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Reconstruction of Heron's Formulas for Calculating the Volume of Vessels

Abstract: Heron's formulas for "pithoid" and "spheroid" pithos, as formulas of volume of the truncated paraboloid and $\frac{3}{4}$ of ellipsoid of revolution, have been reconstructed in the present study. Research has proven that the ovoid body of Roman narrow-necked light-clay D-type amphorae is a special case of "pithoid". With the help of geometrical approximation by conic sections of the profile of these amphorae, it is shown that in antiquity there really were amphorae projected on the basis of conic sections. Their volumes could be calculated with the help of ancient mathematical formulas. The results obtained testify to the influence of antique mathematics on morphology and technology of amphorae design. Formulas for the volume of simple solids of revolution defined by conic sections can be applied in modern calculations of volumes of some types of antique amphorae and in reconstructing their profiles.

Introduction

Studies of the trade history of the ancient states situated on the northern shore of the Black Sea are a major task of the socio-economic history of the ancient world. One of the leading directions of the ancient trade appears to be trade in commodities being transported in sharp-bottomed amphorae. Therefore, assessment of the volume is closely connected with the quantitative characteristics of the proper amphorae. Vessels manufactured in different places differ in their morphological and technical attributes. Their analysis allows origin, places of manufacture, and chronology of amphorae to be established in many cases. The size of a vessel is the most general of the attributes forming classes for Black Sea amphorae. Thus the volume of amphorae is the most indicative attribute (Brašinskij 1984, 73; Monakhov 1992, 166) on which the vessel's height directly depends. Metric and volumetric standards existed for amphorae in antiquity. For example, a little about amphorae design is known from Athenaeus (XI, 784), where an amphora manufacturing standard is mentioned in diadoh Kassandr's order for exporting Menda wine from Kassandrii. An amphora standard has been made by the outstanding sculptor Lisipp (Grace 1949, 178). It is the certificate of creation of the metrological standard of the amphorae ordered by the government which has been concerned in ordering trading operations and in the control over them. To set the capacity of an amphora, even ancient standards were able to use geometrical designs and mathematical calculations. Mathematical formulas would be the key to the geometry of a vessel in such cases. Archimedes (3rd century BC) had already found a method for calculating the volume of an arbitrary vessel, if it was a solid of revolution. However, the method of exhaustion estimation of the upper and lower bounds when carrying out the integration processes is a laborious method. The application of formulas was easier and more convenient.

Amphorae are solids of revolution. Formulas for the calculation of the volume of some simple solids of revolution were well-known in antiquity. Ancient Egyptians could calculate the volume of a sphere correctly, and Democritus (5th–4th centuries BC) was the first to calculate the volume of a cone correctly (van der Varden 1959, 44–45; 192). However, until now, it was not known that there was a formula for calculating the volume of an ovoid amphorae body during antiquity. Formulas by Heron of Alexandria (1st century AD) for the volume of "pithoid" and "spheroid" pithos are known (Hultsch 1864, 202–203). However, Heron did not specify the meaning of some terms in these formulas. Therefore, the geometry of these vessels and the exact meaning of the formulas have remained unclear. We have assumed that they refer to vessels with an ovoid body.

The main goal of our research was mathematically reconstructing Heron's formulas and finding geometrical conformity for "pithoid" and for "spheroid" pithos. Another important aspect of our research was the proof that geometrical design and mathematical calculations were used to create amphorae standards of a definite capacity in antiquity.

Fig. 1.

A mathematical model was constructed and volumes of 3rd-century AD Roman light-clay narrow-necked amphorae of D-type from Tanais have been used to test the reconstructed formula for the "pithoid" pithos and to provide preliminary proof of amphorae design. The standard size of the amphorae and a large quantity of intact specimens allowed a statistically reliable investigation to be carried out *(Fig. 1)*.

Our method consisted of geometric approximation of the inner contour of an amphora profile by second degree polynomials. We have determined a mathematical model for light-clay narrow-necked amphorae as a compound solid of revolution, composed of three simple solids of revolution defined by second degree polynomials describing conic sections.

Reconstruction of Heron's Formulas

Reconstruction of Heron's Formula for the "Pithoid" Volume

Heron of Alexandria's (1st century AD) formula for calculating the "pithoid" volume V_{Heron_pit} (Hultsch 1864, 202; Heiberg 1976, 98–101) is:

$$V_{Heron\_pit} = \frac{11}{14} \cdot \left(\frac{d_{\max} + d_{\min}}{2} \right)^2 \cdot H \tag{1}$$

M. Lang first suggested applying this formula to calculations in the 1950s. She attempted to prove (Lang 1952, 18) that height H, maximal $d_{\max}$ and minimal $d_{\min}$ "pithoid" diameters are specified in an ancient inscription and the volume of a vessel can be calculated using Heron's formula with their help. The Russian researcher I. B. Brašinsky later assumed that Heron's formula for "pithoid" could be used to calculate the volumes of some types of sharply ground amphorae too (Brašinsky 1984, 72–74). He offered the following interpretation for the parameters in Heron's formula: $d_{\max}$ – maximal diameter of amphora body, and $d_{\min}$ – diameter of amphora neck (mouth); H – amphora height. However, the physical meaning of the formula has remained obscure because the shape of the amphora type called "pithoid" remained unclear. The Russian researcher S. J. Monakhov suggested updating Heron's formula for "pithoid" (Monakhov 1986, 106–114). He attempted to explain the physical meaning of Heron's formula as the volume of the cylinder. For such a cylinder, the diameter of the base is equal to the average of the internal diameter of the neck $d_{\min}$ and the maximal internal diameter $d_{\max}$, and the height of the cylinder is equivalent to amphora height H.

Neither researcher specified a geometrical equivalent of the pithoid form.

A pithos is by definition a large clay vessel whose basic part has an ovoid form. Heron introcuced the new term "pithoid" instead of the widely-used "pithos", which means a definite type of the vessel. When "pithoid" is used, it can be assumed that one means the calculation of ovoid part of the body.

In our opinion, a parabola describes the conic section of an ovoid adequately. The solid of revolution created by a parabola was first described by Archimedes. He determined formulas for calculating the volume of a "conoid" (paraboloid of revolution) and a "spheroid" (ellipsoid of revolution: a spheroid is a special case of ellipsoid in modern mathematics) (Archimedes 1962, 168; 170; 508). Archimedes' paraboloid volume of revolution segment is expressed by the formula:

$$V_{par} = \frac{1}{2} \pi \cdot r^2 \cdot H = \frac{1}{8} \pi \cdot d^2 \cdot H \tag{2}$$

However, two diameters are present in Heron's formula for "pithoid": maximal and minimal. We have assumed that the formula describes the truncated paraboloid of revolution. The formula for its volume is:

$$V_{trunc\_par} = \frac{\pi}{4} \cdot \frac{d_{\max}^2 + d_{\min}^2}{2} \cdot H \approx \frac{11}{14} \cdot \frac{d_{\max}^2 + d_{\min}^2}{2} \cdot H \tag{3}$$

The formula for the volume of the truncated paraboloid almost coincides with Heron's formula for "pithoid". Probably, the formula for "pithoid" was corrupted before reaching Heron, or has been corrupted since. If it is the same formula, we can interpret the physical meaning of Heron's formula for "pithoid" as the volume of the truncated paraboloid, in which case the term "pithoid" amphora should be applied to amphorae with paraboloid bodies.

Reconstruction of Heron's Formula for the Volume of "Spheroid" Pithos

Heron's formula for "spheroid" pithos (Hultsch 1864, 203; Heiberg 1976, 100–101) has an obvious error, it is a second degree polynomial, though a third degree polynomial is expected for volumetric calculations:

$$V_{Heron\_sph} = \frac{11}{21}\left(\frac{3}{2}d + H\right)^2 \quad (4)$$

It is improbable that Heron applied an obviously erroneous formula in the 1st century AD, after the fundamental works of Archimedes on the calculation of volumes of various solids of revolution. This mistake most likely arose when the manuscript was later copied. The formula for the volume of an ellipsoid of revolution was deduced by Archimedes. He termed an ellipse a "spheroid". In modern notation, the formula of ellipsoid volume is:

$$V_{ell} = \frac{4}{3}\pi \cdot r^2 \cdot h = \frac{\pi}{3} d^2 \cdot h = \frac{\pi}{6} d^2 \cdot H \quad (5)$$

where r is the minor ellipsoid semiaxis, h the major ellipsoid semiaxis, d the minor ellipsoid axis, and H the major ellipsoid axis.

If it is assumed that Heron's "spheroid" pithos represents not a whole ellipsoid, but $\frac{3}{4}$ of its volume – i.e. a truncated ellipsoid – then the formula will be as follows:

$$V_{tranc\_ell} = \frac{3}{4} \cdot V_{ell} = \frac{\pi}{6} \cdot \frac{3}{2} d^2 \cdot h = \frac{11}{21}\left(\frac{3}{2} d^2 \cdot h\right) \quad (6)$$

This formula is very similar to Heron's formula for "spheroid" pithos (4). Therefore it is quite possible that it is the same formula, but it has been passed down corrupted, whether intentionally or otherwise. If it really is the same formula, it is possible to interpret Heron's formula for "spheroid" pithos as $\frac{3}{4}$ ellipsoid volume.

The Question of Authorship of Heron's Formulas

Heron's Alexandrian works are a collection of formulas with problems in their application in many respects. He did not give proofs of his formulas. Some of Heron's numerical examples are still found in cuneiform texts. His best-known formula (that of the definition of the area of a triangle) goes back to the greatest scientist of antiquity, Archimedes. The central themes of Archimedes' mathematical works were the analysis of volumes and the areas of various surfaces. Archimedes calculated the volume of a sphere and of a spherical segment, and also the volume of ellipsoids and paraboloids of revolution (Archimedes 1962). There were other scientists in antiquity who studied conic sections before Archimedes. Concepts of conic sections first appeared in the works of the Greek mathematician Menaechmus (4th century BC). Aristaeus the Elder, a senior contemporary of Euclid, wrote "Five Books Concerning Solid Loci" about conic sections. In ancient mathematics, "solid loci" were the conic sections arising from the intersection of the surface of a circular cone by a plane. Antique mathematics defined conic sections as the result of intersecting a circular conical surface with a plane. Planes were perpendicular to one of their rectilinear generatrix. The parabola is a section of a cylinder the ellipse a section of an acute-angled cone, and one of the two branches of a hyperbola is a section of an obtuse-angled cone. Archimedes used the same names for conic sections. Archimedes considered the surfaces formed by rotation of an ellipse, a parabola and a hyperbola around their axes of symmetry. The first of these surfaces Archimedes named "spheroid", i.e. similar to the sphere, the second and third, "conoids", i.e. similar to a cone. He named the surface of rotation of the section of a cylinder "rectangular conoid", and the surface of rotation of section of an obtuse-angled cone "obtuse-angled conoid". In modern mathematics, Archimedes's rectangular conoids are known as paraboloids of revolution, and his obtuse-angled conoids are cavities of double-cavity hyperboloids of revolution. Archimedes calculated the volumes of some segments of conoids and spheroids in his "On Conoids and Spheroids" (Archimedes 1962, 168; 170; 508). All his formulae were proved by mathematical arguments and coincide with the modern formulae.

The other great scientist of antiquity, Apollonius of Perga (3rd–2nd centuries AD), was engaged in conic sections after Archimedes and before Heron. Apollonius coined the terms still used today: "hyperbola", "parabola", "ellipse" (Rozenfeld 2004, 18–20). Apollonius developed methods of antique mathematics in the field of conic sections considerably; however, he did not calculate volumes of solids of revolution directly. It is most probable that the formula for the volume of a "pithoid" as truncated paraboloid was deduced by Archimedes, and then borrowed by Heron. However, corresponding manuscripts by Archimedes have not yet been discovered, although his last work, "On Floating Bodies", considers segments of a paraboloid of revolution directly.

Mathematical Model of the Narrow-necked Light-clay D-type Amphorae

We have chosen light-clay narrow-necked amphorae of D-type from Tanais from the 3rd century AD *(Fig. 2)* to test our hypothesis that ancient craftsmen could use geometrical and mathematical calculations to create amphorae for a given capacity. The amphorae have an ovoid body and represent one of the most widespread types of vessels among light-clay narrow-necked amphorae in the area of the Northern Black Sea coast. Many scientists believe that these amphorae were basically used for transportation and storage. A large quantity of these vessels was discovered in Tanais complexes which were lost in a fire in the middle of the 3rd century AD. The large number of intact copies and their standard form has allowed statistically authentic research to be carried out.

It is impossible to measure the volume of the damaged vessels precisely, and the proportion of undamaged vessels is low. Therefore, scientists have developed ways to calculate the volume of amphorae as solids of revolution. Their capacity can be computed from drawings of the pottery profiles, provided the profiles are complete and the drawings represent vessels with interiors that are surfaces of revolution (Louise / Dunbar 1995). The method is based on the observation that a three-dimensional vessel can be reconstructed from its profile by revolving it around the axis of rotation. Several mathematical appoaches are available to describe a profile curve. The profile is completely specified when any of the following representation functions is known, where s is the arc length measured along the profile; $x(s)$ the cartesian distance of the point s on the profile from the axis of rotation of the vessel (Wilcock / Shennan 1975b, 17–31; Wilcock / Shennan 1975a); $\theta(s)$ the angle of the tangent at the point s on the profile measured relative to the x-axis (Leese / Main 1983, 171–180; Main 1986); and $\kappa(s)$ the curvature at

Fig. 2.

the point s on the profile (Leymarie / Levine 1988; Mokhtarian / Bober 2003). To obtain the mathematical expression of a curve approximating the profile it is also possible to use the computer program Maple. This allows the method of the least squares polynomial approximant of the data to be used (Govoruhin / Tsibulin 2001, 219). Thus for each amphora, the function is constructed in such a way that the polynomial function and the profile line coincide approximately. On the basis of these profile functions, it is possible to obtain volumes of vessels as figures of rotation from the formula (Karasik / Smilansky 2006):

$$V = \pi \int_0^f [r(s)]^2 \frac{dy}{ds} \, ds \qquad (7)$$

Several improvements of this formula were used by different researchers to calculate the volume of various vessels and solids of revolution independently (Gray 1997, 457–480; Govoruhin / Tsibulin 2001, 88). But we are interested in the conic sections of Archimedes. Therefore, we have chosen a method of geometrical approximation by second degree polynomials of the inner contour of the amphora profile. Our method is related to methods for analyzing vessels solely on measurable attributes (Orton 1980; Whallon / Brown 1982).

Being responsible for the volume, the containing part of the typical narrow-necked light-clay D-type amphora, like most of the sharp-bottomed amphorae, is structurally composed of three segments: body, shoulders, and neck. In the context of the proposed mathematical model, the boundaries of the solids of revolution were brought to conformity with the boundaries of these constructional segments of the amphora. For each of the structural segments a curve was constructed which describes a conic section, already studied by Archimedes: parabola, ellipse, and hyperbola. The parabola proved to correspond most accurately to the profile of the ovoid body of the narrow-necked light-clay D-type amphorae, i.e. the parabola approximates the body profile, and the body itself shows a correlation with a paraboloid of revolution. However, inside the body of the narrow-necked light-clay amphorae there is as a rule a small flat bottom. Therefore, not only the paraboloid of revolution but also the truncated paraboloid was brought to conformity with the body. The equation for the parabola approximating the segment of the body is (8). The ellipse corresponds most accurately to the profile of the shoulders of the narrow-necked light-clay D-type amphorae, while half of the truncated ellipsoid of revolution conforms to the shoulders. The equation for the ellipse approximating the segment of the shoulders is (9). The profile of the neck is most accurately approximated by a parabola; however, the corresponding solid of revolution is not a classical paraboloid, since the axis of revolution of the body does not coincide with the axis of symmetry, but rather, the axis is on the outer side. The equation of the parabola approximating the segment of the neck is (10).

$$y = \frac{H}{(R^2 - r^2)} \cdot (x^2 - r^2) \quad (8)$$

$$y = H + h_1 \cdot \sqrt{\frac{R^2 - x^2}{R^2 - r_1^2}} \quad (9)$$

$$y = H + h_1 + h_2 \cdot \left(\frac{x - r_1}{r_2 - r_1}\right)^2 \quad (10)$$

where R – inner radius of the widest part of the body; r – inner radius of the bottom of the body; r_1 – inner radius of the lower part of the neck; r_2 – inner radius of the upper part of the neck (at the level top of handles attachment); H – inner height (depth) of the body; h_1 – height of the shoulders; h_2 – height of the neck. An approximation of the full inner profile of the typical narrow-necked light-clay amphora of the D-type by the second degree polynomials is presented in *Fig. 3*.

Having achieved stereometric conformity for each of the amphora segments and the approximating lines, we can now determine the formulae for calculating corresponding volumes. Thus, the volume of the body V_1 is calculated by the formula of the truncated paraboloid volume (11), the volume of the shoulder part V_2 using the formula of the truncated ellipsoid volume (12), and the volume of the neck V_3 by the following formula (13).

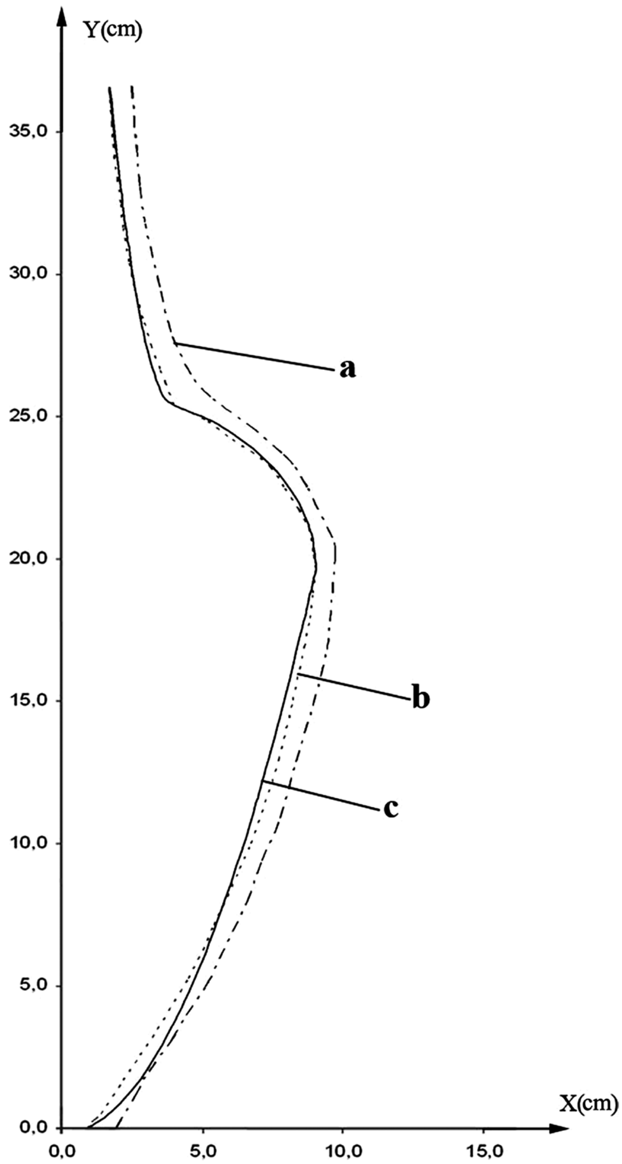

Fig. 3. Amphora T-89-XIV, No. 293; a – outside profile, b – inner profile, c – the second-degree polynomials that approximate the inner profile.

$$V_1 = \frac{1}{2} \pi (R^2 + r^2)\, H \quad (11)$$

$$V_2 = \frac{1}{3} \pi (2R^2 + r_1^2)\, h_1 \quad (12)$$

$$V_3 = \pi h_2 \left(r_1^2 + \frac{1}{2}(r_1 - r_2)^2 - \frac{4}{3} r_1 (r_1 - r_2)\right) \quad (13)$$

The volume of the whole amphora corresponds to the sum of the calculated volumes:

$$V_{cal\_i} = V_{1i} + V_{2i} + V_{3i} \quad (14)$$

V_{cal_i} – total calculated volume of amphora no. *i*; V_{1i} – calculated volume of the body of amphora no. *i*; V_{2i} – calculated volume of the shoulder part of amphora no. *i*; V_{3i} – calculated volume of the neck of amphora no. *i*.

Practical Implementation

The method of volume calculation, based on the developed mathematical model of the narrow-necked light-clay D-type amphorae dated to the 3rd century AD, was verified on the amphorae from the collection of the Tanais Archaeological Museum-Preserve. The samples investigated comprised 65 well-preserved amphorae with intact bodies. The samples represented amphorae in the range 2.6–5.0 liter. The typical thickness of amphorae walls was taken into account for calculation of inner parameters. The average thickness of the walls of a body and shoulders were assumed to be 0.7 cm, the walls of the lower part of the neck 1.5 cm, the walls of the upper part of the neck 0.8 cm, and the inner radius of the bottom of the body 0.9 cm. The upper boundary of the liquid-holding part of the neck was determined by the level of the top attachment of handles (Iljaščenko 2006, 189). We compared the calculated volumes of examined amphorae to the volumes measured by water and millet. The partially published volumes of amphorae measured by water have been given by S. M. Iljaščenko (2006, 193–199), the chief curator of the Tanais Archeological Museum – Reserve. The maximum difference of the calculated volumes from the measured did not exceed 6%. The average difference of the calculated volumes from the measured did not exceed 3%. The obtained result shows the high degree of conformity of the suggested mathematical model to real narrow-necked light-clay amphorae of the D-type from Tanais.

The result proves the existence of amphorae in the antiquity designed on the basis of conic sections. Studying the design of the narrow-necked light-clay D-type amphorae has several interesting consequences. We found a high linear correlation (r = 0.97) between the size of the maximal diameter D of amphorae and the designed volume V_{cal}. The property of linear dependence of the volume of amphorae from their maximal diameter could have been used during antiquity for the operative control of the volume of amphorae for wholesale deliveries, for example, and also for the fast calculation of the general capacity of transported wine. It was also established that two standards existed for the amphorae in question. They had identical proportions, but were based on two differing systems of measurement: Roman and Greek. The calculated volume of the "Roman" standard is equal to 3.2 liter, nearly 1 *congius* (3.24 liter), and the "Greek" standard contained 3.6 liter. It appeared, that the majority of the investigated amphorae adhere to the "Roman" standard. This shows that the basic part of the narrow-necked light-clay D-type amphorae was made in the provinces on the Black Sea coast with strong political and cultural influences from Rome.

Conclusion

In conclusion, Heron's formulas for "pithoid" and "spheroid" pithos were successfully reconstructed. They represent formulas for the volume of the truncated paraboloid and $\frac{3}{4}$ of an ellipsoid of revolution respectively. In this paper it is shown that amphorae designed on the basis of conic sections existed in antiquity. The volume of this kind of amphorae can be calculated with the help of volume formulas for solids of revolution formed by conic sections. Research has proven that the ovoid body of the narrow-necked light-clay D-type amphorae is a special case of "pithoid". However, it is impossible to assert that all antique amphorae with ovoid body can be described as a "pithoid". Some of those amphorae which archaeologists consider ovoid cannot be described as a paraboloid, and must be considered as an ellipsoid. It is necessary to check a body of concrete amphorae type for conformity with the truncated paraboloid of revolution preliminarily, before drawing a conclusion about conformity of an amphora to "pithoid".

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