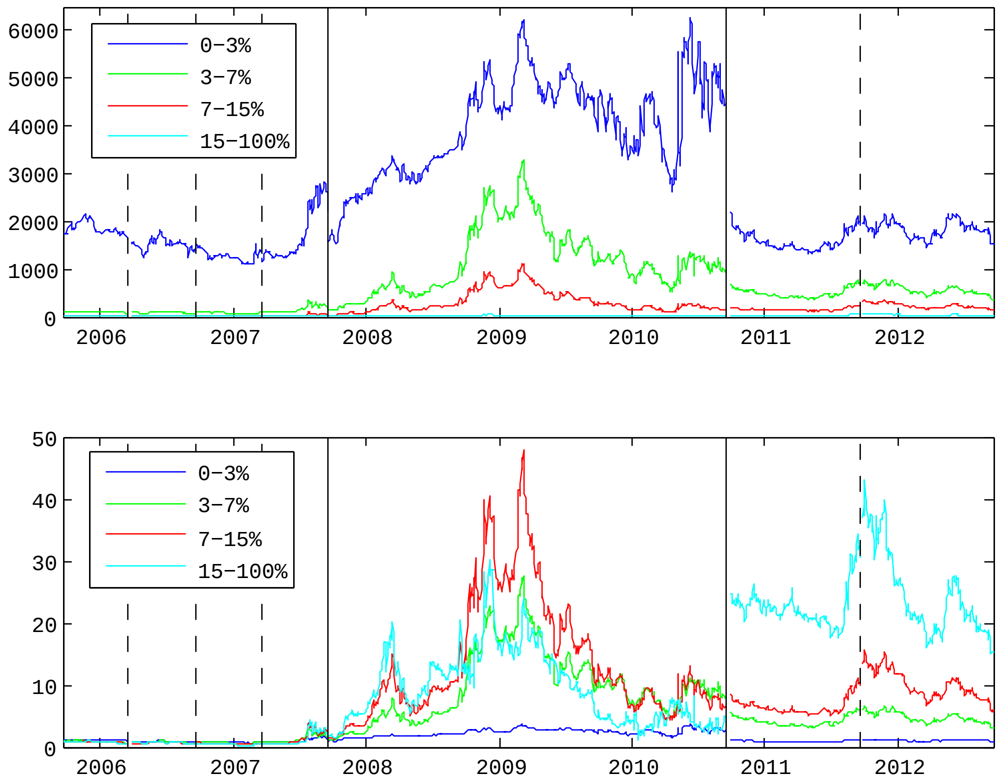

Figure 2.1: Graphs for the Time Series of the CDX North America Investment Grade CDO Tranche Model Premia

The upper graph shows the time series of unified CDO tranche premia computed from the portfolio model for the period from September 2005 until September 2012. The dashed vertical division lines indicate the roll of one CDX index to the next. The straight lines indicate the rolls for the CDX 9 index and simultaneously mark the borders of pre-crisis, crisis and post-crisis periods in our data sample. The lower panel shows the same premia but divided by the first values of their respective time series. Values of the upper panel are reported in basis points. Values of the lower panel are normalized to the first observation of the respective time series.

	Correlations											
	3–7	7–15	15–100	Mean	SD	Min.	Med.	Max.	Skew.	Kurt.	Serial corr.	N
0–3 Tranche	0.862	0.736	-0.006	2668.40	1361.06	1059.23	1960.04	6271.22	0.80	2.27	0.999	1706
3–7 Tranche		0.960	0.330	686.59	638.48	57.69	515.66	3302.73	1.51	5.05	0.999	1706
7–15 Tranche			0.497	199.15	198.74	7.65	162.75	1126.05	1.81	6.70	0.998	1706
15–100 Tranche				17.39	15.67	0.34	13.17	67.81	0.63	2.38	0.998	1706
Δ 0–3 Tranche	0.439	0.466	-0.035	-0.12	138.42	-2322.54	1.18	2102.15	-1.31	116.80	0.063	1705
Δ 3–7 Tranche		0.874	0.219	0.14	48.01	-453.48	-0.01	387.07	-0.07	21.51	0.074	1705
Δ 7–15 Tranche			0.567	0.07	15.80	-115.09	-0.02	144.42	0.62	20.11	0.136	1705
Δ 15–100 Tranche				0.01	1.53	-8.50	0.00	33.16	6.41	139.23	-0.020	1705

Table 2.1: Summary Statistics for the Levels and First Differences of the CDX North America Investment Grade CDO Tranche Premia

This table reports summary statistics for unified CDO tranche premia computed from the top model. The model was calibrated to the original, observed data and the statistics are computed from the whole time series ranging from September 2005 until September 2012. Values are reported in basis points.

Each CDX index comprises the 125 most liquid single names during its on-the-run time period. Figure 2.2 plots the 5% and 95% quantiles, the mean and the median of the cross-section of CDS premia. Until the outbreak of the crisis, the overall CDS premium level was comparably low, with the 95% quantile moving below 150 basis points and the mean below 60 basis points. The outbreak of the financial crisis in summer 2007 led to a considerable increase in overall CDS premium levels which peaked at the beginning of 2009. The level of the 95% quantile reached more than 1500 basis points, indicating that during that time, the market priced high default probabilities for the single names with the lowest credit quality among the index constituents. The median CDS premium reached a maximum of more than 200 basis points. This level does not reflect a very high default probability but signals that in the overall market perception, protection sellers took high premia for single names with a good credit quality, which may suggest that correlated defaults were priced in CDS markets.

Table 2.2 shows the corresponding summary statistics[5]. In line with Table 2.1, the observed CDS time series exhibit high serial correlations which are considerably reduced by taking first differences. In addition, the time series of all considered cross-sectional statistics are highly correlated. The finding suggests that CDSs follow overall market movements, which again indicates that correlated defaults are likely to be priced in the CDS market.

Interest rate data are the last missing piece that we need for the calibration of the top-down approach. We compute discount factors from the Svensson parameters published on the website of the Federal Reserve. The parameters reflect the interest rate term structure of US Treasuries that are considered to be the most liquid interest rate products in the US market. Therefore, they are ideally suited to our purposes. An explanation of the related methodology can be found in *Gürkaynak et al. (2006).*

[5] Descriptive statistics for the pre-crisis, crisis and post-crisis periods are reported in the appendix.

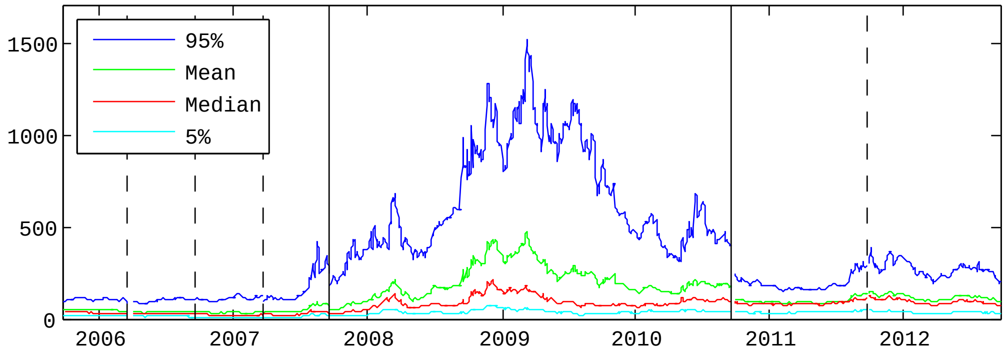

Figure 2.2: CDS Time Series of the CDX North America Investment Grade Index Constituents

The graph shows daily cross-sectional statistics for the CDX index constituents. The dashed vertical division lines indicate the roll of one CDX index to the next. The straight lines indicate the rolls for the CDX 9 index and simultaneously mark the borders of pre-crisis, crisis and post-crisis periods in our data sample. Values are reported in basis points.

	Correlations														
	SD	Min.	5	Med.	95	Max.	Mean	SD	Min.	Med.	Max.	Skew.	Kurt.	Serial corr.	N
Mean	0.934	0.648	0.788	0.859	0.955	0.908	131.69	91.09	30.46	109.35	477.61	1.27	4.35	0.999	1706
SD		0.446	0.584	0.653	0.941	0.986	200.64	210.67	23.21	86.56	982.81	1.30	3.63	0.997	1706
Min.			0.822	0.801	0.548	0.414	18.23	9.31	0.00	19.10	45.35	0.20	2.26	0.994	1706
5				0.961	0.633	0.586	30.68	13.99	6.81	33.04	77.58	0.10	2.58	0.999	1706
Med.					0.710	0.642	73.82	37.74	16.85	78.66	214.73	0.47	3.12	0.999	1706
95						0.895	385.00	313.25	76.06	273.67	1515.47	1.33	3.88	0.999	1706
Max.							1633.29	1833.96	125.42	585.74	8861.85	1.34	3.90	0.993	1706
Δ Mean	0.791	0.100	0.557	0.691	0.715	0.648	0.03	5.87	-64.15	-0.02	49.93	-0.93	28.08	0.232	1705
Δ SD		-0.001	0.174	0.268	0.412	0.934	0.01	23.65	-404.46	0.03	289.86	-2.94	91.24	0.012	1705
Δ Min.			0.292	0.177	0.058	-0.014	0.00	2.27	-38.52	0.00	38.48	-0.26	169.08	-0.138	1705
Δ 5				0.706	0.420	0.097	0.01	1.02	-6.06	-0.01	9.13	0.75	14.39	0.229	1705
Δ Med.					0.555	0.173	0.02	2.82	-20.92	-0.03	25.91	0.48	16.17	0.242	1705
Δ 95						0.302	0.06	22.58	-227.62	-0.01	176.99	-0.35	21.56	0.213	1705
Δ Max.							-0.01	296.20	-5800.93	0.09	3615.09	-3.72	122.65	-0.070	1705

Table 2.2: Summary Statistics for the Levels and First Differences of the Cross-Section of CDX Index Constituents

This table reports summary statistics for the cross-sectional statistics time series of CDX index constituents. Values are reported in basis points.

2.4.3 Calibration Results

In the following we present the results of the calibration procedure introduced in Section 2.3. First, as a preparation for the CDS study, we discuss the calibrated parameters of the top model and the associated goodness of fit. Afterwards, we turn towards the question of which default factors are priced in CDS markets and investigate the calibrated CDS parameters.

2.4.3.1 Portfolio Level

Jump Size and Volatility Parameters We applied the proposed calibration procedure to each CDX index. Table 2.3 presents the resulting parameter values. The jump size parameters $\gamma_i, i = 1, 2, 3$ are comparably stable across the different CDX indices. The value of the first parameter γ_1 ranges from 0.0042 to 0.0083. Given the fact that each constituent is weighted with $1/125 = 0.008$ in a CDX index, the jump size allows the interpretation of the first factor to model the default of a single name while the recovery rates vary from 47.5% to 0%[6]. For this reason, we refer to the first factor as the single default factor of the portfolio. The jump size parameter of the second factor lies between 0.0592 and 0.0792 which are clearly above the jump sizes reported for the single default factor. Thus, a jump of the second factor induces a default that corresponds with the simultaneous default of 9.11 single names given a recovery rate of 0%, or 18 firms with a recovery rate of 49.38%. Since a CDX index is composed of single names from 12 different industries, the average number of single names per industry is equal to 10.42. Consequently, the second factor can be interpreted as an industry default factor. This interpretation implies that correlated defaults are priced in CDOs because more than 10 firms default at the same time in case the second factor jumps. In contrast, if there was no default correlation priced at all, more than one single name could not default at the same time in the model. The jump sizes of the third factor range from 0.3459 to 0.4044. A default event of the third factor may thus wipe out more than 40% of the portfolio capital. This means that more than 50 single names with recovery rate 0% would default at the same time if the third factor jumps. This seems to be a very unlikely event because it is tantamount to an extremely severe economic crisis with consequences beyond the scope of the last financial crisis in 2008. We interpret this factor as modeling a systemic default event and therefore call it the systemic default factor. The standard errors of the jump size parameters show that

[6] Or even less than 0%. In that rare case, two single names default if the first factor jumps.

the calibration worked particularly well in all cases. Furthemore, they are in line with *Longstaff and Rajan (2008)*, who report similar values.
The stability observed for jump size parameters across CDX indices does not apply to the volatility parameters $\sigma_i, i = 1, 2, 3$. In the pre-crisis and crisis periods, the σ_i are relatively stable with the exception of σ_1, which drops for the CDX 8 index. However, the picture changes for the post-crisis period, where the σ_i can take values that are more than three times larger than before. The finding suggests that despite decreased premium levels, and hence lower default risk, the market environment exhibited a high degree of uncertainty. In this context, it seems plausible that the market perception of default volatility has changed because of the financial crisis.

Priced Defaults by Portfolio Risk Factors Since we are interested in the question of how the importance of the factors has changed relative to each other on the one side, and how this translates to the portofolio loss distribution on the other side, we have plotted the priced defaults resulting from the top model in Figure 2.3. Before the crisis, more than 80% of priced defaults can be attributed to the single default factor. The industry and the systemic factor played a negligible role during that period. Furthermore, only around 2% of the portfolio capital were expected to default under the risk-neutral measure. Afterwards, the financial crisis changed the situation dramatically. At the beginning of 2009, the market expected almost 10% of the portfolio capital to default. Roughly 80% of these defaults were priced by the industry and the systemic risk factor, with the former clearly the more dominant of the two. The market situation eased in 2010 and the expected default mass declined to 4%, half of which can be attributed to the single default factor. The index roll from CDX 9 to 15 saw a sharp increase in the time series of systemic defaults. This increase might be explained by the new index composition or the new tranche borders, which may have stimulated a greater awareness of systemic risk in the senior tranche. After the crisis, the systemic risk factor accounts for more than 40% of priced defaults while the proportion between the first and the second factor remains relatively stable. Summing up, the single default factor played the major role during the pre-crisis period whilst the correlated default factors dominated afterwards. Thus, the market perception of correlated defaults and particularly systemic risk has changed during and after the financial crisis.

	Volatility parameters			Jump size parameters		
	First	Second	Third	First	Second	Third
CDX5	0.1872	0.2115	0.1573	0.0045	0.0592	0.3459
	(0.0006)	(0.0001)	(0.0007)	(0.0000)	(0.0000)	(0.0006)
CDX6	0.2901	0.1928	0.1573	0.0048	0.0619	0.3459
	(0.0032)	(0.0004)	(0.0007)	(0.0000)	(0.0001)	(0.0002)
CDX7	0.3522	0.1960	0.1573	0.0048	0.0611	0.3459
	(0.0008)	(0.0002)	(0.0009)	(0.0000)	(0.0000)	(0.0004)
CDX8	0.0006	0.2003	0.1573	0.0043	0.0602	0.3459
	(0.5597)	(0.0001)	(0.0005)	(0.0000)	(0.0000)	(0.0003)
CDX9	0.1343	0.2003	0.1573	0.0042	0.0602	0.3459
	(0.0371)	(0.0085)	(0.0115)	(0.0001)	(0.0005)	(0.0150)
CDX15	0.7130	0.0000	0.4044	0.0079	0.0729	0.4044
	(0.0022)	(0.8375)	(0.0013)	(0.0000)	(0.0001)	(0.0002)
CDX17	0.4740	0.4550	0.3834	0.0083	0.0656	0.3834
	(0.0101)	(0.0054)	(0.0037)	(0.0001)	(0.0003)	(0.0013)

Table 2.3: Top Model Parameter Estimates for the CDX North America Investment Grade Indices

This table reports parameter estimates for the top model. The volatility parameters are $\sigma_i, i = 1, 2, 3$ and the jump size parameters are $\gamma_i, i = 1, 2, 3$. Standard errors are in parantheses and are computed according to *Gallant (1975)*.

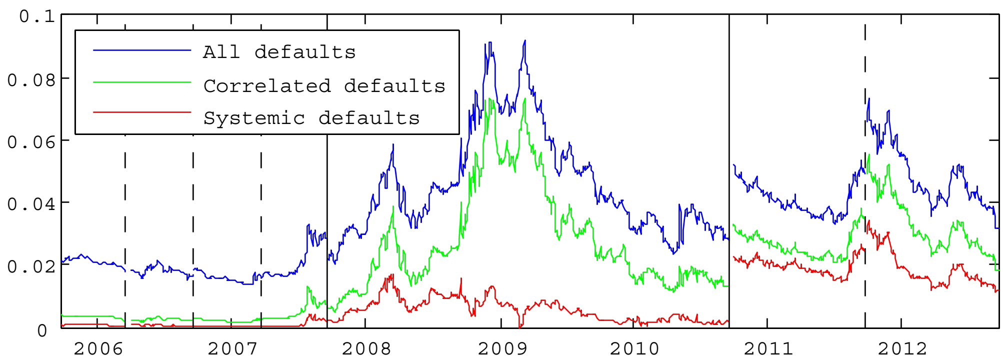

Figure 2.3: Time Series of Priced Defaults in CDX CDO Tranches

The graph shows the priced defaults in CDX CDO tranches computed from the top model. All defaults refer to the factors $i = 1, 2, 3$, and correlated defaults refer to the factors $i = 2, 3$ combined. Systemic defaults refer to factor $i = 3$ only. The maximum possible default loss is equal to the portfolio notional with amount 1.

Goodness of Fit To assess the overall model fit, we examine the fit of model premia with regard to observed premia of the data set. A typical measure that is applied in this context is the so-called root mean square error (RMSE). The value of the RMSE reports the absolute deviation between observed and model premia. In our study the RMSE would measure the difference between observed and model premia in basis points. Although this would provide a high degree of comparability across tranches, one major drawback lies in the explanatory power, since a deviation of e.g. 10 basis points for the equity tranche is relatively lower than a deviation of 10 basis points for the senior tranche. In order to satisfy this circumstance for CDO tranches, we introduce the root mean square relative error (RMSRE) that is defined as follows:

$$RMSRE_{t,T}^{p} = \sqrt{\sum_{\tau=t}^{T} \left(\frac{c_{\tau}^{p} - c_{\tau}^{*p}}{c_{\tau}^{*p}} \right)^{2}} \qquad (2.26)$$

The advantage of the RMSRE lies in its comparability across tranches: if the model premia of an equity tranche and a senior tranche are both calibrated with a deviation of 10% then they can be considered to fit the data equally well. Another favorable property for our study is the robustness of the RMSRE towards a change in quotation conventions. As mentioned earlier, the quotation convention for some on-the-run tranches in the data set suddenly changes from running to upfront, which would directly lead to a sharp increase or drop in RMSEs. Since this does not hold for RMSREs they are clearly better suited for our purposes. Furthermore, the RMSRE is closely related to the error in problem (2.21) and thus coincides with the top model calibration. However, there is a slight drawback associated with the RMSRE: if the observed premium c_t^{*p} is close to zero it takes huge values and distorts a further analysis. Therefore we have to erase outliers.

Figure 2.4 plots the mean RMSRE across tranches. During the pre- and post-crisis periods, the mean RMSRE ranges from roughly 1% to 10% with high fluctuations. For a senior tranche with a premium of 50 basis points this translates to a maximum deviation of 5 basis points which can be considered a good model fit. However, during the crisis the RMSRE takes higher values and has partially higher fluctuations than in the other periods. This might be considered a weakness of the model but given the fact that, during the crisis, observed premia are usually subject to greater distortion owing to market uncertainty, the model fit in the crisis is still good.

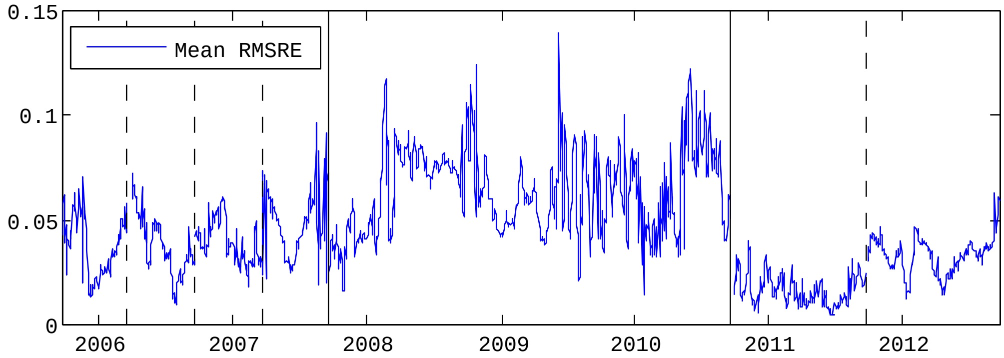

Figure 2.4: Time Series of Mean RMSRE across CDO Tranches

The graph shows the time series of the mean RMSRE across CDO tranches adjusted for outliers. The RMSRE refers to observed CDO tranche premia.

2.4.3.2 Single-Name Level

As an outcome of top model calibration, we know that the CDO market prices correlated defaults, especially during and after the crisis. But are these risks also perceived in CDS markets? With the help of our top-down model, we are able to answer this question empirically by analyzing the calibrated values of the q_k^p and θ_t^k.

Test Setup for Portfolio Sensitivities In the first step, we hypothesize how correlated defaults are reflected in CDS prices within our model. For this reason, let us assume the following two cases: first, a single name with a very high CDS premium in the cross-section of the portfolio. Second, a single name with a low CDS premium. For the first single name, a high value of the associated CDS suggests a high default probability and favors an early default time. For the single name with the low CDS premium, the opposite relation holds: since a default is unlikely it is expected that other single names will default beforehand, if at all. Therefore, the low CDSs should be priced in the senior tranche $p = 4$ whose capital is only affected by defaults that occur last of in a portfolio. One could also argue that single names with a very low CDS premium may only default in catastrophic scenarios such as natural disasters or very severe economic crises. In this case, they should also exhibit a high sensitivity towards the senior tranche. Accordingly, single names with high CDS premia should be priced in the equity tranche since they are expected to default as one of the first portfolio constituents.

To test whether these considerations hold empirically, we formulate the regression equation

$$q^p = a_p + b_p \cdot f^* + \varepsilon^p, \tag{2.27}$$

$$q^p = \begin{pmatrix} q_1^p \\ q_2^p \\ \vdots \\ q_K^p \end{pmatrix}, f^* = \begin{pmatrix} E(f_t^{*1}) \\ E(f_t^{*2}) \\ \vdots \\ E(f_t^{*K}) \end{pmatrix},$$

where K denotes the total number of single names in the portfolio and ε^p a zero mean error term. The notion behind this regression is to determine the relationship between the size of CDS f^* and the sensitivities q^p toward a tranche p. If b_p is positive, we can conclude that high CDSs are priced in p. If it is negative, this holds for low CDSs.

We compute the q^p and f^* for each single CDX index and regress them according to equation (2.27) per period. That means that the first regression refers to the parameters q_k^p and f^* of CDX 5 to 8 combined, the second regression to CDX 9 and the third regression to CDX 15 und 17 combined. The packages assure that there is an equal amount of data involved in each regression and thus they have comparable explanatory power.

From our notion that only single names with high CDS premia are priced in the equity tranche, we retrieve the following hypothesis:

Hypothesis 1. *The tranche sensitivity q_k^1 is higher for high CDS premia than for low CDS premia.*

Empirically, we can consider hypothesis 1 to hold if the regression parameter b^1 is positive and significant.

Accordingly, since single names with high CDS premia are supposed to default first, they should have no influence on the pricing of the senior tranche, and the respective tranche sensitivity should equal zero. Therefore we retrieve:

Hypothesis 2. *The tranche sensitivity q_k^4 is higher for low CDS premia than for high CDS premia.*

The validity of hypothesis 2 can be verified by a negative parameter b^4.

Test Setup for Idiosyncratic Risk Factors To capture potential pricing deviations between observed and CDO-induced CDS premia, we introduced idiosyncratic risk factors. Since we are interested in finding out which CDSs they particularly apply to, we conduct the following regression with respect to the calibrated idiosyncratic intensities θ_t^k:

$$\theta = a + b \cdot f^* + \varepsilon \tag{2.28}$$

$$\theta = \begin{pmatrix} E(\theta_t^{*1}) \\ E(\theta_t^{*2}) \\ \vdots \\ E(\theta_t^{*K}) \end{pmatrix}. \tag{2.29}$$

where ε indicates a zero mean error term. If idiosyncratic risk factors are especially present in high CDS premia, b should be positive. In the opposite case that they are present in low CDS premia, b should be negative. If the CDS level is in no way related to the amount of idiosyncratic risk, b should be close to zero.

Test Results for Pre-Crisis Period The regression results for the three periods are presented in Table 2.4. It can be seen that the only tranche sensitivities for which the regression coefficient b_p is significant during the pre-crisis periods, are the equity and the junior mezzanine tranche with $p = 1$ and $p = 2$. The coefficient for the junior mezzanine tranche is higher than for the equity tranche, which is fairly surprising because from hypotheses 1 and 2 the coefficient should decrease with the seniority of the CDO tranche. The scatter plots of Figure 2.5 reveal that this is due to outliers with a comparably low CDS premium that lead to a high b_2. The plots also show that the sensitivities q^2 are higher than q^1 for the highest CDS premia of the portfolio. The finding suggests that hypothesis 1 holds and that the highest CDS premia are priced in the most junior tranches. Regarding the senior mezzanine $p = 3$ and the senior tranche $p = 4$, there is no evidence that the single names follow a particular default order since the coefficients b_p are not significant. This is not surprising because correlated defaults were less important in the pricing of CDOs during the pre-crisis period and hence they were not perceived by the single-name CDS market.

Test Results for Crisis Period Later, though the CDO market apparently attached greater importance to correlated defaults owing to the outbreak of the financial crisis. Table 2.4 shows that both hypotheses 1 and 2 hold in the CDS market because the values of the coefficients b_1 and b_4 are both significant and they exhibit the expected sign. This translates to the single default factor being priced in single names with high CDS premia and systemic risks being priced in low CDSs. Thus, the CDS market follows the suggested default order, especially if both single and correlated default risks play an important and visible role in the CDO market. Only the coefficients b_2 and b_3 are not in line with the results for the equity and the senior tranche. $b_2 > b_1$ holds because of an outlier and b_3 is not significant. A possible explanation lies in the high volatility of the industry factor which accounts for most of the risk inherent in the junior and senior mezzanine tranches. Since the time series of those tranches and the CDSs that should — from a theoretical viewpoint — be in line with them are very volatile, the calibrated tranche sensitivities do not take the suggested values. Therefore, we cannot conclude that the default order holds for the mezzanine tranches.

Test Results for Post-Crisis Period Almost the same picture applies to the post-crisis period with the difference that all b_p are strongly significant. Hypotheses 1 and 2 hold but again b_2 and b_3 take the highest values. Our findings show that correlated defaults were not priced in CDS markets during the pre-crisis period; only single defaults were. The higher relevance of industry and systemic risk for CDO portfolios during the crisis was anticipated by participants in the CDS market who considered possible correlated defaults in their CDS trades.

Test Results for Idiosyncratic Risk Factors The relevance of the idiosyncratic price effect for single-name CDSs is revealed in Table 2.4 and Figure 2.5. Idiosyncratic risk factors prevailed during all three time periods of our analysis but were especially dominant during the crisis period. Furthermore, idiosyncratic risk increases with the level of CDS premia. Single names which are under high financial distress are very volatile on top of the single default factor that prices them. Similarly, not all dynamics observed on the CDS market automatically influence the time series of the equity tranche. A possible explanation could include illiquidity or diversification effects that act as a buffer between high CDSs and the equity tranche premium. However, these effects have a much weaker influence on low CDSs and the senior tranche premium. Thus, low CDS premia mainly follow overall market movements which can be characterized in terms of the systemic risk factor.

Goodness of Fit The presence of idiosyncratic risk factors implies that CDS premia cannot be fully explained by the top model and the tranche sensitivities. A perfect match is only possible with the idiosyncratic extension of CDS model premia that we introduced in equations (2.9) and (2.11). Nevertheless, it is still interesting to assess the model fit with tranche sensitivities, if only because it reveals how well they can explain observed CDS premia. We measure the deviations between observed and model premia with the RMSRE. Figure 2.6 plots the RMSRE time series for the cross-section of CDS premia. It shows that more than one half of CDS premia are mispriced by less than 20%, which is tolerable for various applications. However, the top 5% of deviations take very large values that are even above 100%. In these cases, the idiosyncratic risk factors are required. The graph also shows that the mispricings are comparably higher in the crisis period than outside of it because the top-down model has its difficulties capturing the overall high market volatility at that time. But the high deviations could also be attributed to high levels of idiosyncratic risk caused by increased illiquidity in the markets. Despite some outliers, the overall model fit can be regarded as good enough to conclude that the tranche sensitivities have high explanatory power to back the findings of our analysis.

		Tranche sensitivities				
		q^1	q^2	q^3	q^4	θ_t
Pre-Crisis	a_p, a	0.0027	0.0015	0.0090	0.0087	-0.0005
	p-Value(a_p, a)	0.0001	0.3727	0.0000	0.0000	0.0000
	b_p, b	1.2261	1.5074	-0.2282	-0.1524	0.2369
	p-Value(b_p, b)	0.0000	0.0000	0.2883	0.0636	0.0000
	R^2	0.2107	0.0652	0.0023	0.0069	0.4552
Crisis	a_p, a	0.0030	0.0004	0.0071	0.0089	-0.0070
	p-Value(a_p, a)	0.0320	0.8553	0.0017	0.0000	0.0000
	b_p, b	0.2323	0.3469	0.0577	-0.0502	0.7823
	p-Value(b_p, b)	0.0000	0.0000	0.3176	0.0007	0.0000
	R^2	0.2589	0.2522	0.0084	0.0933	0.7922
Post-Crisis	a_p, a	0.0047	-0.0054	-0.0017	0.0105	-0.0008
	p-Value(a_p, a)	0.0006	0.0165	0.4260	0.0000	0.0000
	b_p, b	0.3107	1.2431	0.9000	-0.2351	0.1415
	p-Value(b_p, b)	0.0036	0.0000	0.0000	0.0009	0.0000
	R^2	0.0336	0.1683	0.1050	0.0433	0.2862

Table 2.4: Regression Results for Single-Name CDSs during the Pre-Crisis, Crisis and Post-Crisis Periods

This table reports results for the regressions of the single-name parameters q_k^p and θ_t^k against the respective absolute CDS mean levels. Regression data comprise pre-crisis, crisis and post-crisis period CDX data.

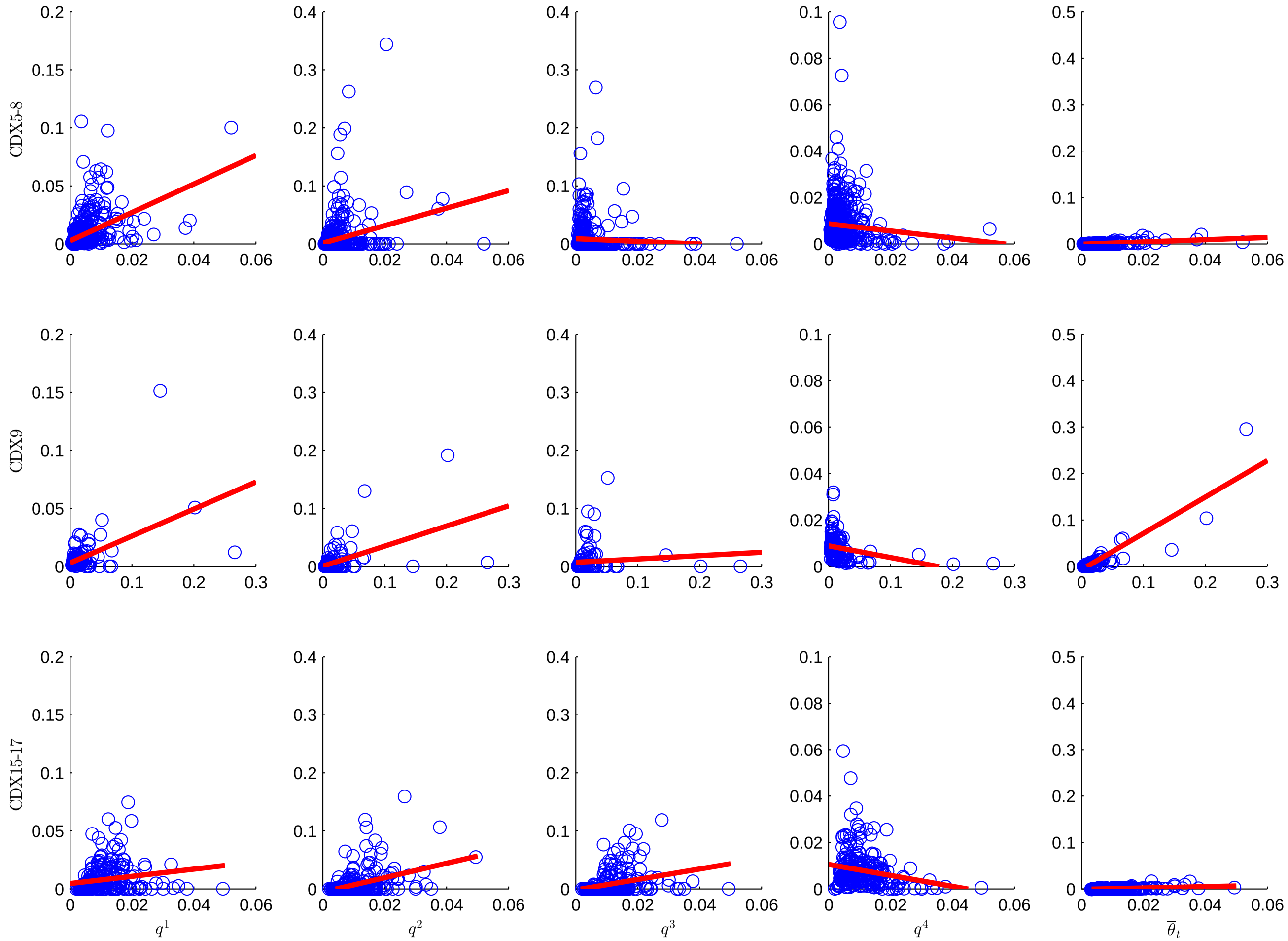

Figure 2.5: Regression Results for Single-Name CDSs during the Pre-Crisis, Crisis and Post-Crisis Period

The graphs show scatter plots for the regressions of the single-name parameters q_k^p and θ_t^k against the respective absolute CDS mean levels. Regression data comprise pre-crisis, crisis and post-crisis period CDX data.

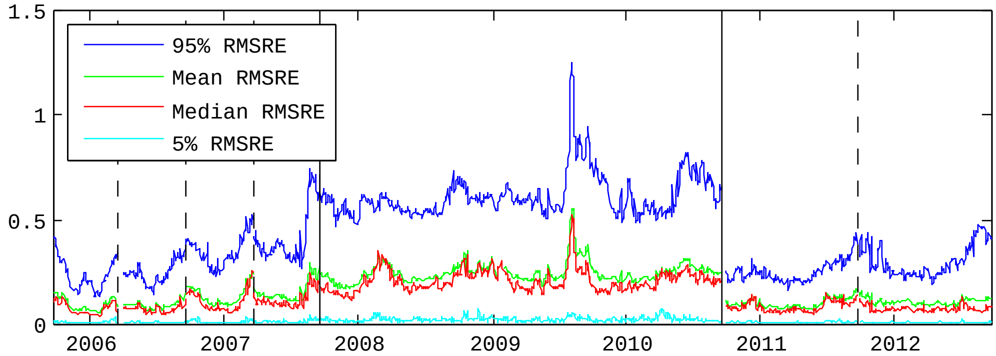

Figure 2.6: Time Series of Mean RMSRE across CDS Premia

The graph shows the time series of the mean RMSRE without idiosyncratic risk across CDS premia. The RMSRE refers to observed CDS premia.

2.5 Conclusion

In this paper, we studied the impact of correlated default factors on CDS premia. Therefore, we first recapitulated the top model of *Longstaff and Rajan (2008)* to model CDO tranche premia. Afterwards, we derived a cash flow based top-down approach that links theoretical CDS model premia to any kind of CDO model. A special feature of our framework lies in the ability to allow for idiosyncratic risk factors that are not priced in the CDO portfolio. With the help of these risk factors, empirically observed CDS premia can be perfectly matched by our model.

The top-down model was calibrated to an extensive CDX data set that covers daily tranche and CDS quotes from September 2005 until September 2012. We found that before the outbreak of the financial crisis in 2007, the influence of correlated default factors on CDS premia was only minor. However, the financial crisis led to a dramatic increase in those factors which accounted for most of the expected defaults in CDO and CDS markets at that time. After the crisis, the market situation eased, leading to an overall lower default risk level but still with high importance attached to the correlated default factors. Accordingly, we found correlated default factors to be priced in CDS markets when they were particularly high, that is, during and after the financial crisis. Our analysis revealed that the prices of single names with a low pricing level are mainly driven by correlated default factors. Furthermore, we found idiosyncratic risk to be priced in CDS markets in the whole data set but especially during the financial crisis. Single names with high CDS premia were in particular subject to high levels of idiosyncratic risk.

Appendix 2.A Tables

	Correlations											
	3–7	7–15	15–100	Mean	SD	Min.	Med.	Max.	Skew.	Kurt.	Serial corr.	N
0–3 Tranche	0.818	0.842	0.853	1593.06	376.29	1059.23	1503.54	2827.16	1.28	4.49	0.996	480
3–7 Tranche		0.986	0.939	110.58	50.81	57.69	97.89	365.22	2.59	9.92	0.994	480
7–15 Tranche			0.971	19.55	14.32	7.65	15.47	93.59	2.87	11.28	0.990	480
15–100 Tranche				1.25	1.04	0.34	0.97	6.63	3.02	12.04	0.985	480
Δ 0–3 Tranche	0.719	0.737	0.596	0.99	52.78	-295.43	-1.64	307.52	0.28	11.61	0.177	479
Δ 3–7 Tranche		0.912	0.739	0.07	9.81	-61.94	-0.35	87.17	0.76	25.27	0.162	479
Δ 7–15 Tranche			0.881	0.03	2.78	-16.08	-0.02	26.68	1.36	32.15	0.144	479
Δ 15–100 Tranche				0.00	0.23	-1.46	0.00	1.51	0.37	25.40	0.103	479

Table 2.5: Summary Statistics for the Levels and First Differences of the CDX North America Investment Grade CDO Tranche Premia during the Pre-Crisis Period

This table reports summary statistics for unified CDO tranche premia computed from the top model. The model was calibrated to the original, observed data and the statistics are computed from the time series ranging from September 2005 until September 2007. Values are reported in basis points.

	Correlations											
	3–7	7–15	15–100	Mean	SD	Min.	Med.	Max.	Skew.	Kurt.	Serial corr.	N
0–3 Tranche	0.819	0.690	0.328	3985.24	1030.54	1515.93	4120.93	6271.22	-0.15	2.31	0.998	742
3–7 Tranche		0.957	0.654	1158.30	690.56	116.71	1070.02	3302.73	0.78	3.02	0.998	742
7–15 Tranche			0.814	317.72	231.23	25.80	234.69	1126.05	1.27	3.91	0.998	742
15–100 Tranche				14.92	9.34	2.03	13.12	47.47	0.85	3.35	0.996	742
Δ 0–3 Tranche	0.396	0.537	0.164	3.89	182.21	-1944.84	6.69	2102.15	1.45	53.60	0.070	741
Δ 3–7 Tranche		0.905	0.333	1.11	70.07	-453.48	3.08	387.07	-0.01	10.56	0.078	741
Δ 7–15 Tranche			0.605	0.15	22.71	-115.09	0.18	144.42	0.38	10.47	0.150	741
Δ 15–100 Tranche				0.00	1.52	-8.50	0.05	12.40	0.83	16.93	0.000	741

Table 2.6: Summary Statistics for the Levels and First Differences of the CDX North America Investment Grade CDO Tranche Premia during the Crisis Period

This table reports summary statistics for unified CDO tranche premia computed from the top model. The model was calibrated to the original, observed data and the statistics are computed from the time series ranging from September 2007 until September 2010. Values are reported in basis points.

	Correlations											
	3–7	7–15	15–100	Mean	SD	Min.	Med.	Max.	Skew.	Kurt.	Serial corr.	N
0–3 Tranche	0.868	0.817	0.580	1716.06	218.26	1330.79	1726.76	2186.54	0.15	1.99	0.997	484
3–7 Tranche		0.893	0.814	534.68	101.00	352.34	512.60	795.25	0.50	2.30	0.997	484
7–15 Tranche			0.795	195.50	59.40	119.23	182.72	367.29	0.79	2.71	0.998	484
15–100 Tranche				37.20	8.40	23.49	34.77	67.81	1.34	4.32	0.998	484
Δ 0–3 Tranche	0.918	0.840	0.792	-1.37	45.75	-161.71	-0.92	201.65	-0.01	5.31	0.022	483
Δ 3–7 Tranche		0.852	0.692	-0.67	18.74	-73.08	-0.99	80.91	0.07	5.01	0.006	483
Δ 7–15 Tranche			0.891	-0.13	8.78	-35.82	-0.25	86.28	1.86	23.83	0.007	483
Δ 15–100 Tranche				-0.03	1.55	-6.68	-0.06	10.21	0.42	9.43	-0.025	483

Table 2.7: Summary Statistics for the Levels and First Differences of the CDX North America Investment Grade CDO Tranche Premia during the Post-Crisis Period

This table reports summary statistics for unified CDO tranche premia computed from the top model. The model was calibrated to the original, observed data and the statistics are computed from the time series ranging from September 2010 until September 2012. Values are reported in basis points.

	Correlations														
	SD	Min.	5	Med.	95	Max.	Mean	SD	Min.	Med.	Max.	Skew.	Kurt.	Serial corr.	N
Mean	0.920	0.797	0.897	0.853	0.871	0.844	43.48	11.66	30.46	39.68	92.55	2.07	7.08	0.996	480
SD		0.685	0.713	0.621	0.907	0.950	47.64	24.06	23.21	40.70	166.19	2.35	8.92	0.994	480
Min.			0.844	0.772	0.612	0.662	7.14	1.85	4.04	6.97	14.88	1.17	4.88	0.995	480
5				0.941	0.637	0.687	12.90	4.20	6.81	13.39	25.16	0.81	3.57	0.996	480
Med.					0.538	0.618	28.04	6.40	16.85	27.00	47.27	0.55	2.60	0.997	480
95						0.751	123.06	52.24	76.06	108.81	417.74	3.19	13.22	0.994	480
Max.							365.00	215.47	125.42	297.83	1392.62	1.78	6.75	0.992	480
Δ Mean	0.789	0.219	0.581	0.722	0.829	0.523	0.04	1.35	-8.53	-0.04	8.88	0.22	17.17	0.441	479
Δ SD		0.039	0.278	0.336	0.728	0.882	0.13	3.09	-23.66	0.03	24.45	0.10	30.52	0.345	479
Δ Min.			0.383	0.261	0.141	0.023	0.00	0.39	-3.80	0.00	2.71	-1.58	34.06	-0.121	479
Δ 5				0.588	0.352	0.173	0.01	0.38	-2.10	-0.01	3.55	1.96	25.58	0.202	479
Δ Med.					0.487	0.110	0.00	0.82	-4.55	-0.04	6.36	1.16	16.74	0.113	479
Δ 95						0.475	0.32	8.55	-61.04	-0.06	71.63	0.93	28.12	0.331	479
Δ Max.							0.90	35.43	-380.18	0.17	350.26	-1.14	56.35	0.170	479

Table 2.8: Summary Statistics for the Levels and First Differences of the Cross-Section of CDX Index Constituents during the Pre-Crisis Period

This table reports summary statistics for the cross-sectional statistics time series of CDX index constituents ranging from September 2005 until September 2007. Values are reported in basis points.

	Correlations														
	SD	Min.	5	Med.	95	Max.	Mean	SD	Min.	Med.	Max.	Skew.	Kurt.	Serial corr.	N
Mean	0.915	0.520	0.727	0.900	0.920	0.866	204.25	90.12	47.83	183.57	477.61	0.75	3.08	0.999	742
SD		0.304	0.516	0.714	0.869	0.968	385.41	202.21	57.67	386.07	982.81	0.36	2.46	0.996	742
Min.			0.569	0.603	0.447	0.249	22.54	7.73	0.00	21.21	45.35	0.29	3.90	0.990	742
5				0.921	0.489	0.563	38.32	11.69	16.62	37.69	77.58	0.53	3.20	0.999	742
Med.					0.710	0.720	92.79	35.47	27.92	83.30	214.73	0.82	3.46	0.999	742
95						0.767	653.58	300.38	177.54	544.66	1515.47	0.65	2.28	0.999	742
Max.							3225.57	1790.45	350.55	3423.20	8861.85	0.38	2.86	0.992	742
Δ Mean	0.784	0.134	0.583	0.705	0.696	0.637	0.17	8.33	-52.99	0.22	49.93	-0.18	11.19	0.239	741
Δ SD		0.041	0.189	0.273	0.381	0.931	0.39	34.28	-404.46	0.40	289.86	-1.57	42.23	0.005	741
Δ Min.			0.300	0.173	0.079	0.020	0.01	3.32	-38.52	0.00	38.48	-0.32	85.75	-0.145	741
Δ 5				0.719	0.432	0.105	0.02	1.39	-6.06	-0.01	9.13	0.61	9.20	0.262	741
Δ Med.					0.562	0.176	0.08	3.93	-20.92	0.00	25.91	0.33	9.65	0.264	741
Δ 95						0.273	0.30	32.41	-227.62	0.58	176.99	-0.13	10.74	0.209	741
Δ Max.							4.07	435.97	-5800.93	4.64	3615.09	-2.35	57.94	-0.078	741

Table 2.9: Summary Statistics for the Levels and First Differences of the Cross-Section of CDX Index Constituents during the Crisis Period

This table reports summary statistics for the cross-sectional statistics time series of CDX index constituents ranging from September 2007 until September 2010. Values are reported in basis points.

	Correlations														
	SD	Min.	5	Med.	95	Max.	Mean	SD	Min.	Med.	Max.	Skew.	Kurt.	Serial corr.	N
Mean	0.955	0.164	0.669	0.941	0.964	0.930	107.94	17.58	84.21	103.13	162.35	0.58	2.24	0.998	484
SD		-0.064	0.458	0.814	0.982	0.954	69.11	17.71	41.79	69.99	111.24	0.05	1.77	0.998	484
Min.			0.631	0.356	-0.026	0.083	22.61	6.38	9.59	23.93	35.59	-0.37	2.07	0.997	484
5				0.837	0.497	0.534	36.59	4.97	24.96	36.56	52.86	0.31	3.17	0.998	484
Med.					0.850	0.833	90.13	12.50	72.41	85.33	129.59	0.83	2.62	0.998	484
95						0.930	233.01	55.69	153.45	226.82	385.68	0.35	2.13	0.998	484
Max.							450.04	111.10	275.21	457.66	691.41	0.17	1.97	0.998	484
Δ Mean	0.924	0.367	0.744	0.858	0.865	0.688	-0.04	2.14	-9.39	-0.08	12.27	0.45	7.27	0.294	483
Δ SD		0.206	0.648	0.762	0.856	0.851	-0.02	1.89	-9.02	-0.09	10.83	0.39	7.92	0.293	483
Δ Min.			0.298	0.335	0.247	0.039	-0.04	0.93	-2.89	-0.06	8.59	1.82	18.73	-0.067	483
Δ 5				0.675	0.608	0.514	-0.01	0.76	-2.83	0.00	3.21	0.21	6.07	0.069	483
Δ Med.					0.701	0.571	-0.03	1.93	-6.46	-0.07	9.04	0.48	5.76	0.120	483
Δ 95						0.628	-0.09	7.38	-31.46	-0.24	36.82	0.47	6.77	0.284	483
Δ Max.							-0.43	13.91	-114.33	-0.78	85.28	-0.36	16.12	0.162	483

Table 2.10: Summary Statistics for the Levels and First Differences of the Cross-Section of CDX Index Constituents during the Post-Crisis Period

This table reports summary statistics for the cross-sectional statistics time series of CDX index constituents ranging from September 2010 until September 2012. Values are reported in basis points.

Chapter 3

CCDS:
Accurate and Approximate Pricing*

Abstract

In this paper, we analyze the pricing of contingent credit default swaps (CCDS) which provide protection against default losses in derivative transactions. In a framework with both asset and interest rate risk, we obtain a meaningful semi-analytical solution for CCDS prices with an interest rate swap as underlying. Our model yields three major contributions: (I) CCDS have several properties that fundamentally differ from those of CDS, despite the similar nature between both instruments. (II) We propose a simple approximate pricing formula for CCDS which is computed by the prices of observable traded assets, i.e. the CDS quote, the value of a swaption and a zero-bond. (III) We find that the approximate formula always overestimates the correct values and converges for a financial institution with low default risk.

* This chapter is based on the working paper *"Contingent CDS: Accurate and Approximate Pricing"* by Koziol and Schön (2014).

3.1 Introduction

To manage credit risk of industrial firms or financial institutions, credit default swaps (CDS) have established and are frequently used instruments for protection purposes. An important characteristic of plain vanilla CDS is that they provide a fixed compensation equal to the loss given default of the defaulting company's liabilities. Hence, a loan or a corporate bond can be perfectly hedged when the loss given default of the loan or corporate bond, respectively, coincides with the protection volume of the CDS. However, market participants often do not have a fixed exposure to a financial institution such as in case of a loan or corporate bond but exhibit a dynamic exposure such as in case of a swap contract. Once a financial institution has signed a swap contract with another market participant, who is concerned about the credit risk of the financial institution, a static CDS protection will fail to protect against a potential default. This is due to the fact that the market participant loses the (positive) swap value in case of a default of the financial institution. Since the swap value at default apparently depends on the interest rate term structure at this point in time, the market participant does not know in advance about the exposure at default. As a consequence, she cannot perfectly protect herself against a default of her swap counterpart with a CDS, because the required CDS volume, i.e. the exposure at default, is unknown prior to a default.
As a reaction to this hedging challenge, contingent credit default swaps (CCDS) have been introduced. According to *Brigo and Pallavicini (2006)*, the major difference between CCDS and CDS refers to the amount of protection in case of default. While classical CDS provide the loss given default of a representative loan of the underlying company, a CCDS pays the (positive) value of an arbitrary financial claim. In case of our example with a swap contract between a financial institution and another market participant, a CCDS pays the value of the swap once the financial institution defaults. As a result, CCDS now provide the market participant with a perfect protection against default risk of the financial institution. In case of a default, the market participant, on the one hand side, loses the (positive) swap value as the financial institution will no longer satisfy its obligations. On the other side, she obtains the swap value from the CCDS so that the total position is perfectly hedged.
While the pricing of CDS is well understood, the pricing of CCDS cannot be carried out in an analogous way due to the stochastic exposure over time. In this paper, we aim at rigorously analyzing the pricing of CCDS. For this purpose, we formulate a structural credit risk model for financial institutions that accounts for both asset losses and interest rate risk. In particular, both the credit event of a financial institution as well as the exposure of a CCDS written on a swap are crucially driven by interest rate

risk. Our framework adapts the *Merton (1974)* model to the typical characteristics of financial institutions where a bank's assets have a larger time to maturity than its liabilities. Hence, both losses of the assets of the financial institution and an unfortunate interest rate development can trigger a default of the financial institution. Imposing a geometric Brownian motion for asset values and interest rate uncertainty according to *Vasicek (1977)*, we obtain tractable pricing formulae from a change of the usual risk-neutral measure to the risk-neutral forward measure.

This framework yields the following three major contributions:

(I) In contrast to CDS, the prices of CCDS are no longer monotonous in the short rate, its volatility and the correlation between assets and interest rates of the financial institution.

(II) We propose a simple approximate pricing formula for CCDS with a swap exposure. This approximation for the CCDS price is the product out of the classical CDS quote and the value of a corresponding swaption divided by the according zero-bond price. Hence, the major advantage of this formula is that it only consists of market prices of traded assets.

(III) We analyze the accuracy of our approximate pricing formula. We find that the approximation is an overestimation for the correct value. For very low default probabilities, the approximate formula converges towards the correct CCDS price. For higher default probabilities, the difference can be relevant so that knowledge of a sophisticated structural model is necessary.

The remainder of the paper is organized as follows. In Section 3.2, we introduce a structural default model with interest rate risk. Exact valuation formulae for swaptions, CDS and CCDS as well as the approximate valuation formula for CCDS are retrieved in Section 3.3. We investigate the pricing behaviour of the approximate CCDS valuation formula in Section 3.4. Section 3.5 concludes.

3.2 The Model

CCDS protect against a credit event of an arbitrary counterparty. In contrast to classical CDS, the protection refers to the current value of a pre-specified derivative such as interest rate swaps, swaptions and spread options. Hence, a CCDS eliminates the counterparty risk of OTC derivatives. In our analysis, we focus on interest rate swaps as a representative of a very common underlying used by financial institutions to manage interest rate risk. *Brigo and Pallavicini (2006)* give an overview about further types of derivatives underlying a CCDS.

We set up our model in the context of *Merton (1974)* and extend it for interest rate risk. This extenstion will provide us with an illustrative modeling of the default event where both asset value and interest rate changes can cause bankruptcy as it is true for financial intermediaries.

3.2.1 Interest Rate Process

Let r_t denote the short rate that is subject to interest rate risk in our model. In accordance with *Vasicek (1974)* we assume that r_t follows the mean-reversion process

$$dr_t = \kappa \cdot (\theta - r_t)dt + \sigma_r dZ^{\mathbb{Q}}_{1,t}, \tag{3.1}$$

where κ is the speed of reversion, θ the level of the long-term mean, σ_r the instantaneous standard deviation and $dZ^{\mathbb{Q}}_{1,t}$ the increments of a standard Wiener process under the risk-neutral measure $\mathbb{Q}$ for which the money market account is the numeraire. The well-known solution $B(r_t, t, T)$ to bond prices at time t with maturity in T reads

$$B(r_t, t, T) = E^{\mathbb{Q}}_t \left[\exp\left(-\int_t^T r_u du \right) \right] = \exp(-K(t,T) \cdot r_t + D(t,T)), \tag{3.2}$$

$$K(t,T) = \frac{1 - \exp(-\kappa \cdot (T-t))}{\kappa}, \tag{3.3}$$

$$D(t,T) = \left(\theta - \frac{\sigma_r^2}{2\kappa^2} \right) (K(t,T) - (T-t)) - \frac{\sigma_r^2 \cdot K^2(t,T)}{4\kappa}, \tag{3.4}$$

Despite the well-known drawbacks associated with the Vasicek model such as negative interest rates or constant volatility (see e.g. *Chan et al. (1992)*) the Vasicek model has its merits for our application due to the following reasons: First, the model has a high analytical tractability as well as a relatively low parameterization which allows for an illustrative choice for our purposes. Second, we primarily need interest rate risk at all but not a specific type of model. For this reason, we prefer to obtain illustrative semi-closed form solutions within the Vasicek dynamics rather than only numerical results within a more advanced interest rate model.

To price arbitrary claims CL_t at time t in this setup, we need to compute the expectation of the terminal value CL_T of the claim divided by the realization $\exp(\int_t^T r_u du)$ of the money market account:

$$CL_t = E_t^{\mathbb{Q}} \left[\exp\left(-\int_t^T r_u du \right) \cdot CL_T \right]. \tag{3.5}$$

While the terminal value CL_T only depends on the short rate r_T at time T, the value $\exp(\int_t^T r_u du)$ of the money market account is driven by the entire path of the short rate from t to T. At first glance, this path dependency requires numerical solutions which prevent a meaningful analysis based on closed-form representations. Fortunately, a more convenient representation avoiding a costly evaluation of the path dependency is possible by applying stochastic calculus. We can erase the path dependency by changing the measure from the risk-neutral measure $\mathbb{Q}$ to the risk-neutral forward measure $\mathbb{Q}^T$ under which forward bond prices with maturity T are martingales. This approach leads to the general valuation formula where the expected terminal payoff under the new measure $\mathbb{Q}^T$ is discounted with the corresponding bond price $B(r_t, t, T)$:

$$CL_t = B(r_t, t, T) \cdot E_t^{\mathbb{Q}^T}[CL_T]. \tag{3.6}$$

In the Vasicek model, this change of measure can be achieved by the following transformation:

$$dZ_{1,t}^{\mathbb{Q}^T} = \sigma_r K(t,T)dt + dZ_{1,t}^{\mathbb{Q}}. \tag{3.7}$$

As a result, the dynamics of r_t under $\mathbb{Q}^T$ are specified by

$$dr_t = \left[\kappa \cdot (\theta - r_t) - \sigma_r^2 \cdot K(t,T)\right] dt + \sigma_r dZ_{1,t}^{\mathbb{Q}^T}. \tag{3.8}$$

Furthermore, we can easily obtain a solution of stochastic differential equation (3.8). The interest rate r_T at time T from the perspective of time t is identical in distribution to

$$r_T \stackrel{d}{=} \mu_{r,t}^{\mathbb{Q}^T} + \sigma_{r,t} \cdot \tilde{X}, \tag{3.9}$$

$$\mu_{r,t}^{\mathbb{Q}^T} = e^{-\kappa(T-t)} \left(r_t + \theta \cdot (e^{\kappa(T-t)} - 1) - \frac{\sigma_r^2}{\kappa} \sinh\left[\kappa \cdot (T-t)\right] \right), \tag{3.10}$$

$$\sigma_{r,t}^2 = \sigma_r^2 \cdot \left(\frac{1 - e^{-2\kappa(T-t)}}{2\kappa} \right), \tag{3.11}$$

where $\tilde{X}$ is a standard-normally distributed random variable. A proof of equations (3.7) to (3.11) can be found in Appendices 3.A and 3.A.

3.2.2 Asset Value Process

As both the asset value of the financial institution and the interest rates may impact the default probability, we need to account for them in CCDS valuation. Typically, lower interest rates improve refinancing conditions of borrowers and therefore increase their repayment probability. For this reason, we assume that the asset value of the financial institution is negatively related to interest rates, expressed by a non-positive correlation parameter $\rho \leq 0$. We will loosen this assumption later when we investigate the general CCDS pricing formula. The dynamics of A_t of the financial institution read:

$$dA_t = \sigma_A A_t \cdot \left(\rho dZ_{1,t}^{\mathbb{Q}} + \sqrt{1-\rho^2} dZ_{2,t}^{\mathbb{Q}} \right), \tag{3.12}$$

where σ_A is the volatility and ρ the correlation parameter through which we correlate A_t with r_t. $dZ_{1,t}^{\mathbb{Q}}$ and $dZ_{2,t}^{\mathbb{Q}}$ denote the increments of two independent standard Wiener processes under the initial risk-neutral measure $\mathbb{Q}$ from which the first one was already used in equation (3.1) for the short rate process.

Notice that A_t does not denote the asset value in the classical sense but its forecast value towards the time horizon T_A. For example, the asset value at time $T < T_A$ follows from its forecast discounted with the corresponding bond value:

$$\text{Asset value in } T = A_T \cdot B(r_T, T, T_A). \tag{3.13}$$

This characteristic will become important when we introduce the default condition of the financial institution in which unfavorable interest rate movements can also lead to a default.

The change of measure in equation (3.7) also affects the dynamics of A_t. As it is only related to the increments $dZ_{1,t}^{\mathbb{Q}}$ of the first Wiener process, $dZ_{2,t}^{\mathbb{Q}^T} = dZ_{2,t}^{\mathbb{Q}}$ holds. The dynamics of A_t are then given by

$$dA_t = -\rho\sigma_r\sigma_A \cdot K(t,T) \cdot A_t dt + \sigma_A A_t(\rho dZ_{1,t}^{\mathbb{Q}^T} + \sqrt{1-\rho^2} dZ_{2,t}^{\mathbb{Q}^T}), \tag{3.14}$$

with the solution

$$A_T = A_t \exp(\mu_{A,t}^{\mathbb{Q}^T} + \sigma_A \cdot \sqrt{T-t} \cdot (\rho\tilde{X} + \sqrt{1-\rho^2}\tilde{Y})), \tag{3.15}$$

$$\mu_{A,t}^{\mathbb{Q}^T} = \rho\sigma_r\sigma_A \cdot \left[\frac{1}{\kappa^2} - \frac{T-t}{\kappa} - \frac{\exp(-\kappa \cdot (T-t))}{\kappa^2}\right] - \frac{\sigma_A^2}{2} \cdot (T-t), \tag{3.16}$$

where $\tilde{X}, \tilde{Y}$ denote independent standard-normally distributed random variables.

3.2.3 Default Condition

The classical *Merton (1974)* approach considers the asset value of a counterparty to be the single driver of default. For financial institutions, such as the counterparty of the interest rate swap, there are two potential sources for a default: an unfavourable asset value and/or unfavourable interest rates. We assume that the financial institution defaults at time T if the value of its assets falls below the value of its liabilities. Since A_T denotes the asset value forecast with respect to the horizon T_A, the asset value at time T, i.e. the discounted asset value forecast at time T reads $A_T \cdot B(r_T, T, T_A)$ and is subject to asset as well as interest rate risk. We assume that the financial institution will have fixed short-term liabilities with amount L whose maturity is denoted as T_L. Furthermore, the values of the maturities are chosen such that $T < T_L < T_A$ holds. The default condition at time T can then be written as

$$A_T \cdot B(r_T, T, T_A) < L \cdot B(r_T, T, T_L) \Leftrightarrow \text{def}[r_T, A_T] \tag{3.17}$$

It is now clear how interest rates can induce a default. As $T_L < T_A$, the predicted asset value A_T is discounted more strongly than the short-term liabilities L. Hence, the financial institution will default for sufficient interest rates increases. That way, even favorable movements of A_t can be offset by interest rates.

So far we have not discussed the role of the entities that sell the fixed payer swap to the financial institution and the CCDS protection. In line with *Sorensen and Bollier (1994)*, the credit risk of both counterparties in an transaction has to be taken into account. In our analysis this would translate to three entities that are subject to default. In order to have clear effects and a meaningful interpretation of our pricing formulae, we do not consider the credit risks of the market participant that sells the interest rate swap as well as the CCDS protection seller. Since *Arora et al. (2012)* pinpoint that the effect on CDS spreads is vanishingly small, we can abstract from credit risk for sell-side institutions, i.e. the market participant selling the interest rate swap and the CCDS protection seller. Thus, our model setup may not perfectly match theoretical considerations on two-way credit risk but it can be justified by the negligibility of empirical evidence.

3.3 Valuation Approach

We use the model introduced in the previous section to evaluate CCDS, CDS and swaptions. Note that the following valuation formulae are independent from the particular stochastics within the model and that they represent a general setting which allows for other possible specifications of interest and asset value forecast risk.

3.3.1 Swaption Valuation

With the definitions of the default condition and the asset and interest rate processes, we are able to value the instruments that are of our interest — swaptions, CDS and CCDS. At first glance it might seem counterintuitive why we should value swaptions rather than swaps. This is because in case of default, the CCDS only pays the positive swap value which can be characterized as a swaption component.
We assume that the financial institution manages the interest rate risk of its balance sheet and buys a fixed payer swap from the market participant. Therefore, the market participant holds a fixed receiver swap long. In order to manage counterparty risk, the market participant seeks to buy protection against a default loss in the fixed receiver swap which might only occur when the financial institution defaults in case the fixed the receiver swap has a positive value to the market participant. Therefore, the possible default loss clearly exhibits an option character with regard to the fixed receiver swap. In order to value the associated receiver swaption, we first need to define its inner value at maturity T:

$$ISW[r_T] = \max\left[s\sum_{T_n=T+1}^{T_S} B(r_T,T,T_n) - (1 - B(r_T,T,T_S)), 0\right]. \tag{3.18}$$

We can now represent the value of the swaption at time t as the expectation of (3.18) under $\mathbb{Q}^T$ discounted with a zero-bond:

$$SW_t = B(r_t,t,T)\cdot E_t^{\mathbb{Q}^T}\left[ISW[r_T]\right]. \tag{3.19}$$

3.3.2 CDS Valuation

CDS classically require frequent payments of CDS premia unitl a protection payment is made or the CDS matures. In order to have a clear and simple timeline, we restrict all payments to the initial and terminal dates $t = 0$ and $t = T$, respectively. Hence, we have one upfront payment that serves as the protection leg and one possible default payment, the default leg at time T. Let φ denote the recovery rate. The upfront payment CDS_t can then be valued by

$$CDS_t = (1-\varphi) \cdot B(r_t, t, T) \cdot E_t^{\mathbb{Q}^T} \left[\text{def}[r_T, A_T] \right], \tag{3.20}$$

which is the fair value of the CDS in t. Simply put, the value of the CDS is equal to the discounted payment $1-\varphi$ that will occur with the default probability $E_t^{\mathbb{Q}^T} \left[\text{def}[r_T, A_T] \right]$.

3.3.3 CCDS Valuation

The CCDS differs from the CDS as it no longer pays a plain vanilla notional in case of default but the value of an interest rate swap. As the CCDS insures the swap-selling market participant against losses she might sustain from the swap, the only scenario in which it will pay out is when the swap has a positive value upon default of the financial institution. This option character explains why the CCDS is composed of a swaption and a default component. In line with the pricing of the CDS, we have an upfront payment $CCDS_t$ at time $t = 0$ and a potential default payment at time T. Therefore, the fair value at time t is

$$CCDS_t = (1-\varphi) \cdot B(r_t, t, T) \cdot E_t^{\mathbb{Q}^T} \left[ISW[r_T] \cdot \text{def}[r_T, A_T] \right]. \tag{3.21}$$

In general, all three pricing equations (3.19) — (3.21) cannot be solved analytically. Nevertheless, their semi-analytical representations allow for a straightforward numerical computation which turns out to be fast and accurate.
In addition to the generally valid but abstract pricing formula (3.21), we aim at representing the CCDS price using liquid market instruments. The beauty of this approach is that it is free of any model-risk. In order to obtain an alternative pricing formula we use the well-known covariance rule $Cov[X, Y] = E[X \cdot Y] - E[X] \cdot E[Y]$ and rewrite equation (3.21):

$$\begin{aligned} CCDS_t = (1-\varphi) \cdot B(r_t, t, T) \cdot &\left(E_t^{\mathbb{Q}^T} \left[ISW[r_T] \right] \cdot E_t^{\mathbb{Q}^T} \left[\text{def}[r_T, A_T] \right] \right) \\ &+ (1-\varphi) \cdot B(r_t, t, T) \cdot Cov_t^{\mathbb{Q}^T} \left[ISW[r_T, s], \text{def}[r_T, A_T] \right] \\ &= CDS_t \cdot \frac{SW_t}{B(r_t, t, T)} \\ &+ (1-\varphi) \cdot B(r_t, t, T) \cdot Cov_t^{\mathbb{Q}^T} \left[ISW[r_T, s], \text{def}[r_T, A_T] \right] \end{aligned} \tag{3.22}$$

Thus, the CCDS price is composed of the CDS, swaption and bond price and the covariance between the default condition and the inner value of the swaption which can be interpreted as a further correction term. It is now straightforward to distinguish between the components in CCDS valuation that are free of model risk and the one for which models need to be utilized, at least in theory. For this reason, we define an approximate value $\widehat{CCDS}_t$ for the correct CCDS price that only accounts for the traded assets but neglects the covariance term:

$$\widehat{CCDS}_t = \frac{CDS_t \cdot SW_t}{B_t}. \tag{3.23}$$

Obviously, formula (3.23) has an advantageous property: It is simply the product of a CDS and a swaption price with maturity of the CCDS divided by the corresponding zero-bond price $B_t := B(r_t, t, T)$. Thus, its representation is completely model-free and fully relies on the prices of liquid market instruments.

Practically, it would be of great interest to know whether there are certain situations in which the covariance term in (3.22) can be neglected or whether it can be ignored at all. That way, one would not need to implement any additional model for the computation of the covariance term and the CCDS price could be approximated by the information priced in bond, swaption and CDS markets. We derive the following important property for CCDS that provide protection against default losses in fixed receiver swaps[7]:

$$Cov_t^{\mathbb{Q}^T}[ISW[r_T], \text{def}[r_T, A_T]] \leq 0. \tag{3.24}$$

From the covariance property (3.24) and equation (3.22) we can see that the approximate formula always overestimates the correct price for CCDS with a fixed receiver interest rate swap as an underlying. This property represents an important aspect for practitioners as they will not run the risk of underrating the true risks inherent to a receiver CCDS with this approximation. Therefore, the approximate formula can serve as an indicator of default risk in derivative transactions, even in turbulent market environments.

To this point, we have assumed that the correlation may only take non-positive values, $\rho \leq 0$. We can now loosen this assumption by taking the monotonicity properties of $\text{def}[r_T, A_T]$ into consideration. As can be easily shown from inserting the definitions of

[7] A proof can be found in Appendix 3.A.

r_T and A_T, the default condition $\text{def}[\mu_{r,t}^{\mathbb{Q}^T} + \sigma_{r,t} \cdot \tilde{X}, A_t \exp(\mu_{A,t}^{\mathbb{Q}^T} + \sigma_A \cdot \sqrt{T-t} \cdot (\rho\tilde{X} + \sqrt{1-\rho}\tilde{Y}))]$ is increasing in $\tilde{X}$ as long as

$$\rho < \frac{\sigma_r}{\sigma_A\sqrt{T}} \cdot [K(t,T_A) \cdot J(t,T_A) - K(t,T_L) \cdot J(t,T_L)], \quad (3.25)$$

$$J(t,T) = \sqrt{\frac{1-\exp(-2\kappa \cdot (T-t))}{2\kappa}}$$

holds. Since r_T is also increasing in $\tilde{X}$, we must have a positive relation between $\text{def}[r_T, A_T]$ and r_T in this case. However, if ρ is sufficiently greater than zero, restriction (3.25) is violated and consequently, the monotonicity of the default condition inverts. In this case, the covariance may take positive values and (3.24) does not hold. The approximate formula (3.23) then underestimates the correct value of a CCDS.

When regarding the pricing representations for CCDS, CDS and swaptions derived in this section, we can easily see that all formulae hold for every arbitrage-free term structure model that provides information about the short rate and expectations under the required measure. Hence, the assumed Vasicek dynamics are just for illustration purposes when computing specific values.

A_0	σ_A	T_A	L	T_L	r_0	κ	θ	σ_r	ρ	s	T_S	T
100	0.03	10	65	2	0.03	0.3	0.05	0.01	0	0.05	10	1

Table 3.1: Initial Parameter Set

The table shows the initial parameter set for our structural model. The parameter values imply a one-year risk-neutral forward default probability of 3.20%

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3.4 Value Analysis

In order to gain an understanding of the issues associated with CCDS, we accomplish a comparative-static analysis to

- see whether CCDS properties are in line with
  well-known properties of CDS
- analyze the accuracy of the approximate formula.

This analysis is carried out in a realistic parameter setting. The aim is to produce default probabilities that are in line with observable default rates.
The values of the initial parameter set are listed in Table 3.1. The asset forecast volatility $\sigma_A = 0.03$ accords with the common volatility level observed for financial institutions. As the considered financial institution of our model performs term transformation regarding assets and liabilities, T_A is chosen to be 10 years and much larger than the lifetime of liabilities that have to be paid in the near future. We set their maturity to $T_L = 2$ years. r_0 is chosen below the long-term mean θ. The parameters κ, θ as well as σ_r approximately match the values reported by *Chan et al. (1992)*. We suppose that the asset value forecast of the financial institution exhibits no correlation with the short rate leading to $\rho = 0$. The swap rate s equals the long-term mean θ and it is thought that the swap will expire in $T_S = T_A = 10$ years from the CCDS valuation date $t = 0$. As $r_0 < \theta$ one could interpret the value of s to be the fair rate of a swap that was closed before t. Finally, we set $A_0 = 100$ and choose the value $L = 65$ so that it provides a one-year risk-neutral forward default probability of 3.20%. Since *Hull et al. (2005)* indicate that risk-neutral default probabilities are approximately 5 times as high as their corresponding physical default probabilities, our model-implied default probability is in line with the weighted average default rate for financial institutions of 0.69% reported by Standard and Poor's (2013) for the time period from 1981 to 2012. We will vary some of these values in the following subsections in order to assess their impact on CCDS prices.

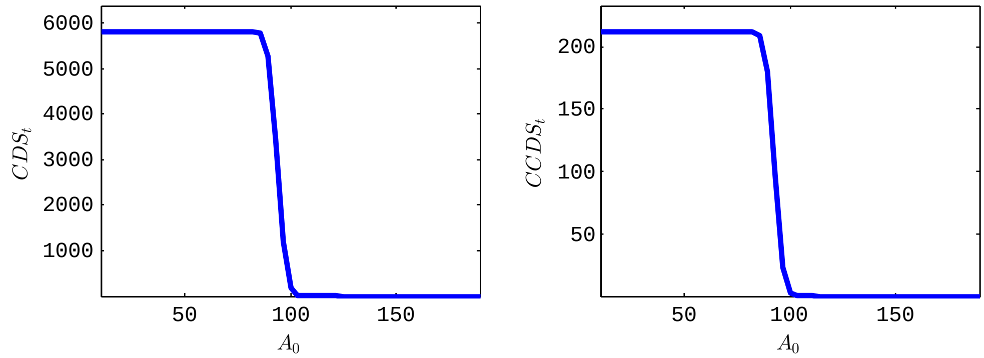

Figure 3.1: Impact of A_0 on CDS and CCDS Price

The panels show CDS_t and $CCDS_t$ for varying A_0 while all other parameters are held constant. Prices are reported in basis points.

3.4.1 CCDS and CDS Prices

In general, CCDS are very illiquid and bespoke contracts as they have a very specific underlying. No noteworthy indices and markets have emerged for CCDS yet. Nevertheless, market participants are still interested in managing their credit exposure especially with regard to the value of their outstanding derivatives.

A naive approach to hedge these risks would be to buy swaption-many CDS contracts due to similiar pricing characteristics with respect to A_0. As both panels of Figure 3.1 show, the CDS and the CCDS behave in the exact same manner if A_0 increases and the other parameters remain constant. This property would justify the naive hedging approach.

However, as we show in the following, CCDS and CDS also exhibit important differences in their respective pricing behaviors making a naive hedge highly doubtful. The left panel of Figure 3.2 shows the CDS price for different interest rate scenarios. The plot shows that increasing interest rates lead to higher CDS prices in our model. On the one hand side, higher interest rates imply a lower zero-bond price which according to equation (3.20) should make the price cheaper. However, the effect is overcompensated by the default condition (3.17) that is triggered in more cases. Economically, this is because higher interest rates worsen the refinancing conditions for the financial institution and therefore make a default more likely. Thus, higher interest rates imply higher default probabilities and higher CDS prices.

According to equation (3.21), the default condition plays an important role for the valuation of CCDS. For this reason, when interest rates are increasing from a very low starting point close to zero, the right panel of Figure 3.2 shows that the CCDS

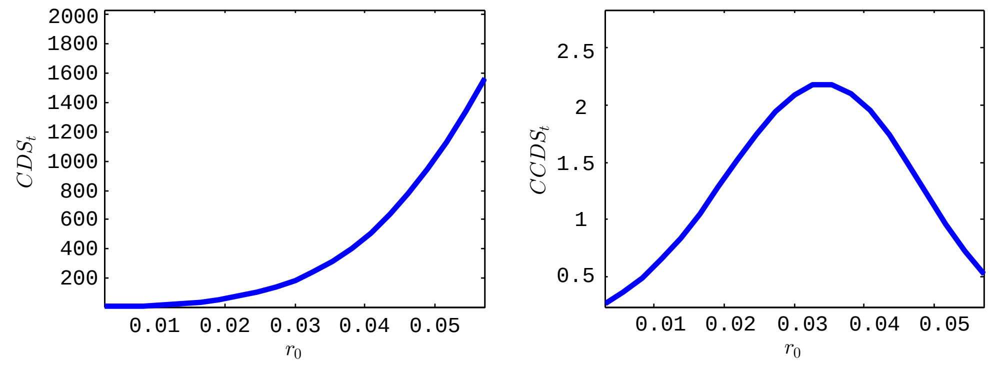

Figure 3.2: Impact of r_0 on CDS and CCDS Price

The panels show CDS_t and $CCDS_t$ for varying r_0 while all other parameters are held constant. Prices are reported in basis points.

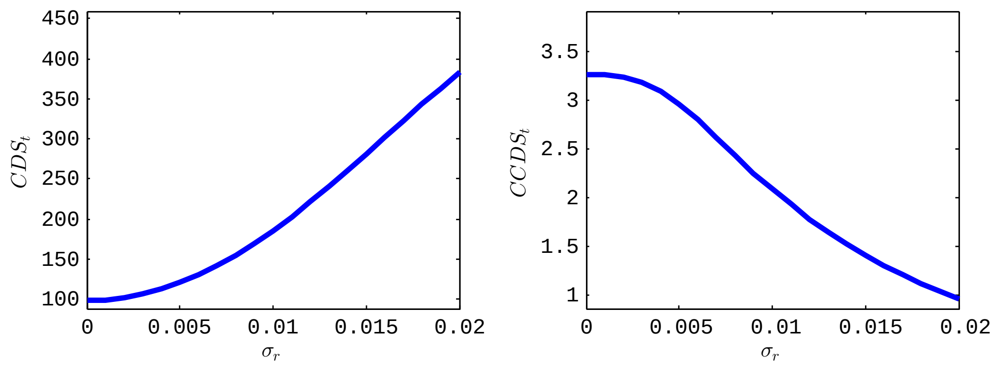

Figure 3.3: Impact of σ_r on CDS and CCDS Price

The panels show CDS_t and $CCDS_t$ for varying σ_r while all other parameters are held constant. Prices are reported in basis points.

price increases in r_0. This property is in line with increasing CDS prices for those parameters. However, when r_0 is high, the CCDS price declines in r_0. This effect, that clearly distinguishes CCDS from CDS prices, can be explained by the behavior of the swaption component inherent to the CCDS. Since a CCDS provides default insurance against a fixed receiver swap long, the value of the underlying declines with increasing interest rates thus implying a declining CCDS price. To sum up, the CCDS price with regard to r_0 is driven by two effects: the default likelihood that increases the price for low r_0 and the swaption price that decreases it for high r_0 because of a low exposure.

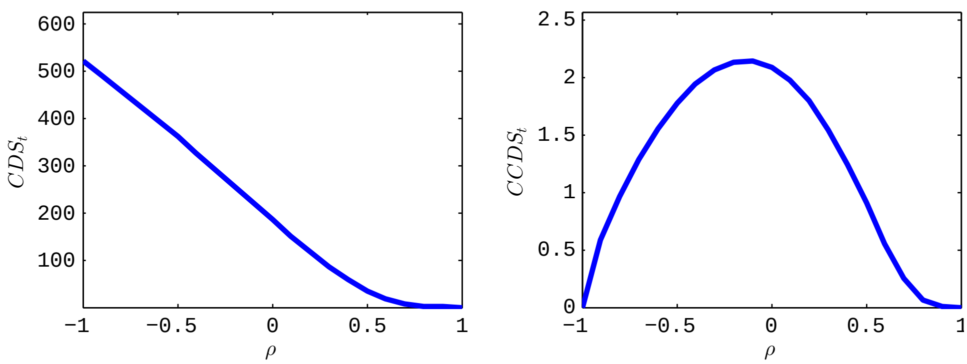

Figure 3.4: Impact of ρ on CDS and CCDS Price

The panels show CDS_t and $CCDS_t$ for varying ρ while all other parameters are held constant. Prices are reported in basis points.

The parameter σ_r determines the amplitude of the stochastic increments in equation (3.1). Figure 3.3 shows the related pricing graphs for CDS and CCDS. As can be seen from the left panel, the CDS price increases in σ_r. This is because of the fact that its payoff function is convex in r_0 as seen in Figure 3.2. Remarkably, we find the opposite relationship for CCDS. Figure 3.3 illustrates that CCDS decline with a higher σ_r. This is a consequence from a concave payoff function of the CCDS in σ_r seen in Figure 3.2. A further major difference between the prices of CDS and CCDS refers to the relationship to the correlation ρ. As the left panel of Figure 3.4 shows, the CDS price declines in the correlation ρ between the asset forecast A_t and the short rate r_t. Let us assume that $\rho = -1$ holds. That way, the short rate and the asset forecast value are perfectly correlated and always developing into the opposite direction. Now, if A_T is low then r_T is high. Both variables imply a higher default probability because a low A_T makes a shortfall of the asset value under L more likely and higher short rates worsen the refinancing conditions for the financial institution. However, for $\rho = 1$, the opposite is the case: When A_T is low, then r_T is low as well leading to higher zero-bond prices. Consequently, the negative effect from a lower asset forecast value is compensated by the advantages of lower short rates. Therefore, CDS quotes decline with ρ because a higher correlation allows for a better hedge of low asset value forecasts with favorable lower short rates. For the CCDS, we again find a different shape. In Figure 3.4, the CCDS quote first increases and then tends to zero for correlations from -1 to 1. This behavior of the CCDS price results from the interplay between the swaption component and the default condition. For $\rho = -1$, A_T declines for increasing r_T. In this situation, we have a high default probability associated with a low swaption value close to zero. Since the present value of a payoff in case of default, i.e. the swaption value, is nearly zero, the CCDS is also close to zero for $\rho = -1$. For a high correlation $\rho = 1$, the hedging effects between assets and short rates imply a low default probability. Hence, the CCDS such as the CDS in this case tend to zero. Summing up, since the CCDS is close to zero for both $\rho = -1$ and $\rho = 1$, it must attain its maximum for an intermediate correlation ρ between -1 and 1.

3.4.2 CCDS and Approximate CCDS Price

In order to correctly value a CCDS, we use our model that accounts for both asset value and interest rate risk. From (3.25), we have a simple approximation $\widehat{CCDS}_t$ out of observable prices of traded assets that upward estimates the correct value of the CCDS. In the following analysis, we focus on the differences between $\widehat{CCDS}_t$ and $CCDS_t$, i.e. the accuracy of (3.23).

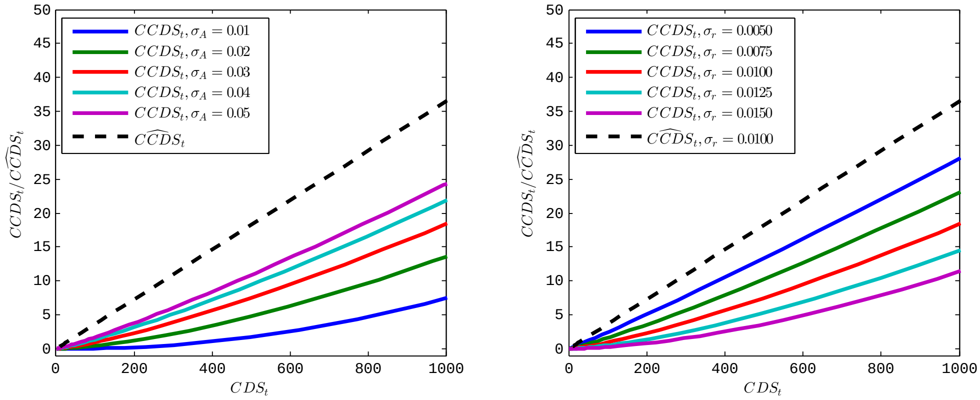

Figure 3.5: Impact of $\boldsymbol{\sigma_A}$ and $\boldsymbol{\sigma_r}$ on CCDS and Approximate CCDS Price

The panels show $CCDS_t$ and $\widehat{CCDS}_t$ in dependence of CDS_t for varying σ_A and σ_r and increasing A_0. Prices are reported in basis points.

The left panel of Figure 3.5 shows the CCDS price in dependence of the CDS price for varying A_0. This computation is carried out for various volatilities σ_A of the asset value. The approximate CCDS price obviously increases linearly in CDS_t because a rise in A_0 only affects the default condition but neither the swaption nor the bond price. Furthermore, the correct price always lies below the approximate price because since inequation (3.24) holds. Hence, the approximate $\widehat{CCDS}_t$ quote converges to the true value $CCDS_t$ when the default risk, i.e. CDS_t, is low.
When regarding different levels for the volatility of the asset forecast, we observe that the correct CCDS price moves towards the approximate price for increasing σ_A. The higher asset value volatility makes the approximation more accurate since it only affects the default condition and therefore dilutes the importance of the short rate for the covariance term. As can be seen from the right panel of Figure 3.5 the opposite holds for the short rate volatility σ_r that is varied for increasing A_0. When σ_r increases, the influence of the short rate on the covariance term, i.e. the distance between the approximate and the correct price, rises. Thus, we have two ways how the volatility parameters of our model influences the goodness of the approximate formula: it is improved by higher σ_A but it deteriorates for higher σ_r.

3.5 Conclusion

In this paper, we have focused on the valuation of CCDS that have a fixed receiver interest rate swap as an underlying. A reasonable CCDS price requires the consideration of a structural model with both asset and interest rate risk. When applying appropriate techniques for change of measures, we obtain tractable pricing formulae for CCDS. Moreover, we can propose an approximate formula for CCDS that is composed of a CDS, a swaption and a zero-bond which can be received from the prices of liquid market instruments. This formula overestimates the correct price of a CCDS. For financial institutions with low CDS quotes, the approximation tends to the true CCDS value. Despite the similar nature between CCDS and CDS, they exhibit different properties. This outcome justifies the use of sophisticated structural models for the pricing of CCDS especially when credit risk is severe.

Appendix 3.A Proofs

3.A.1 Distribution of $FB(T, T, T_F)$ under $\mathbb{Q}^T$

The path dependency in the expectation operator can be erased by changing the probability measure from the risk-neutral measure $\mathbb{Q}$ to the risk-neutral forward measure $\mathbb{Q}^T$.

Proof: We change the measure by representing the forward bond price as

$$FB(t, T, T_F) = B(r_t, t, T_F)/B(r_t, t, T). \tag{3.A.1}$$

From Itō's lemma we obtain

$$\begin{aligned} dFB(t, T, T_F) = {} & \sigma_r^2 K(t, T) \left[K(t, T) - K(t, T_F)\right] FB(t, T, T_F)dt \\ & + \sigma_r \left[K(t, T) - K(t, T_F)\right] FB(t, T, T_F) dZ_{1,t}^{\mathbb{Q}}. \end{aligned} \tag{3.A.2}$$

We now apply Girsanov's theorem, in order to make $FB(t, T, T_F)$ a martingale under $\mathbb{Q}^T$:

$$dZ_{1,t}^{\mathbb{Q}^T} = \sigma_r K(t, T)dt + dZ_{1,t}^{\mathbb{Q}} \tag{3.A.3}$$

which results in the following forward bond price dynamics under $\mathbb{Q}^T$:

$$dFB(t, T, T_F) = \sigma_r \left[K(t, T) - K(t, T_F)\right] FB(t, T, T_F) dZ_{1,t}^{\mathbb{Q}^T}. \tag{3.A.4}$$

Using Itō's lemma, Itō's isometry and the transformation

$$X(t, T, T_1) = \log\left[FB(t, T, T_F)\right], \tag{3.A.5}$$

it can be shown that the following equation holds:

$$\begin{aligned} FB(T, T, T_F) &= \frac{B(t, t, T_F)}{B(t, t, T)} \exp\left[-\frac{1}{2} \cdot Z(t, T, T_F) + \sqrt{Z(t, T, T_F)}\varepsilon_1^{\mathbb{Q}^T}\right], \\ Z(t, T, T_F) &= \sigma_r \int_t^T \left[K(u, T) - K(u, T_F)\right]^2 du \end{aligned} \tag{3.A.6}$$

where $\varepsilon_1^{\mathbb{Q}^T}$ is standard-normally distributed. □

3.A.2 Distribution of r_T under $\mathbb{Q}^T$

The properties of the distribution of r_T under the risk-neutral forward measure $\mathbb{Q}^T$ are derived in accordance with *Mamon (2004)*.

Proof: After applying Girsanov's theorem, the short rate process under $\mathbb{Q}^T$ is given by:

$$dr_t = \left[\kappa \cdot (\theta - r_t) - \sigma_r^2 \cdot K(t,T)\right] dt + \sigma_r dZ_{1,t}^{\mathbb{Q}^T}, \tag{3.A.7}$$

with its solution

$$r_T = r_t + \int_t^T \left[\kappa \cdot (\theta - r_u)\right] du - \sigma_r^2 \cdot \int_t^T K(u,T) du + \sigma_r \cdot \int_t^T dZ_{1,u}^{\mathbb{Q}^T}. \tag{3.A.8}$$

Applying the expectation operator on both sides and taking the partial derivative for t, we obtain the following differential equation

$$\frac{\partial \mu_{r,t}^{\mathbb{Q}^T}}{\partial t} = \kappa \cdot \left(\theta - \mu_{r,t}^{\mathbb{Q}^T}\right) - \sigma_r^2 \frac{\partial K(t,T)}{\partial t}. \tag{3.A.9}$$

Given the boundary condition $\mu_{r,t}^{\mathbb{Q}^T} = r_t$, this equation solves to

$$\mu_{r,t}^{\mathbb{Q}^T} = e^{-\kappa(T-t)} \left(r_t + \theta \cdot (e^{\kappa(T-t)} - 1) - \frac{\sigma_r^2}{\kappa} \sinh\left[\kappa \cdot (T-t)\right] \right). \tag{3.A.10}$$

The variance of the short rate can be derived by applying Itō's isometry, resulting in

$$\sigma_{r,t}^2 = \sigma_r^2 \cdot \left(\frac{1 - e^{-2\kappa(T-t)}}{2\kappa} \right) \tag{3.A.11}$$

which is the same variance as under $\mathbb{Q}$. Summing up, r_T is normally distributed with mean $\mu_{r,t}^{\mathbb{Q}^T}$ and variance $\sigma_{r,t}^2$ under $\mathbb{Q}^T$. □

3.A.3 $\boldsymbol{Cov_t^{\mathbb{Q}^T}[ISW[r_T], \mathrm{def}[r_T, A_T]] \leq 0}$

The CCDS price in equation (3.22) is overestimated by its approximation if and only if the covariance term takes non-positive values. In the following we prove that this condition holds for receiver swaps as long as (3.25) is valid. The proof holds for arbitrary distributional assumptions of r_T and A_T as long as the indicator function $\mathrm{def}[r_T, A_T]$ is weakly monotonically increasing in r_T.

Proof: Let

$$\widehat{ISW}[\tilde{X}] = ISW[\mu_{r,t}^{\mathbb{Q}^T} + \sigma_{r,t} \cdot \tilde{X}], \tag{3.A.12}$$

$$\widehat{\mathrm{def}}[\tilde{X}, \tilde{Y}] = \mathrm{def}[\mu_{r,t}^{\mathbb{Q}^T} + \sigma_{r,t} \cdot \tilde{X}, A(\tilde{X}, \tilde{Y})], \tag{3.A.13}$$

$$A(\tilde{X}, \tilde{Y}) = A_t \exp(\mu_{A,t}^{\mathbb{Q}^T} + \sigma_A \cdot \sqrt{T-t} \cdot (\rho\tilde{X} + \sqrt{1-\rho}\tilde{Y}))$$

and

$$\Delta\widehat{ISW}[\tilde{X}] = \widehat{ISW}[\tilde{X}] - E_t^{\mathbb{Q}^T}[\widehat{ISW}[\tilde{X}]], \tag{3.A.14}$$

$$\Delta\widehat{\mathrm{def}}[\tilde{X}, \underline{Y}] = \widehat{\mathrm{def}}[\tilde{X}, \underline{Y}] - E_t^{\mathbb{Q}^T}[\widehat{\mathrm{def}}[\tilde{X}, \tilde{Y}]|\underline{Y}]. \tag{3.A.15}$$

Then the conditional covariance depending on the fixed value $\underline{Y}$ can be computed by

$$Cov_t^{\mathbb{Q}^T}[\widehat{ISW}[\tilde{X}], \widehat{\mathrm{def}}[\tilde{X}, \tilde{Y}]|\underline{Y}] \tag{3.A.16}$$

$$= \int_{-\infty}^{\infty} \Delta\widehat{ISW}[x] \cdot \Delta\widehat{\mathrm{def}}[x, \underline{Y}] \cdot f(x)dx.$$

It can be easily shown that $\Delta\widehat{ISW}[x]$ and $\Delta\widehat{\mathrm{def}}[x, \underline{Y}]$ are monotonous and that the former decreases and the latter increases in x. Thus, the integration interval $(-\infty, \infty)$ in (3.A.16) can be devided into three intervals $S_1 = (-\infty, v]$, $S_2 = (v, w]$ and $S_3 = (w, \infty)$ with $v < w$. We now distinguish the following two cases:

Case 1: v and w are chosen so that

$$\mathrm{sign}\left(\Delta\widehat{ISW}[x]\right) = \begin{cases} 1, & \text{for } x \in S_1 \bigcup S_2 \\ -1, & \text{for } x \in S_3. \end{cases} \tag{3.A.17}$$

and

$$\Delta\widehat{\mathrm{def}}[x,\underline{Y}] = \begin{cases} h^-, & \text{for } x \in S_1 \\ h^+, & \text{for } x \in S_2 \bigcup S_3. \end{cases} \tag{3.A.18}$$

Then

$$\begin{aligned}
&Cov_t^{\mathbb{Q}^T}[\widehat{ISW}[\tilde{X}], \widehat{\mathrm{def}}[\tilde{X},\tilde{Y}]|\underline{Y}] \qquad (3.A.19)\\
&= \int_{-\infty}^{v} \underbrace{\underbrace{\Delta\widehat{ISW}[x]}_{>0} \cdot \underbrace{h^-}_{<0}}_{<0} \cdot f(x)dx + \int_{v}^{\infty} \Delta\widehat{ISW}[x] \cdot h^+ \cdot f(x)x \\
&\leq \int_{-\infty}^{v} \underbrace{\underbrace{\Delta\widehat{ISW}[x]}_{>0} \cdot \underbrace{h^+}_{>0}}_{>0} \cdot f(x)dx + \int_{v}^{\infty} \Delta\widehat{ISW}[x] \cdot h^+ \cdot f(x)dx \\
&= h^+ \cdot \int_{-\infty}^{\infty} \Delta\widehat{ISW}[x] \cdot f(x)dx \\
&= h^+ \cdot \int_{-\infty}^{\infty} (\widehat{ISW}[x] - E_t^{\mathbb{Q}^T}[\widehat{ISW}[x]]) \cdot f(x)dx \\
&= h^+ \cdot \left(E_t^{\mathbb{Q}^T}[\widehat{ISW}[\tilde{X}] - E_t^{\mathbb{Q}^T}[\widehat{ISW}[\tilde{X}]] \right) \\
&= 0.
\end{aligned}$$

Case 2: v and w are chosen so that

$$\text{sign}\left(\Delta\widehat{ISW}[x]\right) = \begin{cases} 1, & \text{for } x \in S_1 \\ -1, & \text{for } x \in S_2 \bigcup S_3. \end{cases} \tag{3.A.20}$$

and

$$\Delta\widehat{\text{def}}[x, \underline{Y}] = \begin{cases} h^-, & \text{for } x \in S_1 \bigcup S_2 \\ h^+, & \text{for } x \in S_3. \end{cases} \tag{3.A.21}$$

Then

$$\begin{aligned} & Cov_t^{\mathbb{Q}^T}[\widehat{ISW}[\tilde{X}], \widehat{\text{def}}[\tilde{X}, \tilde{Y}]|\underline{Y}] \qquad (3.A.22) \\ &= \int_{-\infty}^{w} \Delta ISW[x] \cdot h^- \cdot f(x)dx + \int_{w}^{\infty} \underbrace{\underbrace{\Delta\widehat{ISW}[x]}_{<0} \cdot \underbrace{h^+}_{>0}}_{<0} \cdot f(x)dx \\ &\leq \int_{-\infty}^{w} \Delta\widehat{ISW}[x] \cdot h^- \cdot f(x)dx + \int_{w}^{\infty} \underbrace{\underbrace{\Delta\widehat{ISW}[x]}_{<0} \cdot \underbrace{h^-}_{<0}}_{>0} \cdot f(x)dx \\ &= h^- \cdot \int_{-\infty}^{\infty} \Delta\widehat{ISW}[x] \cdot f(x)dx \\ &= h^- \cdot \int_{-\infty}^{\infty} (\widehat{ISW}[x] - E_t^{\mathbb{Q}^T}[\widehat{ISW}[x]]) \cdot f(x)dx \\ &= h^- \cdot \left(E_t^{\mathbb{Q}^T}[ISW[\tilde{X}] - E_t^{\mathbb{Q}^T}[ISW[\tilde{X}]]\right) \\ &= 0. \end{aligned}$$

Both cases yield the result that $Cov_t^{\mathbb{Q}^T}[\widehat{ISW}[\tilde{X}], \widehat{\text{def}}[\tilde{X}, \tilde{Y}]|\underline{Y}] \leq 0$. In order to compute the total covariance $Cov_t^{\mathbb{Q}^T}[\widehat{ISW}[\tilde{X}], \widehat{\text{def}}[\tilde{X}, \tilde{Y}]]$ we need to determine the value of the covariance between the conditional expectations of the inner swap value and the default condition. It can be easily shown that the following equation holds with respect to stochastic $\underline{Y}$:

$$Cov_t^{\mathbb{Q}^T}[E_t^{\mathbb{Q}^T}[\widehat{ISW}[\tilde{X}]|\tilde{Y} = \underline{Y}], E_t^{\mathbb{Q}^T}[\widehat{\text{def}}[\tilde{X}, \tilde{Y}]|\tilde{Y} = \underline{Y}]] = 0 \tag{3.A.23}$$

due to the invariance of $E_t^{\mathbb{Q}^T}[\widehat{ISW}[\tilde{X}]|\tilde{Y} = \underline{Y}]$ for $\underline{Y}$. By applying the law of total covariance, we can write:

$$
\begin{aligned}
&Cov_t^{\mathbb{Q}^T}[\widehat{ISW}[\tilde{X}], \widehat{\text{def}}[\tilde{X}, \tilde{Y}]] \\
&= \int_{-\infty}^{\infty} Cov_t^{\mathbb{Q}^T}[\widehat{ISW}[\tilde{X}], \widehat{\text{def}}[\tilde{X}, \tilde{Y}]|y] \cdot f(y)dy \\
&+ Cov_t^{\mathbb{Q}^T}[E_t^{\mathbb{Q}^T}[\widehat{ISW}[\tilde{X}]|\tilde{Y} = \underline{Y}], E_t^{\mathbb{Q}^T}[\widehat{\text{def}}[\tilde{X}, \tilde{Y}]|\tilde{Y} = \underline{Y}]]
\end{aligned}
\tag{3.A.24}
$$

Since the second term is always zero according to (3.A.23) and the integral $\int_{-\infty}^{\infty} Cov_t^{\mathbb{Q}^T}[\widehat{ISW}[\tilde{X}], \widehat{\text{def}}[\tilde{X}, \tilde{Y}]|y] \cdot f(y)dy$ is non-positive due to (3.A.19) and (3.A.22) the total term must be negative:

$$
Cov_t^{\mathbb{Q}^T}[\widehat{ISW}[\tilde{X}], \widehat{\text{def}}[\tilde{X}, \tilde{Y}]] \leq 0. \tag{3.A.25}
$$

□

Chapter 4

How Market Model Choice Affects the CVA*

Abstract

The Basel II/III accords aim at an improved stability of the financial system. A key aspect is the credit valuation adjustment (CVA) that accounts for counterparty credit risk in derivatives transactions. As the CVA for a specific derivative trade with a counterparty is not observable on the market, its value needs to be computed from a market and a default model. In this paper, we investigate the model risk that is associated with market models in the context of CVA. We find that the CVA can be highly dependent on model choice. This has important implications for practitioners in financial institutions and regulators because the amount of required economic capital can vary to a notable extent.

* This chapter is based on the working paper *"How Market Model Choice Affects the CVA"* by Schön (2015).

4.1 Introduction

To erase the counterparty credit risk component of derivatives trades, central clearing houses can call market participants for collateral that is used for compensating losses. Although this exchange-based mechanism works well for standardized contracts such as futures, it cannot be generally applied to highly complex derivatives because of their very bespoke nature and the resulting illiquidity. In order to overcome this drawback in over-the-counter markets, the Basel II/III accords stipulate market pariticipants to account for credit risk in their OTC trades and to back those trades with collateral.
Because of their high illiquidity, it is usually not possible to observe a market price for these derivatives and to appropriately correct them for counterparty credit risk. It is therefore necessary to use a sophisticated market model and a default model to compute the price that is adjusted for any losses that might occur upon default, known as credit valuation adjustment (CVA). The amount of these default losses again depends on the potential future exposure (PFE) of a single derivative transaction which needs to be computed with the help of a market model since it is not observable (see e.g. *De Prisco and Rosen (2005)*). However, there is no widely-accepted standard market model within each asset class that can be used for PFE computations and consequently, the adjusted price could be strongly model-dependent and therefore subject to model risk. An implication of choosing a non-suitable market model could be an underestimation of possible default losses. In this case, the computed value of a derivative would be too high entailing that a counterparty would need to back a transaction with too much collateral which is costly. On the other hand, an overestimation of possible default losses would lead to an adjusted price that is too low implying that less collateral is posted than actually needed making the trade more risky. Therefore, banks have a clear incentive to compute the adjusted price properly by choosing an appropriate market model which on the one hand saves cost of collateral but on the other leads to an adequate valuation of counterparty credit risk.
In this paper, we examine the dependence of CVA on the choice of the market model. We conduct our analysis on equity and interest rate markets since they both represent important asset classes for standard derivatives such as equity options and interest rate swaps. For assessing model risk, we follow the approach introduced by *Branger et al. (2012)*. We assume that a sophisticated model represents the true market model in each of both asset classes and generate prices from these two models. Afterwards, less complex market models are calibrated to the generated prices. They are then used for simulating the PFE profiles of standard securities in each market: call and put options in the equity market, bonds, floaters, swaps, swaptions, caplets and floorlets in the

interest rate market. In the last step, we compute the CVA from these profiles and carve out to which extent it is dependent on model choice.

Both analyses are carried out independently of each other since equity dynamics are not relevant for the valuation of the interest rate instruments that we consider in our analysis. Furthermore, the same holds for the equity instruments with respect to interest rate models since only a complex equity model without a stochastic interest rate component is assumed to reflect the true equity market.

Our analysis is based on memorable contributions in the academic literature. *Bakshi et al. (1997)* analyze models for European call and put options empirically. They find that a stock price diffusion process with stochastic volatility and jumps (SVJ) performs best in pricing and hedging, compared to the *Black and Scholes (1973)*, the *Heston (1973)* and the *Amin and Ng (1993)* models which either exhibit constant volatility, stochastic volatility (SV) or stochastic volatility and stochastic interest rates (SVSI). For assessing model dependence on interest rate markets, we implement the *Heath et al. (1992)* (HJM) framework since it provides an arbitrage-free modelling of the whole term structure and many important models such as the continuous-time version of the *Ho and Lee (1986)* and the *Hull and White (1993)* model are embedded within it. Based on the findings of *Bühler et al. (1999)* we use the single factor version of the HJM framework and implement three interest rate models by specifying their volatility functions according to *Amin and Morton (1994)*. In short, we find that the properties of future distributions are highly model dependent for equity as well as interest rate instruments. The CVA itself exhibits a high model dependence for the equity derivatives, but shows no mentionable deviations for the interest rate instruments across the three considered interest rate models.

The paper is organized as follows. In Section 4.2 we provide a short summary of the HJM framework, the volatility functions and the interest rate instruments that are to be analyzed. Section 4.3 treats the equity models and instruments. The model dependence of CVA is studied in Section 4.4. Section 4.5 concludes.

4.2 Interest Rate Models and Instruments

In this section, we present the interest rate models and give a brief description of the assessed interest rate instruments. There are many models avaiblable for the valuation of interest rate instruments, beginning with *Vasicek (1977)* and followed by *Brennan and Schwartz (1979)*, *Cox et al. (1986)* and many others. These models have in common that the future state of the term structure is retrieved from the state of a short-term interest rate that is complemented by a long-term interest rate in some models. Although this modelling approach is very intuitive, it has its drawbacks with respect to matching the term structure of a given state. Furthermore, in most cases, it only allows for normal term structure shapes and is not capable of capturing important characteristics such as slope, curvature and the volatility term structure. A framework that overcomes these drawbacks is the forward rate framework introduced by *Heath et al. (1992)* (HJM). In contrast to the aforementioned short rate models, the HJM framework allows for a simultaneous and arbitrage-free modelling of the complete term structure by imposing certain restrictions on the drift term of the forward rates. It can also cure the lack of capturing slope and curvature movements when enough risk factors are involved. The most important reason why we stick to the HJM framework however, lies in its generality. Some important interest rate models, e.g. the *Hull and White (1993)* and the continuous-time version of the *Ho and Lee (1986)* model, turn out to be special cases of the HJM framework when the volatility function is chosen appropriately. For these reasons, the HJM framework provides a good environment for our analysis with regard to interest rate markets. In the following, we give a brief summary of the HJM framework, outline its implementation and the volatility specifications according to *Amin and Morton (1994)* for the continuous-time *Ho and Lee (1986)*, the exponential *Vasicek (1977)* and the *Hull and White (1993)* model. Since our implementation of the *Hull and White (1993)* model allows for a time-dependent volatility calibration, it is the most general of the three models considered in our analysis. Therefore, we assume it to be the true market model in the context of this paper.

4.2.1 Continuous HJM Framework

According to *Heath et al. (1992)*, the arbitrage-free dynamics of an instantaneous forward rate $f(t,T)$ under the risk-neutral measure $\mathbb{Q}$ are given by the stochastic differential equation

$$df(t,T) = \sigma(t,T,f(t,T)) \int_t^T \sigma(t,s,f(t,T))ds dt + \sigma(t,T,f(t,T))dz(t), \tag{4.1}$$

where, in general, the forward rate $f(t,T)$ has the volatility $\sigma(t,T,f(t,T))$ that can depend on the valuation date t, the maturity T and $f(t,T)$. $dz(t)$ denotes the increment of a one-dimensional Brownian motion. Although the HJM approach allows for a finite number of risk factors, we only stick to the one-dimensional version since according to the empirical results of *Bühler et al. (1999)* it is sufficient to price standard interest rate derivatives like we do in our paper.

As can be easily seen from equation (4.1), the dynamics of $f(t,T)$ are particularly dependent on the definition of the volatility function $\sigma(t,T,f(t,T))$. One way to define $\sigma(t,T,f(t,T))$ lies in the numerical calibration to observable market data, such as the current term and volatility structures, but also in the calibration to market prices of interest rate derivatives such as swaptions, caps and floors. Although this approach can lead to very reasonable calibration and pricing results, it does not allow for an insightful interpretation of the volatility function because of its purely numerical specification. A pre-specified structure of $\sigma(t,T,f(t,T))$ represents a better way that requires less parameters. Therefore, we follow the approach of *Amin and Morton (1994)* and slightly modify their volatility function by including time-dependence for the volatility parameter σ_0 and ignoring the possibility of including the current level of $f(t,T)$:

$$\sigma(t,T,f(t,T)) = [\sigma_0(t) + \sigma_1 \cdot (T-t)] \cdot \exp[-\kappa \cdot (T-t)]. \tag{4.2}$$

We ignore $f(t,T)$ in equation (4.2) because it can lead to non-reasonable derivative prices when using a non-recombining tree with only a few time steps. This may seem avoidable, for example by increasing the number of steps. However, since we need to rely on a very fast valuation for derivatives in the context of the Monte Carlo simulation outlined in Section 4.4, we exclude $f(t,T)$ from the specification of $\sigma(t,T,f(t,T))$ in order to reduce the number of time steps in the tree.

4.2.2 Discrete HJM Framework

The dynamics of the forward rate $f(t,T)$ are completely defined by equations (4.1) and (4.2) in a continuous-time setting. However, an infinite number of instantaneous forward rates on the one side and time continuity on the other are not workable in practical situations. So we need to discretize the HJM framework in order to make it work. In the following, we describe which steps are necessary for discretization according to *Amin and Morton (1994)*.

First, we discretize the instantaneous forward rates to an observable forward rate curve $f^d(t,t_{j-1},t_j)$ whose forward rates cover an equally spaced and complete time grid with

$t_J = T$ and $t_j = t + j \cdot \frac{T-t}{J}, j \in \mathbb{N}_0$. In this representation, the first forward rate $f^d(t, t, \frac{T-t}{J})$ marks the spot rate and

$$b^d(t, T_b) = \exp\left(-\sum_{t \le t_j \le T_b} f^d(t, t_{j-1}, t_j) \cdot h_j\right). \tag{4.3}$$

defines the zero-bond price at time t with maturity T_b and $h_j = t_j - t_{j-1}$. Note that if T_b does not match the time grid, we linearly interpolate missing forward rates from the model.

The discretization of the instantaneous forward rates is straighforward. However, the discretization of the time continuity entails a pitfall with an important mathematical implication. First, let $\alpha(t, t_i, f^d(t, t_i))$ define the discretized drift term of the stochastic differential equation in (4.1), where $t = t_0 < t_1 < \ldots < t_{i-1} < t_i \le T$ denotes a non-equally-spaced time grid with $I + 1$ points in time and $k_i = t_i - t_{i-1}$. (4.1) then reads

$$\begin{aligned} f^d(t + k_i, t_i) = f^d(t, t_i) + \alpha(t, t_i, f^d(t, t_i))k_i \\ \pm \sigma(t, t_i, f^d(t, t_i))\sqrt{k_i}, \forall i : t_i - t > 0. \end{aligned} \tag{4.4}$$

The up and the down state in (4.4) are both reached with a probability of $\frac{1}{2}$. The problem with this representation is that the drift term $\alpha(t, t_i, f^d(t, t_i))$ is not simply the discrete time version of the drift term in (4.1) because then, zero-bond prices would not be martingales in the model. In Appendix 4.A.1, we derive the following definition for the drift terms in the discrete time setting satisfying a general martingale condition for zero-bonds:

$$\sum_{j=1}^{L} \alpha(t, t_j, f^d(t, t_j))h_j = \frac{1}{k_i} \ln\left(\cosh\left(\sqrt{h_j}\sum_{j=1}^{L} \sigma(t, t_j, f^d(t, t_j))h_j\right)\right). \tag{4.5}$$

The drift terms $\alpha(t, t_j, f(t, t_j)$ can be computed successively in each time step by setting $L = 1, \ldots, J$.

The definition of the time grid t_i allows step sizes that do not necessarily exhibit the same length. Although this feature is not required in the Monte Carlo setup outlined in Section 4.4 because there, only a fixed step size of one week is applied in the simulation, it is very useful in derivatives valuation that would otherwise be more costly. For each simulated path of the term structure we computed prices of derivatives with a non-recombining tree. The time grid of the tree is chosen in accordance with *Amin and*

Morton (1994) who define them to be linearly increasing with the last step having double the size of the first one. With the help of complete induction, it can be shown that the time grid satisfying these conditions is defined by

$$k_i = k_1 \cdot \left(1 + \frac{i-1}{I-1}\right), k_1 = \frac{2T}{3I}. \tag{4.6}$$

The number of steps in the tree is crucial for the precision of computed derivative prices. Precision increases with the number of steps but also involves a more costly computation. *Bühler et al. (1999)* find that seven time steps are sufficient to achieve accurate derivative prices. For this reason, we choose seven steps in the non-recombining tree.

4.2.3 Interest Rate Models

The advantage of the HJM approach lies in its versatility. By giving a concrete specification of equation (4.2), in some cases the approach reduces to well-known interest rate models. Table 4.1 shows the volatility specifications for the interest rate models that we consider for our analysis along with the calibrated parameters. The continuous-time

	$\sigma(t, T, f(t, T))$	σ_0	σ_1	κ
HW	$\sigma_0(t) \cdot \exp[-\kappa \cdot (T-t)]$			0.1000
EV	$\sigma_0 \cdot \exp[-\kappa \cdot (T-t)]$	0.0033		0.0448
HL	σ_0	0.0021		

Table 4.1: HJM Volatility Functions of Interest Rate Models and Calibrated Model Parameters

The table shows the HJM volatility functions of the three interest rate models considered in our analysis: the continuous-time version of the *Ho and Lee (1986)* model (HL) and the exponential *Vasicek (1977)* model (EV) according to *Amin and Morton (1994)* and the *Hull and White (1993)* model (HW). Furthermore, the calibrated model parameters for the risk-neutral measure are presented. The HW model was calibrated to 15 swap rates and 10 swaption prices observed on 9/17/2014 in the German market. From this model, we generated 10 swaption prices to which we calibrated the other two interest rate models. For readability reasons, we do not present the values of $\sigma_0(t)$.

version of the *Ho and Lee (1986)* model (HL) exhibits a constant volatility parameter $\sigma_0 = 0.0021$. In contrast, σ_0 is clearly higher for the exponential *Vasicek (1977)* model (EV) since it is exponentially dampened with parameter $\kappa = 0.0448$. The function $\sigma_0(t)$ of the *Hull and White (1993)* model (HW) was specified according to *Henrard (2012)* as piecewise-constant on succeeding time intervals, where we choose the time intervals to be equidistant with length one year. The specification of the HW volatility function makes it the most flexible interest rate model in the context of this paper. For this reason, the HW model is assumed to be the true market model.

4.2.4 Interest Rate Instruments

For assessing CVA model risk, we consider plain vanilla interest rate instruments that are most common in the markets: zero-bonds, coupon bonds, floaters, swaps, swaptions, caplets and floorlets. Their valuation formulae are outlined in the following.

Zero-Bond The price of a zero-bond at time t with maturity T and notional 1 is expressed by the value of the expected discount factor $D(t, T)$:

$$ZB(t,T) = E_t\left[D(t,T)\right], \tag{4.7}$$

$$D(t,T) = \exp\left(-\int_t^T r(\tau)d\tau\right). \tag{4.8}$$

The expectation operator in (4.7) is evaluated by computing the short rate from the forward rate term structure in each tree node of the discrete HJM framework. In some cases, when the time grid of the tree does not match the time grid of the forward rate structure, $r(t)$ needs to be interpolated.

Coupon Bond Let N_B denote the number of coupon payments with payment dates $t_i = t_1 \ldots, t_{N_B}$. Furthermore, let c_B denote the coupon payment measured in percent. The value of a coupon bond with notional 1 can then be computed according to

$$B(t, t_{N_B}) = E_t\left[c_B \sum_{i=1}^{N_B} D(t,t_i) + D(t, t_{N_B})\right]. \tag{4.9}$$

Floater The cash flows of a floater can be replicated by investing the nominal at the spot rate and reinvesting it at the future spot rate as soon as the maturity of the current spot rate has been reached. Since the future value of the spot rate can be expressed by the forward rate $f^d(t_{i-1}, t_{i-1}, t_i)$, the today's value of the floater reads

$$F(t, t_{N_F}) = E_t\left[\sum_{i=1}^{N_F} f^d(t_{i-1}, t_{i-1}, t_i) \cdot D(t, t_i) + D(t, t_{N_F})\right] \tag{4.10}$$

where N_F denotes the number of floater payments and $t_i = t_1, \ldots, t_{N_F}$ their respective payment dates.

Swap Swaps are standard interest rate derivatives that are used for managing interest rate risk. A common usage is to exchange variable interest rate payments, like from spot rates whose future value is uncertain, against fixed payments until the maturity $t_{N_{fix}}$ of the swap is reached. The value $S(t, t_{N_{fix}})$ of such a fixed-receiver swap at time t is computed from

$$S(t, t_{N_{fix}}) = E_t \left[CF(t, t_{N_{fix}}) \right], \tag{4.11}$$

$$CF(t, t_{N_{fix}}) = c_S \sum_{i=1}^{N_{fix}} D(t, t_i) - \sum_{i=1}^{N_{fl}} f^d(t_{i-1}, t_{i-1}, t_i) \cdot D(t, t_i). \tag{4.12}$$

As can be seen from (4.11) and (4.12), the swap payments can be replicated by a coupon bond long with maturity $t_{N_{fix}}$ and a floater short with maturity $t_{N_{fl}}$. Typically $t_{N_{fix}} = t_{N_{fl}}$ holds but $N_{fix} \neq N_{fl}$.

Swaption An instrument that allows its buyer to enter into a fixed-receiver swap for low interest rates is called fixed-receiver swaption. Its value at time t is given by

$$\text{Swaption}(t, t_1, t_{N_{fix}}) = E_t \left[D(t, t_1) \cdot \max \left[CF(t_1, t_{N_{fix}}), 0 \right] \right], \tag{4.13}$$

where t_1 marks the maturity of the swaption and $t_{N_{fix}}$ the maturity of the underlying fixed-receiver swap with $t < t_1 < t_{N_{fix}}$.

Floorlet Swaps and swaptions can provide full protection against changes in interest rate levels. In some cases, however, it is more purposeful to hedge against certain interest rate levels. Floors are broadly used instruments that insure its buyer against low interest rate levels by paying the difference to a pre-defined threshold. Payments can occur periodically until the maturity of the floor. The instrument that has the same payoff as a floor for one period is called floorlet. Its value is defined by

$$\text{Floorlet}(t, t_2) = E_t \left[D(t, t_2) \cdot \max \left[K - f^d(t_1, t_1, t_2), 0 \right] \right]. \tag{4.14}$$

K marks the strike of the floorlet and t_2 its maturity. The payment of the floorlet is fixed at the beginning of its spot rate period t_1 with $t_1 < t_2$ and the buyer receives the payment $\max \left[K - f^d(t_1, t_1, t_2), 0 \right]$ at t_2.

Caplet The opposite holds for caplets. Since they insure against high interest rate levels, their values are given by equation

$$\text{Caplet}(t, t_2) = E_t \left[D(t, t_2) \cdot \max \left[f^d(t_1, t_1, t_2) - K, 0 \right] \right] \tag{4.15}$$

with the same notations that are used for floorlets.

4.3 Equity Models and Instruments

Black and Scholes (1973) along with *Merton (1973)* (BSM) pioneered the valuation of European equity options. One of their main assumptions lies in the stochastic behavior of stock returns. Although even today their model is highly relevant in many practical situations, the event of the Black Monday 1987 showed that the assumption of a flat volatility of stock returns does not hold for real option markets. Many traders bypassed this problem by computing a volatility surface from observable option prices that can still be used with the BSM formulae. However, the incompleteness of the BSM formulae attracted the attention of researchers. *Heston (1993)* improved the BSM approach by assuming that the equity return volatility is neither static nor deterministic but stochastic. His two factor model is able to capture volatility smiles and skews and therefore, represents the first important extension to BSM in the literature. For this reason, we further denote this model as stochastic volatility model (SV). Another important extension was introduced by *Amin and Ng (1993)*. In their model, the equity return volatility and the short-term interest rate are assumed to be stochastic. We call it the stochastic volatility and stochastic interest rate model (SVSI). *Bakshi et al. (1997)* (BCC) present a framework that is capable of modelling stochastic volatility and stochastic jumps in stock returns as well as stochastic interest rates. Their model can be condensed to one of the three equity models mentioned earlier. They find that a model with stochastic volatility and jumps performs best among all models tested in their analysis and that it is not remarkably improved by adding a stochastic interest rate factor. For this reason, we implement the BCC model with stochastic volatility and jumps (SVJ) and assume it to be the true market model. In the following, we outline the general BCC approach that includes the SV, SVSI and SVJ models.

4.3.1 Continuous BCC Framework

We examine a nondividend-paying stock with price $S(t)$ whose dynamics under the risk-neutral measure are given by

$$\frac{dS(t)}{S(t)} = [r(t) - \lambda \cdot \mu_J]dt + \sqrt{v(t)}dz_1(t) + J(t)dq(t). \qquad (4.16)$$

In (4.16), $r(t)$ denotes the continuous-time riskless short rate, $dz_1(t)$ the increment of a Wiener process, $J(t)$ the percentage jump size of the Poisson jump counter $q(t)$ with yearly jump intensity λ. The time-related properties of q_t are defined by $\mathrm{P}[dq(t) = 1] = \lambda dt$ and $\mathrm{P}[dq(t) = 0] = 1 - \lambda dt$. $v(t)$ marks the volatility of the diffusion part of the stock returns that is assumed to follow a *Cox et al. (1985)* mean-reversion process defined by

$$dv(t) = [\theta_v - \kappa_v \cdot v(t)]dt + \sigma_v\sqrt{v(t)}dz_2(t). \qquad (4.17)$$

θ_v marks the mean-reversion level, κ_v the speed of adjustment and σ_v the volatility of volatility. The Wiener increments $dz_2(t)$ are correlated with $dz_1(t)$ via
$Cov\,[dz_1(t), dz_2(t)] = \rho dt$ with correlation parameter ρ. The dependence between both Wiener processes satisfies the common observation that the volatility of equities increases when their prices drop. Therefore, as we present later, the correlation parameter ρ is typically smaller than zero.
The dynamics of the riskless short rate also follow a *Cox et al. (1985)* process with

$$dr(t) = [\theta_r - \kappa_r \cdot r(t)]dt + \sigma_r\sqrt{r(t)}dz_3(t), \qquad (4.18)$$

where θ_r marks the mean-reversion level, κ_r the speed of reversion and σ_r the volatility. The Wiener increments $dz_3(t)$ are assumed to be independent of other stochastic variables.
The percentage jump size $J(t)$ is i.i.d. over time following the lognormal distribution

$$\ln[1 + J(t)] \sim \mathcal{N}\left(\ln[1 + \mu_J] - \frac{1}{2}\sigma_J^2, \sigma_J^2\right) \qquad (4.19)$$

with mean parameter μ_J and volatility parameter σ_J.
The parameter values for the three models are presented in Table 4.2. Analogously to the calibration of the interest rate models, we choose the SVJ model as reference model because it exhibits the most important stochastic properties. The parameters related to

	$S(0)$	$v(0)$	θ_v	κ_v	σ_v	ρ	$r(0)$
SVJ	100	0.040	0.040	2.030	0.380	−0.570	0.020
SVSI	100	0.050	0.041	0.665	0.524	−0.672	0.001
SV	100	0.040	0.039	1.226	0.362	−0.614	0.016
	λ	μ_J	σ_J	θ_r	κ_r	σ_r	
SVJ	0.590	−0.050	0.070				
SVSI				0.005	0.513	0.026	
SV							

Table 4.2: Calibrated BCC Model Parameters

The table shows the model parameters for the SV, SVSI and SVJ models. The parameters of the SVJ model are retrieved from *Bakshi et al. (1997)*. In line with *Branger et al. (2012)*, we computed 25 call prices from the SVJ model to which we calibrated the SVSI and the SV model.

the jump component of the SVJ model imply an average jump size of nearly -5% with a frequency of $\lambda = 0.590$ jumps per year. In other words, the equity market is assumed to react negatively on rare events as it can be observed for real markets in turmoil periods. Furthermore, the correlation between the stock price and the volatility increments is clearly below zero with a value of $\rho = -0.570$. Again, this parameter setting is in line with real markets where volatility tends to increase when stock prices drop but not when stock prices rise. We generate call prices from the SVJ model to which the other two models are calibrated afterwards. The calibrated correlation $\rho = -0.614$ of the SV model is clearly lower than for the SVJ model because it compensates the missing jump component by emphasizing downsize risk. The reversion speed κ_v is also clearly lower implying that the future volatility has a larger volatiltiy itself with more up- and downside risk. This also holds for the SVSI model. The correlation parameter ρ of the SVSI model takes the lowest value among the considered models. This can be attributed to the fact that the stochastic interest rate feature leads to overall higher call prices that need to be compensated by a higher downside risk to match the prices generated by the SVJ model.

4.3.2 Equity Instruments

Call In general, the fair price of a European call option on the stock price $S(t)$ is given by

$$C(t, t_C) = E_t \left[D(t, t + t_C) \cdot \max[S_{t_C} - K, 0] \right], \tag{4.20}$$

where t_C marks the time to maturity of the call in the point of time t and K its strike price. It can be shown that under the assumed BCC framework, the semi-analytical solution to this equation is represented by

$$
\begin{aligned}
C(t,t_C) = S(t) \cdot \Pi_1(t, t_C, S(t), r(t), v(t)) \\
- K \cdot D(t, t + t_C) \cdot \Pi_2(t, t_C, S(t), r(t), v(t)).
\end{aligned}
\tag{4.21}
$$

The functions $\Pi_j, j = 1, 2$ are defined by

$$
\begin{aligned}
&\Pi_j(t, t_C, S(t), r(t), v(t)) \\
&= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re \left[\frac{\exp(-i\phi \ln[K]) f_j(t, t_C, S(t), r(t), v(t), \phi)}{i\phi} \right] d\phi,
\end{aligned}
\tag{4.22}
$$

where f_1 and f_2 are complex exponential functions. The call price (4.21) is easily obtained with the help of a numeric integrator and the price of the corresponding put from the put-call parity.

4.4 Monte Carlo Simulation and Results

The central question of this paper is whether model choice has a significant impact on CVA. Hence, we investigate the model dependency of equity and interest rate instruments that are most commonly traded in the markets. The first subsection outlines the Monte Carlo setup of our analysis, the specifics of the investigated instruments and a definition of CVA. In the second section, we present the distributional properties for the selected instruments in dependence of model choice and carve out differences they can cause in CVA values.

4.4.1 Monte Carlo Setup and CVA Definition

In order to compute the CVA, one needs to know the distributions of derivatives prices in the future. Since the associated distributional properties are highly complex, it is suitable to apply Monte Carlo (MC) methods (*De Prisco and Rosen (2005)*). The paths of the risk factors are simulated in the first step and the price paths of derivatives are computed accordingly. Typically, the distribution of future derivatives prices is retrieved by carrying out the simulation of the risk factors under the physical measure. This way, the paths are supposed to follow observable time series of e.g. interest rates, equity indices, commodity prices and counterparty default risk. Afterwards, derivatives prices are computed under the risk-neutral measure. But as *Gregory (2009)* and *Ghamami and Zhang (2014)* point out, the CVA reflects the market price of counterparty credit risk and all related simulations and computations should be carried out under the risk-neutral measure. Therefore, we choose to simulate risk factors under the risk-neutral measure as well.

Another important aspect that is related to simulation deals with *roll-off risk* which is the risk that important distributional properties, such as the price at a payment date, are not captured by a MC method if its step size is too large. For instance, a payment date of a floater could be left out from simulation which could lead to an overly smoothed exposure profile. Therefore, the ideal step size would be one day which is highly costly to evaluate. To make our simulation more efficient, we choose a step size of one week and the payment dates to match the simulation dates. This way, we fully eliminate roll-off risk.

The MC simulation is performed under the following conditions: the number of paths is chosen to be $J = 10,000$ which leads to a maximum distance of 5 bps to the 95% confidence bounds of the CVA in our MC application. As can be seen from Tables 4.3 and 4.4, this is sufficient to compare the investigated CVA values. The maturity of all derivative contracts, if not stated otherwise, is chosen to be $T = 5$ years where valuation starts at $t = 0$. Since we simulate on a weekly basis, the total number of steps per path is 260. The constant yearly default intensity of the counterparty is $\lambda = 3\%$.

The specifics of the examined derivative contracts are as follows. We simulate the paths for out-of-the-money (OTM), at-the-money (ATM) and in-the-money (ITM) European call and put options with strike prices $K_{ITM} = 80$, $K_{ATM} = 100$ and $K_{OTM} = 120$ for the calls and in reverse order for the puts. The payments of all interest rate instruments are chosen to occur on a yearly basis if not stated otherwise. The coupon is that of a five year par bond priced in $t = 0$. The payments of the caplet and the floorlet are fixed in $t_1 = 4$ and are due in $t_2 = 5$ and the strike K is set to the coupon of the par bond for the caplet and $K = 2.25\%$ for the floorlet. The maturity of the swap, which is fairly priced in $t = 0$, is set to $t_{N_{fix}} = 10$. This way, the exposure profile of the swap can be studied more easily since it is much clearer than for a swap expiring in five years. The swaption expires at $t_1 = 5$ and has a forward swap rate of $c_S = 2.25\%$ with $t_1 = 5$ and $t_{N_{fix}} = 10$ as underlying. The notional of all interest rate instruments or their respective underlying, where applicable, is set to 100.

Potential Future Exposure Let ω_j denote the j^{th} simulated path with $j = 1, \dots, J$, where J is the corresponding number of paths. Furthermore, let $t_0 < t_1 < \dots < t_K$

denote the MC time grid with K being the number of time steps. Since a financial entity can only experience a loss if its derivatives position i to a counterparty h has positive value, the potential future exposure of that position is defined according to *De Prisco and Rosen (2005)* by

$$PFE_i^h(\omega_j, t_k) = \max\left[V_i^h(\omega_j, t_k), 0\right]. \tag{4.23}$$

$V_i^h(\omega_j, t_k)$ is the mark-to-future value of derivative i at path ω_j and time t_k. The set $\overline{PFE_i^h} = \{PFE_i^h(\omega_j, t_k) | j = 1, \ldots, J, k = 0, \ldots, K\}$ is called PFE profile and contains all possible future exposures of derivative i related to counterparty h. Together with the default risk properties of h, it provides the basis for the CVA computations.

Expected Exposure An important CVA measure that is derived from $\overline{PFE_i^h}$ is called expected exposure and is computed in line with *De Prisco and Rosen (2005)* as

$$EE_i^h(t_k) = \frac{1}{J} \sum_{j=1}^{J} PFE_i^h(\omega_j, t_k). \tag{4.24}$$

$EE_i^h(t_k)$ quantifies the mean of the cross section of paths at time t_k from the perspective of time t_0. Its values provide a graphical illustration that can be easily interpreted. For example, if $EE_i^h(t_k)$ is low, then the expected amount of money that can be lost in a derivatives transaction due to default is low, too. Therefore, it indicates the amount of possible losses a financial entity experiences if its counterparty h defaults at t_k.

Credit Valuation Adjustment (CVA) An important building block in the Basel II/III accords is the credit valuation adjustment whose value represents the expected default loss of a derivative i with counterparty h. Its value is used for adjusting the value of a default-free derivative for the default risk associated with counterparty h. In other words, if the CVA is subtracted from the default-free price of a derivative i then one receives its defaultable price with respect to counterparty h. There are several definitions of the CVA in the literature. However, in the following, we propose a new definition that is based on the framework of *Lando (1998)* for reduced-form default models. The advantage of this definition lies in its relationship to valuation formulae that are commonly used in the literature for pricing default risky instruments, such as credit default swaps or defaultable bonds, and therefore allows for the applicability of established valuation methodologies.

Let λ_t^h denote the intensity of a Cox process that can only jump once. If a jump occurs, then h defaults which triggers a loss in the outstanding derivatives notional if it is positive to the financial entity. Otherwise, no losses can occur. Then, the CVA is computed as the expected discounted default losses across the simulated paths of $\overline{PFE_i^h}$:

$$CVA_i^h(t,t_k) = E_t\left[\int\limits_{\xi=t}^{t_k} D(t,\xi)\lambda_\xi^h \exp\left(-\int\limits_{\psi=t}^{\xi}\lambda_\psi^h d\psi\right) PFE_i^h(\omega_j,\xi)d\xi\right], \tag{4.25}$$

where $\lambda_\xi^h \exp(-\int_{\psi=t}^{\xi}\lambda_\psi^h d\psi)$ marks the probability that counterparty h defaults exactly at time ξ and $D(t,\xi)$ the discount factor for the time period from t until ξ.

4.4.2 Analysis of Potential Future Exposures

Expected Exposure and Distribution Properties The distributions from the MC simulation are rich of information which need to be condensed in order to draw meaningful conclusions from them. Their properties are reflected by the mean[8], the standard deviation and the quantiles to a satisfactory extent. The distribution properties of the at-the-money put are plotted in Figure 4.1. The figure reveals that the properties are highly model-dependent. The quantile $Q_{90\%}(t)$ shows the strongest increase over time for the SVSI model, followed by the SVJ and the SV model with clearly lower values. The upside risk of the put is more pronounced for the SVSI model since stochastic interest rates may lead to a higher NPV of the strike price at the valuation date. This effect overcompensates the risk of downside jumps that are priced with the SVJ model. However, as can be seen for the other measures, the SVJ model generates the highest values followed by the SVSI and the SV model. For the great majority of simulation paths, the jump feature of the SVJ model leads to higher put prices, since its negative jump size leads to a higher ad hoc moneyness of the put option. Therefore, the lowest $EE(t)$ is associated with the SV model.

The future distributions of the coupon bond are outlined in Figure 4.2. The distributions that are generated from all three models exhibit spikes at the coupon payment dates since for CVA analysis, the bond must be valued according to the clean price convention. The pull-to-par effect is also a characteristic that is retrieved from all four models. But the distributions from the HW and the EV model have a higher variance than from the HL model which makes upside and downside outliers more

[8] In our analysis, the mean is equivalent to $EE_i^h(t_k)$.

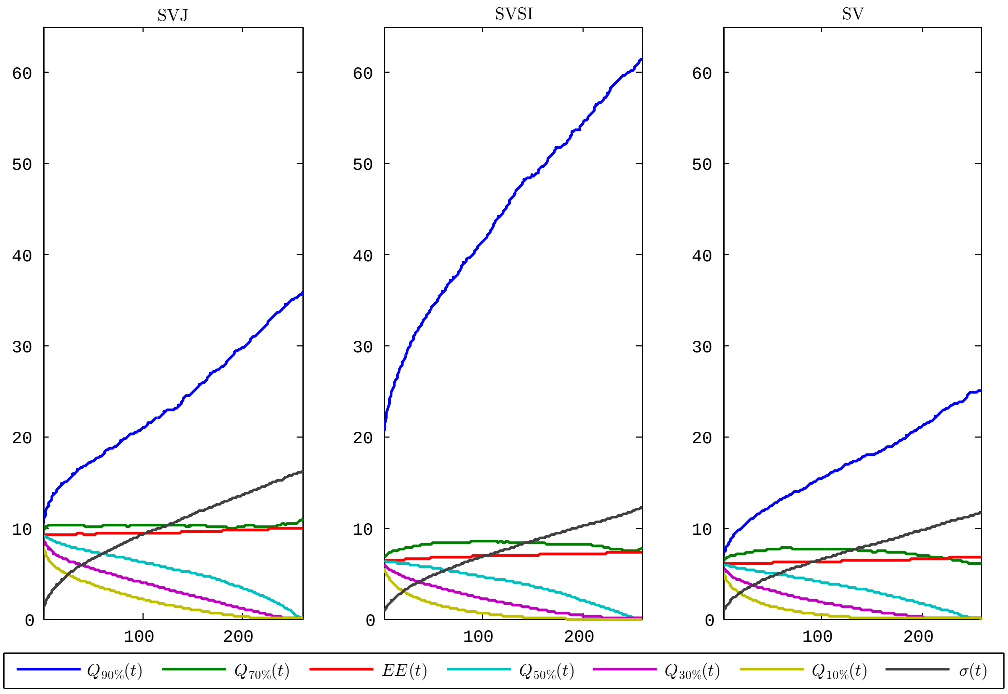

Figure 4.1: Expected Exposure and Distribution Properties of At-the-Money Put

The graphs show $EE(t)$, the distribution quantiles $Q_{90\%}(t)$, $Q_{70\%}(t)$, $Q_{50\%}(t)$, $Q_{30\%}(t)$, $Q_{10\%}(t)$ and the standard deviation $\sigma(t)$ of the cross section of Monte Carlo paths of the at-the-money put option for all t. Paths are simulated on a weekly basis from the SVJ, SVSI and SV model over a period of five years. The simulation of risk factors and the computation of derivative prices are carried out under the risk-neutral measure.

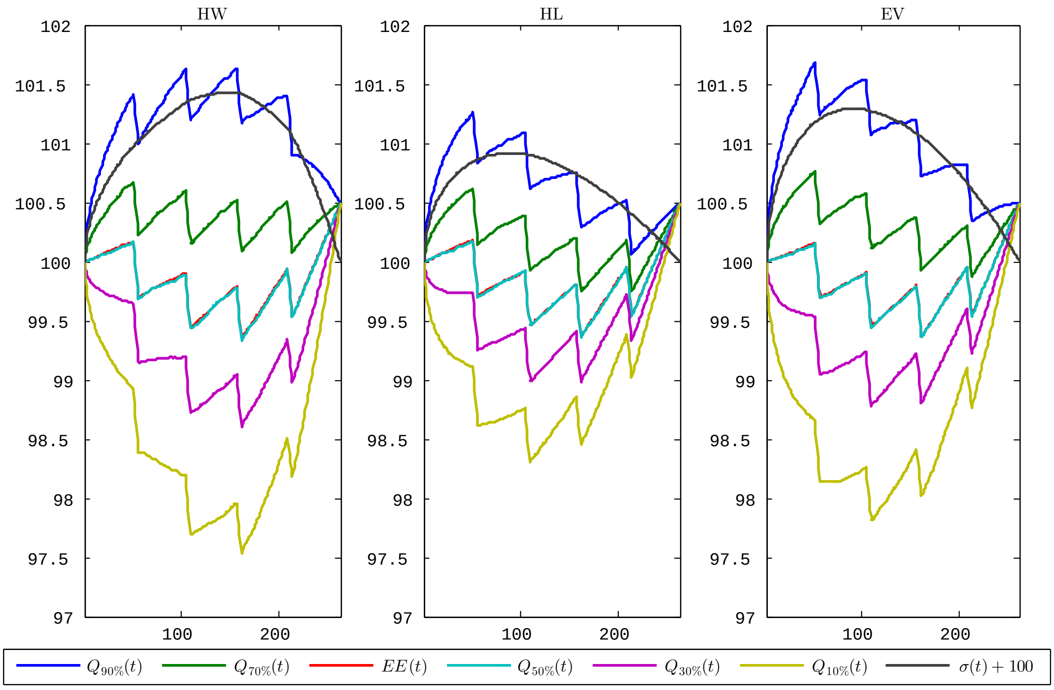

Figure 4.2: Expected Exposure and Distribution Properties of Coupon Bond

The graphs show $EE(t)$, the distribution quantiles $Q_{90\%}(t)$, $Q_{70\%}(t)$, $Q_{50\%}(t)$, $Q_{30\%}(t)$, $Q_{10\%}(t)$ and the standard deviation $\sigma(t)$ of the cross section of Monte Carlo paths of the coupon bond for all t. Paths are simulated on a weekly basis from the HW, HL and EV model over a period of five years. The simulation of risk factors and the computation of derivative prices are carried out under the risk-neutral measure.

likely. Furthermore, the HW model shows the highest variance for observations after 200 weeks because its piecewise-constant volatilities have increasing values over time. The pull-to-par effect predominates after 60 weeks for the other three models and is not compensated by their volatility specifications.
Similar conclusions can be drawn for the floorlet whose distribution properties are plotted in Figure 4.3. For the HL model, the floorlet expires out-of-the-money in the majority of cases. For the HW and the EV model however, it can take high values at expiration. The fact that the standard deviation $\sigma(t)$ at maturity is higher than $Q_{90\%}(t)$ resides in the high values of outliers. The sample shows that model choice has a strong impact on potential future exposures and that in extreme cases, contracts may mostly expire out-of-the-money for a specific model whereas it can take comparably high values for others. Further figures that are related to the other instruments can be found in the appendix.

CVA In post-trade processing, it is crucial to compute the defaultable value of a derivative because it is processed in further computations which define the amount of economic capital that is backed behind a portfolio of (derivative) securities. This defaultable value is obtained by subtracting the CVA as defined in equation (4.25) from the default-free market price of a derivative. Consequently, the higher the CVA, the lower the value of the derivative. But there is an important catch. The CVA is not observable on the market and consequently, market models need to be employed for its computation. This circumstance is important in practical applications — by choosing a specific market model, a financial institution may receive a lower CVA than in other cases with all important implications towards economic capital and risk management. Table 4.3 presents the CVA for call and put options in dependence of the market model. The SVJ and the SV model produce the lowest CVA values for all call options. When comparing the results of the SV and the SVJ model, the jump feature of the latter seems to have a negligible impact on the valuation of call options. However, the reason for this lies in the calibrated parameter values of the SV model: its correlation parameter is clearly lower than the one of the SVJ model. The same holds for κ_v implying a higher dispersion of future volatility levels. Both parameter values compensate the jumps occuring with an average negative jump size in the SVJ model. The SVSI model produces the highest CVA values because of stochastic interest rates that lead to lower discounted values of strike prices compared to the other two models. The opposite holds for the put options: the SVSI model produces the lowest CVA values because of lower discounted strike values. The values of the SV model are slightly above the ones

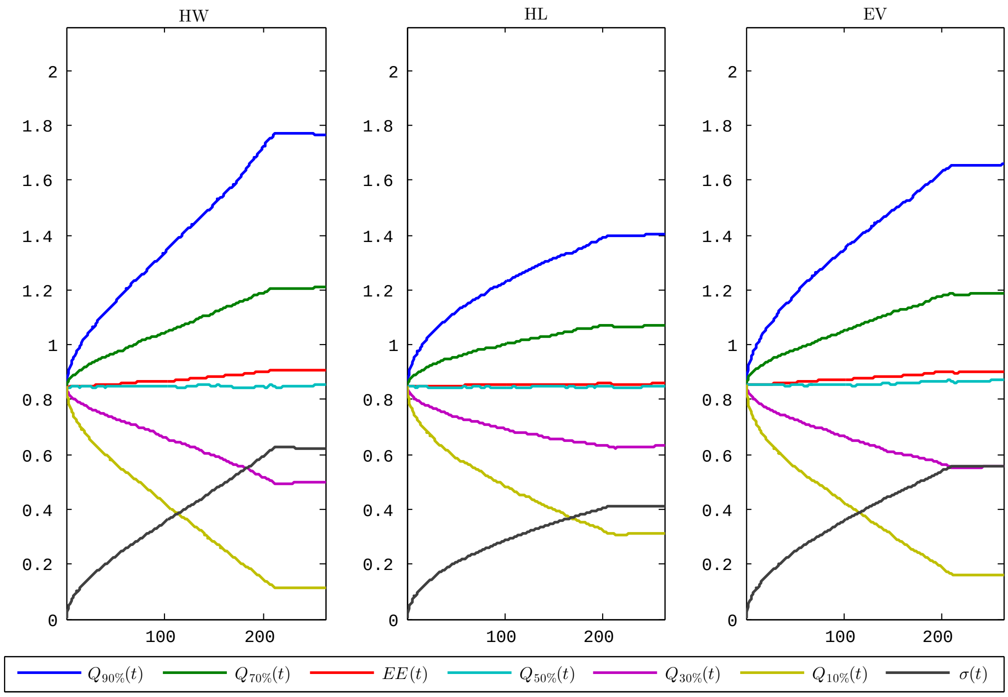

Figure 4.3: Expected Exposure and Distribution Properties of Floorlet

The graphs show $EE(t)$, the distribution quantiles $Q_{90\%}(t)$, $Q_{70\%}(t)$, $Q_{50\%}(t)$, $Q_{30\%}(t)$, $Q_{10\%}(t)$ and the standard deviation $\sigma(t)$ of the cross section of Monte Carlo paths of the floorlet for all t. Paths are simulated on a weekly basis from the HW, HL and EV model over a period of five years. The simulation of risk factors and the computation of derivative prices are carried out under the risk-neutral measure.

	Call			Put		
	OTM	ATM	ITM	OTM	ATM	ITM
SVJ	134.773	255.426	428.943	47.434	125.164	255.757
	(2.4719)	(3.5038)	(4.4074)	(1.2641)	(2.2572)	(3.2990)
SVSI	167.723	293.858	466.544	42.177	109.691	223.802
	(2.9460)	(3.8826)	(4.6770)	(1.2209)	(2.2260)	(3.3282)
SV	136.721	254.456	422.770	51.866	130.835	260.385
	(2.5811)	(3.6232)	(4.5301)	(1.2372)	(2.2397)	(3.3001)

Table 4.3: CVA of Equity Options

The table presents the CVA of out-of-the-money (OTM), at-the-money (ATM) and in-the-money (ITM) call and put options as valued with the SV, SVSI and SVJ model. The distance to the 95% confidence bounds of the simulation are reported in parantheses. The confidence bounds were retrieved under the normality assumption and give a rough indication of the MC precision. The constant yearly default intensity of the counterparty is $\lambda = 3\%$. Values are reported in basis points.

of the SVJ model which can be attributed to the values of ρ and κ_v. To summarize the findings, the CVA produced from the SV model is close to the CVA of the SVJ model. However, the CVA values of the SVSI model clearly deviate from the results of the SVJ model. Therefore, we conclude that the CVA can be highly model-dependent for equity options.

The CVA values for interest rate instruments are shown in Table 4.4. Surprisingly, the CVA values show no mentionable model dependence across all instruments. This is in clear contrast to the model dependence of the distribution properties which show a high variation across models. The reason lies in the high dependence of the CVA on the expected exposure $EE_i^h(t_k)$ in our simulation. As we assumed no correlation between the default intensity λ and the market risk factor, the default characteristics are the same for all three model setups. And since $EE_i^h(t_k)$ is not related to tail risk, the differences between CVA values are negligible.

4.5 Conclusion

The usage of CVA in the financial industry has strongly increased in the past years because of the requirements defined in the Basel II/III accords. The main benefit of CVA should lie in its stabilizing effect on the financial system since the default of counterparties may cause less severe losses in financial portfolios. However, we find

	Bond	Caplet	Floater	Floorlet	Swap	Swaption
HW	1392.723	12.596	4.872	0.044	0.189	0.109
	(0.3417)	(0.0857)	(0.1024)	(0.0915)	(0.5242)	(0.3220)
HL	1392.971	12.225	4.873	0.042	0.158	0.067
	(0.2485)	(0.0678)	(0.0763)	(0.0713)	(0.3980)	(0.2365)
EV	1392.783	12.502	4.873	0.044	0.170	0.086
	(0.3566)	(0.0852)	(0.1110)	(0.0899)	(0.4866)	(0.2797)

Table 4.4: CVA of Interest Rate Instruments

The table presents the CVA of the coupon bond, caplet, floorlet, floater, swap and swaption as valued with the HW, EV and HL model. The distance to the 95% confidence bounds of the simulation are reported in parantheses. The confidence bounds were retrieved under the normality assumption and give a rough indication of the MC precision. The constant yearly default intensity of the counterparty is $\lambda = 3\%$. Values are reported in basis points.

that the computation of the CVA of equity options can be subject to high model risk with important consequences with respect to the amount of economic capital.

Appendix 4.A Proofs

4.A.1 Proof of equation (4.5)

In the following, we derive equation (4.5) in accordance with *Amin and Morton (1994)*.

Proof: Since the price of a zero-bond according to equation (4.7) is required to meet the martingale condition $E_t[D(t+k_i,T)D(t,t+k_i)] = \exp[-\int_t^T f(t,\tau)d\tau]$, in the discrete time setting with instantaneous forward rates $f(t,T)$

$$\begin{aligned}
&\exp\left[-\int_t^T f(t,\tau)d\tau\right] \qquad (4.26)\\
&= \frac{1}{2}\exp\left[-\int_t^T f(t,\tau) + k_i\alpha(t,\tau,f(t,\tau)) + \sqrt{k_i}\sigma(t,\tau,f(t,\tau))d\tau\right]\\
&+ \frac{1}{2}\exp\left[-\int_t^T f(t,\tau) + k_i\alpha(t,\tau,f(t,\tau)) - \sqrt{k_i}\sigma(t,\tau,f(t,\tau))d\tau\right]
\end{aligned}$$

needs to hold, which can be simplified to the condition

$$\begin{aligned}
1 &= \frac{1}{2}\exp\left[-\left(k_i\int_t^T \alpha(t,\tau,f(t,\tau))d\tau + \sqrt{k_i}\int_t^T \sigma(t,\tau,f(t,\tau))d\tau\right)\right] \qquad (4.27)\\
&+ \frac{1}{2}\exp\left[-\left(k_i\int_t^T \alpha(t,\tau,f(t,\tau))d\tau - \sqrt{k_i}\int_t^T \sigma(t,\tau,f(t,\tau))d\tau\right)\right].
\end{aligned}$$

After applying some algebra, equation (4.27) can be reformulated as

$$\exp\left[-k_i\int_t^T \alpha(t,\tau,f(t,\tau))d\tau\right] = \cosh^{-1}\left[\sqrt{k_i}\int_t^T \sigma(t,\tau,f(t,\tau))d\tau\right]. \qquad (4.28)$$

By assuming discrete forward rates $f^d(t,t_{j-1},t_j)$, we arrive at equation (4.5). □

Appendix 4.B Figures

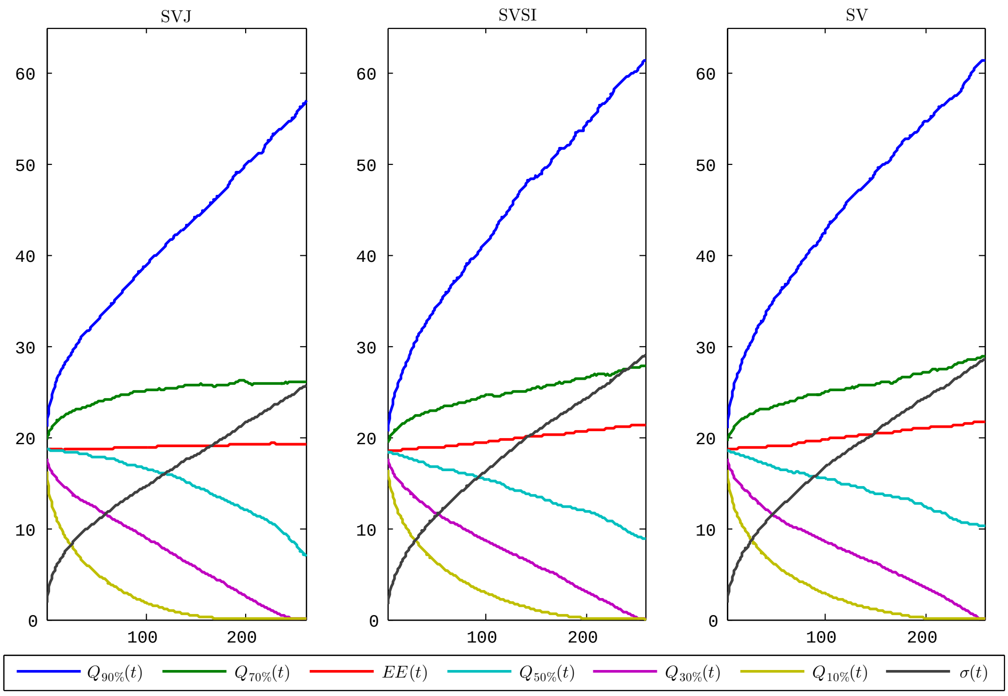

Figure 4.4: Expected Exposure and Distribution Properties of At-the-Money Call

The graphs show $EE(t)$, the distribution quantiles $Q_{90\%}(t)$, $Q_{70\%}(t)$, $Q_{50\%}(t)$, $Q_{30\%}(t)$, $Q_{10\%}(t)$ and the standard deviation $\sigma(t)$ of the cross section of Monte Carlo paths of the at-the-money call option for all t. Paths are simulated on a weekly basis from the SVJ, SVSI and SV model over a period of five years. The simulation of risk factors and the computation of derivative prices are carried out under the risk-neutral measure.

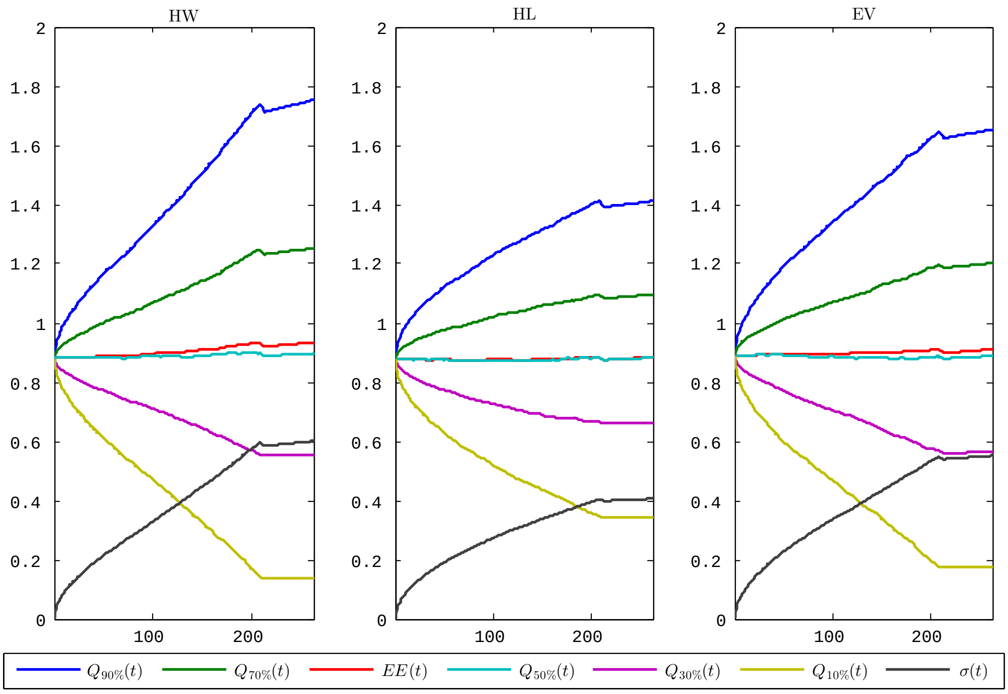

Figure 4.5: Expected Exposure and Distribution Properties of Caplet

The graphs show $EE(t)$, the distribution quantiles $Q_{90\%}(t)$, $Q_{70\%}(t)$, $Q_{50\%}(t)$, $Q_{30\%}(t)$, $Q_{10\%}(t)$ and the standard deviation $\sigma(t)$ of the cross section of Monte Carlo paths of the caplet for all t. Paths are simulated on a weekly basis from the HW, HL and EV model over a period of five years. The simulation of risk factors and the computation of derivative prices are carried out under the risk-neutral measure.

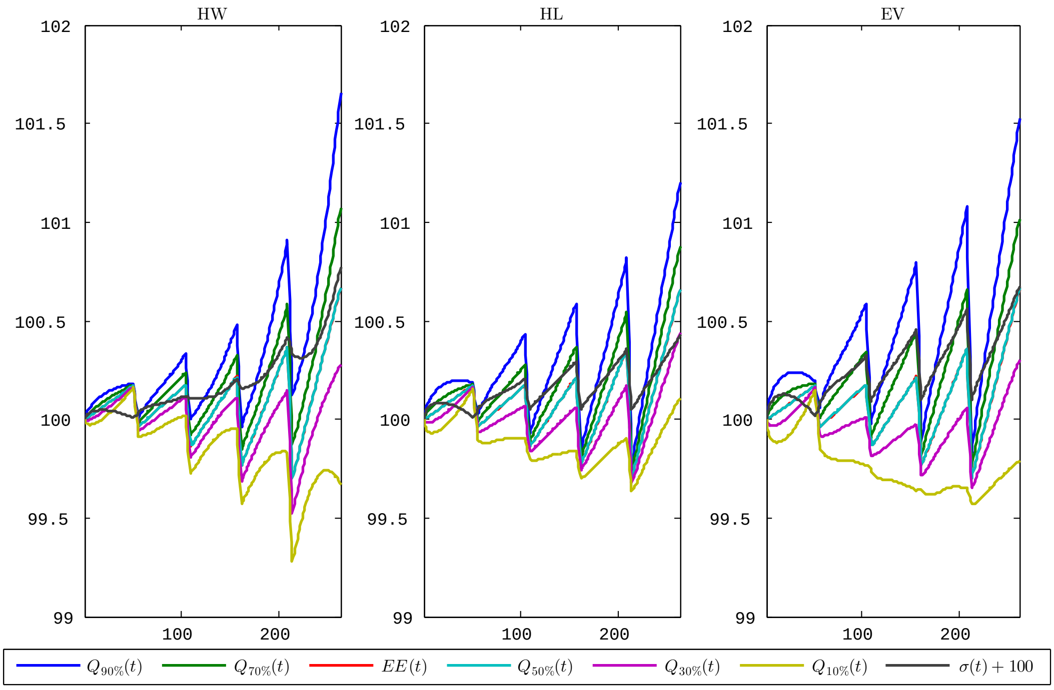

Figure 4.6: Expected Exposure and Distribution Properties of Floater

The graphs show $EE(t)$, the distribution quantiles $Q_{90\%}(t)$, $Q_{70\%}(t)$, $Q_{50\%}(t)$, $Q_{30\%}(t)$, $Q_{10\%}(t)$ and the standard deviation $\sigma(t)$ of the cross section of Monte Carlo paths of the floater for all t. Paths are simulated on a weekly basis from the HW, HL and EV model over a period of five years. The simulation of risk factors and the computation of derivative prices are carried out under the risk-neutral measure.

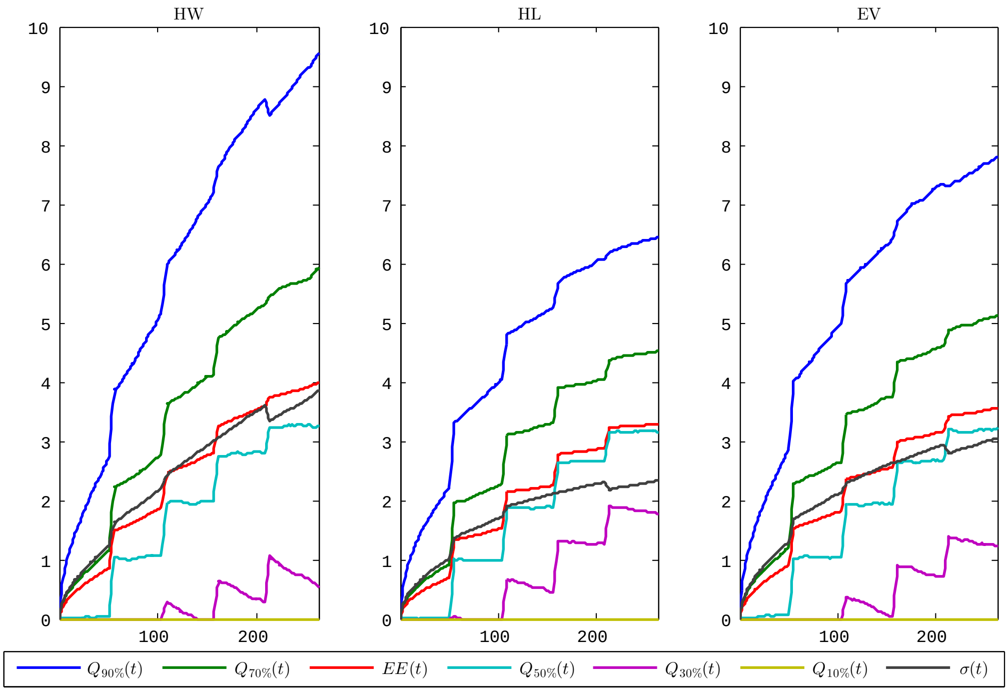

Figure 4.7: Expected Exposure and Distribution Properties of Swap

The graphs show $EE(t)$, the distribution quantiles $Q_{90\%}(t)$, $Q_{70\%}(t)$, $Q_{50\%}(t)$, $Q_{30\%}(t)$, $Q_{10\%}(t)$ and the standard deviation $\sigma(t)$ of the cross section of Monte Carlo paths of the swap for all t. Paths are simulated on a weekly basis from the HW, HL and EV model over a period of five years. The simulation of risk factors and the computation of derivative prices are carried out under the risk-neutral measure.

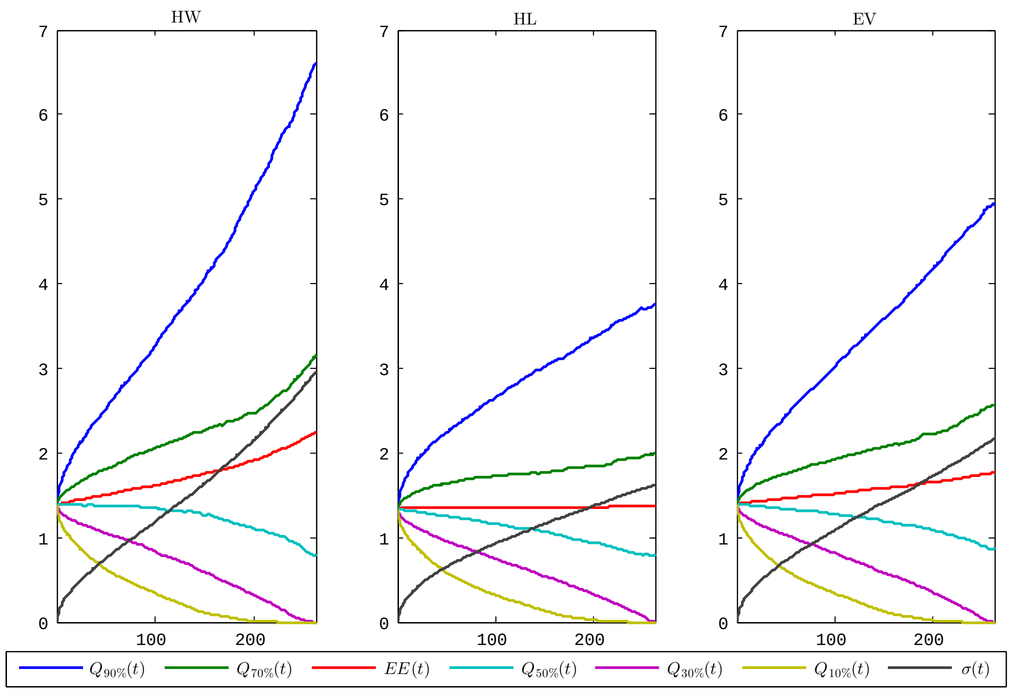

Figure 4.8: Expected Exposure and Distribution Properties of Swaption

The graphs show $EE(t)$, the distribution quantiles $Q_{90\%}(t)$, $Q_{70\%}(t)$, $Q_{50\%}(t)$, $Q_{30\%}(t)$, $Q_{10\%}(t)$ and the standard deviation $\sigma(t)$ of the cross section of Monte Carlo paths of the swaption for all t. Paths are simulated on a weekly basis from the HW, HL and EV model over a period of five years. The simulation of risk factors and the computation of derivative prices are carried out under the risk-neutral measure.

Chapter 5

Summary and Conclusion

This thesis contains three essays on empirical and theoretical finance that analyze default risky instruments.

In Chapter 2, we devote ourselves to the question where correlated default factors can be detected in the underlying CDS market. Hence, we investigate a very large CDX data set that comprises daily index CDS, CDO tranche and CDS data from the years 2005 - 2012. We divide this data set into a pre-crisis, a crisis and a post-crisis period and draw the following conclusions: before the financial crisis, more than 80% of the observed default risk was caused by the single default factor and only 20% could be attributed to correlated default factors. This picture changed dramatically during the crisis, when correlated default factors accounted for more than 80% of default risk whereas the remaining 20% are due to the single default factor. After the crisis, the fraction of the correlated default factors was still above 50%. Accordingly, correlated default factors played a negligible role in the pricing of CDS before the crisis. However, during and after the crisis, correlated default factors were highly immanent to CDS on firms with a high creditworthiness. Thus, we can draw the conclusion that especially firms with a low default probability are likely to default in groups during a financial crisis or in catastrophic scenarios. In contrast, firms with a high default probability are less affected by correlated default factors since often, they are already in financial distress that is independent from macroeconomic factors and the state of other firms.

The pricing of CCDS contracts is analyzed in Chapter 3. CCDS insure against default losses in derivatives transactions. As a prominent example, we consider CCDS that have IR swaps as an underlying. We set up our model in the context of a structural model according to *Merton (1974)* that accounts for correlated asset and interest rate factors. Consequently, the counterparty of the model may not only default because of asset risk but also because of unfavorable interest rate movements that can worsen refi-

nancing conditions and lead to a default. We derive semi-analytical valuation formulae for CCDS, CDS, swaptions and zero-bonds that can easily be evaluated numerically. Furthermore, we derive a model-free approximate formula for the valuation of CCDS that only relies on observable market prices of a CDS, a swaption and a zero-bond. Subsequently, we carry out a comparative-static analysis yielding the following important results: CCDS fundamentally differ from CDS in their pricing behavior. An increasing initial short rate and interest rate volatility level lead to higher CDS prices because of a higher probability of high interest rates that are more likely to cause a default of the counterparty. However, the CCDS price declines for an increasing interest rate volatility because the underlying swaption is out-of-the-money. Thus, the swap has values close to zero upon default. Furthermore, we find that our approximate formula always overestimates the correct CCDS price which makes it a useful tool for time-critical practical situations.
An alternative way of accounting for the default risk of a complex financial instrument is provided by subtracting the CVA from its default-free value. We analyze the model dependence of CVA in Chapter 4. Hence, we implement three equity and three interest rate models that are based on the *Bakshi et al. (1997)* and the *Heath et al. (1992)* framework respectively. We find that in general, CVA-related computations are highly model dependent for equity instruments. For the interest rate models though, the CVA is far less model dependent than in the equity case. These findings have important implications for practitioners and regulators alike: by choosing a market model with suitable properties for CVA computation, financial institutions are able to overvalue their positions backing them with less economic capital than truly necessary. Hence, regulators are well advised to take a close look to the applied market models in order to assure the stability of an important cornerstone of the financial system.

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