On Information Asymmetry as a Source of Value: Intermediation, Auctions and Information

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Contents

Introduction and Overview

Why are there market makers, where a bargainer has limited information about the reservation prices of other buyers and sellers? What are the conditions such that a bargainer prefers the market maker over direct bilateral or multilateral trade, even if full information about his peer's reservation prices may be available in direct negotiations?

Within this work, four papers address these questions. Each paper analyses how bargainers determine prices on different platforms, where buyers and sellers reveal their respective offers for a good or a service to each other. A special focus lies upon the distribution of information between the bargaining parties and their inability of precise valuation. Each platform's efficiency is analysed in detail. Additionally, each paper introduces a market maker's market, where bid and ask prices for a good or service are quoted. Despite a bid or an ask price, no more information is revealed to a seller or a buyer. Each paper compares the efficiency of the market maker and the platform market and develops conditions when the market maker is preferred by the bargainers. When these conditions are satisfied, then a market with limited information is Pareto efficient over a market design where full information may be available.

A bargainer's inability of precise valuation is an important ingredient to this work. While a buyer (or a seller) can arrive at an individual reservation price, that buyer can not determine whether that price is high or low compared to some unknown average valuation and how his reservation price compares to the other bargainers' reservation prices. This statement is valid until the bargainers reveal their reservation prices or full information is available. Assuming a buyer with a certain valuation of a good, that buyer can only determine or estimate his valuation imprecision with some effort.

Following, each paper's individual focus on this topic will be summarised.

Paper 1

In order to trade a good or service, one needs at least two traders; one buyer and one seller. They engage in bilateral negotiations, which can be modelled as a double auction. In this paper, the buyer and the seller simultaneously reveal their respective offers. If the buyer is willing to pay a higher price than the seller requires, the trade is successful at a price that is between the seller's requirement and the buyer's offer.

The first paper analyses this bilateral bargaining procedure[1]. It shows that a double

[1] We work with a bilateral bargaining model that was introduced by Flood and Dresher (1952) and refined by Myerson and Satterthwaite (1983) as well as Chatterjee and Samuelson (1983). In their work, the authors assume reservation prices to be distributed on an interval $[\underline{v}, \overline{v}]$, with $0 \leq \underline{v} < \overline{v} < \infty$ (and often $\underline{v} = 0$). This distribution is common knowledge in the sense of Aumann (1976). In the present work, this assumption is relaxed by allowing reservation prices to be imprecisely distributed around some unknown average valuation.

auction[2] is most efficient, when a buyer and a seller are aware of their respective reservation prices. In this case, they make offers that are equal to their reservation prices. As a result, a double auction generates most profit for the bargainers when full information is available.

In contrast, a market maker quotes bid and ask prices in the Dealer's Market. He reveals nothing else to the traders. Thus information is limited regarding a buyer's and a seller's reservation prices. However, the paper proves that the market maker is more efficient than direct bilateral trade when he sets his fee scheme accordingly. This fee scheme is non-restrictive and generates a positive gain for the market maker on each round-trip transaction.

Two examples illustrate the first paper's theory. First, the Headhunter Game provides a numerical example that explains why an employer and a job candidate may be in favour of engaging a recruitment firm rather than taking part in bilateral salary negotiations.

The second example considers a corporation that wants to sell a fixed share of its owner rights. This corporation's management may either try to sell the share privately by negotiating and placing it with an individual or an institution, such as a venture capitalist. The corporation's management may, on the other hand, hire an investment banker to place the corporation's shares in an IPO. The example provides conditions that allow the IPO to be the first preference of all parties. It is proven that the average share price is below the bargainers' average valuation when they negotiate bilaterally. An investment banker can exploit this fact by underpricing the IPO.

Paper 2

The second paper expands the first paper's theory. Here, not only one seller, but a group of sellers bargains over the price of a good or service with one buyer. The paper models these negotiations as a reverse auction. It shows that the sellers loose profit when they place their bids without coordinating them. When all sellers commit to a shared bid strategy, their individual as well as their shared profit is maximised. Without coordinating their bids, each seller's profit converges to zero with an increasing group size of bidding sellers.

Furthermore, the second paper introduces a market maker under information asymmetry. This market maker can be the most efficient trading partner for all parties under the condition that his fee scheme is set appropriately. At the same time, his inventory level can be kept at a decent size.

The paper provides an example, where the parties' preference for a market maker over a reverse auction is illustrated. It shows how a firm chooses to place a bond on the capital market rather than meeting a financing agreement directly with an investor.

[2] The term double auction is commonly used in more present literature, such as by Gibbons (1992). A double auction and bilateral bargaining often are used as synonyms.

Our model has testable implications. Consider the Industrial Revolution in the 18th and early 19th century. Then, workers were not organised and the competition for jobs in the labour force was high. As a result, each worker had to sell his time and labour for a lower salary that his competitors. This process necessarily lead to extremely low wages. The rise of labour unions and upcoming political support helped the labour force to ensure a better coordination of their negotiations with an employer. Our model explains that an employer was able to exploit the unorganised labour force. Further, our model implies that coordinated negotiations with an employer are beneficial for workers. In addition, the paper's model may be used to calculate a union's optimal salary negotiation strategy.

Paper 3

A market with a group of buyers bargaining with a group of sellers is a further generalisation of the trading model introduced in paper 1 and 2. In this model, all sellers simultaneously reveal their offers on a platform. Buyers arrive one after another and buy at the lowest offer available at that moment, if that offer does not exceed that buyer's reservation price. Thus, there is full price information available on the analysed platform. Real world phenomena, such as the Amazon online market platform serve as examples for this market design.

In this paper, the focus lies upon the efficiency of the bargaining procedure and analyses its properties in detail. Furthermore, it proves that there is a significant proportion of buyers and sellers that are not matched by this procedure.

Alternatively, a buyer (seller) may consult a dealer. The parties are given the option to buy from (sell to) that dealer. He quotes each party an individual price and hides this information from that party's peers. The paper shows that when the dealer's prices are set accurately, all parties prefer the dealer over the direct trading platform.

In particular, the paper shows that the market design is a major determinant to allocate resources optimally and thus is in accordance with Roth (2008).

Paper 4

The last paper also deals with two different market designs. Namely, direct negotiations of the bargaining parties in contrast to a concept where a market maker intermediates between the negotiators. The paper focuses on a practical financing decision and compares two different forms of debt financing: bank loans and the public placement of bonds by an investment banker.

First of all, the bilateral negotiation process until achieving a satisfactory loan agreement is modelled and analysed in detail. During this process a firm opens its books to potential creditors and provides full information. Nevertheless, the negotiating parties can not value the firm precisely. A firm's management updates its estimation of valuation imprecision during the negotiation process in a Bayesian way. However, the paper proves that an investment banker who operates under information asymmetry may Pareto dominate financing with bank loans.

The paper's theory is accompanied by two numerical examples of negotiation processes. Using these examples, we show how an investment banker under information asymmetry can be Pareto efficient over direct loan negotiations.

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Information Asymmetry Allows Investment Bankers to Underprice IPOs Achieving a more Efficient Allocation than Raising Venture Capital under Full Information

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Abstract

This paper models investment banking under information asymmetry, when the investors are unable to precisely value the newly issued shares. It provides a solution for the IPO underpricing puzzle. Under reasonably general conditions, we show that using investment banking under information asymmetry Pareto dominates raising venture capital under full information. In our model social welfare increases when investors are less precise in valuing the newly issued shares. Investors may use a portfolio of seasoned shares, with precise market prices, to span the risk and the return of the IPO shares. In order to compete with precisely valued investment opportunities, an investment banker underprices imprecisely valued newly issued shares. Thereby, an IPO generates additional wealth over raising venture capital that compensates the issuing firm for the IPO underpricing.

We calculate a unique Nash equilibrium for a version of the bargaining model of Chatterjee and Samuelson (1983), with less restrictive assumptions and under a variety of information sets. In our model, the valuation of the bargainers is imprecise. We show that both the regulators and the intermediaries may optimally restrict access to full information in order to achieve a better allocation and to generate social wealth. In summary, asymmetric information and valuation imprecision may create wealth.

Key words: Investment Banking, IPO Underpricing, IPO Long-term Under-Performance, Venture Capital, Information Asymmetry, Imprecise Valuation, Bilateral Monopoly Bargaining Model, Double Auction, Pareto Efficient Market, Naive Agents, JEL Classifications: G10, G14, G38, D44

Contents

1 Introduction

An immediate concern with information asymmetry is that it harms the market. For example, the seminal work of Akerlof (1970) shows that the market under information asymmetry can break down. Additionally, Stiglitz and Weiss (1981) show that intermediaries in the credit market will optimally ration credit under information asymmetry. A regularity implication might be that full information markets should be promoted to increase social wealth compared to markets with asymmetric information. This paper shows that traders who value an asset imprecisely may profit from information asymmetry.

In our analysis, we consider a two-player double auction with valuation imprecision. Our model is an extension of the work of Chatterjee and Samuelson (1983) on bilateral monopolies. In addition, we consider different settings of information and rationality. As opposed to Akerlof (1970) and Stiglitz and Weiss (1981), this paper introduces conditions under which information asymmetry is preferred over full information. To do so, we install a market maker under information asymmetry. When that market maker sets his prices reasonably, then the traders prefer his market over a double auction under full information.

We apply our findings to a firm wanting to sell a share of its owner rights. The paper provides conditions for an IPO under information asymmetry being optimal. Thereby our model provides an explanation for the IPO underpricing puzzle.

From 1980 to 2001, an investor buying shares at an IPO just prior to the first day of trading and holding them until the market closes that day, would have been able to sell those shares at an average of 18.8% above the price at which the issuing firm sold them (see Welch and Ritter (2002)). If the same investor held these shares for a period of three years, his investment would have underperformed the CRSP value-weighted market index by 23.4%. In addition, these three-year IPO investments would have underperformed investments in seasoned companies with the same market capitalization and book-to-market ratio by 5.1%. However, this long-term IPO underperformance does not explain the one-day IPO underpricing, as an investor may solely choose a short investment horizon. For a detailed literature survey of IPO underpricing see Ljungqvist (2004).

Some of the more successful theories of IPO underpricing rely on asymmetric information. In particular, the following four asymmetric information explanations for IPO underpricing are noteworthy. Baron (1982) presents an IPO model where underpricing is used to induce optimal selling effort by an investment banker who is better informed about demand conditions than the issuing firm. Welch (1989) introduces a model with an equilibrium in which higher-valued firms use underpricing to signal their quality. Rock (1986) models a winner's curse which may be remedied by underpricing. Benveniste and Spindt (1989) propose a model in which underpricing is used to encourage investors to reveal their private information.

None of the above models stablishes that information asymmetry leads to a more efficient allocation than that under full information. In contrast, Diamond and Verrecchia (1991) show that a reduction of information asymmetry by revealing information to the public can reduce a firm's cost of capital. Thereby large firms disclose more information because their profit from that effect is higher than that of small firms. Diamond and Verrecchia (1991) note that the lowest cost of capital occurs with a certain level of information asymmetry.

Where Diamond and Verrecchia (1991) analyse securities that are already publicly traded, we analyse the process that leads to the decision to conduct an IPO, rather than placing a firm's shares privately. In this process, it is easier for an investment banker to pursue a firm to conduct an IPO when there is no full information between firm and investors. When firms and investors are in favour of an IPO, then disclosing information to the public may have a positive effect on a firm's profit. However, we concentrate on a firm's decision between placing its shares privately or publicly. Thereby we explore market conditions in which a regulator or market participants may prefer to promote information asymmetry over full information. To do so, let us start with the introduction of a simple bargaining procedure.

Kilgur et al. (2011) define a bargaining procedure as a set of rules for two bargainers making offers in order to reach a mutually satisfactory agreement. Myerson and Satterthwaite (1983) introduce a bilateral bargaining procedure where a buyer's and a seller's valuation is random and independent. They show that there is no ex post efficient bargaining strategy for both players in that bilateral monopoly. Myerson and Satterthwaite (1983) also prove that when two bargainers haggle, they inevitably miss some feasible trades, because they exaggerate their offers (in opposite directions) in order to maximize their expected returns. Saran (2011) further analyses this issue and shows that naive traders may increase efficiency over strategic players. Kilgur et. al (2011) analyse three procedures that induce honest offers and thereby increase efficiency in bargaining. However, their procedures do not achieve maximum efficiency.

In the above literature each trader has a reservation price, $V \geq 0$, which is distributed on the known interval $[0, v]$ (see Chatterjee (1983) for an example). Often that interval is restricted to $[0, 1]$, as for instance in the double auction considered by Gibbons[1]. In this paper, these constraints are relaxed as we allow more general intervals for reservation prices. In addition, these intervals are unknown to the traders.

Reservation prices depend on an individual's taste and preferences which can be expressed by a utility function. Even though the reservation prices of two individuals are not necessarily equal, both individuals are assumed to precisely value the assets in the economy. In valuing financial assets, that generate positive future cash flows, individual taste and prefer-

[1]See Gibbons (1992), pages 158ff.

ences regarding cash receipts is not relied on in finance literature. In our model, individuals disagree over the value of a (financial) asset due to their valuation imprecision. We consider symmetrically distributed imprecision in valuation.

In addition, pricing functions in the bargaining literature usually allow linear offer strategies, which is only a subset of all feasible strategies. Our study considers all feasible strategies. Furthermore, bilateral trading literature commonly assumes that each player knows her own valuation and the distribution of valuations for both players. Thus, in order to formulate a detailed offer strategy, each player may compare her valuation with its distribution. However, valuation imprecision implies that individuals do not have a valuation benchmark[2]. Assuming a buyer with a certain valuation of a good, that buyer might determine or estimate his valuation imprecision with some effort. However, he is not able to determine whether his valuation is high or low compared to the average valuation because ex-ante he has no benchmark. In this paper we model valuation imprecision by assuming that the bargainers are aware of the common distribution of their valuation imprecision. They however have no indication, whether their reservation price is above or below average.

More generally, our bargaining model can be considered as a Bayesian game with (un)known common prior[3]. These games have thoroughly been studied. Conditions for the existence of equilibrium strategies have been established by Nikaido and Isoda (1955) for instance. Studies of Bayesian games are usually conducted in abstract terms. Our modelled bargaining game is more practical, as it provides concrete formulas and advice on how bargainers should best set their prices. At the same time, our model is more abstract and thus realistic than those in the bargaining literature discussed above.

We show that in our double auction bargaining model under imprecise valuation, rational and non-cooperative bargainers inevitably miss feasible trades. We prove that ex-ante, a naive offer strategy (which is also the only available strategy under full information), is the most efficient one. However, efficiency may be further increased under asymmetric information. That is, an intermediary (such as an investment banker) may introduce a market mechanism under information asymmetry that is more efficient than a double auction under full information.

In section 2, we provide a detailed analysis of the two-player double auction under different sets of information and rationality. Section 3 studies a market with a dealer who

[2] Assume for instance that a player's valuation is uniformly distributed on the interval [0,100]. Then, in the mentioned bargaining literature an individual with the reservation price of 50 knows that her valuation is exactly average. Therefore that player can implement a linear bidding strategy of $a50 + b$. In our model, a player with a reservation price of 50 does not have a benchmark to determine whether that valuation is high or low. Therefore, she optimally implements bidding strategy $50s$. This factor s represents all feasible strategies, whereas a linear response contains only a subset of all possible strategies when the distribution of the valuations is known.

[3] See Harsanyi (1967) for reference.

intermediates between a buyer and a seller. Pareto efficiency of intermediation in markets with asymmetric information versus the double auction under full information is considered in section 4. That section also presents a numerical example of our theory that shows how information asymmetry Pareto dominates full information. Section 5 applies our model to capital markets and presents a solution for the IPO underpricing puzzle. Section 6 concludes.

2 A Two-Player Double Auction

We consider a seller S, who owns an indivisible good or a financial security, such as a share of stock, and a buyer B. These players individually value this asset as V_S and V_B, respectively. A seller S will sell the asset if and only if the deal price is not below her valuation V_S, whereas a buyer B buys the asset if and only if the deal price is not above his valuation V_B.

We model the sale as a two-person single-stage non-cooperative game of trading a single indivisible asset, as Chatterjee and Samuelson (1983). The buyer and the seller each make a sealed offer. If the buyer's offer O_B exceeds the seller's offer O_S then the good is traded at a price P from the interval $[O_S, O_B]$.

Let us introduce valuation imprecision to this two-player double auction. Assume that the two parties that value a good independently may over- or underestimate its value by an iid random imprecision factor that is uniformly distributed on the interval $[1-\alpha, 1+\alpha]$, with valuation imprecision $\alpha \in [0, 1)$[4]. Due to the lack of a valuation benchmark, neither party knows whether it underestimates or overestimates the value of the asset. The bargainers just know that their valuation is on the interval $[(1-\alpha)V, (1+\alpha)V]$, with some unknown average valuation $V > 0$.

At the beginning of our double auction game, the buyer and the seller reveal their offer to each other simultaneously. If the buyer's offer O_B is at least as high as the seller's O_S, then the deal is successful. We model relative negotiation skills of the parties by the factor $k \in [0, 1]$. The successful bargaining price is $P = \frac{kO_B + (1-k)O_S}{2} \in [O_S, O_B]$.

In the above equation, when the seller possesses supreme negotiation skills $k = 1$. In contrast $k = 0$ defines the buyer to be the most skilful negotiant. Thus $k = 1/2$ represents equal negotiation skills.

When market participants behave naively in our model, then they make offers at their

[4] A more abstract framework for the party's imprecision may be considered. In such a framework, valuation imprecision would not necessarily be identically and uniformly distributed. For instance, a buyer's imprecision may be uniformly distributed on the interval $[b_1, b_2]$ and a seller's imprecision uniformly distributed on $[s_1, s_2]$. We discuss this point in more detail in several proofs. Some propositions are proven for this more general framework and are then applied to the double auction as defined here. That approach is taken in proposition 1, for instance. Furthermore, uniform distributions of imprecision may be exchanged for other distributions, e.g. a (log-)normal distribution. Our research suggests that symmetric distributions of valuation imprecision are sufficient in order to obtain similar results as proven in this paper. In summary, we focus on providing a realistic bargaining model, while maintaining a sufficient degree of abstraction.

actual reservation prices. In that case, $O_B = V_B$ and $O_S = V_S$. Rationally behaving market participants strategically determine their respective offers O_S and O_B depending on their reservation prices. That is, a rational seller's offer is given by $O_S = sV_S$ and a rational buyer's offer is $O_B = bV_B$, with the scalars s and b. The buyer and the seller may determine their offer strategies b and s such that their individual expected profits are maximised. A buyer's profit is the difference between his reservation price and the deal price P. Thus, a buyer's profit is represented by the formula $P_B = V_B - P$. Likewise, a seller's profit is $P_S = P - V_S$.

To analyse this game let us first calculate the probability of a successful deal in the following proposition.

Proposition 1. *The deal probability p_d is given by*

$$p_d = \frac{\boldsymbol{1}_{s_1' \leq b_2'}}{bs\Delta b\Delta s}\Big(b_2'(min(b_2', s_2') - s_1') - b_1'(max(b_1', s_1') - s_1') \\ -\frac{1}{2}\left(min(b_2', s_2')^2 - max(b_1', s_1')^2\right)\Big),$$

where $b_1' = bb_1$, $b_2' = bb_2$, $s_1' = ss_1$ and $s_2' = ss_2$. When the conditions $b_1 = s_1$, $b_2 = s_2$, $b \in [b_1/b_2, 1]$, $s \in [1, b_2/b_1]$ and $b_2/b_1 > s/b$ hold, then deal probability simplifies to

$$p_d = \frac{(bb_2 - sb_1)^2}{2bs\Delta b^2}.$$

Proof: See the Appendix. □

The above formula shows that for the offer strategies $b = s = 1$, the deal probability is 0.5. The proposition also shows that reducing b, or increasing s, reduces the deal probability. That is, when the buyer decreases his offer price (reducing b), then the probability of the deal being successful decreases. Similarly, an increase in the seller's price (increasing s) decreases the deal probability.

In our model, the buyer's (seller's) valuation imprecision is uniformly distributed on the interval $[b_1, b_2] = [1-\alpha, 1+\alpha]$ ($[s_1, s_2] = [1-\alpha, 1+\alpha]$). As a result, the formula for the deal probability simplifies significantly when the conditions as stated in the above proposition are satisfied. Following proposition 2, we will show that these conditions arise naturally.

Let us now analyse expected profit of the two players.

Proposition 2. *Let offer strategies be bounded by $b \in [b_1/b_2, 1]$, $s \in [1, b_2/b_1]$ and $b_2/b_1 > s/b$. Then, in a two-person double auction a buyer's expected profit as a function of his offer strategy b is*

$$\boldsymbol{E}(P_B)(b) = \frac{1}{\Delta b\Delta s}\left((1-kb)\int_{s_1}^{s_2}\int_{b_1}^{b_2}\boldsymbol{1}_{Deal}\ x\ dxdy - (1-k)\,s\int_{s_1}^{s_2}\int_{b_1}^{b_2}\boldsymbol{1}_{Deal}\ y\ dxdy\right).$$

A seller's expected profit as a function of the offer strategy s is

$$\boldsymbol{E}(P_S)(s) = \frac{1}{\Delta b \Delta s}\left(kb\int_{s_1}^{s_2}\int_{b_1}^{b_2} \boldsymbol{1}_{Deal}\ x\ dxdy + ((1-k)s-1)\int_{s_1}^{s_2}\int_{b_1}^{b_2} \boldsymbol{1}_{Deal}\ y\ dxdy\right).$$

The integrals are

$$\int_{s_1}^{s_2}\int_{b_1}^{b_2} \boldsymbol{1}_{Deal}\ x\ dxdy = \frac{b\,\mathrm{b_2}^3}{3\,s} - \mathrm{b_1}\left(\frac{\mathrm{b_2}^2}{2} - \frac{\mathrm{b_1}^2\,s^2}{6\,b^2}\right)$$
$$\int_{s_1}^{s_2}\int_{b_1}^{b_2} \boldsymbol{1}_{Deal}\ y\ dxdy = \frac{b^2\,\mathrm{b_2}^3}{6\,s^2} - \mathrm{b_1}^2\left(\frac{\mathrm{b_2}}{2} - \frac{\mathrm{b_1}\,s}{3\,b}\right).$$

Proof: See the Appendix. □

The above formulas show that the two players' profits are dependent on their relative negotiation skills and their respective offers. Therefore both, the buyer's and seller's offer strategy b and s affect their expected profits. Expected profit of both players must be greater than or equal to zero, otherwise the player with a loss would refuse to trade. We assume that when the gain is zero, the players are willing to trade.

An upper bound for the buyer's offer strategy b is 1. That is, when $b > 1$, then the buyer's offer exceeds his reservation price and his expected profit is negative. The same line of reasoning shows that the seller's offer strategy s has the lower bound of 1.

If the seller's offer strategy s is greater than b_2/b_1, then the seller's minimum offer price exceeds b_2. As the buyer's offer strategy b is bounded by one, his maximum offer price is b_2. In this case the seller's price exceeds the buyer's offer with probability 1 and the deal fails deterministically. Therefore an upper bound for the seller's offer strategy is $s \leq b_2/b_1$. Similar arguments leads to b_1/b_2 being a lower bound for the buyer's offer strategy. Therefore the feasible offer strategies are $b \in [b_1/b_2, 1]$ and $s \in [1, b_2/b_1]$. While the traders apply feasible offer strategies, the deal probability simplifies to

$$p_d = \frac{(bb_2 - sb_1)^2}{2bs\Delta b^2},$$

according to proposition 1.

Let us now focus on the welfare effect of double auctions.

Proposition 3. *The sum of the buyer's and the seller's expected profit is*

$$\boldsymbol{E}(P_S)(s) + \boldsymbol{E}(P_B)(b) = \frac{6}{\Delta b \Delta s}\left(\frac{b^2}{s^2}b_1^2 + \left(\frac{2b}{s} - \frac{b^2}{s^2}\right)b_2^2 - \frac{2s}{b}b_1^3 - 3b_1b_2\Delta b\right).$$

If there is a positive deal probability in a double auction, then the sum of the two players' profits is increasing when the buyer increases his offer strategy b. The sum of the players' profits is decreasing when the seller increases her offer strategy s.

Proof: See the Appendix. □

Proposition 3 shows that the sum of the players' expected profits decreases when they selfishly pursue individually optimal offer strategies. That is, when the buyer lowers the price that he is willing to pay for the good or the seller raises the price that she expects from trade, then the sum of their profits diminishes.

The two players' strategies must not be too extreme, i.e. $s/b < b_2/s_1$. Otherwise according to proposition 1, the deal probability is zero. Thus reasonable response strategies are always within the above bounds. The total wealth in a double auction shrinks when players optimise their individual profits. Consequently, an increase in the expected profit of one party lowers the combined wealth of the two parties and thus diminishes the other party's profit significantly.

A rational buyer maximizes his expected profit. That is, the buyer maximizes $\mathbf{E}(P_B)$ by optimizing his offer strategy $b_{opt}(s)$ as a best response to the seller's strategy s. Similarly, a rational seller optimises her offer strategy $s_{opt}(b)$ as a function of the buyer's strategy b.

Solving the first order condition, an optimum strategy for each player, based on the other player's strategy can be calculated. An equilibrium is a set of offer strategies (b, s) such that no player profits from changing her strategy. From here onwards we assume that the buyer and the seller have equally strong negotiation skills (that is, $k = 1/2$). Then the optimal response strategies for the two players are characterised in the following proposition.

Proposition 4. *Optimal strategies for the buyer $b_{opt}(s)$ and the seller $s_{opt}(b)$ are given by*

$$b_{opt}(s) = \frac{1}{18\ b_2}\left(B(s, b_1, b_2) - \frac{(3\,b_1\,s - 4\,b_2)\,(15\,b_1\,s + 4\,b_2)}{B(s, b_1, b_2)} - 3b_1 s + 4b_2\right)$$

$$s_{opt}(b) = \frac{1}{18\ b_1}\left(B(b, b_2, b_1) - \frac{(3\,b_2\,b - 4\,b_1)\,(15\,b_2\,b + 4\,b_1)}{B(b, b_2, b_1)} - 3b_2 b + 4b_1\right),$$

where A and B are

$$A(s, b_1, b_2) := \sqrt{189\,{b_1}^4 s^4 + 432\,{b_1}^3 s^3 b_2 + 3168\,{b_1}^2 s^2 {b_2}^2 + 1280\,b_1\,s{b_2}^3 + 256\,{b_2}^4}$$

$$B(s, b_1, b_2) := \sqrt[3]{216\,{b_1}^3 s^3 + 1404\,{b_1}^2 s^2 b_2 + 288\,{b_2}^2 b_1\,s + 64\,{b_2}^3 + 27\,b_1\,sA(s, b_1, b_2)}.$$

Proof: See the Appendix. □

The formulas above hold while $b \in [b_1/b_2, 1]$ and $s \in [1, b_2/b_1]$. This condition implies two properties. First, the seller's lowest offer is at least as high as the buyer's minimal offer.

Furthermore, deal probability is greater than zero. As a result, each party has non-negative expected profit from participating in the double auction.

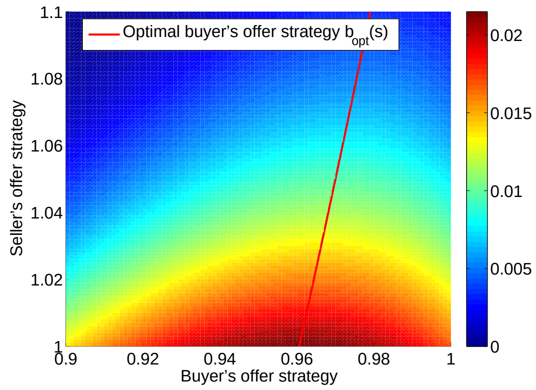


Fig. 1 – Buyer's optimal offer strategy $b_{opt}(s)$

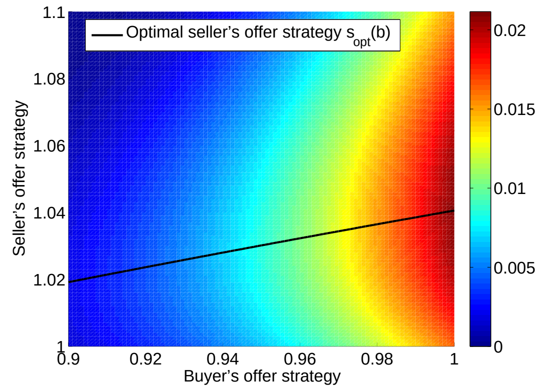


Fig. 2 – Seller's optimal offer strategy $s_{opt}(b)$

Let us analyse optimal offer strategies. Figures 1 and 2 show the expected profit of the two players as a function of their strategies b and s. In these figures, we have numerically illustrated the formulas with an imprecision rate of 10 %. On the x-axis, the buyer's offer strategy b is drawn, the seller's offer strategy s is displayed by the y coordinate. The coloured area in figure 1 (figure 2) shows the buyer's (the seller's) expected profit for each set of strategies (b, s). The lines in each figure represent the two players' optimal response strategies. The optimal buyer's response strategy is represented by a red line in figure 1, while the optimal seller's strategy is shown by a black line in figure 2.

When the seller makes an offer close to her reservation price, then the buyer gains from this lower offer. This may be observed towards the bottom of figure 1, where the maximum buyer's expected profit is shown in red. In contrast, the buyer's profit decreases as the seller increases her offer. This may be seen towards the top of figure 1 (shown in dark blue), where the buyer's profit is almost zero.

Figure 2 shows that the seller's profit increases when the buyer is willing to pay a higher price. This can be observed on the right of that figure, where the seller's expected profit is maximal (indicated in red). In contrast, when the buyer reduces his offer, then the seller's benefit in the double auction is reduced. This can be seen on the left of figure 2 (which is coloured in dark blue).

In the above extreme scenarios, one party's expected profit is close to zero. To ensure a well functioning market, both players need to choose their offer strategies b and s such that they generate sufficiently high, non-zero profit for the other party.

The optimal buyer's response strategy is an increasing function in seller's strategy s. Similarly, the same holds for the seller's optimal response as a function of the buyer's strategy.

Therefore the two party's optimal response strategies are strategic complements. The remainder of this section presents a more detailed analysis of the players' optimal response strategies and their effect on the double auction's efficiency.

2.1 Full Information

In this section we analyse bargaining behaviour of the players under mutual full information in the sense of Aumann (1976). This means that each player knows the reservation price of her counter party, knows that the counter party knows, and so forth. Given full information, the parties do not need to submit sealed bids, as each party knows the reservation price of his counter party. As a consequence, offers are equal to the parties' reservation prices.

In fact, full information and naive behaviour of the parties imply the same bargaining strategy. To illustrate this fact, suppose that each side's offer strategy is naive. Then each trader makes an offer at his reservation price. That is, a player's offer is not the best reaction on the latter side's anticipated behaviour. This is the same situation as that under full information. Therefore full information and naive behaviour induce equivalent offer strategies.

Let us calculate the deal probability under full information.

Proposition 5. *In a double auction, when there is full information and there are two players with uniform iid valuation imprecision on the interval* $[1-\alpha, 1+\alpha]$*, then the deal probability is* 0.5 *for any imprecision parameter* $0 < \alpha < 1$.

Proof: See the Appendix. □

When there is full information in a double auction, on average half of the deals fail. In our model, trade occurs if and only if both parties benefit from it. This occurs, when the seller's offer does not exceed the buyer's offer. Thus naive offer strategies allow for all feasible trades that are of mutual benefit for both parties.

The next proposition derives the two players' expected profits as a linear function of their valuation imprecision.

Proposition 6. *In a double auction, when there is full information and there are two players with uniform iid valuation imprecision on the interval* $[1-\alpha, 1+\alpha]$*, then the players' expected profits are equal. This expected profit is a linear function of the two parties' imprecision and is given by the following formula:* $\boldsymbol{E}(P_{B,S}) = \alpha/6$.

Proof: See the Appendix. □

Proposition 6 shows that each player may expect a profit equal to 1/6 of the valuation imprecision α. This immediately implies that valuation imprecision is wealth increasing.

Intuitively, two individuals who are endowed with supreme valuation abilities, will arrive at the same value for an asset and find trade unsatisfactory. Mathematically speaking, players expect to profit from a double auction if and only if there is imprecision in valuation, that is $\alpha > 0$. Furthermore, imprecision and expected profit are positively correlated. Therefore, a higher imprecision in valuation causes an increased benefit for both players. Without valuation imprecision, the expected profit in a double auction is zero. Let us formally state this intuition in the following lemma.

Lemma 1. *In a double auction, when there is full information and there are two players with uniform iid valuation imprecision on the interval* $[1-\alpha, 1+\alpha]$*, then both market participants profit from a higher valuation imprecision. Higher imprecision generates a higher expected profit and is socially wealth increasing. When there is no imprecision, the expected profit is zero for both players.*

Proof: See the Appendix. □

So far, we have analysed double auctions under full information. As explained above, naive players who do not hide their valuation strategically, also play under full information. In sections 2.2 and 2.3, we analyse double auctions where a strategic player trades with a naive party.

2.2 A Rational Buyer and a Naive Seller

This section considers double auctions with a rational buyer who determines his offer strategy in order to maximize his expected profit. In contrast, there is a revealed naive seller who makes an offer equal to her reservation price. Let us analyse the buyer's optimal offer strategy within this double auction setting in the following proposition.

Proposition 7. *In a two-player double auction with a rational buyer and a naive seller, the buyer's optimal offer strategy is*

$$b_{opt}(1) = \frac{1}{18\ b_2}\left(B(1,b_1,b_2) - \frac{(3\,b_1\ - 4\,b_2)\,(15\,b_1\ + 4\,b_2)}{B(1,b_1,b_2)} - 3b_1 + 4b_2\right).$$

Proof: See the Appendix. □

From the buyer's optimal strategy we derive the probability of bargaining success.

Proposition 8. *In a two-player double auction with a rational buyer and a naive seller, the deal probability is*

$$p_d = \frac{(b_{opt}(1)b_2 - b_1)^2}{2b_{opt}(1)\Delta b^2}.$$

Proof: See the Appendix. □

Les us now analyse the players' expected profit.

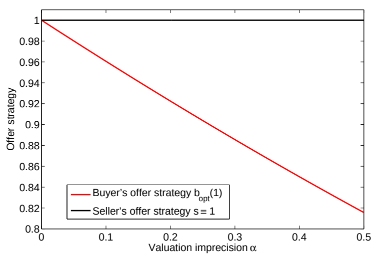
Proposition 9. *In a two-player double auction with a rational buyer and a naive seller, the buyer's expected profit is*

$$\boldsymbol{E}\left(P_B\right) = \frac{\left({b_{opt}}^{-1} - 1/2\right)\left(3\,{b_2}^2{b_{opt}}^2\left(b_2\,b_{opt} - b_1\right) - \,{b_2}^3{b_{opt}}^3 + \,{b_1}^3\right)}{6\left(b_2 - b_1\right)^2 b_{opt}}$$
$$+ \frac{-3\,b_2\,b_{opt}\left({b_2}^2{b_{opt}}^2 - {b_1}^2\right) + 2\,{b_2}^3{b_{opt}}^3 - 2\,{b_1}^3}{12\left(b_2 - b_1\right)^2 b_{opt}}$$

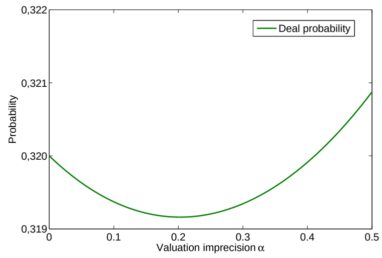
and the seller's expected profit is

$$\boldsymbol{E}\left(P_S\right) = \frac{b_2^3 b_{opt}^3 - b_1\left(b_1^2 - 3b_2 b_{opt}(b_2 b_{opt} - b_1)\right)}{12 b_{opt}(b_2 - b_1)^2}.$$

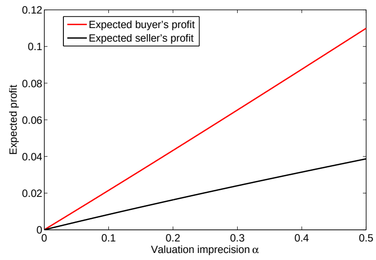
Proof: See the Appendix. □



(a) The offer strategies



(b) The deal probability



(c) The expected profits

Fig. 3 – A two-player double auction with a rational buyer and a naive seller

Propositions 7 - 9 are illustrated in figure 3. Figure 3 (a) shows the buyer's offer strategy as a function of imprecision. The seller's naive behaviour is represented by her offer strategy $s \equiv 1$. This means that her offer, independent of imprecision, is given by her reservation price. The buyer's offer strategy is a strictly decreasing function of imprecision. Therefore, higher imprecision causes the buyer to make an offer that is a smaller fraction of his reservation price. One might expect that the buyer's optimal offer strategy causes a decrease in the probability of bargaining success. This however is not the case, as can be seen from figure 3 (b).

Figure 3 (b) shows that the deal probability decreases until valuation imprecision approaches approximately 20%. For imprecision values higher than 20%, the deal probability is strictly increasing. The deal probability in the above example is between 31.9% and 32.1% for an imprecision parameter below 50%. When the different double auction settings are compared in section 2.5, we will show that a deal probability within the above range is not significantly affected by the level of valuation imprecision.

Figure 3 (c) shows the players' expected profits. Due to strategic behaviour, the buyer has a higher expected profit than the seller in this figure. Moreover, the expected profits of the two players ($\mathbf{E}(P_B)$ and $\mathbf{E}(P_S)$) are both strictly increasing functions of the valuation imprecision α. This means that a higher valuation imprecision increases each parties' expected profit. The seller's expected profit is positive; its maximum is 4% of the average good's valuation and it is always lower than the buyer's expected profit. The buyer's expected profit exceeds the seller's, with a maximum of approximately 11% of the average good's valuation. The buyer profits more than the seller, because the strategic behaviour of the buyer gives him an advantage over the naive seller. However, both parties have a positive profit from the double auction.

This section provided an analysis of double auctions with a strategic buyer and a naively behaving seller. Contrary, double auctions with a strategic seller and a naive buyer are analysed in the following section.

2.3 A Rational Seller and a Naive Buyer

In this section, there is a revealed naive buyer, who makes an offer equal to his reservation price. In contrast, the seller determines her offer strategically. Within this double auction setting we analyse the seller's optimal offer strategy as follows.

Proposition 10. *In two-player double auctions with a rational seller and naive buyer, the seller's optimal offer strategy is*

$$s_{opt}(1) = \frac{1}{18\ b_1}\left(B(1,b_2,b_1) - \frac{(3\,b_2\ - 4\,b_1)\,(15\,b_2\ + 4\,b_1)}{B(1,b_2,b_1)} - 3b_2 + 4b_1\right).$$

Proof: See the Appendix. □

Given the above optimal strategy, let us derive the probability of bargaining success.

Proposition 11. *In two-player double auctions with a rational seller and naive buyer, the deal probability is*

$$p_d = \frac{(b_2 - s_{opt}(1)b_1)^2}{2s_{opt}(1)\Delta b^2}.$$

Proof: See the Appendix. □

The players' expected profits may be calculated as follows.

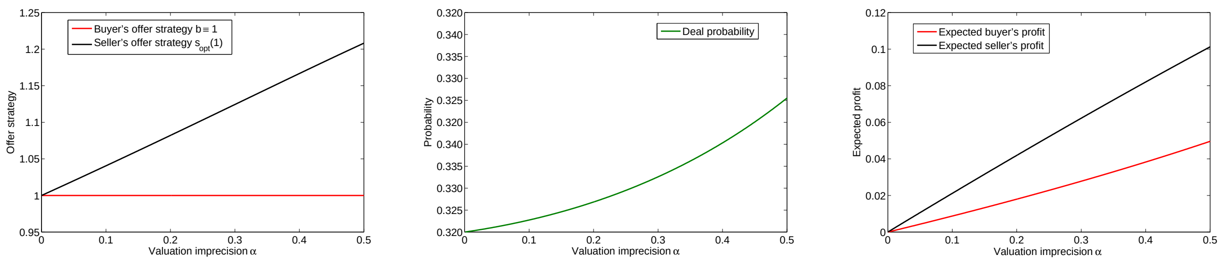
Proposition 12. *In two-player double auctions with a rational seller and a naive buyer, the seller's expected profit is*

$$\boldsymbol{E}\left(P_S\right) = \frac{\left(0.5 - {s_{opt}}^{-1}\right)\left(3\, b_2 \left({b_2}^2 - {b_1}^2 {s_{opt}}^2\right) - 2\, {b_2}^3 + 2\, {b_1}^3 {s_{opt}}^3\right)}{6\left(b_2 - b_1\right)^2 s_{opt}} + \frac{3\, {b_2}^2 \left(b_2 - b_1\, s_{opt}\right) - \, {b_2}^3 + \, {b_1}^3 {s_{opt}}^3}{12\left(b_2 - b_1\right)^2 s_{opt}}$$

and the buyer's expected profit is

$$\boldsymbol{E}\left(P_B\right) = \frac{-b_1^3 s_{opt}^3 - b_2\left(b_2^2 - 3 b_1 s_{opt}(b_2 - b_1 s_{opt})\right)}{12 s_{opt}(b_2 - b_1)^2}.$$

Proof: See the Appendix. □



(a) The offer strategies (b) The deal probability (c) The expected profits

Fig. 4 – A two-player double auction with a rational seller and a naive buyer

The results of propositions 10 - 12 are illustrated in figure 4. Figure 4 (a) shows the players' offer strategies as a function of their valuation imprecision. Obviously $b \equiv 1$ holds, as the buyer is a naive player in this double auction setting. That is, the naive buyer makes an offer at his reservation price. In contrast, the seller makes a strategic offer that maximizes her expected profit. Consequently, the seller makes an offer $s_{opt}(1)$, which is an increasing function of the valuation imprecision. That is, a higher imprecision in valuation causes the seller to make a higher offer compared to her valuation.

Figure 4 (b) shows the deal probability as a function of the valuation imprecision. In a double auction with a rational seller and a naive buyer, the deal probability is strictly increasing in imprecision. It rises from 32% to 33.25% for $0 < \alpha \leq 0.5$. As in the case of double auctions with a rational buyer and a naive seller from section 2.2, in the present double auction setting, the deal probability changes slightly as a function of imprecision. However, in this case, it increases monotonically as valuation imprecision rises.

Figure 4 (c) shows the two parties' expected profits in a double auction with a rational seller and a naive buyer. Both parties' expected profits ($\mathbf{E}\left(P_B\right)$ and $\mathbf{E}\left(P_S\right)$) are strictly positive and increasing functions of valuation imprecision α. That is, on average the trade is profitable for both players. Furthermore, each party has a higher profit from the double auction as the imprecision increases.

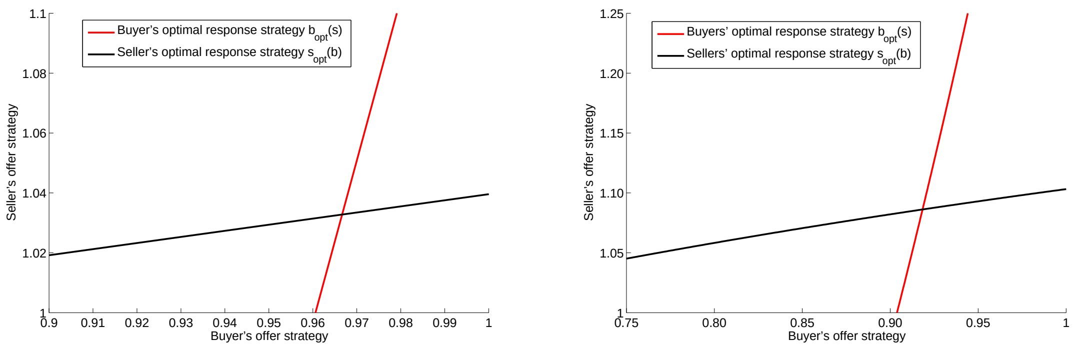
In this example, the buyer's maximum expected profit is 5% of the good's average value. The seller's expected profit exceeds the buyer's profit and is at the most 10% of the average good's value. That is, the seller's profit is roughly twice as high as the naive buyer's profit.

Let us now analyse a double auction where both, the buyer and the seller, behave strategically.

2.4 A Rational Buyer and a Rational Seller

In this section we analyse a two-player double auction with two strategically playing individuals. That is, both, the buyer and the seller optimize their offer strategies such that their individual expected profit is maximized. We explain this concept with some examples in the first place. Later in this section, the findings from the examples will be generalised.

In proposition 4, we calculated the optimal buyer's and seller's offer strategy as a function of the other party's offer strategy. Figure 5 graphs these combined strategies. In figure 5 (a) we use a valuation imprecision of $\alpha = 10\%$, while in figure 5 (b) the valuation imprecision is $\alpha = 25\%$. In both cases there is exactly one set of equilibrium offer strategies. In the equilibrium neither the buyer nor the seller profits from a change in his offer strategy $b_{opt}(s)$ or $s_{opt}(b)$. It is a Nash equilibrium in pure strategies under rational expectations.



(a) Valuation imprecision $\alpha = 10\%$ (b) Valuation imprecision $\alpha = 25\%$

Fig. 5 – Optimal response offer strategies and their equilibrium

Let us replicate the equilibria from figure 5 for different imprecision levels $\alpha = 5\%$, 10%, 20%, 25% and 50%. The results are summarized in table 1. In this example, there is always exactly one set of equilibrium offer strategies (b_{opt}, s_{opt}). The buyer's offer strategy b_{opt} is a decreasing function of the imprecision level, while the seller's offer strategy increases as valuation imprecision α rises. As the valuation imprecision increases, the buyer asks for more of a discount and the seller increases her price. This results in a higher expected profit for both parties.

Table 1 – Properties of a double auction with a rational buyer and a rational seller

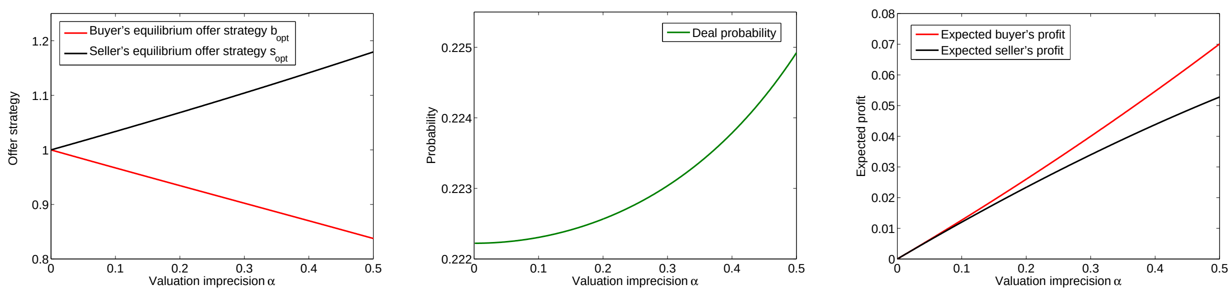
	b_{opt}	s_{opt}	$\mathbf{E}(P_B)$	$\mathbf{E}(P_S)$	p_d
$\alpha = 5\%$	0.9834	1.0168	0.63%	0.61%	22.22%
$\alpha = 10\%$	0.9670	1.0337	1.27%	1.20%	22.23%
$\alpha = 20\%$	0.9345	1.0684	2.60%	2.33%	22.25%
$\alpha = 25\%$	0.9184	1.0862	3.29%	2.87%	22.27%
$\alpha = 50\%$	0.8376	1.1795	7.01%	5.28%	22.49%

A surprising property of the equilibrium offer strategies is that they are not symmetrically distributed. That is, in equilibrium the buyer has a higher profit than the seller. Note that this analysis covers profit in absolute terms. When profit is calculated relative to each player's valuation, then the results are reversed (the seller has a higher profit than the buyer in equilibrium).

So far, we have analysed double auctions using fixed values for the valuation imprecision. In all our examples there is exactly one equilibrium set of strategies (b_{opt}, s_{opt}). Let us generalize these examples in the following proposition in order to rigorously show that there is exactly one equilibrium for any feasible imprecision parameter α.

Proposition 13. *For each imprecision parameter $0 < \alpha < 1$, there is exactly one set of equilibrium offer strategies (b_{opt}, s_{opt}).*

Proof: See the Appendix. □



(a) The offer strategies (b) The deal probability (c) The expected profit

Fig. 6 – Two-player double auction with a rational buyer and a rational seller

Figure 6 (a) shows the players' offer strategies in equilibrium. As the imprecision increases, the buyer decreases his offer (compared to his valuation) and the seller increases her offer (compared to her valuation).

Figure 6 (b) shows that the deal probability increases as imprecision increases. However, because the deal probability varies only in the small range between 22.2% and 22.5%, its fluctuation with the imprecision level is almost imperceivable.

It can be observed in figure 6 (c) that a higher imprecision level leads to higher expected profits. The buyer's expected profit is higher than the seller's. However, both parties profit from participation in this double auction. Their profit increases as the valuation imprecision α increases.

The next section analyses relative efficiency of the double auction settings, which were introduced in sections 2.1 - 2.4.

2.5 The Downside of Rationality

In the last four sections we discussed four different settings of double auctions. In section 2.1 we analysed double auctions under full information. We showed that this setting corresponds to the setting of naive players. Thereby their offers reflect exactly their valuation. Sections 2.2 and 2.3 discussed the double auction settings with exactly one market participant (in section 2.2 the buyer and in section 2.3 the seller) being fully rational whereas the other participant behaves naively. The rational acting market participant's strategy is to adjust the reservation price in order to maximize the expected profit. The naively acting market participant's reservation price equals exactly that party's valuation. The market setting where buyer and seller behave fully rational was discussed in section 2.4. For each valuation imprecision there is a unique equilibrium in bidding strategies.

This section closes the analysis of double auctions by comparing the different market settings. We focus on the question, which setting is most preferable for the players.

A buyer's and a seller's expected profit in each double auction setting has been calculated in propositions 6, 9 and 12. Applying equilibrium strategies to the formulas from proposition 2 returns the players' expected profit in the market setting where they both place their bids strategically.

Figure 7 draws a buyer's expected profit in the different double auction market settings. It shows that there is a clear ranking order for the market settings that is independent of valuation imprecision. The most preferable market setting for a buyer is being the only rational trader. His second best alternative is a double auction under full information. The setting where the players both act rational is the buyer's third best alternative, followed by the setting where just the seller behaves rational.

The seller's profit in the double auction market settings are drawn in figure 8. There is also a clear ranking order regarding the seller's preferences. The highest expected profit is obtained if only the seller behaves rationally. Just like the buyer, the seller expects the second highest profit in a double auction under full information. The third and fourth best alternatives are the fully rational and buyer rational settings, respectively.

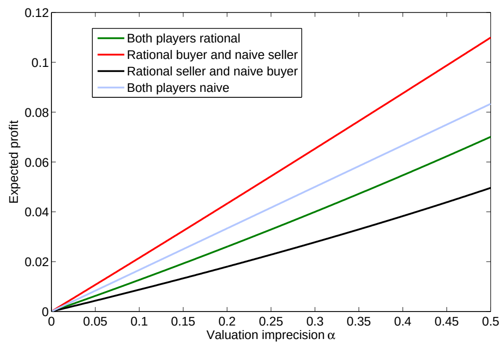


Fig. 7 – Buyer's expected profit in different market settings

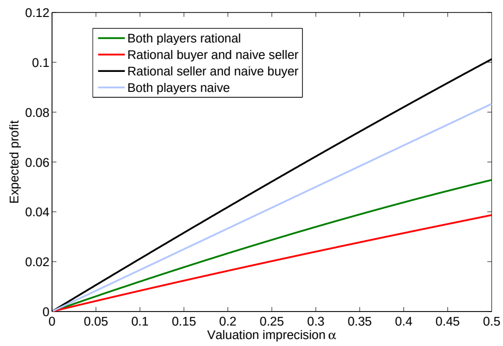


Fig. 8 – Seller's expected profit in different market settings

Both parties have the full information market setting as their second priority. Thus full information dominates the setting where both parties place their bids strategically, which is priority three for both players. The naive setting is therefore more preferable for both parties than placing strategic bids. Consequently, both parties would profit from committing to the naive setting. However, given the counter party's naive strategy, one party then could adjust its offer strategy such that its expected profit is maximized (and thereby the other party's profit is minimized). Each party anticipating rational behaviour from the counterpart needs to behave rationally as well, in order to at least get priority three. Thus the parties break their commitment for the naive strategy, if they do not trust each other.

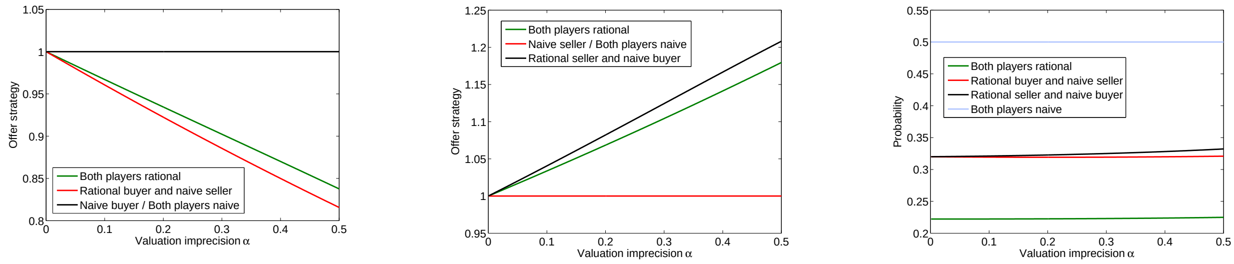
The different double auction settings therefore are a typical example of a prisoner's dilemma[5]. Compared to bidding strategically, the buyer as well as the seller would profit if they would commit to placing their bids naively. However, anticipating rational behaviour of their counterpart leads to a market setting where both parties are profiting less from trade than they could. In other words they share a smaller profit-pie in equilibrium as they could by placing naive bids.

In contrast to zero sum games, that are Pareto efficient according to von Neumann (1928), a double auction is a non-zero sum game. Furthermore, the sum greater zero that may be distributed between a buyer and a seller is dependent on their offer strategies. In fact, the sum is decreasing when the parties' offers diverge from their reservation prices. Thus individual

[5]See e.g. Flood and Dresher (1952) for reference

best responses are not Pareto efficient. Opposed to Cournot competition (2001 (org. 1838)), naive strategies (which are equivalent to honest behaviour) develop more efficient allocations in a double auction.

There may be procedures to implement honesty, as for instance shown by Kilgur et. al (2011). Further, Schelling (1960) proposes to include criteria such as cultural background into the strategy decision. While the trading mechanism of a double auction is not altered and players behave rationally in terms of individual profit maximisation, we conclude that in equilibrium otherwise feasible trades are missed. In conclusion, the equilibrium bidding strategy turns out to be not Pareto efficient.



(a) Buyer's offer strategy in different market settings

(b) Seller's offer strategy in different market settings

(c) Deal probability in different market settings

Fig. 9 – Properties of different market settings

Figure 9 compares further properties of the different market settings. The functions are calculated from proposition 2 and optimal offer strategies as discussed in this section. Figures 9 (a) and (b) show the offer strategies the buyer and the seller choose in each market setting. Both parties choose the most extreme strategies when they are the rational individual and their counterpart behaves naively. Deal probabilities in each market setting are summarized in figure 9 (c). The highest deal probability of 50% is given under full information. A deal probability of approximately 22.5% is achieved in the rational setting. This is the lowest possible deal probability. When one of the players behaves rationally and the counterpart naively, a deal probability of approximately 32% is achieved in both cases.

Figure 10 shows the buyer's expected profit in the different market settings, with a valuation imprecision of 10%. That profit is indicated by the colour scale. The ranking order of the buyer's preferred double auction settings can be observed from that figure. The buyer prefers the setting, where he behaves strategically and his counterpart naively. Followed by the setting, where both players place their bids naively. The third (fourth) best alternative is the setting, where both parties place their bids strategically (the buyer plays naively and the seller places her bid strategically). Likewise, the seller's preferences can be observed in figure 11.

If there were costs associated with the participation in a double auction, which are higher than either player's expected profit, then neither the buyer nor the seller would use this platform. For instance, if the valuation imprecision α is 10 %, then a seller's expected profit is 1.20% if both players place their bids strategically. As a result the double auction is not attractive for a seller when her costs for market participation exceed her expected profit of 1.20%.

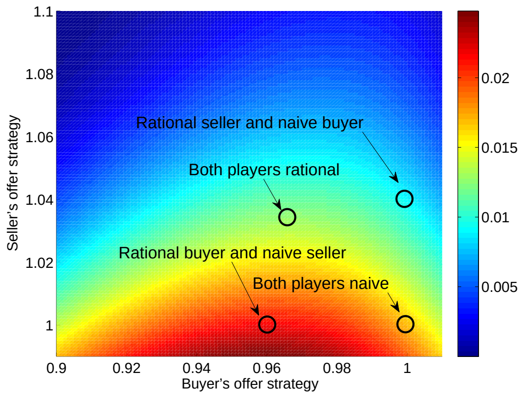


Fig. 10 – Buyer's expected profit in different market settings

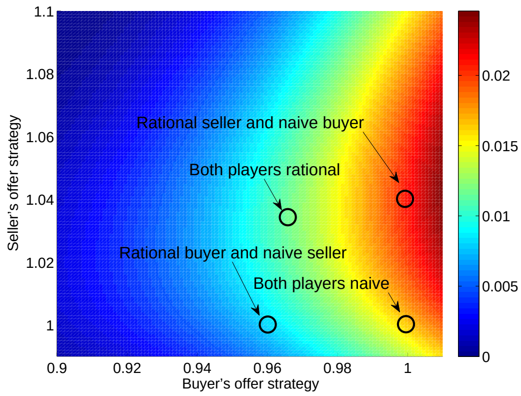


Fig. 11 – Seller's expected profit in different market settings

After a detailed analysis of double auctions, the next section is concerned with a market maker's market. We will show that even without costs for bilateral trade in a double auction and under non-restrictive conditions, this dealer is preferable for both market participants.

3 The Dealer's Market

In the Dealer's Market, there is a dealer present. The dealer has experience regarding the good and knows its average value. Thus he can value the good precisely. He acts as a market maker and charges a bid-ask spread. The dealer offers to buy the good for its average value multiplied by $1 - f$ and offers to sell the good for its average value multiplied by $1 + f$, with fee $f > 0$. The bid-ask spread guarantees the dealer a positive profit on each round-trip transaction, given by $2f > 0$. Hence, he deterministically profits from his strategy.

Buyers and sellers do not know the average valuation. Therefore they are unaware whether the dealer truly shows them prices $(1 - f)V$ and $(1 + f)V$, respectively. This means that the parties need to trust the intermediary to charge truthful prices. Consequently, the intermediary needs to be endowed with exogenous reputation capital such that the bargainers consider him trustworthy.

The dealer pursues the strategy to install an environment under information asymmetry. In the Dealer's Market buyers and sellers do not interact. They solely communicate with the dealer and choose whether to accept his offer, or not. In this sense there is information asymmetry in the Dealer's Market. Asymmetric information is important to the success of the dealer's strategy. This means that a buyer and a seller should either consult the dealer or choose to trade in a double auction. Additionally, a buyer and a seller can first bargain in a double auction and, in case they are unsuccessful, they may consult the dealer in the next step. While this sequential strategy is beneficial for a buyer and a seller, the dealer is left with a Lemons problem: buyers with a low reservation price and sellers with a high reservation price. Thus the dealer suffers from adverse selection. Installing a beneficial fee strategy consequently becomes more complicated for the dealer under full information because then the players may engage in bilateral negotiations prior to consulting the dealer. Then the dealer's strategy may even collapse.

We start to formally analyse properties of the Dealer's Market by calculating the deal probability.

Proposition 14. *When $f \leq \alpha$, then the probability that the buyer makes a gain from the dealer's offer is given by $p = (\alpha - f)/(2\alpha)$. The same is true for the seller.*

Proof: See the Appendix. □

Obviously, the deal probability in the Dealer's Market is a strictly decreasing linear function of the dealer's fee f. For $f = 0$ deal probability is exactly 50 %. It decreases linearly for an increasing dealer fee f. For $f = \alpha$, the deal probability is zero. We find that there will be no successful deals in the Dealer's Market if the fee f exceeds the maximum players' valuation imprecision α.

The next proposition calculates a buyer's and a seller's profit in the Dealer's Market.

Proposition 15. *When $f < \alpha$, then the buyer has a positive expected profit from consulting the dealer. It is given by $\boldsymbol{E}(P_D(B)) = (\alpha - f)^2/(2\Delta b)$. The same is true for the seller's expected profit, that is given by $\boldsymbol{E}(P_D(S)) = (\alpha - f)^2/(2\Delta b)$.*

Proof: See the Appendix. □

We find that both the buyer and the seller have positive expected profit from participating in the Dealer's Market if the condition $f < \alpha$ is satisfied. The higher the market participants' valuation imprecision α, the more likely is it that they participate in the Dealer's Market. Furthermore, the attractiveness of the Dealer's Market increases when the dealer lowers his fee f.

The next section compares the double auction and the Dealer's Market. It provides non-restrictive criteria, when the players prefer the Dealer's Market.

4 The Downside of Full Information

This section analyses the relative attractiveness of double auctions and the Dealer's Market. Section 4.1 introduces an optimal dealer strategy to Pareto dominate double auctions. The introduction of less restrictive assumptions in section 4.2 allows us to show that Pareto efficiency of the Dealer's Market can be achieved even in more general frameworks. Section 4.3 presents a numerical example on how a dealer Pareto dominates bilateral negotiations.

4.1 Pareto Efficiency of Information Asymmetry

In this section we establish upper bounds for the dealer's fee such that the Dealer's Market is Pareto efficient over the different double auction settings that were introduced in section 2.

First, the next proposition analyses in which cases the deal probability in the Dealer's Market is higher than that of a double auction.

Proposition 16. *Let p_d be the deal probability in a double auction. Then the deal probability in the Dealer's Market exceeds that of a double auction if $f < \alpha(1 - 2p_d)$.*

Proof: See the Appendix. □

The attractiveness of the Dealer's Market increases when the dealer reduces his fee. Furthermore, the probability for deal success increases when the fee is reduced. Proposition 16 establishes an upper bound for the dealer's fee such that the Dealer's Market offers a higher probability of deal success than a double auction. That upper bound is $f < \alpha(1 - 2p_d)$. While f is below this bound, success probability in the Dealer's Market is higher than that in a double auction.

The next 5 propositions establish upper bounds for the dealer's fee, such that the Dealer's Market is Pareto efficient over the different double auction settings. The following proposition develops a general formula for a higher expected profit in the Dealer's Market compared to a double auction.

Proposition 17. *Let $\boldsymbol{E}\left(P_{(\cdot)}\right)$ be a player's expected profit in a double auction. Then that player's expected profit in the Dealer's Market exceeds that profit if $f < \alpha - 2\sqrt{\alpha \boldsymbol{E}\left(P_{(\cdot)}\right)}$.*

Proof: See the Appendix. □

Proposition 17 provides a general formula for an appropriate dealer's fee strategy. Given a player's expected profit in a double auction, the dealer may determine his fee according to proposition 17 in order to Pareto dominate that double auction.

Let us concentrate on the different double auction settings as introduced in section 2. First, we will develop a dealer's strategy to Pareto dominate double auctions under full information.

Proposition 18. *Compared to a two-player double auction under full information, the players prefer the Dealer's Market if* $f < \alpha \left(1 - \sqrt{\frac{2}{3}}\right)$.

Proof: See the Appendix. □

Proposition 19. *Let* $\boldsymbol{E}(P_B)$ *be the buyer's expected profit in a double auction with a rational buyer and a naive seller. Then both players prefer the Dealer's Market over the double auction if* $f < \alpha \; - \; 2\sqrt{\alpha \boldsymbol{E}(P_B)}$.

Proof: See the Appendix. □

Proposition 20. *Let* $\boldsymbol{E}(P_S)$ *be the seller's expected profit in a double auction with a rational seller and a naive buyer. Then both players prefer the Dealer's Market over the double auction if* $f < \alpha \; - \; 2\sqrt{\alpha \boldsymbol{E}(P_S)}$.

Proof: See the Appendix. □

Proposition 21. *Let* $\boldsymbol{E}(P_B)$ *be the buyer's expected profit in a double auction with rational players. Then both players prefer the Dealer's Market over that double auction if* $f < \alpha \; - \; 2\sqrt{\alpha \boldsymbol{E}(P_B)}$.

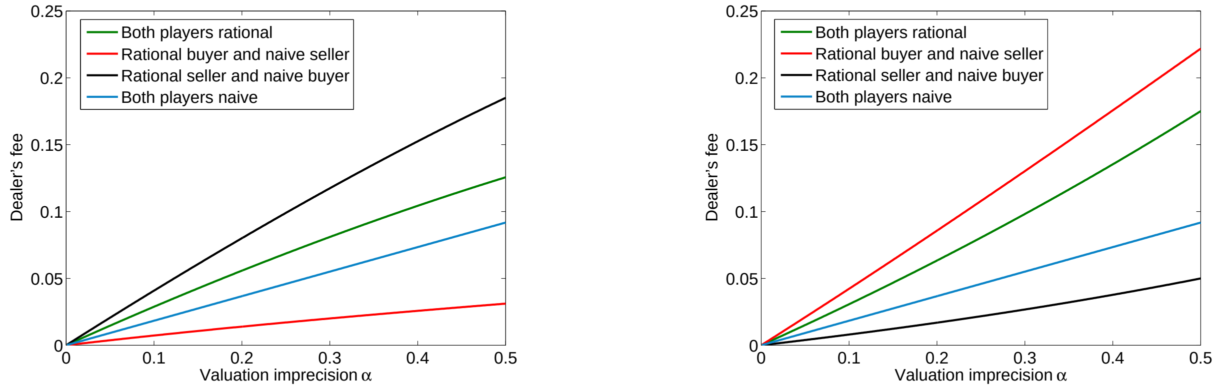
Proof: See the Appendix. □

Propositions 18 - 21 determine upper bounds for the dealer's fee such that the Dealer's Market is preferred over different double auction settings by both the buyer and the seller. If the dealer's fee f reaches the upper bound, then one of the market participants is indifferent between the Dealer's Market and a double auction, and the other party prefers the Dealer's Market. Therefore this upper bound f_{max} is the highest fee the dealer can charge for the Dealer's Market to be preferred over a double auction by the buyer and the seller. Secondly, this fee maximizes the dealer's profit.

Figure 12 shows this maximum fee that the dealer can charge each market participant, such that this player prefers the Dealer's Market over a double auction. The maximum fee $f_{max}(B)$ for the buyer to be indifferent is shown in figure 12 (a). Accordingly, figure 12 (b) shows the maximum dealer's fee $f_{max}(S)$ such that the seller is indifferent between the Dealer's Market and a double auction.

Figure 13 shows the fee f_{max} as a function of valuation imprecision. This is the upper bound for a dealer's fee such that a buyer and a seller are indifferent between a double auction and the Dealer's Market.

It can be seen that a higher valuation imprecision leads to a higher fee f_{max}. This is true for each double auction setting. As a result, the dealer profits from an increase in the players' valuation imprecision.



(a) $f_{max}(B)$ such that the buyer is indifferent

(b) $f_{max}(S)$ such that the seller is indifferent

Fig. 12 – Maximum fee such that the buyer or the seller is indifferent between the Dealer's Market and different double auction settings

In section 2.5 it was shown that a double auction with rational players is the unique Nash equilibrium of the four double auction settings. When a buyer and a seller place their bids rationally in a double auction, then there is exactly one such optimal strategy for either player. In particular, figure 13 shows the dealer's fee that makes the Dealer's Market exactly as favourable as double auctions in the bidding strategy equilibrium. That is, when both parties place their bids strategically. The buyer and the seller prefer the Dealer's Market over a double auction if the dealer's fee is below or equal to the fee f_{max}.

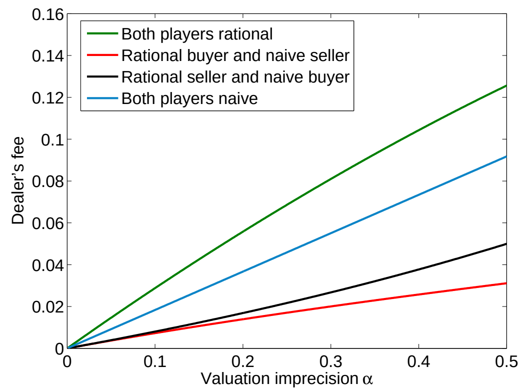


Fig. 13 – f_{max} such that both market participants are indifferent between the Dealer's Market and different double auction settings

We have analysed optimal dealer's fee schemes that allow the Dealer's Market to be Pareto efficient over the four double auction settings. The next theorem summarises this analysis and shows when the Dealer's Market is Pareto efficient over double auctions in bidding equilibrium. That is, when both players place their bids strategically.

Theorem 1. *Let the buyer and the seller be rational bidders in a two-player double auction. Let* $\boldsymbol{E}(P_B)$ *be the buyer's expected profit in that double auction. Then the Dealer's Market is Pareto efficient over the double auction if and only if*

$$0 < f < \alpha \; - \; 2 \, \sqrt{\alpha \boldsymbol{E}(P_B)}.$$

In this case the market under information asymmetry Pareto dominates the market under full information.

Proof: See the Appendix. □

The above theorem presents an equivalent condition for the Dealer's Market to be Pareto efficient over a double auction in bidding equilibrium. The theorem states that when the condition $0 < f < \alpha \; - \; 2 \, \sqrt{\alpha \mathbf{E}(P_B)}$ is satisfied, then all parties favour the Dealer's Market over a double auction in bidding equilibrium.

The condition is closely linked to the dealer's fee strategy. By applying a reasonable fee strategy, the dealer can therefore influence the players' market preferences. When he sets his fee accordingly, the buyer and the seller will prefer the Dealer's Market over bilateral negotiations.

Propositions 18 - 21 and theorem 1 are true for the players' ex-ante decisions. In fact, parties have to decide on their market preference first and choose either market according to their ex-ante preference. It is of importance for our model that players face an "either or" decision between a double auction and the Dealer's Market. Otherwise a buyer and a seller might bargain in a double auction in the first place. If their bilateral bargaining attempt is unsuccessful, they might consult the dealer in their attempt of successful bargaining. This strategy encourages adverse selection and thus disfavours the dealer. Consequently, the dealer should create an environment of information asymmetry, where he hides the respective reservation price of a buyer and a seller from each other.

Table 2 presents a numerical example to illustrate this section's analysis. It shows a buyer's and a seller's strategic options as a normal-form game. For this example, we set valuation imprecision to a maximum of $\alpha = 10\%$ and the dealer charges a fee of $f = 1.5\%$[6]. When bilateral trade in a double auction is the only option to the bargainers, then the naive strategy's profit is highest with $\alpha/6 \approx 1.67\%$. Playing naively is however not dominant. The equilibrium strategy in a double auction is that both players place their bids strategically. Their profit then is 1.20% (seller) and 1.27% (buyer). The dealer offers each party an expected profit of 1.81%. He dominates double auctions with rational players and further dominates

[6]The maximal dealer's fee for Pareto dominance of the Dealer's Market is $f = 10\% \cdot (1 - \sqrt{2/3}) \approx 1.84\%$, according to proposition 18.

bilateral trading with naively behaving players. Therefore both players prefer the Dealer's Market over a double auction. Thus, the Dealer's Market is Pareto efficient.

Table 2 – Illustration of the players' strategic options and their expected outcome. Valuation imprecision is $\alpha = 10\%$ and the dealer's fee is $f = 1.5\%$

		Buyer's strategy		
		Rational	Naive	Dealer
	Rational	1.20% / 1.27%	2.12% / 0.87%	
Seller's strategy	Naive	0.83% / 2.15%	1.67% / 1.67%	
	Dealer			1.81% / 1.81%

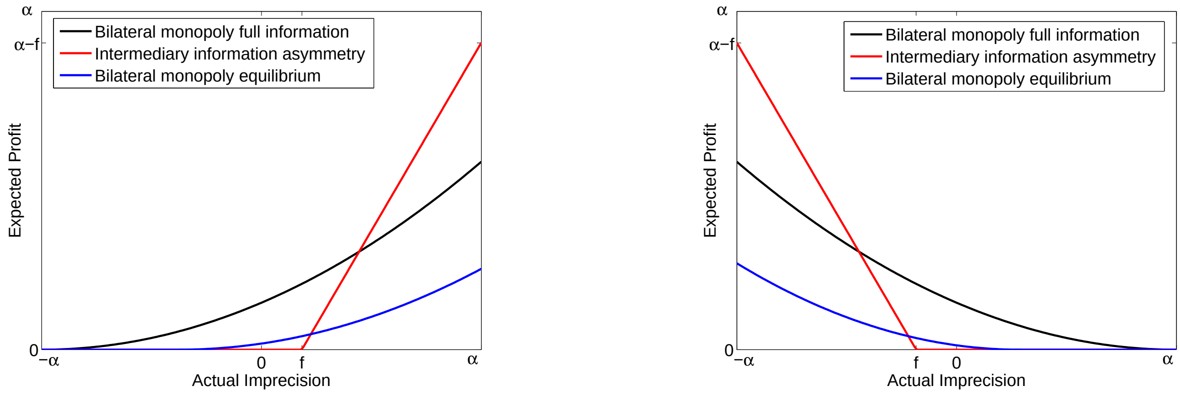
Table 3 calculates the maximum dealer's fee f_{max} that the dealer may charge the buyer and the seller. It shows that, independent of the valuation imprecision α, the dealer can charge the seller a higher fee than the buyer and still the Dealer's Market Pareto dominates double auctions. The optimal price the dealer offers the seller is thus $V(1 - f_{max}(S))$. The optimal price he charges the buyer is $V(1 + f_{max}(B))$. When a dealer applies this fee scheme, then his earnings per round trip transaction are $f_{max}(B) + f_{max}(S)$. That asymmetric fee strategy can further increase the dealer's earnings.

Table 3 – Preference of Dealer's Market over a double auction

	Buyer $f_{max}(B)$	Seller $f_{max}(S)$	Pareto Dominance f_{max}
$\alpha = 5\%$	1.46%	1.51%	1.46%
$\alpha = 10\%$	2.28%	3.07%	2.28%
$\alpha = 20\%$	5.58%	6.33%	5.58%
$\alpha = 25\%$	6.86%	8.05%	6.86%
$\alpha = 50\%$	12.56%	17.51%	12.56%

Figure 14 analyses the relative attractiveness of the Dealer's Market and the double auction. Assume a player, say the buyer, has some reservation price. Then the buyer's imprecision $z \in [-\alpha, \alpha]$ is below (or above) the average valuation. On the basis of z, the expected buyer's profit may be calculated.

In figures 14 (a) and (b), the players' actual valuation imprecision z is drawn on the x-axis. Expected profit, given z is drawn on the y-axis. Thus the area between each function and the x-axis represents a player's expected profit (not conditioned on z). Both players' profits are drawn for naive players and in bidding equilibrium, where both players place their bids strategically. This is compared to expected profit in the Dealer's Market.



(a) Buyer's expected profit

(b) Seller's expected profit

Fig. 14 – Expected players' profits as a function of their actual imprecision

When the buyer has a relatively high reservation price, then the Dealer's Market is most preferable for him. For an actual valuation imprecision below f, a buyer's expected profit in the Dealer's Market is zero. This can be seen by the red line in figure 14 (a), where a buyer's profit is zero for an actual imprecision below f. In this case a double auction generates a higher expected profit. This can be seen in figure 14 (a), when the black line (double auction with naive players) and the blue line (double auction with rational players) are above the red line (profit in the Dealer's Market).

In figure 14 (b) the opposite is true. The seller profits most from the Dealer's Market, if her reservation price is minimal. In fact, an actual valuation imprecision above $-f$ causes zero profit in the Dealer's Market, whereas the expected profit in a double auction still exceeds zero.

As shown in section 2, double auctions under full information always dominate those with rational individuals, because in a double auction the expected profit is always higher under full information. If the profit in the Dealer's Market on average is higher than the average profit in a double auction, then the Dealer's Market is Pareto efficient. In figure 14 the dealer's fee is $f^* = \alpha(1 - \sqrt{2/3})$. As shown in proposition 18, this fee allows for the players' indifference between double auctions under full information and the Dealer's Market. As a result, the areas under the red and black graphs in figure 14 are identical. For fees lower than f^*, the Dealer's Market is more efficient than double auctions under full information and we have Pareto efficiency of information asymmetry over full information. In this case, the area under the red line is bigger than the area under the black line, in figures 14 (a) and (b).

We have shown that the Dealer's Market under information asymmetry can be Pareto efficient over double auctions, even those under full information. By relaxing our assumptions, the next section generalizes this result in a variety of aspects.

4.2 Pareto Efficiency in more General Frameworks

This section analyses a more general framework of double auctions than presented in section 2. It proves that the dealer can determine a suitable fee strategy such that his fee scheme is Pareto efficient over the more general double auction framework.

In the prior sections, a buyer and a seller are assumed to have "true" identical distributions of valuation imprecision. Further, the players are aware of these "true" distributions of imprecision. However, in a more general framework, both players anticipate their own distribution of valuation imprecision and their negotiant's imprecision distribution. In addition, true imprecision distributions and anticipated distributions are not identical in general. This matter will be discussed in the remaining part of this section.

Let us model the conditions given above. Assume the buyer anticipates his valuation imprecision to be represented by a random variable X_B and the seller's valuation as anticipated by the buyer is given by the random variable Y_B. Now the buyer may determine his optimal offer strategy $b \leq 1$, as shown in proposition 4. Analogously, the seller anticipates her and the buyer's valuation imprecision to be represented by the random variables X_S and Y_S, respectively. Like the buyer, the seller may determine her optimal offer strategy $s \geq 1$ from her anticipations.

As in double auctions from section 2, rational behaviour disfavours both parties.

Theorem 2. *Assume the true buyer's valuation imprecision is represented by the random variable X. Likewise, the seller's true valuation imprecision is represented by the random variable Y. Assume that $\boldsymbol{P}(X > Y) > 0$ (otherwise the deal probability is zero deterministically). Let the buyer's anticipated optimal offer strategy be $b \leq 1$ and the seller's anticipated optimal offer strategy be $s \geq 1$.*

Then the buyer's and seller's expected profit is highest when they apply offer strategies $b = s = 1$.

Proof: See the Appendix. □

The above theorem implies that a buyer and a seller are better off if they place their offers naively. Rational behaviour therefore harms the buyer and the seller in general terms. According to theorem 2, the more general double auction in bidding equilibrium is at most as efficient as a double auction with naive individuals. Thus the profit-pie (sum of the profits of the players) in equilibrium is smaller than the profit-pie naively behaving players share. The inefficiency of the bidding equilibrium can be exploited by the dealer.

Theorem 3. *If a buyer and a seller behave rationally in a double auction, the dealer may find a suitable fee strategy such that his earnings are positive and the players prefer engaging the dealer. Thus the Dealer's Market Pareto dominates that double auction.*

Proof: See the Appendix. □

From theorems 2 and 3 general results about double auctions and the Pareto dominance of the Dealer's Market can be derived. In double auctions rational behaviour of the players harms both parties. Thus, generalising double auctions, the players would profit more if their offer price is equal to their reservation price. This naive strategy however, is not an equilibrium. When a seller's strategy is fixed, then a buyer's attempt to increase his share of the profit-pie shrinks the overall profit-pie at the same time. The buyer's strategy therefore ensures him maximal profit and the seller is left with a smaller share of a smaller profit-pie. The same effect is true for a strategic seller. In equilibrium, a buyer and a seller then share a smaller profit-pie than that, which would have resulted from the naive strategy.

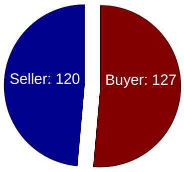
In the Dealer's Market, the profit-pie has maximum size. The overall profit may even be bigger than in a double auction with naive individuals[7]. The dealer's optimal strategy is to offer the bargainers more profit in absolute terms than they would expect from the smaller profit-pie in a double auction. This way, there is some proportion of the bigger profit-pie left that he can have for himself.

As an example, this is illustrated in figure 15. It provides the profits from a double auction with valuation imprecision of $\alpha = 10\%$ and an average valuation that is given by 10.000. In this example, the dealer's fee is $f = 0.25\%$.

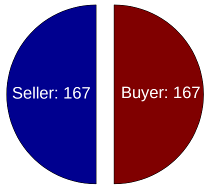
Figure 15 (a) shows the expected profit of a rational buyer and a rational seller in the double auction: the buyer's expected profit is 127 and that of the seller is 120. Thus they share a profit-pie of size 247. When the parties behave naively, as shown in figure 15 (b), each player gets half of the profit-pie. The naive strategy generates a profit of 167 per player. That is, the profit-pie has a size of 334. In comparison to placing their bids strategically, the naive strategy thus results in a bigger profit-pie.

Proposition 18 additionally states that the Dealer's Market can in fact Pareto dominate double auctions under full information. This means that the dealer may install a fee structure such that the profit-pie in the Dealer's Market is bigger than the biggest pie available in a double auction; even bigger than the pie that full information offers. As a consequence, a buyer and a seller always prefer the Dealer's Market ex-ante, even if full information was available in a double auction. This fact can be observed by comparing figures 15 (b) and (c). The Dealer's Market offers a buyer and a seller each a profit of 238. At the same time, the dealer's gain is 50 per round-trip transaction. Thus, the Dealer's Market offers a profit-pie of size 526. Therefore, the profit in the Dealer's Market exceeds that from a double auction with naive or strategic players.

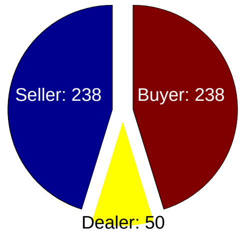
[7] When the dealer's fee is below $\alpha(1-\sqrt{2/3})$, then the profit-pie in the Dealer's Market is bigger than that of a double auction with naive individuals, according to proposition 18.



(a) A double auction with rational players with a total profit of 247



(b) A double auction with naive players with a total profit of 334



(c) The Dealer's Market with a total profit of 526

Fig. 15 – Size of the profit-pie in double auctions and the Dealer's Market. Average valuation is 10.000, valuation imprecision is $\alpha = 10\%$ and the dealer's fee is 0.25%

So far, we have presented a theory that explains when information asymmetry can be Pareto efficient over full information. Let us illustrate the theory by a practical example.

4.3 Example: The Headhunter Game

In this game the job candidate is asked to propose the salary S she expects to earn. The candidate gets hired with salary S if the employer is willing to pay at least the proposed salary S. Otherwise the candidate is not hired.

In our model, the candidate can be considered the seller who offers her time and skills to the buyer (the employer). Both players have a certain reservation price. The employer does not need to adjust his reservation price as the level of salary exclusively depends on the candidate's proposed salary S. The candidate can ask the employer for a higher salary S as she truly requires. This strategy is risky because the probability of not getting hired increases. However, the strategy's possible benefit is a higher compensation. In fact, a *reasonable* increase of the job candidate's minimum salary requirement is optimal. For a job candidate, this reasonable increase can be modelled within the framework of a double auction with rational seller, as in section 2.3. Furthermore, the negotiation skill parameter satisfies $k = 0$, because the salary only depends on her proposed salary. Negotiation skills are irrelevant.

In our example, we set the average salary to $50,000$€ and the maximum valuation imprecision to $\alpha = 20\%$. Then reservation prices are uniformly distributed on $[40,000; 60,000]$€. Assume the candidate to have a salary requirement of $49,000$€ and the employer to have a reservation price of $54,000$€. If the candidate pursues an optimal offer strategy according to proposition 4[8], then her optimal strategy is to increase her salary requirement by

[8] Here $k = 0$, whereas in the proposition $k = 1/2$.

$s_{opt}(1) = 14.57\%$. In fact, her optimal proposed salary then is $S = 56,139.30$€. When the candidate acts upon the optimal strategy, her proposed salary exceeds the employer's reservation price and the candidate is not hired. In a double auction, negotiations therefore fail.

Now consider a recruitment firm (the dealer) that has experience and sufficient market expertise to know that the average salary for the job is $50,000$€. Moreover, the recruiter's fee strategy is asymmetric and the recruitment fee is fully charged to the employer. Let that fee be 4% of the average salary[9].

The recruiter offers the job to the job seeker with a salary of $50,000$€. The job seeker accepts as the offer is above her salary requirement. The employer pays $50,000$€ for salary and $2,000$€ recruitment fee. The employer's total expenses are below his reservation price of $54,000$€. When she places the candidate successfully, then the recruiter earns $2,000$€ in fees.

It is worth mentioning that the employer and the employee profit from this even more, when the employment relationship holds longer, as the recruiter's fee needs to be paid only once. With the argument of longer relationships, the recruiter may even charge a higher fee.

In this numerical example, direct negotiations are unsuccessful but the recruitment firm is able to place the candidate successfully. Therefore all players prefer hiring the recruitment firm. Even if direct negotiations lead to success it is ex-ante optimal for all players to hire the recruitment firm.

However, it is necessary that the recruiter limits information between both parties. Otherwise players call upon the recruiter's services only when direct negotiations were unsuccessful. Then the recruiter is consulted by lemons to a greater extent: employers who pay small salaries and job-seekers with high salary requirements. Due to adverse selection, the recruiter's expected earnings would be smaller. Therefore the recruiter benefits from information asymmetry between employer and employee.

The Headhunter game has presented an example for the theory of this article. It showed how a Dealer's Market (i.e. the headhunter) is preferred by all parties compared to direct loan negotiations. The next section presents an application on initial public offerings (IPOs). We will show that our theory can be applied to solve the IPO underpricing puzzle.

5 A Solution for the IPO Underpricing Puzzle

In this section we apply our theory to capital markets and present a solution for the IPO underpricing puzzle. Let us introduce our capital market model and adopt it to our theory.

[9]The employer's expected profit from that double auction is 2.07%, according to proposition 2. Then proposition 17 calculates a maximum fee of 7.13% that the recruiter may charge the employer. When the recruiter does not exceed that fee, then ex-ante his services are preferable over bilateral salary negotiations.

First, we focus on a firm that has the alternative of raising private or public equity capital. In our model, the firm may be viewed as the seller of its equity securities. In section 5.1 we concentrate on the firm's alternatives. Section 5.2 provides an analysis of an investor's options on how to buy equity capital of a firm. Conditions for an IPO to be preferred by all parties are established in section 5.3. This section further shows that an IPO is underpriced on at least average. Moreover it provides conditions, when an IPO is underpriced deterministically. Finally, section 5.4 discusses a numerical example for the Pareto dominance of an IPO and its underpricing.

5.1 Alternatives for raising Capital: IPO and Venture Capital

Assume a corporation wants to sell a fixed share S of its owner rights. This corporation's management then may try to sell the share S privately by negotiating and placing it with an individual or an institution, such as a venture capitalist. The corporation's management further may hire an investment banker to place the corporation's share in an IPO. According to our model, the firm and the venture capitalist are unable to precisely value the firm's share. Each party makes an estimation error that is uniformly distributed around an average valuation V for the share.

Let us first model negotiations for private placement, for example with a venture capitalist: the firm's management's valuation is V_M and a venture capitalist's valuation is V_C. The management wants to raise at least the amount V_M for the corporation's share S. The venture capitalist is willing to pay at the most V_C for it. Then V_M and V_C are the players' reservation prices. During price negotiations, the parties simultaneously reveal their offers O_M and O_V to their counter party. The deal price for the corporation's share then is $P = kO_M + (1-k)O_V$ (for some k on the interval $[0,1]$), if the venture capitalist's offer is higher than or equal to the management's offer. Otherwise the deal is unsuccessful. As described in the introduction to section 2, the parameter k represents the players' negotiation skills. The offers O_M and O_V are derived by the offer strategies and the reservation prices of the negotiators. Negotiations between a corporation and a venture capitalists are consequently an example of bilateral negotiations as discussed in section 2.4, where both parties behave strategically.

We showed in section 4, that a dealer (an investment banker), may determine his fee strategy such that the seller (a firm), prefers the Dealer's Market over bilateral negotiations with the buyer (a venture capitalist). In this case, the investment banker would sell the share S to the public. When the investment banker's fee strategy is chosen adequately, the firm chooses employing the investment banker to issue an IPO over bilateral negotiations with a venture capitalist.

5.2 The Investor's Alternatives

Assume an investor is interested in a certain share of a firm that is considering an IPO. Then the investor faces three options: the investor may (a) engage in bilateral negotiations with the firm. Alternatively it can (b) wait for the IPO to be issued and buy the newly issued shares. Finally, the investor may (c) invest in a portfolio that spans the issuing firm's shares, that is, the investor buys a duplicating portfolio that generates returns and bares risk exactly as the firm's newly issued shares.

As in the previous section 5.1, case (a) can be modelled as a double auction, where an investor and the firm strategically place their bids. We have already shown in section 4 that an investment banker with reasonable fee strategy dominates that double auction by issuing an IPO. Thus case (a) is dominated by case (b).

Since the stock market provides an exact stock price for each asset in the spanning portfolio, the value of the spanning portfolio V is known with certainty in case (c). In other words, there is perfect valuation for the spanning portfolio and thus also perfect valuation for the firms share.

The following section proves that, in order to dominate option (c), the investment banker (option (b)) needs to underprice the IPO. Otherwise the investor prefers the spanning portfolio.

5.3 An IPO under Asymmetric Information dominates raising Capital from a Venture Capitalist with Full Information

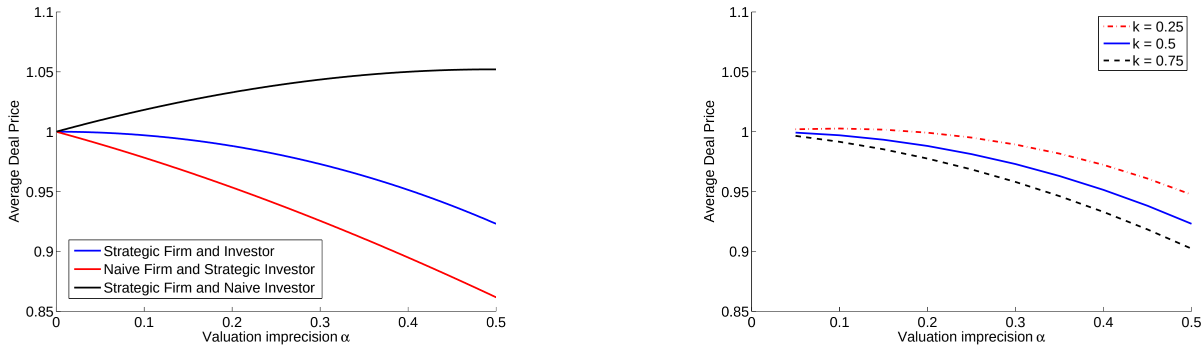
This section formalises the prior argumentations and proves under which conditions an IPO is the dominant alternative for an investor and the firm. We furthermore present conditions that cause underpricing of an IPO. First, an analysis of the average prices in bilateral negotiations between a firm and a venture capitalist is conducted. It will be shown that this price is below the average valuation V. This means that even in bilateral negotiations we find underpricing. To understand this, we first calculate the average price resulting from bilateral negotiations.

Proposition 22. *When bilateral negotiations between a firm and a venture capitalist are successful, the average deal price is*

$$\boldsymbol{E}(P|\text{Success}) = \frac{1}{(b\mathrm{b}_2 - s s_1)^2}\left(kb\left[\frac{b\,\mathrm{b}_2{}^3}{3\,s} - \mathrm{b}_1\left(\frac{\mathrm{b}_2{}^2}{2} - \frac{\mathrm{b}_1{}^2\,s^2}{6\,b^2}\right)\right]\right.$$
$$\left. + (1-k)s\left[\frac{b^2\,\mathrm{b}_2{}^3}{6\,s^2} - \mathrm{b}_1{}^2\left(\frac{\mathrm{b}_2}{2} - \frac{\mathrm{b}_1\,s}{3\,b}\right)\right]\right).$$

Proof: See the Appendix. □

Proposition 22 gives a formula for the average deal price in case of a bargaining success. In section 2 we established optimal offer strategies (b, s) for both players in different bilateral negotiation settings. Using these strategies makes it possible to calculate the average deal price in the different bilateral negotiation settings discussed in section 2.



(a) Average deal prices in bilateral negotiations ($k = 1/2$)

(b) Average deal prices in bilateral negotiations with strategically behaving players

Fig. 16 – Average deal prices in bilateral negotiations

Figure 16 shows these prices as a function of valuation imprecision. When both players place their offers naively, the average deal price is V, according to proposition 22. Figure 16 (a) analyses the remaining bilateral negotiation alternatives[10]. When the firm is behaving strategically and the investor naively, there is some overpricing that increases with valuation imprecision. It reaches approximately 5% of V at the most. When the investor behaves strategically and the firm naively, then the investor reduces his offer, whereas the firm's offer and reservation price are identical. This has the effect, that the average price is below V and decreasing in valuation imprecision. That price reaches down to approximately 86% of V, which is equivalent to 14% underpricing. When one player behaves strategically and the other naively, then the strategic player can shift the average price to his advantage. When both players behave strategically (this is indicated by the blue graph) the average deal price is decreasing in valuation imprecision and always below the average valuation V. In this case, there is significant underpricing of up to approximately 7% of V. This property of bilateral negotiations is unexpected, as one might reason that the price behaves symmetrically when both parties place their bids strategically. This effect has been explained in section 2, where it was shown that a buyer has the upper hand in bilateral negotiations. That section showed that a buyer's profit is higher than that of a seller when they both bid strategically.

Figure 16 (b) analyses the influence of negotiation skills on the average deal price in bilateral negotiations with strategic individuals. The parameter k represents the relative

[10]Without loss of generality we set $V = 1$ in this figure

negotiation skills of the parties. When $k = 0$, the deal price is given by the firm's offer. This means that the firm has superior negotiation skills compared to the investor. When $k = 1$ the opposite is true. Then the deal price is given by the investor's offer and his negotiation skills exceed the firm's. If $k = 1/2$, both parties share the same negotiation skills. The graph shows that when the investor's negotiation skills are better than the firm's, the deal price decreases, and vice versa. If the firm gains the upper hand in bilateral negotiations (for $k = 0.25$), the deal price is approximately constant at V. The average deal price even diminishes for a valuation imprecision above 20%. We conclude that in bilateral negotiations with strategic individuals, there is underpricing, even when the firm's negotiation skills exceed that of the investor.

The next proposition shows that an IPO with asymmetric information dominates bilateral negotiations of a firm and a venture capitalist under full information. However, the investment banker needs to underprice an IPO in order to compete with the capital market which provides a spanning portfolio.

Proposition 23. *Let $\boldsymbol{E}(P_S)$ be the firm's expected profit in bilateral negotiations with strategic players. Define*

$$f_{max} := \alpha \; - \; 2\,\sqrt{\alpha \boldsymbol{E}(P_S)}.$$

When the investment banker offers to buy a firm's shares for $B_d > V(1 - f_{max})$ and offers to sell the shares at the price S_d within the bounds $B_d < S_d < V$, then an IPO is Pareto efficient.

Proof: See the Appendix. □

Proposition 23 develops a pricing strategy such that the IPO is Pareto efficient. The intuition behind the proposition is that, while the investment banker does not exaggerate his fee, the IPO is Pareto efficient over bilateral negotiations between a firm and a venture capitalist. We can infer from proposition 23 that the investment banker offers the IPO shares to investors below their average valuation V.

The next lemma shows an interesting property of an IPO which can be derived from proposition 23.

Lemma 2. *When the investment banker applies a fee strategy as in proposition 23, IPOs are underpriced.*

Proof: See the Appendix. □

Lemma 2 states that IPOs are underpriced. In order to attract investors for the IPO, the investment banker needs to compete with an efficient market, according to proposition

23. Prices in the efficient market are exact and therefore the spanning portfolio for the IPO shares has the exact price V. To attract investors, the investment banker needs to price the IPO shares below that value V. In other words, the investment banker underprices the IPO shares. Otherwise investors will reject the investment banker's offer.

Note that when a spanning portfolio for the firm's shares does not exist, the investor looses option (c), the exactly priced duplicating portfolio. Then the investment banker's strategy to attract an investor and a firm to the IPO is simpler.

Proposition 24. *Assume there is no spanning portfolio for the firm's shares. Let $\boldsymbol{E}(P_B)$ and $\boldsymbol{E}(P_S)$ be an investor's and a firm's expected profits in bilateral negotiations with strategic players, respectively. Define*

$$f_b := \alpha \; - \; 2\sqrt{\alpha \boldsymbol{E}(P_B)}$$
$$f_s := \alpha \; - \; 2\sqrt{\alpha \boldsymbol{E}(P_S)}.$$

When the investment banker offers to buy a firm's shares for $B_d > V(1-f_s)$ and offers to sell the shares at price S_d, with $B_d < S_d < V(1+f_b)$, then an IPO is Pareto efficient.

Proof: See the Appendix. □

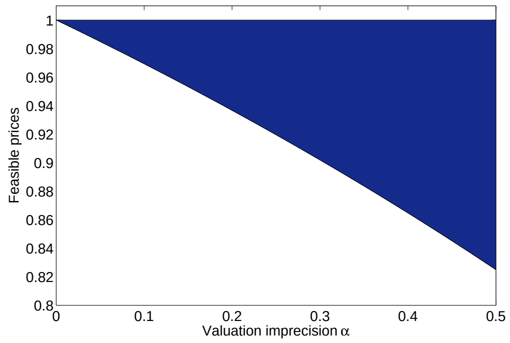
When there is no spanning portfolio for the firm's shares, it is simpler for the investment banker to install a fee structure such that an investor and a firm are in preference of an IPO. Dropping the spanning portfolio assumption, IPOs are still underpriced on average, but to a smaller extent.

Lemma 3. *Assume there is no spanning portfolio for the firm's shares. When the investment banker applies a fee scheme as in proposition 24, IPOs are underpriced on average.*

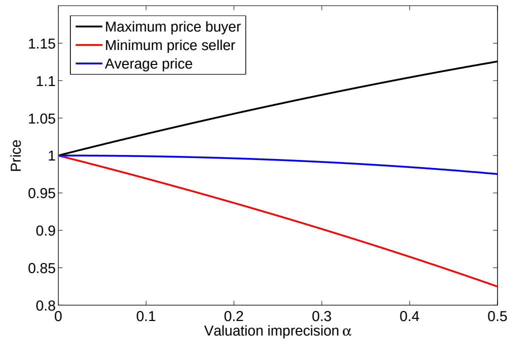
Proof: See the Appendix. □

In general it is possible that IPOs may be overpriced if the assumption of the existence of a spanning portfolio is dropped. Lemma 3 however states, that on average they are underpriced. Furthermore, section 2 proved that a firm's profit is lower than that of an investor in bilateral negotiations. Section 4 discussed that the investment banker may therefore install an asymmetric fee structure to make his market Pareto efficient over bilateral negotiations. In fact, to successfully compete with bilateral negotiations, the investment banker may offer the firm a lower profit than an investor, because a firm's gain in bilateral negotiations is lower than an investor's. This asymmetric offer strategy is responsible for the IPO to be underpriced on average. We conclude that IPO underpricing is a robust property. However, an IPO is deterministically underpriced when a spanning portfolio exists.

The underpricing of IPOs is illustrated in figure 17. Using propositions 23 and 24 we computed feasible IPO prices for different valuation imprecision.



(a) Feasible prices with spanning portfolio



(b) Price borders without spanning portfolio

Fig. 17 – Feasible price strategies of the dealer

Figure 17 (a) shows feasible IPO prices with the spanning portfolio hypothesis, that is an investor's option (c). The blue area in this figure represents feasible investment banker's price strategies. It can be seen that IPOs are always underpriced. The intensity of the underpricing is the dealer's choice and depends on several factors. For example, when a firm is more inclined to sell its shares, it is easier for the investment banker to satisfy the firm with a lower price. As a result, the IPO can be underpriced to a greater extent, if the investment banker does not change his fee. When a firm needs more encouragement for an IPO, the investment banker may offer a higher price. While the investment banker's earnings are constant, the initial shares become more costly. They are, however, still underpriced in order to dominate the investor's spanning portfolio alternative.

Figure 17 (b) shows price bounds without the option of a spanning portfolio. The minimum price the firm demands is illustrated by the red line. It is equal to the lower price bound in figure 17 (a). The investment banker needs to offer the firm a higher price than this lower bound. When there is no spanning portfolio, the investment banker may charge the investor a price higher than V. The maximum price he may charge the investor is illustrated by the black line in figure 17 (b). The investment banker has more freedom in pricing and thus possibly more fee earnings when imprecision increases. In this case however, IPOs are not necessarily underpriced: the investment banker may set an IPO price from the whole spectrum between $1 - f_s$ (the red line) and $1 + f_b$ (the black line). Actual pricing may be dependent on numerous parameters, such as the necessity for sellers to raise capital and investors to buy those shares. The size of the IPO market and the supply of investment capital, negotiation skills and the investment banker's minimum fee requirement are further factors that determine the actual IPO price.

5.4 Numerical Example: IPO Underpricing

For the numerical example we allow a share of a company to have average value $V = 100$. Assume maximum valuation imprecision is 20%. Then the players' valuations are uniformly distributed on $[80, 120]$. However, the players do not know the valuation interval. Let the management's valuation be $V_M = 95$ and the venture capitalist's valuation be $V_C = 105$. Both players behave strategically. Therefore they adjust their reservation prices by certain factors s and b, respectively. The firm's management increases its valuation by a certain percentage and the venture capitalist reduces his valuation by a certain percentage. Table 1 on page 21 gives these optimal offer strategies $s = 1.0684$ and $b = 0.9345$. Thus the management's offer price is $O_M \approx 101.50$ and the venture capitalist's offer price is $O_C \approx 98.12$. In this case, the firm's management demands more for the corporation's share than the venture capitalist is willing to pay. That is, with strategically behaving players the deal is unsuccessful, even though with naive behaviour it would have taken place.

Let there be an investment banker with market expertise. In this case the investment banker knows the average valuation V. She strategically conducts an IPO and offers the firm a price of B_d for its shares. Further, the investment banker offers to sell the shares to the investor at price S_d. Proposition 23 introduces the bounds for these prices. Accordingly[11], the lower bound of the offer to the firm is $B_d > V(1 - f_{max}) \approx V(1 - 0.0635) = 0.9365V$. An investor needs to be priced within the bounds $0.9365V < B_d < S_d < V$.

Assume the investment banker's strategy is $B_d = 0.97V = 97$ and $S_d = 0.99V = 99$. In this case the investment banker sells the share for 99 and the firm raises 97 for the share after the investment banker's fee. In fact, 97 exceeds the management's valuation and the management agrees to sell. Further, 99 is less than the investor's valuation and consequently he agrees to buy the share. The firm thus earns an additional value of 2 compared to its minimum requirement. The investor accepts the offer that represents a gain of 6 compared to his imprecise valuation. As the deal is successful, the investment banker earns a fee of 2 for successful placement. The IPO is underpriced, since it is sold for 99 and its precise value is 100. The investor may therefore generate an additional gain of 1, if the shares are later priced precisely in the stock market.

The analysis of the example shows that employing an investment banker and conducting an IPO is preferable over raising funds from a venture capitalist. In addition, in the example above it is critical to hide information from the firm and the venture capitalist and to create information asymmetry. In fact, in a world of full information, the valuation of the firm and the venture capitalist would be common knowledge. Then direct negotiations would

[11] According to table 1 on page 21, the firm's (the seller's) expected profit in bilateral negotiations is 2.33%. From that profit, a maximum fee $f_{max} \approx 6.35\%$ can be calculated with the formula that proposition 23 provides.

be successful and the dealer would not conduct the IPO. As a result, both, a firm and investors would only go to the investment banker when direct negotiations fail. Thus, when all valuations are known, the IPO is not the preferable solution. An IPO is Pareto dominant only when investor and firm are unaware of their counterpart's valuation under information asymmetry. The investment banker thus faces a Lemons problem under full information. To avoid this adverse selection, the investment banker systematically hides information from the firm and the investors. This strategy ensures ex-ante Pareto efficiency of the IPO over bilateral negotiations. In other words, the IPO is the first priority of investors and a firm in this case.

6 Conclusion

In contrast to common bilateral trading literature the model that has been developed in this paper also considers imprecise valuation. This means that each player has a reservation price for a good, a service or a share of a firm. Although each player knows that his valuation is imprecise, a player can not determine whether his reservation price is high or low compared to the other player's reservation price, as he has no benchmark. Considering bilateral trade with imprecise valuation, the players' optimal offer strategies were calculated and implications for the market's efficiency were analysed. Furthermore, the advantage of a market maker over bilateral trade was discussed. Finally, it was shown that an IPO under information asymmetry may be Pareto efficient over direct negotiations under full information. This application further presented an explanation for IPO underpricing.

We have shown that in a two-player double auction, the bargainers have the highest expected profit when they behave naively. Naive behaviour leads the parties to make offers at their reservation price. In fact, we proved that naive behaviour and full information lead to equivalent strategies in a double auction. Even though naive behaviour is optimal, it is not an equilibrium strategy. The equilibrium, under which both players strategically determine their offer strategies, generates less expected profit than that which is generated under the naive strategy. Strategic behaviour maximises individual profit, however it leads to the reduction of the set of feasible trades. A party's rational behaviour thus harms the other party significantly. Section 2 calculated the optimal bidding strategies within double auctions and the players' resulting profits explicitly.

Section 3 introduced a dealer who quotes bid and ask prices. This is the only information that is revealed to a buyer or a seller. As the parties' reservation prices remain sealed, there is information asymmetry in the Dealer's Market. A buyer's and a seller's profit as well as the dealer's gain in the Dealer's Market were analysed in detail in that section.

Double auctions and the Dealer's Market were compared in section 4. That section introduced conditions on the parties' market preferences. When a dealer sets his prices reasonably, then all parties are in favour for the Dealer's Market. This significant section helped to understand why traders may prefer the Dealer's Market under information asymmetry over a double auction under full information.

We showed in section 4 that a dealer's strategy can Pareto dominate a double auction with a rational buyer and a rational seller under non-restrictive conditions. Even when a double auction is most efficient (when full information is available), the dealer may set his fee low enough such that the Dealer's Market under information asymmetry is Pareto efficient over the double auction.

In summary, double auctions with two rational players are suboptimal. When a dealer is not involved, a double auction with naive players is the most efficient option. However, employing a dealer generates the highest gain for buyer and seller as long as the dealer's fee is set reasonably. When the surplus of wealth that a dealer generates is shared among all parties, the Dealer's Market is efficient and information asymmetry Pareto dominates full information. At last, section 4 showed that this major result is true, even when our model is further generalised.

In our model, the difference between a player's reservation price and the deal price is that player's profit. Imprecise valuation causes reservation prices to diverge from deal prices and thereby leads to an increased gain from trade. Furthermore, the dealer may receive more fees when deal and reservation prices diverge. Therefore all parties gain from increasing imprecision in valuation.

Our theory may be applied to salary negotiations. We modelled a two-player headhunter game as a two-player double auction. We showed that the employer and the employee profit from hiring a recruitment firm, despite paying fees for that service.

Comparing double auctions and a Dealer's Market naturally leads us to the capital market in section 5, where we tied our theory to a firm that is raising equity capital and an investor that faces several options of investing in the firm's shares. With the developed strategy we proved that a dealer (i.e. an investment banker) can use information asymmetry and an adequate fee structure such that firm and investor prefer an IPO over their other alternatives. When there is a portfolio that spans the firm's share, the IPO is significantly underpriced. However, when we drop this condition, then IPOs may be overpriced. However, in this case, they are still underpriced on average. IPO underpricing thus is robust with respect to the existence of a spanning portfolio. A numerical example was presented to support this theory.

Our underpricing model has testable implications that are distinct from other information asymmetry IPO models. In particular, Baron (1982) is consistent with the one-day abnormal returns in IPOs. In that IPO model, underpricing is used to induce an optimal selling effort by an investment banker who is better informed about demand conditions than the issuing firm. However, Baron (1982) is not consistent with the long-term IPO underperformance, as both, the firm and the investment banker have compatible incentives to attract long-term investors. In Welch (1989) higher valued firms use underpricing to signal their quality. Long-term underperformance of the supposedly higher valued firms is therefore inconsistent with that model. Rock (1986) is not necessarily inconsistent with long-term underperformance. However, in that model, the investment banker has an incentive to reduce information asymmetry. A testable implication of our model is, that empirical investigation of investment banking should show that an investment banker does not assert effort to mitigate information asymmetry and to remove valuation uncertainty ex-ante. That is, the valuation uncertainty of an investor and the issuing firm is the key to generate wealth from an IPO.

Our paper relates to two philosophical ideas. First, we predict that the economies which use the IPO process as a significant method for financing their corporate output generate more wealth than those that use more private financing. Not only public corporations have easier access to raising capital due to limited liability, but also the IPO process is subject to less failure than privately raising equity capital. Secondly, there is an alternative behavioural finance philosophy behind our arguments. We showed that equity investors gain when their reservation price is higher than the IPO price. This gain may exist due to valuation imprecision or explained psychologically. In either case, an investment banker operating under information asymmetry, is able to finance more projects than privately raising equity in bilateral negotiations. IPO investors immediately gain from IPO underpricing. Our model is also consistent with investors' reservation prices adjusting to post-IPO information, revealed in market places. Therefore the IPO process may optimistically finance net present value positive projects that perform relatively worse than privately financed ones. That is, there is long-term IPO underperformance. We believe that as long as these projects are net present value positive, an economy that uses the IPO process prominently generates more wealth than economies that do not.

Our analysis applies to any market, where players can not value an asset precisely. For example it can be used to analyse auctions or "buy it now"-offers on eBay or optimal pricing strategies for sellers on Amazon. When more players are present on both sides, a group of buyers may bargain with a group of sellers on a platform with a certain design. As we did with the two-player double auction, that platform's efficiency can be analysed and compared to an intermediary's market design. Interesting capital market applications, as for instance the analysis of treasury bond auctions, develop naturally.

7 Appendix

Proof of Proposition 1: Let a buyer's and a seller's valuation be $i_B \stackrel{d}{=} unif[b_1, b_2]$ and $i_S \stackrel{d}{=} unif[s_1, s_2]$ with $b_1 \leq b_2$ and $s_1 \leq s_2$. Then the deal probability is

$$p_d = P(i_B \geq i_S) \qquad = \begin{cases} 0 & b_1 \leq b_2 \leq s_1 \leq s_2 \\ 1 & s_1 \leq s_2 \leq b_1 \leq b_2 \\ \frac{1}{2\Delta b \Delta s}(b_2 - s_1)^2 & b_1 \leq s_1 \leq b_2 \leq s_2 \\ \frac{1}{2\Delta b \Delta s}\left((b_2 - s_2)\Delta s + \frac{1}{2}\Delta s^2\right) & b_1 \leq s_1 \leq s_2 \leq b_2 \\ \frac{1}{2\Delta b \Delta s}\left(2\Delta b \Delta s - (s_2 - b_1)^2\right) & s_1 \leq b_1 \leq s_2 \leq b_2 \\ \frac{1}{2\Delta b \Delta s}\left(2\Delta b(b_1 - s_1) + \Delta b^2\right) & s_1 \leq b_1 \leq b_2 \leq s_2. \end{cases}$$

Merging the cases above, we obtain

$$p_d = \frac{\mathbf{1}_{s_1 \leq b_2}}{\Delta b \Delta s}\left(b_2(min(b_2, s_2) - s_1) - b_1(max(b_1, s_1) - s_1) - \frac{1}{2}\left(min(b_2, s_2)^2 - max(b_1, s_1)^2\right)\right).$$

From that formula we derive the special case that is considered in the proposition. □

Proof of Proposition 2: We calculate expected profit of the buyer. The seller's expected profit is obtained analogously.

$$\begin{aligned} \mathbf{E}\left(P_B\right) &= \mathbf{E}\left(\mathbf{1}_{Deal}\left(V_B - P\right)\right) = \mathbf{E}\left(\mathbf{1}_{Deal}\left(V_B - kbV_B - (1-k)sV_S\right)\right) \\ &= \frac{1}{\Delta b \Delta s}\int_{s_1}^{s_2}\int_{b_1}^{b_2} \mathbf{1}_{Deal}\left(x - kbx - (1-k)sy\right)\ dxdy \\ &= \frac{1}{\Delta b \Delta s}\left((1-kb)\int_{s_1}^{s_2}\int_{b_1}^{b_2} \mathbf{1}_{Deal}\ x\ dxdy - (1-k)s\int_{s_1}^{s_2}\int_{b_1}^{b_2} \mathbf{1}_{Deal}\ y\ dxdy\right) \end{aligned}$$

We calculate the two integrals separately in order to keep the terms more clear:

$$\begin{aligned} \int_{s_1}^{s_2}\int_{b_1}^{b_2} \mathbf{1}_{Deal}\ x\ dxdy &= \int_{s_1}^{s_2}\int_{b_1}^{b_2} \mathbf{1}_{\{bx \geq sy\}}\ x\ dxdy \\ &= \int_{b_1}^{(b/s)b_2}\int_{(s/b)y}^{b_2} x\ dxdy = \frac{b\,\mathrm{b_2}^3}{3\,s} - \mathrm{b_1}\left(\frac{\mathrm{b_2}^2}{2} - \frac{\mathrm{b_1}^2\,s^2}{6\,b^2}\right) \\ \int_{s_1}^{s_2}\int_{b_1}^{b_2} \mathbf{1}_{Deal}\ y\ dxdy &= \int_{b_1}^{(b/s)b_2}\int_{(s/b)y}^{b_2} y\ dxdy = \frac{b^2\,\mathrm{b_2}^3}{6\,s^2} - \mathrm{b_1}^2\left(\frac{\mathrm{b_2}}{2} - \frac{\mathrm{b_1}\,s}{3\,b}\right). \end{aligned}$$

□

Proof of Proposition 3: Adding up buyer's and seller's expected profits that were calculated in proposition 2 yields

$$Sum(b,s) = \mathbf{E}\left(P_S\right)(s) + \mathbf{E}\left(P_B\right)(b) = \frac{6}{\Delta b \Delta s}\left(\frac{b^2}{s^2}b_1^2 + \left(\frac{2b}{s} - \frac{b^2}{s^2}\right)b_2^2 - \frac{2s}{b}b_1^3 - 3b_1b_2\Delta b\right).$$

We take the first derivative, as a function of b, of the sum of profits

$$\begin{aligned}\frac{\partial Sum(b,s)}{\partial b} &= b_2^3\left(\frac{2}{s} - \frac{2b}{s^2}\right) + b_1^3\left(\frac{2s}{b^2} - \frac{2s^2}{b^3}\right) = b_2^3\frac{2s-2b}{s^2} + b_1^3\frac{2sb-2s^2}{b^3}\\ &= b_2^3\frac{2}{s^2}\left(s-b\right) + b_1^3\frac{2s}{b^3}\left(b-s\right).\end{aligned}$$

In order to be strictly increasing in offer strategy b, the term needs to be greater zero. This is true if and only if

$$\frac{\partial Sum(b,s)}{\partial b} > 0 \iff b_2^3\frac{2}{s^2}\left(s-b\right) > b_1^3\frac{2s}{b^3}\left(s-b\right) \iff b_2^3/b_1^3\frac{b^3}{s^3} > 1 \iff \frac{b_2}{b_1} > \frac{s}{b}.$$

When $\frac{b_2}{b_1} \leq \frac{s}{b}$ holds, deal probability is zero according to proposition 1. Then expected profit of both parties is zero. Consequently $\frac{b_2}{b_1} > \frac{s}{b}$ guarantees deal probability to be greater zero. Analogous arguments apply for the sum of profits to be decreasing in the seller's offer strategy s. □

Proof of Proposition 4: Due to the length of the formulas we used a computer algebra system to optimize the expected profits. We programmed it to take the first derivatives of respective expected profits to find respective maximal points. The mathematics thus is straightforward. □

Proof of Proposition 5: The imprecision of both players has the same distribution. Therefore we have $b_1 = s_1$ and $b_2 = s_2$. There is full information in the two-player double auction. Thus the players' offers are given by their valuations, i.e. their offer strategies are $b = 1$ and $s = 1$. We refer to proposition 1, where we established a general formula for the deal probability. Under the imposed restrictions, the formula reduces to[12]

$$\begin{aligned}p_d &= \frac{\mathbf{1}_{s_1 \leq b_2}}{\Delta b \Delta s}\left(b_2(min(b_2,s_2) - s_1) - b_1(max(b_1,s_1) - s_1) - \frac{1}{2}\left(min(b_2,s_2)^2 - max(b_1,s_1)^2\right)\right)\\ &= \frac{1}{\Delta b^2}\left(b_2(b_2 - b_1) - \frac{1}{2}\left(b_2^2 - b_1^2\right)\right) = \frac{1}{\Delta b^2}\left(\frac{1}{2}b_2^2 - b_1b_2 + \frac{1}{2}b_1^2\right) = \frac{\frac{1}{2}\Delta b^2}{\Delta b^2} = \frac{1}{2}.\end{aligned}$$

□

[12] It is not necessary for valuation imprecision to be symmetrically distributed. Therefore the proposition holds in particular for more general frameworks than considered here.

Proof of Proposition 6: The buyer's expected profit is

$$\mathbf{E}\left(P_B\right) = \frac{1}{\Delta b \Delta s}\frac{1}{bs}\left(\left(\frac{1}{b}-\frac{1}{2}\right)\int_{s_1'}^{s_2'}\int_{b_1'}^{b_2'}\mathbf{1}_{Deal} b\ dbds - \frac{1}{2}\int_{s_1'}^{s_2'}\int_{b_1'}^{b_2'}\mathbf{1}_{Deal} s\ dbds\right).$$

The players' naive offer strategy is given by $b = s = 1$. Inserting this into the formula above yields

$$\begin{aligned}
\mathbf{E}\left(P_B\right) &= \frac{1}{\Delta b \Delta s}\left(\left(1-\frac{1}{2}\right)\int_{s_1}^{s_2}\int_{b_1}^{b_2}\mathbf{1}_{Deal} b\ dbds - \frac{1}{2}\int_{s_1}^{s_2}\int_{b_1}^{b_2}\mathbf{1}_{Deal} s\ dbds\right)\\
&= \frac{1}{2\Delta b \Delta s}\left(\int_{s_1}^{s_2}\int_{b_1}^{b_2}\mathbf{1}_{Deal}\left(b-s\right)\ dbds\right)\\
&= \frac{1}{2\Delta b \Delta s}\left(\int_{s_1}^{s_2}\int_{s}^{b_2} b-s\ dbds\right)\\
&= \frac{1}{2\Delta b \Delta s}\left(\int_{s_1}^{s_2}\frac{1}{2}b_2^2 - b_2 s - \frac{1}{2}s^2\ ds\right)\\
&= \frac{1}{4\Delta b \Delta s}\left(s_2 b_2^2 - b_2 s_2^2 + s_1^2 b_2 - s_1 b_2^2 + \frac{1}{3}(s_1^3 - s_2^3)\right)\\
&\overset{\substack{s_1 = b_1\\ s_2 = b_2}}{=} \frac{1}{4\Delta b^2}\left(\frac{1}{3}b_2^3 + b_1^2 b_2 - b_1 b_2^2 - \frac{1}{3}b_2^1\right) \overset{\substack{b_1 = 1-\alpha\\ b_2 = 1+\alpha}}{=} \frac{1}{6}\alpha
\end{aligned}$$

The proof for the seller's expected profit can be conducted analogously. □

Proof of Lemma 1. A player's expected profit as introduced in proposition 6 is zero for $\alpha = 0$ and increasing in α. Both parties profit from higher imprecision, so it is furthermore wealth increasing. □

Proof of Proposition 7: Proposition 4 introduced a general formula for $b_{opt}(s)$ as a function of s. The seller's offer strategy in a double auction with a rational buyer and a naive seller is given by the offer strategy $s = 1$. The calculation of b_{opt} as a function of $s = 1$ finishes the proof:

$$b_{opt}(1) = C(1, b_1, b_2) := \frac{1}{18\ b_2}\left(B(1, b_1, b_2) - \frac{(3\ b_1\ - 4\ b_2)\,(15\ b_1\ + 4\ b_2)}{B(1, b_1, b_2)} - 3b_1 + 4b_2\right)$$

□

Proof of Proposition 8: Proposition 1 established a general formula for the deal probability. In a double auction with a rational buyer and a naive seller the players' offer strategies are $b_{opt}(1)$ and $s = 1$. Adding up these facts we find

$$\begin{aligned} p_d &= \frac{\mathbf{1}_{s_1' \leq b_2'}}{\Delta b \Delta s} \left(b_2'(min(b_2', s_2') - s_1') - b_1'(max(b_1', s_1') - s_1') - \frac{1}{2}\left(min(b_2', s_2')^2 - max(b_1', s_1')^2\right)\right) \\ &= \frac{1}{b_{opt}\Delta b^2} \left(b_{opt} b_2 (b_{opt} b_2 - b_1) - 0 - \frac{1}{2}\left(b_{opt}^2 b_2^2 - b_1^2\right)\right) \\ &= \frac{1}{b_{opt}\Delta b^2} \left(b_{opt}^2 b_2^2 - b_{opt} b_1 b_2 - \frac{1}{2} b_{opt}^2 b_2^2 + \frac{1}{2} b_1^2 \right) = \frac{1}{2 b_{opt} \Delta b^2} \left(b_{opt} b_2 - b_1\right)^2 . \end{aligned}$$

□

Proof of Proposition 9: A general formula for both players' expected profit was established in proposition 2. Adding the assumptions of a naive seller ($s = 1$) and a rational buyer ($b = b_{opt}(1)$) to the formula creates the terms as stated in the proposition. □

Proof of Proposition 10: Proposition 4 states the formula for $s_{opt}(b)$. Within the framework of double auctions with a rational seller and a naive buyer, the buyer's offer strategy is given by $b = 1$. This leads to

$$s_{opt}(1) = C(1, b_2, b_1) = \frac{1}{18\ b_1} \left(B(1, b_2, b_1) - \frac{(3\ b_2\ - 4\ b_1)\,(15\ b_2\ + 4\ b_1)}{B(1, b_2, b_1)} - 3b_2 + 4b_1 \right) .$$

□

Proof of Proposition 11: In proposition 1 the general formula for the deal probability was established. In the rational seller and naive buyer double auction setting the buyer does not adjust his offer, i.e. $b = 1$. In proposition 10 the form for $s_{opt}(1)$ was analysed. Adding up these findings finishes the proof:

$$\begin{aligned} p_d &= \frac{\mathbf{1}_{s_1' \leq b_2'}}{\Delta b \Delta s} \left(b_2'(min(b_2', s_2') - s_1') - b_1'(max(b_1', s_1') - s_1') - \frac{1}{2}\left(min(b_2', s_2')^2 - max(b_1', s_1')^2\right)\right) \\ &= \frac{1}{s_{opt}\Delta b \Delta b} \left(b_2 (b_2 - s_{opt} b_1) - 0 - \frac{1}{2}\left(b_2^2 - s_{opt}^2 b_1^2\right)\right) \\ &= \frac{1}{s_{opt}\Delta b^2} \left(\frac{1}{2} b_2^2 - s_{opt} b_1 b_2 + \frac{1}{2} s_{opt}^2 b_1^2 \right) = \frac{1}{2 s_{opt} \Delta b^2} \left(b_2 - s_{opt} b_1\right)^2 \end{aligned}$$

□

Proof of Proposition 12: A general formula for both players' expected profit was established in proposition 2. Adding the assumptions of a naive buyer ($b = 1$) and a rational seller ($s = s_{opt}(1)$) to the formula creates the terms as stated in the proposition. □

Proof of Proposition 13: Outlining a buyer's optimal strategy on the x-axis and a seller's optimum strategy on the y-axis (see figure 5 for reference), optimal strategies are given by

$b_{opt}(s)$ and $s_{opt}(b)$, respectively. From prior analysis we know $b_{opt}(s) \in [b_2/b_2, 1]$ and $s_{opt}(b) \in [1, b_2/b_1]$. Due to Nash (1951), there is at least one equilibrium in mixed strategies. The set of strategies is bounded, closed and convex. As players measure their profit in monetary units, players have linear utility. Therefore, there is at least one equilibrium strategy according to Nikaido-Isoda[13]. Therefore it remains to be proven that this equilibrium is unique.

Optimal strategies are characterized by the coordinates $(b_{opt}(s), s)$ and $(b, s_{opt}(b))$. An equilibrium thus is given if the two equations $b_{opt}(s) = b$ and $s_{opt}(b) = s$ hold. With Proposition 4 this is equivalent to

$$b_{opt}(s) = C(s, b_1, b_2) = \frac{1}{18\ b_2}\left(B(1, b_1, b_2) - \frac{(3\ b_1\ - 4\ b_2)\,(15\ b_1\ + 4\ b_2)}{B(1, b_1, b_2)} - 3b_1 + 4b_2\right) = b,$$
$$s_{opt}(b) = C(b, b_2, b_1) = \frac{1}{18\ b_1}\left(B(1, b_2, b_1) - \frac{(3\ b_2\ - 4\ b_1)\,(15\ b_2\ + 4\ b_1)}{B(1, b_2, b_1)} - 3b_2 + 4b_1\right) = s.$$

Therefore we have an equilibrium if and only if the set of the following equations holds:

$$b = C\left(C(b, b_2, b_1), b_1, b_2\right) \tag{1.1}$$
$$s = C\left(C(s, b_1, b_2), b_2, b_1\right). \tag{1.2}$$

Equation (1.1) is independent of s and equation (1.2) is independent of b. Thus each equation is dependent only on a buyer's or a seller's strategy, respectively. The solution of the set of equations (1.1) and (1.2) is complex and an extensive analytical representation. As an alternative, fixed point iteration is applied in this proof. The iteration finds exactly one equilibrium for any imprecision parameter $0 < \alpha < 1$ and starting points b and s within the borders $\frac{1-\alpha}{1+\alpha} < b < 1 < s < \frac{1+\alpha}{1-\alpha}$.

An equilibrium must be within the borders $\frac{1-\alpha}{1+\alpha} < b < 1$ and $1 < s < \frac{1+\alpha}{1-\alpha}$. Otherwise the expected profit of either party becomes negative with probability one or the deal probability is zero. Offer strategies that are not within these bounds are therefore infeasible. Let D be the domain of b and s, i.e. $D := \{(b, s) : \frac{1-\alpha}{1+\alpha} < b < 1 < s < \frac{1+\alpha}{1-\alpha}\}$. We will prove that fixed point iteration converges for any starting value within D. In order to prove the convergence of the iteration, we make use of a tool from analytic mathematics, the Banach fixed-point theorem.

We need to show that the Banach fixed-point theorem can be applied to this situation. For that purpose the two functions

$$f: \qquad b \mapsto b_{opt}\left(s_{opt}(b)\right)$$
$$g: \qquad s \mapsto s_{opt}\left(b_{opt}(s)\right)$$

[13]See Nikaido and Isoda (1955) for reference

have to be contractions. This holds if and only if $(f,g)(D) \subset D$. If (f,g) is a contraction, the Banach fixed-point theorem states that both functions have exactly one fixed point within D. This means that there is exactly one buyer's offer strategy b^* such that $f(b^*) = b_{opt}\left(s_{opt}(b^*)\right) = b^*$ and exactly one seller's offer strategy s^* such that $g(s^*) = s_{opt}\left(b_{opt}(s^*)\right) = s^*$. The set of unique fixed points (b^*, s^*) solves the set of equations (1.1) and (1.2). This observation then finishes the proof.

Thus it remains to show that f and g are contractions: This is true if (1) $f([\frac{1-\alpha}{1+\alpha}, 1]) \subseteq [\frac{1-\alpha}{1+\alpha}, 1]$; (2) $g([1, \frac{1+\alpha}{1-\alpha}]) \subseteq [1, \frac{1+\alpha}{1-\alpha}]$; and (3) $\left|\frac{\partial f}{\partial b}(d_1)\right|, \left|\frac{\partial g}{\partial s}(d_2)\right| < 1$ for all $(d_1, d_2) \in D$. An analysis of (1) shows that $f([\frac{1-\alpha}{1+\alpha}, 1]) = b_{opt}(s_{opt}([\frac{1-\alpha}{1+\alpha}, 1]))$. This is the optimal response strategy of the buyer, if he anticipates that the seller places her bid as the best response of a feasible buyer's strategy. An optimal seller's strategy as a response to a feasible buyer's strategy is always feasible. Therefore it follows that $s_{opt}([\frac{1-\alpha}{1+\alpha}, 1]) \subseteq [1, \frac{1+\alpha}{1-\alpha}]$. That is, $f([\frac{1-\alpha}{1+\alpha}, 1]) \subseteq b_{opt}([1, \frac{1+\alpha}{1-\alpha}])$. The optimal buyer's strategy as a response to a feasible seller's strategy is always a feasible strategy and therefore on the interval $[\frac{1-\alpha}{1+\alpha}, 1]$. In summary we have $f([\frac{1-\alpha}{1+\alpha}, 1]) \subseteq [\frac{1-\alpha}{1+\alpha}, 1]$.

Likewise, case (2) is necessarily a feasible strategy and therefore a subset of the demanded interval.

Considering (3), figures 18 and 19 show the first derivatives of f and g, respectively. It can be seen that both derivatives are smaller than 1 in absolute terms.

Therefore according to the Banach fixed-point theorem there is exactly one fixed point (b^*, s^*) in D. This is the unique Nash equilibrium. □

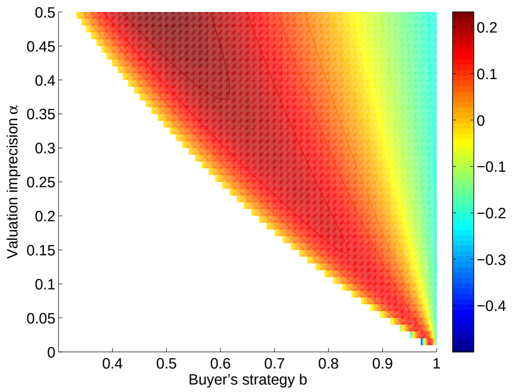


Fig. 18 – First derivative of f

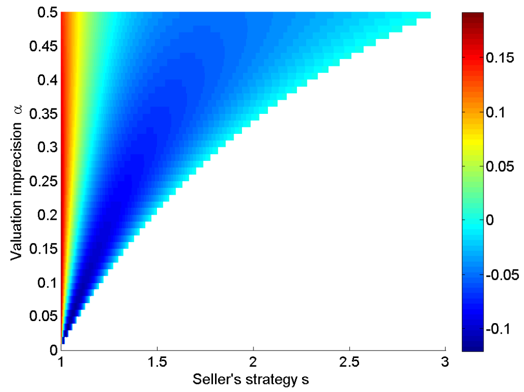


Fig. 19 – First derivative of g

Proof of Proposition 14: The buyer suffers from valuation imprecision that is uniformly distributed on $[1-\alpha, 1+\alpha]$. He generates a gain if his valuation exceeds $1+f$. The probability of this event is $p = \frac{\alpha - f}{2\alpha}$. The argument for the seller works analogously. □

Proof of Proposition 15: As $f < \alpha$, the buyer's expected profit is

$$\begin{aligned}\mathbf{E}\left(P_D(B)\right) &= \frac{1}{\Delta b}\int_{1+f}^{b_2} b - 1 - f \; db = \frac{1}{\Delta b}\left(\frac{1}{2}\left((1+\alpha)^2 - (1+f)^2\right) - (1+f)(\alpha - f)\right) \\ &= \frac{1}{\Delta b}\left(\frac{1}{2}\alpha^2 + \frac{1}{2}f^2 - \alpha f\right) = \frac{1}{2\Delta b}\left(\alpha - f\right)^2.\end{aligned}$$

The product is zero if and only if at least one of the factors is zero. As f and α are both non-negative, the only possibility for expected profit to be zero is $f = \alpha$. Furthermore

$$\mathbf{E}\left(P_D(B)\right) > 0 \Longleftarrow 0 < f < \alpha.$$

The upper bound for the buyer's expected profit is $\alpha/4$. The proof for the seller's expected profit is performed along the lines of the proof for the buyer. □

Proof of Proposition 16: In order to prove the proposition, the inequation

$$p_d < (\alpha - f)/(2\alpha)$$

must hold. A rearrangement of that equation shows that is equivalent to $f < \alpha(1 - 2p_d)$. □

Proof of Proposition 17: The expected profit in the Dealer's Market needs to be greater than in a double auction. That is true if and only if

$$\begin{aligned}\mathbf{E}\left(P_D(B)\right) > \mathbf{E}\left(P_{(\cdot)}\right) &\Longleftrightarrow \frac{1}{2\Delta b}(\alpha - f)^2 > \mathbf{E}\left(P_{(\cdot)}\right) \\ &\Longleftrightarrow \frac{1}{2\Delta b}(\alpha - f)^2 - \mathbf{E}\left(P_{(\cdot)}\right) > 0 \\ &\Longleftrightarrow f^2 - 2\alpha f + \alpha^2 - 4\alpha\mathbf{E}\left(P_{(\cdot)}\right) > 0.\end{aligned}$$

The term $f^2 - 2\alpha f + \alpha^2 - 4\alpha\mathbf{E}\left(P_{(\cdot)}\right)$ equals zero for

$$\begin{aligned}f_{1,2} &= \frac{2\alpha \pm \sqrt{4\alpha^2 - 4(\alpha^2 - 4\alpha\mathbf{E}\left(P_{(\cdot)}\right))}}{2} \\ &= \alpha \pm \frac{\sqrt{4\alpha^4 - 4\alpha^2 + 16\alpha\mathbf{E}\left(P_{(\cdot)}\right)}}{2} = \alpha \;\pm\; 2\sqrt{\alpha\mathbf{E}\left(P_{(\cdot)}\right)}.\end{aligned}$$

Section 4 analysed the players' expected profits in double auctions. This analysis showed that $f_1\alpha - 2\sqrt{\alpha\mathbf{E}\left(P_{(\cdot)}\right)} > 0$. When we set $f = 0$ in the term above, then it is positive if and only if

$$\alpha - 4\mathbf{E}\left(P_{(\cdot)}\right) > 0.$$

This inequation holds as shown in section 4. A polynomial of second order has at most 2 roots. In the above case, these are given by $f_{1,2}$. The term $f^2 - 2\alpha f + \alpha^2 - 4\alpha\mathbf{E}\left(P_{(\cdot)}\right)$ exceeds zero for $f = 0$ and its roots are $0 < f_1 \leq f_2$. Therefore we conclude that

$$f^2 - 2\alpha f + \alpha^2 - 4\alpha\mathbf{E}\left(P_{(\cdot)}\right) > 0$$

for all $f < f_1$. Thus the Dealer's Market is Pareto dominant. □

Proof of Proposition 18: Proposition 6 states that in a double auction under full information both players' expected profit is $\frac{1}{6}\alpha$. In proposition 17 a sufficient condition for the Dealer's Market to dominate a double auction was established. Merging these propositions finishes the proof:

$$f < \alpha \; - \; 2\sqrt{\alpha\mathbf{E}\left(P_{(\cdot)}\right)} = \alpha \; - \; 2\sqrt{\frac{1}{6}\alpha^2} = \alpha\left(1 - \sqrt{\frac{2}{3}}\right).$$

□

Proof of Proposition 19: Proposition 9 introduced the players' expected profit in a double auction with a rational buyer and a naive seller. It was shown that the buyer has higher expected profit than the seller in this market setting. Thus if the buyer prefers the Dealer's Market over a double auction, then the seller shares this preference. A sufficient and necessary condition for the Dealer's Market to be preferred was established in proposition 17. Combining these arguments leads to the inequation $f < \alpha \; - \; 2\sqrt{\alpha\mathbf{E}\left(P_B\right)}$ as a sufficient condition for a double auction with a rational buyer and a naive seller to be dominated by the Dealer's Market. □

Proof of Proposition 20: The statement of this proposition can be proven analogously to proposition 19. □

Proof of Proposition 21: It was shown that within this double auction setting the buyer's profit is higher than the seller's. Thus the proof of this proposition can be conduced analogously to that of proposition 19. □

Proof of Theorem 1: In order to show that the Dealer's Market Pareto dominates a double auction that is in bidding equilibrium, we have to prove that the dealer, the seller and the buyer prefer the Dealer's Market over a double auction:

In a double auction the dealer generates no gain. He profits from the Dealer's Market if and only if his fee is greater zero. Therefore the dealer prefers the Dealer's Market a double auction if his fee is greater zero.

A buyer's expected profit in a double auction in bidding equilibrium is greater than the seller's. As a result, the seller is in preference of the Dealer's Market when the buyer has this preference too. Thus it is sufficient that a buyer prefers the Dealer's Market over the double auction. Proposition 20 states that the buyer prefers the Dealer's Market over double auctions in bidding equilibrium if and only if $f < \alpha - 2\sqrt{\alpha \mathbf{E}(P_B)}$. In summary, when $0 < f < \alpha - 2\sqrt{\alpha \mathbf{E}(P_B)}$, then all players prefer the Dealer's Market over a double auction in bidding equilibrium. That is, the Dealer's Market Pareto dominates that double auction. □

Proof of Theorem 2: Let f and g be the densities of a buyer's and a seller's valuation imprecision, respectively. Let $b_1 < b_2$ be the buyer's lowest and highest possible imprecision. Likewise $s_1 < s_2$ are the seller's lowest and highest imprecision. The deal price is denoted by P. A buyer's and a seller's offer strategies are represented by $b < 1$ and $s > 1$, respectively. Then the buyer's expected profit is

$$
\begin{aligned}
\mathbf{E}(P_B) &= \int_{b_1}^{b_2} \int_{s_1}^{s_2} f(x)g(y)\mathbf{1}_{Deal}\,(x - P)\ dydx \\
&= \int_{b_1}^{b_2} \int_{s_1}^{s_2} f(x)g(y)\mathbf{1}_{bx>sy}\,(x - (sy + k(bx - sy)))\ dydx.
\end{aligned}
$$

Likewise, the seller's expected profit is

$$
\begin{aligned}
\mathbf{E}(P_S) &= \int_{b_1}^{b_2} \int_{s_1}^{s_2} f(x)g(y)\mathbf{1}_{Deal}\,(P - y)\ dydx \\
&= \int_{b_1}^{b_2} \int_{s_1}^{s_2} f(x)g(y)\mathbf{1}_{bx>sy}\,(sy + k(bx - sy) - y)\ dydx.
\end{aligned}
$$

These profits accumulate to

$$
\mathbf{E}(P_B) + \mathbf{E}(P_S) = \int_{b_1}^{b_2} \int_{s_1}^{s_2} f(x)g(y)\mathbf{1}_{bx>sy}\,(x - y)\ dydx.
$$

The function $\mathbf{1}_{bx>sy}$ is obviously decreasing for decreasing b and increasing s. Thus the sum of a buyer's and a seller's profit is decreasing for decreasing b and increasing s. Individual profit optimisation, represented by $b < 1$ and $s > 1$, therefore negatively affects the sum of profits. The accumulated profit is highest when the buyer and the seller apply a naive offer strategy and make offers at their reservation prices. That is, $b = s = 1$. □

Proof of Theorem 3: We calculate the difference between the rational and naive strategies:

$$\begin{aligned}\Delta\mathbf{E}(P_B+P_S) &= \int_{b_1}^{b_2}\int_{s_1}^{s_2} f(x)g(y)\left(\mathbf{1}_{x>y}-\mathbf{1}_{bx>sy}\right)(x-y)\ dydx \\ &= \int_{b_1}^{b_2}\int_{s_1}^{s_2} f(x)g(y)\mathbf{1}_{y\in x[b/s,1]}\,(x-y)\ dydx.\end{aligned}$$

This is the positive profit the dealer may distribute among a buyer and a seller, after he deducts his fee. □

Proof of Proposition 22: The average price is

$$\begin{aligned}\mathbf{E}(P) &= \mathbf{E}\left(\mathbf{1}_{Deal}P\right) = \mathbf{E}\left(\mathbf{1}_{Deal}\frac{kP_B+(1-k)P_S}{2}\right) \\ &= \frac{1}{2}\mathbf{E}\left(\mathbf{1}_{Deal}kP_B+(1-k)P_S\right) = \frac{k}{2}\mathbf{E}\left(\mathbf{1}_{Deal}P_B\right)+\frac{1-k}{2}\mathbf{E}\left(\mathbf{1}_{Deal}P_S\right) \\ &= \frac{kb}{2}\mathbf{E}\left(\mathbf{1}_{Deal}V_B\right)+\frac{(1-k)s}{2}\mathbf{E}\left(\mathbf{1}_{Deal}V_S\right) \\ &= \frac{kb}{2\Delta b^2}\int_{b_1}^{b_2}\int_{b_1}^{b_2}\mathbf{1}_{Deal}x\ dxdy+\frac{(1-k)s}{2\Delta b^2}\int_{b_1}^{b_2}\int_{b_1}^{b_2}\mathbf{1}_{Deal}y\ dxdy.\end{aligned}$$

Both integrals were computed in proposition 2. Let p_d be the probability of bargain success as in proposition 1. In case of feasible offer strategies (which necessarily is the case when the bargain is successful), the deal probability simplifies to

$$p_d = \frac{(bb_2-ss_1)^2}{2\Delta b^2}.$$

The expected deal price conditioned on the event of bargaining success therefore is

$$\begin{aligned}\mathbf{E}(P|\text{Success}) &= \frac{\mathbf{E}(P)}{p_d} \\ &= \frac{1}{(bb_2-ss_1)^2}\left(kb\int_{b_1}^{(b/s)b_2}\int_{(s/b)b_1}^{b_2} x\ dxdy+(1-k)s\int_{b_1}^{(b/s)b_2}\int_{(s/b)b_1}^{b_2} y\ dxdy\right) \\ &= \frac{1}{(bb_2-ss_1)^2}\left(kb\left[\frac{b\,{\mathrm{b}_2}^3}{3\,s}-\mathrm{b}_1\left(\frac{{\mathrm{b}_2}^2}{2}-\frac{{\mathrm{b}_1}^2\,s^2}{6\,b^2}\right)\right]\right. \\ &\qquad \left.+(1-k)s\left[\frac{b^2\,{\mathrm{b}_2}^3}{6\,s^2}-{\mathrm{b}_1}^2\left(\frac{\mathrm{b}_2}{2}-\frac{\mathrm{b}_1\,s}{3\,b}\right)\right]\right).\end{aligned}$$

□

Proof of Proposition 23: A lower bound for the dealer's offer to the firm B_d is established in proposition 20 and given by $B_d > V(1-f_{max})$. The investor has the alternative of an exactly priced spanning portfolio with value V. To attract the investor for the IPO, the investment banker needs to offer the shares for a value $S_d < V$ to an investor. Now, the investment banker's gain is positive, when the price an investor pays exceeds the amount the firm receives, that is $B_d < S_d$. In summary, we have $V(1 - f_{max}) < B_d < S_d < V$. Following that pricing strategy, the IPO is Pareto efficient compared to bilateral negotiations, as all players prefer the IPO. □

Proof of Lemma 2: Proposition 23 showed that the shares are offered to the investor for a price $S_d < V$, that is below the value of the spanning portfolio. Therefore the IPO is underpriced. □

Proof of Proposition 24: Proposition 23 introduces the lower bound for the investment banker's offer to the firm. Dropping the spanning portfolio hypothesis, the only remaining investor's alternative is to negotiate bilaterally. According to proposition 17, the expected profit of this option is dominated by the intermediary's offer when $S_d < V(1 + f_b)$. As a result, the investment banker's strategy is Pareto efficient because all parties prefer an IPO over bilateral negotiations. □

Proof of Lemma 3: In section 4 we showed that the intermediary can exaggerate his offer to the firm to a greater extent than his offer to the investor. Figure 17 (b) on page 42 further illustrates this fact. Thus the average IPO share price is below V and thus and IPO is underpriced on average. □

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On the Pareto Efficiency of a Market Maker over Reverse Auctions in Equilibrium

Johannes Seemüller

Abstract

This paper studies a reverse auction market under the "independent private value model" assumption, where auction participants value the auctioned good imprecisely. In our model, there is a unique multilateral bidding strategy that maximises each seller's profit. At the same time, that strategy generates the highest possible profit for the seller-group. When there is no mechanism that commits sellers to that strategy, a seller may increase her individual profit by pursuing a unilateral bidding strategy. When sellers follow that unilateral strategy, then the reverse auction generates less profit for each seller than that under the multilateral strategy. We calculate these strategies and expected profits explicitly. A dealer can exploit this inefficiency of a reverse auction by providing market maker services. His strategy under information asymmetry is Pareto efficient over the auction market under non-restrictive conditions. The dealer can maintain a reasonable inventory size, even when there are many sellers.

Key words: Reverse Auction, Valuation Imprecision, Independent Private Value Model, Optimal Pricing Strategy, Market Maker, Bid-Ask Spread, Pareto Efficiency, Bond Underwriting. JEL Classifications: C72, D44, D47, G12

Contents

1 Introduction

Typically, an auction is originated by a seller. This paper considers the reverse case, when auctions serve to find a suitable price for an initiating buyer. Such an auction that is set up by a buyer is refereed to as a reverse auction. Then the buyer specifies the auction type and notifies suppliers to submit their bids.

A popular example for an Internet auction platform is eBay, where dealers and private individuals offer their goods. In the form of *Buy it now*-offers, eBay has implemented reverse auctions into its trading platform as a buyer can choose among several sellers' offers. Amazon also allows private sellers and businesses to offer their goods on its trading platform. Both, eBay and Amazon address the end-user of the buy side. Additionally to serving the end-user, (reverse) auctions are increasingly used in the procurement side of supply chains, as Huhy and Roundy (2004) note. They refer to Covisint and Fast Buyer as examples of business-to-business solutions and product providers for the automobile industry. Both platforms are founded by OEMs and provide online auction services. Furthermore, governments are required to initiate reverse auctions to award contracts among competing bidders. Another example are publicly offered corporate bonds or treasury bills. Their prices are often determined by reverse auctions as well.

A seller who wants to sell a specific good on eBay or Amazon can observe other sellers' offers and thereby gains information on his expected sale success. On the contrary, in a procurement auction, a seller who places a bid can not compare it to those of the other sellers. In this paper we study such reverse auctions, where each seller's bid is hidden from further bidding sellers.

A lot of research has already been done in auction theory, so this introduction serves to embed our research into works which have previously been published within this field.

"Many of the world's most important markets are auction markets", as Milgrom and Weber (1982) note in their contribution to auction theory. Since then the popularity of auctions to determine the prices for goods or services has further increased. This development has been enforced by the rise of the Internet where platforms for electronic commerce and trade allow an efficient determination of prices and thus simplify the exchange of goods and services. These virtual platforms allow the allocation process to be less cost intensive than conventional trading.

Milgrom and Weber (1982) prove the existence of a unique equilibrium in bidding strategies in English, first- and second-price auctions. They also analyse the effect of entry fees and allow the seller to set a reserve price. Maskin and Riley (1984) analyse the effect of risk averse buyers on the auction type a seller prefers. Lebrun (1999) analyses first-price auctions in the asymmetric n bidder case. He considers bidders with valuations that are not identically distributed and he allows bidders to have different offer strategies. He develops restrictions that allow an equilibrium in bidding strategies. Further asymmetric bidding strategies are discussed by Lebrun (1998) and (1999) and Maskin and Riley (2000b). Bidding in combination with signalling is studied by Maskin and Riley (2000a) and Rodriguez (2000). Bikchandani and Riley (1991) as well as Milgrom and Weber (1982) study the influence of common values on bidding strategies.

In particular, procurement auctions have been the subject of recent studies. Holt (1990) analyses bidding strategies for contracts in different auction procedures. Dasgupta and Spulber (1990) as well as Chen (2001) particularly address procurement auctions. The impact of the announcement of the buyer's reservation price is studied by Carey (1993). Gallien and Wein (2001) consider procurement auctions with capacity constraints. Teich et al. (2001) study the design of electronic auctions, where they mix elements of auctions and negotiations to a new market procedure. Jin and Wu (2002) further address the supply chain coordination of electronic markets, whereas Seshadri and Zemel (2001) focus on supply chains in general. Compte and Jehiel (2002) analyse the impact of competition in procurement auctions. The procurement of options is studied by Schummer and Vohra (2003). Reiß and Schöndube (2003) and Brosig and Reiß (2007) study sequential procurement auctions, where similar auctions are conducted in time overlapping intervals. They analyse the bidders' participation and their strategies theoretically and empirically. Reiß and Schöndube (2010) further analyse equilibria and revenue equivalence in their sequential procurement auction model.

The above examples of reverse auction literature mainly focus on supply chain applications. Huhy and Roundy (2004) note that the first-price reverse auction bidding strategy, corresponding to the lowest payment by the buyer, has the same expected payment as the (unique) bidding strategy of the second-price reverse auction. In the second-price reverse auction, by comparison, an extension of Vickrey (1961) shows an analogous result in the first-price auction that bidding one's own cost is a dominant strategy of every seller. Huhy and Roundy (2004) further study the impact of the buyer's reserve price. They show that this reserve price is eliminating the multiplicity of bidding strategies and also the associated risk of very high costs, as well as maximizing the buyer's cost. Such benefits are consistent with a recent trend in the automobile industry. More buyers are setting reserve prices when they originate auctions, following a recommendation of the trading platform Covisint.

Suppose that a government sets up an auction to procure a certain service or that an automobile plant wants to buy a new press. It is plausible to assume that the costs of potential service providers are independent and identically distributed. In this paper, we study single-unit single-period sealed-bid first-price reverse auctions in which bidders are symmetric and have independent and identically distributed private costs. The buyer has a random reserve price that is independent of the bidders' private costs. Our model thus is similar to Huhy and Roundy (2004). In addition, in our model the buyer does not reveal his reserve price to bidding sellers.

In auction literature each trader often has a reservation price, $V \geq 0$, which is distributed on the interval $[0, v]$. See Chatterjee (1983) for an example. Often that interval is restricted to $[0, 1]$, as for instance in the double auction considered by Gibbons (1992)[1]. In this paper, these constraints are relaxed.

The above literature often assumes that each player knows the distribution of the other players' valuations. Thus each player may compare her valuation to these distributions, in order to formulate detailed offer strategies. However, valuation imprecision implies that individuals do not have a valuation benchmark.

In this paper we model valuation imprecision by assuming that the buyer and the bidding sellers are aware only of their own reservation price and the common distribution of valuation imprecision. They however have no indication, whether their reservation price is above or below average and how it compares to the other players' valuations[2].

Our model can be considered as a Bayesian game with common, but unknown prior[3]. Studies of Bayesian games are usually conducted abstractly, as for instance by Nikaido and Isoda (1955). Our reverse auction model is more practical, as it provides concrete formulas and advice on how sellers should best set their prices. Regarding valuation imprecision, our model is more abstract and realistic than those of the double auction literature discussed above.

A detailed analysis of the reverse auction is undertaken in section 2. Different levels or rational bidding strategies are discussed and their effect on the platform's efficiency is analysed in detail. We show that in the reverse first-price auction with reserve price, there is a symmetric Nash equilibrium in pure bidding strategies. Each of these strategies corresponds to a distinct expected payment of the buyer. When all sellers commit to a multilateral bidding strategy, their individual expected profit is higher than that in bidding equilibrium. Under that strategy, the profit of all sellers is increasing if the number of sellers increases. When all sellers commit to this multilateral bidding strategy then it is preferable over individual profit maximisation. Section 3 introduces and analyses a market maker's strategy of quoting bid-ask prices. The market maker's strategy and the reverse auction are compared in section 4. That section further provides conditions for the Pareto efficiency of the market maker's strategy compared to the double auction. A market maker exploits inefficiencies in the auction

[1]See Gibbons (1992), pages 158ff.

[2]Assume for instance that a player's valuation is uniformly distributed on the interval [50,150]. When that distribution is known to the players, then an individual with the reservation price of 101 knows that her valuation is almost average. In our model, a player with a reservation price of 101 does not have a benchmark to determine whether that valuation is high or low. That player only knows that his valuation is imprecise and distributed around some unknown average valuation.

[3]See Harsanyi (1967) for reference.

market and successfully competes with that market. When the market maker applies a reasonable price scheme, then his offers are more attractive to a buyer and the sellers than the participation in an auction. Thereby this paper shows that the market maker's price strategy is Pareto efficient over that auction. Section 5 concludes the findings of this paper.

2 The Reverse Auction

We consider a platform market where sellers offer an indivisible and indistinguishable good. Each seller independently quotes a price that she sells the good for. A buyer observes the sellers' offers and buys at the lowest price if his reservation price exceeds the lowest price. The platform thus is similar to *Buy it now*-offers on eBay. Another example is the Amazon market platform, where the same good can be bought from different sellers with different prices. Further, an automobile manufacturer that plans to procure a new machine may set up a reverse auction to find a suitable supplier. Then suppliers may bid to win the contract.

Let us rigorously model this reverse auction market. Several owners (sellers) of the good disclose their offers on a platform. A buyer observes the sellers' offers and decides whether to buy at the lowest offer or not. When the best offer, i.e. the minimum of the sellers' offers is below the buyer's reservation price, then the good is traded for the price P of that lowest offer. In case of a successful trade, either party's profit is given by the difference between their reserve price and the deal price P.

We assume that the parties suffer from valuation imprecision regarding their reservation prices[4]. We model these imprecise valuations as independent and uniformly distributed random variables with unknown mean valuation $V > 0$. When there are n sellers, then the $i-th$ seller's reservation price is $V_{S_i} \overset{d}{=} unif[s_1, s_2]V$, with $0 < s_1 < 1 < s_2$. The buyer's reservation price is denoted by $V_B \overset{d}{=} unif[b_1, b_2]V$, with $0 < b_1 < 1 < b_2$. In our model valuation imprecision is symmetrically distributed around 1. Furthermore, the buyer's and the sellers' valuation imprecision is identically distributed. As a result, a valuation imprecision parameter $0 < \alpha < 1$ is sufficient to model imprecision in our market. That is, $s_1 = b_1 = 1 - \alpha$ and $s_2 = b_2 = 1 + \alpha$.

We distinguish two stages of the market: Stage 1 can be regarded as the initializing of the trading platform. Each player knows that he suffers from valuation imprecision and knows that the other players know, knows that they do and so forth. Thus there is mutual full information in the sense of Aumann (1976). The players furthermore know the imprecision's distribution, know that the others know, etcetera. The players, however, are not aware whether their reservation price is below or above average, because they do not have a bench-

[4]When the good is a machine or a similar product, then the term valuation imprecision may be misleading: Different suppliers have different production costs and therefore they have different prices they require. Thus, in that case, valuation imprecision also may be caused by different costs of production or manufacturing.

mark to compare their price with. If for instance a seller's reservation price is 160 and she knows the distribution of the imprecision, she still can not determine neither over- nor undervaluation. Stage 1 therefore can be regarded as the initializing of the platform because there are no reference offers present from which over- or undervaluation could be derived. Thus although the reservation prices of the sellers are not identical, while they share homogeneous offer strategy, their expected profit is the same ex ante. Profit is furthermore dependent on each seller's offer strategy: Assume there are two sellers. The first seller places an offer twice her reservation price. The second seller places an offer that exceeds her reservation price by 10%. Then it is likely that the offer of seller two is lower than that of the first seller. Therefore a seller's offer strategy influences her profit when her bid is successful, her deal probability and eventually the expected profit of all sellers.

After the initialisation of the platform, each seller observes the other sellers' offers. This is the case on eBay or Amazon, when a good is traded there for some time. Sellers then can compare their reservation price to the offers of sellers that previously placed offers on the platform. A seller who enters the platform thereby is enabled to calculate her individual deal success probability and expected profit, as a function of the offers she observes and her offer strategy. This post-initialisation phase is characterised as stage 2. That stage is not reached in a variety of auctions, such as a single procurement auction. In this paper, the initialising stage 1, where offers are sealed, is analysed.

A sellers profit is the difference between her reservation price V_S and her offer O_S, that is $O_S - V_S$. A positive profit therefore requires that her offer exceeds a seller's reservation price. Her offer strategy further is a function of her reservation price. We model a seller's offer strategy as a scalar $s \geq 1$, that is $O_S = sV_S$. The scalar s necessarily is greater than or equal to 1 because a seller's profit would be negative otherwise. That strategy furthermore is the only feasible strategy. A seller does not know the distribution of the reservation price. She only is aware of the distribution of valuation imprecision. Therefore a reasonable offer strategy must only be dependent on a seller's valuation. Rational behaving sellers determine their offer strategies s in order to maximize expected profit.

When a seller's offer strategy exceeds the bound s_2/s_1, then that seller overbids the buyer's reservation price with probability 1. Thus, that seller's probability to win the auction is 0. Furthermore, a seller's offer strategy should exceed 1, as otherwise her offer is below her reservation price. In this case, a seller's profit would be negative. In summary, feasible offer strategies are within the interval $s \in [1, s_2/s_1]$. If not stated otherwise, sellers are assumed to bid feasibly.

Let us analyse different possibilities of profit maximisation. We start with level 1 optimisation, where all sellers agree on the same offer strategy s.

2.1 Level 1 Rationality

In this section, we analyse optimal sellers' bidding strategies when they agree on a multilateral offer strategy s. This means that all sellers pursue the same offer strategy and commit to it. Then the sellers can multilaterally optimise their profit as a function of their offer strategy s.

Note that, while no mechanism is established to commit each seller to this strategy, a seller can optimise her offer strategy, as a function of the other sellers' multilateral offer strategy. This section, however, studies multilateral optimisation, where all sellers commit to the same offer strategy. We refer to this strategy as level 1 rationality.

We start with a lemma that will be of great benefit in the remainder of this paper.

Lemma 1. *Let $X_1, X_2, ..., X_n$ be iid random variables, with $X_1 \stackrel{d}{=} unif[x_1, x_2]$. Then the cdf of $min(X_1, X_2, ..., X_n)$ is given by $M(x) = 1 - \left(\frac{x_2 - x}{x_2 - x_1}\right)^n$. The pdf of $min(X_1, X_2, ..., X_n)$ is given by $m(x) = n\frac{(x_2-x)^{n-1}}{(x_2-x_1)^n}$.*

Proof: See the Appendix. □

If needed, the notation of the functions M and m will be expanded in an intuitive way. Then we may for instance write $M(x, x_1, x_2, n)$ instead of $M(x)$.

This section calculates and analyses important properties of the sellers' unilateral offer strategy. The next proposition introduces each seller's probability to win the auction. It further calculates the probability that there is an offer below the buyer's reservation price. That probability can be characterized as the buyer's success probability.

Proposition 1. *When n sellers pursue a multilateral offer strategy $s \geq 1$, then each seller's probability to win the auction is*

$$\boldsymbol{P}(D_S)(s) = \frac{F(s_2) - F(ss_1)}{s^n \Delta s^{n+1}}.$$

The buyer's success probability is

$$\boldsymbol{P}(D_B)(s) = \frac{s_2 - ss_1}{\Delta s} + \frac{s}{n+1}\left(\left(\frac{s_2(s-1)}{s\Delta s}\right)^{n+1} - 1\right).$$

Each seller's probability to win the auction converges to zero for $n \to \infty$. The buyer's deal probability converges to $\frac{s_2 - ss_1}{\Delta s}$ for $n \to \infty$. To simplify the above notation, the function F is defined as

$$F(x) = \frac{(ss_2 - x)^n(-ns_2 - s_2 + ss_2 + nx)}{n(n+1)}.$$

Proof: See the Appendix. □

The proposition calculates a formula for a seller's and the buyer's success probability in the auction. The buyer's success probability exceeds a seller's probability to place the lowest bid. In fact, the buyer is successful in the auction if and only if exactly one seller is successful. Therefore the probability that the buyer is successful and the probability that exactly one seller is successful are equal. Due to the same offer strategy of all sellers, each seller has the same probability of winning the auction ex ante. When there is an infinitely high number of sellers, then the success probability of a single seller converges to zero. The buyer profits from an increasing number of sellers because the probability that his reservation prices exceeds the lowest offer increases. The success probability of each seller and the buyer is a decreasing function in the sellers' multilateral offer strategy s. The intuition behind this observation is that, when sellers increase their offers, then the probability that the buyer's reservation prices exceeds the offer of at least one seller decreases

The next proposition calculates the expected profit in the auction. These formulas are the key for the sellers' multilateral profit optimisation offer strategy.

Proposition 2. *When there are n sellers with multilateral offer strategy s, then the ex ante expected profit of each seller is*

$$\boldsymbol{E}\,(P_S)\,(s) = \frac{(s-1)\ \left(\ A(s_2) - A(ss_1)\ \right)}{s^{n+1}\ \ \Delta s^{n+1}}.$$

The function A is

$$A(x) = (s_2\,s - x)^n \left(-x s_2\,n^2 + x^2 n^2 - 2\,x s_2\,n + 2\,x s_2\,sn + x^2 n - {s_2}^2 sn - 2\,{s_2}^2 s \right.$$
$$\left. + 2\,{s_2}^2 s^2\right)/(n(n+1)(n+2))\,.$$

The expected profit of all sellers is

$$\boldsymbol{E}\,(S)\,(s) = \frac{n(s-1)\ \left(\ A(s_2) - A(ss_1)\ \right)}{s^{n+1}\ \ \Delta s^{n+1}}.$$

This can be interpreted as expected gain of the seller-group.

Proof: See the Appendix. □

We have shown in proposition 1 that individual deal probability decreases with increasing size of the seller-group. Combined with proposition 2 we further find that each seller's expected profit decreases when the size of the seller-group increases. That is reasonable because when there are more sellers, then the number of competitors increases. Thus each seller's expected profit decreases. In contrast, valuation imprecision has positive effect on each seller's expected profit. The sellers thus profit from a higher imprecision in their valuations. Figure 1 illustrates these properties later in this section.

The profit of the seller-group increases with increasing size of the group. When the seller-group is large, then the probability that one seller underbids the buyer's reserve price increases. This has the effect that the profit of the seller-group is affected positively by its size. However, the expected profit of the seller-group is bounded, as the next proposition shows.

Proposition 3. *The expected profit of the seller-group is convergent in group size. This means that the profit an additional seller adds to the group's profit converges to zero with increasing group size. The maximum expected group profit is*

$$\boldsymbol{E}(S)(s) \overset{n\to\infty}{\longrightarrow} \frac{s_1}{\Delta s}(s-1)(s_2 - ss_1).$$

For infinitely large seller-group, the optimal multilateral offer strategy is $s^1_{opt} = 1/(1-\alpha)$.

Proof: See the Appendix. □

The proposition states that there is an upper bound for the profit of the seller-group. That bound is $\frac{s_1}{\Delta s}(s-1)(s_2 - ss_1)$. For a sufficiently large seller-group, the optimal multilateral offer strategy s^1_{opt} thus can be determined with moderate effort by finding the maximum of that upper bound as a function of s. According to proposition 3, this optimal level 1 multilateral offer strategy is $s^1_{opt} = 1/(1-\alpha)$. This means that it is optimal for a seller to bid her reservation price multiplied by $s^1_{opt} = 1/(1-\alpha)$ (as long as all sellers commit to this strategy).

We summarize the major properties of each seller's profit in the next proposition.

Proposition 4. *Let all sellers pursue a multilateral offer strategy s. Then the expected profit of each seller is strictly increasing in valuation imprecision and strictly decreasing in the number of sellers in the market.*

Proof: See the Appendix. □

Proposition 4 proves that the expected profit of a seller is increasing in valuation imprecision. That is, when the players suffer from a higher valuation imprecision, then their expected profit exceeds that, which results from an enhanced valuation ability. When more sellers place bids on the good, then the probability that a specific seller places the lowest bid decreases. As a result, each seller's expected profit decreases.

When the number of sellers is finite, then finding the optimal multilateral offer strategy is more difficult than in the above case that was analysed in proposition 3. Further, there may be more than one offer strategy that generates maximum expected profit. The next theorem proves however that, independent of the number of sellers and the valuation imprecision, there is exactly one optimal multilateral offer strategy.

Theorem 1. *Let there be $n \in \mathbb{N}$ sellers with an identical offer strategy. When they place their bids level 1 rationally, then there is exactly one optimal strategy s^1_{opt}. An upper bound for the optimal offer strategy is $s^1_{opt} \leq 1/s_1$.*

Proof: See the Appendix. □

When all sellers pursue the same multilateral offer strategy, then there is exactly one optimal offer strategy s^1_{opt}, according to theorem 1. That optimal strategy maximizes each seller's individual profit, given that all sellers follow that same strategy.

This section so far analysed the optimal sellers' multilateral offer strategy and proved major properties of that strategy and its implications on the sellers' profit. Next, we focus on that strategy's effects on the buyer. We start with the buyer's expected profit in the auction.

Proposition 5. *The buyer's expected profit as a function of the sellers' multilateral offer strategy s is*

$$\boldsymbol{E}\left(P_B\right)(s) = \frac{n}{2s^n \Delta s^{n+1}}\left(B(s_2) - B(s s_1)\right),$$

where the function B is defined as

$$\begin{aligned} B(x) &= \frac{2\,\mathrm{s}_2\left(s\,\mathrm{s}_2 - x\right)^{n+1}\,(s-1)}{n+1} - \frac{\left(s\,\mathrm{s}_2 - x\right)^{n+2}}{n+2} - \frac{{\mathrm{s}_2}^2\left(s\,\mathrm{s}_2 - x\right)^n (s-1)^2}{n} \\ &= (s s_2 - x)^n \left(\frac{2\,\mathrm{s}_2\left(s\,\mathrm{s}_2 - x\right)(s-1)}{n+1} - \frac{\left(s\,\mathrm{s}_2 - x\right)^2}{n+2} - \frac{{\mathrm{s}_2}^2 (s-1)^2}{n}\right). \end{aligned}$$

Proof: See the Appendix. □

The proposition above allows us to analyse the influence that sellers have on the buyer's profit. The buyer profits when more sellers participate in the auction. In fact, more participating sellers have two positive effect for the buyer. First of all, the probability that one seller underbids the buyer's reserve price increases. This affects the probability that the auction is successful positively. Secondly, the value of the expected lowest offer of the seller-group decreases when more sellers bid in the auction. The buyer's profit, which is the difference between his reserve price and the lowest bid, thereby increases on average. However, if the seller-group raises their offer strategy s, then in particular the lowest offer of the seller-group increases as well. This has negative effect on the buyer's profit.

As shown in proposition 5, the buyer's expected profit increases if the number of bidding sellers increases. However, that profit is bounded, according to the next proposition.

Proposition 6. *When there are infinitely many sellers, then the buyer's expected profit as a function of the sellers' multilateral offer strategy s is*

$$\boldsymbol{E}(B)(s) = \frac{(s_2 - s s_1)^2}{2s\Delta s}.$$

Proof: See the Appendix. □

When there are infinitely many sellers, then the above proposition proves that the buyer's profit is dependent on 2 variables: the sellers' multilateral offer strategy s and the players' valuation imprecision α. The buyer's profit is decreasing in the sellers' offer strategy, as in the case of finitely many sellers. The buyer's profit increases with increasing valuation imprecision, when the sellers' offer strategy is constant. However, sellers select their optimal offer strategy as a function of imprecision. That strategy s increases with valuation imprecision. As discussed above, a higher sellers' offer strategy affects the buyer's profit negatively. That is, the effects of valuation imprecision on the buyer's expected profit need to be analysed in more detail.

We summarize the above stated verbal analysis of the buyer's profit in the next proposition.

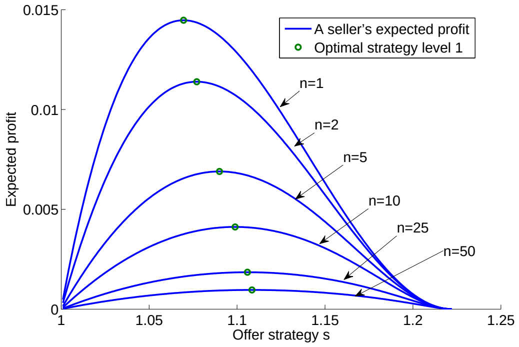
Proposition 7. *The expected profit of the buyer is increasing in the number of sellers and decreasing in the sellers' multilateral offer strategy s.*

Proof: See the Appendix. □

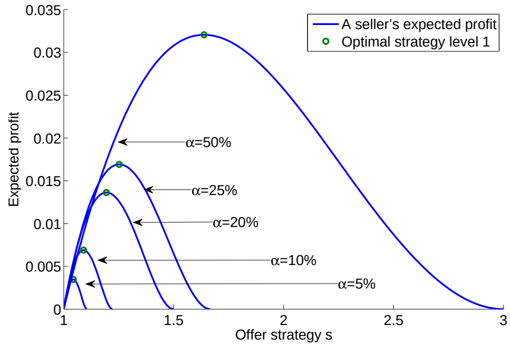
Proposition 1 states that there is exactly one optimal level 1 offer strategy s^1_{opt} for valuation imprecision $0 < \alpha < 1$ and an arbitrary number of sellers. That is, even when the sellers do not agree on an offer strategy, they mutually choose the same optimal offer strategy s^1_{opt}, in level 1 rationality. Proposition 4 argues that if there are more sellers in the market, then they affect each seller's expected profit negatively. The intuition behind this fact is, that each seller's probability to place the lowest bid decreases when more sellers bid in the auction. Proposition 4 furthermore shows that valuation imprecision is a valuable property in the auction market: the higher the sellers' valuation imprecision, the higher their expected profit.

This section proved the intuition, that a buyer profits when more sellers place their bids in the auction. He further profits when each seller places her offer comparably low. As for a seller, valuation imprecision is beneficial for a buyer.

Let us illustrate this section's analysis with some figures. Figure 1 illustrates major properties of the reverse auction under level 1 rationality. The propositions 2 to 4 established the basis for these two figures. On the x-axis, the sellers' multilateral offer strategy s is shown. The y-axis illustrates the expected profit of a seller as calculated in proposition 2.



(a) Optimal level 1 strategy for valuation imprecision $\alpha = 10\%$ and different numbers of sellers n



(b) Optimal level 1 strategy for $n = 5$ sellers and different valuation imprecision α

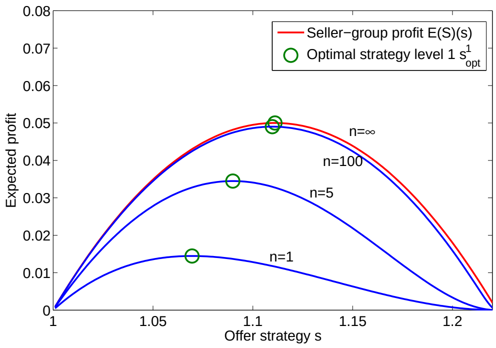
Fig. 1 – A seller's expected profit

It can be seen that the expected profit increases until s_{opt} is reached. Optimal multilateral offer strategy is marked with a circle. Offer strategies that exceed the optimal strategy ($s > s_{opt}$) lead to a lower expected profit and therefore are inefficient strategies. Proposition 1 proves that this optimal level 1 multilateral offer strategy s_{opt} is unique. This can be seen in figure 1 (a) and (b), where markets with different numbers of sellers and different valuation imprecision always have exactly one optimal multilateral offer strategy.

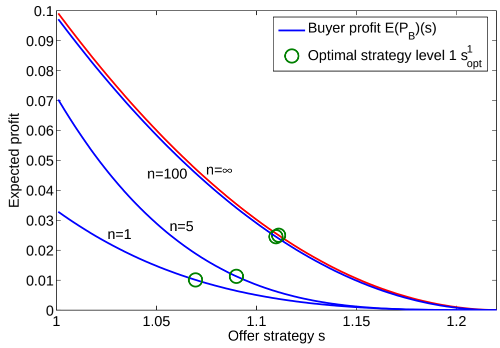
Figure 1 (a) analyses the influence of the number of sellers on the expected profit of a single seller. In this figure, valuation imprecision α is fixed at 10%. Proposition 4 states that an increasing number of sellers leads to a decrease in a seller's expected profit. This intuitive finding can be observed in figure 1 (a), where auctions with an increasing number of sellers between 1 and 50 are illustrated. The figure shows that the expected profit of a seller is decreasing in the number of sellers for any multilateral offer strategy s. Note that the decrease is not linear in the number of sellers n. It furthermore can be observed that the optimal offer strategy s_{opt} is increasing in the number of sellers n. That is, under level 1 rationality it is optimal for each seller to increase her offer as more sellers bid in the auction.

Figure 1 (b) fixes the number of sellers to $n = 5$ and shows the effect of different valuation imprecision α on the optimal level 1 multilateral offer strategy s_{opt} and a seller's expected profit. Higher valuation imprecision is a main driver for profit. In fact, a seller's expected profit is approximately linearly increasing in valuation imprecision. Further, the optimal level 1 offer strategy is increasing in the number of sellers. This means that it is level 1 rational for a seller to increase her offer when the valuation imprecision α increases. The optimal offer strategy s_{opt} is approximately linearly increasing in the number of sellers, just as a seller's expected profit.

Figure 2 shows the expected profit of the group of all sellers and compares that group's



(a) Expected profit of the seller-group as a function of groupsize



(b) Expected profit of the buyer as a function of the number of sellers

Fig. 2 – Expected profits as a function of the sellers' offer strategy for fixed valuation imprecision $\alpha = 10\%$.

profit to that of the buyer. For this illustration valuation imprecision is set to $\alpha = 10\%$.

Figure 2 (a) analyses the effect of the size of the seller-group on its expected profit. It can be seen that the expected group-profit is increasing in the number of sellers, that is, each additional seller increases the expected profit of the seller-group. In comparison, figure 1 (a) showed that each individual seller's expected profit is decreasing in the seller-group size. Therefore the group size affects the group profit positively, but individual profit negatively. The seller-group's expected profit converges with increasing group size. As a result, an additional seller influences the expected group profit less, if the seller-group is bigger. This observed convergence has been shown in proposition 3. The figure shows that a group of 100 sellers is already close to the convergence state.

Figure 2 (b) shows the buyer's expected profit as a function of the sellers' multilateral offer strategy s. Proposition 7 showed that the buyer's expected profit is increasing in the number of sellers and furthermore decreasing in their multilateral offer strategy. This can be observed in the figure. For each number of sellers ($n = 1, 5, 10, 100, \infty$), the buyer's expected profit is decreasing in the sellers' offer strategy s. That is, when they increase their offer price, then the buyer's profit decreases. However, the higher the number of sellers n, the higher the buyer's expected profit. This fact is independent on the sellers' offer strategy s.

In figure 2 (b), the green circle on each solid line shows the sellers' optimal multilateral level 1 offer strategy s^1_{opt}. It can be seen that the buyer profits from an increase in the number of sellers when they place their bids level 1 optimally. Proposition 6 showed that the buyer's profit converges with an increasing seller-group size. That proposition can be observed in the figure, where the expected profit of the buyer for $n = 100$ sellers is already close to the limit profit, for $n \to \infty$.

It is a necessary condition in proposition 2 that all sellers pursue the same offer strategy

s^1_{opt}. They choose their strategy such that each seller's profit is maximized. Thereby they also maximise the profit of the seller-group. We call that strategy rationality level 1.

Assume that a single seller knows the multilateral profit optimizing offer strategy s^1_{opt} and anticipates that the other sellers place their bids accordingly. Then she may decide to optimize her expected profit given this uniform offer strategy of the other sellers. We call this behaviour rationality level 2. In other words, rationality level 2 means that all sellers mutually agree on the profit optimizing offer strategy s^1_{opt}. One seller does not commit to this multilateral strategy and optimizes her profit as a best response to the remaining sellers' multilateral offer strategy. The next section discusses the reverse auction under this level 2 rationality.

2.2 Level 2 Rationality

In the last section all sellers pursue the same optimal multilateral offer strategy s^1_{opt}. When each seller knows the other sellers' optimal offer strategy s^1_{opt}, then a single seller can optimize her expected profit given the other sellers' offer strategies s^1_{opt}. That strategy puts the single seller in an advantageous position as she knows the other sellers' multilateral level 1 offer strategy and optimally reacts upon it. That strategy is called level 2 optimisation. In this section, this level 2 strategy and its effects on bidding sellers and the buyer are analysed and interpreted.

We start with the deal probability for the unilateral behaving seller.

Proposition 8. *When there are $n-1$ sellers with a multilateral offer strategy $s \in [1, s_2/s_1]$, then for a seller with the unilateral offer strategy s' the probability of winning the auction is*

$$\boldsymbol{P}(D_S)(s,s') = \begin{cases} 1 & , \textit{for } s' < s_1/s_2 \\ \frac{F(s_2)-F(ss_1)}{s's^{n-1}\Delta s^{n+1}} + \frac{s_2\, s_1\, (s-s')-1/2\, s_1{}^2 s^2 + 1/2\, s'^2 s_1{}^2}{\Delta s^2 s'} & , \textit{for } s_1/s_2 \le s' < s \\ \frac{F(s_2)-F(s's_1)}{s's^{n-1}\Delta s^{n+1}} & , \textit{for } s \le s' \le s_2/s_1 \\ 0 & \textit{otherwise.} \end{cases}$$

The function F is defined as in proposition 1.

Proof: See the Appendix. □

When a seller with unilateral offer strategy increases that strategy s', then this seller's offer price increases. As a result, the probability that this seller's offer exceeds that of the other sellers increases. In this case, the unilateral bidding seller's probability to win the auction decreases. This effect can be observed in the formula above, where the seller's increase in s' affects her success probability negatively.

When the multilateral bidding $n-1$ sellers increase their offer strategy s, then their offer prices increase. As a result, the probability that these sellers' offers exceed the unilateral bidding seller's offer increases. In this case, the unilateral bidding seller's probability to win the auction increases. This effect can be observed in the formula above, where the unilateral bidding seller's probability to win the auction increases when the remaining $n-1$ sellers increase their offer strategy s.

In the next proposition the buyer's deal probability is calculated.

Proposition 9. *When there are $n-1$ sellers with the multilateral offer strategy $s \in [1, s_2/s_1]$ and one seller with a unilateral offer strategy $s' < s_2/s_1$, then the probability that the buyer receives a suitable offer is*

$$\boldsymbol{P}(D_B)(s,s') = \begin{cases} 1 & \text{, for } s' < s_1/s_2 \\ \frac{s_2 - s_1 s'}{\Delta s} - \frac{s_1(s-s')(s's_2 - s_1(s+s'))}{2\Delta s^2 s'} - \frac{G(s_2)-G(ss_1)}{\Delta s^{n+1} s^{n-1} s' n(n+1)} & \text{, for } s_1/s_2 \leq s' \leq s \\ \frac{s_2 - s_1 s}{\Delta s} - \frac{s}{n} + \frac{(ss_2 - s's_1)^n}{\Delta s^n s^{n-1} n} - \frac{G(s_2)-G(s's_1)}{\Delta s^{n+1} s^{n-1} s' n(n+1)} & \text{, for } s < s' \leq s_2/s_1. \end{cases}$$

The function G, to simplify the terms above, is defined as

$$G(x) := (s\,\mathrm{s}_2 - x)^n \;(n\,x + s\,\mathrm{s}_2 - \mathrm{s}_2\,\mathrm{s}' - n\,\mathrm{s}_2\,\mathrm{s}').$$

Proof: See the Appendix. □

When the $n-1$ sellers' increase their multilateral offer strategy s or the seller with unilateral offer strategy s' increases her offer, then offer prices rise. In particular, the price of the lowest offer rises. As a result, the probability that the buyer's reservation price exceeds the sellers' lowest offer decreases. Accordingly, the probability that the auction is successful decreases. This can be seen from the above proposition, where an increase in s or s' affects the auction success negatively.

When unilateral offer strategy and multilateral offer strategy are identical (that is, $s = s'$), then the formula of proposition 9 simplifies to that of proposition 1, where all sellers bid multilaterally.

After formulas for the success probabilities have been introduced, the bidding sellers' and the buyer's expected profits in the auction is calculated. We start with the expected profit of the seller that bids unilaterally.

Proposition 10. *When $n-1$ sellers pursue the offer strategy $s \in [1, s_2/s_1]$, then the expected* *profit of the seller with unilateral offer strategy* $s' \in [1, s_2/s_1]$ *is*

$$\boldsymbol{E}_{s' \geq s}\left(P_S\right)(s, s') = \frac{(s'-1)\ \left(\ A(s_2) - A(s's_1)\ \right)}{s'^2\ s^{n-1}\ \Delta s^{n+1}}$$
$$\boldsymbol{E}_{s' < s}\left(P_S\right)(s, s') = \frac{(s'-1)\ \left(\ A(s_2) - A(ss_1)\ \right)}{s'^2\ s^{n-1}\ \Delta s^{n+1}} + \frac{2\ s_1^3\ (s'^3 - s^3) + 3\ s_1^2 s_2 (s^2 - s'^2)}{6s'^2 \Delta s^2}.$$

The expected profit of the seller with unilateral offer strategy s' *can be summarized as*

$$\boldsymbol{E}\left(P_S\right)(s, s') = \mathbb{1}_{s' \geq s} \boldsymbol{E}_{s' \geq s}\left(P_S\right) + \mathbb{1}_{s' < s} \boldsymbol{E}_{s' < s}\left(P_S\right).$$

The function A *is defined as in proposition 2.*

Proof: See the Appendix. □

The above proposition calculates the expected profit of the unilateral bidding seller who is aware of the other sellers' multilateral offer strategy. In particular, we have $\mathbf{E}\left(P_S\right)(s, s) = \mathbf{E}\left(P_S\right)$. That is intuitive, because when the unilateral bidding seller pursues the offer strategy $s' = s$, then her expected profit is the same as the other sellers' profit calculated in proposition 2.

Before we determine the optimal seller's unilateral offer strategy s', we calculate the buyer's expected profit.

Proposition 11. *When* $n-1$ *sellers pursue a multilateral offer strategy* $s \in [1, s_2/s_1]$ *and one seller a unilateral offer strategy* $s' \in [1, s_2/s_1]$*. Then the buyer's expected profit is*

$$\boldsymbol{E}(P_B)(s, s') = \begin{cases} \frac{(s_2 - s's_1)^3}{6\Delta s^{n+1} s^{n-1} s'}\left((s\Delta s)^{n-1} - (s_2(s-1))^{n-1}\right) + \frac{(n-1)(H(s_2) - H(ss_1))}{3\Delta s^{n+1} s^{n-1} s'} & \text{, for } s' \leq s \\ \frac{(s_2 - s's_1)^3}{6\Delta s^{n+1} s^{n-1} s'}\left((ss_2 - s's_1)^{n-1} - (s_2(s-1))^{n-1}\right) + \frac{(n-1)(H(s_2) - H(s's_1))}{3\Delta s^{n+1} s^{n-1} s'} & \\ + \frac{(n-1)(s_2 - s's_1)(I(s's_1) - I(ss_1))}{2\Delta s^{n+1} s^{n-1} s'} & \text{, for } s < s'. \end{cases}$$

The functions H *and* I*, that simplify the terms above, are*

$$I(y) := -\frac{\left(y - s\,\mathrm{s}_2\right)^2 \left(s\,\mathrm{s}_2 - y\right)^n}{n+2} - \frac{\mathrm{s}_2{}^2 \left(s\,\mathrm{s}_2 - y\right)^n (s-1)^2}{n} - \frac{2\,\mathrm{s}_2\ \left(y - s\,\mathrm{s}_2\right)\ \left(s\,\mathrm{s}_2 - y\right)^n\ (s-1)}{n+1}$$
$$H(y) := \frac{\left(y - s\,\mathrm{s}_2\right)^3 \left(s\,\mathrm{s}_2 - y\right)^n}{n+3} + \frac{\mathrm{s}_2{}^3 \left(s\,\mathrm{s}_2 - y\right)^n (s-1)^3}{n}$$
$$+ \frac{3\,\mathrm{s}_2 \left(y - s\,\mathrm{s}_2\right)^2 \left(s\,\mathrm{s}_2 - y\right)^n\ (s-1)}{n+2} + \frac{3\,\mathrm{s}_2{}^2\ \left(y - s\,\mathrm{s}_2\right)\ \left(s\,\mathrm{s}_2 - y\right)^n (s-1)^2}{n+1}.$$

Proof: See the Appendix. □

When the $n-1$ sellers increase their multilateral offer strategy s or the seller with unilateral offer strategy s' increases her offer strategy, then their offer prices rise. That is, then the probability that these offers exceed the buyer's reservation price increases. As a result, it is less likely that the auction is successful. Furthermore, in case of a successful auction, the expected difference between the buyer's reserve price and the winning seller's bid is lower if sellers have higher offer strategies. In this case, the buyer's expected profit in the auction decreases. This can be seen from the above proposition, where an increase in s and s' affect the auction success negatively and vice versa.

When unilateral offer strategy and multilateral offer strategy are identical (that is, $s = s'$), then the formula from proposition 11 simplifies to that of proposition 5, where all sellers bid multilaterally.

The next theorem shows that there is exactly one optimal level 2 offer strategy. This optimal strategy is lower than the optimal level 1 strategy.

Theorem 2. *When $n-1$ sellers choose an optimal multilateral offer strategy s^1_{opt} and the $n-th$ seller is aware of their strategy, then that seller may optimise her offer strategy s^2_{opt} accordingly*[5]*. There is exactly one such optimal level 2 strategy. That strategy has the property $s^2_{opt} \leq s^1_{opt}$.*

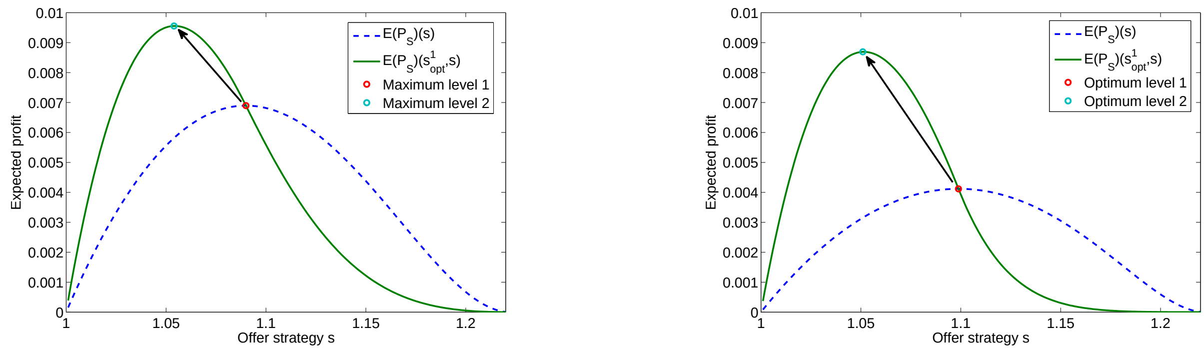
Proof: See the Appendix. □

Theorem 1 proved that there is exactly one optimal level 1 strategy for all sellers. When all sellers but one commit to that strategy, then that seller can optimise her offer by exploiting her knowledge of the other sellers' strategy. Theorem 2 states that this unilateral behaving seller has exactly one such optimal offer strategy s^2_{opt}. The theorem further shows that the optimal level 2 strategy is lower than the optimal level 1 strategy. Thus in these optima, the unilateral bidding seller's offer is lower than the average offer of the other sellers.

When a seller pursues a lower strategy than the other sellers, then this seller's offer is on average lower than those of the other sellers. Accordingly, this seller has a higher chance to win the auction by placing the lowest bid. Thus the unilateral seller with offer $s^2_{opt} \leq s^1_{opt}$ has a higher chance to win the auction. Furthermore that seller's expected profit is higher than that of the other sellers. If this was not true, then that seller could choose offer strategy s^1_{opt}, just as the other sellers. That strategy would give her a profit equal to that of the other sellers.

In summary, the seller that does not commit to the multilateral level 1 strategy is in advantage over sellers that commit to this strategy. We call this offer strategy level 2 rationality. It is a reasonable label because a seller pursuing the level 2 strategy takes the optimal level 1 strategy into her consideration and places her bid accordingly. Thereby she optimises her offer under the constraint that the other sellers commit to the level 1 strategy.

[5] In this theorem, the unilateral strategy s^2_{opt} is a seller's best response strategy, when the other sellers pursue offer strategy s^1_{opt}. In this paper we may also use the notation s^2_{opt} as a seller's optimal response when the other sellers pursue some feasible offer strategy s. When we do so, the different use will be mentioned.



(a) $n = 5$ sellers (b) $n = 10$ sellers

Fig. 3 – Optimal level 2 strategy for fixed valuation imprecision $\alpha = 10\%$

The unilateral bidding seller on average uses a lower offer than the multilateral bidding sellers. That fact affects the buyer's expected profit positively, as the next proposition shows.

Proposition 12. *The buyer profits when a seller places her bid level 2 optimally.*

Proof: See the Appendix. □

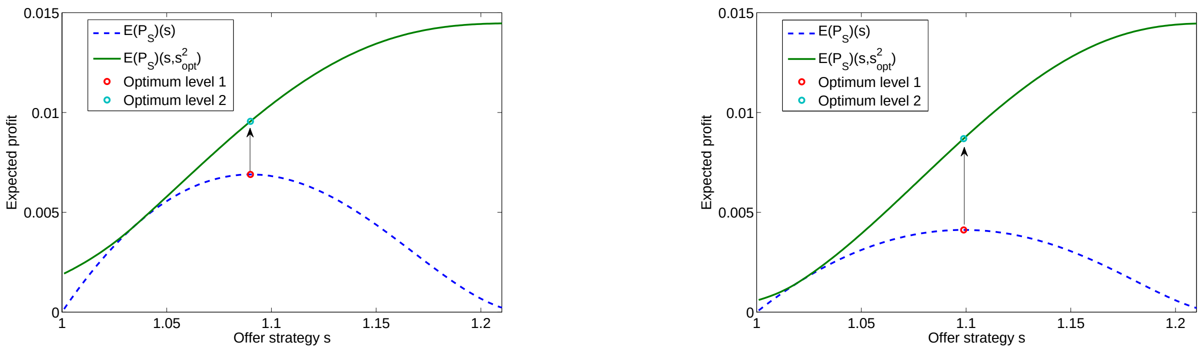
The buyer profits from the unilateral offer strategy of a single seller, as proven by the proposition above. When a seller pursues the optimal level 2 strategy, then her offer strategy is lower than that of the other sellers, as shown in theorem 2 ($s^2_{opt} \leq s^1_{opt}$). As a result, the average lowest offer decreases. This lowest offer affects the buyer's expected profit from the auction positively. However, the unilateral bidding seller profits from her strategy. However, the sellers who commit to the multilateral level 1 offer strategy suffer from a decreasing profit.

This section provided the reader with the major properties of the level 2 bidding strategy. Next, a detailed numeric and graphic analysis of this section's result follows.

The main properties of this section's propositions are illustrated in figure 3 and 4. For a numerical illustration, the valuation imprecision is fixed at $\alpha = 10\%$. The left graph of both figures shows the analysis for $n = 5$ sellers, whereas the right shows the analysis for $n = 10$ sellers. This enables the reader to clearly see different properties of the expected profit and its dependence on the size of the seller-group.

The dashed lines in figure 3 (a) and (b) are the same as in figure 1 (a): namely the expected profit of a single seller when all sellers pursue the multilateral offer strategy s, as indicated on the x-axis. The properties of this multilateral offer strategy have thoroughly been discussed in figure 1 (a).

Level 2 strategy and its influence on the seller who pursues this strategy can be observed in figures 3 (a) and (b). These figures show the expected profit of the single seller who pursues



(a) $n = 5$ sellers

(b) $n = 10$ sellers

Fig. 4 – Optimal level 2 strategy for fixed valuation imprecision $\alpha = 10\%$.

the unilateral offer strategy s' (indicated on the x-axis) assuming the other sellers commit to multilateral optimal level 1 strategy s^1_{opt}. That seller's expected profit is illustrated as the solid line in figure 3.

In figure 3 (a) the optimal level 1 strategy is $s^1_{opt} = 1.0900$. By applying the unilateral offer strategy $s^2_{opt} = 1.0543$, a seller can increase her expected profit from 0.69% to 0.96%, i.e almost by 50%. Analysing figure 3 (b), it can be seen that a seller can increase her expected profit by approximately 50%, when this unilateral bidding seller applies the optimal level 2 bidding strategy.

Figure 4 is rather similar to figure 3, but takes a different point of view. Analogous to figures 3 and 1 (a) it shows the expected level 1 profit for $n = 5$ and $n = 10$ sellers. This profit is illustrated as the dashed lines in the figure. In contrast to previous figures, this figure additionally shows a seller's optimal level 2 response, when that seller anticipates the other sellers to have bidding strategy s, as indicated on the x-axis.

The solid line shows a seller's expected profit in level 2 optimum, given the other sellers' level 1 offer strategy as shown on the $x-$axis. Figure 4 thus shows the increase in expected profit a single seller can achieve by not committing to the multilateral offer strategy of the other sellers. Obviously the unilateral level 2 strategy generates at least as much profit as the multilateral level 1 strategy. This is indicated in the figures, as the solid line (optimal level 2 response strategy) is at least as high as the dashed line (level 1 strategy). It can be seen that there is exactly one level 1 offer strategy such that level 2 strategy is exactly as good as level 1 strategy. This offer strategy will later be characterizes as the level 3 strategy.

The above properties for the example with valuation imprecision $\alpha = 10\%$ are true while $0 < \alpha < 1$. Further numerical examples are presented in table 1, where the number of sellers is fixed at $n = 5$ and valuation imprecision varies. The case $\alpha = 10\%$ was discussed in detail above, where optimal level 1 strategy is $s^1_{opt} = 1.0900$ and optimal level 2 response

$n = 5$	Level 1 rationality			Level 2 rationality		
	s^1_{opt}	$\mathbf{E}(P_S)(s^1_{opt})$	p_d	s^2_{opt}	$\mathbf{E}(P_S)(s^1_{opt}, s^2_{opt})$	p_d
$\alpha = 5\%$	1.0434	0.3465%	8.2935%	1.0264	0.4840%	18.9685%
$\alpha = 10\%$	1.0900	0.6896%	8.3001%	1.0543	0.9556%	18.8505%
$\alpha = 20\%$	1.1940	1.3630%	8.3154%	1.1156	1.8577%	18.6028%
$\alpha = 25\%$	1.2523	1.6917%	8.3242%	1.1492	2.2860%	18.4727%
$\alpha = 50\%$	1.6379	3.2060%	8.3788%	1.3596	4.1410%	17.7741%

Table 1 – Properties of level 1 and 2 offer strategies with $n = 5$ sellers

strategy is $s^2_{opt} = 1.0543$. It can be seen from the table that optimal level 1 strategy is increasing in imprecision, as is optimal level 2 response strategy. The optimal response strategy is approximately $(1+s^1_{opt})/2$, independent of valuation imprecision. By applying level 2 optimization, the seller approximately doubles her chances to win the auction, compared to level 1 offer strategy. Her expected profit furthermore is increasing in imprecision. Thus valuation imprecision affects her profit positively.

Our analysis showed that for each seller it is optimal to apply the level 2 offer strategy. That is, it is optimal for each seller not to commit to a multilateral offer strategy and thereby to put herself in a better position than sellers who commit to a multilateral offer strategy. If all sellers are aware of this advantage and want to profit from it, eventually no seller commits to the multilateral level 1 strategy and all sellers apply the level 2 offer strategy. As the game is symmetric, the optimal level 2 strategy is the same for each seller. Thus, all sellers apply the same level 2 offer strategy. It therefore can be interpreted as another multilateral strategy all sellers commit to. However, that offer strategy must fulfil the axiom that no seller can put herself in a better position than another seller by applying a different offer strategy. This is equivalent to the condition that this multilateral strategy and a seller's optimal response upon it are identical. Then no seller profits from applying a different offer strategy than the other sellers. The next section studies this equilibrium strategy.

2.3 Level 3 Rationality

In this section all sellers place their bids fully rationally. That is, each seller anticipates the other sellers' bidding strategies and places a best response bid, accordingly. Then a seller's profit is dependent on the other sellers' offer strategies and on her own strategy.

The auction is symmetrical. Therefore no seller will have a different offer strategy than the other sellers. This implies that the sellers indirectly commit to placing their bids multilaterally, as in section 2.1. In contrast to the optimal level 1 strategy, an individual's deviation from the multilateral bidding strategy can not have a positive effect on a seller's profit, when all sellers place their bids level 3 optimally.

More formally, let s^3_{opt} be a multilateral offer strategy when all sellers place their bids level 3 optimally. Then each seller can not increase her profit when she applies another offer strategy $s \in [1, s_2/s_1]$. This is represented by the formula

$$\mathbf{E}\left(P_S\right)\left(s^3_{opt}, s^3_{opt}\right) \geq \mathbf{E}\left(P_S\right)\left(s^3_{opt}, s\right)\ \forall s \in [1, s_2/s_1].$$

This section analyses this level 3 offer strategy. We prove that there is exactly one such optimal level 3 strategy and calculate it explicitly. The buyer profits when the sellers place their bids level 3 optimally, whereas the sellers' expected profit decreases in comparison to level 1 and level 2 strategies.

The next theorem proves main properties of the level 3 optimum.

Theorem 3. *There is exactly one optimum level 3 offer strategy s^3_{opt}. This level 3 optimum is a Nash equilibrium in pure strategies. In equilibrium $s^3_{opt} \leq s^2_{opt}$ holds.*

Proof: See the Appendix. □

The above theorem proves that there is exactly one level 3 strategy. That bidding strategy therefore is the unique bidding equilibrium. When all sellers apply this offer strategy and all sellers anticipate that the other sellers apply it, then there is no seller that can increase her profit by choosing an offer strategy that is different from the equilibrium strategy.

The optimal level 1 strategy exceeds the level 2 optimum, which exceeds the optimal level 3 offer strategy. Therefore the average offers are lowest when the sellers apply the level 3 equilibrium bidding strategy. As a result, the probability that the auction is successful is highest in that equilibrium. When a seller wins the auction, then this seller's profit is lowest when bids are placed level 3 optimally. The buyer profits from the sellers' equilibrium strategy in two ways. First of all, the probability that the auction is successful increases and secondly the winning bid is lower on average.

The next propositions analyse the equilibrium bidding strategy and its implications for a large number of sellers.

Proposition 13. *The optimal level 3 bidding strategy s^3_{opt} converges to 1 for seller-group size $n \to \infty$. The expected profit of an individual seller thereby converges to zero.*

Proof: See the Appendix. □

When the number of sellers increases, then their offer strategy becomes more aggressive. Proposition 13 states that in the limit, the sellers' equilibrium offer strategy $s^3_{opt} = 1$. Then each seller's expected profit is zero because then a seller's reservation and offer price are identical. When more sellers bid in the auction, then the negative effects on each bidding seller increase. That is, the sellers' offers cannibalize their individual profit, which converges to zero.

The effects of the number of bidding sellers on the group-profit are analysed in the proposition below.

Proposition 14. *In bidding equilibrium the expected profit of the seller-group converges to zero for seller-group size $n \to \infty$.*

Proof: See the Appendix. □

When we combine the statements from propositions 13 and 14, we can conclude that an increasing number of bidding sellers harms both, each seller's profit and the profit of the seller-group at the same time. These profits converge to zero for infinitely many bidding sellers. The sellers' competition thus becomes so aggressive that all profit on the sell side finally diminishes.

On the contrary, the buyer, who initiates the auction, profits from the increasing competition among a bigger seller-group. This statement is proven in the next proposition.

Proposition 15. *The buyer's expected profit converges to the maximal valuation imprecision α for $n \to \infty$ sellers, when they place their bids optimally level 3.*

Proof: See the Appendix. □

Proposition 15 shows that a buyer profits from more bidding sellers in the auction. His expected profit converges to the maximal valuation imprecision α. The buyer's expected profit thus is bounded by valuation imprecision. When the parties' valuation abilities are less precise, then a buyer's expected profit from the auction increases. That is, the buyer profits from a raise in valuation imprecision.

We close the analysis of the properties of the level 3 equilibrium bidding strategy by calculating the auction success probability for infinitely many bidding sellers in the proposition below.

Proposition 16. *The probability that the auction is successful converges to* 1 *for infinitely many sellers who place their bids optimally level 3.*

Proof: See the Appendix. □

The above proposition shows that the buyer who initialises an auction has a probability of 1 that his reservation price exceeds the lowest bid, when infinitely many sellers are bidding.

$n = 5$	Level 1 rationality		Level 2 rationality		Level 3 rationality	
	s^1_{opt}	$\mathbf{E}(P_S)(s^1_{opt})$	s^2_{opt}	$\mathbf{E}(P_S)(s^1_{opt}, s^2_{opt})$	s^3_{opt}	$\mathbf{E}(P_S)(s^3_{opt})$
$\alpha = 5\%$	1.04	0.35%	1.03	0.48%	1.02	0.22%
$\alpha = 10\%$	1.09	0.69%	1.05	0.96%	1.04	0.44%
$\alpha = 20\%$	1.19	1.36%	1.12	1.86%	1.08	0.88%
$\alpha = 25\%$	1.25	1.69%	1.15	2.29%	1.10	1.11%
$\alpha = 50\%$	1.64	3.21%	1.36	4.14%	1.27	2.24%

Table 2 – Properties of level 1 to 3 offer strategies with $n = 5$ sellers

That is, then the probability that the auction is successful is 1. For the buyer, this property is desirable because he can be sure to get a suitable offer that generates positive profit.

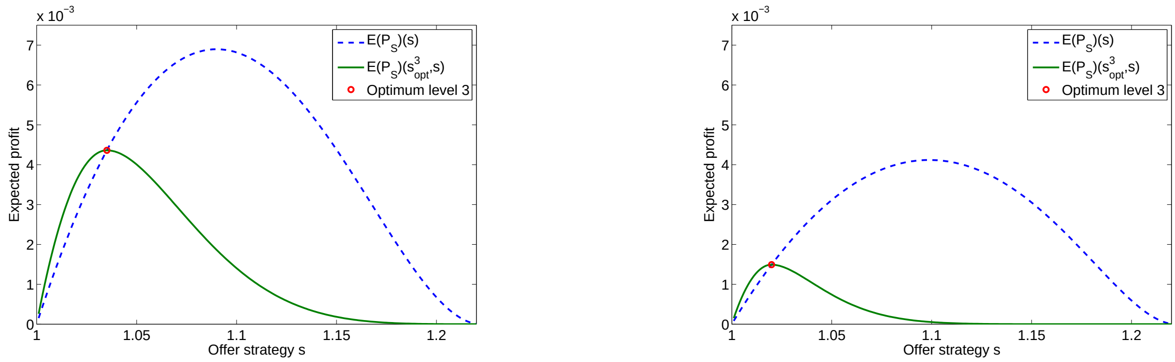
One of infinitely many sellers places the winning lowest bid. That bid however does not generate a profit for the winning seller because the winning seller's reservation and offer prices are identical. In conclusion, more bidding sellers harm each other, while their cannibalising offer strategy is highly profitable for the buyer.

We continue with numerical examples and illustrations of the optimal level 3 offer strategy and its implications for the buyer and the sellers. Table 2 shows the optimal level 1 to 3 offer strategies as a function of valuation imprecision α. In the numerical example, the number of bidding sellers is set to $n = 5$.

Optimal level 1 and level 2 offer strategies were discussed thoroughly in the last section. It was shown that it is of advantage for each seller to pursue the level 2 strategy, as long as all other sellers pursue on optimal level 1 strategy. However, when all sellers behave rationally and optimize their offer optimally level 2, then their combined individual strategies lead to the level 3 optimum s^3_{opt}. Table 2 analyses major properties of these level 3 optima. The level 3 optimum is lower or equal to the level 2 optimum, which does not exceed the level 1 optimum. That is, $s^3_{opt} \leq s^2_{opt} \leq s^1_{opt}$.

As a result of the level 3 optimization the sellers' expected profit decreases, compared to optimal the level 2 and level 1 bidding strategies. Although optimal level 3 bidding is the equilibrium bidding strategy, it is not of benefit for each seller or the seller-group, compared to levels 1 and 2 bidding strategies.

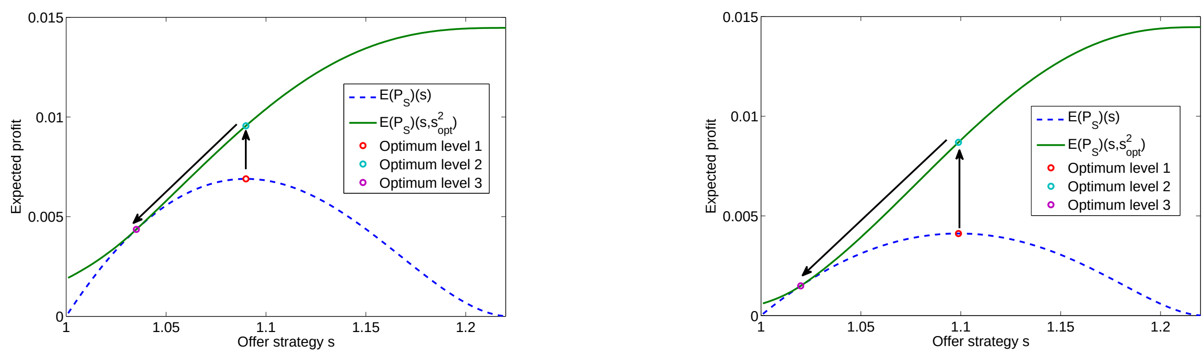
Figure 5 shows the expected profit as a function of the bidding strategy for a fixed valuation imprecision $\alpha = 10\%$ and $n = 5$ sellers in figure 5 (a) ($n = 10$ sellers in figure 5 (b)). In each figure, the dotted line shows the expected profit of a seller, when all sellers pursue multilateral offer strategy s as indicated on the x-axis. The solid line shows a seller's profit who applies unilateral bidding strategy s (as indicated on the x-axis), while the remaining sellers bid



(a) $n = 5$ sellers (b) $n = 10$ sellers

Fig. 5 – Optimal level 3 equilibrium strategy for fixed valuation imprecision $\alpha = 10\%$

optimally level 3. There is exactly one point of intersection, where both strategies share the same expected profit. That point of intersection is marked with a red dot. It is the level 3 optimal bidding strategy s^3_{opt}. When the sellers apply this offer strategy, then a seller can not increase her profit by applying a strategy that is different from s^3_{opt}. This can be observed in the figure, as the solid line (that is, the profit of the unilateral bidding seller) reaches its maximum when her offer strategy is s^3_{opt}. Compared to the level 1 optimal strategy, all sellers' expected profit decreases when they apply the optimal level 3 bidding strategy. However, the level 3 optimum is the unique equilibrium bidding strategy.



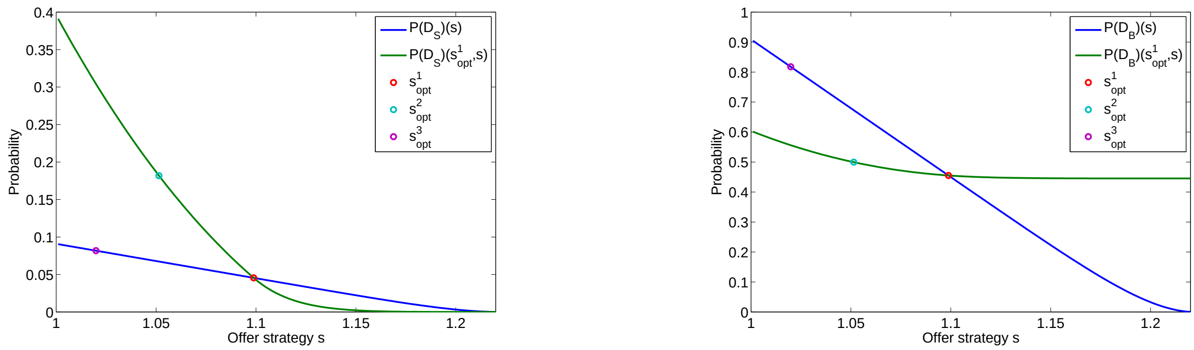
(a) $n = 5$ sellers (b) $n = 10$ sellers

Fig. 6 – Optimal level 3 equilibrium strategy for fixed valuation imprecision $\alpha = 10\%$

Figure 6 shows a single seller's profit for the bidding strategies levels 1, 2 and 3. The dashed line shows a seller's expected profit, when all sellers apply a bidding strategy according to the x-coordinate of the figure. The solid line shows the expected profit (level 2) that a seller can achieve, when she anticipates that the other sellers multilaterally place their bids according to the offer strategy as indicated on the x-axis.

When a single seller knows the other sellers' multilateral offer strategy, then this single seller can increase her expected profit by optimizing her offer strategy accordingly. The arrow from level 1 to level 2 optimum shows a seller's potential profit increase. There is always exactly one multilateral strategy, where unilateral optimization does not put a single seller in favour of the other sellers, that is the level 3 optimum. The left downward-arrow indicates the change of profit from the level 2 to the level 3 optimum. In the level 3 optimum, each seller's profit is lower than in the level 1 and 2 optima.

In bigger markets each seller's expected profit shrinks in all bidding optima. This can be seen by the comparison of figure 6 (a) and (b). All optima are lower in the right figure, where the seller-group consists of 10 sellers, compared to 5 sellers on the left. The optimal bidding strategies in the left figure generate a higher profit than those when more sellers bid in the auction.



(a) Deal probability of a seller

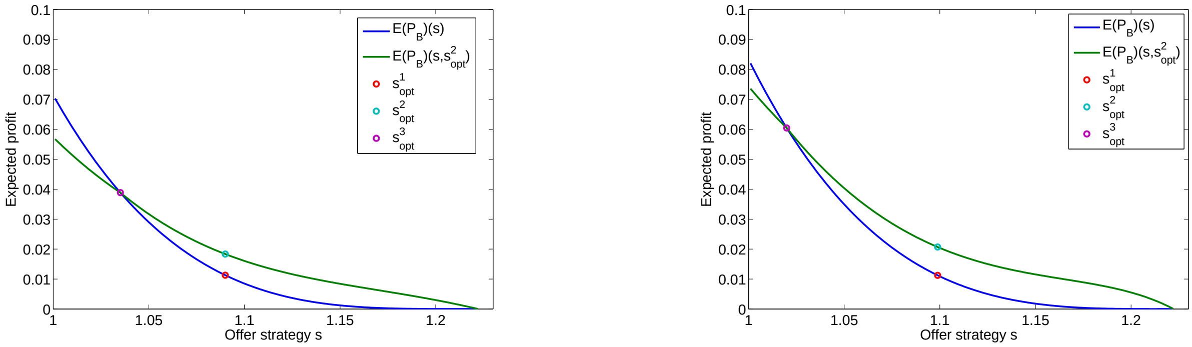
(b) Deal probability of the buyer

Fig. 7 – Deal probabilities for $n = 10$ sellers and $\alpha = 10\%$

Figure 7 illustrates the success probability for a seller in figure 7 (a) and the buyer in figure 7 (b). The probabilities are based on the formulas developed in propositions 8 and 9. The blue line shows the deal probabilities when all sellers commit to a multilateral offer strategy. The green line shows the deal probability under the constraint that a seller applies offer strategy as indicated on the x-axis, whereas the remaining sellers multilaterally pursue offer strategy s^1_{opt}. For these examples, $n = 10$ sellers and valuation imprecision $\alpha = 10\%$ where chosen.

Under multilateral level 1 optimum an individual seller has approximately a probability of 5% for deal success, as 7 (a) shows. Unilateral optimisation allows a seller to increase her success probability to almost 20%. In optimal level 3 bidding strategy, the deal probability is slightly below 10%. In terms of expected profit, the level 3 bidding equilibrium has a lower expected profit than the optimal level 1 strategy. In contrast, the deal probability increases from the optimal level 1 to level 3 bidding strategies. The level 3 optimal strategy is therefore preferable compared to the level 1 optimum in terms of success probability.

Figure 7 (b) shows the buyer's deal probability. That is, the probability that one seller places a successful bid. It can be seen that the level 2 strategy leads to a higher success probability (approximately 55%) compared to the level 1 strategy (approximately 50%). Further, the level 3 optimisation includes a higher deal probability of over 80%. The buyer thus profits from the sellers' level 3 bidding equilibrium compared to the level 1 and 2 bidding strategies.



(a) $n = 5$ sellers

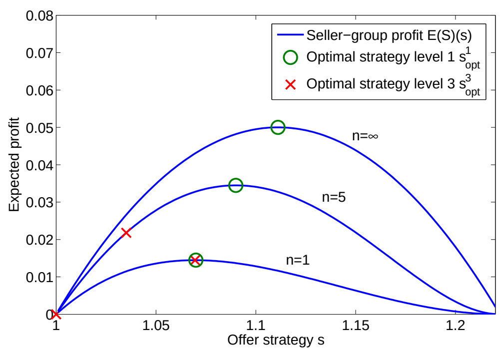
(b) $n = 10$ sellers

Fig. 8 – Buyer's expected profit in offer strategies levels 1 to 3

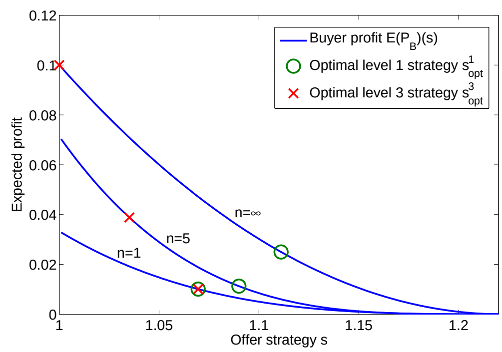
Proposition 11 introduced the buyer's expected profit under level 2 rationality. This formula can be adapted to the level 3 rationality by applying $s' = s$ to the formula. As already mentioned, that simplified formula is the one introduced in proposition 5. Additionally, we have proven $s^3_{opt} \leq s^2_{opt} \leq s^1_{opt}$ in theorem 2 and 3. That is, applying offer strategies 1 to 3, the average sellers' offers decrease. Therefore the buyer's expected profit is increasing in offer strategies 1 to 3. That fact is illustrated in figure 8, where markets with $n = 5$ and $n = 10$ sellers are analysed. The buyer's expected profit in a multilateral strategy is illustrated in blue. The green line shows the expected profit of the buyer, given the optimal according unilateral response strategy s^2_{opt}. Both figures show that the buyer has a positive profit from the level 1 bidding optimum. When a seller bids level 2 optimally, then the buyer's profit increases. The buyer's profit further increases when all sellers place their bids level 3 optimally. These observation are intuitive because the sellers' bids are lowest when they bid optimally level 3.

Figure 9 (a) illustrates the effects of level 3 bidding on the profit of the seller-group. For this illustration, the valuation imprecision is fixed at $\alpha = 10\%$. The solid lines in the figure show the expected profit of the seller-group as a function of their multilateral offer strategy s, as is indicated on the x-axis. Thereby each solid line shows the expected profit of the sellers group for a different group size. It can be seen that group profit is increasing in group size and convergent for $n \to \infty$ sellers. For each group size, the optimal multilateral level 1 bidding strategy is market with a circle. So far, the figure is identical to figure 2, where the sellers' multilateral strategy was analysed in detail. It can be seen that the optimal level 1 offer strategy that maximises each seller's profit at the same time maximises the profit of the seller-group.

Additionally, the figure shows the optimal level 3 strategies for different seller-group sizes. These are marked with a cross. For $n = 1$ sellers, levels 1 and 3 strategies are identical, as a single seller can optimise her strategy without taking a second, third, ... seller's offer into consideration. For seller-group size $n = 5$, there is a loss of the expected profit from offer strategy level 1 to level 3. In this case, the optimal strategy $s^1_{opt} = 1.09$ changes to $s^3_{opt} = 1.04$. This represents a reduction from approximately 3.45% to 2.20% in the seller-group's profit. The highest profit decrease is achieved when infinitely many sellers place their bids. In the limit the optimal level 1 strategy is $s^1_{opt} = 1/s_1 = 10/9$, according to proposition 3. Furthermore, the optimal level 3 strategy is $s^3_{opt} = 1$. In the latter case the expected group profit diminishes to zero.



(a) Expected profit of the seller-group for different numbers of sellers



(b) Expected profit of the buyer for different numbers of sellers

Fig. 9 – Change in profit when sellers bid rationally level 1 and 3. Valuation imprecision $\alpha = 10\%$

Figure 9 (b) shows the effects of optimal level 1 and level 3 strategies on the buyer. Section 2.1 proved that the expected profit of the buyer decreases when sellers increase their offers. A buyer's profit increases when the seller-group size increases. These properties can be observed in figure 9 (b), where the buyer's expected profit for $n = 1, 5, \infty$ sellers is shown. In the optimal level 1 strategy the buyer has positive expected profit, that is increasing in the number of sellers. The optimal level 3 strategy is lower or equal to the optimal level 1 strategy, i.e. $s^3_{opt} \leq s^1_{opt}$. Therefore the buyer profits from the sellers' level 3 bidding strategy.

This fact can be observed in the figure. When there is $n = 1$ seller present, then the level 1 and 3 optimisation strategies are equivalent. In this case the buyer therefore is indifferent to the seller's bidding strategies. When more sellers bid in the auction, then level 1 and 3 bidding is different, that is $s^3_{opt} \leq s^1_{opt}$. Therefore the buyer's profit changes, when the sellers bid differently. In the figure this can be seen at the example of $n = 5$ sellers, where the buyer's profit increases when sellers change their bidding strategy from level 1 to level 3. When the number of bidding sellers increases, then this effect increases. In the limit of infinitely many sellers, where the optimal offer strategy is $s^3_{opt} = 1$, the buyer's profit is maximal. Then it is $\mathbf{E}(P_B) = \alpha$ (which is 10% in this example). Valuation imprecision is the upper bound for the buyer's expected profit. This shows that valuation imprecision is beneficial for the buyer. That is, when valuation imprecision increases, then the upper bound for the buyer's expected profit increases. In this case, the auction is more attractive for the buyer, when the parties valuation abilities are limited to an even greater extent.

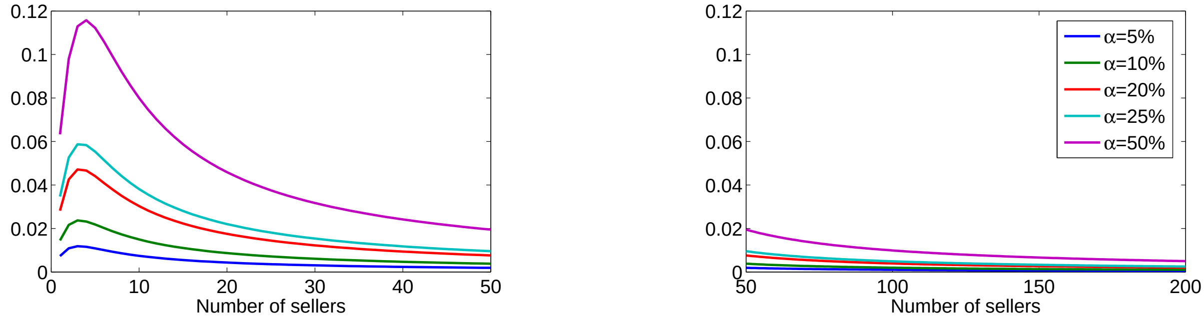
The next section summarizes the properties of the sellers' bidding strategies.

2.4 Summary: The Downside of Unilateralism

This section so far introduced the reverse auction, where the initiating buyer hides his reservation price. Different bidding strategies of the sellers were discussed. While the bidding strategies level 1 and 2 are no equilibria, we have proven that the optimal level 3 bidding strategy is an equilibrium bidding strategy. When the sellers bid rationally, they place their bids level 3 optimal. This optimal bidding strategy is unique. When the sellers place their bids according to this strategy, then opposed effects on the buyer and the sellers are achieved.

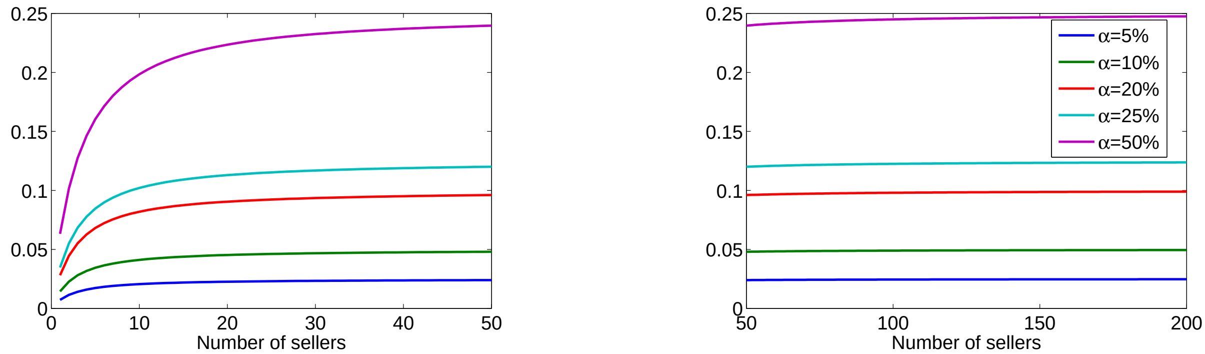
The buyer profits from the sellers' equilibrium bidding strategy in a variety of aspects. First of all, when the sellers place their bids according to their equilibrium bidding strategy, then the probability that the auction is successful increases, compared to optimal level 1 bidding strategy. Thereby the buyer's expected profit increases. Furthermore, he profits from an increasing number of bidding sellers and from increasing valuation imprecision. In fact, when there are infinitely many sellers, then buyer's expected profit is maximal and given by the players' valuation imprecision α. Therefore the buyer has an incentive to increase the valuing parties' valuation imprecision and to attract an increasing number of bidding sellers. In summary, the buyer profits in numerous aspects from the design of this auction market in equilibrium.

In contrast to the buyer, seller that bid in the auction have the highest expected profit when all sellers mutually agree to place their bids level 1 optimal. As a result, all sellers then share the same expected profit. A single seller may increase her expected profit by applying a unilateral level 2 bidding strategy. By placing her bid unilaterally, each seller



(a) $0 < \text{Groupsize} \leq 50$ (b) $50 \leq \text{Groupsize} \leq 200$

Fig. 10 – All sellers' expected profit in level 3 optimum as a function of group size



(a) $0 < \text{Groupsize} \leq 50$ (b) $50 \leq \text{Groupsize} \leq 200$

Fig. 11 – All sellers' expected profit in level 1 optimum as a function of group size

has the possibility to increase her profit by not committing to the optimal level 1 bidding strategy. Therefore that level 1 optimum is not stable. These considerations lead to the level 3 bidding strategy. When all sellers place their bids according to the optimal level 3 strategy, then no seller can increase her profit by breaking the commitment. The advantage of that strategy is its stability, as no seller profits from altering her bidding strategy. Moreover, the probability to place a winning bid increases when the sellers indirectly commit to place their bids according the level 3 optimum. However, each seller's expected profit is lower in equilibrium bidding strategy than if sellers would commit to the level 1 optimum strategy.

The seller-group profits from an increasing group size in level 1 optimum. In contrast, the expected group profit converges to zero in the level 3 optimum. These properties are illustrated in figure 10 and 11. Figure 10 shows in particular that there is an optimal number of bidding sellers to optimise the seller-group profit. This optimum is dependent on the valuation imprecision.

Each seller would profit when all sellers place their bids according to the optimal level 1 strategy. Therefore the seller-group could employ a mechanism that commits each seller to that bidding strategy. When no such mechanism is installed, then level 3 bidding is the unique equilibrium bidding strategy, which has numerous downsides for the sellers and many positive effects for the buyer who initiates the auction.

Let us introduce a numerical example.

2.5 Example: Private Bond Placement

Assume that five investors bid on a firm's bond with volume €1 billion and the maturity of one year. Each of the five investors offers a coupon that he requires from the issuing firm at maturity. The investor with the lowest coupon requirement wins the bid. Each investor has a reserve value for the coupon. In our example, these reserve values are uniformly distributed on the interval $[2\%, 6\%]$ of the bond volume. Thus the average coupon requirement is 4% and the minimum (maximum) requirement is 0.5 lower (higher) than the average. In fact, this represents a valuation imprecision of 50%. The investors only know their own coupon requirement and that their common valuation imprecision is 50%.

In the example, the investors' reserve values are

$$(V_1, V_2, V_3, V_4, V_5) = (2.5, 3, 4, 5, 5.5)mio.$$

The reserve value of the firm is 4.5% of the bond volume, that is €4.5mio.

When the investors commit to optimal multilateral offer strategy level 1, then they increase their reserve values by the factor $s^1_{opt} = 1.64$. Their offers then are

$$(O_1, O_2, O_3, O_4, O_5) = (4.10, 4.92, 6.56, 8.20, 9.02)mio.$$

The lowest bid is €4.1mio, which is below the firm's reserve value of €4.5mio. Thus the firm accepts that offer. The bidding investor's profit is €1.6mio and the firm's profit is €0.4mio.

Now, let us consider the influence of rationality level 2. The second investor knows that the four other investors commit to the offer strategy $s^1_{opt} = 1.64$. Then that investor's optimal strategy is to increase her valuation by $s^2_{opt} = 1.36$. The investors' offers thus are

$$(O_1, O_2, O_3, O_4, O_5) = (4.10, 4.08, 6.56, 8.20, 9.02)mio.$$

The lowest bid is €4.08mio, which is below the firm's reserve value of €4.5mio. Thus the firm accepts that offer. The second investor's profit is €1.08mio and the firm's profit is €0.42mio. Level 2 rationality thus puts the second investor in favour over the first investor. Further, the issuing firm profits from the second seller's strategy.

The optimal level 3 strategy avoids the possibility that one investor can create an advantage for himself over the other sellers. That equilibrium offer strategy is that all sellers increase their reserve value by $s^3_{opt} = 1.27$. The sellers' equilibrium bids thus are

$$(O_1, O_2, O_3, O_4, O_5) = (3.18, 3.81, 5.08, 6.35, 6.99) mio.$$

The lowest bid, offered by investor 1, is €3.18mio, which is below the firm's reserve value of €4.5mio. Thus the firm accepts that offer. The winning investor's profit is €0.68mio and we calculate that the firm's profit is €1.32mio. The equilibrium strategy and the level 1 strategy produce the same winner of the auction. That winner is unknown to the sellers ex ante. However, the winning investor's profit diminishes in equilibrium. In contrast, the issuing firm's profit increases when the sellers apply the equilibrium bidding strategy.

The next section introduces a dealer's offer strategy.

3 The Dealer's Market

The Dealer's Market in this section is similar to the Dealer's Market in Seemüller (2013). Although there might be some redundancy, its definition will be recapitulated at this point.

In the Dealer's Market, there is a dealer present. The dealer is experienced in trading and therefore knows the average value of a good. That is, he values the good precisely. He acts as market maker and charges a bid-ask spread. The dealer offers the seller a price which is the good's average value multiplied by $1 + f_s$ and he offers the buyer a price which is the average value multiplied by $1 + f_b$, with fees $f_b \geq f_s$. The bid-ask spread generates a positive profit on each round-trip transaction for the dealer, while he maintains $f_b - f_s \geq 0$. Hence, he deterministically profits from his strategy on a round-trip transaction.

Buyers and sellers do not know the average valuation of a good. Therefore a seller and a buyer are unaware whether the dealer truly shows them the prices $(1 + f_s)V$ and $(1 + f_b)V$, respectively. Thus the parties need to trust the intermediary to charge truthful prices. Consequently, the intermediary needs to be indulged with exogenous reputation capital such that the bargainers consider him trustworthy.

The dealer pursues the strategy to install an environment under information asymmetry. In the Dealer's Market buyers and sellers do not interact. They solely communicate with the dealer and choose whether to accept his offer, or not. In this sense there is information asymmetry in the Dealer's Market. Asymmetric information is important to the success of the dealer's strategy. This means that the buyer and the seller should either consult the dealer or choose an alternative trading platform (e.g. an auction). Otherwise buyers and sellers may first bargain on a free trading platform and only consult the dealer if their auction is unsuccessful. While this sequential strategy is beneficial for buyers and sellers, the dealer is left with a lemons problem. Buyers with low reservation prices and sellers with high reservation prices. Thus the dealer suffers from adverse selection. Installing a beneficial fee strategy consequently becomes more complicated under full information as the dealer's strategy may even collapse otherwise.

We start to formally analyse the properties of the Dealer's Market by calculating the deal probability.

Proposition 17. *Assume that $-\alpha < f_b, f_s < \alpha$. Then the probability that a buyer profits from and thus accepts the dealer's offer is $p_b = (\alpha - f_b)/(2\alpha)$. The probability that a seller profits and thus accepts the dealer's offer is $p_s = (\alpha + f_s)/(2\alpha)$.*

Proof: See the Appendix. □

A buyer (seller) accepts the dealer's offer if his reservation price is higher (lower) than the dealer's offer. Therefore the probability for a successful deal increases when the dealer reduces his offer to a buyer. Further, that probability increases, when the dealer raises his offer to a seller.

The next proposition calculates a buyer's and a seller's expected profit in the Dealer's Market.

Proposition 18. *When $f_b < \alpha$ or $-\alpha < f_s$ then a buyer's or a seller's expected profit in the Dealer's Market is positive. In this case a buyer's expected profit is*

$$\boldsymbol{E}(P_D(B)) = (\alpha - f_b)^2 /(2\Delta s)$$

and a seller's expected profit is

$$\boldsymbol{E}(P_D(S)) = (f_s + \alpha)^2/(2\Delta s).$$

Proof: See the Appendix. □

The above proposition shows that the dealer can attract buyers and sellers to his market when he maintains reasonable fees $f_b < \alpha$ and $-\alpha < f_s$. If, in addition, the price he charges a buyer exceeds his offer to a seller (that is, $f_b - f_s > 0$), then the dealer's profit on each round-trip transaction is positive.

The next section analyses the relative attractiveness of the reverse auction and the Dealer's Market.

4 The Advantage of Bid-Ask Prices

This section compares the reverse auction and the Dealer's Market. It will be shown under which circumstances the Dealer's Market is preferable for all market participants over their reverse auction alternative. Afterwards an example is presented to numerically illustrate the theoretic discussion.

4.1 The Pareto Dominance of a Market Maker under Information Asymmetry

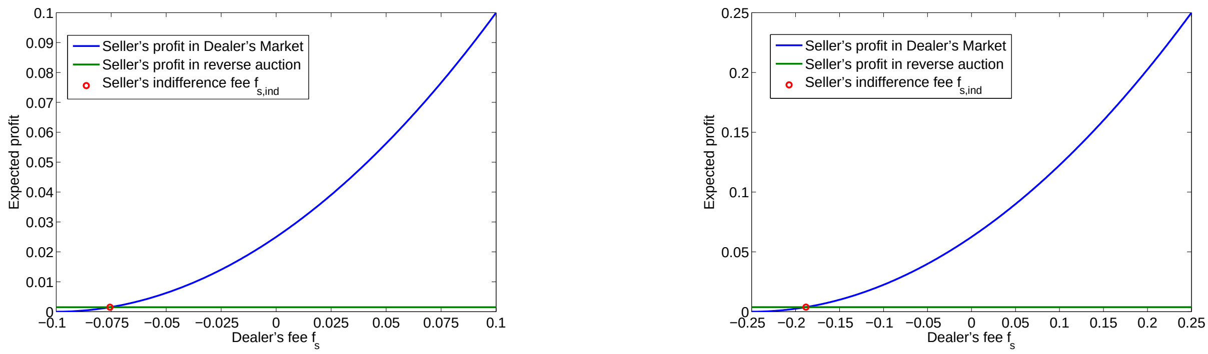
We introduce indifference fees $f_{b,ind}$ and $f_{s,ind}$ with the following properties: If the dealer fee is below $f_{b,ind}$, then buyer prefers the Dealer's Market over his reverse auction alternative. Analogously, sellers prefer the Dealer's Market over their alternative to bid in the reverse auction, if the dealer's fee exceeds the indifference fee $f_{s,ind}$. The theorem proves a major property of these indifference fees.

Theorem 4. *The buyer and sellers prefer the Dealer's Market over the reverse auction, when the condition $f_{s,ind} < f_s < f_b < f_{b,ind}$ holds. Then the dealer's earnings are greater zero. As a result, the Dealer's Market Pareto dominates the reverse auction, i.e. the market under information asymmetry Pareto dominates the reverse auction.*

Proof: See the Appendix. □

When the condition $f_{s,ind} < f_s < f_b < f_{b,ind}$ is satisfied, then all players are in preference of the Dealer's Market and the dealer has positive earnings on each round-trip transaction. Then the Dealer's Market is Pareto efficient over the reverse auction alternative.

The seller's indifference fee as a function of valuation imprecision α is shown in figure 12. For the analysis the number of sellers is fixed to $n = 10$ and the valuation imprecision α is set to 10% in figure 12 (a) and 25% in figure 12 (b). The x-axis shows the dealer's fee. A seller's expected profit in the Dealer's Market and the reverse auction is illustrated on the y-axis. In both cases a seller's profit is comparably low when she bids in the reverse auction. In contrast, her profit is increasing in the dealer's fee. When the dealer's fee exceeds the indifference fee, then the seller prefers the Dealer's Market over bidding in the auction. In both figures, the indifference fee is almost at the lower bound for the dealer's fee: for $\alpha = 10\%$ ($\alpha = 25\%$), the indifference fee is $f_{s,ind} = -7.56\%$ ($f_{s,ind} = -18.82\%$). When the scaling in the figures 12 (a) and (b) is not regarded, then the indifference fees and the expected profits seem to be almost the same in both figures. The analysis of the indifference fee as a proportion of valuation imprecision in fact shows that $f_{rel} = -75.6\%$ (for $\alpha = 10\%$) and $f_{rel} = -75.3\%$ (for $\alpha = 25\%$). It can be assumed that the seller's indifference fee is almost linear in valuation imprecision.



(a) Valuation imprecision $\alpha = 10\%$

(b) Valuation imprecision $\alpha = 25\%$

Fig. 12 – A seller's expected profit in the reverse auction and in the Dealer's Market for $n = 10$ sellers

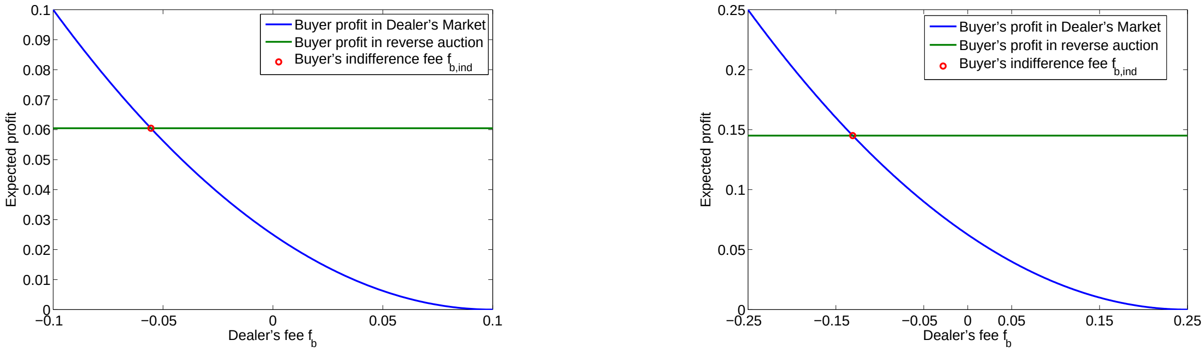
The buyer's preferences as a function of the valuation imprecision $\alpha$is shown in figure 13, where $n = 10$ sellers is fixed and valuation imprecision is set to $\alpha = 10\%$ and $\alpha = 25\%$ in figures 13 (a) and (b), respectively. As in figure 12, the x-axis shows the dealer's fee and a buyer's expected profit is drawn on the y-axis.

The buyer's expected profit decreases with increasing dealer's fee. Intuitively, a higher fee implies a higher price. That higher price influences the buyer's profit negatively. When valuation imprecision is set at $\alpha = 10\%$, then the buyer's indifference fee is $f_{b,ind} = -5.55\%$. That is, all fees below this indifference fee puts the buyer in favour for the Dealer's Market over setting up a reverse auction. In the case $\alpha = 25\%$, the indifference fee is $f_{b,ind} = -13.08\%$. As a result, the dealer needs to offer a buyer a lower price when the valuation imprecision increases.

As in the above analysis of a seller's indifference fee (when the scaling of the figure is disregarded) it can be seen that this fee is almost a linear function in valuation imprecision. This becomes even clearer when we calculate the indifference fees as a proportion of valuation imprecision: For the valuation imprecision $\alpha = 10\%$, the relative indifference fee is 55.5%. In the case of $\alpha = 25\%$ that relative figure is 52.3%. The buyer's indifference fee therefore is almost a linear function.

The buyer's and the seller's indifference fee as a function of seller-group size is analysed in figure 14. The figure compares the expected profit in the Dealer's Market and in a reverse auction. In that figure valuation imprecision is held constant at $\alpha = 10\%$. The dealer's fee is shown on the x-axis. The y-axis illustrates the expected profit.

Figure 14 (a) shows a seller's indifference fee as a function of the number n of the sellers. Previous analysis proved that a seller's expected profit in a reverse auction decreases in the number of bidding sellers. This fact can be observed in the figure. An increasing seller



(a) Valuation imprecision $\alpha = 10\%$

(b) Valuation imprecision $\alpha = 25\%$

Fig. 13 – The buyer's expected profit in the reverse auction and in the Dealer's Market for $n = 10$ sellers

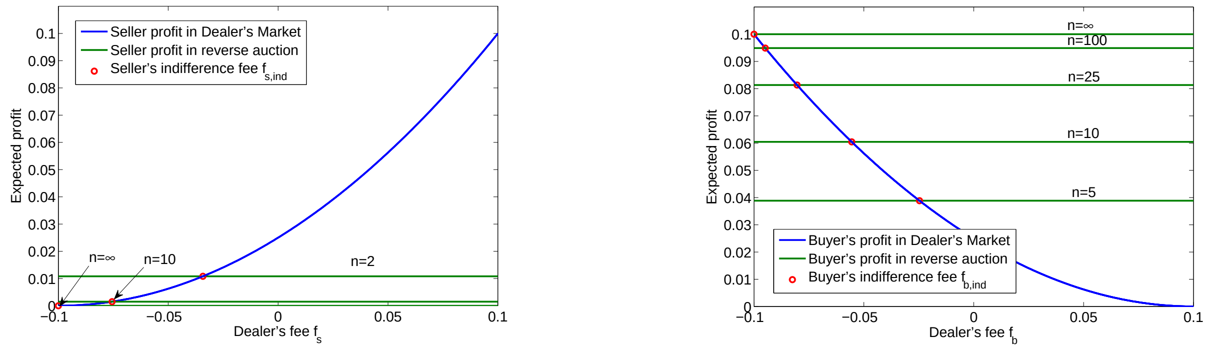
number therefore affects a seller's indifference fee negatively. It decreases from -3.42% (for $n = 2$ sellers) down to -7.56% (for $n = 10$ sellers) down to -10% in the limit of infinitely many sellers. A sellers always accepts a fee above her indifference fee. As a result, the dealer can generate higher earnings, when more sellers are present.

Figure 14 (b) shows a buyer's indifference fee as a function of the number $n$of the sellers. Previous analysis proved that a buyer's expected profit in a reverse auction is an increasing function in the number of bidding sellers. This fact can be observed in the figure. As a result, an increasing number of sellers affects a buyer's indifference fee negatively. That fee decreases from -2.47% (for $n = 5$ sellers) down to -5.55% (for $n = 10$ sellers) down to -8.04% (for $n = 25$ sellers) down to 9.49% (for $n = 100$ sellers) down to -10% in the limit of infinitely many sellers. A buyer always accepts fees below her indifference fee. As a result, the dealer may charge the buyer a smaller fee and thus a lower price, when more sellers are present. This affects the dealer's earnings negatively.

When infinitely many sellers are present, then a buyer's and a seller's indifference fee is -10%. Then there is no pair of fees (f_b, f_s), such that the dealer can attract both, the buyer and the seller to the Dealer's Market while maintaining positive earnings. The dealer then can not supply an environment that is Pareto efficient over the reverse auction.

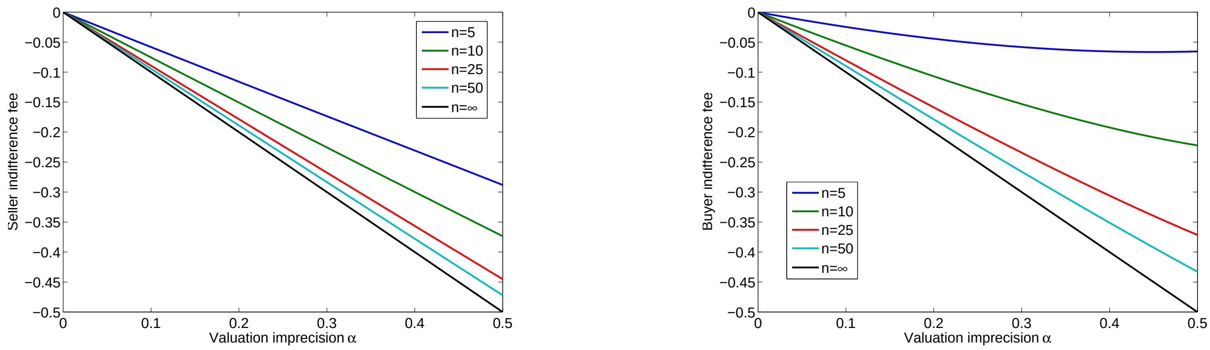
Next, we analyse when the dealer can create an environment that Pareto dominates a reverse auction, while maintaining positive dealer's earnings. The constraints that influence the players' indifference fees are the number of sellers n and the valuation imprecision α.

Figures 15 (a) and (b) show a seller's and a buyer's indifference fees, respectively. These indifference fees are drawn as functions of valuation imprecision. The figures show that the indifference fee is decreasing in valuation imprecision for both the buyer and the seller. It is furthermore decreasing in the number of sellers. However, the figures suggest that the



(a) Seller's profit (b) Buyer's profit

Fig. 14 – The players' expected profit for different numbers of seller and valuation imprecision $\alpha = 10\%$



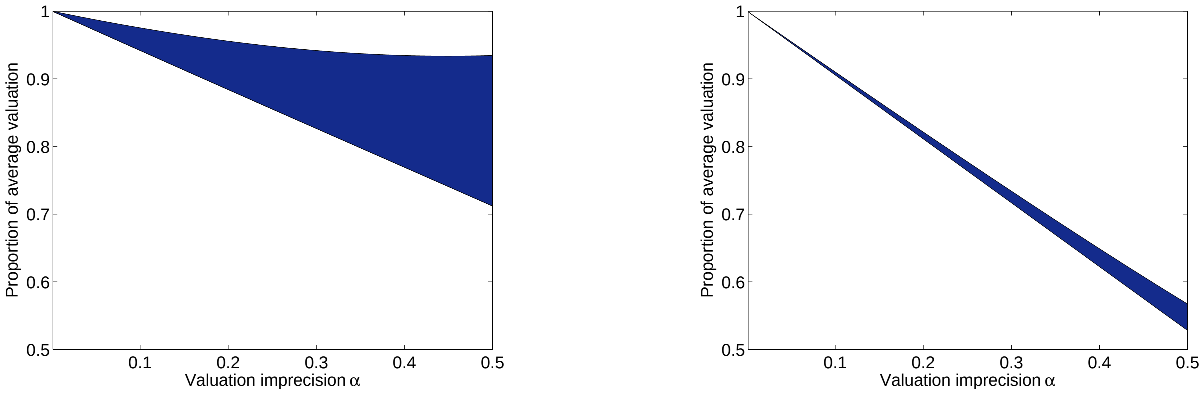
(a) Indifference fee seller $f_{s,ind}$ (b) Indifference fee buyer $f_{b,ind}$

Fig. 15 – The players' indifference fees as functions of valuation imprecision

indifference fee of the seller usually is lower than the buyer's indifference fee. Within these bounds the dealer can determine his fees such that all parties are in favour of the Dealer's Market.

When $\alpha = 0$ or $n = \infty$, then the dealer can not offer a fee structure that Pareto dominates the reverse auction. For $\alpha = 0$ there is no valuation imprecision and thus all parties value the good precisely. That is, the buyer's and the seller's reservation prices are identical. This does not allow positive profit from bargaining. As a result, valuation imprecision is necessary for the Dealer's Market to be Pareto efficient. When $n = \infty$, then a buyer's and a seller's indifference fee is $-\alpha$. In other words, their indifference fees are identical. The dealer therefore can maintain no positive bid-ask spread. He therefore can not install a fee structure that attracts a buyer and a seller to his market and at the same time allows him positive earnings.

The influence of the dealer's fees on the average price is illustrated in figures 16 (a) and



(a) Feasible bid-ask prices for $n = 5$ sellers

(b) Feasible bid-ask prices for $n = 50$ sellers

Fig. 16 – Feasible bid-ask prices for a Pareto efficient Dealer's Market

	$n = 5$		$n = 10$		$n = 50$		$n = \infty$
	$f_{b,ind}$	$f_{s,ind}$	$f_{b,ind}$	$f_{s,ind}$	$f_{b,ind}$	$f_{s,ind}$	$f_{b,ind} = f_{s,ind}$
$\alpha = 5\%$	-1.29%	-2.92%	-2.82%	-3.78%	-4.51%	-4.73%	-5%
$\alpha = 10\%$	-2.47%	-5.82%	-5.55%	-7.56%	-8.99%	-9.45%	-10%
$\alpha = 20\%$	-4.43%	-11.61%	-10.70%	-15.08%	-17.87%	-18.90%	-20%
$\alpha = 25\%$	-5.21%	-14.48%	-13.08%	-18.82%	-22.26%	-23.62%	-25%
$\alpha = 50\%$	-6.55%	-28.81%	-22.22%	-37.35%	-43.28%	-47.21%	-50%

Table 3 – Buyer's and seller's market indifference

(b) for markets with $n = 5$ and $n = 50$ sellers, respectively. The blue area in both figures represents valid bid-ask prices such that all parties are in preference of the Dealer's Market. That area is non-empty, so the dealer can determine a fee structure for the Pareto dominance of the Dealer's Market. It can be seen that the area for $n = 50$ sellers is smaller than that for markets with $n = 5$ sellers. In fact, this area converges to zero for $n \to \infty$, as discussed in the previous paragraph. This further can be observed in table 3. That table shows that a seller's and a buyer's indifference fees are the same, when infinitely many sellers are present. Thus, the dealer can not price the good in a way such that the Dealer's Market dominates a double auction. Table 3 illustrates further exact values of the dealer's price bounds. As can be seen from figure 16 and the table, for valuation imprecision that exceeds zero, the dealer can buy the good from the seller and sell it to the buyer for a higher price. The dealer's earnings are determined by a positive bid-ask spread. The next Theorem summarises these observations.

Theorem 5. *Assume that the valuation imprecision* $0 < \alpha < 1$ *and that the number of sellers* *is $0 < n < \infty$. Then the dealer can always determine fees f_b and f_s, such that (a) the dealer has positive earnings per round-trip transaction and (b) buyer and seller prefer the Dealer's Market over a reverse auction. Then the Dealer's Market Pareto dominates a reverse auction.*

Proof: See the Appendix. □

The above theorem proves that the dealer can determine bid-ask prices such that all players are in preference for his market. Then a buyer and a seller prefer the Dealer's Market over trading in a reverse auction. With this strategy, the dealer is able to maintain positive earnings per round-trip transaction.

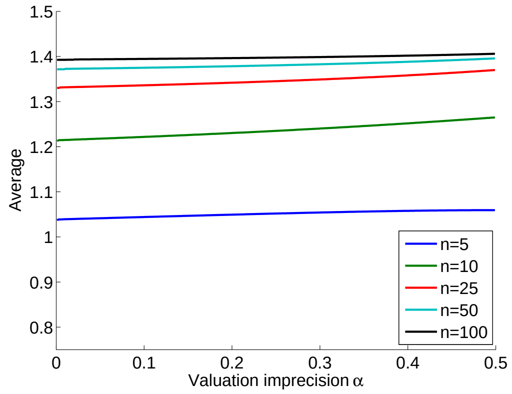
Consider now the reverse auction, where one initiating buyer and a certain number n of bidding sellers are present. The dealer competes with this auction, where there is just one buyer but a possibly high number of sellers. When the dealer acts as a market maker, then he buys from every prospective seller and sells to every prospective buyer. When prospective sellers outnumber prospective buyers, as in the reverse auction, then the dealer puts himself at risk of building a large inventory. This is however not the case, as we will analyse in the next proposition.

Proposition 19. *On average $n\frac{\alpha+f_s}{2\alpha}$ sellers accept the dealer's offer.*

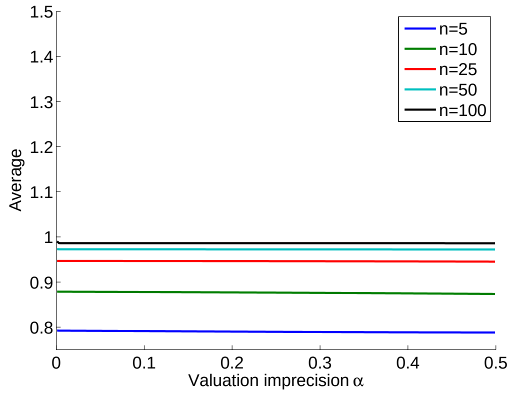
Proof: See the Appendix. □

Proposition 19 calculates the expected number of sellers that accept the dealer's offer. That number is an important decision parameter for the dealer. There are n prospective sellers versus 1 prospective buyer. That is, the dealer faces the danger of high inventory levels if he attracts too many sellers. Considering the inventory level, the dealer should choose his seller fee f_s such that sellers prefer the Dealer's Market over bidding in the auction. At the same time the number of sellers that accept the dealer's offer should be limited. Furthermore the buyer fee f_b should attract the buyer to the Dealer's Market and maximise the probability that he accepts the dealer's offer. While applying this strategy, the dealer's inventory is held to a minimum.

We give a numerical illustration of the above proposition. Figures 17 and 18 analyse the proposed dealer's fee strategy in detail. In these figures, the dealer applies the maximum seller fee such that a seller is marginally in favour of the Dealer's Market over bidding in the auction. The buyer is charged the minimum fee, such that the dealer's profit remains positive. Then the number of sellers that accept the dealer's offer is minimised and the probability of a buyer to accept the dealer's offer is maximised. At the same time the dealer's earnings remain positive and all players are in favour of the Dealer's Market. This strategy has two positive effects: the Dealer's Market is Pareto dominating the double auction and the dealer's inventory is minimised.



(a) The average number of sellers that accept the dealer's offer



(b) The average number of buyers that accept the dealer's offer

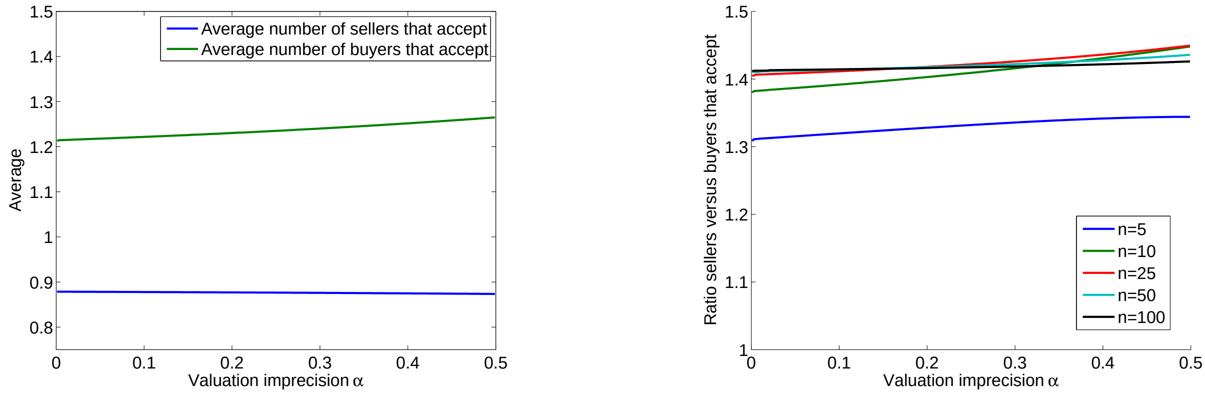
Fig. 17 – Average number of players that accept the dealer's offer

Figure 17 shows the expected number of sellers and buyers that accept the dealer's offer. Valuation imprecision is drawn on the x-axis, whereas the average number of accepting buyers or sellers is shown on the y-axis.

Figure 17 (a) shows the expected number of sellers that accept the dealer's offer for $n = 5, 10, 25, 50$ and 100 sellers. That expected number is almost constant in valuation imprecision. However, a higher valuation imprecision affects that number slightly positive. The main determinant is the number of sellers in the market as more potential sellers sell more often to the dealer than fewer sellers. The expected number of sellers that accept the dealer's offer ranges from approximately 1.05 ($n = 5$ sellers) to approximately 1.40 ($n = 100$ sellers). Although that value increase with the number of sellers, it is bounded in a range approximately between 1 and 1.5.

The expected number of buyers that accept the dealer's offer (which is equivalent to the probability that the buyer accepts the dealer's offer) is shown in figure 17 (b). For a fixed number of sellers, that number is relatively constant in valuation imprecision. The influence of valuation imprecision is limited to a minimum, such that it almost cannot be observed in the figure. The expected number of buyer's that accept the dealer's offer is increasing in the number of sellers n. Where the number is approximately 0.8 when there are $n = 5$ sellers, it increases up to almost 1 when $n = 100$ sellers are present. That figure thus ranges from approximately 0.8 up to 1.

Figure 18 (a) shows the expected number of sellers and buyers that accept the dealer's offer when $n = 10$ sellers are present. It can be seen that both numbers are almost constant in valuation imprecision α. This proposition is supported by figure 18 (b), where the quotient of sellers and buyers that accept the dealer's offer is illustrated. When $n = 5$, then the ratio is relatively stable at approximately 1.3 sellers per buyer. For $n = 10, 25, 50$ and 100 sellers,



(a) Expected number of sellers versus buyers for $n = 10$ sellers

(b) Ratio of sellers versus buyers

Fig. 18 – Properties of buyer and seller acceptance

the ratio is almost independent of imprecision and further stable as a function of n. The ratio is approximately 1.4 sellers per buyer.

In summary, while the dealer applies an optimal fee structure, he can attract buyers and sellers to his market. Furthermore, his offer strategy is Pareto efficient over the reverse auction market. Applying optimal fee structure, the dealer's profit is positive. At last we find that his inventory level is bounded at a low level because the ratio of sellers per buyer that accept his offer is relatively stable at approximately 1.4 sellers per buyer.

From a buyer's and a sellers' perspective the Dealer's Market is the dominant alternative over the reverse auction, when the dealer is trustworthy and applies a moderate fee scheme.

To illustrate the above findings, the private bond placement example from section 2.5 is further extended.

4.2 Example: Bond Placement by an Investment Banker

We continue the example from section 2.5 on page 90 and apply it to our theory. The investors' reserves prices are

$$(V_1, V_2, V_3, V_4, V_5) = (2.5, 3, 4, 5, 5.5)mio.$$

As in section 2.5, the reserve value of the firm is 4.5% of the bond volume, that is €4.5mio.

Investors and the firm may now decide between negotiations within a double auction or to consult an intermediary. Investors and the firm are aware of their common valuation imprecision of 50%, as in section 2.5. Ex ante, each investor's expected profit from the double auction is 2.24% according to proposition 2 on page 68. That profit was calculated in table

2 on page 83. According to proposition 5 on page 70, we calculate an expected ex ante profit of 15.99% for the firm when it initiates a double auction.

In order to compete with the double auction, the intermediary must choose his fees adequately. These fees are calculated in table 3 on page 97. Accordingly, the intermediary needs to charge fees $-28.81\% < f_s < f_b < -6.55\%$ to investors and the firm. While he maintains fees within these bounds, the firm and investors are ex ante in preference of the dealer.

In our example, the intermediary charges fees $f_s = -26\%$ and $f_b = -16\%$. This means that he offers investors a coupon of volume €$4mio(1 + f_s) =$€$2.96mio$. The firm is obliged to pay €$4mio(1 + f_b) =$€$3.36mio$ at maturity. The first investor accepts the intermediary's offer, as it exceeds that investor's reservation price. The firm accepts his offer as well, because it is below the firm's reservation price.

In summary, the first investor gains a coupon of volume €$2.96mio$, which represents a gain of €$0.46mio$ compared to her reserve value. The firm agrees to pay €$3.36mio$ at maturity, which is €$1.14mio$ less than its maximum coupon volume. Finally, the intermediary generates a gain of €$0.40mio$. Each party gains from the intermediaries strategy.

Compared to the example in section 2.5, the winning investor's profit is reduced from €$0.68mio$ in the double auction to €$0.46mio$ from the intermediary's strategy. However, the investor is not aware of this fact. Ex ante, it is each investor's best option to consult the intermediary.

5 Conclusion

We have added imprecise valuation to a model of reverse auctions. In an auction, bidding sellers and the initiating buyer are aware of their own valuation imprecision that results from their inability of precise valuation. In case of a procurement auction, different costs for the procured good or service lead to different reserve values. Due to valuation imprecision, each bidding seller can not determine whether her valuation is high or low and how it compares to the other sellers' reservation prices. To achieve a positive profit, each seller's bid exceeds that seller's reserve value.

We proved when all sellers apply a multilateral optimal bidding strategy and they commit to it, then the profit of the seller-group is maximised. At the same time, that strategy maximises the profit of each seller. An increase in the seller-group's size also increases the group's profit. The optimal multilateral bidding strategy was proven to be unique.

A single seller profits from breaking the commitment of the multilateral offer strategy. That is, when a seller knows that the other sellers play the optimal multilateral strategy, then it is individually optimal to break that commitment and to apply an optimal unilateral response strategy. While this strategy increases a seller's individual profit, it harms the other sellers.

There is a unique equilibrium bidding strategy. Its characterisation is that no seller can profit from applying a strategy that differs from that equilibrium. Therefore it is a stable strategy. In equilibrium all sellers pursue the same offer strategy. However, it is not the most efficient strategy. In equilibrium, each seller's profit is lower than that under the multilateral optimal bidding strategy. Furthermore, a greater number of sellers has negative effect on each seller's profit and on that of the seller-group. Both converge to zero for infinitely many sellers. While the equilibrium bidding strategy is stable, its consequences for the profit of the bidding sellers is negative. The sellers can increase their profit by installing a mechanism that ensures that all sellers commit to the multilateral optimal bidding strategy.

The sellers' equilibrium bidding strategy is of benefit for the buyer who initiates the auction. In equilibrium, the average bid of a seller is lower. The average bid furthermore decreases when more sellers bid in the auction. Consequently, the buyer's profit is positively affected by the sellers' bidding equilibrium and the number of bidding sellers. When the seller-group commits to their optimal bidding strategy, then the profit of the buyer is lowest.

The model has testable implications. Consider the Industrial Revolution in the 18th and early 19th century. Then, workers were not organised and the competition for jobs in the labour force was high. As a result, each worker had to sell his time and labour for a lower salary that his competitors. This process necessarily lead to extremely low wages. The rise of labour unions and upcoming political support helped the labour force to ensure a better coordination of their negotiations with an employer. Our model explains that an employer was able to exploit the unorganised labour force. Further, our model implies that coordinated negotiations with an employer are beneficial for workers. In addition, the paper's model may be used to calculate a union's optimal salary negotiation strategy.

A further application is the situation of crofters or producers of milk. Our model explains that if they do not coordinate their price strategies, their profit is lower than the profit they could achieve with coordinated pricing. The optimal price strategy can be calculated with the formulas that the paper provides.

A market maker quotes bid and ask prices for the good or service. When the bid-ask spread is positive, then his strategy is profitable for him. We have shown that the market maker can adjust his quotes such that the buyer and the sellers prefer the market maker's bid-ask prices over the reverse auction. Then the market maker is Pareto efficient over the reverse auction.

When there are $n > 1$ sellers and one buyer, then the dealer's inventory needs to be considered. This paper analysed the ratio of sellers per buyer that accept the dealer's offer. When the dealer optimises that ratio, then the inventory level is sufficiently bounded.

We presented an example to illustrate our theory. A firm that wants to issue a bond has the option to initiate a reverse auction, where interested investors bid the coupon payment that they require the firm to pay. We showed that investors earn less when they do not commit to their optimal bidding strategy. The firm that issues the bond gains more when investors do not commit to a common bidding strategy. It was shown that an investment banker that offers intermediation services can successfully places the bond. His services are beneficial for all parties ex ante.

In our model, $n \geq 1$ sellers bid in an auction that a buyer initiated. When the lowest sellers' offer is below the buyer's reservation price, then the auction is successful and the good is traded at that price. Thus, ex post at the most one seller benefits from that market design. More generally, $n \geq 1$ sellers offer their goods to $m \geq 1$ buyers. Consider for instance the Amazon trading platform, where sellers disclose their offer prices for a good. Buyers sequentially arrive and buy that good at its lowest price, when a buyer's reserve value exceeds that lowest price. This is just one example of an extension of our model.

6 Appendix

Proof of Lemma 1. The cdf of $min(X_1, X_2, ..., X_n)$ is given by

$$\begin{aligned} M(x) &= \mathbf{P}\left(min(X_1, X_2, ..., X_n) \leq x\right) = 1 - \mathbf{P}\left(min(X_1, X_2, ..., X_n) > x\right) \\ &= 1 - \mathbf{P}\left(X_i > x, i = 1, ..., n\right) = 1 - \mathbf{P}\left(X_1 > x\right)^n = 1 - \left(\frac{x_2 - x}{x_2 - x_1}\right)^n . \end{aligned}$$

The pdf is given by the first derivative of $M(x)$. □

Proof of Proposition 1. The proof uses lemma 1, where the pdf of the minimum of the seller's offers $min\{S_1, S_2, ..., S_n\}$ was established. The buyer's deal success probability is

$$\begin{aligned}
\mathbf{P}(D_B)(s) &= \int_{s_1}^{s_2} \frac{1}{\Delta s}\mathbf{P}(min\{S_1, S_2, ..., S_n\} \leq x)\ dx \\
&= \frac{1}{\Delta s}\int_{ss_1}^{s_2} 1 - \left(\frac{ss_2 - x}{s\Delta s}\right)^n\ dx \\
&= \frac{1}{\Delta s}\left((s_2 - ss_1) - \frac{1}{s^n\Delta s^n}\left[-\frac{1}{n+1}(ss_2 - x)^{n+1}\right]_{ss_1}^{s_2}\right) \\
&= \frac{1}{\Delta s}\left((s_2 - ss_1) + \frac{1}{s^n\Delta s^n(n+1)}\left((ss_2 - s_2)^{n+1} - (ss_2 - ss_1)^{n+1}\right)\right) \\
&= \frac{1}{\Delta s}\left((s_2 - ss_1) + \frac{1}{s^n\Delta s^n(n+1)}\left((s_2(s-1))^{n+1} - (s\Delta s)^{n+1}\right)\right) \\
&= \frac{s_2 - ss_1}{\Delta s} + \frac{(s_2(s-1))^{n+1}}{s^n\Delta s^{n+1}(n+1)} - \frac{(s\Delta s)^{n+1}}{s^n\Delta s^{n+1}(n+1)} \\
&= \frac{s_2 - ss_1}{\Delta s} + \frac{s}{n+1}\left(\left(\frac{s_2(s-1)}{s\Delta s}\right)^{n+1} - 1\right)
\end{aligned}$$

The term $\left(\frac{s_2(s-1)}{s\Delta s}\right)$ is zero for $s = 1$ and one for $s = s_2/s_1$. Further,

$$0 \leq \left(\frac{s_2(s-1)}{s\Delta s}\right) \leq 1,$$

while the sellers' offer strategy is feasible, i.e. within the interval $s \in [1, s_2/s_1]$. Thus the buyer's deal probability converges either to zero ($s \in [1, s_2/s_1)$, or is constant for $s = s_2/s_1$. The factor $s/(n+1)$ converges to zero for $n \to \infty$. As a result the buyer's deal probability converges to $\frac{s_2 - ss_1}{\Delta s}$ for $n \to \infty$. In proposition 8 a more general proof for a seller's profit will be shown. In order to avoid redundancy, we refer to that proof here. By allowing $s' = s$, a single seller's deal probability under multilateral strategy is obtained. Considering $n \to \infty$ shows that the seller's deal probability converges to zero. □

Proof of Proposition 2. Seemüller (2013) established the expected profit of a seller in a bilateral monopoly. In a double auction the buyer's profit is not affected by his offer. He thus has no motivation to adjust his offer on the platform. The buyer's behaviour thus is analogous to the case of bilateral monopoly with a rational seller in Seemüller (2013). Further, the deal price in the double auction is given by the lowest offer of the sellers. A single seller's expected profit is thus given by

$$
\begin{aligned}
\mathbf{E}\left(P_S\right)(s) &= \frac{1}{s\Delta s}\int_{ss_1}^{ss_2} \mathbf{1}_{Deal}\ \left(x - x/s\right)\right)\, dx \\
&= \frac{s-1}{s^2\Delta s}\int_{ss_1}^{ss_2} \mathbf{P}\left(x \leq min\left(S_2, S_3, ..., S_n\right),\ x \leq B\right)\ x\ dx \\
&= \frac{s-1}{s^2\Delta s}\int_{ss_1}^{s_2} \left(\frac{ss_2 - x}{s\Delta s}\right)^{n-1} \frac{b_2 - x}{\Delta b}\ x\ dx =: \frac{(s-1)\left(A(s_2) - A(ss_1)\right)}{s^{n+1}\ \Delta s^{n+1}},
\end{aligned}
$$

where A is the primative of $(ss_2 - x)^{n-1}\,(b_2 - x)\ x$. The probability that a certain seller's offer is the lowest is obtained from lemma 1. Also note that the upper bound of the integral is s_2. That is, when a seller's offer exceeds s_2, then the buyer's reservation price will be lower than that offer with probability 1.

All sellers have the same offer strategy. It follows that the distribution of the sellers' offers is identical. Thus ex ante the probability for a seller to make the lowest offer is $1/n$. The sellers' expectations on placing the lowest bid are homogeneous ex ante. Therefore the expected profit of all sellers is the sum of the expected profit of a single seller. □

Proof of Proposition 3. When the number of bidding sellers increases, then the lowest reservation price in the group converges to the lowest valuation possible, which is s_1. Assume the lowest reservation price is above s_1. Then, at some point, a seller with reservation price below the currently lowest reservation price will join the seller-group. This argument is valid for all valuations that exceed s_1. Thus the sellers' lowest reservation price converges to s_1 for the number of sellers $n \to \infty$. The buyer buys from the seller with the lowest offer. All sellers pursue multilateral offer strategy s. Therefore the seller with the lowest reservation price places the lowest offer ss_1. All other sellers do not sell their good. The seller with lowest reservation price makes the profit $ss_1 - s_1$, that is the difference between her offer price and her reservation price. The probability that the buyer's valuation exceeds the seller's offer is $\frac{s_2 - ss_1}{\Delta s}$. As the buyer's and the sellers' valuations are independent, the expected profit of the seller-group is $\mathbf{E}\left(S\right)(s) = \frac{1}{\Delta s}(s_2 - ss_1)(ss_1 - s_1) = \frac{s_1}{\Delta s}(s_2 - ss_1)(s-1)$. The optimal offer strategy s is found by solving the first order condition.

$$
\frac{\partial \mathbf{E}\left(S\right)(s)}{\partial s} = \frac{s_1}{\Delta s}(s_2 - ss_1 - s_1(s-1)) = 0
$$

if and only if $s = 1/s_1 = 1/(1-\alpha)$, which is a maximum of the sellers' profit. □

Proof of Theorem 1. We rearrange the formula for a seller's expected profit.

$$\begin{aligned}
\mathbf{E}\left(P_S\right)(s) &= \frac{1}{\Delta s}\int_{s_1}^{s_2} \mathbf{1}_{Deal}\ (xs - x)\ dx \\
&= \frac{s-1}{\Delta s}\int_{s_1}^{s_2/s} \mathbf{P}\left(sx < min\{sS_2, ..., sS_n\}, xs < B\right)\ x\ dx \\
&= \frac{s-1}{\Delta s}\int_{s_1}^{s_2/s} \mathbf{P}\left(x < min\{S_2, ..., S_n\}\right) \frac{s_2 - sx}{\Delta s}\ x\ dx \\
&= \frac{s-1}{\Delta s^2}\int_{s_1}^{s_2/s} \mathbf{P}\left(x < min\{S_2, ..., S_n\}\right)(s_2 - sx)\ x\ dx.
\end{aligned}$$

The probability $\mathbf{P}\left(x < min\{S_2, ..., S_n\}\right)$ in the term above is independent of s. To simplify the notation, that probability will be referred to as $\mathbf{P}(x)$ within this proof. That is, we have a formula for a seller's expected profit. That is,

$$\mathbf{E}\left(P_S\right)(s) = \frac{s-1}{\Delta s^2}\int_{s_1}^{s_2/s} \mathbf{P}(x)(s_2 - sx)\ x\ dx$$

Feasible offer strategies are defined on the interval $s \in [1, s_2/s_1]$. A seller's expected profit is zero for $s \in \{1, s_2/s_1\}$ and positive otherwise. Furthermore the term for the seller's expected profit is continuous. According to Rolle there is at least one profit maximum on the interval $s \in (1, s_2/s_1)$. To prove that this maximum is unique, we take the first derivative of the seller's profit and show that its root is unique. By the Leibnitz formula, the first derivative of a seller's expected profit is

$$\begin{aligned}
\frac{\partial \mathbf{E}\left(P_S\right)(s)}{\partial s} &= \frac{1}{\Delta s^2}\left(\int_{s_1}^{s_2/s} \mathbf{P}(x)(s_2 - sx)\ x\ dx + (s-1)(-\int_{s_1}^{s_2/s} \mathbf{P}(x)x^2\ dx)\right) \\
&= \frac{1}{\Delta s^2}\left(\int_{s_1}^{s_2/s} \mathbf{P}(x)x\left((1-2s)x + s_2\right)\ dx\right).
\end{aligned}$$

The factor in the integrand that is dependent on the sellers' offer strategy is $(1-2s)x + s_2$. It is a polynomial of first order. The second factor is $\mathbf{P}(x)x$. That term is positive and independent of the offer strategy s.

A necessary condition of the optimum is that $\frac{\partial \mathbf{E}(P_S)(s)}{\partial s}$ is zero. As the factor $\mathbf{P}(x)x$ is positive, it is necessary that the factor $(1-2s)x+s_2$ is negative within the integration bounds $[s_1, s_2/s]$. The root of that factor is $x_0 = s_2/(2s-1)$. It is the upper integration bound, when $s = 1$. For $s = 1/s_1$, its root is given by the lower integration bound. Therefore $s = 1/s_1$ is an upper bound for the seller's optimal offer strategy.

The first derivative of the sellers' profit is positive in the integration bounds $[s_1, x_0]$ and negative on the integration interval $[x_0, s_2/s]$. The first factor of the integrand is independent of s. The second factor is decreasing in the sellers' offer strategy s. Furthermore, the root x_0 is decreasing in the sellers' offer strategy. As a result, the integral decreases in s on the integration interval $[s_1, x_0]$. On the interval $[x_0, s_2/s]$ the integral is zero for $s \in \{1, s_2/s_1\}$ and negative otherwise. Due to its continuity and Rolle, it has one extremal point, where it reaches its minimum.

The above arguments show that the first derivative of the sellers' expected profit is decreasing in the sellers' offer strategy s. That is, until the minimum on the first derivative is achieved. That minimum is lower than zero because the integral starts positive with $s = 1$ and by Rolle there must be a root because the seller's expected profit has a maximum. When the first derivative increases after the minimum, there can not be another root. For $s = s_2/s_1$, the derivative is zero. If there were another root s^*, then is could not be zero for $s = s_2/s_1$, because it is increasing on $[s^*, s_2/s_1]$. As a result, the root of the derivative of a seller's profit is unique. Therefore there is exactly one offer strategy that maximises a seller's profit unilaterally. $\square$

Proof of Proposition 4. First it will be shown that $\mathbf{E}(P_S)$ is strictly decreasing in the number of sellers. Proposition 1 established the formula for expected profit under multilateral Level 1 rationality. We rearrange the formula and obtain

$$\mathbf{E}(P_S)(s) = \frac{s-1}{s^{n+1}\Delta s^n} \int_{ss_1}^{s_2} (ss_2 - x)^{n-1} \frac{b_2 - x}{\Delta b}\ x\ dx \ = \frac{s-1}{s^2 \Delta s} \int_{ss_1}^{s_2} \left(\frac{ss_2 - x}{s\Delta s}\right)^{n-1} \frac{b_2 - x}{\Delta b}\ x\ dx.$$

The only remaining term that is dependent on n is $\left(\frac{ss_2 - x}{s^n \Delta s}\right)^{n-1}$. Note that $ss_1 < x < s_2$, which allows to establish the inequality

$$0 < \left(\frac{ss_2 - x}{s\Delta s}\right)^{n-1} = \left(\frac{ss_2 - x}{ss_2 - ss_1}\right)^{n-1} < q^{n-1}, \text{ where } 0 < q < 1.$$

q^n is strictly decreasing in n while $0 < q < 1$. Therefore the only term that is dependent on n is strictly decreasing in the number of sellers. Additionally, the integral is positive. In summary, a seller's expected profit is decreasing in the number of bidding sellers. Analogous arguments hold for a strictly increasing profit as a function of valuation imprecision. $\square$

Proof of Proposition 5. This setting is a modification of a result from Seemüller (2013). By allowing $k = 0$ and introducing n sellers, we calculate the formula for the buyer's expected profit. The pdf of the lowest sellers' offer thereby is obtained from lemma 1:

$$\begin{aligned}
\mathbf{E}(P_B)(s) &= \int_{ss_1}^{s_2} m(x)\mathbf{P}(x \leq B)\mathbf{E}(P_B|x \leq B)\ dx \\
&= \int_{ss_1}^{s_2} \frac{n(ss_2-x)^{n-1}}{s^n \Delta s^n} \frac{s_2 - x}{\Delta s} \frac{1}{s_2-x} \int_x^{s_2} y - x dy\ dx \\
&= \frac{n}{2s^n \Delta s^{n+1}} \int_{ss_1}^{s_2} (ss_2 - x)^{n-1} (x - s_2)^2\ dx \\
&= \frac{n}{2s^n \Delta s^{n+1}} \left(B(s_2) - B(ss_1)\right),
\end{aligned}$$

where the function A is the primative of the integral, given by

$$\begin{aligned}
B(x) &= \frac{2\,\mathrm{s_2}\,(s\,\mathrm{s_2}-x)^{n+1}\,(s-1)}{n+1} - \frac{(s\,\mathrm{s_2}-x)^{n+2}}{n+2} - \frac{\mathrm{s_2}^2\,(s\,\mathrm{s_2}-x)^n\,(s-1)^2}{n} \\
&= (ss_2 - x)^n \left(\frac{2\,\mathrm{s_2}\,(s\,\mathrm{s_2}-x)\,(s-1)}{n+1} - \frac{(s\,\mathrm{s_2}-x)^2}{n+2} - \frac{\mathrm{s_2}^2\,(s-1)^2}{n}\right).
\end{aligned}$$

□

Proof of Proposition 6. In the limit of infinitely many sellers, there is a seller with the lowest offer $s_1 s$. Then the buyer's expected profit is

$$\begin{aligned}
\mathbf{E}(B)(s) &= 1/\Delta s \int_{ss_1}^{s_2} (x - ss_1)\ dx \\
&= 1/\Delta s \left(s_2^2 - s^2 s_1^2/2 - ss_1(s_2 - ss_1)\right) = (s_2 - ss_1)^2/2/\Delta s.
\end{aligned}$$

□

Proof of Proposition 7. First it will be shown that the buyer's expected profit decreases for an increasing offer strategy s. Proposition 5 showed that the buyer's expected profit is

$$\mathbf{E}(P_B)(s) = \frac{n}{2s^n \Delta s^{n+1}} \left(B(s_2) - B(ss_1)\right),$$

where $B(x)$ is the primary of the integrand $(ss_2 - x)^{n-1}(s_2 - x)^2$. The integrand is positive and strictly decreasing (as a function of x) within the integration borders $ss_1 \leq x \leq s_2$. Thus $B(x)$ is strictly increasing and positive within these bounds. Therefore the difference $B(s_2) - B(ss_1)$ is strictly decreasing in s, while $1 \leq s \leq s_2/s_1$ holds. Therefore the second factor of the buyer's profit, that is $B(s_2) - B(ss_1)$, is decreasing in the sellers' offer strategy. The first factor $\frac{n}{2s^n \Delta s^{n+1}} = \frac{1}{s^n} \frac{n}{2\Delta s^{n+1}}$ obviously is decreasing in s, while s is within the feasible region. Combining both arguments shows that the buyer's expected profit is decreasing in the sellers' offer strategy.

It remains to be prove that the buyer's expected profit is increasing in the number of sellers n. The proof will be lead verbally. When an additional seller bids in the auction, then there is a positive probability that this seller places a bid that is lower than all previous bids. If the previous lowest offer exceeds the buyer's reservation price, then the lowest bid that an additional seller places directly increases the buyer's profit. When the previous lowest bid exceeds the buyer's reservation price, then there is a positive probability that the buyer accepts to buy at the bid price that an additional seller places. Therefore an additional seller always increases a buyer's expected profit.

$\square$

Proof of Proposition 8. We calculate the probability that the seller with the offer strategy $s_1/s_2 \leq s' \leq s_2/s_1$ wins the auction.

$$\begin{aligned}
\mathbf{P}(D_S)(s,s') &= \mathbf{P}(S' \leq min(S_1, S_2, ..., S_{n-1}), S' \leq B) = \frac{1}{s'\Delta s}\int_{ss_1}^{ss_2} \mathbf{1}_{Deal} \;\; dx \\
&= \frac{1}{s'\Delta s}\int_{ss_1}^{ss_2} \mathbf{P}\left(x \leq min\left(S_1, S_2, ..., S_{n-1}\right),\; x \leq B\right)\; dx \\
&= \frac{1}{s'\Delta s}\int_{ss_1}^{s_2} \left(\frac{ss_2 - x}{s\Delta s}\right)^{n-1} \frac{b_2 - x}{\Delta b}\; dx + \mathbb{1}_{\{s'<s\}}\frac{1}{s'\Delta s}\int_{s's_1}^{ss_1} \frac{b_2 - x}{\Delta b}\; dx \\
&= \frac{1}{s's^{n-1}\Delta s^{n+1}}\int_{max(ss_1, s's_1)}^{s_2} \left(ss_2 - x\right)^{n-1}\left(s_2 - x\right)\; dx + \mathbb{1}_{\{s'<s\}}\frac{1}{s'\Delta s^2}\int_{s's_1}^{ss_1} s_2 - x\; dx.
\end{aligned}$$

The primative of $f(x) = \left(ss_2 - x\right)^{n-1}\left(s_2 - x\right)$ is given by

$$F(x) = \frac{\left(ss_2 - x\right)^n\left(-ns_2 - s_2 + ss_2 + nx\right)}{n\left(n+1\right)}.$$

This observation ends the proof for the case $s_1/s_2 \leq s' \leq s_2/s_1$. The remaining cases are obvious. $\square$

Proof of Proposition 9. Proposition 1 calculated the formula for the buyer's deal probability when the sellers' offer strategy is multilateral. Assume first $s' \leq s$. Then

	$x \leq s's_1$	$s's_1 < x \leq ss_1$	$ss_1 < x \leq s_2$
$\mathbf{P}\left(x < smin\{S_1, S_2, ..., S_{n-1}\}\right)$	1	1	< 1
$\mathbf{P}\left(x < s'S_n\right)$	1	< 1	< 1

With analogous steps as in the proof of proposition 1 and using lemma 1 and the independence of the sellers' offers we obtain the desired result:

$$
\begin{aligned}
\mathbf{P}(D_B)(s,s') &= \mathbf{P}\left(B \geq smin\{S_1,S_2,...,S_{n-1}\} \cup B \geq s'S_n\right)\\
&= 1-\mathbf{P}\left(B < smin\{S_1,S_2,...,S_{n-1}\} \cap B < s'S_n\right)\\
&= 1-\frac{1}{\Delta s}\int_{s_1}^{s_2}\mathbf{P}\left(x < smin\{S_1,S_2,...,S_{n-1}\} \cap x < s'S_n\right)\,dx\\
&= 1-\frac{1}{\Delta s}\int_{s_1}^{s_2}\mathbf{P}\left(x < smin\{S_1,S_2,...,S_{n-1}\}\right)\mathbf{P}\left(x < s'S_n\right)\,dx\\
&= 1-\frac{s's_1-s_1}{\Delta s}-\frac{1}{\Delta s}\int_{s's_1}^{ss_1}\mathbf{P}\left(x < s'S_n\right)\,dx\\
&\quad -\frac{1}{\Delta s}\int_{ss_1}^{s_2}\mathbf{P}\left(x < smin\{S_1,S_2,...,S_{n-1}\}\right)\mathbf{P}\left(x < s'S_n\;dx\right)\\
&= 1-\frac{s_1(s'-1)}{\Delta s}-\frac{1}{\Delta s}\int_{s's_1}^{ss_1}\frac{s's_2-x}{s'\Delta s}\,dx\\
&\quad -\frac{1}{\Delta s}\int_{ss_1}^{s_2}\left(\frac{ss_2-x}{s\Delta s}\right)^{n-1}\frac{s's_2-x}{s'\Delta s}\,dx\\
&= \frac{s_2-s_1s'}{\Delta s}-\frac{1}{\Delta s^2 s'}\left(s's_2s_1(s-s')-\frac{s^2s_1^2-s'^2s_1^2}{2}\right)-\frac{\int_{ss_1}^{s_2}(ss_2-x)^{n-1}(s's_2-x)\,dx}{\Delta s^{n+1}s^{n-1}s'}\\
&= \frac{s_2-s_1s'}{\Delta s}-\frac{s_1(s-s')}{\Delta s^2 s'}\left(s's_2-s_1(s+s')/2\right)-\frac{G(s_2)-G(ss_1)}{\Delta s^{n+1}s^{n-1}s'n(n+1)},
\end{aligned}
$$

where

$$G(x) := (s\,\mathrm{s}_2-x)^n\,\left(n\,x+s\,\mathrm{s}_2-\mathrm{s}_2\,\mathrm{s}'-n\,\mathrm{s}_2\,\mathrm{s}'\right).$$

Now let $s' > s$. Then

$$
\begin{aligned}
\mathbf{P}\left(D_B\right)(s,s') &= 1-\frac{1}{\Delta s}\int_{s_1}^{s_2}\mathbf{P}\left(x < smin\{S_1,S_2,...,S_{n-1}\}\right)\mathbf{P}\left(x < s'S_n\right)\,dx\\
&= 1-\frac{ss_1-s_1}{\Delta s}-\frac{\int_{ss_1}^{s's_1}\mathbf{P}\left(x < smin\{S_1,S_2,...,S_{n-1}\}\right)\,dx}{\Delta s}\\
&\quad -\frac{1}{\Delta s}\int_{s's_1}^{s_2}\mathbf{P}\left(x < smin\{S_1,S_2,...,S_{n-1}\}\right)\mathbf{P}\left(x < s'S_n\right)\,dx\\
&= \frac{s_2-s_1s}{\Delta s}-\frac{\int_{ss_1}^{s's_1}(ss_2-x)^{n-1}\,dx}{\Delta s^n s^{n-1}}-\frac{G(s_2)-G(s's_1)}{\Delta s^{n+1}s^{n-1}s'n(n+1)}\\
&= \frac{s_2-s_1s}{\Delta s}-\frac{(ss_2-ss_1)^n-(ss_2-s's_1)^n}{\Delta s^n s^{n-1}n}-\frac{G(s_2)-G(s's_1)}{\Delta s^{n+1}s^{n-1}s'n(n+1)}\\
&= \frac{s_2-s_1s}{\Delta s}-\frac{s}{n}+\frac{(ss_2-s's_1)^n}{\Delta s^n s^{n-1}n}-\frac{G(s_2)-G(s's_1)}{\Delta s^{n+1}s^{n-1}s'n(n+1)}.
\end{aligned}
$$

The case $s_2/s_1 < s'$ is obvious. □

Proof of Proposition 10. Let the sellers' offer strategy be s and one seller's offer strategy be s'. Then that seller's expected profit is

$$\begin{aligned}
\mathbf{E}\left(P_S\right)(s,s') &= \frac{1}{s'\Delta s}\int_{s's_1}^{s's_2} \mathbb{1}_{Deal}\ (x - x/s')\ dx = \frac{s'-1}{s'^2\Delta s}\int_{s's_1}^{s's_2} \mathbf{P}(Deal)\ x\ dx \\
&= \frac{s'-1}{s'^2\Delta s}\int_{s's_1}^{s's_2} \mathbf{P}\left(x \le min(S_2,S_2,...,S_n),\ x \le B\right)\ x\ dx \\
&= \frac{s'-1}{s'^2\Delta s}\left(\mathbb{1}_{s'<s}\int_{s's_1}^{ss_1} \frac{b_2 - x}{\Delta b}\ x\ dx + \int_{max(s,s')s_1}^{b_2}\left(\frac{ss_2 - x}{s\Delta s}\right)^{n-1}\frac{b_2 - x}{\Delta b}\ x\ dx\right) \\
&= \frac{s'-1}{s'^2\Delta s}\left(\mathbb{1}_{s'<s}\int_{s's_1}^{ss_1} \frac{b_2 - x}{\Delta b}\ x\ dx + \frac{1}{s^{n-1}\Delta s^n}\int_{max(s,s')s_1}^{s_2}\left(ss_2 - x\right)^{n-1}(b_2 - x)\ x\ dx\right) \\
&= \frac{s'-1}{s'^2\Delta s}\left(\mathbb{1}_{s'<s}\frac{2\ s_1^3\ (s'^3 - s^3) + 3\ s_1^2 s_2(s^2 - s'^2)}{6\Delta s} + \frac{A(s_2) - A(max(s,s')s_1)}{s^{n-1}\Delta s^n}\right).
\end{aligned}$$

The primative A is defined as in proposition 2. □

Proof of Proposition 11. Let $m(x)$ be the pdf of $s\min\{S_1, S_2, ..., S_{n-1}\}$ and $f(x)$ be the pdf of $s'S_n$. Then

$$\begin{aligned}
\mathbf{E}(P_B)(s,s') &= \int_{ss_1}^{ss_2}\int_{s's_1}^{s's_2} m(y)f(x)\mathbf{P}(B>min(x,y))\mathbf{E}(P_B|B>min(x,y))\ dxdy \\
&= \int_{ss_1}^{ss_2}\int_{s's_1}^{s's_2} m(y)f(x)\mathbb{1}_{x\le y}\mathbf{P}(B>x)\mathbf{E}(P_B|B>x)\ dxdy \\
&\quad + \int_{ss_1}^{ss_2}\int_{s's_1}^{s's_2} m(y)f(x)\mathbb{1}_{x>y}\mathbf{P}(B>y)\mathbf{E}(P_B|B>y)\ dxdy \\
&= \int_{max(ss_1,s's_1)}^{s_2}\int_{s's_1}^{y} m(y)f(x)\mathbf{P}(B>x)\mathbf{E}(P_B|B>x)\ dxdy \\
&\quad + \int_{ss_1}^{s_2}\int_{max(y,s's_1)}^{s_2} m(y)f(x)\mathbf{P}(B>y)\mathbf{E}(P_B|B>y)\ dxdy \\
&= \int_{max(ss_1,s's_1)}^{s_2}\int_{s's_1}^{y}(n-1)\frac{(ss_2-y)^{n-2}}{(s\Delta s)^{n-1}}\frac{1}{s'\Delta s}\frac{s_2-x}{\Delta s}\frac{s_2-x}{2}\ dxdy \\
&\quad + \int_{ss_1}^{s_2}\int_{max(y,s's_1)}^{s_2}(n-1)\frac{(ss_2-y)^{n-2}}{(s\Delta s)^{n-1}}\frac{1}{s'\Delta s}\frac{s_2-y}{\Delta s}\frac{s_2-y}{2}\ dxdy \\
&= \frac{n-1}{2\Delta s^{n+1}s^{n-1}s'}\left(\int_{max(ss_1,s's_1)}^{s_2}\int_{s's_1}^{y}(ss_2-y)^{n-2}(s_2-x)^2\ dxdy\right) \\
&\quad + \frac{n-1}{2\Delta s^{n+1}s^{n-1}s'}\left(\int_{ss_1}^{s_2}\int_{max(y,s's_1)}^{s_2}(ss_2-y)^{n-2}(s_2-y)^2\ dxdy\right) \\
&= \frac{n-1}{6\Delta s^{n+1}s^{n-1}s'}\left(\int_{max(ss_1,s's_1)}^{s_2}(ss_2-y)^{n-2}\left((s_2-s's_1)^3-(s_2-y)^3\right)dy\right) \\
&\quad + \frac{n-1}{2\Delta s^{n+1}s^{n-1}s'}\left(\int_{ss_1}^{s_2}(s_2-max(y,s's_1))(ss_2-y)^{n-2}(s_2-y)^2\ dy\right) \\
&\overset{s'\le s}{=} \frac{n-1}{6\Delta s^{n+1}s^{n-1}s'}\int_{ss_1}^{s_2}(ss_2-y)^{n-2}(s_2-s's_1)^3-(ss_2-y)^{n-2}(s_2-y)^3\ dy \\
&\quad + \frac{n-1}{2\Delta s^{n+1}s^{n-1}s'}\int_{ss_1}^{s_2}(ss_2-y)^{n-2}(s_2-y)^3\ dy \\
&= \frac{(n-1)(s_2-s's_1)^3}{6\Delta s^{n+1}s^{n-1}s'}\int_{ss_1}^{s_2}(ss_2-y)^{n-2}\ dy \\
&\quad - \frac{n-1}{6\Delta s^{n+1}s^{n-1}s'}\int_{ss_1}^{s_2}(ss_2-y)^{n-2}(s_2-y)^3\ dy \\
&\quad + \frac{n-1}{2\Delta s^{n+1}s^{n-1}s'}\int_{ss_1}^{s_2}(ss_2-y)^{n-2}(s_2-y)^3\ dy \\
&= \frac{(s_2-s's_1)^3}{6\Delta s^{n+1}s^{n-1}s'}\left((s\Delta s)^{n-1}-(s_2(s-1))^{n-1}\right)+\frac{(n-1)(H(s_2)-H(ss_1))}{3\Delta s^{n+1}s^{n-1}s'}.
\end{aligned}$$

Now let $s < s' \leq s_2/s_1$. Following analogous steps as in the first case we get

$$\begin{aligned}
\mathbf{E}(P_B)(s,s') &= \frac{n-1}{6\Delta s^{n+1}s^{n-1}s'}\left(\int_{s's_1}^{s_2}(ss_2-y)^{n-2}\left((s_2-s's_1)^3-(s_2-y)^3\right)dy\right)\\
&\quad+\frac{n-1}{2\Delta s^{n+1}s^{n-1}s'}\left(\int_{ss_1}^{s's_2}(s_2-s's_1)(ss_2-y)^{n-2}(s_2-y)^2\,dy\right)\\
&\quad+\frac{n-1}{2\Delta s^{n+1}s^{n-1}s'}\left(\int_{s's_1}^{s_2}(ss_2-y)^{n-2}(s_2-y)^3\,dy\right)\\
&=\frac{(n-1)(s_2-s's_1)^3}{6\Delta s^{n+1}s^{n-1}s'}\left(\int_{s's_1}^{s_2}(ss_2-y)^{n-2}\,dy\right)\\
&\quad-\frac{n-1}{6\Delta s^{n+1}s^{n-1}s'}\left(\int_{s's_1}^{s_2}(ss_2-y)^{n-2}(s_2-y)^3\,dy\right)\\
&\quad+\frac{(n-1)(s_2-s's_1)}{2\Delta s^{n+1}s^{n-1}s'}\left(\int_{ss_1}^{s's_1}(ss_2-y)^{n-2}(s_2-y)^2\,dy\right)\\
&\quad+\frac{n-1}{2\Delta s^{n+1}s^{n-1}s'}\left(\int_{s's_1}^{s_2}(ss_2-y)^{n-2}(s_2-y)^3\,dy\right)\\
&=\frac{(s_2-s's_1)^3}{6\Delta s^{n+1}s^{n-1}s'}\left((ss_2-s's_1)^{n-1}-(s_2(s-1))^{n-1}\right)\\
&\quad+\frac{(n-1)(H(s_2)-H(s's_1))}{3\Delta s^{n+1}s^{n-1}s'}\\
&\quad+\frac{(n-1)(s_2-s's_1)(I(s's_1)-I(ss_1))}{2\Delta s^{n+1}s^{n-1}s'}.
\end{aligned}$$

The primatives H and I are defined as in the proposition. $\square$

Proof of Theorem 2. First it will be shown that the unilateral offer strategy in its optimum does not exceed the optimal multilateral strategy, i.e. $s^2_{opt} \leq s^1_{opt}$. To do so, we rearrange the formulas for the sellers' expected profit. First, the formula for the expected profit of a seller is calculated, when all sellers pursue multilateral offer strategy $s \in [1, s_2/s_1]$. As all sellers have the same expected profit, we refer to an arbitrary seller as seller 1. Before the formula is introduced, we define the event

$$A(s) := \text{"Seller 1 with offer strategy } s \text{ places winning bid"}$$

Then the expected profit of that seller is

$$\mathbf{E}(P_{S_1})(s) = \mathbf{P}(A(s))\mathbf{E}(\text{Profit of seller 1 with offer strategy } s|A(s))(s).$$

Note that the probability in the formula above deceases when the sellers increase their offer strategy s. In contrast, when a seller places the winning bid, then her expected profit increases when her offer strategy s was higher.

Secondly, we rearrange the formula of a seller's expected profit, when that seller places her bid unilaterally with strategy $s' \in [1, s_2/s_1]$, while the other sellers pursue multilateral offer strategy $s \in [1, s_2/s_1]$. To simplify that formula, we define the event

$$B(s, s') := \text{"The unilateral bidding seller with offer strategy } s' \text{ places the winning bid.} \\ \text{The other sellers pursue multilateral offer strategy } s.\text{"}$$

Then the expected profit of the unilaterally bidding seller is

$$\mathbf{E}(P_S)(s, s') = \mathbf{P}(B(s, s'))\mathbf{E}(\text{Profit of seller with unilateral offer strategy } s' \mid B(s, s'))(s, s').$$

Note that the probability in the formula above deceases when the unilaterally bidding seller increases her offer strategy s'. In contrast, when that seller places the winning bid, then her expected profit increases when her offer strategy s' was higher.

When we assume that a seller placed the winning bid and then calculate her expected profit as a function of her offer strategy, then this expectation is independent of the other sellers' offer strategies. That is to say, that the other sellers' offer strategies only change the probability that a player places the winning bid. Thus, the expectations in both formulas above are equal. That is,

$$\mathbf{E}(\text{Profit of seller with unilateral offer strategy } s' \mid B(s, s'))(s, s') \\ = \mathbf{E}(\text{Profit of seller 1 with offer strategy } s' | A(s'))(s').$$

When we analyse the probabilities in the formulas above, we conclude that

$$\mathbf{P}(B(s, s')) < \mathbf{P}(A(s')),$$

while $s < s'$. That is, when the unilaterally bidding seller places a higher average offer than the other sellers ($s' > s$), then that seller's probability to win the auction is lower than a multilaterally bidding seller's chance to do so.

When all sellers pursue optimal multilateral offer strategy s^1_{opt}, then this strategy generates the highest profit for each seller. Now assume that a unilaterally bidding seller chooses an offer strategy that exceeds the multilateral strategy. That is $s' > s^1_{opt}$. Then that seller's probability to win the auction is lower than the chance of a multilaterally bidding seller. Further, her expected profit when she wins the auction is lower than if she placed her bid with strategy $s' = s^1_{opt}$. That is, when $s' > s^1_{opt}$, then the unilaterally bidding seller's profit is

$$\begin{aligned}
\mathbf{E}(P_S)(s^1_{opt}, s') &= \mathbf{P}(B(s^1_{opt}, s'))\mathbf{E}(\text{Profit of seller with unilateral offer strategy } s' \, |B(s^1_{opt}, s'))(s^1_{opt}, s') \\
&= \mathbf{P}(B(s^1_{opt}, s'))\mathbf{E}(\text{Profit of seller 1 with offer strategy } s'|A(s'))(s') \\
&< \mathbf{P}(A(s'))\mathbf{E}(\text{Profit of seller 1 with offer strategy } s'|A(s'))(s') \\
&= \mathbf{E}(P_{S_1})(s') < \mathbf{E}(P_{S_1})(s^1_{opt}).
\end{aligned}$$

The inequation shows that a unilateral bidding seller can not increase her profit when she has an offer strategy that exceeds the optimal strategy s^1_{opt} that the other sellers pursue. In summary, a unilaterally bidding seller has a lower expected profit when her offer strategy exceeds that of the multilaterally bidding sellers. As a result, an optimal level 2 offer strategy must not exceed the optimal level 1 strategy. That is, $s^2_{opt} \leq s^1_{opt}$.

Next, we show that the optimal level 2 strategy is unique. As the optimal strategy is within $(1, s^1_{opt}]$, the further analysis will exclusively consider that case. For $s' \leq s$, the level 2 bidding seller's expected profit is

$$\begin{aligned}
\mathbf{E}\left(P_S\right)(s, s') &= \frac{s'-1}{s'^2 \Delta s}\left(\mathbb{1}_{s'<s} \frac{2\ s_1^3\ (s'^3 - s^3) + 3\ s_1^2 s_2 (s^2 - s'^2)}{6\Delta s} + \frac{1}{s^{n-1}\Delta s^n}\left(s s_2 - y\right)^{n-1}\left(b_2 - y\right)\, y(s_2 - s s_1)\right) \\
&= \frac{(s'-1)s_1^2}{6\ s'^2 \Delta s^2}\left(\mathbb{1}_{s'<s}\left(2\ s_1\ (s'^3 - s^3) + 3\ s_2(s^2 - s'^2)\right) + \frac{6}{s^{n-1}\Delta s^{n-1} s_1^2}\left(s s_2 - y\right)^{n-1}\left(b_2 - y\right)\, y(s_2 - s s_1)\right) \\
&= c_1 \frac{(s'-1)}{s'^2}\left(\mathbb{1}_{s'<s}\left(2\ s_1\ (s'^3 - s^3) + 3\ s_2(s^2 - s'^2)\right) + c_2\right),
\end{aligned}$$

where c_1 and c_2 are positive constants as functions of the unilateral offer strategy s' given by

$$\begin{aligned}
c_1 &:= s_1^2/\Delta s^2/6 \\
c_2 &:= \frac{6}{s^{n-1}\Delta s^{n-1} s_1^2}\left(s s_2 - y\right)^{n-1}\left(b_2 - y\right)\, y(s_2 - s s_1).
\end{aligned}$$

The expected profit for $s' = 1$ is zero. The expected profit for $s' = s$ is greater zero. This case represents multilateral level 1 optimization strategy which is positive while $s > 1$.

The first derivative of expected profit in unilateral level 2 optimization is

$$\begin{aligned}
\frac{\partial \mathbf{E}\left(P_S\right)(s, s')}{\partial s'} &= c_1 \frac{\left(2\, c_2 - c_2\, \mathrm{s}' + 6\, s^2\, \mathrm{s}_2 - 4\, s^3\, \mathrm{s}_1 - 2\, \mathrm{s}_1\, \mathrm{s}'^3 + 4\, \mathrm{s}_1\, \mathrm{s}'^4 - 3\, \mathrm{s}_2\, \mathrm{s}'^3 - 3\, s^2\, \mathrm{s}_2\, \mathrm{s}' + 2\, s^3\, \mathrm{s}_1\, \mathrm{s}'\right)}{\mathrm{s}'^3} \\
&= c_1 \frac{(4\, \mathrm{s}_1)\, \mathrm{s}'^4 + (-2\, \mathrm{s}_1 - 3\, \mathrm{s}_2)\, \mathrm{s}'^3 + \left(\mathrm{s}_1\, s^3\, 2 - \mathrm{s}_2\, s^2\, 3 - c_2\right)\, \mathrm{s}' - \left(\mathrm{s}_1\, s^3\, 4 - \mathrm{s}_2\, s^2\, 6 - 2\, c_2\right)}{\mathrm{s}'^3}.
\end{aligned}$$

The second derivative as a function of s' is

$$\frac{\partial^2 \mathbf{E}\left(P_S\right)\left(s, s'\right)}{\partial^2 s'} = c_1 \frac{\left(4\,\mathrm{s}_1\right)\,\mathrm{s}'^4 + \left(-\mathrm{s}_1\, s^3\, 4 + \mathrm{s}_2\, s^2\, 6 + 2\, c_2\right)\,\mathrm{s}' + \left(\mathrm{s}_1\, s^3\, 12 - \mathrm{s}_2\, s^2\, 18 - 6\, c_2\right)}{\mathrm{s}'^4}.$$

The second derivatives is zero if and only if

$$\begin{aligned} 0 &= 2\,\mathrm{s}_1\,\mathrm{s}'^4 + \left(3\, s^2\, \mathrm{s}_2\ - 2\, s^3\, \mathrm{s}_1\ + c_2\right) s' - 9\, s^2\, \mathrm{s}_2 + 6\, s^3\, \mathrm{s}_1 - 3\, c_2 \\ &=: a\mathrm{s}'^4 + bs' + c, \end{aligned}$$

where $a := 2s_1,\ b := 3\, s^2\, \mathrm{s}_2\ - 2\, s^3\, \mathrm{s}_1\ + c_2$ and $c := -9\, s^2\, \mathrm{s}_2 + 6\, s^3\, \mathrm{s}_1 - 3\, c_2$. As $s_2 > 0$, $a > 0$. Furthermore $b > 0$ because $3s^2 s_2 - 2s^3 s_1 > 3s^2 s_1 - 2s^3 s_1 = s_1 s^2(3 - 2s) > 0$, while $s < 3/2$. Assume $c \geq 0$. Then $a\mathrm{s}'^4 + bs' + c > 0$ while $s' > 0$. Then the second derivative has no positive roots. Assume $c < 0$. Then $a\mathrm{s}'^4 + bs' + c$ has exactly one positive root, because $a\mathrm{s}'^4 > 0$ and bs' are positive and increasing in s'. In this case the second derivative therefore has exactly one root. Therefore for $s' > 0$ and thus all s' in the feasible region $1 \leq s' < s$, the second derivative has at the most one root. Therefore there is exactly one maximum of the expected profit in $1 \leq s' \leq s^1_{opt}$. □

Proof of Proposition 12. Theorem 2 showed that $s^2_{opt} \leq s^1_{opt}$ holds. The unilateral bidding seller places a lower bid than the other sellers on average. This lower bid has positive effect on the buyer's expected profit. This effect can be observed from buyer's expected profit as a function of the bidding strategies, that was introduced in proposition 11. That function is increasing when the offer strategy decreases. □

Proof of Theorem 3. Assume the sellers' multilateral strategy is $s = s_2/s_1$. Then each seller's expected profit is zero. Then a single seller may pursue some strategy $s' < s$ to gain a positive expected profit. Therefore at the level 3 optimum, the optimal sellers' multilateral strategy is lower than s_2/s_1. For the multilateral strategy s to be optimal level 3, i.e. no seller benefits by pursuing the unilateral strategy $s' \neq s$, it must hold that the multilateral strategy s is the maximum in level 2 optimization. This means that $\frac{\partial \mathbf{E}(P_S)(s,s')}{\partial s'}(s, s) = 0$ must hold. Proposition 10 introduced the formula for the expected profit of a seller with a unilateral strategy s', while the other sellers pursue strategy s. This formula was rewritten in the proof of theorem 2 and is of great value in this proof.

$$
\begin{aligned}
\frac{\partial \mathbf{E}(P_S)(s,s')}{\partial s'}(s,s) =& \frac{6\, s^{1-n}\, y\, \left(\mathrm{s}_2 - s\,\mathrm{s}_1\right)\, \left(s\,\mathrm{s}_2 - y\right)^{n-1} \left(\mathrm{s}_2 - \mathrm{s}_1\right)^{1-n}\, \left(\mathrm{s}_2 - y\right)}{s^2\, {\mathrm{s}_1}^2} \\
& - \frac{\left(6\, s\,\mathrm{s}_2 - 6\, s^2\,\mathrm{s}_1\right)\, (s-1)}{s^2} \\
& - \frac{12\, s^{1-n}\, y\, \left(\mathrm{s}_2 - s\,\mathrm{s}_1\right)\, \left(s\,\mathrm{s}_2 - y\right)^{n-1} \left(\mathrm{s}_2 - \mathrm{s}_1\right)^{1-n}\, \left(\mathrm{s}_2 - y\right)\, (s-1)}{s^3\, {\mathrm{s}_1}^2} \\
=& \frac{6\; y\, \left(\mathrm{s}_2 - s\,\mathrm{s}_1\right)\, \left(s\,\mathrm{s}_2 - y\right)^{n-1}\, \left(\mathrm{s}_2 - y\right)}{\Delta s^{n-1} s^{n+1}\, {\mathrm{s}_1}^2} - \frac{6s\left(\mathrm{s}_2 - s\,\mathrm{s}_1\right)\, (s-1)}{s^2} \\
& - \frac{12\, y\, \left(\mathrm{s}_2 - s\,\mathrm{s}_1\right)\, \left(s\,\mathrm{s}_2 - y\right)^{n-1}\, \left(\mathrm{s}_2 - y\right)\, (s-1)}{s^{n+2}\, \Delta s^{n-1} {\mathrm{s}_1}^2},
\end{aligned}
$$

is zero if

$$
\begin{aligned}
& 0 = \frac{y\left(s\,\mathrm{s}_2 - y\right)^{n-1}\, \left(\mathrm{s}_2 - y\right)}{\Delta s^{n-1} s^{n+1}\, {\mathrm{s}_1}^2} - \frac{s\,(s-1)}{s^2} - \frac{2y\left(s\,\mathrm{s}_2 - y\right)^{n-1}\, \left(\mathrm{s}_2 - y\right)\, (s-1)}{s^{n+2}\, \Delta s^{n-1} {\mathrm{s}_1}^2} \\
\Longleftrightarrow\ & 0 = sy\left(s\,\mathrm{s}_2 - y\right)^{n-1}\, \left(\mathrm{s}_2 - y\right) - s^{n+1}\,(s-1)\,\Delta s^{n-1} s_1^2 - 2y\left(s\,\mathrm{s}_2 - y\right)^{n-1}\, \left(\mathrm{s}_2 - y\right)\, (s-1) \\
\Longleftrightarrow\ & 0 = y(s_2 - y)(s\,\mathrm{s}_2 - y)^{n-1}\,(s - 2(s-1)) - s^{n+1}\,(s-1)\,\Delta s^{n-1} s_1^2 \\
\Longleftrightarrow\ & 0 = y(s_2 - y)(s\,\mathrm{s}_2 - y)^{n-1}\,(2-s) - s^{n+1}\,(s-1)\,\Delta s^{n-1} s_1^2.
\end{aligned}
$$

To keep the notation clearer, the last term is defined as $f(s)$. Note that $f(1) > 0$. Assume the derivative of f as a function of s is negative. Then there is at the most one $s > 1$, such that $f(s) = 0$ can hold. This implies that the first derivative of the expected profit $\frac{\partial \mathbf{E}(P_S)(s,s')}{\partial s'}(s,s)$ has at the most one root, which means that there is at the most one maximum. In theorem 2 it was shown that there is a maximum. Thus there is exactly one multilateral strategy s, such that unilateral optimization s' is not preferable compared to multilateral optimization.

If remains to be shown that $f'(s) < 0$, while $1 \leq s \leq s_2/s_1$.

$$\begin{aligned}
\frac{\partial f}{\partial s} &= y(s_2 - y)\left(s_2(n-1)(ss_2 - y)^{n-2}(s-2) - (ss_2 - y)^{n-1}\right) - \Delta s^{n-1}s_1^2\left((n+1)s^n(s-1) + s^n\right) \\
&= y(s_2 - y)(ss_2 - y)^{n-2}\left(s_2(n-1)(s-2) - (ss_2 - y)\right) - \Delta s^{n-1}s_1^2 s^n\left((n+1)(s-1) + 1\right) \\
&\leq y(s_2 - y)(ss_2 - y)^{n-2}\left(s_2(n-1)(s-2) - (ss_2 - y)\right) \\
&\quad - (s_2 - y)(ss_2 - y)^{n-2}s^n\left((n+1)(s-1) + 1\right) \\
&= (s_2 - y)(ss_2 - y)^{n-2}\left(y\left(s_2(n-1)(s-2) - (ss_2 - y)\right) - s^n\left((n+1)(s-1) + 1\right)\right) \\
&= (s_2 - y)(ss_2 - y)^{n-2}\left(s_2nsy - 2s_2ny - s_2sy + 2s_2y - ss_2y + y^2 - ns^{n+1} + ns^n - s^{n+1}\right) \\
&= (s_2 - y)(ss_2 - y)^{n-2}\left(ss_2y(n-2) - 2s_2y(n-1) + s^n(-ns + 1 - s)\right) \\
&= (s_2 - y)(ss_2 - y)^{n-2}\left(s_2y(sn - 2s - 2n + 2) + s^n(-ns + 1 - s)\right) \\
&= (s_2 - y)(ss_2 - y)^{n-2}\left(s_2y(\ n\underbrace{(s-2)}_{<0} + 2\underbrace{(1-s)}_{<0}\ ) + s^n\underbrace{(-ns + 1 - s)}_{<0}.\right) < 0
\end{aligned}$$

The derivative is negative and therefore there is exactly one optimum level 3. It remains to be shown that the level 3 optimisation leads to a Nash equilibrium. Assume all sellers optimise their strategy multilateral level 1. Then each seller can optimise her offer strategy, given the multilateral strategy of the remaining sellers. Thus there is no equilibrium. This statement is true unless unilateral optimisation of a single seller leads towards the same optimal strategy as a multilateral offer strategy. This is characterised by level 3 optimisation. In this case no seller benefits from applying a unilateral strategy unequal to the multilateral strategy. Therefore level 3 optimisation is an equilibrium strategy.

At last, it will be shown that for the equilibrium strategy $s^3_{opt} \leq s^2_{opt}$ holds: Assume the following proposition (p) holds: "The optimal level 2 response strategy s^2_{opt} as a function of the other sellers' multilateral strategy s is increasing in the multilateral strategy". Let now $s^3_{opt} > s^1_{opt}$. Then the expected profit of the seller with the unilateral strategy s^2_{opt} increases by applying unilateral strategy. Thus this case is not possible, as the multilateral strategy exceeds its optimal response strategy s^2_{opt}. Let $s^2_{opt} < s^3_{opt} < s^1_{opt}$. Then the optimal level 2 response strategy to the multilateral strategy s^3_{opt} is lower than the optimal response strategy to s^1_{opt}. Therefore s^3_{opt} can not be a Nash equilibrium. At last, just the case $s^3_{opt} \leq s^2_{opt}$ remains to be possible. Finally proposition (p) remains to be shown: Theorem 2 showed $\mathbf{E}(P_S)(s,1) = 0$ and $\mathbf{E}(P_S)(s,s) \geq 0$ and $\mathbf{E}(P_S)(s,s') > 0$ for $s' \in (1,s)$ (while $s \neq 1$). It further was shown that $\frac{\partial^2 \mathbf{E}(P_S)(s,s')}{\partial^2 s'}(s,s') < 0$. Then just one maximum exists. If the expected profit's first derivative is furthermore decreasing in the multilateral strategy s, then the level

2 optimum also decreases with s' (and thus increases in s):

$$\frac{\partial \mathbf{E}(P_S)(s,s')}{\partial s'} = \frac{c_1}{s'^3}\left(s^3(2s_1 s' - 4s_1) - s^2(s_2 s' + 6s_2) + c\right).$$

The addends $(2s_1s' - 4s_1)$ and $-(s_2s' + 6s_2)$ dependent on s are both negative. The expected profit's first derivative is decreasing in s as all powers of s are positive while s is positive. Thus proposition (p) and therefore the theorem holds. □

Proof of Proposition 13. In theorem 3, the necessary condition

$$0 = \frac{y\,(s\,\mathrm{s}_2 - y)^{n-1}\,(\mathrm{s}_2 - y)}{\Delta s^{n-1} s^{n+1}\,\mathrm{s}_1{}^2} - \frac{s\,(s-1)}{s^2} - \frac{2y\,(s\,\mathrm{s}_2 - y)^{n-1}\,(\mathrm{s}_2 - y)\,(s-1)}{s^{n+2}\,\Delta s^{n-1}\mathrm{s}_1{}^2}$$

was established for an offer strategy s to be optimal level 3. The first and third addend converges to zero for $n \to \infty$. That is,

$$\frac{y\,(s\,\mathrm{s}_2 - y)^{n-1}\,(\mathrm{s}_2 - y)}{\Delta s^{n-1} s^{n+1}\,\mathrm{s}_1{}^2} - \frac{s\,(s-1)}{s^2} - \frac{2y\,(s\,\mathrm{s}_2 - y)^{n-1}\,(\mathrm{s}_2 - y)\,(s-1)}{s^{n+2}\,\Delta s^{n-1}\mathrm{s}_1{}^2} \overset{n\to\infty}{\to} -\frac{s\,(s-1)}{s^2}.$$

The limit $-\frac{s(s-1)}{s^2}$ is zero for $s = 1$. Therefore the optimal offer strategy is $s^3_{opt} = 1$.

In proposition 1 the expected profit of a seller under multilateral strategy s was calculated. That profit is

$$\mathbf{E}\,(P_S)\,(s) = \frac{(s-1)\;\left(\;A(s_2) - A(ss_1)\;\right)}{s^{n+1}\;\Delta s^{n+1}}.$$

This formula shows that the offer strategy $s = 1$ generates a profit of 0 for a seller. □

Proof of Proposition 14. Proposition 13 showed that the optimal level 3 offer strategy in the limit is $s^3_{opt} = 1$. In proposition 2 the seller-group profit under multilateral offer strategy was calculated. That formula is

$$\mathbf{E}\,(S)\,(s) = \frac{n(s-1)\;\left(\;A(s_2) - A(ss_1)\;\right)}{s^{n+1}\;\Delta s^{n+1}}.$$

That profit is zero for $s = s^3_{opt} = 1$. Furthermore, the limit of a zero sequence is zero. Therefore the sellers' group profit under optimal level 3 strategy is zero for a group size $n \to \infty$. □

Proof of Proposition 15. Proposition 13 shows the optimal level 3 strategy $s^3_{opt} \to 1$ for $n \to \infty$. According to proposition 6 the buyer's profit in the limit is

$$\mathbf{E}(B)(s) = \frac{(s_2 - ss_1)^2}{2s\Delta s} \overset{s=1}{=} \alpha.$$

$\square$

Proof of Proposition 16. Proposition 1 introduced the formula for the buyer's deal probability. That formula was dependent on the sellers' multilateral offer strategy and the number of sellers. Proposition 13 showed that in level 3 optimum and for infinitely many sellers, the optimal strategy is $s^3_{opt} = 1$. When we combine these statements we obtain

$$\mathbf{P}(D_B)(s) = \frac{s_2 - ss_1}{\Delta s} + \frac{s}{n+1}\left(\left(\frac{s_2(s-1)}{s\Delta s}\right)^{n+1} - 1\right) \overset{n\to\infty}{\longrightarrow} \frac{s_2 - 1 \cdot s_1}{\Delta s} = 1.$$

The probability that the auction is successful converges to 1. That proves the proposition. $\square$

Proof of Proposition 17. The proof of proposition 17 is shown by Seemüller (2013) in the discussion of the *Dealer's Market.* $\square$

Proof of Proposition 18. The proof of proposition 18 is shown by Seemüller (2013) in the discussion of the *Dealer's Market.* $\square$

Proof of Proposition 19. According to proposition 17 each seller accepts the dealer's offer with probability $p_s = (\alpha + f_s)/(2\alpha)$. Sellers determine their reservation prices independently. Thus the expected number of acceptances is binomially distributed with the success parameter p_d. Expectation thus is $np_d = n\frac{\alpha+f_s}{2\alpha}$. $\square$

Proof of Theorem 4. The dealer's profit per round-trip transaction is $f_b - f_s > 0$. Therefore the dealer profits from every round-trip transaction if $f_s < f_b$. The buyer prefers the Dealer's Market over the reverse auction, while $f_b < f_{b,ind}$. The seller prefers the Dealer's Market over the reverse auction, while $f_s > f_{s,ind}$. As a result the Dealer's Market is Pareto efficient under the given conditions. $\square$

Proof of Theorem 5. The theorem is a summary of the analysis of this section. The proof is delivered in the section's argumentation above. $\square$

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On the Pareto Efficiency of an Intermediary
Over a Multiple Buyer and Seller Auction Market

Johannes Seemüller

Abstract

We introduce two markets where a single non-divisible good is traded. In the Buyers' Market, buyers and sellers directly trade on a platform under full price information. Secondly, a dealer offers his services in the Dealer's Market under information asymmetry. Profit optimizing buyers and sellers may choose one of these markets to trade the good. We show that both markets can generate a positive expected profit for all traders, so they benefit from participating in these markets. Furthermore, under generally non-restrictive conditions, an optimal dealer's pricing strategy causes the market participants to prefer the Dealer's Market over the Buyers' Market. In this case, the market under information asymmetry Pareto dominates the market under full price information. This property is noteworthy, as numerous authors such as Akerlof (1970) or Stiglitz and Weiss (1981) found information asymmetry a factor that leads towards inefficient market allocations.

Key words: Auction Market, Market Design, Intermediary, Full Information, Information Asymmetry, eBay, Amazon, Pareto Efficient Market, Imprecise Valuation, Underpricing. JEL Classifications: C72, D44, D47, D82

Contents

1 Introduction

How markets allocate resources depends on the institutions that govern transactions and a market's design. There are, however, markets where the price system operates, but perfect resource allocations are not obtained. The Nobel price committee recognized that a centralized clearinghouse can improve market efficiency by implementing certain market procedures. In particular, Roth and Shapley, who have been awarded the 2012 Nobel price for economic sciences[1], have illuminated how markets operate. Their research was involved in designing institutions that help markets function better. They built the groundwork that has led to the emergence of the branch of economics known as market design.

In this paper, we study a market where buyers and sellers directly trade on a platform that is similar to Amazon and the buy-it-now option of eBay. Sellers disclose their prices for an homogeneous good or service. Buyers arrive one after another and decide to buy at the lowest price. The efficiency of this platform is analysed and compared to a dealer's market design. On the platform full price information is available, whereas a dealer operates under information asymmetry. We introduce conditions when the Dealer's Market is preferable for all parties.

Empirical studies of online trading platforms are performed for instance by Roth and Ockenfels (2002). Wang et al. (2004) in particular study buy-it-now offers and present statistics on their popularity. Accordingly, eBay introduced its buy-it-now service in 2000, which has been adopted subsequentially by 45% of eBay's U.S. auctions by the end of its first year. These buy-it-now auctions accounted for 29% of gross merchandise sales on eBay in 2003. The empirical work of Park and Bradlow (2003) further shows that the buy-it-now feature is an important element in auction design.

The benefits of eBay's buy-it-now option have been discussed in recent literature. Mathews (2003) shows that the buy-it-now option is beneficial to bidders who want to buy their product in a shorter span of time than others. That impatience does not reflect the limited cognitive resources that most consumers appear to apply in making decisions, as Ratchford (1982) and Mehta et al. (2003) note.

Studies have empirically and experimentally analysed the influence of transaction costs on online auctions. For instance, List and Lucking-Reiley (2002) found in a field experiment that cognitive costs influence a bidder's strategic behaviour.

Seemüller (2013b) introduces a model on bilateral trade, where bargainers value the traded good or service imprecisely. In that model, individuals may behave rationally or naively and it further allows for different negotiation skills of the traders. Seemüller (2013a) generalizes his paper on markets with an arbitrary number of sellers who trade with a single buyer. He

[1] See in Nobelprize.org (2012) for reference

shows that a dealer under information asymmetry can improve the efficiency of these markets. The present paper is a further generalisation of his trading model and allows for an arbitrary number of buyers who buy from the same number of sellers. Where Seemüller focuses on optimal offer strategies in the referred papers, the present paper's analysis concentrates on the platform's efficiency.

In our analysis, we compare a market under full price information and a market under information asymmetry. Within our framework of profit maximizing individuals, we add imprecise valuation. Each individual has a certain reservation price for the good. These reservation prices are random and identically and independently distributed. Based on her reservation price, each seller assigns an offer price to the good, such that this seller's expected profit is maximized. As a result, a seller's offer price is also random. Buyers arrive one after another. Each buyer accepts to buy the good or service at the present lowest price, if that price does not exceed his reservation price.

Both analysed markets are wealth increasing for all traders. However, we show under which conditions the market under information asymmetry is preferable to the market under full price information.

Section 2 introduces the framework of the market under full price information (the Buyers' Market) and analyses its properties in detail. Section 3 discusses the dealer's pricing strategy under information asymmetry in the Dealer's Market. Furthermore, it analyses the traders' profit in this market. The market preferences of the traders are discussed in section 4. Section 5 concludes.

2 The Buyers' Market

We model a platform market similar to buy-it-now offers on eBay, where a single indivisible good is traded. Several owners (sellers) of the good disclose their offer prices on the platform. Potential buyers can observe the sellers' offers and decide whether to buy at each seller's offer or not. Denote P_1 as the lowest available sellers' offer. In our model, a buyer buys a good at that lowest price P_1, when that buyer's reserve price exceeds P_1. The good then is traded at that price[2]. When a subsequent buyer enters the market, then that buyer observes the remaining sellers' offers. Similar to the prior buyer, the good is sold at the lowest available price, when that price does not exceed the present buyer's reserve value. This procedure continues until no further buyer arrives. When a deal is successful, then a buyer's profit is determined by the difference between the deal price and that buyer's reserve value. A seller's profit is given by the deal price.

[2]The event that a buyer's reserve price and a seller's offer are equal is a zero set. It thus has probability zero. Therefore the events (a) a buyer's reserve price exceeds a seller's offer; and (b) a seller's offer exceeds the buyer's reserve price; add up to a probability of 1.

In our model the parties suffer from valuation imprecision. Their reservation prices thus are not identical. We model imprecision as independent uniformly distributed random variables. When there are n sellers, then the i-th seller's reservation price is given by V_{S_i}. If there are m buyers, then the j-th buyer's reservation price is given by V_{B_j}. Due to the lack of a valuation benchmark, neither party knows whether he underestimates or overestimates the value of the asset. They just know that their valuation is uniformly and symmetrically distributed around some average valuation $V > 0$.

In order for a seller to expect a positive profit trade, her offer must be lower than her reservation price. Otherwise her expected profit is zero, or even negative if her offer exceeds her reservation price. In our model, each seller's offer strategy is to adjust her reservation price by a certain factor $s > 1$. Then her offer is sV_S. With this strategy, each seller can determine her offer sV_s such that it maximizes her expected profit. This offer strategy is a special case of Chatterjee's model Chatterjee and Samuelson (1983).

When there is one buyer and one seller in the auction, then the model simplifies to a two-player double auction. In this auction, optimal offer strategies are studied by Seemüller (2013b). Optimal strategies in the one buyer and n seller case are further studied by Seemüller (2013a).

We distinguish two types of the Buyers' Market: Level 1 can be regarded as the initializing of the trading platform. Each player knows that he suffers from valuation imprecision and knows that the other players know, knows that they do and so forth. The players furthermore know the imprecision's distribution, know that the others know, etcetera. Thus there is mutual full information in the sense of Aumann (1976). The players, however, are not aware whether they over- or undervalue the good because they have no respective benchmark. When, for instance a seller values the good at 160 and knows the imprecision's distribution, she still cannot determine neither over- nor undervaluation. This can be regarded as the initializing of the platform as there are no reference offers present from which over- or undervaluation can be derived.

Although the sellers' reservation prices are not identical, their expected profit is the same ex ante. Profit is further dependent on each seller's individual offer strategy: Assume there are 2 sellers. The first seller places an offer twice her reservation price. The second seller places an offer that is 10% higher than her reservation price. Then it is likely that the offer of seller two is lower than seller one's. That is, each seller's offer strategy influences individual and other sellers' deal probability and expected profit. Underlying rational sellers, they apply a homogeneous equilibrium offer strategy as can be derived from Chatterjee Chatterjee and Samuelson (1983).

The Buyers' Market level 2 is in post initialization phase, where sellers can observe offers from previous sellers. Post initialization thus allows sellers to compare their individual reservation price to their peers' prices. This a seller to install a profit optimizing offer strategy, that is conditioned on the other sellers' offers. As a result, offer strategies are non-homogeneous in the level 2 post initialization phase because a newly arriving seller conditions her offer strategy on the previous sellers' strategies.

This paper studies the level 1 initializing phase. That focus allows us to make use of the sellers' homogeneous offer strategy.

In our model, the sellers' offers O_S and the buyers' reservation prices V_B have the same uniform distribution. In mathematical terms, this is represented by the equation $V_B \overset{d}{=} O_S \overset{d}{=} unif[s_1, s_2]V = unif[1-\alpha, 1+\alpha]\mathrm{V}$, with valuation imprecision $\alpha \in (0, 1)$ and some average valuation $V > 0$. Further we assume that the number of arriving buyers and offering sellers is identical. That framework allows for analytical solutions in decently sized markets.

Let us start with an analysis of the Buyers' Market's properties.

2.1 The Buyers' Market Properties

In this section we analyse major properties of the Buyers' Market. We start with a lemma which will prove worthwhile for numerous proofs throughout the paper.

Lemma 1. *Let $X_1, X_2, ..., X_n$ be iid random variables, with $X_1 \overset{d}{=} unif[x_1, x_2]$. Then the cdf of $min(X_1, X_2, ..., X_n)$ is given by $M(x) = 1 - \left(\frac{x_2 - x}{x_2 - x_1}\right)^n$. The pdf of $min(X_1, X_2, ..., X_n)$ is given by $m(x) = n\frac{(x_2-x)^{n-1}}{(x_2-x_1)^n}$.*

Proof: See the Appendix. □

If needed, the notation of the functions M and m is expanded in an intuitive way. Then we may for instance write $M(x, x_1, x_2, n)$ instead of $M(x)$. This lemma is a key ingredient for the calculation of expected prices and deal probabilities in the Buyers' Market. Before we calculate these, the next proposition introduces the cumulative distribution function of the above minimum, given the prior deal history.

Proposition 1. *Let $X_1, X_2, ..., X_n$, $Y_1, Y_2, ..., Y_i$ and $Z_1, Z_2, ..., Z_j$ be iid random variables, with $X_1 \overset{d}{=} unif[x_1, x_2]$. Conditioned on $Y_k \geq min(X_1, X_2, ..., X_n) \geq Z_l$ for all $k = 1, ..., i$ and $l = 1, ..., j$, the cdf of $min(X_1, X_2, ..., X_n)$ is*

$$F_{i,j}(x) = \frac{\sum_{k=1}^{n-1+i}\sum_{l=1}^{j}\binom{n-1+i}{k}\binom{j}{l}x^{n-1+i-k}x_1^{j-l}(-1)^{k+j-l}\frac{1}{k+l+1}(x^{k+l+1} - x_1^{k+l+1})}{\sum_{k=1}^{n-1+i}\sum_{l=1}^{j}\binom{n-1+i}{k}\binom{j}{l}x^{n-1+i-k}x_1^{j-l}(-1)^{k+j-l}\frac{1}{k+l+1}(x_2^{k+l+1} - x_1^{k+l+1})}$$

The corresponding pdf is

$$f_{i,j}(x) = \frac{(x-y)^{n-1+i}(y-x_1)^j \ dy}{\sum_{k=1}^{n-1+i}\sum_{l=1}^{j}\binom{n-1+i}{k}\binom{j}{l}x^{n-1+i-k}x_1^{j-l}(-1)^{k+j-l}\frac{1}{k+l+1}(x_2^{k+l+1} - x_1^{k+l+1})}.$$

Proof: See the Appendix. □

The above proposition calculates the minimum of the n sellers' offers. Compared to lemma 1, that proposition is conditioned on the event that prior buyers accepted and declined sellers' offers. Explicitly, in proposition 1 j buyers traded on the market and i buyers did not find a suitable offer.

Next, we analyse the probability that the $k-$th arriving buyer finds a suitable offer. Then that buyer's deal is successful. We calculate the probability of the complementary event, that is, that the $k-$th deal is unsuccessful $p_d(k)$. In other words, the number $p_d(k)$ represents the probability that all sellers' offers exceed the k-th buyer's reservation price.

We first calculate the probability $p_d(1)$ that the first deal is unsuccessful.

Proposition 2. *The probability that the first deal is unsuccessful is given by*

$$p_d(1) = \frac{1}{n+1}.$$

Proof: See the Appendix. □

The above formula shows that the probability that the first deal is unsuccessful is independent of valuation imprecision α. In fact, it is only dependent of the number of sellers n in the marketplace. When there are infinitely many sellers, then the probability that the first buyer rejects to trade is zero. When the market has the minimum possible size (i.e. $n = 1$), then trade success probability is 0.5. That is, the first buyer has probability of at least 0.5 that he finds a suitable offer. An higher number of sellers increases that probability. With an increase in the number of sellers, the first buyer's deal success probability converges to 1.

In the next propositions, we calculate the probability that the second buyer rejects trade.

Proposition 3. *The probability that the second deal is unsuccessful is given by*

$$p_d(2) = (2n+3)/((n+1)(n+2)).$$

Proof: See the Appendix. □

As in proposition 2, we observe that the deal probability is dependent on market size, whereas the imprecision's distribution does not affect it. Furthermore, for infinitely many sellers, the probability that the second buyer rejects trade is zero. That is, that buyer finds a suitable offer with probability 1.

It becomes increasingly time intensive to calculate the probability that the k-th buyer rejects trade when k increases. This can be seen from the proof of proposition 3. To calculate the probability $p_d(k)$, 2^{k-1} different integrals, each with up to 2^{k-1} integrands, need to be calculated. This can in particular be observed in the proof of proposition 3, where we calculate the probability that the third buyer rejects trade.

Proposition 4. *The probability that the third deal is unsuccessful is given by*

$$p_d(3) = \frac{3n^2 + 12n + 10}{(n+1)(n+2)(n+3)}.$$

Proof: See the Appendix. $\square$

The above proposition shows that the probability that the third arriving buyer finds a suitable offer is increasing in the number of offering sellers. That probability further converges to 1 for infinitely many sellers.

As mentioned prior to proposition 4 and as can in particular be observed in the proof of that proposition, the calculations of the deal reject probabilities is of increasing complexity. The next proposition lists the probabilities that the $k-th$ buyer rejects trade. The calculations of these formulas are omitted in this paper to save on space. Note that in order to calculate the probability that the 13th buyer rejects trade, it is necessary to calculate $2^{13-1} = 4,096$ different integrals, each with up to $4,096$ integrands. That is, approx. 16 million integrals have to be calculated. Despite the number and length of the integrals, each is (apart from time and effort) straightforward to solve. Up to this date, general formulas apart from the list below, can not be obtained with standard computers within reasonable computation time.

Proposition 5. *The probability that the* $4th$, $5th, ..., 13th$ *buyer rejects trade is*

$$p_d(4)\binom{n+4}{n}4! = 4n^3 + 30n^2 + 66n + 39$$

$$p_d(5)\binom{n+5}{n}5! = 5n^4 + 60n^3 + 245n^2 + 385n + 176$$

$$p_d(6)\binom{n+6}{n}6! = 6n^5 + 105n^4 + 680n^3 + 1,980n^2 + 2,455n + 905$$

$$p_d(7)\binom{n+7}{n}7! = 7n^6 + 168n^5 + 1,575n^4 + 7,245n^3 + 16,709n^2 + 17,213n + 5,244$$

$$\begin{aligned} p_d(8)\binom{n+8}{n}8! = {} & 8n^7 + 252n^6 + 3,220n^5 + 21,350n^4 + 77,728n^3 \\ & + 150,066n^2 + 132,664n + 34,111 \end{aligned}$$

$$\begin{aligned} p_d(9)\binom{n+9}{n}9! = {} & 9n^8 + 360n^7 + 6,006n^6 + 54,054n^5 + 283,017n^4 + 863,648n^3 \\ & + 1,444,754n^2 + 1,122,771n + 250,425 \end{aligned}$$

$$p_d(10)\binom{n+10}{n}10! = 10n^9 + 495n^8 + 10,440n^7 + 122,220n^6 + 866,287n^5 + 3,798,446n^4$$

$$+ 10,066,846n^3 + 14,955,830n^2 + 10,419,777n + 2,129,527$$

$$p_d(11)\binom{n+11}{n}11! = 11n^{10} + 660n^9 + 17,160n^8 + 253,107n^7 + 2,327,413n^6$$

$$+ 13,789,050n^5 + 52,535,054n^4 + 123,712,231n^3$$

$$+ 167,594,909n^2 + 102,992,233n + 26,263,013$$

$$p_d(12)\binom{n+12}{n}12! = 12n^{11} + 858n^{10} + 26,950n^9 + 488,567n^8 + 5,645,402n^7$$

$$+ 43,314,695n^6 + 222,582,744n^5 + 755,718,615n^4 + 1,606,727,220n^3$$

$$+ 2,047,731,583n^2 + 1,016,664,378n + 543,306,757$$

$$p_d(13)\binom{n+13}{n}13! = 13n^{12} + 1,092n^{11} + 40,755n^{10} + 890,180n^9 + 12,611,931n^8$$

$$+ 121,362,534n^7 + 806,312,138n^6 + 3,691,200,773n^5$$

$$+ 11,308,692,054n^4 + 22,562,585,539n^3$$

$$+ 24,689,032,812n^2 + 17,959,414,927n$$

Proof: A computer was programmed to automatically calculate these probabilities. □

The last four propositions calculated probabilities that the 1st, 2nd, ..., 13th buyer refuses to buy at the sellers' lowest offer. These probabilities are increasing, because a buyer who enters the market has a positive chance that the offer he receives is higher than that of previous buyers. As a result, the probability that a buyer who enters the market later refuses to trade is higher when that buyer enters the market comparably late.

When the market size increases, that is, when more sellers present their offers on the platform, then there are more offers that are comparably low. Therefore the probability that the $k-$th buyer who enters the market rejects to trade decreases when more sellers place offers for the good. That is, then the probability that a particular buyer finds a suitable offer increases with the number of offering sellers.

The next two propositions calculate average prices of successful deals.

Proposition 6. *The expected price for the first successful deal is*

$$\boldsymbol{E}(Pr_1) = s_1 + \frac{1}{n+1}\Delta s$$

Proof: See the Appendix. □

Proposition 7. *Let $k \in \mathbb{N}$ with $k \leq n$. Then the expected price of the $k-$th successful deal is*

$$\boldsymbol{E}(Pr_k) = \frac{n+1-k}{n+1}s_1 + \frac{k}{n+1}s_2$$

Proof: See the Appendix. □

Proposition 6 calculates the price of the first successful deal. That price is dependent on the lowest possible deal, the spread between the highest and the lowest offer and the number of sellers in the market. The first two criteria are dependent on valuation imprecision α. When α increases, then the lowest possible offer decreases and the spread between that offer and the best offer increases. An increase in valuation imprecision therefore reduces the first deal price. With regards to deal price, an increase in valuation imprecision therefore is a good property for the first buyer and affects a seller's expected price negatively. More sellers in the market also reduce the price of the first successful deal. The first buyer therefore profits from an increasing number of sellers in the market. In contrast, the price the seller with the lowest offer achieves, is affected negatively when more sellers are in the market place.

The influence of valuation imprecision and the number of seller on the price of the first successful deal are also true for prices of subsequent deals, as proposition 7 shows: More sellers in the market always reduce the deal price. When we calculate the price of the k-th successful deal, then proposition 7 shows that these prices are increasing in k. That is, the prices of later deals exceed the prices of deals that are closed earlier. The intuition behind the formula is, that a buyer who arrives earlier, gets better offers than a buyer that arrives later, when the best offers are gone. When $k < (n+1)/2$, then the price of the k-th deal is lower than average valuation. Then valuation imprecision influences the deal price positively. When $k > (n+1)/2$, then the price of the $k-$the deal exceeds average valuation and a higher valuation imprecision increases the price of that deal. When $k = (n+1)/2$, then the price is the average valuation and it is not influenced by valuation imprecision. In summary, prices below average valuation are reduced by increasing valuation imprecision, whereas prices that exceed average valuation, are increased by higher valuation imprecision.

Next, we calculate the expected profit of a buyer that has knowledge of the last successful deal price and the number of remaining sellers.

Proposition 8. *Assume the prior buyer's deal was successful and priced at x_k and there are m remaining offers from sellers. Then the buyer who enters the market next, has expected profit of*

$$\boldsymbol{E}(P)(x_k, m) = \frac{(s_2 - x_k)^2}{2\Delta s}\frac{m}{m+2}.$$

Proof: See the Appendix. □

Proposition 8 allows to calculate the expected profit of buyers that enter the market at a certain stage: When there are m remaining offers from sellers and the last deal was successful and priced at x_k, then the proposition introduces a formula to calculate the profit of a buyer who enters the market in this stage. That profit is dependent on the number of remaining offers m. When more offers are available, then a buyer's profit exceeds that when there are less offers. When the previous successful deal was prices at x_k, then all remaining offers exceed that price. Therefore that price influences expected profit of a subsequent buyer. That buyer's price is at least as high as x_k. When x_k is comparatively high, then a buyer's expected profit shrinks. In contrast, a lower previous price allows for a higher expected profit for subsequent buyers. When $x_k < V$ ($x_k > V$), then an increase in valuation imprecision has positive (negative) effect on a buyer's expected profit. That is, when the last successful deal was priced below average valuation, then valuation imprecision has positive effect of subsequent buyers, whereas a previous price above average has negative effect for subsequent buyers' expected profits.

When the present lowest offer is common knowledge, then a buyer's expected profit before he determines his reserve price is calculated in the next proposition.

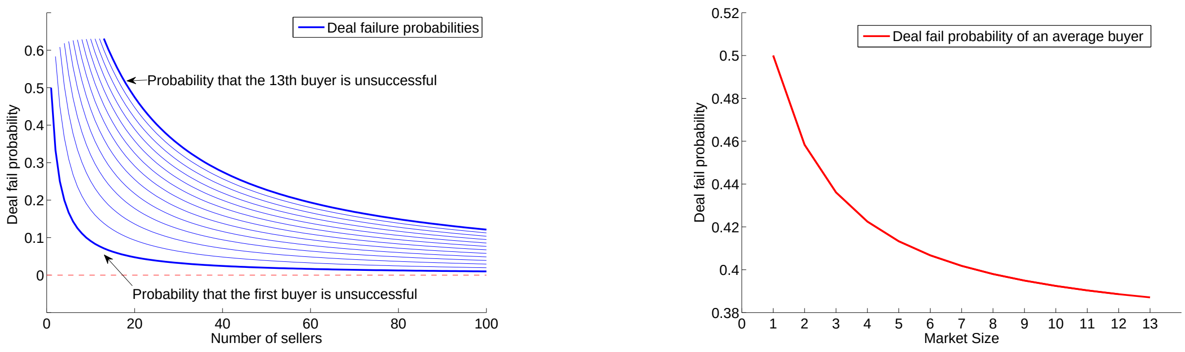
Proposition 9. *Assume the present lowest offer is x. On entering the market, a buyer's expected profit then is*

$$\boldsymbol{E}(P)(x) = \frac{(s_2 - x)^2}{2\Delta s}.$$

Proof: See the Appendix. □

The proposition allows the calculation of a buyer's expected profit before that buyer assigns a reserve price, but after he knows the present lowest offer. When the price of that offer is known to be x, then a buyer's expected profit is $\mathbf{E}(P)(x)= \frac{(s_2-x)^2}{2\Delta s}$. That is, expected profit is higher, when the present lowest offer is low. Intuitively, then a buyer has a higher chance that his reserve price exceeds the lowest offer, which increases a buyer's expected profit. When the present lowest offer $x < V$ ($x > V$), then valuation imprecision has positive (negative) effect on a buyer's expected profit. That is, when the lowest available offer is below average valuation, then valuation imprecision has positive effect on a buyer's profit, whereas an offer above average has negative effect for a buyers' expected profit.

Propositions 2 - 5 are illustrated in figure 1 (a). It calculates deal failure probabilities as a function of the number of sellers. It can be seen in the figure that the probability that the $k+1-th$ deal is unsuccessful exceeds the probability that the $k-th$ deal is unsuccessful. That is, a buyer entering the market gets offers at least as good as subsequent buyers. Furthermore, a buyer's deal is successful with positive probability, such that subsequent buyer's deal failure probability increases. When the number of sellers increases, then the price of the $k - th$ lowest offer drops, according to proposition 7. Therefore, the $k-th$ buyer has higher chances to find an offer that exceeds his valuation, when more sellers present their offers. This can



(a) Deal failure probabilities for deal 1,2,...,13

(b) Average rate of unsuccessful deals

Fig. 1 – Summary of deal failure probabilities as a function of market size

be observed in the figure, as all deal failure probabilities decrease in the number of sellers. Deal failure probabilities are drawn for the 1st, 2nd, ..., 13th buyer who enters the market. Propositions 2 - 5 presented formulas to calculate these probabilities.

Figure 1 (b) shows the average probability that a deal is unsuccessful as a function of market size (i.e. the fraction of unsuccessful deals in a market of size three would be given by $(p_d(1)+p_d(2)+p_d(3))/3$). It can be seen that this probability is a strictly decreasing function. Average deal failure probability is 0.5 for market size 1 (i.e. when there is one buyer and one seller present) and reaches down to 0.3871 for market size 13. However, it is not clear whether average deal failure probability converges to zero or some value greater zero for big markets. A convergence to zero would imply that on average each deal is successful, when the number of market participants approaches infinity. Then each individual would profit from entering the market with probability 1. When the average deal failure probability converges to some value v greater than zero, then on average, each $v-th$ deal is unsuccessful.

The average rate of unsuccessful deals, in a market with n buyers and n sellers is calculated by the term $(p_d(1)+p_d(2)+...+p_d(n))/n$. therefore it is necessary to calculate all probabilities $p_d(1), p_d(2), ..., p_d(n)$. As discussed in proposition 4, these calculations are time sensitive for big markets. It is therefore wise to shift to a quicker technique to analyse the behaviour of the rate of unsuccessful deals for big markets.

We ran Monte Carlo simulations for markets of size greater 13. Due to the Law of Large Numbers, the average of these simulations converges to the actual ratio of unsuccessful deals. Each simulation represents the average of 100.000 sample markets for each market size and therefore consists of a decent sample size such that the simulation's results are sufficiently close to the actual rate of unsuccessful deals.

The red line in figure 2 is the analytic solution for the ratio of unsuccessful deals, as shown in figure 1 (b). Its derivation and properties were discussed thoroughly. The blue lines show

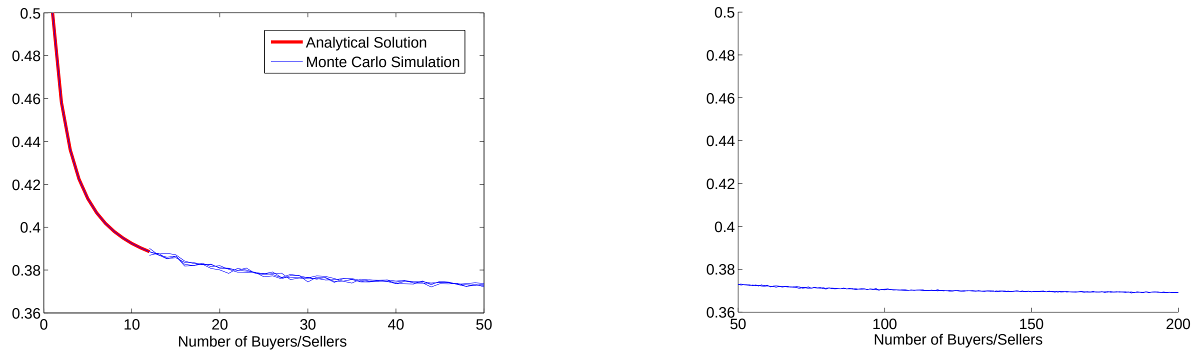


Fig. 2 – Rate of unsuccessful deals as a function of market size

Monte Carlo simulations of the average rate of unsuccessful deals for markets that have a size that exceed 13. When we combine the analytical calculations with the Monte Carlo solutions, it can be summarized that the rate of unsuccessful deals is a strictly decreasing function in the market size that converges to a value greater 0.36.

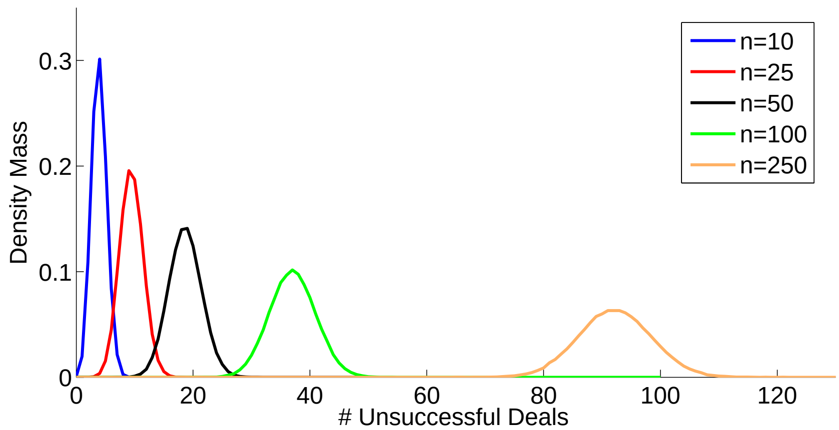


Fig. 3 – Simulated distribution of unsuccessful deals for different market sizes

Figure 3 analyses the simulation of deal failures in more detail. It shows the distribution of unsuccessful deals in markets with sizes 5, 10, 25, 50, 100 and 250. It can be seen that distributions are approximately symmetrically distributed. Furthermore with increasing market size, deal failures seem to be close to a normal distribution.

In order to compare the distributions for different market sizes, we normalized the distributions of unsuccessful deals in figure 4. On the $x-$axis deal failure rates are drawn. It can be seen that the mean values of deal failures are strictly decreasing in market size. That is in accordance with figure 2, that illustrated that deal failure rates are decreasing in market size.

A second property, that can be seen from figure 4, is a strict decrease in variance with

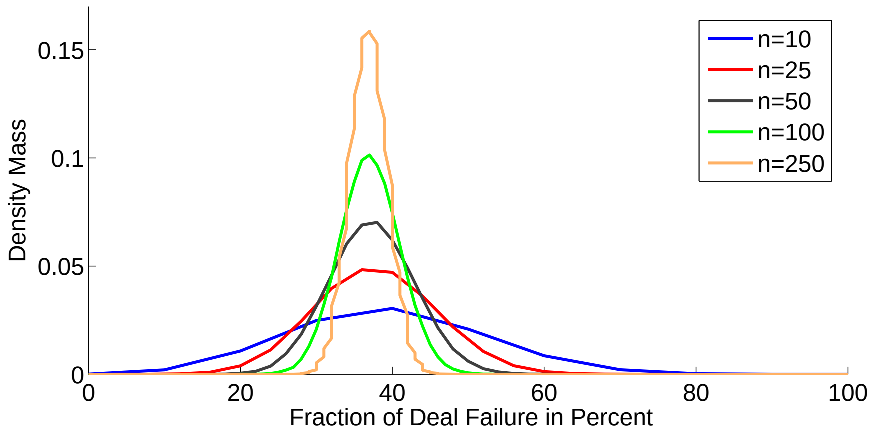


Fig. 4 – Simulated distribution of unsuccessful deals as a fraction of market size

Market Size	10	25	50	100	250
Mean of deal failure	39.2834%	37.7727%	37.3151%	37.0565%	36.8897%
Variance of deal failure	1.6532%	0.6381%	0.3149%	0.1586%	0.0625%

Table 1 – Key properties of the simulated deal failure distribution

increasing market size. This statement is supported by the intuition that in markets with size 10, there are some samples where 0 or 10 deal successes occur. When there are more market participants, then there are fewer samples where such extremes can be observed. Observations tend to be gathered around the average value of deal failures in bigger markets. This fact suggests that average deal failure rate does not only converge, it moreover is increasingly likely that this rate is actually observed in an arbitrary sample of a Buyers' Market.

Market Size	1	2	3	4	5	6
Average deal failure (in %)	50	45.83	43.61	42.25	41.33	40.67

Market Size	7	8	9	10	11	12	13
Average deal failure (in %)	40.18	39.80	39.49	39.24	39.04	38.86	38.71

Table 2 – Average deal failure probability

Decreasing variances in deal failure probabilities is supported by the calculation of deal failure variances for different market sizes, which are noted in table 1. The table shows that deal failure variance decreases with increasing market size. Furthermore this decrease is a reciprocal function in market size. That means for instance, that doubling market size has the effect that variance decreases by the factor 0.5. Despite convergence to a certain deal failure rate above 36%, we conclude that variance shrinks reciprocal in market size. Therefore the probability that a Buyers' Market in fact possesses the average deal failure rate converges to 1 for big markets. In summary, figure 4 shows that (although number of simulations per market size is constantly 100.000) the Monte Carlo simulations are closer together, the bigger the market. This simulation's behaviour also is explained by shrinking variance for bigger markets.

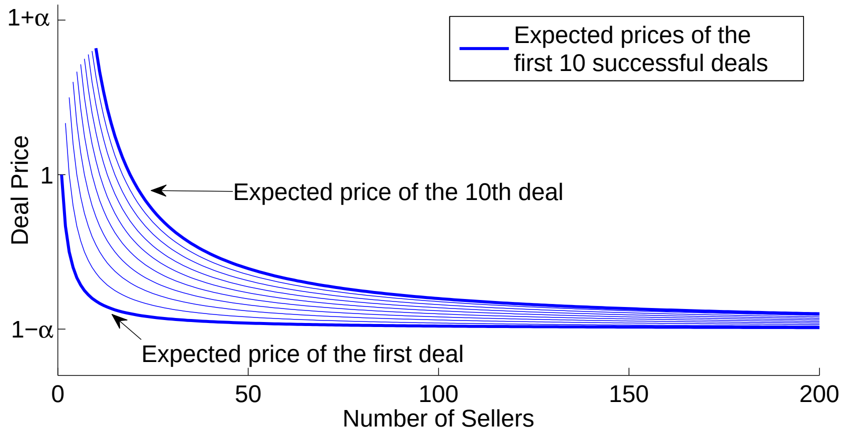


Fig. 5 – Average prices of the first 10 deals as a function of market size.

The average price of the $k-$th deal was calculated in proposition 7. Figure 5 illustrates the behaviour of the prices of the first 10 deals as a function of the number of sellers. When there are more sellers in the market, then there are more offers. These offers are uniformly distributed on the interval $[1-\alpha, 1+\alpha]V$. Thus, when there are more sellers, then there are more comparatively low offers. As a result, deal prices diminish. This effect can be observed in figure 5. On the $x-$axis the number of sellers is drawn and the $y-$axis shows the expected deal price[3]. The figure shows that the deal price of the 1st, 2nd, ..., 10th deal is decreasing in the number of sellers. In the limit of infinitely many sellers, the price of each deal converges to $1-\alpha$, which is the lowest possible offer.

The next two paragraphs will discuss expected profits of the market participants. Let us start with the expected profit of a seller.

2.2 The Sellers' Profit

The profit of a seller is given by her offer if the seller's offer leads to a successful deal. Otherwise the seller's profit is zero. A seller has no benchmark to compare her offer to.

[3]Without loss of abstraction, we set $V = 1$ in that figure.

Therefore she is not aware whether her offer is above or below average and how it compares to other sellers' offers.

A seller's expected profit is given by the product of that seller's probability that the deal is successful and her reservation price. To determine deal probability, a seller assumes that she is an average seller with an average offer, as she has no benchmark to compare her offer to. That is, her deal probability is the average probability of the seller with the lowest, second lowest, third lowest, ..., n-th lowest offer.

We sort the sellers such that the first seller's offer S_1 is the lowest offer and the offer of the k-th seller is the k-th lowest offer. In order to calculate the profit of an average seller, the expected profit of the k-th seller for $k = 1, ..., n$ needs to be calculated.

We start with the calculation of the expected profit of the seller with the lowest offer as a function of the number of buyers. Without loss of abstraction, we set average valuation $V = 1$ in the following sections.

Proposition 10. *The expected profit of the seller with the lowest offer is*

$$\boldsymbol{E}(P_{S_1}) = 1 - \frac{n+2+n\alpha}{(n+1)(n+2)}.$$

Proof: See the Appendix. □

Proposition 10 shows that the profit of the seller with the lowest offer depends on the number of buyers and the valuation imprecision α. That profit is decreasing in α. That is, a higher valuation imprecision has negative effect on the first seller's profit. More buyers are beneficial for the seller. When there are more buyers, then the probability that one buyer's reservation price exceeds the seller's offer increases. As a consequence the probability of a successful deal for the seller and her expected profit increase. For infinitely many buyers, her expected profit converges to 1 (i.e. V).

The next two propositions calculate the expected profits of the sellers with the second and third lowest offers.

Proposition 11. *The expected profit of the seller with the second lowest offer is*

$$\begin{aligned}\boldsymbol{E}(P_{S_2}) = &\sum_{k=1}^{n-1} \frac{1}{k^2(k+1)} \left(s_2 - \frac{2\Delta s}{k+2}\right) - \sum_{k=1}^{n-1} \frac{1}{kn(n+1)} \left(s_2 - \frac{2\Delta s}{n+2}\right) \\ &+ \sum_{k=1}^{n-1} \frac{1}{k(k+1)^2} \left(s_2 - \frac{\Delta s}{k+2}\right) - \sum_{k=1}^{n-1} \frac{1}{k(k+1)(n+1)} \left(s_2 - \frac{\Delta s}{n+2}\right).\end{aligned}$$

Proof: See the Appendix. □

Proposition 12. *The expected profit of the seller with the third lowest offer is*

$$
\begin{aligned}
&\boldsymbol{E}(P_{S_3}) \\
&= \sum_{k=1}^{n-2} \sum_{l=k+1}^{n-1} \frac{2}{l-k} \left(\frac{1}{(l-1)l(l+1)} \left[2s_2 - \frac{6\Delta s}{l+2} \right] \left(\frac{1}{k} - \frac{1}{l} \right) \right. \\
&+ \frac{1}{l(l+1)} \left[s_2 - \frac{2\Delta s}{l+2} \right] \left(\frac{1}{k(k+1)} - \frac{2}{l(l+1)} + \frac{1}{k(l-k+1)} - \frac{1}{(l+1)(l-k+1)} \right) \\
&+ \left. \frac{1}{l+1} \left[s_2 - \frac{\Delta s}{l+2} \right] \left(\frac{1}{k(k+1)(l-k+1)} - \frac{2}{l(l+1)(l+2)} - \frac{1}{(l+1)(l+2)(l-k+1)} \right) \right) \\
&- \sum_{k=1}^{n-2} \sum_{l=k+1}^{n-1} \frac{2}{l-k} \left(\frac{1}{(n-1)n(n+1)} \left[2s_2 - \frac{6\Delta s}{n+2} \right] \left(\frac{1}{k} - \frac{1}{l} \right) \right. \\
&+ \frac{1}{n(n+1)} \left[s_2 - \frac{2\Delta s}{n+2} \right] \left(\frac{1}{k(k+1)} - \frac{2}{l(l+1)} + \frac{1}{k(l-k+1)} - \frac{1}{(l+1)(l-k+1)} \right) \\
&+ \left. \frac{1}{n+1} \left[s_2 - \frac{\Delta s}{n+2} \right] \left(\frac{1}{k(k+1)(l-k+1)} - \frac{2}{l(l+1)(l+2)} - \frac{1}{(l+1)(l+2)(l-k+1)} \right) \right).
\end{aligned}
$$

Proof: See the Appendix. □

The above two propositions calculate formulas for the profit of the sellers with the second and third lowest offers. These profits are increasing in the number of buyers and decreasing in valuation imprecision. That is, more buyers increase the probability that a buyer's offer exceeds these sellers' offers. The increased deal probability has positive effect on a seller's profit. Both sellers have expected profit of 1 for $n \to \infty$ buyers. That is, in the limit each seller's offer is successful eventually.

Figure 6 (a) shows the expected profits of the first three sellers as a function of the number of buyers, as calculated in propositions 10 - 12. The $y-$axis shows their expected profits and the number of buyers is drawn on the $x-$axis. For this example, we used a valuation imprecision $\alpha = 10\%$[4].

The figure shows that each seller's expected profit is increasing in the number of buyers. The profit of the seller with the lowest offer exceeds that of the seller with the second lowest offer and her profit exceeds that of the seller with the third lowest offer. This is intuitive because the second offer can be successful only after the lowest offer is sold. The same relation holds between the second and the third lowest offer.

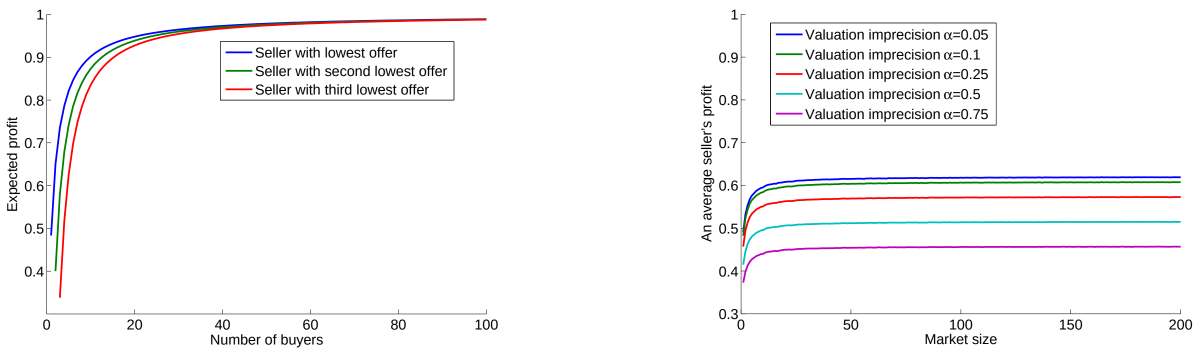
Figure 6 (a) further shows that the seller's expected profits converge to 1 for the number of buyers $n \to \infty$. That is, for a high number of buyers, each seller's profit approaches 1 up to an arbitrarily small distance.

[4]Note that the sellers' expected profit are negatively affected by increasing valuation imprecision. However, this effect is comparably small in contrast to the effect that the market size has on the sellers' expected profits.

In this section we calculated expected profits of the first sellers with the first three lowest offers. To calculate the expected profit of an average seller in a market with size n, the formulas for the profits of all sellers need to be calculated.

The formulas the of the profits for the first, second and thirds sellers, as seen in propositions 10 - 12, develop increasingly long terms. Their length further increases exponentially for the expected profit of the seller with the $k-$th lowest offer, with $3 < k \leq n$. Thus it is not efficient to calculate these profits analytically. A more efficient method is a Monte Carlo simulation to calculate all sellers' profits and an average seller's profit.

In our Monte Carlo simulation we ran 50,000 samples per market size $n = 1, ..., 200$ and used valuation imprecision $\alpha \in \{0.05, 0.1, 0.25, 0.5, 0.75\}$. That is 50 million samples, which allows for a sufficiently high significance of the simulation.



(a) Profit of the sellers with the 3 lowest offers (b) Simulated average profit of a seller

Fig. 6 – Sellers' profits as a function of market size

Figure 6 (b) shows the Monte Carlo simulation's results. The $x-$axis shows the market size and on the $y-$axis an average seller's profit is drawn. The average profit increases for each valuation imprecision and they are all bounded. Valuation imprecision has negative effect on an average seller's expected profit.

Valuation imprecision α	0.05	0.1	0.25	0.5	0.75
Average seller's profit ($n = 1$)	0.49	0.48	0.46	0.41	0.37
Average seller's profit ($n = 200$)	0.62	0.61	0.57	0.51	0.46

Table 3 – An average seller's profit for small and big markets as a function of valuation imprecision

The effect of the market size and valuation imprecision on an average sellers profit are summarized in table 3. It can be seen that bigger markets increase a seller's profit for each valuation imprecision. Independent of the market size, valuation imprecision influences a seller's expected profit negatively.

The next section calculates the profit of a buyer.

2.3 The Buyers' Profit

In this section, a buyer's profit is analysed. When the first buyer enters the market, then that buyer can buy at the lowest price of all sellers' offers. A buyer that enters the market in a later stage therefore has a lower range of available offers and the lowest offer is at least as high as the offer for the first buyer. When a buyer enters the market in a later stage, then that buyer's expected profit is lower than that of a prior buyer.

We calculate the expected profit of the first, second and third buyers in this section. Expected profits of subsequent buyers will be calculated with a Monte Carlo simulation. Let us start with the profit of the first buyer.

Proposition 13. *The expected profit of the first buyer as a function of the number of sellers n is*

$$\boldsymbol{E}(P_{B_1}) = \alpha \frac{n}{n+2}.$$

Proof: See the Appendix. □

Proposition 13 calculates the expected profit of the first buyer that enters the market. It is dependent on the valuation imprecision and the number of available offers from sellers. Both parameters have a positive influence on the first buyer's profit.

When valuation imprecision increases, then expected profit also rises. In fact, the first buyer's expected profit is linear as a function of valuation imprecision α. When there is no valuation imprecision, then the first buyer's expected profit is zero. Expected profit of the first buyer exceeds that of subsequent buyers. When there is no valuation imprecision, then all buyers therefore have zero profit in the Buyers' Market.

The number of sellers n also has positive effect on the first buyer's expected profit. Intuitively, more sellers make more offers the buyer can choose from. This increases the chance of particularly good offers for the buyer and his profit increases.

In the limit of infinitely many sellers, the first buyer's expected profit is the valuation imprecision α. That buyer's expected profit therefore is bounded by valuation imprecision.

The next 2 propositions calculate the expected profits of the second and third buyers.

Proposition 14. *The expected profit of the second buyer as a function of the number of* *sellers n is*

$$\boldsymbol{E}(P_{B_2}) = \alpha \frac{n^3 + 2n^2 - n}{(n+3)(n+1)(n+1)}.$$

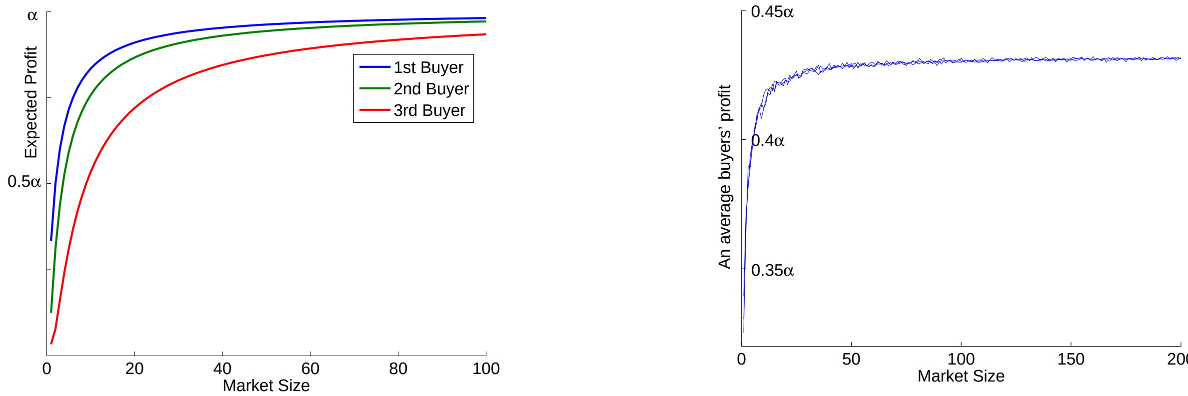
Proof: See the Appendix. □

Proposition 15. *The expected profit of the third buyer as a function of the number of sellers n is*

$$\boldsymbol{E}(P_{B_3}) = \alpha \frac{n^5 + 4n^4 + 2n^3 - 12n^2 + 7n + 6}{(n+1)^2(n+2)(n+3)(n+4)}.$$

Proof: See the Appendix. □

The expected profit of the second and third buyers are both increasing in valuation imprecision α and the number of sellers n. The expected profit of the second buyer exceeds that of the third buyer because the probability that the second buyer accepts the lowest offer is greater zero. Then the third buyer is left with offers that are higher than those of the second buyer. Both expected profits converge to α for $n \to \infty$. That is, in the limit it is insignificant when a buyer enters the market. Expected profit further is a linear function in valuation imprecision. That is, an increase in imprecision has positive effect on the buyers' expected profits. Additionally, doubling valuation imprecision has the effect that expected profit also is doubled.



(a) Profit of the first 3 buyers (b) Simulated average profit of a buyer

Fig. 7 – Buyers' profits as a function of market size

Figure 7 (a) shows the expected profits of the first three buyers. On the x-axis the number of sellers are drawn. The y-axis illustrates expected profits. The first buyers' profit exceeds the second buyer's profit, which exceeds the third buyer's profit. Buyers that arrive later thus have lower expected profit. This is true for any number of sellers. When the market size increases, then each buyer's profit converges to α. The maximum valuation imprecision therefore is an upper bound for a buyer's expected profit, independent of the time of a buyer's market entry.

The calculations of the second and third buyers' expected profits in propositions 14 and 15 developed progressively extensive terms. It is therefore more productive to shift to quicker techniques for the analysis of the k-th buyers' expected profit, with $k > 3$. We performed Monte Carlo simulations to analyse expected profits for buyers $4, 5, ...$ up to the number of sellers n. An average buyer is not aware whether he enters the market comparatively early or late. Therefore the expected profit of the average buyer is the mean profit of the first, second, third, ..., $n-$th buyer.

That simulated profit of an average buyer is illustrated in figure 7 (b). The x-axis shows the number of buyers and sellers. The simulated expected average profit of a buyer is drawn on the y-axis. That profit is increasing in the market size. A greater market size implies that there are more sellers. That is, there are more comparatively low offers. On the other hand, there are more buyers in a greater market. Buyers who arrive late might get high remaining offers. The effect of more low offers is stronger as can be observed in the figure; the average buyer benefits from a bigger market size. An average buyer's profit furthermore converges to approximately 0.43α. That is, when the market becomes infinitely large, then an average buyer's profit is approximately 0.43α.

An average buyer's profit therefore is a linear function in valuation imprecision α. When α is increased by a certain factor, then average profit also is increases by that exact factor. Doubling imprecision thus doubles a buyer's expected profit. Therefore valuation imprecision is beneficial for buyer's and higher valuation imprecision leads to higher expected profits. When there is no imprecision, that is, when each party values the good precisely, then a buyer's expected profit from trade is zero.

Let us next introduce a dealer and analyse his pricing strategy.

3 The Dealer's Market

On Dealer's Market, there is a dealer present. The dealer has past experience regarding the good and therefore knows its average value. He therefore has precise valuation. He acts as market maker and charges a bid-ask spread: The dealer offers to sell the good at a buyer's reservation price multiplied by $1 - f_B$ and offers to buy the good at a seller's offer multiplied by $1 - f_S$.

Proposition 16. *When $f_S > f_B$, then the dealer profits on average from his pricing strategy.*

Proof: See the Appendix. □

Buyers and sellers each get different prices, dependent on their price expectations. The dealer thus needs to be experienced in order to find out the player's true price expectations. Otherwise the bargainers may lie to the dealer about their true price expectations.

The dealer pursues the strategy to install an environment under information asymmetry. On Dealer's Market buyers and sellers do not interact. They solely communicate with the dealer and choose whether to accept his offer, or not. Otherwise they might complain about the dealer's pricing strategy, because each player gets a different price, dependent on a player's valuation. In this sense there is information asymmetry on Dealer's Market. Asymmetric information is important to the success of dealer's strategy. This means that buyer and seller should either consult the dealer or choose trade on the Buyers' Market. Otherwise buyer and seller first bargain on the Buyers' Market. In case they are unsuccessful, they may consult the dealer in the next step. While this sequential strategy is beneficial for buyer and seller, the dealer is left with a lemons problem: Buyers with low reservation price and sellers with high reservation price. Thus the dealer suffers from adverse selection. Installing a beneficial fee strategy consequently becomes more complicated under full information as dealer's strategy may collapse otherwise.

The next propositions calculate a buyer's and seller's expected profit in Dealer's Market. Let us start with a buyer's profit.

Proposition 17. *When $f_B > 0$, then a buyer's expected profit in the Dealer's Market is positive. Then it is given by $\boldsymbol{E}(P_B) = f_B$*

Proof: See the Appendix. □

Proposition 17 shows that the buyer has a positive expected profit from participating in Dealer's Market if the condition $f_B > 0$ is met. The higher the dealer chooses the discount f_B, the more attractive Dealer's Market becomes to the buyer.

We continue with the calculation of a seller's profit.

Proposition 18. *When $f_S < 1$, then a seller's expected profit in the Dealer's Market is positive. When that condition is met, a seller's profit is*

$$\boldsymbol{E}(P_S) = 1 - f_S.$$

Proof: See the Appendix. □

The above proposition shows that dealer's pricing strategy f_S influences a seller's expected profit in Dealer's Market. That pricing strategy affects a seller's expected profit negatively. That is, a higher discount f_S reduces a seller's profit, whereas a lower discount increase that profit.

In summary, buyers profit from high dealer's discounts, whereas sellers prefer lower discounts.

In the next section, we analyse the relative attractiveness of the Buyers' Market and the Dealer's Market.

4 The Downside of Full Information

This section compares the attractiveness of the Buyers' and the Dealer's Market. Conditions that allow the Dealer's Market to Pareto dominate the Buyers' Market will be established. We start with a buyer's market preferences.

Proposition 19. *When a buyer has a choice between a profit of x and participation in the Dealer's Market, then the buyer prefers the Dealer's Market, when $f_B > x$.*

Proof: See the Appendix. □

The above proposition calculates a criterion that allows the Dealer's Market to Pareto dominate an opportunity, where a buyer gets a fixed profit x. That is, when the dealer's discount f_B exceeds a buyer' alternative, then the buyer prefers the Dealer's Market over that alternative.

A buyer's preference of the Dealer's Market over the Buyers' Market is analysed in the next proposition.

Proposition 20. *Independent of market size, a buyer's expected profit in the Dealer's Market exceeds expected profit on the Buyers' Market, when $f_B > 0.43\alpha$.*

Proof: See the Appendix. □

Proposition 20 shows that, when $f_B > 0.43\alpha$, then the Dealer's Market is more preferable than the Buyers' Market for a buyer. That is, when the dealer's discount on a buyer's valuation f_B is sufficiently high, then a buyer is in preference for the dealer's offer compared to the Buyers' Market.

The necessary dealer's discount increases with a higher valuation imprecision. When the buyer values the good less precise, then the dealer needs to reduce his offer in order to attract that buyer to his market.

The inequation $f_B > 0.43\alpha$ holds for any market size because it considers the limit for market size $n \to \infty$. For a smaller market size, f_B may be below that lower bound. That is, then the dealer's discount may be smaller. The exact lower bound for dealer's discount can be seen in figure 7 (b). For markets of size 1, the lower bound is $f_B > \alpha/3$, according to proposition 13. The lower bound for the dealer's discount is therefore increasing in the market size. This fact implies that a dealer needs to give higher discounts when more buyers and sellers are present.

Next, we focus on a seller's preferences.

Proposition 21. *When a seller has a choice between a profit of profit x and participation in the Dealer's Market, then the seller prefers the Dealer's Market, when $f_S < 1 - x$.*

Proof: See the Appendix. □

When a seller can choose between a profit of x on a market and selling the good in the Dealer's Market, then a seller prefers the latter market, when $f_S < 1 - x$ holds. That is, the dealer's discount on a seller's offer must not exceed $1 - x$. When the dealer's discount is to high, then the seller may not be satisfied with that offer and thus prefer the market with profit x.

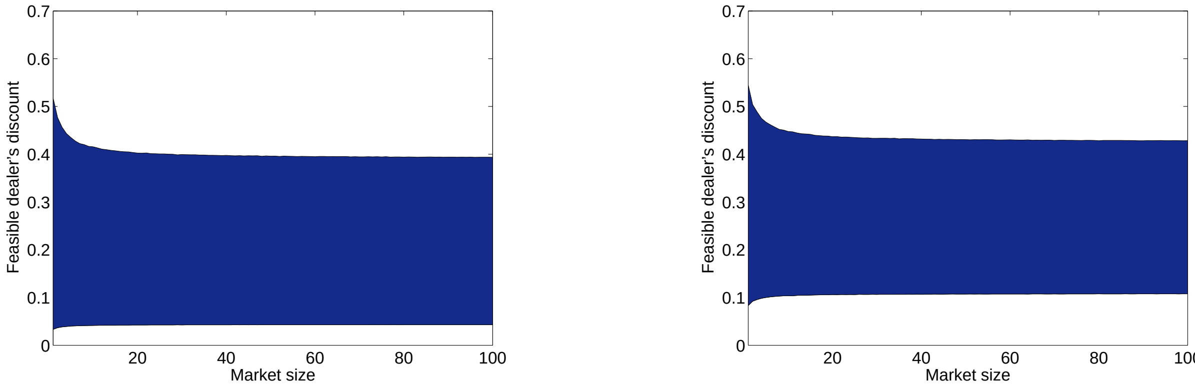
Section 2.2 analysed a seller's profit in the Buyers' Market as a function of valuation imprecision α and market size n. That profit is illustrated in figure 6 (b) and table 3. When we define that profit to be x, then proposition 21 allows us to calculate, when a seller prefers the Dealer's Market over the Buyers' Market.

Valuation imprecision α	0.05	0.1	0.25	0.5	0.75
Lower bound for $f_{B,min}$	0.022	0.043	0.108	0.215	0.323
Upper bound for $f_{S,max}$	0.38	0.39	0.43	0.49	0.54

Table 4 – Bounds for dealer's fee such that Dealer's Market is Pareto efficient

Table 4 shows the maximum (minimum) dealer's discount such a seller (buyer) is in preference of the Dealer's Market. It can be seen that higher valuation imprecision leads to a higher dealer's discount that allows him to attract market participants to trade in the Dealer's Market. Independent of valuation imprecision and market size, the discount for a buyer may be lower than a seller's discount. Therefore the dealer can choose buyer's and seller's discounts such that both players are in favour for his market. At the same time, the dealer furthermore generates positive expected earnings.

Feasible dealer's discounts are illustrated in figure 8. In this figure, the x-axis shows the market size. The dealer's discount is illustrated on the y-axis. The blue area shows feasible dealer's discount strategies. The lower bound in that area represents the lower bound for the discount a buyer demands. The discount for a buyer's offer therefore needs to exceed that bound. The upper bound of the blue area represents the maximum discount a seller allows. The discount that a dealer applies to a seller's offer thus must be lower than that bound. For each market size, the dealer may choose discounts for a buyer f_B and the seller f_S. When the



(a) Valuation imprecision $\alpha = 10\%$

(b) Valuation imprecision $\alpha = 25\%$

Fig. 8 – Simulated feasible dealer's discounts for Pareto dominance of the Dealer's Market over the Buyers' Market

dealer pays attention to the condition $f_B < f_S$, then his expected profit is positive according to proposition 16. In the figure, the blue area is non-empty. Therefore the dealer may choose discounts that attract buyer's and seller's to the Dealer's Market and generate a positive gain at the same time.

Figure 8 (a) draws feasible discounts for valuation imprecision $\alpha = 10\%$. When the market is smaller, then the dealer has a greater variety of discount strategies. In fact, for market size 1, he may choose $0.033 < f_B < f_S < 0.517$. When the market size is arbitrarily big, then the dealer's discount strategy restricts to approximately $0.043 < f_B < f_S < 0.39$.

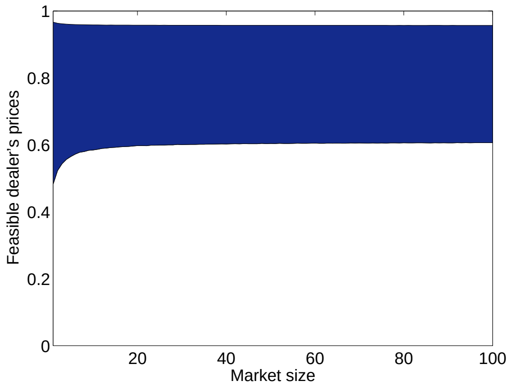
Figure 8 (b) draws feasible discounts for valuation imprecision $\alpha = 25\%$. When the market is smaller, then the dealer has a greater variety of discount strategies. In fact, for market size 1, he may choose $0.083 < f_B < f_S < 0.542$. When the market size is arbitrarily big, then the dealer's discount strategy restricts to approximately $0.108 < f_B < f_S < 0.54$.

When valuation imprecision increases, then the dealer may choose more extreme discounts, as can be seen from this example.

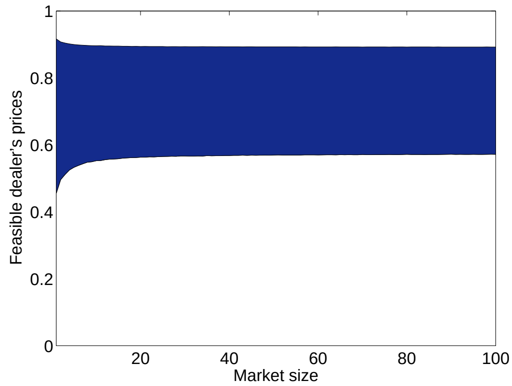
From feasible dealer's discounts, feasible deal prices can be calculated. These are illustrated in figure 9. In that figure, the x-axis shows the market size. The dealer's feasible prices are illustrated on the y-axis. The blue area shows feasible deal prices as a function of the Buyers' Market size. That is, while prices are within the blue area, buyers and sellers prefer the Dealer's Market over the Buyers' Market.

The lower bound in that blue area represents the highest discount that a seller accepts. When prices are lower, then sellers are not in favour for the Dealer's Market. While prices exceed that lower bound, sellers favour the Dealer's Market over the Buyers' Market.

The upper bound of the blue area represents the maximum price that a buyer is willing



(a) Valuation imprecision $\alpha = 10\%$



(b) Valuation imprecision $\alpha = 25\%$

Fig. 9 – Simulated feasible deal prices for Pareto dominance of the Dealer's Market over the Buyers' Market

to accept. When the dealer's offer exceeds that bound, then buyers are in favour for the Buyers' Market. In order to attract buyers, the dealer offers the good for a price within the blue area. Then buyers accept and the dealer's offer dominates their expected profit in the Buyers' Market.

When the dealer discounts a seller's offer higher than that of a buyer, then the dealer expects a positive gain from his strategy.

Figure 9 (a) draws feasible discounts for valuation imprecision $\alpha = 10\%$. When the market is smaller, then the dealer has a greater set of possible discount strategies. In fact, for market size 1, he may choose discounts on the interval $[0.483, 0.967]$. When the market size is arbitrarily big, then the dealer's discount strategy restricts to approximately $[0.61, 0.957]$.

Figure 8 (b) draws feasible discounts for valuation imprecision $\alpha = 25\%$. When the market is smaller, then the dealer has a greater set of discount strategies. In fact, for market size 1, he may choose discounts on the interval $[0.458, 0.917]$. When the market size is arbitrarily big, then the dealer's discount strategy restricts to approximately $[0.46, 0.892]$.

When valuation imprecision increases, then dealer's discounts may increase, as can be seen from this example.

5 Conclusion

This paper presented a detailed analysis of a realistic auction market, the Buyers' Market. In this market, sellers reveal their offer prices for a good and the buyers arrive one after another. Each buyer buys the good if the present lowest offer does not exceed that buyer's reservation price. Buyers and sellers therefore have full price information on this market. The Amazon market platform and buy-it-now auctions on eBay are examples, where such a market design is installed.

The deal success rate in the Buyers' Market increases with market size. That is, when there are more buyers and sellers, then a higher fraction of deals is successful. In minimal markets with one buyer and one seller, every second deal is unsuccessful. The success rate however converges to 0.64 for big markets. The variance of the success rate decreases in the market size. That is, if the market size increases, it becomes more certain that a particular market actually possesses the deal success rate of 0.64. Therefore, we conclude that the platform becomes more attractive, when more individuals use it for their transactions.

Furthermore, it was shown that a greater number of sellers generate a higher profit for each buyer. This follows from the fact, that when there are more sellers, the price of the lowest offer decreases. This reduction of the lowest offer represents more profit for a buyer.

In case of one buyer and one seller, the average buyer's profit is one third of the valuation imprecision (i.e. $\frac{1}{3}\alpha$). That profit converges up to 0.43α for big markets. Thus, an average buyer's profit is linearly dependent on the valuation imprecision α. Therefore, a doubling of the valuation imprecision doubles a buyer's profit. A buyer therefore profits from valuation imprecision and has a profit of zero when the traders value the good precisely.

When there is an equal number of buyers and sellers, then an average seller's profit is increasing in the number of traders. For a comparably low valuation imprecision of 5%, an increase in the market size from 1 to 200 increases a seller's profit from 0.49 to 0.62 (as a proportion of the average valuation). A comparably high valuation imprecision of 75% generates a seller's profit of 0.37 for markets with only one buyer and one seller. This profit increases up to 0.46 for big markets with a size of 200 buyers and sellers. In summary, an average seller's profit is increasing in the market size and decreasing in valuation imprecision.

In the Dealer's Market, a dealer buys the good from a seller and then sells it to a buyer. He offers to buy the good at a discount from the seller and offers it to the buyer at a price that is lower than that buyer's reservation price. It is critical that the dealer hides his price quotes to a trader from other traders, as each trader gets a different price offer. That is, the dealer needs to maintain information asymmetry between all buyers and sellers.

The dealer can set his pricing strategy in a way that the buyer and the seller expect positive profit from the dealer's offer. In addition, his strategy allows him a positive gain on average. Therefore the dealer may set pricing strategies such that all parties profit from participation in the Dealer's Market.

A buyer's and a seller's profit in the Dealer's Market can exceed their expected profit on the Buyers' Market. In other words, when the dealer sets his discount strategy adequately, then buyers and sellers prefer the Dealer's Market over the Buyers' Market. These discounts can be set non-restrictively for any valuation imprecision and market size. At the same time, his discount strategy allows the dealer a positive gain. Then the Dealer's Market Pareto dominates the Buyers' Market. That is, the market under information asymmetry Pareto dominates the market under full price information.

Our research suggests that the above results hold for further generalisations of the Buyers' and Dealer's Market. Simulations show that valuation imprecision may be normally distributed (instead of uniformly) with the same implications on the deal success rate. We believe that symmetrically distributed valuation imprecision is a sufficient condition for most of the established characteristics of the Buyers' Market. Furthermore, the properties of the Buyers' Market with an unequal number of buyers and sellers may be an interesting field for further research. In this case, the dealer's inventory additionally demands attention. Although some of these topics have been touched in this paper, a detailed study of the mentioned market abstractions may be worthwhile.

6 Appendix

Proof of Lemma 1: The cdf of $min(X_1, X_2, ..., X_n)$ is given by

$$\begin{aligned} M(x) &= \mathbf{P}\left(min(X_1, X_2, ..., X_n) \leq x\right) = 1 - \mathbf{P}\left(min(X_1, X_2, ..., X_n) > x\right) \\ &= 1 - \mathbf{P}\left(X_i > x, i = 1, ..., n\right) = 1 - \mathbf{P}\left(X_1 > x\right)^n = 1 - \left(\frac{x_2 - x}{x_2 - x_1}\right)^n. \end{aligned}$$

The pdf is given by the first derivative of $M(x)$. □

Proof of Proposition 1: We calculate

$$\begin{aligned} F_{i,j}(x) &= \frac{\mathbf{P}\left(min(X_1, X_2, ..., X_n) < x | Y_k \geq min(X_1, X_2, ..., X_n) \geq Z_l \forall k = 1, ..., i \ l = 1, ..., j\right)}{\mathbf{P}\left(Y_k \geq min(X_1, X_2, ..., X_n) \geq Z_l \forall k = 1, ..., i \ l = 1, ..., j\right)} \\ &= \frac{\int_{x_1}^{x} m(y) \mathbf{P}\left(Y > y\right)^i \mathbf{P}\left(Y < y\right)^j \ dy}{\int_{x_1}^{x_2} m(y) \mathbf{P}\left(Y > y\right)^i \mathbf{P}\left(Y < y\right)^j \ dy} \\ &= \frac{\frac{n}{(x-x_1)^n} \int_{x_1}^{x} (x-y)^{n-1} (x-y)^i (y-x_1)^j \ dy}{\frac{n}{(x-x_1)^n} \int_{x_1}^{x_2} (x-y)^{n-1} (x-y)^i (y-x_1)^j \ dy} \\ &= \frac{\int_{x_1}^{x} (x-y)^{n-1+i} (y-x_1)^j \ dy}{\int_{x_1}^{x_2} (x-y)^{n-1+i} (y-x_1)^j \ dy} \\ &= \frac{\sum_{k=1}^{n-1+i} \sum_{l=1}^{j} \binom{n-1+i}{k} \binom{j}{l} x^{n-1+i-k} x_1^{j-l} (-1)^{k+j-l} \int_{x_1}^{x} y^{k+l} \ dy}{\sum_{k=1}^{n-1+i} \sum_{l=1}^{j} \binom{n-1+i}{k} \binom{j}{l} x^{n-1+i-k} x_1^{j-l} (-1)^{k+j-l} \int_{x_1}^{x_2} y^{k+l} \ dy} \\ &= \frac{\sum_{k=1}^{n-1+i} \sum_{l=1}^{j} \binom{n-1+i}{k} \binom{j}{l} x^{n-1+i-k} x_1^{j-l} (-1)^{k+j-l} \frac{1}{k+l+1} (x^{k+l+1} - x_1^{k+l+1})}{\sum_{k=1}^{n-1+i} \sum_{l=1}^{j} \binom{n-1+i}{k} \binom{j}{l} x^{n-1+i-k} x_1^{j-l} (-1)^{k+j-l} \frac{1}{k+l+1} (x_2^{k+l+1} - x_1^{k+l+1})} \end{aligned}$$

The density is the first derivative of the distribution function. Using the First Fundamental

Theorem of Calculus and the Binomial Theorem we derive the density.

$$
\begin{aligned}
f_{i,j}(x) = \frac{\partial F_{i,j}(x)}{\partial x} &= \frac{(x-y)^{n-1+i}(y-x_1)^j \ dy}{\int_{x_1}^{x_2}(x-y)^{n-1+i}(y-x_1)^j \ dy} \\
&= \frac{(x-y)^{n-1+i}(y-x_1)^j \ dy}{\sum_{k=1}^{n-1+i}\sum_{l=1}^{j}\binom{n-1+i}{k}\binom{j}{l}x^{n-1+i-k}x_1^{j-l}(-1)^{k+j-l}\frac{1}{k+l+1}(x_2^{k+l+1}-x_1^{k+l+1})}.
\end{aligned}
$$

□

Proof of Proposition 2: We use lemma 1 to obtain first deal's unsuccess probability.

$$
\begin{aligned}
p_d(1) &= \mathbf{P}\left(B > \ min(S_1, ..., S_n)\right) = \frac{1}{\Delta b}\int_{b_1}^{b_2} \mathbf{P}\left(x > \ min(S_1, ..., S_n)\right) \ dx \\
&= \frac{1}{\Delta b}\int_{s_1}^{b_2}\left(\frac{s_2-x}{s_2-s_1}\right)^n \ dx = \frac{1}{\Delta b^{n+1}}\int_{s_1}^{s_2}(s_2-x)^n \ dx = \frac{\Delta s^{n+1}}{\Delta b^{n+1} \ (n+1)} = \frac{1}{n+1}.
\end{aligned}
$$

□

Proof of Proposition 3: Before we prove the proposition's statement, we introduce the notation for a binary vector $v \in [0,1]^k$. It describes concrete deal success and unsuccess structures, e.g. $v = (1,0,0)$ means that the first deal is successful, whereas deal two and three are unsuccessful. The unsuccess probability of the second deal $p_d(2)$ is thus represented by the sum $p_d(2) = \mathbf{P}((1,0)) + \mathbf{P}((0,0))$, that is the sum of the independent probabilities of the two events '*Fist deal successful, second deal unsuccessful*' and '*First deal unsuccessful, second deal unsuccessful.*' We calculate $\mathbf{P}((1,0))$ and $\mathbf{P}((0,0))$ separately:

$$
\begin{aligned}
\mathbf{P}((0,0)) &= \int_{s_1}^{s_2} m(x_1, s_1, n)\mathbf{P}(B < x_1)\mathbf{P}(B < x_1) \ dx_1 = \int_{s_1}^{s_2} n\frac{(s_2-x_1)^{n-1}}{\Delta s^n}\left(\frac{x_1-s_1}{\Delta s}\right)^2 \ dx_1 \\
&= \frac{n}{\Delta s^{n+2}}\int_{s_1}^{s_2}(s_2-x_1)^{n-1}(x_1-s_1)^2 \ dx_1 = \frac{n\Delta s^n(2s_2^2-4s_1s_2+2s_1^2)}{n(n+1)(n+2)\Delta s^{n+2}} = \frac{2}{(n+1)(n+2)} \\
\mathbf{P}((1,0)) &= \int_{s_1}^{s_2} m(x_1, s_1, n)\mathbf{P}(B > x_1)\int_{x_1}^{s_2} m(x_2, x_1, n-1)\mathbf{P}(B < x_2) \ dx_2 dx_1 \\
&= \int_{s_1}^{s_2} n\frac{(s_2-x_1)^{n-1}}{\Delta s^n}\frac{s_2-x_1}{\Delta s}\int_{x_1}^{s_2}(n-1)\frac{(s_2-x_2)^{n-2}}{(s_2-x_1)^{(n-1)}}\frac{x_2-s_1}{\Delta s} \ dx_2 dx_1 \\
&= \frac{n(n-1)}{\Delta s^{n+2}}\int_{s_1}^{s_2}(s_2-x_1)\int_{x_1}^{s_2}(s_2-x_2)^{n-2}(x_2-s_1) \ dx_2 dx_1 \\
&= \frac{n(n-1)}{n(n+1)\Delta s^{n+2}}\int_{s_1}^{s_2}(s_2-x_1)(s_2-x_1)^{(n-1)}(s_2+nx_1-ns_1-x_1)dx_1 \\
&= \frac{n(n-1) \ \Delta s^{n+1}(2s_2 n-2ns_1+s2-s1)}{n(n-1)(n+1)(n+2)\Delta s^{n+2}} = \frac{2n+1}{(n+1)(n+2)}
\end{aligned}
$$

The probability's sum is given by $p_d(2) = \frac{2n+3}{(n+1)(n+2)}$. □

Proof of Proposition 4: As in the proof of proposition 3, we use the notation $p_d(3) = \mathbf{P}((0,0,0)) + \mathbf{P}((0,1,0)) + \mathbf{P}((1,0,0)) + \mathbf{P}((1,1,0))$ and calculate each probability separately:

$$\begin{aligned}
\mathbf{P}((0,0,0)) &= \int_{s_1}^{s_2} m(x_1, s_1, n)\mathbf{P}(B < x_1)^3 \; dx_1 = \int_{s_1}^{s_2} n \frac{(s_2 - x_1)^{n-1}}{\Delta s^n} \left(\frac{x_1 - s_1}{\Delta s}\right)^3 \; dx_1 \\
&= \frac{n}{\Delta s^{n+3}} \int_{s_1}^{s_2} (s_2 - x_1)^{n-1} (x_1 - s_1)^3 \; dx_1 \\
&= \frac{n}{\Delta s^{n+3}} \frac{(s_2 - s_1)^n \left(-18\, s_2{}^2 s_1 + 6\, s_2{}^3 - 6\, s_1{}^3 + 18\, s_2\, s_1{}^2\right)}{(n+3)\,(n+2)\,(n+1)\,n} = \frac{6}{(n+3)\,(n+2)\,(n+1)} \\
\mathbf{P}((0,1,0)) &= \int_{s_1}^{s_2} m(x_1, s_1, n)\mathbf{P}(B < x_1)\mathbf{P}(B > x_1) \int_{x_1}^{s_2} m(x_2, x_1, n-1)\mathbf{P}(B < x_1) \; dx_2 dx_1 \\
&= \int_{s_1}^{s_2} n \frac{(s_2 - x_1)^{n-1}}{\Delta s^n} \frac{x_1 - s_1}{\Delta s} \frac{s_2 - x_1}{\Delta s} \int_{x_1}^{s_2} (n-1) \frac{(s_2 - x_2)^{n-2}}{(s_2 - x_1)^{n-1}} \frac{x_2 - s_1}{\Delta s} \; dx_2 dx_1 \\
&= \frac{n(n-1)}{\Delta s^{n+3}} \int_{s_1}^{s_2} (x_1 - s_1)(s_2 - x_1) \int_{x_1}^{s_2} (s_2 - x_2)^{n-2}(x_2 - s_1) \; dx_2 dx_1 \\
&= \frac{n(n-1)}{n(n-1)\Delta s^{n+3}} \int_{s_1}^{s_2} (x_1 - s_1)(s_2 - x_1)^n (s_2 + n x_1 - n s_1 - x_1) \; dx_1 \\
&= \frac{n(n-1)}{n(n-1)\Delta s^{n+3}} \frac{\Delta s^{n+1} \Delta s^2\,(3\,n+1)}{(n+3)\,(n+2)\,(n+1)} = \frac{3\,n+1}{(n+3)\,(n+2)\,(n+1)} \\
\mathbf{P}((1,0,0)) &= \int_{s_1}^{s_2} m(x_1, s_1, n)\mathbf{P}(B > x_1) \int_{x_1}^{s_2} m(x_2, x_1, n-1)\mathbf{P}(B < x_2)^2 \; dx_2 \; dx_1 \\
&= \int_{s_1}^{s_2} n \frac{(s_2 - x_1)^{n-1}}{\Delta s^n} \frac{s_2 - x_1}{\Delta s} \int_{x_1}^{s_2} (n-1) \frac{(s_2 - x_2)^{n-2}}{(s_2 - x_1)^{n-1}} \left(\frac{x_2 - s_1}{\Delta s}\right)^2 \; dx_2 \; dx_1 \\
&= \frac{n(n-1)}{\Delta s^{n+3}} \int_{s_1}^{s_2} (s_2 - x_1) \int_{x_1}^{s_2} (s_2 - x_2)^{n-2}(x_2 - s_1)^2 \; dx_2 \; dx_1 \\
&= \frac{n(n-1)}{\Delta s^{n+3}} \frac{6\Delta s^{n+3}}{(n-1)n(n+2)(n+3)} = \frac{6(n+1)}{(n+1)(n+2)(n+3)} \\
\mathbf{P}((1,1,0)) &= \int_{s_1}^{s_2} m(x_1, s_1, n)\mathbf{P}(B > x_1) \int_{x_1}^{s_2} m(x_2, x_1, n-1)\mathbf{P}(B > x_2) \\
&\quad \int_{x_2}^{s_2} m(x_3, x_2, n-2)\mathbf{P}(B < x_3) \; dx_3 \; dx_2 \; dx_1 \\
&= \int_{s_1}^{s_2} n \frac{(s_2 - x_1)^{n-1}}{\Delta s^n} \frac{s_2 - x_1}{\Delta s} \int_{s_1}^{s_2} (n-1) \frac{(s_2 - x_2)^{n-2}}{(s_2 - x_1)^{n-1}} \frac{s_2 - x_2}{\Delta s} \\
&\quad \int_{s_1}^{s_2} (n-2) \frac{(s_2 - x_3)^{n-3}}{(s_2 - x_2)^{n-2}} \frac{x_3 - s_1}{\Delta s} \; dx_3 \; dx_2 \; dx_1 \\
&= \frac{n(n-1)(n-2)}{\Delta s^{n+3}} \int_{s_1}^{s_2} (s_2 - x_1) \int_{s_1}^{s_2} (s_2 - x_2) \int_{s_1}^{s_2} (s_2 - x_3)^{n-3}(x_3 - s_1) \; dx_3 \; dx_2 \; dx_1 \\
&= \frac{n(n-1)(n-2)}{(n-1)(n-2))\Delta s^{n+3}} \int_{s_1}^{s_2} (s_2 - x_1) \\
&\quad \int_{s_1}^{s_2} (s_2 - x_2)^{n-1}(n x_2 - n s_1 + s_1 + s_2 - 2x_2) \; dx_2 \; dx_1 \\
&= \frac{n(n-1)(n-2)}{(n+1)n(n-1)(n-2)\Delta s^{n+3}} \int_{s_1}^{s_2} (s_2 - x_1)^{n+1}
\end{aligned}$$

$$(2s_2n + s_1 + n^2x_1 - n^2s_1 - s_2 - 2nx_1)\ dx_1$$
$$= \frac{n(n-1)(n-2)\Delta s^{n+3}3(n^2+n-1)}{(n+3)(n+2)(n+1)n(n-1)(n-2)\Delta s^{n+3}} = \frac{3(n^2+n-1)}{(n+3)(n+2)(n+1)}$$

□

Proof of Proposition 6: We calculate the expected price of the first successful deal, using lemma 1.

$$\begin{aligned}\mathbf{E}(Pr_1) &= \int_{s_1}^{s_2} m(s)s\ ds = \frac{n}{\Delta s^n}\int_{s_1}^{s_2}(s_2-s)^{n-1}s\ ds \\ &= \frac{n}{\Delta s^n}\left(\frac{1}{n}\Delta s^n s_1 + \frac{1}{n(n+1)\Delta s^{n+1}}\right) = s_1 + \frac{1}{n+1}\Delta s.\end{aligned}$$

□

Proof of Proposition 7: Let n offers be uniformly distributed on $[x, s_2]$. Then the expected price of the lowest offer is

$$\begin{aligned}\mathbf{E}(min(X_1, X_2, ..., X_n)) &= \int_x^{s_2} m(s, x, s_2)s\ ds = \frac{n}{(s_2-x)^n}\int_x^{s_2}(s_2-s)^{n-1}s\ ds \\ &= \frac{n}{n+1}x + \frac{1}{n+1}s_2.\end{aligned}$$

The pdf required here is obtained from lemma 1. Then the integral is calculated along the lines of the proof of proposition 6. Note that when we set $x = s_1$, we get exactly the formula that was established in proposition 6. The expected price of the first deal is

$$\mathbf{E}(Pr_1) = \frac{n}{n+1}s_1 + \frac{1}{n+1}s_2.$$

The second deal is the minimum of $n-1$ independent random variables that are uniformly distributed on $[\mathbf{E}(Pr_1), s_2]$. By the formula we calculated above, the second deal therefore has expected price

$$\begin{aligned}\mathbf{E}(Pr_2) &= \frac{n}{n+1}\mathbf{E}(P_1) + \frac{1}{n+1}s_2 = \frac{n}{n+1}\left(\frac{n}{n+1}s_1 + \frac{1}{n+1}s_2\right) + \frac{1}{n+1}s_2 \\ &= \frac{n-1}{n+1}s_1 + \frac{2}{n+1}s_2.\end{aligned}$$

Iteratively calculating the next deal prices, the $k-th$ deal price is

$$\mathbf{E}(Pr_k) = \frac{n+1-k}{n+1}s_1 + \frac{k}{n+1}s_2.$$

□

Proof of Proposition 8: Assume the $k-th$ successful deal was priced at x_k. Then the m remaining offers are independently uniformly distributed on the interval $[x_k, s_2]$. When a buyer enters that market, then that buyer's profit is

$$\begin{aligned}
\mathbf{E}(P)(x_k, m) &= \int_{x_k}^{s_2} \int_{s}^{s_2} m(s, x_k, s_2) \frac{1}{\Delta s}(b-s)\ dbds \\
&= \frac{m}{\Delta s(s_2 - x_k)^m} \int_{x_k}^{s_2} (s_2 - x)^{m-1} \int_{s}^{s_2} (b-s)\ dbds \\
&= \frac{m}{\Delta s(s_2 - x_k)^m} \int_{x_k}^{s_2} (s_2 - s)^{m-1} \frac{(s_2 - s)^2}{2}\ ds \\
&= \frac{m}{2\Delta s(s_2 - x_k)^m} \int_{x_k}^{s_2} (s_2 - s)^{m+1}\ ds \\
&= \frac{m(s_2 - x_k)^{m+2}}{2\Delta s(s_2 - x_k)^m (m+2)} = \frac{(s_2 - x_k)^2}{2\Delta s} \frac{m}{m+2}.
\end{aligned}$$

□

Proof of Proposition 9: We calculate a buyer's expected profit, when the lowest present offer is x.

$$\begin{aligned}
\mathbf{E}(P)(x) &= \int_{s_1}^{s_2} \mathbf{1}_{Deal}(b-x)\ db = \int_{x}^{s_2} b - x\ db \\
&= \frac{s_2^2 - x^2 - 2x(s_2 - x)}{2} = \frac{(s_2 - x)^2}{2\Delta s}.
\end{aligned}$$

□

Proof of Proposition 10: The profit of the seller with the lowest offer is

$$\begin{aligned}
\mathbf{E}(P_{S_1}) &= \frac{1}{2\alpha} \int_{s_1}^{s_2} \mathbf{P}\left(\exists\ Buyer\ with\ V_B > x\right) x\ dx \\
&= \frac{1}{2\alpha} \int_{s_1}^{s_2} \left(1 - \mathbf{P}\left(\forall\ Buyers\ have\ V_B \le x\right)\right) x\ dx = \frac{1}{2\alpha} \int_{s_1}^{s_2} x - x \left(\frac{x - s_1}{\Delta s}\right)^n\ dx \\
&= \frac{1}{2\alpha} \left(\frac{1}{2}(s_2^2 - s_1^2) - \left[\frac{1}{n+1} \frac{(x - s_1)^{n+1}}{\Delta s^n} x \right]_{s_1}^{s_2} + \int_{s_1}^{s_2} \frac{1}{n+1} \frac{(x-s_1)^{n+1}}{\Delta s^n}\ dx \right) \\
&= \frac{1}{2\alpha} \left(\frac{4\alpha}{2} - \frac{1}{n+1} \frac{\Delta s^{n+1}}{\Delta s^n} s_2 + \frac{1}{(n+1)(n+2)} \frac{\Delta s^{n+2}}{\Delta s^n} \right) \\
&= \frac{1}{2\alpha} \left(2\alpha - \frac{1}{n+1} \Delta s s_2 + \frac{1}{(n+1)(n+2)} \Delta s^2 \right) = 1 - \frac{1}{n+1} s_2 + \frac{1}{(n+1)(n+2)} \Delta s \\
&= 1 - \frac{n + 2 + n\alpha}{(n+1)(n+2)}.
\end{aligned}$$

□

Proof of Proposition 11: When a seller's offer is uniformly distributed on the interval $[s_1, x]$, then the probability that the k-th buyer buys that offer is

$$\begin{aligned}
\mathbf{P}(k-\text{th buyer buys offer}) &= \frac{1}{x-s_1}\int_{s_1}^{x} \mathbf{P}(V_B < y \ for \ k-1 \ buyers)\mathbf{P}(V_B > y)\ dy \\
&= \frac{1}{x-s_1}\int_{s_1}^{x} \left(\frac{y-s_1}{\Delta s}\right)^{k-1} \frac{s_2-y}{\Delta s}\ dy \\
&= \frac{1}{(x-s_1)\Delta s^k}\int_{s_1}^{x} (y-s_1)^{k-1}(s_2-y)\ dy \\
&= \frac{1}{(x-s_1)\Delta s^k}\left(\frac{1}{k}(x-s_1)^k(s_2-x) + \frac{1}{k(k+1)}(x-s_1)^{k+1}\right) \\
&= \frac{1}{k\Delta s^k}(x-s_1)^{k-1}(s_2-x) + \frac{1}{k(k+1)\Delta s^k}(x-s_1)^k.
\end{aligned}$$

To simplify later calculations, we evaluate the following integral for some $m \in \mathbb{N}$.

$$\begin{aligned}
&\int_{s_1}^{s_2} x(x-s_1)^m(s_2-x)\ dx = -\frac{1}{m+1}\int_{s_1}^{s_2} (x-s_1)^{m+1}(s_2-2x)\ dx \\
&= -\frac{1}{m+1}\left(\left[\frac{1}{m+2}(x-s_1)^{m+2}(s_2-2x)\right]_{s_1}^{s_2} - \frac{1}{m+2}\int_{s_1}^{s_2}(x-s_1)^{m+2}(-2)\ dx\right) \\
&= -\frac{1}{(m+1)(m+2)}\left((s_2-s_1)^{m+2}(s_2-2s_2) + \frac{2}{m+3}(s_2-s_1)^{m+3}\right) \\
&= \frac{s_2\Delta s^{m+2}}{(m+1)(m+2)} - \frac{2\Delta s^{m+3}}{(m+1)(m+2)(m+3)} = \frac{\Delta s^{m+2}}{(m+1)(m+2)}\left(s_2 - \frac{2\Delta s}{m+3}\right).
\end{aligned}$$

Next, we calculate the expected profit of the seller with the second lowest offer.

$$
\begin{aligned}
&\mathbf{E}(P_{S_2})\\
&=\frac{1}{\Delta s}\sum_{k=1}^{n-1}\int_{s_1}^{s_2}\left(\mathbf{P}(k-\text{th buyer buys first offer})\mathbf{P}(\text{One of } n-k \text{ buyers buys second offer})\right) x\,dx\\
&=\frac{1}{\Delta s}\sum_{k=1}^{n-1}\int_{s_1}^{s_2}\left(\left(\frac{1}{k\Delta s^k}(x-s_1)^{k-1}(s_2-x)+\frac{1}{k(k+1)\Delta s^k}(x-s_1)^k\right)\left(1-\frac{(x-s_1)^{n-k}}{\Delta s^{n-k}}\right)\right)x\;dx\\
&=\frac{1}{\Delta s}\sum_{k=1}^{n-1}\int_{s_1}^{s_2}\left(\frac{(x-s_1)^{k-1}(s_2-x)}{k\Delta s^k}-\frac{(x-s_1)^{n-1}(s_2-x)}{k\Delta s^n}+\frac{(x-s_1)^k}{k(k+1)\Delta s^k}-\frac{(x-s_1)^n}{k(k+1)\Delta s^n}\right)x\;dx\\
&=\sum_{k=1}^{n-1}\frac{1}{k\Delta s^{k+1}}\int_{s_1}^{s_2}x(x-s_1)^{k-1}(s_2-x)\;dx-\sum_{k=1}^{n-1}\frac{1}{k\Delta s^{n+1}}\int_{s_1}^{s_2}x(x-s_1)^{n-1}(s_2-x)\;dx\\
&\quad+\sum_{k=1}^{n-1}\frac{1}{k(k+1)\Delta s^{k+1}}\int_{s_1}^{s_2}x(x-s_1)^k\;dx-\sum_{k=1}^{n-1}\frac{1}{k(k+1)\Delta s^{n+1}}\int_{s_1}^{s_2}x(x-s_1)^n\;dx\\
&=\sum_{k=1}^{n-1}\frac{1}{k\Delta s^{k+1}}\frac{\Delta s^{k+1}}{k(k+1)}\left(s_2-\frac{2\Delta s}{k+2}\right)-\sum_{k=1}^{n-1}\frac{1}{k\Delta s^{n+1}}\frac{\Delta s^{n+1}}{n(n+1)}\left(s_2-\frac{2\Delta s}{n+2}\right)\\
&\quad+\sum_{k=1}^{n-1}\frac{1}{k(k+1)\Delta s^{k+1}}\frac{\Delta s^{k+1}}{k+1}\left(s_2-\frac{\Delta s}{k+2}\right)-\sum_{k=1}^{n-1}\frac{1}{k(k+1)\Delta s^{n+1}}\frac{\Delta s^{n+1}}{n+1}\left(s_2-\frac{\Delta s}{n+2}\right).
\end{aligned}
$$

□

Proof of Proposition 12: When the third lowest offer is $x \in [1-\alpha, 1+\alpha]$, then for $1 \leq k < l \leq n$, we can calculate the probability

$$
\begin{aligned}
&\mathbf{P}(\text{The } k-\text{th and } l-\text{th buyers buy the first and second lowest offer}|\text{Third lowest offer is } x)\\
&=\frac{2}{(x-s_1)^2}\int_{s_1}^{x}\mathbf{P}(k-1 \text{ buyers have } V_B<y \text{ and one } V_B>y)\\
&\qquad\qquad\frac{1}{x-y}\int_{y}^{x}\mathbf{P}(l-k-1 \text{ buyers have } V_B<z \text{ and one } V_B>z)\;dz\;dy\\
&=\frac{2}{(x-s_1)^2}\int_{s_1}^{x}\frac{(y-s_1)^{k-1}}{\Delta s^{k-1}}\frac{s_2-y}{\Delta s}\int_{y}^{x}\frac{(z-s_1)^{l-k-1}}{\Delta s^{l-k-1}}\frac{s_2-z}{\Delta s}\;dz\;dy\\
&=\frac{2}{(x-s_1)^2\Delta s^l}\int_{s_1}^{x}(y-s_1)^{k-1}(s_2-y)\int_{y}^{x}(z-s_1)^{l-k-1}(s_2-z)\;dz\;dy\\
&=\frac{2}{(x-s_1)^2\Delta s^l}\int_{s_1}^{x}(y-s_1)^{k-1}(s_2-y)\frac{1}{l-k}\left(\left[(z-s_1)^{l-k}(s_2-z)\right]_y^x-\int_y^x(z-s_1)^{l-k}(-1)\;dz\right)\;dy\\
&=\frac{2}{(x-s_1)^2\Delta s^l(l-k)}\int_{s_1}^{x}(y-s_1)^{k-1}(s_2-y)\left((x-s_1)^{l-k}(s_2-x)-(y-s_1)^{l-k}(s_2-y)\right.\\
&\qquad\qquad\left.+\frac{1}{l-k+1}\left((x-s_1)^{l-k+1}-(y-s_1)^{l-k+1}\right)\right)\;dy\\
&=\frac{2}{(x-s_1)^2\Delta s^l(l-k)}\left((x-s_1)^{l-k}(s_2-x)\int_{s_1}^{x}(y-s_1)^{k-1}(s_2-y)\;dy-\int_{s_1}^{x}(y-s_1)^{l-1}(s_2-y)^2\;dy\right.\\
&\qquad\qquad\left.+\frac{(x-s_1)^{l-k+1}}{l-k+1}\int_{s_1}^{x}(y-s_1)^{k-1}(s_2-y)\;dy-\frac{1}{l-k+1}\int_{s_1}^{x}(y-s_1)^{l}(s_2-y)\;dy\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2}{(x-s_1)^2 \Delta s^l (l-k)} \left((x-s_1)^{l-k}(s_2-x) \left(\frac{1}{k}(x-s_1)^k (s_2-x) + \frac{1}{k(k+1)}(x-s_1)^{k+1} \right) \right. \\
&\qquad - \frac{(x-s_1)^l (s_2-x)^2}{l} - \frac{2(x-s_1)^{l+1}(s_2-x)}{l(l+1)} - \frac{2(x-s_1)^{l+2}}{l(l+1)(l+2)} \\
&\qquad + \frac{(x-s_1)^{l-k+1}}{l-k+1} \left(\frac{1}{k}(x-s_1)^k (s_2-x) + \frac{1}{k(k+1)}(x-s_1)^{k+1} \right) \\
&\qquad \left. - \frac{1}{l-k+1} \left(\frac{1}{l+1}(x-s_1)^{l+1}(s_2-x) + \frac{1}{(l+1)(l+2)}(x-s_1)^{l+2} \right) \right) \\
&= \frac{2}{(x-s_1)^2 \Delta s^l (l-k)} \left(\frac{1}{k}(x-s_1)^l (s_2-x)^2 + \frac{1}{k(k+1)}(x-s_1)^{l+1}(s_2-x) \right. \\
&\qquad - \frac{(x-s_1)^l (s_2-x)^2}{l} - \frac{2(x-s_1)^{l+1}(s_2-x)}{l(l+1)} - \frac{2(x-s_1)^{l+2}}{l(l+1)(l+2)} \\
&\qquad + \frac{1}{k(l-k+1)}(x-s_1)^{l+1}(s_2-x) + \frac{1}{k(k+1)(l-k+1)}(x-s_1)^{l+2} \\
&\qquad \left. - \frac{1}{(l+1)(l-k+1)}(x-s_1)^{l+1}(s_2-x) + \frac{1}{(l+1)(l+2)(l-k+1)}(x-s_1)^{l+2} \right) \\
&= \frac{2}{\Delta s^l (l-k)} \left((x-s_1)^{l-2}(s_2-x)^2 \left(\frac{1}{k} - \frac{1}{l} \right) \right. \\
&\qquad + (x-s_1)^{l-1}(s_2-x) \left(\frac{1}{k(k+1)} - \frac{2}{l(l+1)} + \frac{1}{k(l-k+1)} - \frac{1}{(l+1)(l-k+1)} \right) \\
&\qquad \left. + (x-s_1)^l \left(\frac{1}{k(k+1)(l-k+1)} - \frac{2}{l(l+1)(l+2)} - \frac{1}{(l+1)(l+2)(l-k+1)} \right) \right)
\end{aligned}
$$

In addition note that for $m \in \mathbb{N}$

$$
\int_{s_1}^{s_2} (x-s_1)^m x (s_2-x)^2 = \frac{\Delta s^{m+3}}{(m+1)(m+2)(m+3)} \left[2s_s - \frac{6\Delta s}{m+4} \right]
$$

We calculate the expected profit of the seller with the third lowest offer as follows

$$
\begin{aligned}
&\mathbf{E}(P_{S_3}) \\
&= \frac{1}{\Delta s} \sum_{k=1}^{n-2} \sum_{l=k+1}^{n-1} \int_{s_1}^{s_2} (\mathbf{P}\,(\text{Exactly the } k\text{−th and } l\text{−th buyers buy the first and second lowest offers}) \\
&\qquad\qquad \mathbf{P}\,(\text{One deal at } x \text{ after the second deal}))x\, dx \\
&= \frac{2}{\Delta s} \sum_{k=1}^{n-2} \sum_{l=k+1}^{n-1} \frac{1}{\Delta s^l (l-k)} \int_{s_1}^{s_2} x \left(1 - \frac{(x-s_1)^{n-l}}{\Delta s^{n-l}} \right) \left((x-s_1)^{l-2}(s_2-x)^2 \left(\frac{1}{k} - \frac{1}{l} \right) \right. \\
&\quad + (x-s_1)^{l-1}(s_2-x) \left(\frac{1}{k(k+1)} - \frac{2}{l(l+1)} + \frac{1}{k(l-k+1)} - \frac{1}{(l+1)(l-k+1)} \right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + (x-s_1)^l \left(\frac{1}{k(k+1)(l-k+1)} - \frac{2}{l(l+1)(l+2)} - \frac{1}{(l+1)(l+2)(l-k+1)}\right)\Bigg)\, dx \\
&= \frac{2}{\Delta s}\sum_{k=1}^{n-2}\sum_{l=k+1}^{n-1} \frac{1}{\Delta s^l (l-k)} \int_{s_1}^{s_2} \Bigg(x(x-s_1)^{l-2}(s_2-x)^2 \left(\frac{1}{k}-\frac{1}{l}\right) \\
&\quad + x(x-s_1)^{l-1}(s_2-x)\left(\frac{1}{k(k+1)} - \frac{2}{l(l+1)} + \frac{1}{k(l-k+1)} - \frac{1}{(l+1)(l-k+1)}\right) \\
&\quad + x(x-s_1)^l \left(\frac{1}{k(k+1)(l-k+1)} - \frac{2}{l(l+1)(l+2)} - \frac{1}{(l+1)(l+2)(l-k+1)}\right)\Bigg)\, dx \\
&- \frac{2}{\Delta s}\sum_{k=1}^{n-2}\sum_{l=k+1}^{n-1} \frac{1}{\Delta s^n (l-k)} \int_{s_1}^{s_2} \Bigg(x(x-s_1)^{n-2}(s_2-x)^2 \left(\frac{1}{k}-\frac{1}{l}\right) \\
&\quad + x(x-s_1)^{n-1}(s_2-x)\left(\frac{1}{k(k+1)} - \frac{2}{l(l+1)} + \frac{1}{k(l-k+1)} - \frac{1}{(l+1)(l-k+1)}\right) \\
&\quad + x(x-s_1)^n \left(\frac{1}{k(k+1)(l-k+1)} - \frac{2}{l(l+1)(l+2)} - \frac{1}{(l+1)(l+2)(l-k+1)}\right)\Bigg)\, dx \\
&= \frac{2}{\Delta s}\sum_{k=1}^{n-2}\sum_{l=k+1}^{n-1} \frac{1}{\Delta s^l (l-k)} \Bigg(\frac{\Delta s^{l+1}}{(l-1)l(l+1)}\left[2s_2 - \frac{6\Delta s}{l+2}\right]\left(\frac{1}{k}-\frac{1}{l}\right) \\
&\quad + \frac{\Delta s^{l+1}}{(l(l+1))}\left[s_2 - \frac{2\Delta s}{l+2}\right]\left(\frac{1}{k(k+1)} - \frac{2}{l(l+1)} + \frac{1}{k(l-k+1)} - \frac{1}{(l+1)(l-k+1)}\right) \\
&\quad + \frac{\Delta s^{l+1}}{(l+1)}\left[s_2 - \frac{\Delta s}{l+2}\right]\left(\frac{1}{k(k+1)(l-k+1)} - \frac{2}{l(l+1)(l+2)} - \frac{1}{(l+1)(l+2)(l-k+1)}\right)\Bigg) \\
&- \frac{2}{\Delta s}\sum_{k=1}^{n-2}\sum_{l=k+1}^{n-1} \frac{1}{\Delta s^n (l-k)} \Bigg(\frac{\Delta s^{n+1}}{(n-1)n(n+1)}\left[2s_2 - \frac{6\Delta s}{n+2}\right]\left(\frac{1}{k}-\frac{1}{l}\right) \\
&\quad + \frac{\Delta s^{n+1}}{(n(n+1))}\left[s_2 - \frac{2\Delta s}{n+2}\right]\left(\frac{1}{k(k+1)} - \frac{2}{l(l+1)} + \frac{1}{k(l-k+1)} - \frac{1}{(l+1)(l-k+1)}\right) \\
&\quad + \frac{\Delta s^{n+1}}{(n+1)}\left[s_2 - \frac{\Delta s}{n+2}\right]\left(\frac{1}{k(k+1)(l-k+1)} - \frac{2}{l(l+1)(l+2)} - \frac{1}{(l+1)(l+2)(l-k+1)}\right)\Bigg) \\
&= \sum_{k=1}^{n-2}\sum_{l=k+1}^{n-1} \frac{2}{l-k} \Bigg(\frac{1}{(l-1)l(l+1)}\left[2s_2 - \frac{6\Delta s}{l+2}\right]\left(\frac{1}{k}-\frac{1}{l}\right) \\
&\quad + \frac{1}{l(l+1)}\left[s_2 - \frac{2\Delta s}{l+2}\right]\left(\frac{1}{k(k+1)} - \frac{2}{l(l+1)} + \frac{1}{k(l-k+1)} - \frac{1}{(l+1)(l-k+1)}\right) \\
&\quad + \frac{1}{l+1}\left[s_2 - \frac{\Delta s}{l+2}\right]\left(\frac{1}{k(k+1)(l-k+1)} - \frac{2}{l(l+1)(l+2)} - \frac{1}{(l+1)(l+2)(l-k+1)}\right)\Bigg) \\
&- \sum_{k=1}^{n-2}\sum_{l=k+1}^{n-1} \frac{2}{l-k} \Bigg(\frac{1}{(n-1)n(n+1)}\left[2s_2 - \frac{6\Delta s}{n+2}\right]\left(\frac{1}{k}-\frac{1}{l}\right) \\
&\quad + \frac{1}{n(n+1)}\left[s_2 - \frac{2\Delta s}{n+2}\right]\left(\frac{1}{k(k+1)} - \frac{2}{l(l+1)} + \frac{1}{k(l-k+1)} - \frac{1}{(l+1)(l-k+1)}\right) \\
&\quad + \frac{1}{n+1}\left[s_2 - \frac{\Delta s}{n+2}\right]\left(\frac{1}{k(k+1)(l-k+1)} - \frac{2}{l(l+1)(l+2)} - \frac{1}{(l+1)(l+2)(l-k+1)}\right)\Bigg).
\end{aligned}
$$

$\square$

Proof of Proposition 13: We calculate the profit of the first buyer.

$$\begin{aligned}
\mathbf{E}(P_{B_1}) &= \int_{s_1}^{s_2}\int_{b_1}^{b_2} \mathbf{1}_{b>s} m(s)\frac{1}{\Delta b}(b-s)\ dbds = \int_{s_1}^{s_2}\int_{s}^{b_2} n\frac{(s_2-s)^{n-1}}{(s_2-s_1)^n}\frac{1}{\Delta b}(b-s)\ dbds \\
&= \frac{n}{\Delta b^{n+1}}\int_{s_1}^{s_2}(s_2-s)^{n-1}\int_{s}^{b_2}(b-s)\ dbds = \frac{n}{\Delta b^{n+1}}\int_{s_1}^{s_2}(s_2-s)^{n-1}\frac{b_2^2-s^2-2s(b_2-s)}{2}\ ds \\
&= \frac{n}{\Delta b^{n+1}}\int_{s_1}^{s_2}(s_2-s)^{n-1}\frac{b_2^2+s^2-2sb_2}{2}\ ds = \frac{n}{\Delta b^{n+1}}\int_{s_1}^{s_2}(s_2-s)^{n-1}\frac{(b_2-s)^2}{2}\ ds \\
&= \frac{n}{2\Delta b^{n+1}}\int_{s_1}^{s_2}(s_2-s)^{n+1}\ ds = \frac{n}{2\Delta b^{n+1}}(-\frac{1}{n+2}[(s_2-s)^{n+2}]_{s_1}^{s_2} \\
&= \frac{n}{2\Delta b^{n+1}}\frac{1}{n+2}\Delta b^{n+2} = \frac{\Delta b}{2}\frac{n}{n+2} = \alpha\frac{n}{n+2}
\end{aligned}$$

$\square$

Proof of Proposition 14: We calculate the expected profit of the second buyer. The functions introduced in lemma 1 and propositions 7 and 9 are used for the calculations.

$$\begin{aligned}
\mathbf{E}(P_{B_2}) &= \int_{s_1}^{s_2} m(n,s_1,s_2,x_1)\left[\mathbf{P}(B<x_1)\mathbf{E}(P)(x_1)+\mathbf{P}(B>x_1)\mathbf{E}(P)(x_1,n-1)\ dx_1\right] \\
&= \frac{n}{\Delta s^n}\int_{s_1}^{s_2}(s_2-x_1)^{n-1}\left[\frac{x_1-s_1}{\Delta s}\frac{(s_2-x_1)^2}{2\Delta s}+\frac{s_2-x_1}{\Delta s}\frac{(s_2-x_1)^2}{2\Delta s}\frac{n-1}{n+1}\ dx_1\right] \\
&= \frac{n}{2\Delta s^{n+2}}\left[\int_{s_1}^{s_2}(x_1-s_1)(s_2-x_1)^{n+1}\ dx_1+\frac{n-1}{n+1}\int_{s_1}^{s_2}(s_2-x_1)^{n+2}\ dx_1\right] \\
&= \frac{n}{2\Delta s^{n+2}}\left[\frac{1}{n+2}\int_{s_1}^{s_2}(s_2-x_1)^{n+2}\ dx_1+\frac{(n-1)\Delta s^{n+3}}{(n+1)(n+3)}\right] \\
&= \frac{n\Delta s}{2(n+3)}\left[\frac{1}{n+2}+\frac{n-1}{n+1}\right] = \alpha\frac{n^3+2n^2-n}{(n+3)(n+1)(n+1)}.
\end{aligned}$$

$\square$

Proof of Proposition 15: We calculate the expected profit of the second buyer. The functions introduced in lemma 1 and propositions 7 and 9 are used for the calculations. We continue to use the notation from the proof of proposition 2 and calculate the third buyer's expected profit as the sum of the prior bargaining process. Then

$$\mathbf{E}(P_{B_3}) = \mathbf{E}(P_{B_3})\left[(0,0)\right]+\mathbf{E}(P_{B_3})\left[(0,1)\right]+\mathbf{E}(P_{B_3})\left[(1,0)\right]+\mathbf{E}(P_{B_3})\left[(1,1)\right]$$

We calculate each of these expected profits separately:

$$\mathbf{E}(P_{B_3})\left[(0,0)\right] = \int_{s_1}^{s_2} m(x)\mathbf{P}(B<x)^2\mathbf{E}(P)(x)\ dx = \frac{n}{\Delta s^n}\int_{s_1}^{s_2}(s_2-x)^{n-1}\frac{(x-s_1)^2}{\Delta s^2}\frac{(s_2-x)^2}{2\Delta s}\ dx$$

$$
\begin{aligned}
&= \frac{n}{2\Delta s^{n+3}} \int_{s_1}^{s_2} (s_2 - x)^{n+1}(x - s_1)^2 \ dx = \frac{n}{2(n+2)\Delta s^{n+3}} \int_{s_1}^{s_2} (s_2 - x)^{n+2}(x - s_1) \ dx \\
&= \frac{n}{2(n+2)(n+3)\Delta s^{n+3}} \int_{s_1}^{s_2} (s_2 - x)^{n+3} \ dx = \frac{n}{2(n+2)(n+3)(n+4)} \Delta s
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{E}(P_{B_3})\left[(1,0)\right] &= \int_{s_1}^{s_2} m(x_1, n)\mathbf{P}(B > x_1) \int_{x_1}^{s_2} m(x_2, n-1)\mathbf{P}(B < x_1)\mathbf{E}(P)(x_2, n-2) \ dx_2 dx_1 \\
&= \frac{n}{\Delta s^n} \int_{s_1}^{s_2} (s_2 - x_1)^{n-1} \frac{s_2 - x_1}{\Delta s} \frac{n-1}{(s_2 - x_1)^{n-1}} \\
&\qquad \int_{x_1}^{s_2} (s_2 - x_2)^{n-2} \frac{x_2 - s_1}{\Delta s} \frac{(s_2 - x_2)^2}{2\Delta s} \frac{n-1}{n+1} \ dx_2 dx_1 \\
&= \frac{n(n-1)^2}{2(n+1)\Delta s^{n+3}} \int_{s_1}^{s_2} (s_2 - x_1) \int_{x_1}^{s_2} (s_2 - x_2)^n (x_2 - s_1) \ dx_2 dx_1 \\
&= \frac{n(n-1)^2}{2(n+1)\Delta s^{n+3}} \int_{s_1}^{s_2} (s_2 - x_1) \frac{1}{(n+1)(n+2)} (s_2 - x_1)^{n+2} \ dx_1 \\
&= \frac{n(n-1)^2}{2(n+1)^2(n+2)\Delta s^{n+3}} \int_{s_1}^{s_2} (s_2 - x_1)^{n+3} \ dx_1 \\
&= \frac{n(n-1)^2}{2(n+1)^2(n+2)(n+4)\Delta s^{n+3}} \Delta s^{n+4} = \alpha \frac{n(n-1)^2}{(n+1)^2(n+2)(n+4)}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{E}(P_{B_3})\left[(0,1)\right] &= \int_{s_1}^{s_2} m(x_1, n)\mathbf{P}(B > x_1)\mathbf{P}(B < x_1)\mathbf{E}(P)(x_1, n-1) dx_1 \\
&= \frac{n}{\Delta s^n} \int_{s_1}^{s_2} (s_2 - x_1)^{n-1} \frac{s_2 - x_1}{\Delta s} \frac{x_1 - s_1}{\Delta s} \frac{(s_2 - x_1)^2}{2\Delta s} \frac{n-1}{n+1} \ dx_1 \\
&= \frac{n(n-1)}{2(n+1)\Delta s^{n+1}} \int_{s_1}^{s_2} (s_2 - x_1)^{n+2}(x_1 - s_1) \ dx_1 = \frac{n(n-1)}{2(n+1)(n+3)(n+4)} \Delta s \\
&= \alpha \frac{n(n-1)}{(n+1)(n+3)(n+4)}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{E}(P_{B_3})\left[(1,1)\right] &= \int_{s_1}^{s_2} m(x_1, n)\mathbf{P}(B > x_1) \int_{x_1}^{s_2} m(x_2, n-1)\mathbf{P}(B > x_1)\mathbf{E}(P)(x_2, n-2) \ dx_2 dx_1 \\
&= \frac{n}{\Delta s^n} \int_{s_1}^{s_2} (s_2 - x_1)^{n-1} \frac{s_2 - x_1}{\Delta s} \frac{n-1}{(s_2 - x_1)^{n-1}} \\
&\qquad \int_{x_1}^{s_2} (s_2 - x_2)^{n-2} \frac{s_2 - x_2}{\Delta s} \frac{(s_2 - x_2)^2}{2\Delta s} \frac{n-2}{n} \ dx_2 dx_1 \\
&= \frac{n(n-1)(n-2)}{2n\Delta s^{n+3}} \int_{s_1}^{s_2} (s_2 - x_1) \int_{x_1}^{s_2} (s_2 - x_2)^{n+1} \ dx_2 dx_1 \\
&= \frac{n(n-1)(n-2)}{2n(n+2)\Delta s^{n+3}} \int_{s_1}^{s_2} (s_2 - x_1)^{n+3} dx_1 = \frac{n(n-1)(n-2)}{2n(n+2)(n+4)\Delta s^{n+3}} \Delta s^{n+4}
\end{aligned}
$$

$$= \alpha \frac{(n-1)(n-2)}{(n+2)(n+4)}$$

We calculate the sum of these four terms.

$$\begin{aligned}\mathbf{E}(P_{B_3}) &= \mathbf{E}(P_{B_3})\left[(0,0)\right] + \mathbf{E}(P_{B_3})\left[(0,1)\right] + \mathbf{E}(P_{B_3})\left[(1,0)\right] + \mathbf{E}(P_{B_3})\left[(1,1)\right] \\ &= \alpha \frac{n^5 + 4n^4 + 2n^3 - 12n^2 + 7n + 6}{(n+1)^2(n+2)(n+3)(n+4)}.\end{aligned}$$

□

Proof of proposition 16: On each round-trip transaction, the dealer has income $(1-f_B)V_B$ and has expenses $(1-f_S)O_S$. Then his gain from a round-trip transaction is $P_D = (1-f_B)V_B - (1-f_S)O_S$. Taking the expectation of the gain gives $\mathbf{E}(P_D) = (1-f_B)-(1-f_S) = -f_B + f_S$, which is greater zero if and only of $f_S > f_B$. □

Proof of Proposition 17: A buyer's profit in the Dealer's Market is the difference between his reservation price V_B and the dealer' offer, that is $(1-f_B)V_B$. We calculate that profit.

$$\mathbf{E}\left(P_B\right) = \mathbf{E}\left(V_B - (1-f_B)V_B\right) = f_B.$$

A buyer's profit therefore is greater zero if and only if $f_B > 0$. □

Proof of Proposition 18: A seller's profit is given by

$$\mathbf{E}\left(P_S\right) = (1-f_S)\mathbf{E}(O_S) = 1 - f_S.$$

That profit is greater zero if and only if $f_S < 1$. □

Proof of Proposition 19: According to proposition 17, a buyer's profit in the Dealer's Market is f_B. That profit exceeds profit of x, when $f_B > x$. □

Proof of Proposition 20: Proposition 19 calculated a conditions for the dominance of the Dealer's Market. A buyer's profit on the Buyers' Market is bounded by 0.43α, as shown in section 2.3. When we combine these statements, then a buyer prefers the Dealer's Market over the Buyers' Market when $f_B > 0.43\alpha$. □

Proof of Proposition 21: Proposition 18 calculates a seller's profit in the Dealer's Market. That profit needs to exceed profit of x, that is

$$1 - f_S > x \Longleftrightarrow f_S < 1 - x.$$

□

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On a Firm's Choice of Debt:
Pareto Efficiency of Bond Financing over Bank Loans

Johannes Seemüller

Abstract

This paper focuses on two different forms of debt financing and thereby analyses Bank loans and the public placement of bonds by an investment banker. Under reasonable conditions, bond financing using the services of an investment banker, who operates under information asymmetry, Pareto dominates financing with bank loans, where a firm opens its books to creditors and provides full information. The firm's management and the bank often have different valuations of the firm. We model a Bayesian updating of the management's estimation on their valuation precision based on the loan negotiation process. At times loan negotiations are unsuccessful, resulting in a loss of negotiation costs. Upon successful floatation, an investment banker receives a fee as a percentage of the bond proceeds. Bond transaction costs are incurred exclusively upon successful financing, as opposed to bank loan negotiation costs, which are incurred even when the financing is declined. We show that if the investment banker is able to maintain information asymmetry between investors and the issuing firm, bond financing is optimal.

Key words: Bank Loans, Bond Financing, Investment Banking, Bond Underwriting, Imprecise Valuation, Asymmetric Information, Full Information, Negotiation Costs. JEL Classifications: C11, G10, G14, G38

Contents

1 Introduction

When external sources for capital are considered, a firm may sell equity or borrow. Seemüller (2013) focuses on the choice between raising venture capital and an IPO. In this paper we focus on borrowing capital.

Debt can be separated into private and public debt. The former dominates the public debt market in size. Within the private debt market, firms may choose among different alternatives, such as bank loans and traditional private placements. Some firms borrow mainly from financial intermediaries such as banks and their providers of debt. Others prefer to publicly or privately issue bonds. In 2009, international syndicated lending amounted to $1.8 trillion and firms borrowed another $1.5 trillion in international bond markets[1]. As debt is by far the major source of capital, it is worth asking how firms prefer one sort of debt over another.

Quantitative analysis on the form of debt firms prefer has been undertaken. Recently, Arena (2010) examined possible determinants of a firm's choice between different forms of debt financing. He analysed the most comprehensive sample of U.S. corporate debt issues. His study focuses primarily on the connection between a firm's credit rating and its debt choice. Arena finds that firms with a high credit quality prefer public bond offerings, whereas small firms with good credit quality are more likely to issue traditional private debt. Firms with moderate credit quality prefer bank loans. In fact, Arena finds that poor quality firms preferentially issue 144A debt. That is, 144A debt allows private placements only to trade to and from qualified institutional investors. Arena's findings challenge the conventional view of firms with poor quality credit rating to choose non-bank traditional private placements, as for instance proposed by Mihov (2003). Fenn (2000) and Arena (2010) further suggest that after 1990, 144A bonds may have sequentially replaced traditional private placements of low-quality high risk debt. They find that firms that issue 144A debt usually have higher information asymmetry and lower credit quality than firms issuing traditional non-bank private debt.

Houston and James (2012) investigate the relation between a firm's growth opportunities and its mix of private and public debt claims by the analysis of a panel data set of 250 publicly listed U.S. firms. Besides investigating on financial characteristics as potential factors influencing a firm's debt choice, further determinants were analysed. More recently, Lin et al. (2012) correlated the ownership structure of a firm with its choice between bank debt and public debt. They examined the relation between a borrowing firm's ownership structure and its choice of debt. They use a data set on corporate ownership, control and debt structure of almost 10,000 global firms from 2001 to 2010. Their results are consistent with the hypothesis that firms controlled by large shareholders with excess control rights prefer public debt financing over bank debt.

[1] See Lin et al. (2012)

In addition to statistical explanations, various corporate theories provide a variety of explanations on a firm's choice of debt. Bank debt potentially has the ability to soften costs of information asymmetry compared to traditional private debt. When banks maintain long term relationships with borrowing firms, they may accumulate additional and soft information about these firms. Banks further have significant comparative advantages in monitoring efficiency because they can get access to private information as insiders, as for instance Fama (1985) and Diamond (1984) have shown. Houston and James (1996) prove that the fragmented ownership structure of public debt and the resulting free rider problems weaken an individual bondholder's incentives to engage in costly monitoring. Assuming bondholders were willing to monitor, it would be inefficient as it would involve unnecessary and thus redundant duplication of monitoring costs and efforts. In summary, banks are more efficient in monitoring than the sum of private investors. Contrary, Lin (2012) finds that banks may be more likely to impose strong and intensive monitoring on borrowing firms. Anticipating strict monitoring by banks, firms might prefer bond financing over bank debt as a way of avoiding scrutiny from bank monitoring. This explanation has been suggested by Houston and James (1996) as well as Denis and Mihov (2003) among others.

This paper presents a theoretical approach to this field and concentrates on two major observations. When estimating a firm's value, potential creditors and a firm's management suffer from valuation imprecision, as each party arrives at a different firm value. As valuation always is a subjective matter to some extent, valuation imprecision occurs even if both parties can check a firm's accounts and thus full information is available. Secondly, when a creditor and a firm negotiate about the terms for debt financing, both parties inevitably suffer costs from negotiations and from checking the firm's accounts. Hence, there is a risk involved that loan negotiations are unsuccessful. In this case, costs invested in negotiations are lost on both sides. When an intermediary is hired to place a firm's bonds, he may offer them to interested investors. Again, a firm's management and investors have a certain valuation of the firm. Contrary to the option of checking the firm's accounts, an investor merely is able to estimate the firms value. By pursuing a firm's management to decrease their valuation and the investor to increase her valuation, chances for successful bond placement increase. When an intermediary's fee structure is reasonable, all parties can profit from his bond placement services.

Section 2 of this paper presents our model in detail. Section 3 focuses on private loan financing. In section 4 intermediation and public placement of bonds is discussed. Section 5 compares those different forms of debt financing. Finally, section 6 concludes.

2 The Model

Arrow Debreu prices[2] are the prices of time and state contingent claims which promise one unit of a specific good in a specific uncertain state at a specific date in the future. Such claims were introduced by Arrow and Debreu in their work on general equilibrium theory under uncertainty, to allow agents to trade state and time contingent claims separately. Then the general equilibrium problem with uncertainty can be reduced to a conventional game without uncertainty. In finite state financial models, Arrow-Debreu claims can be viewed as atomic building blocks of more complex multiple state and multiple time dependent financial framework. In fact, Arrow-Debreu prices of time and state contingent claims determine a unique arbitrage-free price system.

Let a firm invest K at the beginning of a one-period model to produce uncertain output with a payoff that is dependent on investment K and the uncertain future. Then expected value of the firm at the end of the period can be interpreted as a function

$$V: \ \mathbb{R}_{\geq} \to \mathbb{R}: \ K \mapsto V(K).$$

Expected present value of the firm, dependent on K and future uncertain cash-flows can be determined using Arrow Debreu prices. In this context the obligation of the firm to pay back the amount K can further be priced with Arrow-Debreu state contingent prices.

Assume each of n agents cannot precisely value a firm, but approximate it. In the process, each agent makes a certain deviation from the firm's Arrow Debreu value V. We model these deviations independent and uniformly distributed on $[1-\alpha, 1+\alpha]$, with $0 < \alpha < 1$. The parameter α represents maximum imprecision of the evaluating parties. Following the model, maximum valuation is $V_{max} = (1+\alpha)V(K)$ and minimum valuation is $V_{min} = (1-\alpha)V(K)$. Obviously average valuation is exactly the Arrow-Debreu value. Let for instance $X_i \stackrel{d}{=} unif[1-\alpha, 1+\alpha]$ be valuation imprecision of player i. Then that player's valuation is $V_i = X_i V(K)$.

We assume that the company's management decides on optimal investment level K^* such that expected company value is maximized. In order to raise necessary capital K^*, management follows two options: (a) Loan negotiations with banks or (b) consulting an investment banker to sell bonds of volume K^*. In case (a) management opens the firm's books for potential investors such that the firm's accounts are fully observed. In this case there is full information available. Deviations in valuation therefore are caused by valuation imprecision of management and potential investors. On the contrary in case (b), when management uses the services of an investment banker, books are not opened for investors. The investment

[2]See Debreu (1959) and Arrow (1964) for reference.

banker acts as an intermediary and makes certain offers to potential investors. Investors can merely approximate the firm's value. Then deviations in valuation on the buy side are caused by information asymmetry. We assume that a firm's management and an investor can agree on terms of financing if management's valuation of the firm is lower than or equal to that of the investor[3].

3 Loan Financing

In case of loan financing, management approaches banks, one potential lender after another. A potential lender checks the company's books and thereafter negotiates with management the loan's terms. If management values the company not higher than a potential creditor, the parties agree on terms for raising investment level K^*. We assume that management negotiates with potential creditors one after another until the first satisfying financing rate is achieved. Furthermore we assume that searching for a creditor, due diligence and negotiations generates a certain cost for management c_M per potential creditor. In our model, each player's valuation is uniformly distributed on

$$[1-\alpha, 1+\alpha]V(K^*) = [(1-\alpha)V(K^*), (1+\alpha)V(K^*)] =: [v_d, v_u].$$

Denote management's valuation by V_M and potential lender k's valuation by V_{I_k}. Probability that management values the company higher than lender k is thus ex ante given by 0.5, independently for each lender. When a negotiation is successful, then the "financing charge" that lender k offers is drawn from a known random variable, which is uniformly distributed on $[\underline{r}, \overline{r}]$, with $0 \leq \underline{r} < \overline{r} < \infty$. We use the term financing charge to account for the loan interest and all other cost imposed by debt covenants.

Management acts fully rational and thus pursues an optimal negotiation strategy. This includes that, given a certain negotiation history, management can determine future negotiation success more precisely. Assume that management has already lead one successful negotiation and is offered financing charge r to finance investment K^*. Then management has to decide to either accept this offer or to approach another potential lender. The latter case costs at least the amount c_M, as at least one more potential lender needs to be approached. However, there is a chance that further negotiations lead to a better financing charge r^* and thus to decreased financing costs r^*K^*. Savings due to better financing conditions add up to $rK^* - r^*K^* = (r - r^*)K^*$. On the other hand, that additional negotiation costs the firm the amount c_M. Therefore after each negotiation management faces the decision whether to

[3]The probability that a firm's management's and an investor's valuation are equal is a zero set. The probability of this event thus is zero. With no loss of generality, we may also say that the parties can agree on terms of financing, when an investor's valuation exceeds that of a firm's management.

close the negotiation process or to conduct further negotiations with a potential creditor. A crucial ingredient for this decision is negotiation history.

The first lemma is needed on the way to calculate expected cost until deal settlement.

Lemma 1. *Let $X_1, X_2, ..., X_n$ be iid random variables, with $X_1 \stackrel{d}{=} unif[x_1, x_2]$. Then the cdf of $min(X_1, X_2, ..., X_n)$ is given by $M(x) = 1 - \left(\frac{x_2 - x}{x_2 - x_1}\right)^n$. The pdf of $min(X_1, X_2, ..., X_n)$ is given by $m(x) = n\frac{(x_2-x)^{n-1}}{(x_2-x_1)^n}$.*

Proof: See the Appendix. □

If needed, the notation of the functions M and m is expanded in an intuitive way. Then we may for instance write $M(x, x_1, x_2, n)$ instead of $M(x)$. The lemma is a key ingredient to next proposition's proof. The proposition introduces a formula for the updated density of valuation imprecision that management calculates during the negotiation process.

Proposition 1. *When n prior negotiations were successful and m were unsuccessful, then the posterior density of a firm's management's valuation imprecision is calculated by the formula*

$$f_{m,n}(x) = \frac{(b-x)^n (x-a)^m}{\sum_{k=0}^{n} \sum_{l=0}^{m} \binom{n}{k}\binom{m}{l}(-1)^{k+m-l} a^{m-l} b^{n-k} \frac{1}{k+l+1}\left(b^{k+l+1} - a^{k+l+1}\right)}.$$

For simplification we use the notation $a := 1 - \alpha$ and $b := 1 + \alpha$.

Proof: See the Appendix. □

Proposition 1 introduces the formula of the updated posterior density of their valuation imprecision that management calculates at each stage in the negotiation process. The formula is dependent on the number of prior successful negotiations m and unsuccessful negotiations n. At the beginning of the negotiation process (that is represented by $m = n = 0$), management believes that their imprecision is uniformly distributed on the interval $[1 - \alpha, 1 + \alpha]$. This is represented by density function $f_{0,0} = 0.5/\alpha$, which is the density function of a random variable that is uniformly distributed on $[1 - \alpha, 1 + \alpha]$. In negotiation process, management knows that m prior negotiations were successful and n prior negotiations were unsuccessful. Management then can optimally update their valuation imprecision density to $f_{m,n}$. When more prior negotiations were successful than unsuccessful, then the probability that management undervalues the firm is above average. That is represented by a shift of management's updated density to the left. On the contrary, when more prior negotiations were unsuccessful than successful, then the probability that management has a valuation above average increases. As a result, the updated density function shifts to the right.

In negotiation process, management decides to accept the best offer from previous successful negotiations or to continue negotiating. The decision management faces is dependent on several factors. As discussed above, prior negotiations and their successes changes the updated density function of valuation imprecision. Negotiation history therefore influences management's decision, whether to accept a given offer or not. Further deciding factors are the financing charge r, the distribution of offers, the volume of the loan K^* and the cost per negotiation c_M:

In fact, in any stage of the negotiation process, management calculates their updated density of valuation imprecision. From that function, management can derive the posterior probability that the next negotiation is successful. Negotiation history thus is a crucial decision factor. Proposition 1 presents the formula for these calculations.

Assume management has a certain financing offer at rate r. When r is above average, then further negotiations lead to better financing charges with a probability that exceeds 0.5. A financing offer r below average implies a lower probability to achieve a better offer in further negotiations.

When the firm wants to raise the amount K^*, then a reduction in financing charges reduces the cost of capital. Now assume that credit volume is below K^*. Then the effect of reduced capital costs looses strength. When the loan size is above K^*, then a change in the financing rate reduces costs of capital more. The loan size K^* therefore is a factor that management takes into consideration for its optimal negotiation strategy.

Now consider cost per negotiation c_M. If that cost is high, then negotiations will be continued when chances to improve financing conditions are promising. Underlie small, or even zero negotiation cost. This favours further negotiations, even when the probability for improved financing conditions is low.

We summarize that negotiation history, best available financing charge r, loan size K^* and cost per negotiation c_M are factors that a firm's management considers to implement optimal negotiation strategy. The theorem formalizes the verbal argumentation above.

Theorem 1. *Assume $n \geq 1$ prior negotiations were successful and $m \geq 0$ were unsuccessful. Let $0 \leq c_M$ be a firm's cost per negotiation and financing charge be uniformly distributed on the interval $[\underline{r}, \overline{r}]$, with $0 \leq \underline{r} < \underline{r} < \infty$. Let the current best financing charge be $r \in [\underline{r}, \overline{r}]$. Then the firm's management closes the negotiation process and accepts charge r if $r < r^*$. The indifference charge r^* is defined by*

$$r^* := \underline{r} + \sqrt{\frac{2\Delta r c_{rel}}{P}},$$

where $P > 0$ is the probability that the next negotiation is successful.

Proof: See the Appendix. □

Theorem 1 gives management a decision rule for accepting the current loan conditions or to conduct further negotiations. That decision rule uses all available information from prior negotiations.

When the actual cost of debt capital rK^* exceeds management's expectations, then the management seeks alternative sources of debt capital. When the actual financing charge r is higher than the indifference cost of debt capital r^*, then a firm's management should conduct further negotiations. If r is below that charge, management should settle the financing deal at rate r. When the loan cost equals management's expected cost of capital, then management is indifferent. As shown in the proof theorem 1, for a continuous distribution density function, the probability of this event is zero.

In addition, theorem 1 shows that when loan negotiation cost are zero, the optimal financing charge is $\underline{r} + \sqrt{\frac{2\Delta r c_{rel}}{P}} = \underline{r}$. Therefore when search and negotiation costs are zero, management stops the search until the best loan conditions are offered. In the remainder of this section this issue is discussed in more detail.

The analysis of management's negotiation strategy provides a decision rule on continuing loan search and negotiations. The next 2 propositions set conditions under which the management will accept a successfully negotiated offer.

Proposition 2. *When relative search and negotiation cost* c_M/K^* *exceeds the bound* $(\overline{r}-\underline{r})/3$, *then a firm's management deterministically accepts the first successfully negotiated offer. The expected financing charge is* $(\overline{r}-\underline{r})/2$ *in this "shortest negotiation process".*

Proof: See the Appendix. □

Proposition 3. *When relative search and negotiation cost* c_M/K^* *exceeds the bound* $(\overline{r}-\underline{r})/12$, *then a firm's management accepts the offer of the first successfully negotiated offer with probability of at least* 0.5.

Proof: See the Appendix. □

The above propositions 2 and 3 set conditions under which management accepts the offer of the first successfully negotiated offer. Proposition 2 states when management always accepts the first financing offer, whereas proposition 3 gives a condition for an acceptance with probability of at least 0.5.

Proposition 2 sets conditions when a firm's management accepts the first successfully negotiated financing offer with certainty, whereas proposition 3 gives a condition for accepting that offer on average. Here, by on average we mean that the probability exceeds 0.5. Obviously the condition of accepting the first offer deterministically is stronger than to accept it on average. The intuition behind this proposition is that the lower bound of $(\overline{r}-\underline{r})/12$ is smaller than the lower bound $(\overline{r}-\underline{r})/3$. Under the first bound the shortest negotiating process is chosen on average while under the second bound it is chosen deterministically.

For the 2 propositions we need the following parameters: (1) Search and negotiation cost as a proportion of loan size $c_{rel} = c/K^*$, and (2) the spread Δ between best and worst financing charges. That is, $\Delta = \overline{r} - \underline{r}$. When relative negotiation costs are higher than one third of the spread of financing charges, that is $c_{rel} \geq \Delta/3$, then proposition 2 states that management always closes the negotiation process after the first successfully negotiated offer. That is, when the negotiation cost are higher, then a firm's management ends negotiations sooner. When the loan size increases, management is willing to conduct more loan negotiations. This proposition also implies that a higher spread in financing charges encourages management to conduct additional negotiations. When financing charges vary by a spread of larger difference in best and worse financing charges, then this increases chances to improve financing conditions by continuing negotiations.

For example, let us consider relative negotiation cost that exceeds the bounds of the financing spread. Assume relative negotiation costs are higher than 1/12 of financing spread, i.e. $c_{rel} \geq \Delta/12$. Then proposition 3 states that management on average closes the negotiation process after the first successfully negotiated offer. When the financing offer of that negotiation exceeds the upper bound set in theorem 1, then negotiations are continued. The inequality $c_{rel} \geq \Delta/12$ has the same implications regarding changes in model parameters as proposition 2. The above argumentation provides us with a lower bound for relative negotiation cost such that management on average accepts the first successfully negotiated offer. That cost is ex ante 1/12 of the spread between best and worst outcome of a successful negotiation. For instance, when the spread is 12%, then a negotiation cost of 1% of loan size (or higher) causes a firm's management to accept the first offer on average. When that relative cost is 1/3 of the spread or higher, then management accepts the first successfully negotiated offer with probability 1. In our example, management deterministically accepts the first offer when the relative negotiation cost exceeds the lower bound $c_{rel} = c/K^* \geq 4\%$. In other words, the first offer is accepted deterministically when the cost per negotiation is higher than 4% of debt capital K^*.

In summary, under the above 2 propositions, the first successfully negotiated offer will accepted with probability of at least 0.5, or deterministically. We have acceptance with probability of at least 0.5 when $\Delta/12 \leq c_{rel} < \Delta/3$ and first offer acceptance with probability 1, while $c_{rel} \geq \Delta/3$.

The next proposition provides a general formula that combines the probability for the shortest negotiation process and a lower bound for the relative negotiation cost.

Proposition 4. *A firm's management optimally chooses the shortest negotiation process with at least probability $p \in [0, 1]$, when relative negotiation cost c_{rel} exceed the lower bound $p^2(\overline{r} - \underline{r})/3$. That is, when relative negotiation cost and spread in financing charges are known,* *then the probability*[4] *that it is optimal to stop the negotiation process after the first successfully negotiated offer is at least* $p^* = \sqrt{\frac{3c_{rel}}{\Delta r}}$.

Proof: See the Appendix. □

The above proposition calculates lower bounds for the relative negotiation cost such that management optimally chooses the shortest negotiation process with at least probability $p \in [0, 1]$. The proposition is in accordance with the formulas from propositions 2 and 3 when we allow $p = 1$ and $p = 0.5$, respectively.

The formula from the proposition may further by rearranged. As a result, we get a bound for the probability that management finds the shortest negotiation process most preferable as a function of financing spread and relative negotiation cost. This bound is, $p^* = \sqrt{\frac{3c_{rel}}{\Delta r}}$. As a result, assume that relative negotiation cost and the spread in financing conditions are known. Then a lower bound for the probability that a firm's management chooses shortest negotiation process can be calculated. This probability is given by the formula above. It can be seen that an increase in relative negotiation cost and a decrease in the financing spread have positive effect on the probability that management closes loan negotiations after the first successful negotiation.

The lower bound for relative negotiation cost is a linear function in the spread of financing conditions. That is, when this spread is zero (then each lender offers the same conditions deterministically) then that lower bound is zero. As a result, a firm's management always optimally chooses the shortest negotiation process. Intuitively, when the spread in financing conditions is zero, then the first offer is the best that can be achieved. Further negotiations thus can not result in a more favourable financing charge.

When the spread in financing conditions doubles, then relative negotiation costs also may double, while the probability that management optimally chooses the shortest negotiation process remains constant.

In the next proposition we discuss the influence of zero negotiation costs on the negotiation process.

Proposition 5. *If and only if negotiation cost is zero, management negotiates indefinitely. When negotiation costs are greater than zero, there is a deterministic end to the negotiation process.*

Proof: See the Appendix. □

Proposition 5 shows that only when negotiations are not costless, then management eventually accepts an offer and closes the deal. Intuitively, proposition 5 means that for every

[4]Note that this formula may exceed a maximum probability of 1. Rigorously, the term $p^* = min(\sqrt{\frac{3c_{rel}}{\Delta r}}, 1)$ is correct.

positive negotiation cost, there is a financing offer, which should optimally be accepted. That is, then additional negotiation costs are no longer justified by potential gains by further negotiations. Theorem 1 provides a formula for calculating the acceptable optimal loan cost, given a financing charge.

When there are no negotiation costs, then there is a chance that the management finds a better financing offer in the future by continuing negotiations. Therefore it is not optimal for the management to end the negotiation process.

The next proposition calculates the average cost for a shortest negotiation process.

Proposition 6. *Assume the firm follows the shortest negotiation process and stops negotiations after $N \in \mathbb{N}$ unsuccessful negotiations. When negotiation cost are c_M, then the firm's expected negotiation costs are*

$$\boldsymbol{E}(C) = c_M \left(ln(N) + \gamma + \mathcal{O}\left(\frac{1}{N^2}\right)\right),$$

where $\gamma \approx 0.5772$ is the Euler Mascheroni constant.

Proof: See the Appendix. □

Proposition 6 allows to calculate the expected costs for loan negotiations that the firm incurs. These costs are dependent on the cost per negotiation. When this cost is higher, then the total costs also rise. Expected negotiation costs in particular is a linear function in cost per negotiation. That is, when that cost is zero, then the negotiation process is costless. Further, doubling cost per negotiation doubles expected cost for the negotiation process.

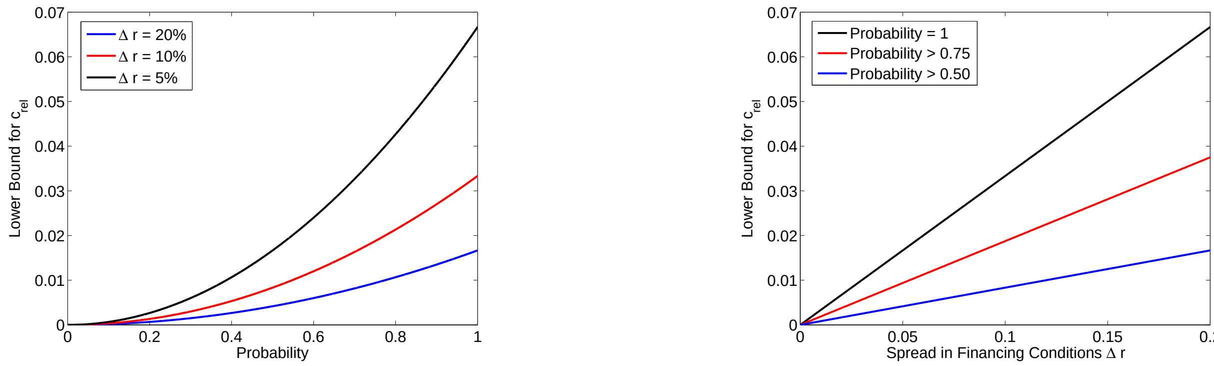
Its cost furthermore depends on the number of unsuccessful negotiations that the management allows until the negotiation process is unsuccessfully stopped. That is, when the management allows a high number of unsuccessful negotiations, then the total costs for the negotiation process increase.

Let us now analyse the implication of our model from a financiers perspective. In particular, we investigate the effect of expected deal settlement cost on providers of debt capital.

Lemma 2. *Given a negotiation cost of c_I for the bank and negotiations cost of $c_{rel} \geq \Delta/3$ for a firm, then on average a bank spends $2c_I$ to close a debt contract.*

Proof: See the Appendix. □

Lemma 2 analyses individual bank negotiations with a credit applicant. The bank receives an acceptable loan application with $p = 1/2$ (our modelling is robust for values $0 < p < 1$) and then makes a financing offer to that applicant. If the firm's relative negotiation costs are greater than $\Delta/3$, then the firm accepts the debt contract. Therefore successful debt contracts are geometrically distributed with $p = 1/2$. The mean of this distribution is 2. That is, a bank on average negotiates with 2 applicants before it successfully closes a deal. Therefore a bank's negotiation costs are $2c_I$. This cost is derived based on the firm's management accepting the first bank loan contract. When $c_{rel} < \Delta/3$ threshold, then the firm's management may reject a few loan offers. In that case, the bank's negotiation costs would be greater than $2c_I$.



(a) Lower bound for c_{rel} as a function probability for shortest negotiation process

(b) Lower bound for c_{rel} as a function of the spread in financing charges

Fig. 1 – Lower bound for relative negotiation cost to cause shortest negotiation process

We analyse the interdependence of the probability that management chooses the shortest negotiation process, a firm's management's relative negotiation cost and the spread in financing charges. A formula for the lower bound of relative negotiation cost as a function of probability and financing spread was established in proposition 4. Figure 1 illustrates this formula.

In figure 1 (a), the x-axis illustrates probability. On the y-axis, the lower bound for relative negotiation cost is drawn. Each of the three lines shows the lower bound for relative negotiation cost, such that the shortest negotiation process is the best negotiation strategy, as a function of the probability for this event. The three lines represent financing spread of 0.05 in blue (that is, $\Delta r = \overline{r} - \underline{r} = 0.05$), a spread of 0.1 (in red) and 0.2 (in black).

When the probability is zero on the x-axis, that is, the shortest negotiation process is chosen with a probability that exceeds zero, then any relative negotiation cost is sufficient for this event. When we observe probability of 1, then a relative negotiation cost of $\Delta r/3$ is necessary to have shortest negotiation process with that probability. This can be seen from the figure, where these values are 1.67% (when $\Delta r = 0.05$), 3.33% (when $\Delta r = 0.1$) and 6.67% (when $\Delta r = 0.2$). When we observe for instance $p = 0.5$, then the lower bound for relative negotiation cost is $\Delta r/12$. This can also been observed from the figure. The bound for relative negotiation cost is linear in financing spread. Furthermore, that bound is increasing in financing spread. This means that, given a certain bound for the probability, a firm's management is less likely to conduct shortest negotiation process when the financing spread increases.

Figure 1 (b) shows the lower bound for relative negotiation costs as a function of financing spread, such that a firm's management chooses the shortest negotiation process at least with a given probability. On the x-axis, the financing spread is drawn. The y-axis shows the lower bound for relative negotiation cost. The three lines determine lower bounds for relative negotiation cost, such that the shortest negotiation process is chosen at least with a given probability. The black line in this figure shows conditions, when the shortest negotiation process is optimal with probability 1. The red (blue) line illustrates this condition for a probability that is at least 0.75 (0.5).

When the spread in financing charges increases, then management is more likely to conduct more negotiations. That is, then the opportunity to obtain better financing conditions by further negotiations increases. When we decrease the certainty of the event that a firm's management finds the shortest negotiation process optimal, then the conditions may be weakened. This can be observed in the figure, as the black (red) line always is above the red (blue) line.

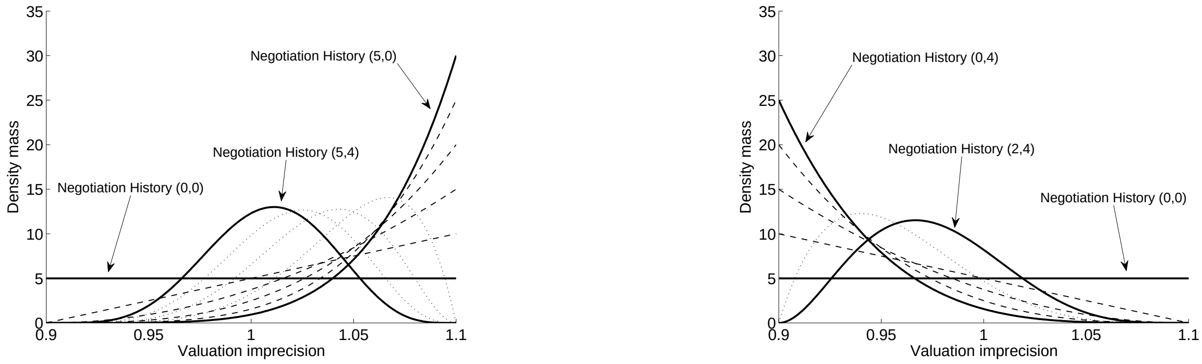
In our model each party suffers from a valuation imprecision. However, the distribution of the valuation imprecision is known by both parties. Let us continue with a numerical simulation, where the maximum valuation imprecision is $\alpha = 10\%$. However, our analysis is robust enough that it holds for any valuation imprecision $0 < \alpha < 1$.

Figure 2 (a) and (b) is drawn for two different negotiation processes. Management's updated estimation of their valuation imprecision in each negotiation process A and B is graphed in figure 2.

The graphs in figure 2 are distribution functions for management's Bayesian updated estimation of valuation imprecision during bank loan negotiations[5].

Figure 2 (a) is based on a relatively less successful sample negotiation process A. In this process the first five negotiations are unsuccessful (indicated by $(j, 0),\ j = 0, 1, ..., 5$). This is followed by four successful negotiations, where management receives financing offers (indicated by $(5, i),\ i = 1, 2, 3, 4$). At the beginning of the negotiations (when there is no negotiation history) management estimates a uniform distribution of imprecision. This is indicated by a horizontal line at negotiation history $(0, 0)$ (i.e. a constant distribution function). During the negotiations, management's loan application is denied 5 times. In that case, management updates its imprecision estimation and concludes that it is likely that it has initially overvalued the firm. This process is indicated by the dashed functions drawn between $(0, 0)$ and $(5, 0)$.

[5] If the value of α is different than 0.1, then the scale of the graphs changes. However, the proportions of the curve do not change. That is, the shape of management's updated estimation of their imprecision is independent of maximal valuation imprecision α.



(a) Imprecision distribution for negotiation process A

(b) Imprecision distribution for negotiation process B

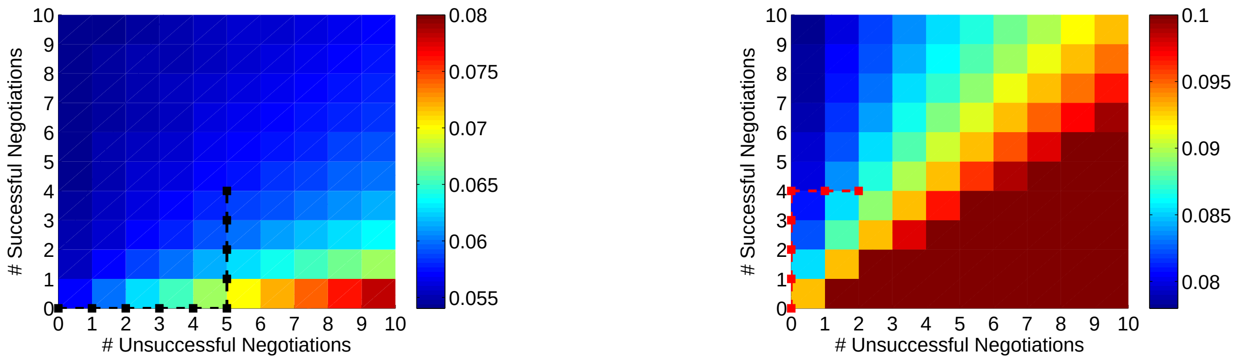
Fig. 2 – Management's Bayesian updated estimation of valuation imprecision during bank loan negotiations

In these dashed functions, the updated distribution mass shifts to the right of the curve. In subsequent negotiations, 4 bank loan applications of this firm are approved. This shifts the firm's updated Bayesian distribution function toward the mean valuation. This is indicated by $(5,4)$ and the process is drawn by the dotted functions between $(5,4)$ and $(5,0)$.

When the negotiation process starts, the probability of receiving a loan approval is ex ante 50%. However, at $(5,4)$, calculating the probability of a loan approval from the evolved density function, results to a Bayesian loan approval probability of 37.7%.

Figure 2 (b) graphs a relatively more successful negotiation process B. In that process the firm receives four loan approvals (indicated by $(0,i)$, where $i = 0,1,...,4$). These four loan approvals are followed by two loan rejections (indicated by $(j,5)$, where $j = 1,2$). At the beginning of the negotiation process (point $(0,0)$), the management has a uniform valuation imprecision on the interval $[1-\alpha, 1+\alpha]$. This is indicated by the uniform distribution function $f \equiv 5$. Note that a different α than 0.1 generates a different uniform distribution, that the integral over the interval $[1-\alpha, 1+\alpha]$ is equal 1. As the negotiation process continues, after $(0,4)$, where four loan applications are approved and none rejected, the management updates its distribution of its Bayesian distribution. They are confident that their estimation has more mass below average valuation. This can be seen by the distribution function with negotiation history $(0,4)$ that has most mass left to 1. Updated distribution functions of the negotiation process $(0,i)$, $i = 1,...3$ are plotted in dashed lines between distribution functions $(0,0)$ and $(0,4)$. Management can be more optimistic for future negotiation success. In sample process B however, the firm's management's loan application is rejected in the next two negotiations (indicated by negotiation history $(2,4)$). That process is graphed by the dotted distribution function $(1,4)$ between the distribution functions $(0,4)$ and $(2,4)$. After 4 loan approvals and

2 rejections, management updates their estimation on valuation imprecision. It calculates that their valuation distribution has high mass close to, but below average valuation. In fact, for the next negotiation, management's updated Bayesian success probability is 77.3%.



(a) Relative negotiation cost 0.001

(b) Relative negotiation cost 0.01

Fig. 3 – Indifference financing charge as a function of negotiation history and cost

Figure 3 shows indifference financing charges r^* as a function of negotiation history and cost. Tables 1 and 2 summarize key properties of negotiation processes A and B, respectively. For the above calculations $r \overset{d}{=} unif[5\%, 10\%]$. In both figures the x-axis illustrates the number of unsuccessful negotiations and negotiation successes are drawn on the y-axis. The colour indicates indifference financing charges for different combinations of successful and unsuccessful negotiations.

In figure 3 (a) the sample negotiation process A is illustrated. In this figure, the underlying indifference financing charges for a relative negotiation cost $c_{rel} = c_M/K^* = 0.001$ are shown. The figure shows that r^* is roughly between 5.5% and 8.0%. Also r^* is increasing in the number of unsuccessful negotiations. This result is not surprising, as an increasing number of unsuccessful negotiations suggests that a firm's management's valuation is comparatively high and therefore ex ante, the probability of future negotiation success decreases. Thus, the probability that negotiation costs are wasted is increasing. As a result, management is willing to accept higher financing charges.

An increasing number of negotiation successes on the contrary indicates ex ante that further negotiations are more likely to be successful as well, leading to a decreasing level of acceptable financing charges.

The black line in figure 3 (a) shows example negotiation process A. That process further can be followed in table 1. In figure 3 (a) the negotiation process starts at $(0, 0)$, which means that there are no previous successful or unsuccessful negotiations. In negotiation process A, the first 5 negotiations are unsuccessful. During that negotiation process, management's updated indifference financing charge rises from 5.70% to 7.00%, as can be seen in the figure on the black line from $(0,0)$ to $(5,0)$. This means that at the initiation of negotiations, management would have found 5.70% an acceptable offer. As the first, second, third, fourth and fifth negotiations fail, management estimates their valuation higher compared to the average of further financiers' valuations. That higher valuation on management's side leads to a diminishing probability for negotiation success. Therefore management is willing to accept a higher financing charge for the loan. In the example, the 6th negotiation is successful. That is represented by the point $(5,1)$ on the plane in figure 3 (a). As a result, management's indifference charge decreases to 6.27%. That is, when for instance an investor's offer is 8%, then management continues negotiations, because that offer exceeds management's upper bound.

Negotiation #	1	2	3	4	5	6	7	8	9
Success	n	n	n	n	n	y	y	y	y
Offer rate	-	-	-	-	-	8.0%	6.1%	6.0%	5.5%
Best offer	-	-	-	-	-	8.0%	6.1%	6.0%	5.5%
Indifference rate	6.00%	6.27%	6.53%	6.77%	7.00%	6.27%	6.00%	5.85%	5.76%
Further Negotiations	y	y	y	y	y	y	y	y	n

Table 1 – Properties of negotiation process A ($c_{rel} = 0.1\%$)

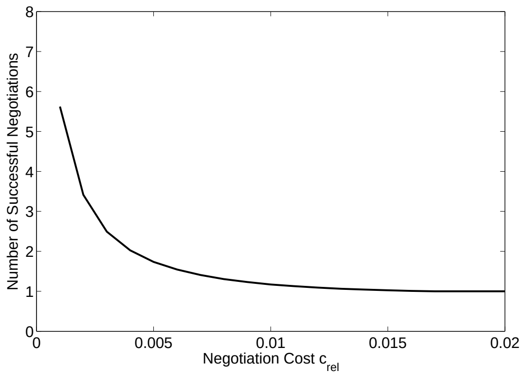
It can be seen from table 1 that the firm's management leads 3 more negotiations, which are all successful in negotiation process A. During these successful negotiations, the management's indifference charge decreases from 6.27% to 6.00% (represented by the point $(5,2)$), to 5.85% (represented by the point $(5,3)$), to 5.76% at the point $(5,4)$. When for instance the second successful negotiation's offer is 6.1%, then this offer is more favourable than the previous best offer. However, that offer exceeds the current management's indifference charge of 6.00%. The investor's offer after the next successful negotiation is 6%. In the previous step, management would have accepted that offer. Due to a further negotiation success, management is more optimistic regarding future negotiations. In fact, management's updated indifference charge is 5.85% and therefore the current offer is declined and further negotiations are conduced. That next negotiation is successful and the investor offers 5.5%. That investor's offer is below management's updated indifference charge, which is 5.76%. Therefore the firm's management accepts that offer.

In summary, management conducted 9 negotiations. Thus the firm's negotiation costs sum up to $9 \cdot c_{rel} = 9 \cdot 0.001K^* = 0.009K^*$, which is 0.8% of credit volume.

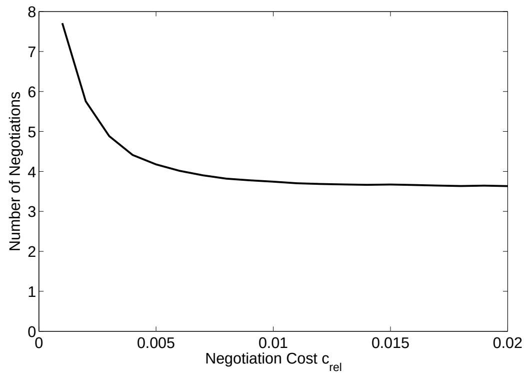
Underlying high negotiation costs, management accepts higher financing charges. That is, the number of negotiations should be reduced due to increased costs. The actual influence of comparatively high negotiation costs are illustrated in figure 3 (b). In this example, we follow negotiation process B with negotiation cost per negotiation that amounts to 1% of loan volume.

Negotiation #	1	2	3	4	5	6
Success	y	y	y	y	n	n
Offer rate	9.0%	9.3%	8.4%	8.6%	-	-
Best offer	9.0%	9.0%	8.4%	8.4%	8.4%	8.4 %
Indifference rate	8.52%	8.24%	8.09%	8.00%	8.35%	8.68%
Further Negotiations	y	y	y	y	y	n

Table 2 – Properties of negotiation process B ($c_{rel} = 1\%$)



(a) Average number of successful negotiations



(b) Average number of negotiations

Fig. 4 – Properties of negotiation process as a function of relative negotiation cost c_{rel}

The figure shows that when the number of unsuccessful negotiations exceeds the number of successful negotiations, then management should accept the best available offer and conduct no further negotiations deterministically. In the plane of maximal 10 (un)successful negotiations, minimum indifference charge is just below 8%. That indifference charge significantly exceeds that from figure 3 (a), where negotiation costs were lower. When relative negotiation cost is 1%, then indifference charge of 10% is reached whenever there are more unsuccessful than successful negotiations. In that case management accepts any given offer.

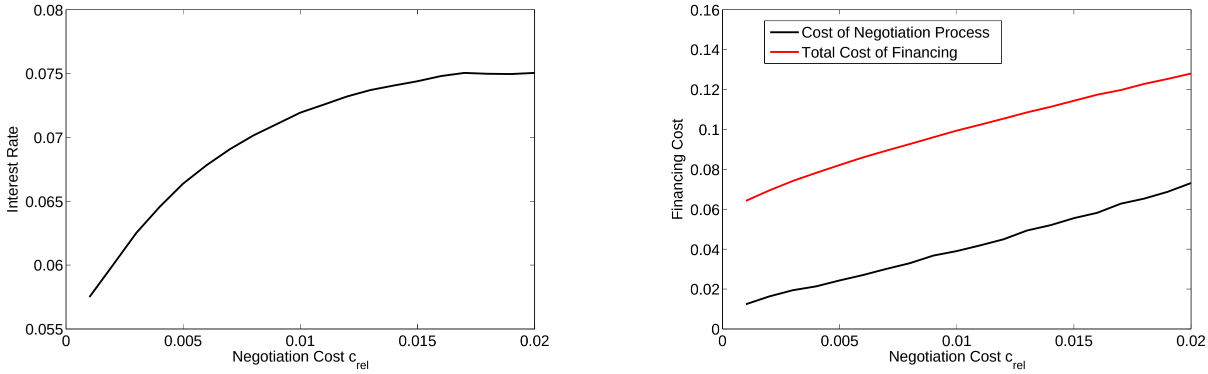
We follow negotiation process B, that is indicated by the red line in figure 3 (b) and summarised in table 2. Negotiations start at the point $(0, 0)$, that indicates that no previous negotiations have been lead. The first negotiation is successful. As a result of this success, management's updated indifference charge is 8.52%. Management however gets offered 9%. That coupon exceeds management's indifference charge. Therefore it is optimal for the management to conduct a further negotiation. That next negotiation is successful in our example and it is optimal that further negotiations are conducted until 4 negotiations were successful. That process is represented by the red line from $(0,0)$ the point $(0,4)$ in figure 3 (b). Due to these successful negotiations, management updates their indifference charge to 8.00%. In our example, the present best offer is a charge of 8.40%. If that was the offer of the first successful negotiation, management would have accepted it. However, in the negotiation process, management updated their assumptions about the success probability of further negotiations. In fact, they estimate relatively well success probability and thus management decline the present offer of 8.40%. In our example the next negotiation fails, represented by a move to the point $(1,4)$ in figure 3 (b). Consequently management's updated indifference charge rises to 8.35%, which still exceeds the present best offer. Therefore it is optimal for the management to conduct further negotiations. In the example the next negotiation is unsuccessful and management increases their updated indifference charge to 8.68%, that is represented by the point $(2,4)$. At this point of the negotiation process, the indifference interest charge of 8.68% exceeds the best offer of 8.35%. Therefore it is optimal to close the negotiation process and accept the best offer, which is 8.35%.

In this negotiation process 6 negotiations were conduced. Therefore the firm's negotiation costs sum up to $6 \cdot 0.01K^* = 0.06K^*$, which amounts to 6% of the loan size.

Probabilities are an important factor to this negotiation model because negotiation success and financing charges are random. That is, the negotiation process, its length and financing conditions depend on numerous interacting random factors. The remainder of this section analyses properties of the negotiation process and expected financing conditions. We have proven that when $c_{rel} \geq \Delta/3$, then a firm's management deterministically closes the negotiation process after the first successful negotiation. This was shown in proposition 2. Financing charges then are the average between the most favourable and the worst possible offer. While relative negotiation costs are within the bounds $\Delta/12 \leq c_{rel} < \Delta/3$, proposition 3 states that management closes the negotiation process after the first negotiation with a probability that exceeds 0.5. Proposition 4 further provided a lower bound for relative negotiation cost as a function of probability such that the shortest negotiation process is preferred. However, while that probability is below 1, ex post management may find it optimal to continue negotiations. When $c_{rel} = 0$, then the firm's management always continues negotiating because the next negotiation always offers positive expectation on better financing conditions for free, according to proposition 5.

Next, we analyse how often management negotiates on average until a financing agreement is met. Furthermore, the average financing charge as a function of relative negotiation cost will be analysed. The number of negotiations, average financing conditions and costs per negotiation finally lead to the total costs associated with loan financing.

The further analysis is based on a Monte-Carlo simulation. 250,000 random negotiation



(a) Average financing charge

(b) Average costs

Fig. 5 – Properties of negotiation process as a function of relative negotiation cost c_{rel}

processes were ran and their average properties are presented. Due to the Law of Large Numbers, that is a decent size to draw sufficiently exact conclusions on analytical mean values.

Figures 4 and 5 show major properties of the negotiation process as a function of negotiation costs. As in the above examples, the distribution of financing charges is $r \overset{d}{=} unif[5\%, 10\%]$. On the x-axis of each figure, relative negotiation costs $c_{rel} = c_M/K^*$ are drawn. We concentrate on negotiation cost that exceed zero because in the latter case, a firm deterministically gets offered best financing conditions for zero cost.

When relative negotiation cost is above or equal to $\Delta r/3$, then it is optimal for a firm's management to always accept the first offer. In this example, $\Delta r/3 = 5/3\% \approx 1.67\%$. This statement can be observed in figure 4, when the management behaves accordingly and accepts the first successfully negotiated offer when negotiation cost exceeds 1.67% of loan size.

In figure 4 (a), average number of successful negotiations until negotiation process is terminated is shown. For low negotiation costs, management continues negotiations even when they received an offer. The number of rejected offers decreases with increasing negotiation costs. From negotiation costs of approx. 1% of loan size, management accepts most offers. Management deterministically accepts the first offer when negotiation costs exceed of 1.67% of loan size.

The number of negotiations of an average negotiation process is decreasing in negotiation costs, as can be seen in 4 (b). In fact the number of negotiations per negotiation process converges to approximately 3.57[6] for high negotiation costs. That is, there are approximately 2.57 unsuccessful negotiations until deal settlement, even when management accepts the first successfully negotiated offer. The explanation is that when management values the company

[6]Apply proposition 6 with $N = 20$.

comparably high, then the negotiation success probability diminishes to a greater extent than the success probability increases as a result of undervaluation.

Figure 5 (a) shows average financing charges for the loan. While negotiation costs are low, management negotiates more and as a result, the financing charges are relatively low at approx. 5.5%. This is almost the minimum charge, which is 5%. When management decides on fewer negotiations when the negotiation costs increase, then the financing charges rise. When management accepts the first offer deterministically, then financing charges are exactly average at 7.5%.

Financing costs are analysed in figure 5 (b). The red line shows costs for the negotiation process. Total cost, which is the sum of negotiation costs and financing charges are illustrated by the black line. Obviously both costs are increasing in negotiation costs. Their increase is approximately linear.

In proposition 4 we calculated a lower bound for the probability that the shortest negotiation process is the most efficient negotiation strategy. This lower bound is a function of the spread in financing charges and relative negotiation cost. Figure 6 shows this lower bound as a function of relative negotiation cost and for different spreads in financing charges.

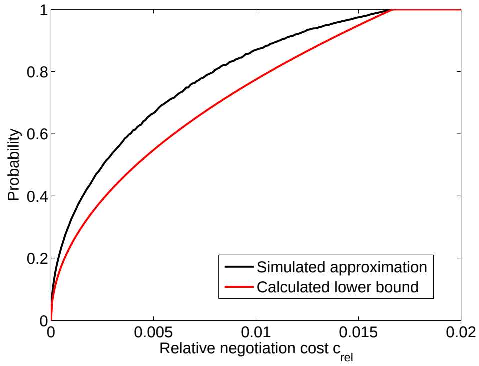
For the simulation of the actual probability 100,000 sample negotiation processes were run for each relative negotiation cost. For the simulation, one needs the probability of a negotiation success when no prior negotiation was successful and m were unsuccessful. That probability can be calculated from proposition 1 and is given by $2/(m+3)$.

In figure 6 (a) a spread of 5% in financing charges is illustrated and in figure 6 (b) that spread is 20%. On the x-axis, the relative negotiation cost is drawn. The y-axis illustrates the probability as a function of negotiation cost. Additionally to the lower bounds that are calculated from proposition 4, the simulated probabilities are illustrated. The simulations are close approximations to the true probability that a firm's management chooses the shortest negotiation process. Intuitively, the simulation therefore exceeds the lower bound of that probability. In both figures, however, the calculated lower bound is close to the true probability.

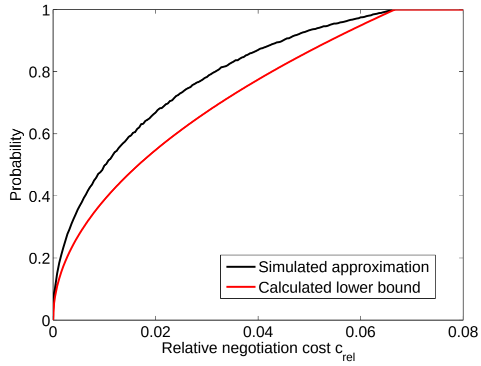
In figure 6 (a), where the spread in financing conditions is 5%, the shortest negotiation process is deterministically preferable, when relative negotiation cost exceeds 1.67%. That can be seen from the figure when both, the simulation and the analytically calculated probability for this event are 1. A probability that exceeds 0.5 is obtained from negotiation cost that exceed $0.05/12\% \approx 0.42\%$ of loan size.

In figure 6 (b), where the spread in financing conditions is 20%, the shortest negotiation process is deterministically preferable, when relative negotiation cost exceeds 16.7%. That can be seen from the figure when both, the simulation and the analytically calculated probability for this event are 1. A probability that exceeds 0.5 is obtained from negotiation cost that exceed $0.2/12\% \approx 1.67\%$ of loan size.

Comparing figure 6 (a) and (b), one can see that the shapes of the lines in both figures are similar. Just the scale of the $x-$axis is different. That is, in figure 6 (a), relative negotiation cost reach up to 0.02, whereas the scale of figure 6 (b) reaches its maximum at 0.08. The latter figure however, illustrates probabilities for a financing spread that is 4 times higher than that in figure 6 (a). As a result one can see that the probability for the shortest negotiation process to be optimal is linear in the financing spread, as these figures are almost indistinguishable from their shape.



(a) Spread in financing charges $\Delta r = 5\%$



(b) Spread in financing charges $\Delta r = 20\%$

Fig. 6 – Calculated lower bound and simulation of the probability that the shortest negotiation process is most efficient negotiation strategy

Now, let us introduce an investment banker's strategy.

4 Intermediation

The placement of debt securities is conducted pursuant to the 1933 Securities Act or to Rule 506 of regulation D., as Arena (2010) notes. According to these rules, firms may issue an unlimited amount of securities to an arbitrary number of accredited investors and up to 35 so called sophisticated investors. Typically firms place private debt with insurance companies, banks, high net worth individuals and private investment firms. Although these firms are still important players, they lost their dominance after the private debt market crunch in the early 1990s, as Carey et al. (1993) note.

An intermediary brings issuer and investor together. They are usually underwriters of banks or private investment firms. Usually private offerings are conduced on a best efforts rather than on firm commitment basis.

Underwriter's compensation is often closely bounded to bond volume and therefore dependent on the underwriter's success and the bond volume he places. Smith (1985) asks, "What determines optimal fee structure?" Baron (1979) analyses contracts in investment banking in general and in particular focuses on contracts under asymmetric information together with Holstrom (1980). In our model we work with fees that are proportional to bond volume.

Section 3 showed that the negotiation process in loan financing is cost intensive for a firm and investors, with negotiation costs c_M and c_I, respectively. An underwriter, such as an investment banker, is assigned to market a firm's debt in the form of bonds in this section. In our model, the bond volume is fixed at K^*. The intermediary operates under a best efforts basis and earns a fixed fee f (that is a proportion of bond volume K^*) when he successfully markets the bonds. Therefore the costs for the bond issuance are deterministic for the firm. A firm's management and investors are aware of costs that are associated with loan financing. In our model, the firm's management is therefore willing to accept a discount $d \geq 0$ on their valuation of the firm. Analogously investors are willing to increase their valuation by a certain amount $u \geq 0$. The investment banker then matches the firm's management's and investor's adjusted valuations. Thereby the parties must not know their respective adjusted valuations. Otherwise they could find a mutually satisfying agreement without the intermediary, who would be redundant. He therefore creates an environment of information asymmetry, where he systematically hides information from either party. In this section we show that this strategy creates positive expected earnings for the underwriter and that it is beneficial for the firm and its investors.

In our model the expected costs for deal settlement in loan financing are known and given as proportions of financing volume, represented by $c_I^* = c_I/K^*$ and $c_M^* = c_M/K^*$. The intermediary's fee is denoted by f and a proportion of bond volume.

The next propositions shows when a firm and its creditor profit from the intermediaries services.

Proposition 7. *The firm profits from intermediation while $d \leq c_M^* - f$ and a creditor profits from intermediation while $u \leq c_I^*$. This means that these players profit from intermediation while discount d (valuation adjustment u) is lower than the deal settlement costs for loan negotiations (minus dealer fee in case of the firm).*

Proof: See the Appendix. □

Proposition 7 determines the conditions under which the firm and lenders profit from intermediation.

A firm profits, while discount d on their valuation is below loan negotiation cost minus investment banker's fee (each given as a proportion of placed debt K^*). Therefore low discount d favours bond issuing over loan financing. A low dealer fee f has the same effect. As an intuitive result, an investment banker can attract more clients when he offers lower fees.

A creditor prefers intermediation over granting a loan, while his increase in valuation is below deal settlement cost for loan negotiations. Investing in a bond does not incur cost for deal settlement for an investor. Thus he is willing to adjust his firm valuation while that adjustment does not exceed deal settlement cost in loan negotiations.

The next proposition calculates the probability for successful bond placement.

Proposition 8. *The probability that an intermediary can match a financier and a borrower is*

$$\frac{1}{2} + (u + d - f)K^*/2/V/\alpha.$$

Proof: See the Appendix. □

Proposition 8 shows the influence of model parameters on the probability for successful bond placement. It is 0.5 plus the product $(u + d - f)K^*/2/V/\alpha$.

Management's valuation discount d thus has positive effect on the success probability. That is, a high discount makes it easier for the investment banker's to successfully place the bond. Therefore the firm's reduction in their valuation favours bond financing negotiations to be successful.

A high investor's valuation adjustment u further increases the success probability for the bond placement. That is, an investor with a higher valuation is more likely to exceed management's valuation than an investor with a lower valuation. When an investor accepts a higher valuation adjustment u, then he thereby affects negotiation success positively.

Low intermediation fee favours the probability that the bond can be placed successfully. The firm pays the underwriter's fee. Therefore it reduces the firm's value. Compared to a high company value, a lower valued firm is less attractive for investors. Therefore higher fees are accompanied with a decrease in the probability that the bond can be placed successfully.

When the bond volume as a fraction of company value (K^*/V) increases, then this affects the intermediary's success probability negatively. K^*/V is a measure of debt to firm value. That is, when the bond volume increases, then that ratio rises. This bares a higher risk and consequently deal success probability decreases. This can be deducted from proposition 8 with an analysis of the term $(u + d - f)K^*/2/V/\alpha = (uK^* + dK^* - fK^*)/2/V/\alpha$. Increasing K^* by keeping all other values fixed needs some care in this case. uK^* and dK^* are constant values management and investors use to adjust their respective valuations. Increasing K^*/V thus solely affects an investment banker's fee fK^*, which is per definition a variable dependent on K^*. It therefore can summarised that a rise in K^*/V affects deal settlement probability negatively.

Valuation imprecision α influences the intermediary's probability for successful bond placement negatively. High valuation imprecision implies that management's and investor's valuations may differ to a greater extent. A firm's Management increases their valuation and investors decrease their valuation. However, given a higher valuation imprecision, the probability that adjusted valuations are compatible decreases. This affects investment banker's success probability negatively. When valuation imprecision is minimal ($\alpha = 0$), then all valuations are equal. As a result, any valuation adjustment that exceeds zero has the effect that a firm's management's and an investor's adjusted valuations are compatible. Then the intermediary may agree with the first investor on terms to place the bond.

The intermediary tries to sell bonds to investors sequentially until mutually satisfactory agreement is met. The next lemma calculates the expected negotiations until the bond is placed.

Lemma 3. *On average the intermediary conducts*

$$\boldsymbol{E}(N) = \frac{\alpha}{2\alpha + (u + d - f)K^*/V}$$

negotiations until the bond is successfully placed.

Proof: See the Appendix. □

Lemma 3 calculates average number of negotiations until the bond is successfully placed by an underwriter. This negotiation process can be interpreted as a geometrical distribution with success probability $p = \frac{1}{2} + (u + d - f)K^*/2/V/\alpha$. The expectation of that process is $\mathbf{E}(N) = 1/p$, as stated in lemma 3.

A high success probability p lowers the expected number of necessary negotiations until the bond is placed. Therefore the interpretations from proposition 8 apply to $\mathbf{E}(N)$: high investor's adjustment, high management's valuation discount, low intermediation fee, low K^*/V and low valuation imprecision have positive effect on the number of necessary negotiations until the investment banker successfully places the bond.

Proposition 8 show the influence of the model's parameters on the intermediary's negotiation success. He earns a fee f for his services. Lemma 3 shows that the negotiation success probability of that results from low fees exceeds that from higher fees. For an intermediary, the upside of high fees are more earnings for successful bond placement. The downside of high fees are more (costly) negotiations until the bond is successfully placed. Therefore there is an optimal adequately chosen fee that maximises an intermediary's expected earnings. When we combine proposition 8 and lemma 3 we can calculate a dealer's expected earnings. These calculations are the first step to determine the dealer's optimal fee strategy.

Proposition 9. *Let dealer's cost (as a proportion of K^*) per negotiation be c_d. Then his expected earnings are*

$$\boldsymbol{E}(D) = K^* \left(f - \boldsymbol{E}(N) c_d\right) = K^* \left(f - c_d/p_d\right) = K^* \left(f - \frac{2\alpha c_d}{\alpha + (u + d - f)\, K^*/V}\right).$$

Proof: See the Appendix. □

Proposition 9 shows that the dealer's expected earnings depend on a number of parameters: his fee f and cost per negotiation c_d. A further determinant is $\mathbf{E}(N)$, the expected number of negotiations until an agreement is met. Lemma 3 shows that the figure $\mathbf{E}(N)$ depends on further model parameters.

The dealer's expected earnings drop when his negotiation costs increase. That is, a higher cost per negotiation affect his earnings negatively. That fact can be seen from the formula, where c_d and expected earnings are negatively connected. This is intuitive, has higher costs usually have negative effect on one's earnings.

Expected negotiations $\mathbf{E}(N)$ is a major determinant to a dealer's expected earnings. In particular, high investor's adjustment, high management's valuation discount, low K^*/V and low valuation imprecision have positive effect on the number of negotiations and therefore for a dealer's expected earnings.

His fee f further determines an intermediary's expected earnings. His earnings are neither strictly increasing nor decreasing as a function of that fee. In fact, there is an optimal fee f_{opt} for which earnings are maximised. The next propositions develop a formula for the optimal dealer's fee strategy. The next proposition shows how the dealer optimally treats the firm and investors.

Proposition 10. *The dealer optimizes his earnings by persuading the firm's management and the investors to maximally discount their valuation ($d = c_M^* - f$) and to maximally increase their valuation ($u = c_I^*$), respectively.*

Proof: See the Appendix. □

Proposition 10 shows that an intermediary tries to persuade management to undervalue their firm as much as possible. At the same time he persuades investors for a maximal increase in their valuation. This strategy allows the intermediary to earn maximal fee, as shown in proposition 10.

The optimal investment banker's fee and its influence on expected earnings are analysed in the next theorem.

Theorem 2. *When the intermediary's auxiliary condition is to require at the most as many negotiations as an investor in bilateral loan negotiations, then his optimal fee and expected earnings in fee optimum are*

$$f_{opt} = min\{u + d + \alpha V/K^* - \sqrt{2c_d\alpha V/K^*}, u + d\}$$

$$\boldsymbol{E}(D) = K^* \cdot \begin{cases} u + d - 2c_d, & c_d \leq \frac{\alpha}{2K^*/V} \\ u + d + \alpha V/K^* - 2\sqrt{2\alpha c_d V/K^*}, & c_d > \frac{\alpha}{2K^*/V}. \end{cases}$$

Proof: See the Appendix. □

Lemma 4. *When the intermediary applies optimal fee f_{opt}, then the intermediation success probability per negotiation is greater or equal* 0.5.

Proof: See the Appendix. □

Theorem 2 calculates the optimal intermediary's fee f_{opt}, such that his expected earnings are maximised. This maximum is calculated under the constraint that the dealer places the bond quicker than bilateral loan negotiations are. That goal is achieved by requiring the intermediation success probability per approach to exceed 0.5. Lemma 4 shows that this requirement is met. By theorem 2 the intermediary's expected earnings are

$$\mathbf{E}(D) = K^* \cdot \begin{cases} u + d - 2c_d, & c_d \leq \frac{\alpha}{2K^*/V} \\ u + d + \alpha V/K^* - 2\sqrt{2\alpha c_d V/K^*}, & c_d > \frac{\alpha}{2K^*/V}, \end{cases}$$

when the intermediary applies optimal fee f_{opt}.

When his negotiation costs are sufficiently low ($c_d \leq \frac{\alpha}{2K^*/V}$), then his expected earnings are $K^*(u + d - 2c_d)$. In this case the earnings can be calculated by 3 parameters - management's and investor's valuation adjustments and the dealer's negotiation cost. Then the statement from proposition 10 becomes more intuitive: The dealer maximises his expected earnings by persuading management for maximal undervaluation and investors for maximal over valuation. Furthermore he profits from minimising his own negotiation costs.

When the intermediary's negotiation costs exceed a critical level ($c_d > \frac{\alpha}{2K^*/V}$), then a lower bound for his expected earnings can be established:

$$\begin{aligned} K^*(u + d + \alpha V/K^* - 2\sqrt{2\alpha c_d V/K^*}) &> K^* \left(u + d + \alpha V/K^* - 2\sqrt{\frac{2\alpha V/K^*\alpha}{2K^*/V}} \right) \\ &= K^*(u + d + \alpha V/K^* - 2\alpha V/K^*) \\ &= K^*(u + d - \alpha V/K^*). \end{aligned}$$

In that case it can be observed that the valuation imprecision α has negative effect on the lower bound of the dealer's expected earnings. Analysing the effect of valuation imprecision in more detail, we see that it also influences the case $c_d \leq \frac{\alpha}{2K^*/V}$: For any fixed negotiation cost, there is a sufficiently high valuation imprecision, such that the case $c_d \leq \frac{\alpha}{2K^*/V}$ is met. This is beneficial for the dealer. That is, when valuation imprecision is sufficiently high, then it stops to influence a dealer's earnings.

Let us formalise a further property of the dealer's earnings in the following proposition.

Proposition 11. *When the intermediary applies optimal fee schedule, then his earnings are strictly positive while* $c_d \leq (u + d)/2$.

Proof: See the Appendix. □

Proposition 11 shows that the intermediary expects positive earnings while his negotiation costs are lower or equal to average valuation adjustments of management and investors. When this condition is met, then he profits from intermediation. An intermediary's skills therefore are a critical issue, as it is of his benefit when he can pursue a firm's management and investors to maximally adjust their original valuations.

The next proposition formalises the influence of the valuation imprecision on an intermediary's earnings.

Proposition 12. *When an intermediary applies optimal fee* f_{opt}*, then his earnings are independent of valuation imprecision while* $c_d \leq \frac{\alpha}{2K^*/V}$. *When* $c_d > \frac{\alpha}{2K^*/V}$*, then valuation imprecision has negative effect on his expected earnings.*

Proof: See the Appendix. □

When the intermediary's negotiation costs are below or equal to $\frac{\alpha}{2K^*/V}$, then proposition 12 states that valuation imprecision and his expected earnings are independent. When his negotiation costs exceed that bound, then the party's valuation imprecision has negative effect on his earnings. The bound $\frac{\alpha}{2K^*/V}$ is dependent on valuation imprecision α. Therefore for any given negotiation cost c_d, there is a sufficiently large valuation imprecision α, such that the intermediary's expected earnings are independent of imprecision. That is, for each negotiation cost c_d, there is some valuation imprecision α^*, such that the imprecision has negative effect on the expected earnings while $\alpha \in [0, \alpha^*]$ and no effect on expected earnings while $\alpha > \alpha^*$. Therefore the intermediary does not need to consider valuation imprecision when it exceeds α^*. In that case, his earnings are independent of valuation imprecision.

A firm and investors profit from intermediation while their valuation adjustments stay within certain bounds. Proposition 7 calculates these bounds explicitly. The intermediary profits from strong valuation adjustments and thus peruses management for strong discount and investors for high adjustment of their valuations. Proposition 11 shows that this behaviour is the most efficient strategy for the intermediary.

Theorem 2, lemma 4 and propositions 11 and 12 build the framework for further analysis on optimal intermediary's strategy. Theorem 2 determines the optimal fee as a function of the model parameters. It furthermore includes the constraint that intermediation needs to be at least as efficient as loan financing with respect to the duration of the negotiation process. Lemma 4 proves that intermediation is more efficient than bilateral loan negotiations under optimal fee f_{opt}. Theorem 2 closes with the calculation of the expected intermediary's gain in fee optimum. That formula is the basis to determine the intermediary's optimal strategy to maximise his gain.

Proposition 11 introduces the condition $c_d < (u+d)/2 \leq (c_M^* + c_I^*)/2$. This means that the intermediary's cost for negotiations are necessary lower than average negotiation costs of management and investors. That is reasonable as an intermediary leads negotiations more often than management or investors and therefore is more skilled and can more easily conduct negotiations. Under this condition proposition 11 states that the intermediary's expected gain is always positive. An intermediary always profits on average from bond marketing when his negotiation cost is sufficiently low.

Proposition 12 shows that there always is a certain border α^*, such that intermediary's expected gain is independent of the distribution of valuation imprecision. That is, an intermediary may profit from intermediation independent of the imprecision's distribution. Therefore, when imprecision exceeds a certain bound α^*, he has no intention to influence management or investors with regards to the accuracy of their valuation precision. This point of view is valid as long as the intermediary possesses only the ability to alter valuations symmetrically. That is, higher valuation imprecision bares the same risk of high over- and undervaluation.

Intuitively speaking, the expected gain of an intermediary increase when bilateral loan negotiations are costly. Furthermore low intermediation costs lead to a higher expected gain for the intermediary. An intermediary tries to cut his own costs on intermediation and persuades management and investors to high valuation adjustments.

The relative advantage of bond financing and intermediation over bilateral loan negotiations is discussed in the next section.

5 The Upside of Information Asymmetry

In this section, we will discuss when bond financing by intermediation is preferred by all parties over loan financing. The investment banker creates an environment of information asymmetry, whereas in loan financing negotiations full information is available. Consequently the preference of intermediation over loan financing implies preference of information asymmetry over full information.

The next theorem states conditions for Pareto efficiency of bond financing.

Theorem 3. *When a firm's management's and investor's valuation adjustments are bounded by $d \leq c_M^* - f$ and $u \leq c_I^*$, respectively and the intermediary's cost per negotiation is bounded by $c_d < (d+u)/2$, then all players prefer bond financing by intermediation over loan financing. That is, then information asymmetry Pareto dominates full information.*

Proof: See the Appendix. □

When the conditions of the theorem are satisfied, then the firm and an investor prefer intermediation over bilateral loan negotiations. Furthermore, then the intermediary has positive expected earnings from his strategy. Therefore all parties find that intermediation dominates bilateral loan negotiations. As a consequence, bond financing by intermediation is Pareto efficient over bilateral loan negotiations. The intermediary holds more information than the other parties and systematically hides information from them. In bond financing information is therefore asymmetrically distributed, compared to bilateral loan negotiations, where full information is available.

The theorem holds, while the intermediary pursues management to increase their valuation by u and bond investor to decrease his valuation by d. His costs must not exceed average valuation adjustments $(u + d)/2$. That is, he either needs ex post low negotiation costs or needs to pursue management and investor to high valuation adjustments. These parties are willing to adjust their valuations until a certain bound is met. When the adjustment exceeds that bound, then the firm and investors prefer bilateral loan negotiations over bond financing.

Next, the previous 2 sample negotiation processes from section 3 are discussed and the intermediary's strategy is illustrated.

Example (Sample process A): In sample process A, management approaches 8 potential lenders until management and lender agree on loan financing conditions in the ninth negotiation with financing charge 5.5%. Cost per negotiation are 0.1% of loan volume. Therefore negotiation costs for the firm add up to 0.9% of loan volume and the investor's negotiation costs add up to 0.2% by lemma 2. If loan volume K^* is 500, then the firm's negotiation costs are 4.5 and cost of capital is 27.5, whereas the investor faces absolute negotiation cost of 0.5 and earns 27.5 on interest.

In our example the intermediary's fee is 0.5% of bond volume. Then management can be pursued to reduce their valuation by up to 0.4% of bond volume, which is 2 in absolute terms. Let management's original valuation be 1005. Then it can be pursued to reduce its valuation to 1003. Furthermore, the intermediary may pursue a bond investor to increase his original valuation by up to 0.2% of bond volume, which is 4.5 in absolute terms. Now let first investor's valuation be 975. Even if the intermediary achieves maximum valuation increase to 979.5, management's and investor's valuation are incompatible. Let the second investor's valuation be 999. The intermediary may pursue the investor for a valuation of 1003.5. Then that investor's valuation is higher than management's valuation and the bond can be placed. In our example, the coupon is 5.5%. Then the costs of capital (and profit from interest) are the same as in loan financing.

The firm pays the dealer a fee for successful bond placement, which is 2.5 and lower than negotiation costs of 4.5 in the loan negotiation case. Investors save negotiation costs of 4.5 compared to loan negotiations. At last, the intermediary faces costs for two negotiations and has earned a fee of 2.5. When the cost per negotiation is lower than management's and investor's, then his negotiation costs are at the most 1 and he earns at least 1.5. Summing up the preferences of all parties, intermediation and bond financing Pareto dominates loan financing in this example.

Example (Sample process B): In sample process B, the firm's management approaches 5 potential lenders until the parties agree on financing conditions after the 6th negotiation. The negotiated financing charge is 8.4%. The costs per negotiation are 1% of loan volume. Therefore the negotiation costs for the firm add up to 6% of loan volume and the investor's negotiation costs add up to 2% by lemma 2. That is, when the loan volume K^* is 200, then the firm's negotiation costs are 12 and the financing charge is 16.8. The investor has absolute negotiation cost of 4 and earns 16.8 on financing charges.

In this example, the intermediary's fee is 1% of bond volume. Then the firm's management can be pursued to reduce their valuation by op to 5% of bond volume, which is 10 in absolute terms. When the firm's management's original valuation is 1005, then it can be pursued to reduce valuation to 995. Furthermore, the intermediary may pursue an investor to overvalue his original valuation by up to 2% of bond volume, which is 4 in absolute terms. When the first investor's valuation is 996, then the intermediary may achieve a maximum valuation increase to 1000. That is, then investor's valuation exceeds management's valuation and the bond can be successfully placed. Note that the investor does not have to adjust her valuation when management adjusts to 995. In our example, the coupon is 8.4%. Then the cost of capital (and profit from capital) is as high as in loan financing.

The firm pays the dealer a fee for bond placement, which is 2 and thus lower than negotiation costs of 12 in loan negotiations. Investors save negotiation costs of 4 compared to loan negotiations. At last, the intermediary has costs for one negotiation and has earned a fee of 2. When the intermediary's negotiation costs are lower than management's and investor's, then his negotiation costs are at the most 1. That is, his earnings are at least 1. Summing up the preferences of all parties in this example, bond marketing with intermediation is Pareto efficient over loan financing.

6 Conclusion

We have presented a model that shows how a firm's management and potential investors optimally behave in loan negotiations. That model's parameters were negotiation costs and - although investors have full insight into the firm's books - valuation imprecision on both sides. When the management's and the investor's valuations are not compatible, then loan negotiations are unsuccessful and negotiation costs are lost on both sides. If these negotiations are successful, the firm's management still may choose to conduct negotiations with further investors as the expected reduction in financing costs may be higher than the costs of further negotiations. The negotiation process is closed when a mutually satisfactory agreement between the firm and an investor is achieved. That process is associated with an expected negotiation cost on both sides.

As an alternative, an intermediary can be hired to market a firm's bonds. He necessarily needs to maintain information asymmetry between a firm and an investor to create an "either-or" decision between bilateral loan negotiations and the intermediary's services. Otherwise, a firm's management and an investor first may conduct bilateral negotiations. When their negotiations fail, the intermediary can be hired as a second alternative. Finally only "lemons" are left for the intermediary. This means firms with high expectations on their value and investors with low valuations seek his services. While the intermediary maintains information asymmetry between the parties, he experiences the full spectrum of valuations. In fact, when he determines his fee wisely, his services can be Pareto efficient over bilateral loan negotiations. Then the firm and investors prefer his services over conducting bilateral loan negotiations. That is, they accept information asymmetry over bilateral loan negotiations under full information.

Our framework can be extended in various ways. For example, the parties' valuation imprecision may be distributed more generally. A log-normally distributed valuation imprecision can be such an intuitive extension. Furthermore, a firm's and an investor's valuation may have a different distribution, as their information on a firm's value is divergent. Additionally, the loan financing charges and the coupon size may be modelled in a more general way. In fact, considering bonds with a maturity that exceeds one period is an interesting extension of our model.

7 Appendix

Proof of Lemma 1: The cdf of $min(X_1, X_2, ..., X_n)$ is given by

$$\begin{aligned} M(x) &= \mathbf{P}\left(min(X_1, X_2, ..., X_n) \leq x\right) = 1 - \mathbf{P}\left(min(X_1, X_2, ..., X_n) > x\right) \\ &= 1 - \mathbf{P}\left(X_i > x, i = 1, ..., n\right) = 1 - \mathbf{P}\left(X_1 > x\right)^n = 1 - \left(\frac{x_2 - x}{x_2 - x_1}\right)^n . \end{aligned}$$

The pdf is given by the first derivative of $M(x)$. □

Proof of Proposition 1: We first calculate the updated distribution function of management's valuation imprecision, given negotiation history.

$$\begin{aligned} F_{m,n}(x) &= \mathbf{P}(V_M < x | n \text{ negotiations successful, } m \text{ negotiations unsuccessful })\\ &= \frac{\mathbf{P}(V_M < x,\ n \text{ negotiations successful, } m \text{ negotiations unsuccessful})}{\mathbf{P}(n \text{ negotiations successful, } m \text{ negotiations unsuccessful })}\\ &= \frac{\int_{1-\alpha}^{x} \frac{1}{2\alpha}\ \mathbf{P}(y < min(X_1, ..., X_n))\mathbf{P}(y > max(X_1, ..., X_m))\ dy}{\int_{1-\alpha}^{1+\alpha} \frac{1}{2\alpha}\ \mathbf{P}(y < min(X_1, ..., X_n))\mathbf{P}(y > max(X_1, ..., X_m))\ dy}\\ &= \frac{\int_{1-\alpha}^{x} \frac{1}{2\alpha}\ \left(\frac{b-y}{b-a}\right)^n \left(\frac{y-a}{b-a}\right)^m\ dy}{\int_{1-\alpha}^{1+\alpha} \frac{1}{2\alpha}\ \mathbf{P}(y < min(X_1, ..., X_n))\mathbf{P}(y > max(X_1, ..., X_m)\ dy}\\ &= \frac{(2\alpha)^{-m-n-1} \int_{1-\alpha}^{x} (b-y)^n (y-a)^m\ dy}{(2\alpha)^{-m-n-1} \int_{1-\alpha}^{1+\alpha} (b-y)^n (y-a)^m\ dy}. \end{aligned}$$

The density is the first derivative of this distribution function. Due to the First Fundamental Theorem of Calculus and the Binomial Theorem we derive the density.

$$\begin{aligned} f_{m,n}(x) &= \frac{\partial F_{m,n}(x)}{\partial x} = \frac{(b-x)^n (x-a)^m}{\int_{1-\alpha}^{1+\alpha} (b-y)^n (y-a)^m\ dy}\\ &= \frac{(b-x)^n (x-a)^m}{\int_{1-\alpha}^{1+\alpha} \sum_{k=0}^{n} \binom{n}{k}(-1)^k y^k b^{n-k} \sum_{l=0}^{m} \binom{m}{l}(-1)^{m-l} y^l a^{m-l}\ dy}\\ &= \frac{(b-x)^n (x-a)^m}{\sum_{k=0}^{n} \sum_{l=0}^{m} \int_{1-\alpha}^{1+\alpha} \binom{n}{k}(-1)^k y^k b^{n-k} \binom{m}{l}(-1)^{m-l} y^l a^{m-l}\ dy}\\ &= \frac{(b-x)^n (x-a)^m}{\sum_{k=0}^{n} \sum_{l=0}^{m} \binom{n}{k}\binom{m}{l}(-1)^{k+m-l} a^{m-l} b^{n-k} \frac{1}{k+l+1}((1+\alpha)^{k+l+1} - (1-\alpha)^{k+l+1})}. \end{aligned}$$

□

Proof of Theorem 1: The probability P that the next negotiation is successful (after n negotiations were successful and m were unsuccessful) can be calculated from proposition 1. Assume that this probability exceeds zero. Otherwise further negotiations are unsuccessful deterministically and are therefore not conducted.

It is a firm's management preference to continue negotiations, when the expected reduction in financing charges exceeds the cost for that negotiation. That is,

$$
\begin{aligned}
&\mathbf{E}\left(\text{Improvement in financing charges}|\text{Negotiation history}\right) > c_{rel}\\
\Longleftrightarrow &\mathbf{P}\left(\text{Deal success}|\text{Negotiation history}\right)\mathbf{P}\left(\text{Better offer than } r\right)\left(r - \mathbf{E}\left(\text{Offer}|\text{Offer} < r\right)\right) > c_{rel}\\
\Longleftrightarrow &P\frac{r-\underline{r}}{\Delta r}\left(\frac{r}{2}-\frac{\underline{r}}{2}\right) > c_{rel}\\
\Longleftrightarrow &\left(r^2 - 2r\underline{r} + \underline{r}^2\right)\frac{P}{2\Delta r} - c_{rel} > 0.
\end{aligned}
$$

The above term is zero for

$$
r_{1,2} = \underline{r} \pm \sqrt{2\Delta r c_{rel}/P}.
$$

When $r = \underline{r}$, then the term is negative while relative negotiation cost exceed zero, that is $c_{rel} > 0$. For $r \to \infty$, that term is positive. As a result, the above term is positive, when $r > r_2 = \underline{r} \pm \sqrt{2\Delta r c_{rel}/P}$. In that case, management favours further negotiations. When $r < \underline{r} \pm \sqrt{2\Delta r c_{rel}/P}$, then further negotiations are rejected. Then the next negotiation's expected improvement in financing conditions is lower than the cost that negotiation. □

Proof of Proposition 2: Take the assumption that the first negotiation is successful with highest financing charge $\overline{r}$. If the first negotiation is successful, then a firm's management estimates their success probability in future negotiations higher than that, when the first negotiation is not successful. Given highest charge $\overline{r}$, a success in a future negotiation deterministically leads to lowered costs for financing. Therefore, in this setting it is the most probable that management continues negotiations. In all other variations negotiation (un)successfulness and financing charges, management is thus less inclined to continue negotiations, compared to the case where the financing charge is maximal at $\overline{r}$. Therefore, if management rejects further negotiations in this case, it will reject further negotiations in any other case as well.

Density of valuation imprecision after the first negotiation was successful is

$$
f_{1,0}(y) = \frac{y-a}{\int_a^b (y-a)\ dy} = \frac{y-a}{(b-a)^2/2} = \frac{y-1+\alpha}{2\alpha^2}
$$

Management rejects further negotiations if expected cost reductions due to better financing conditions are lower than the cost for these negotiations. This is represented by the inequality

$$
\begin{aligned}
&\mathbf{E}(\overline{r}K^* - rK^*|1 \text{ negotiation success}) < c_M \\
&\quad\Longleftrightarrow K^*\mathbf{P}(\text{Negotiation success}|1 \text{ previous negotiation success})\,(\overline{r} - \mathbf{E}(r)) < c_M \\
&\quad\Longleftrightarrow K^*\,(\overline{r} - (\overline{r} + \underline{r})/2) \int_a^b f(x)\mathbf{P}(V_I > x)\;dx < c_M \\
&\quad\Longleftrightarrow K^*(\overline{r} - \underline{r})/2 \frac{1}{4\alpha^3} \int_a^b (x-a)^2\;dx < c_M \\
&\quad\Longleftrightarrow K^*(\overline{r} - \underline{r})/2 \frac{8\alpha^3}{12\alpha^3} < c_M \\
&\quad\Longleftrightarrow (\overline{r} - \underline{r})/3 < c_M/K^*.
\end{aligned}
$$

Therefore relative negotiation cost c_M/K^* higher than $(\overline{r} - \underline{r})/3$ deterministically make management accept the first successful negotiation's financing conditions. On average these negotiation's charges are given by $(\overline{r} + \underline{r})/2$. $\square$

Proof of Proposition 3: Take the assumption that the first negotiation is successful with average financing charge $r^* = (\overline{r} - \underline{r})/2$. If the first negotiation is successful, a firm's management updates their estimation of valuation imprecision in a way that affects further negotiations positively. Given average financing charges $(\overline{r} - \underline{r})/2$ underlines that the first negotiation's offer is average. Management rejects further negotiations if its expected cost reductions due to better financing conditions are lower than the cost for these negotiations. This is represented by the inequality

$$
\begin{aligned}
&\mathbf{P}(\text{Negotiation success}|1 \text{ previous negotiation success})\mathbf{P}(r > r^*)\,(r^* - \mathbf{E}(r|r < r^*)) < c_M/K^* \\
&\quad\Longleftrightarrow \frac{2}{3}\frac{1}{2}\left((\overline{r} + \underline{r})/2 - (\overline{r}/4 + 3\underline{r}/4)\right) < c_M/K^* \\
&\quad\Longleftrightarrow (\overline{r} - \underline{r})/12 < c_M/K^*
\end{aligned}
$$

Therefore on average relative negotiation cost c/K^* higher than $(\overline{r} - \underline{r})/12$ make management accept the first successfully negotiated offer with probability of at least 0.5. $\square$

Proof of Proposition 4: As in the proofs of propositions 2 and 3, we assume that the first negotiation is successful. When the first successful negotiation is not the first negotiation then, due to a worse estimated negotiation success probability, a firm's management is more likely to decide to conduct no further negotiations.

Let $p \in [0, 1]$ be some probability. After the first successful negotiation, a firm's management gets offered a financing charge that is at least as good as $R = \underline{r}(1 - p) + \overline{r}p$ with probability p.

When management gets offered financing charge R after the first negotiation, then it conducts no further negotiations if expected cost reductions due to better financing conditions are lower then the cost for these negotiations. That is,

$$\begin{aligned} K^*\mathbf{P}\left(\text{Negotiation Success}\right)\mathbf{P}\left(\text{Offer} < R\right)\mathbf{E}\left(\text{Offer}|\text{Offer} < R\right) &< c_M \\ \iff \frac{2}{3}\frac{R-\underline{r}}{\Delta r}\frac{R-\underline{r}}{2} &< c_M/K^* \\ \iff \frac{(R-\underline{r})^2}{3\Delta r} &< c_{rel} \\ \iff \frac{p^2\Delta r}{3} &< c_{rel}. \end{aligned}$$

The above calculation are lead along the lines of these from propositions 2 and 3. The formula above establishes a lower bound for relative negotiation cost, such that the probability that a firm's management chooses the shortest negotiation process is at least $p \in [0,1]$. This lower bound is in accordance with propositions 2 and 3, which calculated lower bounds for relative negotiation costs such that management chooses the shortest negotiation process deterministically and with probability of at least 0.5. Rearranging the above equation gives $p^* = \frac{3c_{rel}}{\Delta r}$. □

Proof of Proposition 5: Take the assumption that all prior negotiations were successful. This scenario makes management assume best success probability for future negotiations and therefore dominates all other negotiation processes with respect to future success probability. Taking the limit of this process, management assumes to succeed in future negotiations with probability 1. Assume that a firm's management will not continue negotiations in this optimal negotiation process, then it will not continue negotiations in any other negotiation process as well.

Assume management gets offered financing charges r_1^* and management decides to continue negotiations. Then, after the next successful negotiation, the best financing conditions r_2^* are uniformly distributed on the interval $[\underline{r}, r_1^*]$. Continuing negotiations sufficiently finally gives management an offer $r^* = \underline{r} + \varepsilon$ for all $0 < \varepsilon$. This means that after a sufficient number of successful negotiations, management can receive an offer r^* arbitrary close to best available offer $\underline{r}$.

These described assumptions are the best negotiation results that a firm's management can possibly achieve. Therefore, if management does not continue this negotiation process, it will not continue any other negotiation process. Given rate r^* management rejects further negotiations if and only if expected cost reductions due to better financing conditions are lower than the cost for these negotiations. This is represented by the inequality

$$\begin{aligned}&\mathbf{P}(\text{Negotiation success})\mathbf{P}(r<r^*)\left(r^*-\mathbf{E}(r|r<r^*)\right)<c_M/K^*\\ \iff& 1\cdot\frac{\varepsilon}{\overline{r}-\underline{r}}\left(\underline{r}+\varepsilon-\underline{r}-\varepsilon/2\right)<c/K^*\iff\frac{\varepsilon^2}{2(\overline{r}-\underline{r})}<c_M/K^*\end{aligned}$$

Management may receive offers $r^*=\underline{r}+\varepsilon$ arbitrary close to best offer $\underline{r}$. Therefore for any given negotiation cost $c_M>0$, there will be best offer rate r^* such that management decides against further negotiations. The only cost, where management always decides to continue the negotiation process thus is a negotiation cost of zero. □

Proof of Proposition 6: Let X be a geometrically-like distributed random variable with success probability $p\in[0,1]$, with the difference that all mass that exceeds N lies on zero, that is

$$X:=\begin{cases}Geo(p), & \text{when } Geo(p)\le N\\ 0, & \text{when } Geo(p)>N\end{cases}.$$

Then the expectation of X is

$$\begin{aligned}\mathbf{E}(X)&=\sum_{k=1}^{N}kpq^{k-1}=\sum_{k=0}^{N-1}(k+1)pq^k=\sum_{k=0}^{N-1}kpq^k+\sum_{k=0}^{N-1}pq^k\\&=q\sum_{k=0}^{N-1}kpq^{k-1}+p\sum_{k=0}^{N-1}q^k=q\left(\mathbf{E}(X)-Npq^{N-1}\right)+p\frac{1-q^N}{1-q}\\&=q\mathbf{E}(X)-Npq^N+1-q^N.\end{aligned}$$

We rearrange the equation and obtain

$$\mathbf{E}(X)=\frac{1}{p}\left(1-q^N\right)-Nq^N.$$

Now consider a geometrically-like distributed random variable Y that is defined as follows: Let $Geo(p)$ be geometrically distributed with success probability $p\in[0,1]$. Then we define

$$Y:=\begin{cases}0, & \text{when } Geo(p)\le N\\ N, & \text{when } Geo(p)>N\end{cases}.$$

Then the expectation of Y is

$$\mathbf{E}(Y) = \sum_{k=N+1}^{\infty} Npq^{k-1} = Np\sum_{k=N}^{\infty} q^k = Np\left(\frac{1}{1-q} - \frac{1-q^N}{1-q}\right) = Nq^N$$

Management's negotiation success probability p_v is dependent on its valuation. When that valuation is $v \in [1-\alpha, 1+\alpha]$, then $p_v = (1+\alpha-v)/(2\alpha)$ and $q_v = (v-1+\alpha)/(2\alpha)$. In summary, the expected number of negotiations is

$$\begin{aligned}
\mathbf{E} &= \frac{1}{2\alpha}\int_{1-\alpha}^{1+\alpha} \left(\mathbf{E}(X)(p_v) + \mathbf{E}(Y)(p_v)\right)\ dv \\
&= \frac{1}{2\alpha}\int_{1-\alpha}^{1+\alpha} \frac{1-q_v^N}{p_v}\ dv = \frac{1}{2\alpha}\int_{1-\alpha}^{1+\alpha}\sum_{k=0}^{N-1} q_v^k\ dv \\
&= \frac{1}{2\alpha}\sum_{k=0}^{N-1}\frac{1}{2^k\alpha^k}\int_{1-\alpha}^{1+\alpha}(v-1+\alpha)^k\ dv = \frac{1}{2\alpha}\sum_{k=0}^{N-1}\frac{1}{2^k\alpha^k}\int_0^{2\alpha} x^k\ dx \\
&= \frac{1}{2\alpha}\sum_{k=0}^{N-1}\frac{1}{2^k\alpha^k}\frac{1}{k+1}(2\alpha)^{k+1} = \sum_{k=0}^{N-1}\frac{1}{k+1} = ln(N) + \gamma + \mathcal{O}\left(\frac{1}{N^2}\right),
\end{aligned}$$

where $\gamma \approx 0.5772$ is the Euler Mascheroni constant. The firm's expected negotiation cost is the product of the above expectation and cost per negotiation c_M. □

Proof of Lemma 2: The number of companies an investor negotiates with is geometrically distributed with parameter 1/2. Thus expected number of negotiations is given by $1/p = 2$. Each negotiation involves cost of c_I such that expected cost of deal settlement is $2c_I$. □

Proof of Proposition 7: From management's perspective, intermediation is favoured while

$$V_M - c_M^* \leq V_M - u - f \iff u \leq c_M^* - f,$$

which is exactly the statement of the proposition. An investor's preferences are proven accordingly. □

Proof of Proposition 8: An intermediary can place a bond if and only if

$$\begin{aligned}
V_M - dK^* + fK^* < V_I + uK^* &\iff V_M - V_I + fK^* < uK^* + dK^* \\
&\iff X_M - X_I < (u+d-f)\frac{K^*}{V}.
\end{aligned}$$

Note that valuation imprecision is independently uniformly distributed, i.e.

$$X_M, X_I \stackrel{d}{=} unif[1-\alpha, 1+\alpha].$$

The probability that an intermediary can match a financier and borrower is

$$\mathbf{P}\left(X_M - X_I < (u+d-f)\frac{K^*}{V}\right).$$

To keep notation to a minimum define $t := (u+d-f)\frac{K^*}{V}$. Then

$$\begin{aligned}
\mathbf{P}\left(X_M - X_I < t\right) &= \int_{1-\alpha}^{1+\alpha}\int_{m-t}^{1+\alpha} 1/(4\alpha^2)\ dn\ dm \\
&= 1/(4\alpha^2)\int_{1-\alpha}^{1+\alpha} 1+\alpha+t-m\ dm \\
&= 1/(4\alpha^2)\left((1+\alpha+t)2\alpha - \left((1+\alpha)^2-(1-\alpha)^2\right)/2\right) \\
&= \frac{\alpha+t}{2\alpha},
\end{aligned}$$

which is the formula that is stated in the proposition. □

Proof of Lemma 3: Proposition 8 established negotiation success probability p_d. As the negotiation process is geometrically distributed, the expected number of negotiations until deal settlement is

$$\mathbf{E}(N) = 1/p_d = \frac{\alpha}{2\alpha + (u+d-f)K^*/V}.$$

□

Proof of Proposition 9: An intermediary's expected earnings $\mathbf{E}(D)$ are

$$\begin{aligned}
\mathbf{E}\left(P\right) &= \sum_{k=1}^{\infty} p_d(1-p_d)^{k-1}(fK^* - kc_dK^*) \\
&= K^*\left(f\sum_{k=1}^{\infty} p_d(1-p_d)^{k-1} - c_d\sum_{k=1}^{\infty} p_d(1-p_d)^{k-1}k\right) \\
&= K^*\left(f - c_d\sum_{k=1}^{\infty} p_d(1-p_d)^{k-1}k\right)
\end{aligned}$$

Along the lines of lemma 3 the second sum can be interpreted as the expected value of a geometrically distributed random variable with success parameter p_d. That expectation is given by $1/p_d$. As a result, the expected earnings are

$$\begin{aligned}
\mathbf{E}\left(D\right) &= K^*\left(f - c_d/p_d\right) \\
&= K^*\left(f - \frac{2\alpha c_d}{\alpha + (u+d-f)\,K^*/V}\right)
\end{aligned}$$

□

Proof of Proposition 10: Proposition 9 calculates a formula for dealer's expected earnings. These are determined by $K^* (f - c_d/p_d)$. Obviously expected earnings increase when negotiation success probability p_d increases. Proposition 8 shows that $p_d = \frac{1}{2} + (u + d - f)K^*/2/V/\alpha$, which is increasing in u and d. Therefore a dealer profits from increasing u and d. Therefore the dealer tries to increase these parameters maximally. That maximum is given by the boundary conditions $u \leq c_M$ and $d \leq c_I$. $\square$

Proof of Theorem 2: It is necessary that $X_M - X_I < ((u+d)-f)K^*/V$ for an intermediary to match a borrower with a financier. With proposition 8 we obtain probability of successful matching at a time $p_d = (\alpha + ((u + d) - f)K^*/V)/2/\alpha$. An intermediary's objective is to deliver at least as high negotiation success as capital market success rate. Therefore $p_d \geq 1/2$ is a necessary condition. We derive that the condition implies $f \in (0, u + d)$. Expected intermediary's gain $\mathbf{E}(D)$ and its derivative with respect to fee f are

$$\begin{aligned}
\mathbf{E}\,(D) &= K^* \,(f - c_d/p_d) \\
&= K^* \left(f - \frac{2\alpha c_d}{\alpha + ((u + d) - f)\, K^*/V} \right) \\
\mathbf{E}'\,(D) &= K^* \left(f - \frac{\alpha c_d K^*/V}{(\alpha + ((u + d) - f)\, K^*/V)^2} \right)
\end{aligned}$$

The first derivative is zero for

$$f_{1,2} = \frac{K^*\,(u + d) + V\,\alpha \pm \sqrt{2}\,V\,\sqrt{\frac{K\,\alpha, c_d}{V}}}{K^*} = (u + d) + \alpha V/K^* \pm \sqrt{2 c_d \alpha V/K^*}.$$

There is at the most one $f_{1,2}$ in the feasible region $(0, u + d)$ It is given by

$$f_1 = (u + d) + \alpha V/K^* - \sqrt{2 c_d \alpha V/K^*}.$$

As the constraint $f \in (0, u + d)$ must hold, optimal fee is

$$f_{opt} = min\{(u + d) + \alpha V/K^* - \sqrt{2 c_d \alpha V/K^*}, u + d\},$$

which is the formula stated in the theorem. Expected earnings in fee optimum are obtained by plugging optimal dealer's fee into the formula for his expected earnings. $\square$

Proof of Lemma 4: A negotiation's success probability in fee optimum is

$$p_d = (\alpha + u + d - f_{opt})/2/\alpha \geq 1/2,$$

because $f_{opt} \leq u + d$. $\square$

Proof of Proposition 11: Theorem 2 introduced optimal intermediary's fee

$$f_{opt} = min\{(u+d) + \alpha V/K^* - \sqrt{2c_d \alpha V/K^*}, u+d\}.$$

Assume dealer's fee is $f_{opt} = u + d$. We show that his expected earnings that result from this fee are positive. That is sufficient in order to prove that expected earned fee is always positive under f_{opt}. Applying fee $u + d$ yields expected earnings $\mathbf{E}(D) = u + d - 2c_d$, which is greater zero if and only if $c_d \leq (u + d)/2$. $\square$

Proof of Proposition 12: We analyse the cases $(a): \ c_d \leq \frac{\alpha}{2K^*/V}$ and $(b): \ c_d > \frac{\alpha}{2K^*/V}$. In case (a) intermediary's expected earnings are $(u + d) - 2c_d$. That formula is not dependent on valuation imprecision. In case (b) intermediary's expected earnings and first derivative as a function of α are

$$\begin{aligned}
\mathbf{E}(P) &= (u+d) + \alpha V/K^* - 2\sqrt{2\alpha c_d V/K^*} \\
\mathbf{E}'(P) &= V/K^* - \sqrt{\frac{2c_d}{\alpha K^*/V}}
\end{aligned}$$

Assume the intermediary profits from increasing valuation imprecision. Then

$$\begin{aligned}
\mathbf{E}'(P) > 0 &\Longleftrightarrow V/K^* - \sqrt{\frac{2c_d}{\alpha K^*/V}} > 0 \\
&\Longleftrightarrow V/K^* > \sqrt{\frac{2c_d}{\alpha K^*/V}}.
\end{aligned}$$

As we observe case (b), we conclude

$$\sqrt{\frac{2c_d}{\alpha K^*/V}} > \sqrt{\frac{2}{\alpha K^*/V}\frac{\alpha}{2K^*/V}} = V/K^*$$

Thus for valuation imprecision to be beneficial for the intermediary, it must hold that

$$V/K^* > V/K^*,$$

which is not possible. Therefore valuation imprecision has negative effect on an intermediary's expected earnings. $\square$

Proof of Theorem 3: Proposition 7 proved that within the bounds $u < c_M^* - f$ and $d < c_I^*$, a firm's management and an investor prefer bond financing by intermediation. Furthermore, proposition 11 introduced the condition $c_d < (u + d)/2$ for the intermediary to expect positive earnings. Consequently, while all conditions hold, each party prefers bond financing by intermediation over loan negotiations. $\square$

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Concluding Remarks

When negotiations for a trade are modelled and the negotiating parties have different valuations of the traded object, then literature usually models these different valuations as randomly distributed. These distributions are usually known to the bargaining parties, so each party knows the lowest and highest possible valuations. Gibbons (1992) as well as Chatterjee and Samuelson (1983) model the parties' valuations in that fashion. Accordingly, each trader can benchmark his own valuation to the distribution of his peers' valuations. That information endows a negotiant with a detailed bargaining strategy. As a matter of fact, when a seller's valuation is comparatively high, then that seller sets a price that is close to his valuation. In contrast, a seller with a comparatively low valuation may exaggerate his price to a greater extent.

This work takes a different approach. Each party knows that its valuation is imprecise and is aware of the maximal degree of imprecision. However, there is no valuation benchmark. Hence, a bargainer only knows the percentage of his maximum valuation imprecision compared to an unknown average valuation. In particular, he does not know whether his valuation is above or below average. Thus, although some information is known to each negotiating party, that party's negotiation strategy is limited, compared to the model investigated in numerous papers as mentioned above.

This work comes to the result that rational behaving traders are less efficient than naive bargainers. It can be observed that an intermediary may exploit this inefficiency and offer a higher gain for all traders. At the same time, his price strategy allows him a positive gain as well. While there may be full information when the parties trade without the intermediary, that intermediary maintains an environment of information asymmetry. Thus, when the traders are in preference of the intermediary, they prefer information asymmetry over full information. A noteworthy result, when other literature in this field is considered.

Applicable examples illustrate the theory of this work. For instance, the first paper's theory can be used to explain why a firm that wants to hire an employee should prefer a recruitment firm over direct negotiations with a job applicant. It becomes clear that the recruitment firm's strategy can be more efficient than direct salary negotiations, even if the firm and the job applicant are truthful to each other.

Another application is the IPO of a firm's shares, where an investor's information is limited. In contrast, the firm opens its books to an investor when they directly negotiate over the price of a firm's share. In this case, full information is available. Determinants for a firm's and an investor's preference of an IPO under information asymmetry over their bilateral negotiations under full information are examined. Furthermore, our model presents an explanation for the underpricing of IPOs.

This work compares different market designs, which allow bargainers to directly trade with each other. The efficiency of these platforms, under different information sets, is analysed thoroughly. Additionally, a dealer - operating under information asymmetry - and the attractiveness of his pricing strategy for the bargainers is presented. All papers within this work introduce conditions for the dealer being more efficient than direct trade. In this case, information asymmetry may Pareto dominate full information. There are numerous further market designs that allow for direct trade. Their attractiveness can be compared to an intermediary's pricing strategy. It would be interesting to establish criteria allowing the intermediary to Pareto dominate these market designs as well.

The determinant for profit from trade often is the players' imprecision in their valuations. Throughout this work, valuation imprecision is uniformly and identically distributed. A worthwhile extension of the models presented in each paper would be allowing different, non identical distributions of the players' valuation abilities. When these distributions are symmetrical, we believe that major results in this work remain valid. Our simulations already support this hypothesis.

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