

# Combinatorial description of normal toric schemes over valuation rings of rank one

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Prof. Dr. Wolfgang Rosenstiel

Prof. Dr. Walter Gubler

Prof. Dr. Jürgen Hausen

*To my family, with love.*



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# Introduction

In mathematics, the construction of explicit examples is done not just in order to illustrate the theory but also to find possible counterexamples to open conjectures. In algebraic geometry to do explicit constructions can be very difficult, this lack of examples sometimes obscures the geometry underlying the theory. However, if we restrict to the special case of toric varieties over a field it is easier to do such constructions. The reason for that is the well known dictionary between toric geometry and convex geometry. It allows us to do explicit examples by implementing the combinatorial description into computer algebra systems. This enhances the possibilities of applications and the understanding of algebraic geometry. The cornerstone of these results is a theorem due to Sumihiro [34] which says that every toric variety has a cover by open affine torus invariant subsets. One of the goals of this work is to extend this theorem for toric varieties defined over an arbitrary valuation ring of rank one.

The theory of toric varieties over a field has been very well studied and we can find very good reference for them, see for instance Cox–Little–Schenk [11], Ewald [12], Fulton [13], Kempf–Knudsen–Mumford–Saint-Donat [21] and Oda [27]. In [21] Mumford generalized toric geometry for normal varieties defined over a discrete valuation ring. Ever since, very few approaches have been taken in order to generalize toric geometry over more general bases. The main difficulty is that, for valuation rings with valuation neither trivial nor discrete, the noetherian methods of algebraic geometry are no longer available.

In [17] Gubler introduced and studied toric varieties over arbitrary valuation rings of rank one. A  $\mathbb{T}$ -toric variety  $\mathcal{Y}$  over the valuation ring  $K^\circ$  is an integral separated flat scheme of finite type over  $K^\circ$  such that the generic fiber  $\mathcal{Y}_\eta$  of  $\mathcal{Y}$  contains  $T := (\mathbb{G}_m^n)_K$  and the canonical action of  $T$  over itself extends to an algebraic action of  $\mathbb{T} := (\mathbb{G}_m^n)_{K^\circ}$  on  $\mathcal{Y}$ . Given an admissible fan  $\Sigma \subset \mathbb{R}^n \times \mathbb{R}_+$ , for every cone  $\sigma \in \Sigma$  it is possible to construct an affine  $\mathbb{T}$ -toric scheme  $\mathcal{V}_\sigma$ . Gluing them together we get a  $\mathbb{T}$ -toric scheme  $\mathcal{Y}_\Sigma$ . Note that the combinatorial description of these varieties comes from cones and fans in  $\mathbb{R}^n \times \mathbb{R}_+$ , similarly as in the case of fields and discrete valuation rings. One particular feature in this setting is that tropical geometry provides a very good description of the cone-orbit correspondence, for details see [17, §8] and §1.3.

Toric schemes over an arbitrary base ring  $R$  have been introduced and studied by Rohrer in [31]. He starts with a fan  $\Pi \subset \mathbb{R}^n$  and proceeds as usual, i.e. associating to every cone  $\sigma \in \Pi$  an  $R$ -algebra  $A_\sigma$ . The schemes  $\text{Spec}(A_\sigma)$  can be glued together to get

the scheme  $X_\Pi$ . By construction this toric scheme has an algebraic action of the split torus  $(\mathbb{G}_m^n)_R$ . If we take  $R$  as the valuation ring  $K^\circ$ , this construction yields a special case of Gubler's construction. Actually for a fan  $\Pi$ , we have that  $X_\Pi = \mathcal{Y}_{\Pi \times \mathbb{R}_+}$ . Moreover if we take  $R$  as a field  $K$  and restrict to normal varieties, we get the same objects with the two approaches: We consider  $K$  as a valued field with the trivial valuation, then the special fiber of a  $\mathbb{T}$ -toric variety  $\mathcal{Y}$  is empty, hence it is equal to the generic fiber  $\mathcal{Y}_\eta$  which by Sumihiro's theorem comes from a fan  $\Pi$  and therefore is of the form  $X_\Pi$ .

The difference between these two approaches is that the first one requires in the definition that the torus action should be extended to the whole scheme but in the second one you already start with the combinatorial picture and therefore by definition the torus action is extended to the whole scheme automatically.

The construction made by Gubler depends strongly on the base ring, in particular on the fact that the rank of the valuation is one. It allows the use of the theory of analytic spaces in the sense of Berkovich as well as the use of tropical geometry. In the same way the results of this paper depends on this fact. The question whether these constructions for higher rank valuation rings can be carried out or not remains open.

### Main results

The aim of this work is to classify toric varieties over the valuation ring  $K^\circ$  as introduced by Gubler in [17]. This will be done by proving three theorems. The classification will generalize the correspondence between normal toric varieties and cones, in the affine case, and fans in general.

Let  $K$  be a valued field of rank one,  $K^\circ$  its valuation ring and  $\Gamma$  its value group. Let  $\mathbb{T} = \text{Spec}(K^\circ[M])$  be the split torus over  $K^\circ$ , with  $M$  the character lattice of  $\mathbb{T}$ . In chapter 1, we review the theory of toric varieties over the valuation ring  $K^\circ$ . We show that given a  $\Gamma$ -admissible cone  $\sigma \subset \mathbb{R}^n \times \mathbb{R}_+$  it is possible to construct the algebra  $K[M]^\sigma$  over  $K^\circ$  which gives rise to a normal  $\mathbb{T}$ -toric scheme  $\mathcal{V}_\sigma = \text{Spec}(K[M]^\sigma)$ . If the valuation is trivial or discrete or if the valuation is neither trivial nor discrete and the vertices of  $\sigma \cap (\mathbb{R}^n \times \{1\})$  are contained in  $\Gamma^n \times \{1\}$  then the algebra  $K[M]^\sigma$  is of finite type and the scheme  $\mathcal{V}_\sigma$  is a normal  $\mathbb{T}$ -toric variety over  $K^\circ$ . In this way, we have plenty of examples of  $\mathbb{T}$ -toric varieties over valuation rings of rank one. A natural question to ask is whether all affine normal  $\mathbb{T}$ -toric varieties over  $K^\circ$  are of this form. The first result presented here gives an affirmative answer to this question. In what follows, we assume that the valuation  $v$  is not trivial. Explicitly, we have the following theorem.

**Theorem 1.** *If  $v$  is not a discrete valuation, then the map  $\sigma \mapsto \mathcal{V}_\sigma$  defines a bijection between the set of those  $\Gamma$ -admissible cones in  $\mathbb{R}^n \times \mathbb{R}_+$  for which the vertices of  $\sigma \cap (\mathbb{R}^n \times \{1\})$  are contained in  $\Gamma^n \times \{1\}$  and the set of isomorphism classes of normal affine  $\mathbb{T}$ -toric varieties over the valuation ring  $K^\circ$ .*

With this theorem, we obtain a classification of normal affine  $\mathbb{T}$ -toric varieties defined over a valuation ring of rank one, which extends the standard result known for toric varieties over a field and a discrete valuation ring. Note that if the valuation is discrete or trivial the extra condition on the cones is not needed.

The proof of Theorem 1 is given in chapter 2. We show that given an affine normal  $\mathbb{T}$ -toric variety  $\mathcal{Y} = \text{Spec}(A)$  it is possible to construct a  $\Gamma$ -admissible cone  $\sigma$  such that  $K[M]^\sigma = A$ . First, we construct a semigroup  $S$  from the  $K^\circ$ -algebra  $A$  and then we take the cone generated by it,  $\text{cone}(S)$ . The dual cone  $\sigma := \text{cone}(S)^\vee$  works for our purpose. Furthermore, we can reconstruct  $\sigma$  from the tropicalization of  $\mathcal{Y}_\eta \cap T^\circ$ , see Proposition 1.11. This implies that the cone  $\sigma$  is uniquely characterized by  $\mathcal{Y}$ .

In order to classify normal  $\mathbb{T}$ -toric varieties over the valuation ring  $K^\circ$  which are not affine, it is necessary to generalize the well known Sumihiro's theorem in toric geometry.

**Theorem 2.** *Let  $v$  be a real valued valuation with valuation ring  $K^\circ$  and let  $\mathcal{Y}$  be a normal  $\mathbb{T}$ -toric variety over  $K^\circ$ . Then every point of  $\mathcal{Y}$  has an affine open  $\mathbb{T}$ -invariant neighborhood.*

This result extends Sumihiro's theorem for normal toric varieties over a field [34] to normal  $\mathbb{T}$ -toric varieties over a valuation ring of rank one. The proof is considerably more difficult as in the classical case since the noetherian methods are not available. Instead, we use intersection theory with Cartier divisors for varieties over valuation rings of rank one. This is done in chapter 3. These results follow from the intersection theory with Cartier divisors on admissible formal schemes over  $K^\circ$  developed by Gubler in [19]. We use the notion of PvM-rings in order to study Weil divisors on normal varieties over valuation rings and to associate to every Cartier divisors a cycle of codimension one. With this, in chapter 4 we show that given an open affine subset  $\mathcal{U}_0$  of a normal  $\mathbb{T}$ -toric variety  $\mathcal{Y}$  it is possible to construct a  $\mathbb{T}$ -invariant open subset  $\mathcal{U}$  and a Cartier divisor  $D$  such that  $\mathcal{U}$  contains  $\mathcal{U}_0$  and  $D$  has support  $\mathcal{U} \setminus \mathcal{U}_0$ . Then, we show that the line bundle  $\mathcal{O}(D)$  admits a  $\mathbb{T}$ -linearization which leads to a  $\mathbb{T}$ -equivariant embedding of  $\mathcal{U}$  into a projective  $\mathbb{T}$ -toric variety with a linear action of the torus. Finally, using this fact we prove Theorem 2 in chapter 5.

As a consequence of Theorem 1 and Theorem 2, we obtain our main classification result.

**Theorem 3.** *If  $v$  is not a discrete valuation, then the map  $\Sigma \mapsto \mathcal{Y}_\Sigma$  defines a bijection between the set of fans in  $\mathbb{R}^n \times \mathbb{R}_+$ , whose cones are as in Theorem 1, and the set of isomorphism classes of normal  $\mathbb{T}$ -toric varieties over  $K^\circ$ .*

This theorem extends the classification of normal toric varieties over a field or a discrete valuation ring to the classification of normal  $\mathbb{T}$ -toric varieties over a valuation ring of rank one. Note that if the valuation is discrete or trivial the extra condition on the cones is not needed.

This result allows us to have a better understanding of toric geometry over rank one valuation rings. It is worth to stress the fact that these objects are intimately related to tropical geometry, for instance for a normal affine  $\mathbb{T}$ -toric variety  $\mathcal{V}_\sigma$  the cone  $\sigma$  can be constructed from the tropicalization of the set of potentially integral points  $T^\circ \cap (\mathcal{V}_\sigma)_\eta$  of the generic fibre. Furthermore these toric schemes are used in [17] to generalize results on tropical compactifications of closed subschemes of the torus  $T$  for arbitrary rank one valuation fields, see [17, §12] for more details.

Now, we briefly explain the structure of the thesis. In chapter 2, we introduced the theory of  $\mathbb{T}$ -toric schemes developed by Gubler in [17]. Since we don't need all results proved there and some of the proofs are not enlightening for our purpose, we just quote them and suggest to the interested reader to look at the original source. We start with the definition of a  $\mathbb{T}$ -toric scheme. After the construction of some examples, we state the main propositions which provides, with the help of the tropicalization map, the geometric description in terms of the combinatorics of a  $\Gamma$ -admissible fan. We end this chapter with the description of the projectively embedded  $\mathbb{T}$ -toric varieties with a linear action of the torus. The results of chapter 1 are fundamental for the understanding of this work. In chapter 2, we prove Theorem 1. We construct a  $\Gamma$ -admissible cone  $\sigma$  and then we prove that the  $\mathbb{T}$ -toric variety  $\mathcal{V}_\sigma$  is the original one. The proof of Theorem 2 is considerable longer and therefore is divided in three parts which cover chapters 3, 4 and 5. In chapter 3, we introduce the necessary results concerning intersection theory. These results are very important in order to overcome the problems arising for the non-noetherianness of our setting. With them, we are able to prove that for every affine open subset  $\mathcal{U}_0$  there exist a Cartier divisor  $D$  defined on the smallest  $\mathbb{T}$ -invariant open subset  $\mathcal{U}$  which contains  $\mathcal{U}_0$  such that  $\text{supp}(D) = \mathcal{U} \setminus \mathcal{U}_0$ . Furthermore for every  $t \in T^\circ(K)$  the divisors  $D$  and  $D^t$  are linear equivalent, see Corollary 3.30. In chapter 4, we show that the line bundle  $\mathcal{O}(D)$  on  $\mathcal{U}$  is ample and admits a  $\mathbb{T}$ -linearization. We also show that it is possible to construct a  $\mathbb{T}$ -equivariant immersion into a projective  $\mathbb{T}$ -toric variety. Finally in chapter 5 using the theory of non-necessarily normal projective  $\mathbb{T}$ -toric varieties with a linear action of the torus developed in [17], we end the proof of Theorem 2. Finally, we conclude with the proof of Theorem 3 which give us the bijective correspondence between normal  $\mathbb{T}$ -toric varieties and  $\Gamma$ -admissible fans.

### Notation

For sets, in  $A \subset B$  the equality is not excluded and  $A \setminus B$  denotes the complement of  $B$  in  $A$ . The sets  $\mathbb{Z}_+$ ,  $\mathbb{Q}_+$  and  $\mathbb{R}_+$  denotes the set of non-negative integers, rationals and real numbers respectively. All the rings and algebras are commutative with unity.

For an integral domain  $A$ , we denote by  $\mathbb{Q}(A)$  its quotient field and by  $\mathcal{F}(A)$  the set of fractional ideals. Given elements  $I, J$  of  $\mathcal{F}(A)$  we denote by  $(J : I)$  the  $A$ -module

$$\{x \in \mathbb{Q}(A) \mid xI \subset J\}.$$

Note that it is a fractional ideal as well. By  $P(A) = \text{Ass}_A(\mathbb{Q}(A)/A)$  we denote the set of weakly associated prime ideals of the  $A$ -module  $\mathbb{Q}(A)/A$ , i.e.  $p \in \text{Ass}_A(\mathbb{Q}(A)/A)$  iff  $p \supset \text{ann}(m)$  for some  $m \in \mathbb{Q}(A)/A$  and its a minimal prime over it. Note that any  $p \in P(A)$  is a minimal prime ideal over an ideal of the form  $((a) : (b))$  for  $a, b \in A$  and  $b \notin (a)$ . For a module over a noetherian ring, the weakly associated prime ideals are the same as the associated prime ideals.

$K$  will always denote a rank one valued field  $(K, v)$  with valuation group  $\Gamma := v(K^\times) \subset \mathbb{R}$ . If the valuation is neither trivial nor discrete, then the value group is dense in  $\mathbb{R}$ . Note that  $K$  is not required to be algebraically closed or complete and that its valuation can be trivial. The valuation ring is given by  $K^\circ := \{x \in K \mid v(x) \geq 0\}$  with maximal ideal  $K^{\circ\circ} := \{x \in K \mid v(x) > 0\}$  and residue field  $\tilde{K} := K^\circ/K^{\circ\circ}$ . We denote  $S = \text{Spec}(K^\circ) = \{\eta, s\}$ , where the generic point  $\eta$  correspond to the zero ideal and the special point  $s$  to the maximal ideal  $K^{\circ\circ}$ .

We denote by  $|\cdot| := \exp(-v(\cdot))$  the associated absolute value to the valuation  $v$  on  $K$ . If  $(L, w)$  is a valued field extension of  $(K, v)$ , the absolute value associated to  $w$  on  $L$  will be denoted by  $|\cdot|_w$ .

A seminorm on a ring  $A$  is a function  $p : A \rightarrow \mathbb{R}_+$  which satisfies  $p(1) = 1$ ,  $p(0) = 0$  and for every  $x, y \in A$ , we have

- $p(xy) \leq p(x)p(y)$ ,
- $p(x + y) \leq p(x) + p(y)$ .

If a seminorm satisfies  $p(xy) = p(x)p(y)$  for all  $x, y \in A$  it is called multiplicative. If it satisfies  $p(x^n) = p(x)^n$  for all  $x \in A$ ,  $n \in \mathbb{Z}_+$  it is called power-multiplicative. Given a

$K$ -algebra  $A$ , a seminorm  $p$  on  $A$  extends the norm of  $K$  if  $p(a) = |a| \forall a \in K$ . A norm is a seminorm with trivial kernel. If  $\rho : A \rightarrow \mathbb{R}_+$  is norm,  $A$  is a Banach algebra if it is complete respect the norm topology induced by  $\rho$ . A seminorm  $p$  on  $A$  is bounded if there is a constant  $C > 0$  such that  $p(x) \leq C\rho(x) \forall x \in A$ .

Let  $M$  be a free abelian group of rank  $n$  with dual  $N := \text{Hom}(M, \mathbb{Z})$ . The natural pairing is denoted by  $\langle u, \omega \rangle =: \omega(u) \in \mathbb{Z}$  for  $u \in M, \omega \in N$ . For an abelian group  $G$  the base change is denoted by  $M_G := M \otimes_{\mathbb{Z}} G$ , for instance  $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ . The split torus over  $K^\circ$  with generic fiber  $T = \text{Spec}(K[M])$  is given by  $\mathbb{T} = \text{Spec}(K^\circ[M])$ . Therefore  $M$  can be thought as the character lattice of  $T$  and  $N$  as its group of one parameter subgroups. For  $u \in M$  the corresponding character is denoted by  $\chi^u$ .

For the notions on convex geometry used in this work, we suggest to the reader to look into the Appendix A.

# 1 | Toric schemes over valuation rings

Toric varieties over a field have been studied since the 70's and there are very good references for them, for instance Cox–Little–Schenk [11], Ewald [12], Fulton [13], Kempf–Knudsen–Mumford–Saint-Donat [21] and Oda [27]. Although in these books toric varieties are defined over an algebraically closed field, the main results obtained there hold over any field. The principal feature of these varieties is that their geometry can be understood studying the combinatorics of some objects in convex geometry, namely polytopes, cones and fans.

Some generalizations have been carried out. Toric schemes over discrete valuation rings were studied by Mumford in [21], over arbitrary valuation rings of rank one by Gubler in [17] and over arbitrary rings by Rohrer in [31]. Here we review the main definitions and properties of toric schemes defined over an arbitrary rank one valuation ring. We follow closely the description given in [17]. Over discrete valuation rings more details can be found in the papers by Burgos–Phillipon–Sombra [8], Katz [20], Qu [28] and Smirnov [33].

## 1.1 Affine toric schemes associated to polyhedra

Recall that  $K$  is a valued field of rank one, whose valuation ring is denoted by  $K^\circ$  and its valuation group by  $\Gamma \subset \mathbb{R}$ .

**Definition 1.1.** A  $\mathbb{T}$ -toric scheme over the valuation ring  $K^\circ$  is an integral separated flat scheme  $\mathcal{Y}$  over  $K^\circ$  such that the generic fiber  $\mathcal{Y}_\eta$  contains  $T$  as an open subset and the translation action  $T \times T \rightarrow T$  extends to an algebraic action  $\mathbb{T} \times_{K^\circ} \mathcal{Y} \rightarrow \mathcal{Y}$  over  $K^\circ$ . If it is of finite type, it is called a  $\mathbb{T}$ -toric variety.

Note that if  $\mathcal{Y}$  is a  $\mathbb{T}$ -toric variety then  $\mathcal{Y}_\eta$  is a  $T$ -toric variety over  $K$  and therefore its description can be done by the methods described in the references quoted above. A polyhedron  $\Delta \subset N_{\mathbb{R}}$  is called  $\Gamma$ -rational if it can be written as

$$\Delta := \bigcap_{i=1}^k \{\omega \in N_{\mathbb{R}} \mid \langle u_i, \omega \rangle \geq c_i\}, \quad u_1, \dots, u_k \in M, \quad c_1, \dots, c_k \in \Gamma.$$

It is called *pointed* if  $\Delta$  does not contain affine subspaces of dimension  $> 0$ .

In order to construct examples of  $\mathbb{T}$ -toric schemes, we need to introduce and to study the following algebras associated to  $\Gamma$ -rational polyhedra in  $N_{\mathbb{R}}$ . Let  $\Delta \subset N_{\mathbb{R}}$  be a  $\Gamma$ -rational polyhedron, we define the following subalgebra of the Laurent polynomials:

$$K[M]^{\Delta} := \left\{ \sum_{u \in M} a_u \chi^u \in K[M] \mid v(a_u) + \langle u, \omega \rangle \geq 0 \ \forall \omega \in \Delta \right\}.$$

Some examples of this algebra are the following:

- For  $\Delta = \{0\}$ ,  $K[M]^{\Delta} = K^{\circ}[M]$ .
- For  $\Delta = [0, \infty) \subset \mathbb{R}$  choosing a coordinate  $x$ ,  $K[M]^{\Delta} = K^{\circ}[x]$ .
- For  $\Delta = [0, \lambda] \subset \mathbb{R}$ . With  $\lambda \in \Gamma \setminus \{0\}$ ,  $\Delta$  is  $\Gamma$ -rational. Let  $a \in K$  be such that  $v(a) = \lambda$ . Choosing a coordinate  $x$ ,  $K[M]^{\Delta} = K^{\circ}[x, ax^{-1}]$ .

Note that in these simple examples the algebra  $K[M]^{\Delta}$  is of finite type and flat over  $K^{\circ}$ , since it is  $K^{\circ}$ -torsion free. Therefore by [29, Corollaire 3.4.7] it is of finite presentation. In general we have the following important result, see [17, Proposition 6.7]. For completeness, we reproduce the proof here.

**Proposition 1.2.** *If the valued group  $\Gamma$  is either discrete or divisible in  $\mathbb{R}$ , then the algebra  $K[M]^{\Delta}$  is of finite presentation over  $K^{\circ}$ .*

*Proof.* It is enough to prove that  $K[M]^{\Delta}$  is finitely generated. This follows because every finitely generated flat algebra over an integral domain is of finite presentation, see [29, Corollaire 3.4.7].

If the valuation is discrete, this statement was proved by Mumford in [21]. Actually with  $\sigma = c(\Delta) \subset N_{\mathbb{R}} \times \mathbb{R}_+$  and  $\pi$  an uniformizing parameter for  $K^{\circ}$ ,  $K[M]^{\Delta}$  is generated by the elements  $\pi^k \chi^u$  with  $(u, k) \in S_{\sigma} := \check{\sigma} \cap (M \times \mathbb{Z})$ , which is finitely generated as a semigroup.

If the valuation is divisible, we reduce to the pointed  $\Gamma$ -rational polyhedron case, then the proof given in [3, Proposition 4.11] works. Let  $\sigma_i := \text{LC}_{\omega_i}(\Delta)$  be the local cone of  $\Delta$  at the vertex  $\omega_i$ , then it's shown that  $K[M]^{\Delta}$  is generated by the elements  $\alpha_{ij} \chi^{u_{ij}}$  where  $\{u_{ij}\}_j$  is a finite set of generators of  $\check{\sigma}_i \cap M$  and  $\alpha_{ij} \in K$  are such that  $v(\alpha_{ij}) + \langle u_{ij}, \omega_i \rangle = 0$ .  $\square$

For any  $w \in N_{\mathbb{R}}$ , we define the following  $w$ -weight on  $K[M]$

$$v_w \left( \sum_{u \in M} a_u \chi^u \right) := \min_u \{v(a_u) + \langle u, w \rangle\}, \quad \sum_{u \in M} a_u \chi^u \in K[M].$$

As usual, we can extend this function to a valuation on the fraction field  $K(M)$ . In particular, given a pointed  $\Gamma$ -rational polyhedron  $\Delta$ , we consider the  $w$ -weight associated to each vertex  $w$  of  $\Delta$ . Correspondingly, we have the rings

$$K[M]^w := \{f \in K[M] \mid v_w(f) \geq 0\}$$



which are integrally closed in  $K[M]^\Delta$ , see [17, Proposition 6.10]. With these valuations, we associate to  $\Delta$  the following function

$$v_\Delta(f) := \min_{\omega \in \text{Vert}(\Delta)} \{v_\omega(f)\} \quad \forall f \in K[M]^\Delta,$$

which is not necessarily a valuation.

With this, we prove that  $K[M]^\Delta$  is integrally closed. Basically this follows because of the equality

$$K[M]^\Delta = \bigcap_{\omega \in \text{Vert}(\Delta)} K[M]^\omega.$$

*Remark 1.3.* If  $\Gamma$  is not divisible, then  $K[M]^\Delta$  is not necessarily of finite presentation, as is shown in the following example.

*Example 1.4.* Suppose the value group  $\Gamma \subset \mathbb{R}$  is dense and not divisible. Then there exists an element  $w \in \mathbb{R} \setminus \Gamma$  such that  $nw \in \Gamma$  for some  $n \in \mathbb{Z}_+$ . Therefore the polyhedron  $\Delta := \{\omega\}$  is  $\Gamma$ -rational. Consider the algebra  $K[M]^\Delta$ , with  $M \simeq \mathbb{Z}$ . Explicitly, we have

$$K[M]^\Delta = \left\{ \sum a_u \chi^u \in K[M] \mid v(a_u) + uw \geq 0 \right\}.$$

We claim that  $K[M]^\Delta$  is not finitely generated. For this, we argue by contradiction: We suppose it is finitely generated as a  $K^\circ$ -algebra. Let  $g_1, \dots, g_k$  be  $M$ -homogeneous generators, i.e. of the form  $g_i = a_i \chi^{m_i}$  with  $a_i \in K$  and  $m_i \in \mathbb{Z}$ . Now let  $b \in K$  be such that  $b\chi \in K[M]^\Delta$ . There exist  $\lambda \in K^\circ$  and  $\alpha_1, \dots, \alpha_k \in \mathbb{Z}_+ \setminus \{0\}$  such that

$$b\chi = \lambda g_1^{\alpha_1} \cdots g_k^{\alpha_k}.$$

For some  $j \in \{1, \dots, k\}$  we have  $v_w(g_j^{m_j}) \notin \Gamma$ , otherwise from the previous equation we would get that  $w \in \Gamma$ , which contradicts our assumption. Note that this implies that  $v_w(g_j^{m_j}) > 0$ , in particular  $v_w(g_j) > 0$ . Let

$$\epsilon := \inf_i \{v_w(g_i) \mid v_w(g_i) \neq 0\}.$$

Then for any  $b \in K$  with  $b\chi \in K[M]^\Delta$ , we have

$$v_w(b\chi) = v(b) + w = v(\lambda) + \sum_i \alpha_i v_w(g_i) \geq \epsilon > 0.$$

This inequality contradicts the density of  $\Gamma$  in  $\mathbb{R}$ . We conclude that  $K[M]^\Delta$  is not finitely generated.

If  $v$  is neither discrete nor trivial and  $\Delta$  is a pointed  $\Gamma$ -rational polyhedron then  $K[M]^\Delta$  is finitely generated if and only if the vertices of  $\Delta$  are in  $N_\Gamma$ , see [17, Proposition 6.9].

If  $\Delta$  is a pointed  $\Gamma$ -rational polyhedron, then  $Q(K[M]^\Delta) = K(\check{\sigma}_0 \cap M) = K(M)$ . This follows because  $\sigma_0 := \text{rec}(\Delta)$  contains no affine subspace of dimension  $> 0$ , see [17, Proposition 6.6]. Note that  $K[M]^\Delta$  is naturally  $M$ -graded. Therefore if the valuation  $v$  is neither trivial nor discrete and the vertices of  $\Delta$  are contained in  $N_\Gamma$ , we conclude that  $K[M]^\Delta$  is a normal flat algebra of finite type over  $K^\circ$  with an  $M$ -graduation. If  $v$  is discrete or trivial the extra condition on the vertices of  $\Delta$  is not required.

Geometrically this means that  $\mathcal{U}_\Delta := \text{Spec}(K[M]^\Delta)$  is a separated normal flat scheme of finite type over  $K^\circ$  such that it has an algebraic action of  $\mathbb{T}$  over  $K^\circ$  which extends the translation action of  $T \subset (\mathcal{U}_\Delta)_\eta$  over itself. In the case that  $K[M]^\Delta$  is of finite type, we conclude that  $\mathcal{U}_\Delta$  is a  $\mathbb{T}$ -toric variety.

*Remark 1.5.* For a pointed  $\Gamma$ -rational polyhedron  $\Delta \subset N_\mathbb{R}$ , we can define the  $\mathbb{T}$ -toric scheme  $\mathcal{U}_\Delta$  using  $\sigma := c(\Delta)$ , the closure of the cone generated by  $\Delta \times \{1\}$  in  $N_\mathbb{R} \times \mathbb{R}_+$ , as follows. Define

$$K[M]^\sigma := \left\{ \sum_{u \in \check{\sigma}_0 \cap M} \alpha_u \chi^u \in K[M] \mid cv(\alpha_u) + \langle u, \omega \rangle \geq 0 \quad \forall (\omega, c) \in \sigma \right\}.$$

We have  $K[M]^\sigma = K[M]^\Delta$ , then  $\mathcal{V}_\sigma := \text{Spec}(K[M]^\sigma) = \mathcal{U}_\Delta$ .

Lets see how the geometry of the special fiber can be described. First we will see how the irreducible components of  $(\mathcal{U}_\Delta)_s$  can be characterized.

**Proposition 1.6.** *The reduced induced structure on the special fibre is given by*

$$((\mathcal{U}_\Delta)_s)_{\text{red}} = \text{Spec}(K[M]^\Delta / \{f \in K[M]^\Delta \mid v_\Delta(f) > 0\}).$$

*Proof.* See [17, Lemma 6.13]. □

With this, we get the following important result.

**Proposition 1.7.** *Let  $\Delta$  be a pointed  $\Gamma$ -rational polyhedron in  $N_\mathbb{R}$ . Then there is a bijection between the vertices of  $\Delta$  and the irreducible components of  $(\mathcal{U}_\Delta)_s$ . The irreducible component corresponding to the vertex  $\omega$  is the closed subscheme  $Y_\omega$  of  $\mathcal{U}_\Delta$  given by the prime ideal  $\{f \in K[M]^\Delta \mid v_\omega(f) > 0\}$  of  $K[M]^\Delta$ .*

*Proof.* See [17, Proposition 6.14] □

*Example 1.8.* Let  $\Delta = [0, \lambda]$  with  $\lambda \in \Gamma \setminus \{0\}$ . Then  $A = K[M]^\Delta = K^\circ[x, ax^{-1}] = K^\circ[x, y]/(xy - a)$ , with  $v(a) = \lambda$ . Therefore  $A \otimes_{K^\circ} \tilde{K} = \tilde{K}[x, y]/(xy)$ . We see that  $(\mathcal{U}_\Delta)_s$  has two components which correspond to the vertices of  $\Delta$ .

*Remark 1.9.* Let  $\Delta \subset N_\mathbb{R}$  be a pointed rational polyhedron. If  $K$  is the function field of a normal curve  $Y$  defined over an algebraically closed field  $\mathbb{K}$  of characteristic zero, then the description of the special fiber of  $\mathcal{U}_\Delta$  can also be obtained from the theory of polyhedral divisors developed by Altmann–Hausen in [2]. In order to do this, we consider the polyhedral divisor  $\mathfrak{D} := \Delta \otimes \{y\}$ , for some  $y \in Y$ . Let  $K^\circ := \mathcal{O}_{Y, y}$  be the discrete

valuation ring corresponding to the localization of the structure sheaf of  $Y$  at  $y$ . Note that  $K$  is the fraction field of  $K^\circ$  and  $\mathbb{K}$  its residue field. From [2, §7] we get a  $T$ -variety  $\tilde{X}$  over  $Y$ . Let  $\pi : \tilde{X} \rightarrow Y$  be the canonical morphism and  $\mathbb{K}[\Lambda]$  be defined as in [2, Definition 7.1], then the fiber of this morphism over  $y$  is given by  $\pi^{-1}(y) = \text{Spec}(\mathbb{K}[\Lambda])$ , see [2, Proposition 7.10]. From [2, Proposition 7.3 - (iv)] we see that the irreducible components of  $\pi^{-1}(y)$  are  $T_{\mathbb{K}}$ -toric varieties corresponding to the local cones  $\text{LC}_w(\Delta)$  of  $\Delta$ . On the other hand from [17, Corollary 6.15] this are precisely the irreducible components of the special fiber of  $\mathcal{U}_\Delta$ . Note that the generic fiber of the morphism  $\pi$  coincide with the generic fiber  $(\mathcal{U}_\Delta)_\eta$  of the  $\mathbb{T}$ -toric variety  $\mathcal{U}_\Delta$ . Actually, from the definition of  $\tilde{X}$  follows that the base change  $\tilde{X} \times_Y \text{Spec}(K^\circ)$  is isomorphic to  $\mathcal{U}_\Delta$ . This shows that the constructions are the same for this case.

Finally, in order to show how the geometry of the special fiber depends on the combinatorics of the  $\Gamma$ -rational polyhedron  $\Delta$  and to give an explicit correspondence for the  $\mathbb{T}$ -orbits, we need the tropicalization map which we will introduce here. For further details on tropicalization see the articles by Baker–Payne–Rabinoff [3], Gubler [17] and the draft by Maclagan–Sturmfels [24].

## 1.2 Tropicalization

In this section we are going to give the basic definitions and properties of tropical varieties. Since for the definition of the tropicalization map Berkovich spaces are needed, we start recalling the construction of an analytic space associated to an algebraic variety  $X$  over a valued field  $K$ . Let  $X = \text{Spec}(A)$  be an affine scheme of finite type over  $K$ , the *analytification*  $X^{\text{an}}$  of  $X$  is given by the set of multiplicative seminorms which extend the absolute value of  $K$ , i.e. the maps  $p : A \rightarrow \mathbb{R}_+$  which satisfy

- $p(fg) = p(f)p(g)$
- $p(f + g) \leq p(f) + p(g)$
- $p(a) = |a|$

for all  $f, g \in A$  and all  $a \in K$ . This set is endowed with the weakest topology for which the maps  $f : X^{\text{an}} \rightarrow \mathbb{R}$ , given by  $f(p) := p(f)$ , are continuous for all  $f \in A$ . If  $X$  is a scheme of finite type over  $K$  which is not affine, we consider an affine covering of it, we construct their corresponding analytic spaces and finally we glue them together to obtain  $X^{\text{an}}$ . This space is independent of the chosen affine covering. For a Banach algebra  $\mathcal{A}$ , the Berkovich spectrum  $\mathcal{M}(\mathcal{A})$  is defined as the set of bounded multiplicative seminorms on  $\mathcal{A}$ , provided with the weakest topology defined as above. For details see [5] and [35].

Let  $X$  be a scheme of finite type over  $K$  and  $X^{\text{an}}$  be the corresponding analytic space. An important affinoid subset of  $X^{\text{an}}$  consists of the potentially integral points of  $X$ . This is defined as follows. For simplicity, we consider first the affine case  $X = \text{Spec}(A)$ ,  $X^{\text{an}} = \mathcal{M}(A)$ . Given a point  $p \in X^{\text{an}}$  the quotient field  $L$  of  $A/\{a \in A | p(a) = 0\}$  is endowed with an absolute value induced by  $p$ ,  $|\cdot|_\omega$ , which extends the norm on  $K$ . The

field  $(L, \omega)$  is a valued field extension of  $(K, \nu)$ . The canonical morphism  $A \rightarrow L$  induces an  $L$ -rational point  $P$  on  $X$ . Consequently given an element  $f \in A$  and a seminorm  $p \in X^{\text{an}}$  we can write  $p(f)$  as  $|f(P)|_\omega$ , where  $f(P)$  denotes the image of  $f$  in  $L$ .

The set of potentially integral points is defined as

$$X^\circ := \{p \in X^{\text{an}} \mid p(f) \leq 1 \forall f \in A\}.$$

That is, the set of points for which the image of  $A$  under the canonical map  $A \rightarrow L$  is contained in  $L^\circ$ . Those points induces on  $X$  a potentially integral point, i.e. integral for some valued field extension  $L/K$ . In general the potentially integral points  $X^\circ$  of a scheme  $X$  of finite type is the union of the sets  $U^\circ$  for  $U \in \mathcal{U}$ , where  $\mathcal{U}$  is an open affine covering of  $X$ . The set  $X^\circ$  is independent from the chosen covering.

Consider the split torus  $T = \text{Spec}(K[M])$  over the valued field  $(K, \nu)$ . The *tropicalization map* is defined as:

$$\text{trop}_\nu : T^{\text{an}} \rightarrow N_{\mathbb{R}}, \quad p \mapsto \text{trop}_\nu(p),$$

with  $\text{trop}_\nu(p)$  given by

$$\langle u, \text{trop}_\nu(p) \rangle := -\log(p(\chi^u)).$$

Explicitly, choosing coordinates  $x_1, \dots, x_n$  on the torus  $T$  this map is given by

$$p \mapsto (-\log(p(x_1)), \dots, -\log(p(x_n))).$$

It follows from the definition that the *tropicalization map* is continuous. Let  $X$  be a closed subscheme of  $T$ . We define the tropicalization of the closed subscheme  $X$  or the tropical variety associated to  $X$  as  $\text{Trop}_\nu(X) := \text{trop}_\nu(X^{\text{an}})$ .

We have the following fundamental result of tropical geometry.

**Theorem 1.10. (Bieri-Groves)**  *$\text{Trop}_\nu(X)$  is a finite union of  $\Gamma$ -rational polyhedra in  $N_{\mathbb{R}}$ . If  $X$  is of pure dimension  $d$ , then we may choose all the polyhedra  $d$ -dimensional.*

*Proof.* See [17, Theorem 3.3]. □

Furthermore if  $X$  is connected and  $K$  is complete, algebraically closed or real closed with convex valuation ring then  $\text{Trop}_\nu(X)$  is connected as well, see [10, Theorem 1]. Now, if  $(L, \omega)$  is an algebraically closed valued field extending  $(K, \nu)$  then  $\text{Trop}_\nu(X)$  equals the closure of the set  $\{(-\log|x_1|_\omega, \dots, -\log|x_n|_\omega) \mid \mathbf{x} \in X(L)\}$ . Note that if  $(K, \nu)$  is algebraically closed, then it is just the closure of the valuation map defined on the rational points, i.e. the closure of the set  $\{(v(x_1), \dots, v(x_n)) \mid \mathbf{x} \in X(K)\}$  in  $\mathbb{R}^n$ . This is the usual definition of a tropical variety.

Before we proceed further, we explore another advantage of  $\mathbb{T}$ -toric schemes over valuation rings: there is a reduction map. This map connects the geometry of both fibers and give us the complete description of the scheme. Given a  $\mathbb{T}$ -toric scheme  $\mathcal{X}$ , the reduction map is defined on the set  $T^{\text{an}} \cap \mathcal{X}_\eta^\circ$  of potentially integral points of the generic fibre in the analytic torus. Since every potentially integral point gives rise to a seminorm  $p$  on  $K[M]^\Delta$ , the reduction map  $\pi$  sends this seminorm to the image of the prime ideal  $\{f \in A \mid p(f) < 1\} \subset A$  in  $A \otimes_{K^\circ} \tilde{K}$ . Using the tropical map, the domain of  $\pi$  is given explicitly as the following result shows.

**Proposition 1.11.** *Let  $\Delta$  be a pointed  $\Gamma$ -rational polyhedra, then on the generic fiber  $(\mathcal{U}_\Delta)_\eta =: U_\sigma$  we have  $U_\sigma^\circ \cap T^{an} = \text{trop}_v^{-1}(\Delta)$ .*

*Proof.* See [17, Lemma 6.21]. □

**Proposition 1.12.** *Let  $\Delta$  be a pointed  $\Gamma$ -rational polyhedron in  $N_{\mathbb{R}}$  and let  $\mathcal{U}_\Delta$  be the associated  $\mathbb{T}$ -toric scheme over  $K^\circ$ . Then there is a bijective order reversing correspondence between  $\mathbb{T}$ -orbits  $Z$  of  $(\mathcal{U}_\Delta)_s$  and open faces  $\tau$  of  $\Delta$  given by*

$$Z = \pi(\text{trop}_v^{-1}(\tau)), \quad \tau = \text{trop}_v(\pi^{-1}(Z)).$$

Moreover, we have  $\dim(Z) + \dim(\tau) = n$ .

*Proof.* See [17, Proposition 6.22]. □

Here we can see explicitly how tropical geometry help us to complete the description of the  $\mathbb{T}$ -toric scheme  $\mathcal{U}_\Delta$  in terms of the combinatorics of the pointed  $\Gamma$ -rational polyhedron  $\Delta$ . So far, we have seen that the generic fiber is described by toric geometry, it is the  $T$ -toric variety associated to the cone  $\sigma = \text{rec}(\Delta)$ , and the components of the special fiber are in one-to-one correspondence with the vertices of  $\Delta$ . Furthermore every component of the special fiber is a toric variety over  $\tilde{K}$ . Given a vertex  $\omega$  of  $\Delta$  the corresponding component is a  $\text{Spec}(\tilde{K}[M_\omega])$ -toric variety, with  $M_\omega := \{u \in M \mid \langle u, \omega \rangle \in \Gamma\}$ , for details see [17, Corollary 6.15].

### 1.3 Toric schemes associated to fans

As we have seen, given a pointed  $\Gamma$ -rational polyhedron  $\Delta \subset N_{\mathbb{R}}$ , the scheme  $\mathcal{U}_\Delta$  is a normal  $\mathbb{T}$ -toric scheme over  $K^\circ$ . Recall that this scheme can be written as  $\mathcal{V}_\sigma$ , where  $\sigma = c(\Delta)$ . We would like to associate to a polyhedral complex  $\mathcal{C}$ , made of pointed  $\Gamma$ -rational polyhedra, a  $\mathbb{T}$ -toric scheme by gluing the corresponding  $\mathcal{U}_\Delta$ ,  $\Delta \in \mathcal{C}$ , along their common intersections. Unfortunately there are some problems in the gluing process and to avoid them it is necessary to work with cones in  $N_{\mathbb{R}} \times \mathbb{R}_+$  instead of polyhedra in  $N_{\mathbb{R}}$ . This construction is well behaved for any  $\Gamma$ -admissible fan  $\Sigma$  in  $N_{\mathbb{R}} \times \mathbb{R}_+$ , see Remark 1.13 below.

Let  $\sigma \subset N_{\mathbb{R}} \times \mathbb{R}_+$  be a  $\Gamma$ -admissible cone. There is a bijective correspondence between pointed  $\Gamma$ -rational polyhedra  $\Delta \subset N_{\mathbb{R}}$  and  $\Gamma$  admissible cones  $\sigma$  not contained in  $N_{\mathbb{R}} \times \{0\}$ . This is given by

$$\begin{aligned} \sigma &\mapsto \Delta = \sigma \cap (N_{\mathbb{R}} \times \{1\}), \\ \Delta &\mapsto \sigma = c(\Delta). \end{aligned}$$

A  $\Gamma$ -admissible fan  $\Sigma$  in  $N_{\mathbb{R}} \times \mathbb{R}_+$  is a fan of  $\Gamma$ -admissible cones.

*Remark 1.13.* The previous correspondence can be extended to a correspondence between polyhedral complexes and fans but not in complete generality, an extra condition on those is required. This was pointed out by Burgos–Sombra in [9], see also [17, Remark 7.6]. Concretely there is a bijective correspondence between complete  $\Gamma$ -rational pointed polyhedral complexes in  $N_{\mathbb{R}}$  and complete  $\Gamma$ -admissible fans in  $N_{\mathbb{R}} \times \mathbb{R}_+$ .

Given a  $\Gamma$ -admissible fan  $\Sigma$ , we construct a normal  $\mathbb{T}$ -toric scheme  $\mathcal{Y}_{\Sigma}$  as follows: for each  $\Gamma$ -admissible cone  $\sigma$  we have the normal affine  $\mathbb{T}$ -toric scheme  $\mathcal{V}_{\sigma}$ , then we glue them together along the open subschemes coming from their common faces. The scheme obtained in this way is separated, see [17, Lemma 7.8]. The description of the  $\mathbb{T}$ -toric schemes coming from  $\Gamma$ -admissible fans is given by the combinatorics of the tropical cone, see [17, §8]. Explicitly, we have the following result.

**Proposition 1.14.** *There is a bijective order reversing correspondence between  $\mathbb{T}$ -orbits  $Z$  of the special fiber  $(\mathcal{Y}_{\Sigma})_s$  and open faces  $\tau$  of  $\Sigma$  which are not contained in  $N_{\mathbb{R}} \times \{0\}$ . It is given by*

$$Z = \pi(\text{trop}^{-1}(\tau)), \quad \tau = \text{trop}(\pi^{-1}(Z)).$$

*Proof.* See [17, Proposition 8.8]. □

Finally, it is worth to note that since  $(\mathcal{Y}_{\Sigma})_{\eta}$  and  $(\mathcal{Y}_{\Sigma})_s$  are noetherian topological spaces, then so is  $\mathcal{Y}_{\Sigma}$  although  $K^{\circ}$  is non-noetherian in general. This fact will be very important in chapter 4, see for instance Lemma 3.10.

## 1.4 Projective toric schemes

The projective  $\mathbb{T}$ -toric varieties with a linear action of the torus have a very explicit description. In this section we review how to construct them. These varieties are not necessarily normal, for details see [17, §9]. This construction does not give a classification of all the possible projective toric schemes over  $K^{\circ}$  but just those which have a linear action of the torus, see [17, Proposition 9.8]. Over fields this construction was done by Gelfand–Kapranov–Zelevinsky [15], see also Cox–Little–Schenk [11]. Over discrete valuation rings this construction can be found in the paper by Katz [20].

Let  $A = (u_0, \dots, u_N) \in M^{N+1}$  and  $\mathbf{y} = (y_0 : \dots : y_N) \in \mathbb{P}_{K^{\circ}}^N(K)$ . The height function of  $\mathbf{y}$  is defined as

$$a : \{0, \dots, N\} \rightarrow \Gamma \cup \{\infty\}, \quad j \mapsto a(j) := v(y_j).$$

The action of  $\mathbb{T}$  on  $\mathbb{P}_{K^{\circ}}^N$  is given by

$$(t, \mathbf{x}) \mapsto (\chi^{u_0}(t)x_0 : \dots : \chi^{u_N}(t)x_N).$$

We define the projective toric variety  $\mathcal{Y}_{A,a}$  to be the closure in  $\mathbb{P}_{K^{\circ}}^N$  of the orbit  $T\mathbf{y}$ . The generic fiber  $Y_{A,a}$  is a toric variety respect to the torus  $T/\text{stab}(\mathbf{y})$ . Indeed it follows that  $\mathcal{Y}_{A,a}$  is a  $\mathbb{T}$ -toric variety over  $K^{\circ}$  with respect to the split torus over  $K^{\circ}$  with generic fiber  $T/\text{Stab}(\mathbf{y})$ .

The *weight polytope*  $\text{Wt}(\mathbf{y})$  is defined as the convex hull of  $A(\mathbf{y}) := \{u_j | a(j) < \infty\}$ . The *weight subdivision polytopal complex*  $\text{Wt}(\mathbf{y}, a)$  is obtained from  $\text{Wt}(\mathbf{y})$  with the subdivision given by projecting the faces of the convex hull of  $\{(u_j, \lambda_j) \in M_{\mathbb{R}} \times \mathbb{R}_+ | j = 0, \dots, N; \lambda_j \geq a(j)\}$ .

Defining  $f$  on the set  $A(\mathbf{y})$  by  $f(u_i) := a(i)$  and extending to  $\text{Wt}(\mathbf{y})$  by linearity, we have a convex function  $f$  whose epigraph is equal to the convex hull of  $\{(u_j, \lambda) \in M_{\mathbb{R}} \times \mathbb{R} | \lambda \geq a(j), j = 0, \dots, N\}$ . The function  $f$  is uniquely characterized by its epigraph and the fact that it is convex. By construction the domain of this function is equal to  $\text{Wt}(\mathbf{y})$ .

The *dual complex*  $\mathcal{C}(A, a)$  of  $\text{Wt}(\mathbf{y}, a)$ , is defined as the complete polyhedral complex which has as polyhedra the maximal sets where the function

$$g(\omega) := \min_{j=0, \dots, N} \{a(j) + \langle u_j, \omega \rangle\}$$

is linear. There is an order reversing correspondence between the faces of  $\text{Wt}(\mathbf{y}, a)$  and the polyhedra of  $\mathcal{C}(A, a)$ . Explicitly given a face  $Q$  of  $\text{Wt}(\mathbf{y}, a)$ , we have the following polyhedron of  $\mathcal{C}(A, a)$

$$Q \mapsto \widehat{Q} := \{\omega \in N_{\mathbb{R}} | g(\omega) = \langle u, \omega \rangle + f(u) \forall u \in Q\}.$$

Given a polyhedron  $\sigma$  of  $\mathcal{C}(A, a)$ , we have the following face of  $\text{Wt}(\mathbf{y}, a)$

$$\sigma \mapsto \widehat{\sigma} = \{u \in M_{\mathbb{R}} | g(\omega) = \langle u, \omega \rangle \forall \omega \in \sigma\}.$$

The main result about the  $\mathbb{T}$ -toric schemes obtained in this way is given by the following proposition.

**Proposition 1.15.** *There are a bijective correspondence between*

- (a) *faces  $Q$  of the weight subdivision  $\text{Wt}(\mathbf{y}, a)$ ;*
- (b) *polyhedra  $\sigma$  of the dual complex  $\mathcal{C}(A, a)$ ;*
- (c)  *$\mathbb{T}$ -orbits  $Z$  of the special fiber of  $\mathcal{Y}_{A, a}$ .*

*The correspondences are given explicitly as follows: The face  $Q = \widehat{\sigma}$  is the face of  $\text{Wt}(\mathbf{y}, a)$  spanned by those  $u_j$  with  $x_j \neq 0$  for  $x \in Z$ . The polyhedron  $\sigma$  is given by  $\sigma = \widehat{Q}$  and  $\text{relint}(\sigma) = \text{trop}_v(\{t \in T^{\text{an}} | \pi(t\mathbf{y}) \in Z\})$ . The orbit  $Z$  is equal to*

$$\{\mathbf{x} \in (\mathcal{Y}_{A, a})_s | x_j \neq 0 \Leftrightarrow u_j \in A(\mathbf{y}) \cap Q\} = \{\pi(t\mathbf{y}) | t \in T^{\text{an}} \cap \text{trop}_v^{-1}(\text{relint}(\sigma))\}.$$

*Proof.* See [17, Proposition 9.12.]. □

We will see in chapter 4 that this result is crucial to prove Sumihiro's theorem for normal  $\mathbb{T}$ -toric varieties.





## 2 | The cone of a normal affine toric variety

We recall that  $(K, \nu)$  is a valued field with valuation ring  $K^\circ$ , residue field  $\tilde{K}$  and value group  $\Gamma \subset \mathbb{R}$ . We assume that the valuation is not trivial since for this case the statement of Theorem 1 is reduced to the field situation where it is already known, see [21, ch. I Theorem 1']. The split torus over  $K^\circ$  is  $\mathbb{T} = \text{Spec}(K^\circ[M])$  with generic fiber  $T = \text{Spec}(K[M])$ . The character group of  $T$  is  $M$  with dual  $N = \text{Hom}(M, \mathbb{Z})$ . For an element  $u \in M$ , the corresponding character is denoted by  $\chi^u$ .

As we have seen in the previous chapter a pointed  $\Gamma$ -rational polyhedron  $\Delta \subset N_{\mathbb{R}}$ , with  $\sigma = c(\Delta) \subset N_{\mathbb{R}} \times \mathbb{R}_+$ , induces a normal  $\mathbb{T}$ -toric scheme  $\mathcal{V}_\sigma = \mathcal{U}_\Delta = \text{Spec}(K[M]^\sigma)$ . This is a  $\mathbb{T}$ -toric variety if the valuation is discrete or if the valuation is not discrete and the vertices of  $\Delta$  are in  $N_\Gamma$ . It is natural to ask if every affine  $\mathbb{T}$ -toric variety is of this form. In this chapter we will give an affirmative answer to this question. In fact, we will show that if the valuation is neither trivial nor discrete the isomorphism classes of normal affine  $\mathbb{T}$ -toric varieties are in a bijective correspondence with  $\Gamma$ -admissible cones  $\sigma$  for which the vertices of  $\sigma \cap (N_{\mathbb{R}} \times \{1\})$  are contained in  $N_\Gamma \times \{1\}$ . This gives the proof of Theorem 1.

### 2.1 Construction of the cone

Let  $\mathcal{Y} = \text{Spec}(A)$  be an affine normal  $\mathbb{T}$ -toric variety as in Theorem 1. The algebra  $A$  satisfies the following properties. Since  $\mathbb{T}$  acts on  $\mathcal{Y}$ ,  $A$  has an  $M$ -graduation

$$A = \bigoplus_{m \in M} A_m.$$

Since it is finitely generated, we can choose  $M$ -homogeneous generators  $a_1 \chi^{m_1}, \dots, a_k \chi^{m_k}$  of  $A$ , that is

$$A = K^\circ[a_1 \chi^{m_1}, \dots, a_k \chi^{m_k}] \quad a_i \in K. \quad (2.1)$$

With this representation of the algebra  $A$ , we define the following semigroup

$$S := \{(m, \nu(a)) \in M \times \Gamma \mid a \chi^m \in A \setminus \{0\}\}. \quad (2.2)$$

**Lemma 2.1.** *The semigroup  $S$  is saturated in  $M \times \Gamma$ .*

*Proof.* Let  $(m, \nu(a)) \in M \times \Gamma$  and  $N \in \mathbb{Z}_+$  such that  $N(m, \nu(a)) \in S$ , i.e.  $(a\chi^m)^N \in A$ . Then  $a\chi^m$  satisfies the integral equation

$$x^N - (a\chi^m)^N = 0.$$

By normality of  $A$ , we get  $a\chi^m \in A$ , i.e.  $(m, \nu(a)) \in S$ .  $\square$

Given a subset  $B \subset M \times \Gamma$ , it generates a cone in  $M \times \mathbb{R}$ , namely

$$\text{cone}(B) := \left\{ \sum \alpha_j (n_j, \omega_j) \mid \alpha_j \in \mathbb{R}_+, (n_j, \omega_j) \in B \right\} \subset M_{\mathbb{R}} \times \mathbb{R}.$$

In particular, to the semigroup  $S$  we associate the cone

$$\text{cone}(S) := \left\{ \sum \alpha_j (n_j, \nu(b_j)) \mid \alpha_j \in \mathbb{R}_+, (n_j, \nu(b_j)) \in S \right\}$$

in  $M_{\mathbb{R}} \times \mathbb{R}$ .

**Lemma 2.2.** *Using the homogeneous generators  $a_1\chi^{m_1}, \dots, a_k\chi^{m_k}$  from (2.1), we get*

$$\text{cone}(S) = \text{cone}(\{(0, 1), (m_i, \nu(a_i)), i = 1, \dots, k\}).$$

*Proof.* The inclusion " $\supset$ " is clear. Let  $\sum \alpha_j (n_j, \nu(b_j)) \in \text{cone}(S)$  be as above. Since  $(n_j, \nu(b_j)) \in S$ , we get  $b_j\chi^{n_j} \in A$ . Then (2.1) gives

$$b_j\chi^{n_j} = \lambda^{(j)} (a_1\chi^{m_1})^{l_1^{(j)}} \cdots (a_k\chi^{m_k})^{l_k^{(j)}} \quad \text{for } \lambda^{(j)} \in K^\circ, l_1^{(j)}, \dots, l_k^{(j)} \in \mathbb{Z}_+.$$

This implies

$$\begin{aligned} \nu(b_j) &= \nu(\lambda^{(j)}) + \sum_{i=1}^k l_i^{(j)} \nu(a_i) \\ n_j &= \sum_{i=1}^k l_i^{(j)} m_i, \end{aligned}$$

hence

$$\begin{aligned} \sum_j \alpha_j \left( \sum_{i=1}^k l_i^{(j)} m_i, \nu(\lambda^{(j)}) + \sum_{i=1}^k l_i^{(j)} \nu(a_i) \right) &= \sum_j \alpha_j (0, \nu(\lambda^{(j)})) + \sum_j \sum_i \alpha_j l_i^{(j)} (m_i, \nu(a_i)) \\ &= (0, \kappa) + \sum_i \lambda_i (m_i, \nu(a_i)) \end{aligned}$$

with  $\kappa := \sum_j \alpha_j \nu(\lambda^{(j)})$  and  $\lambda_i := \sum_j \alpha_j l_i^{(j)}$ . This proves the lemma.  $\square$

Using duality of cones in  $M_{\mathbb{R}} \times \mathbb{R}$ , we define  $\sigma := \text{cone}(S)$ . Therefore, again by duality of polyhedral cones we get  $\check{\sigma} = \text{cone}(S)$ , see [13, §1.2]. We have the following property of  $\sigma$ .

**Lemma 2.3.** *The cone  $\sigma$  is  $\Gamma$ -admissible.*

*Proof.* From Lemma 2.2, we have that

$$\sigma = \bigcap_{i=1}^k \{(\omega, s) \in N_{\mathbb{R}} \times \mathbb{R}_+ \mid \langle m_i, \omega \rangle + s\nu(a_i) \geq 0\} \text{ with } m_1, \dots, m_k \in M,$$

which is by definition a  $\Gamma$ -rational cone. To prove that  $\sigma$  is  $\Gamma$ -admissible we just have to show that it doesn't contain a line. Suppose it does, i.e.  $\mathbb{R} \cdot (\omega, t) \subset \sigma$  for some  $(\omega, t) \in N_{\mathbb{R}} \times \mathbb{R}_+$ . Since  $\sigma \subset N_{\mathbb{R}} \times \mathbb{R}_+$ , we must have  $t = 0$ . Otherwise  $(\lambda\omega, \lambda t) \in \sigma$  for all  $\lambda \in \mathbb{R}$  and with  $\lambda = -1$  we would have  $(-\omega, -t) \in N_{\mathbb{R}} \times \mathbb{R}_+$  which is not possible for  $t > 0$ . Therefore the line is of the form  $\mathbb{R} \cdot (\omega, 0) \subset N_{\mathbb{R}} \times \{0\}$ .

On the other hand for any  $a\chi^\mu \in A$ , we have  $(\mu, \nu(a)) \in \text{cone}(S) = \check{\sigma}$ . Then

$$\langle (\mu, \nu(a)), (\lambda\omega, 0) \rangle = \langle \mu, \lambda\omega \rangle \geq 0 \quad \forall \lambda \in \mathbb{R}.$$

In particular with  $\lambda = \pm 1$ , we get  $\langle \mu, \omega \rangle = 0$ , i.e.  $\mu \in \omega^\perp$ . Choosing a basis  $\{u_1, \dots, u_n\}$  for  $M$ , such that  $u_1, \dots, u_{n-1} \in \omega^\perp$ , we get  $A \subset K[\chi^{\pm u_1}, \dots, \chi^{\pm u_{n-1}}]$ . Therefore

$$Q(A) \subset Q(K[\chi^{\pm u_1}, \dots, \chi^{\pm u_{n-1}}]) = K(\chi^{\pm u_1}, \dots, \chi^{\pm u_{n-1}}) \subsetneq K(M).$$

Since  $\mathcal{Y} = \text{Spec}(A)$  has a dense  $T$ -orbit, we have  $Q(A) = K(M)$ . This contradicts our assumption. Hence we conclude that  $\sigma$  doesn't contain any line and therefore it is  $\Gamma$ -admissible.  $\square$

## 2.2 Proof of Theorem 1

With these lemmas, we are ready to prove the main result of this chapter. Define the algebra

$$K[M]^\sigma := \left\{ \sum_{m \in \check{\sigma}_0 \cap M} a_m \chi^m \in K[M] \mid \langle m, \omega \rangle + t \cdot \nu(a_m) \geq 0, \quad \forall (\omega, t) \in \sigma \right\}.$$

By Lemma 2.3 and Remark 1.5 this algebra defines an affine normal  $\mathbb{T}$ -toric scheme. With the previous notation, we have to prove  $K[M]^\sigma = A$ .

Let us see first that  $A \subset K[M]$ . Take any  $a\chi^m \in A$ , since  $(m, \nu(a)) \in S$  then  $(m, \nu(a)) \in \check{\sigma}$ . Therefore by the definition of the dual cone, we have

$$\langle m, \omega \rangle + t \cdot \nu(a) \geq 0 \quad \forall (\omega, t) \in \sigma.$$

Therefore  $a\chi^m \in K[M]^\sigma$ , i.e.  $A \subset K[M]^\sigma$ .

To prove the other inclusion, we take  $a\chi^m \in K[M]^\sigma$ . Then  $(m, \nu(a)) \in \check{\sigma} = \text{cone}(S)$ . By Lemma 2.2, we get

$$(m, \nu(a)) = (0, \kappa) + \sum_{i=1}^k \lambda_i (m_i, \nu(a_i)), \quad \kappa, \lambda_i \geq 0.$$

From this, we get the following system of equations

$$m = \sum_i \lambda_i m_i \tag{2.3}$$

$$\nu(a) = \kappa + \sum_i \lambda_i \nu(a_i) \tag{2.4}$$

Now we show that it is always possible to choose  $\lambda_i \in \mathbb{Q}_+$ . We have to consider the following two cases:

- (a) Suppose that  $\kappa \neq 0$ . Let  $\{b_j\}_{j=1}^s$ , with  $b_j = (b_j^{(1)}, \dots, b_j^{(k)}) \in \mathbb{Q}^k$  ( $j = 1, \dots, s$ ), be a basis in  $\mathbb{Q}^k$  for the solutions of the homogeneous equation associated to (2.3) and let  $\mu \in \mathbb{Q}^k$  be a particular solution for (2.3). The space of solutions is given by

$$\mathbb{L} = \left\{ \mu + \sum_{j=1}^s \rho_j b_j \mid \rho_j \in \mathbb{R}, j = 1, \dots, s \right\}.$$

Since  $\lambda := (\lambda_i) \in \mathbb{R}_+^k$  is a solution of (2.3), there exist  $\rho_j^\circ \in \mathbb{R}$  ( $j = 1, \dots, s$ ) such that

$$\lambda = \mu + \sum_j \rho_j^\circ b_j.$$

Now choose  $\hat{\rho}_j^\circ \in \mathbb{Q}$  close to  $\rho_j^\circ$ , i.e.

$$\rho_j^\circ = \hat{\rho}_j^\circ + \epsilon_j.$$

Then

$$\hat{\lambda} = \mu + \sum_j \hat{\rho}_j^\circ b_j$$

is also a solution of (2.3) which is close to  $\lambda$ . Explicitly, we have

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \hat{\lambda}_1 \\ \vdots \\ \hat{\lambda}_k \end{pmatrix} + \sum_{j=1}^s \epsilon_j b_j.$$

Inserting this in (2.4), we get

$$\begin{aligned} \nu(a) &= \kappa + \sum_i \left( \hat{\lambda}_i + \sum_j \epsilon_j b_j^{(i)} \right) \nu(a_i) \\ &= \kappa + \sum_i \hat{\lambda}_i \nu(a_i) + \sum_i \left( \sum_j \epsilon_j b_j^{(i)} \right) \nu(a_i). \end{aligned}$$

With  $\alpha := \sum_i \left( \sum_j \epsilon_j b_j^{(i)} \right) \nu(a_i)$ , we get

$$\kappa + \alpha = \nu(a) - \sum_i \hat{\lambda}_i \nu(a_i).$$

For  $\epsilon_j$  small enough, we have that  $\kappa + \alpha > 0$ . We conclude that it is possible to choose the coefficients in (2.3) rational, i.e. the element  $(m, \nu(a)) \in \check{\sigma}$  is obtained with the coefficients  $\kappa + \alpha, \hat{\lambda}_i, \quad i = 1, \dots, k$ .

(b) Suppose that  $\kappa = 0$ . In an analogous way, we obtain that

$$\begin{aligned} \alpha &:= \sum_i \left( \sum_j \epsilon_j b_j^{(i)} \right) \nu(a_i) = \nu(a) - \sum_i \hat{\lambda}_i \nu(a_i) \\ &= \sum_j \left( \sum_i b_j^{(i)} \nu(a_i) \right) \epsilon_j. \end{aligned}$$

Then with  $\delta_j := \sum_i b_j^{(i)} \nu(a_i)$ , we get

$$\alpha = \sum_j \delta_j \epsilon_j = \nu(a) - \sum_i \hat{\lambda}_i \nu(a_i).$$

Therefore choosing  $\epsilon_i$  properly, according to the sign of  $\delta_j$ , we can assure that every term in the sum  $\sum_j \delta_j \epsilon_j$  is positive and therefore  $\alpha > 0$ . This shows that it is possible to choose the coefficients in (2.3) rational, i.e. the element  $(m, \nu(a)) \in \check{\sigma}$  is obtained with the coefficients  $\alpha, \hat{\lambda}_i, \quad i = 1, \dots, k$ .

This proves that it is always possible to write  $(m, \nu(a)) = (0, \kappa) + \sum \lambda_i (m_i, \nu(a_i))$  with  $\lambda_i \in \mathbb{Q}_+$  and  $\kappa \in \mathbb{R}_+$ . Let  $N \in \mathbb{Z}_+$  such that  $N\lambda_i \in \mathbb{Z}_+, \quad i = 1, \dots, k$ . Then we get

$$N(m, \nu(a)) = N(0, \kappa) + \sum_i N\lambda_i (m_i, \nu(a_i)).$$

This proves in particular that  $N\kappa \in \Gamma$ . Since  $(0, N\kappa), (m_i, \nu(a_i)) \in S \quad (i = 1, \dots, k)$  and  $N\lambda_i \in \mathbb{Z}_+$ , we conclude  $(Nm, N\nu(a)) \in S$ , because  $S$  is a semigroup. It follows that  $(a\chi^m)^N \in A$ . By normality of  $A$  this implies that  $a\chi^m \in A$ . Therefore  $K[M]^\sigma = A$ . This proves  $\mathcal{Y} = \mathcal{V}_\sigma$  and Theorem 1.



## 3 | Intersection Theory

In this chapter  $K$  will denote a valued field with valuation ring  $K^\circ$ , valued group  $\Gamma \subset \mathbb{R}$  and residue field  $\tilde{K}$ . We recall that a variety over  $K^\circ$  is an integral separated flat scheme of finite type over  $K^\circ$ .

To study the action of the split torus  $\mathbb{T} = (\mathbb{G}_m)_{K^\circ}^n$  on divisors of  $\mathbb{T}$ -toric varieties, results concerning intersection theory of Cartier divisors will be of crucial importance. For varieties over the valuation ring  $K^\circ$ , these results follow from the intersection theory of Cartier divisors on formal admissible schemes over a valuation ring due to Gubler, see [19]. Before state these results, we review some general properties of divisors on normal varieties over valuation rings due to Knaf [23]. For completeness, we recall the definition and basic results of Prüfer  $v$ -multiplication rings. After that, we study intersection theory of Cartier divisors on normal  $\mathbb{T}$ -toric varieties. These results were proved by Gubler in [19] and are presented in an algebraic form in [18].

### 3.1 Prüfer $v$ -multiplication rings

In this section we introduce the theory of Prüfer  $v$ -multiplication rings, which are suitable for the study of normal domains over valuation rings. We will follow closely the paper [23] by Knaf. A good survey by Fontana–Zafrullah can be found in [1].

Let  $A$  be a domain, we denote by  $\mathcal{F}(A)$  the set of fractional ideals of  $A$ . Consider the following operation on  $\mathcal{F}(A)$ :

$$\hat{\phantom{x}} : \mathcal{F}(A) \rightarrow \mathcal{F}(A), \quad I \mapsto \hat{I} := (A : (A : I)). \quad (3.1)$$

A fractional ideal  $I$  which satisfy  $\hat{I} = I$  is called a divisorial ideal. We define the set of finitely generated divisorial ideals by

$$\mathcal{D}_{\text{fin}}^\nu(A) := \left\{ I \in \mathcal{F}(A) \mid I = \hat{I}_0, I_0 \text{ f.g.} \right\}.$$

It is endowed with a semigroup structure using the product

$$(I, J) \mapsto I \cdot J := \widehat{IJ}. \quad (3.2)$$

The same construction without the finiteness restriction on the ideals is denoted by  $\mathcal{D}^\nu(A)$ .

**Definition 3.1.** A domain  $A$  is called a Prüfer  $v$ -multiplication ring or PvM-ring iff  $\mathcal{D}_{\text{fin}}^\nu(A)$  is a group.

*Remark 3.2.* The difference between Prüfer  $v$ -multiplication rings and Prüfer domains as defined in Bourbaki [7, ch.VII §2 Exercise 12] is that the later requires that the localization at any prime ideal is a valuation ring and as we will see, this is not the case for PvM-rings. In particular every Prüfer domain is a PvM-ring.

We denote by  $P(A)$  the set of weakly associated prime ideals of the  $A$ -module  $Q(A)/A$ . Note that any  $p \in P(A)$  is a minimal prime over an ideal of the form  $((a) : (b))$  for some  $a, b \in A$  and  $b \notin (a)$ . The divisorial ideals can be characterized locally on the elements of  $P(A)$  as the next proposition shows.

**Proposition 3.3.** *Let  $A$  be a domain, then for any  $I \in \mathcal{D}^\nu(A)$  we have  $I = \bigcap_{p \in P(A)} IA_p$ . In particular  $A = \bigcap_{p \in P(A)} A_p$ . Furthermore two divisorial ideals are equal if and only if they are equal locally on the elements of  $P(A)$ , i.e.  $\forall I, J \in \mathcal{D}^\nu(A)$  we have  $I = J \Leftrightarrow IA_p = JA_p \forall p \in P(A)$ .*

*Proof.* This follows from [23, §1.1 – §1.2]. □

The next proposition shows how the PvM-rings can be characterized.

**Proposition 3.4.** *Let  $A$  be a domain,  $A$  is a PvM-ring iff for every  $p \in P(A)$  the localization  $A_p$  is a valuation ring and for every  $a, b \in A$ , we have  $(a) \cap (b) \in \mathcal{D}_{\text{fin}}^\nu(A)$ . Furthermore if  $A$  is normal we just require that  $(a) \cap (b)$  is finitely generated for any pairs  $a, b \in A$ .*

*Proof.* See [25, Theorem 3.2 – Theorem 3.3]. □

From this characterization, we get the following important corollary.

**Corollary 3.5.** *Every normal coherent domain is a PvM-ring.*

*Proof.* See [23, §1.3]. □

In order to study further the domain  $A$ , we introduce the  $t$ -operation on the fractional ideals  $\mathcal{F}(A)$  as

$$\mathcal{F}(A) \rightarrow \mathcal{F}(A), \quad I \mapsto I^t := \bigcup_{I_0 \subset I, I_0 \text{ f.g.}} \hat{I}_0.$$

With the product  $(I, J) \mapsto (IJ)^t$ , the set  $\{I \in \mathcal{F}(A) | I^t = I\}$  is a semigroup denoted by  $\mathcal{D}^t(A)$ . The ideals in this semigroup are called  $t$ -ideals. The set of integral  $t$ -ideals can be ordered by inclusion and the maximal elements in this set are called maximal  $t$ -ideals. They are prime ideals in  $A$ . This set is denoted by  $t\text{Max}(A)$ . We denote by  $\text{Val}(A)$  the set of *valued primes* of  $A$ , i.e. the elements  $p \in \text{Spec}(A)$  such that  $A_p$  is a valuation ring. Because of Proposition 3.4, we know that in order to understand the domain  $A$  it is necessary to understand the rings  $A_p$  for every  $p \in P(A)$ . In particular we would like to know if these rings are valuation rings. To answer this question the  $t$ -ideals are a very



good tool, the reason is that every prime ideal  $p$  for which  $A_p$  is a valuation ring is a  $t$ -ideal, see [25, Corollary 4.2].

The PvM-rings can be characterized using  $t$ -ideals.

**Proposition 3.6.** *Let  $A$  be a domain, then  $A = \bigcap_{p \in t\text{Max}(A)} A_p$ . Furthermore  $A$  is a PvM-ring iff  $t\text{Max}(A) \subset \text{Val}(A)$ . Moreover this condition is also equivalent to  $\text{Val}(A) = \text{Spec}(A) \cap \mathcal{D}^t(A)$ .*

*Proof.* This follows from [23, 1.5]. □

The following proposition determine the valued primes in a flat normal domain of finite type over a valuation ring.

**Proposition 3.7.** *Let  $A$  be a normal domain of finite presentation over the valuation ring  $K^\circ$ , then  $\text{Val}(A) = \{p \in \text{Spec}(A) \mid \text{ht}(p) = 1\} \cup \{p_\eta\}$ , where  $p_\eta$  correspond to the generic point of the generic fiber of  $\text{Spec}(A)$ . In particular for every  $p \in P(A)$  the rings  $A_p$  are discrete valuation rings and for every  $p \in \text{Spec}(A \otimes_{K^\circ} \tilde{K})$  minimal the rings  $A_p$  are valuation rings of rank one.*

*Proof.* By [23, Theorem 2.4] a normal flat algebra of finite type over a valuation ring is coherent. Then by Corollary 3.5,  $A$  is a PvM-ring. Hence the first statement follows from [23, Theorem 2.6]. The last statement follows because  $A$  is an algebra of finite type over a rank one valuation ring  $(K^\circ, v)$ . The rings  $A_p$ , for  $p \in A \otimes_{K^\circ} \tilde{K}$ , are valuation rings of constant reduction of  $F|K$  prolonging  $v$ . From [23, page 364] we know that these valuations are extensions of Gauss valuations for a suitable transcendental basis of  $F$  over  $K$ . Now the Gauss valuation of  $K(x_1, \dots, x_n)$  has the same rank as the valuation of  $K$ , which is one, and the extension of a valuation to a finite algebraic field extension also preserves the rank, see [7, ch. VI §8.1 Corollary 1]. □

In particular we have the generalization of the following well known result for normal noetherian domains.

**Proposition 3.8.** *Let  $A$  be a normal flat domain of finite type over  $K^\circ$ , then*

$$A = \bigcap A_p,$$

where the intersection runs through the prime ideals of height one.

*Proof.* This follows from Proposition 3.3, Corollary 3.5 and Proposition 3.7. □

The results obtained so far can be summarized in a geometric way. Given a variety  $\mathcal{Y}$  over  $K^\circ$ , we denote by  $\text{Val}(\mathcal{Y}) := \{p \in \mathcal{Y} \mid \mathcal{O}_{\mathcal{Y},p} \text{ is a valuation ring}\}$  and by  $Y^{(1)} := \{p \in \mathcal{Y}_\eta \mid \dim(\mathcal{O}_{\mathcal{Y},p}) = 1\}$ .

**Proposition 3.9.** *Let  $\mathcal{Y}$  be a normal flat variety over the valuation ring  $K^\circ$ . We denote by  $F := K(\mathcal{Y})$  the field of rational functions on  $\mathcal{Y}$  and by  $\text{Gen}(\mathcal{Y}_s), \text{Gen}(\mathcal{Y}_\eta)$  the set of generic points of the special fiber and generic fiber respectively. Then*

- (a)  $\text{Val}(\mathcal{Y}) = Y^{(1)} \cup \text{Gen}(\mathcal{Y}_s) \cup \text{Gen}(\mathcal{Y}_\eta)$ .
- (b) The rings  $A_p$  for  $p \in Y^{(1)}$  are discrete valuation rings.
- (c) For  $p \in \text{Gen}(\mathcal{Y}_s)$  the valuation rings  $A_p$  are of rank one.

*Proof.* The first statement follows from [23, Theorem 2.6]. Since  $\mathcal{Y}_\eta$  is a normal variety over  $K$ , (b) follows from standard results. The claim (c) follows from Proposition 3.7.  $\square$

## 3.2 Intersection theory with divisors

First note that a variety  $\mathcal{Y}$  defined over the valuation ring  $K^\circ$  has the following properties.

**Proposition 3.10.** *A variety  $\mathcal{Y}$  defined over the valuation ring  $K^\circ$  is a noetherian topological space. Furthermore every irreducible component of the special fibre has dimension  $d := \dim(\mathcal{Y}_\eta)$ . If  $\mathcal{Y}_s$  is non-empty and if  $v$  is non-trivial, then the topological dimension of  $\mathcal{Y}$  is  $d + 1$ . If  $\mathcal{Y}_s$  is empty or if  $v$  is trivial, then  $\mathcal{Y} = \mathcal{Y}_\eta$ .*

*Proof.* The underlying topological space of the scheme  $\mathcal{Y}$  is the union of the underlying spaces of  $\mathcal{Y}_\eta$  and  $\mathcal{Y}_s$ . Since these are noetherian topological spaces the proof of the first statement follows. Since  $\mathcal{Y}$  is flat over  $K^\circ$ , then every component of the special fibre has dimension  $d$ . This implies that the dimension of  $\mathcal{Y}$  is  $d + 1$ . The last claim is clear as for trivial valuations  $\text{Spec}(K^\circ) = \{\eta\}$ .  $\square$

In general the varieties  $\mathcal{Y}$  defined over the valuation ring  $K^\circ$  are non-noetherian, therefore the theory developed by Fulton in [14, Chapter 20] can't be use. Actually a new definition of a cycle is required.

**Definition 3.11.** Let  $\mathcal{Y}$  be a variety over  $K^\circ$ . A *horizontal cycle* on  $\mathcal{Y}$  is a cycle on  $\mathcal{Y}_\eta$ , i.e. a  $\mathbb{Z}$ -linear combination of subvarieties of  $\mathcal{Y}_\eta$ . A *vertical cycle* on  $\mathcal{Y}$  is a  $\mathbb{R}$ -linear combination of subvarieties on  $\mathcal{Y}_s$ . A *cycle* on  $\mathcal{Y}$  is the sum of a horizontal and a vertical cycle. If the closure of every component of the cycle in  $\mathcal{Y}$  have dimension  $k$  (or codimension  $p$  in  $\mathcal{Y}$ ) then the cycle is called a cycle of dimension  $k$  (or codimension  $p$ ). A cycle of codimension 1 is called a Weil divisor.

Given a cycle  $\mathcal{Z} = Z + V$  in  $\mathcal{Y}$ , its support is defined as

$$\text{supp}(\mathcal{Z}) := \text{supp}(Z) \cup \text{supp}(V).$$

If  $Z = \sum m_W W \subset \mathcal{Y}_\eta$ , then  $\text{supp}(Z) = \{\overline{W} \mid m_W \neq 0\} \subset \mathcal{Y}$ . Similarly for  $\text{supp}(V)$ , with  $V \subset \mathcal{Y}_s$  a vertical cycle.

Now we define the pull-back and push-forward of cycles by flat and proper morphisms. Let  $f : \mathcal{Y}' \rightarrow \mathcal{Y}$  be a flat (proper) morphism. The morphisms  $f_\eta : \mathcal{Y}'_\eta \rightarrow \mathcal{Y}_\eta$  and  $f_s : \mathcal{Y}'_s \rightarrow \mathcal{Y}_s$  are flat (proper), as flatness (properness) is preserved under base change. Given a cycle  $\mathcal{Z} = Z + V \subset \mathcal{Y}$  the *pullback* by a flat morphism is defined as

$$f^*(\mathcal{Z}) = f_\eta^*(Z) + f_s^*(V).$$

Similarly for a cycle  $\mathcal{Z}' = \mathcal{D}' + V' \subset \mathcal{Y}'$  the *push-forward* by a proper morphism is defined as

$$f_*(\mathcal{Z}') = (f_\eta)_*(\mathcal{D}') + (f_s)_*(V').$$

Recall that given a flat morphism  $f : Y \rightarrow Y'$  between varieties over a field, the pull-back of a cycle  $Z' \subset Y'$  is defined explicitly by  $f^{-1}(Z')$ , the scheme theoretic inverse image. If  $Z$  has codimension  $k$  then so does  $f^{-1}(Z)$ . Therefore the pullback of cycles of varieties over  $K^\circ$  preserves the codimension. Similarly, given a proper map  $f : Y \rightarrow Y'$  of varieties over a field, the push-forward of a cycle  $Z \subset Y$  is given by

$$f_*(Z) = [K(Z) : K(f(Z))]f(Z)$$

if  $\dim(Z) = \dim(f(Z))$ . In this case  $[K(Z) : K(f(Z))] < \infty$ . Otherwise it is given by  $f_*(Z) = 0$ . Then, the push-forward of cycles preserves the dimension.

Recall that for a Cartier divisor  $D$  on  $\mathcal{Y}$ , its support is defined by

$$\text{supp}(D) := \{y \in \mathcal{Y} \mid \mathcal{O}_{\mathcal{Y},y} \neq \mathcal{O}(D)_y\}.$$

**Definition 3.12.** Given a Cartier divisor  $D$  and a cycle  $\mathcal{Z}$  on  $\mathcal{Y}$ , we say they intersect properly if for every prime component  $W$  of  $\mathcal{Z}$  one has

$$\text{codim}(\text{supp}(D) \cap \overline{W}, \mathcal{Y}) \geq \text{codim}(\overline{W}, \mathcal{Y}) + 1.$$

In particular if  $\mathcal{Z}$  is a Weil divisor, this condition says that the intersection doesn't contain any Weil divisors.

To every Cartier divisor it is possible to associate a Weil divisor. For simplicity, we outline the construction when  $\mathcal{Y}$  is a normal variety over  $K^\circ$ . Let  $D$  be a Cartier divisor on  $\mathcal{Y}$ . Then its associated Weil divisor  $\text{cyc}(D)$  is defined as follows. The horizontal component of  $\text{cyc}(D)$  is the Weil divisor corresponding to the Cartier divisor  $D|_{\mathcal{Y}_\eta}$  on  $\mathcal{Y}_\eta$ . We just need to construct the vertical part of  $\text{cyc}(E)$ . Let  $W$  be an irreducible component of  $\mathcal{Y}_s$ , and let  $\mathcal{V}$  be a sufficiently small open affine subset of  $W$  which contains its generic point  $\eta_W$ . The Cartier divisor  $D$  can be represented in  $\mathcal{V}$  by  $f = h/g$ , with  $f, g \in \mathcal{O}_{\mathcal{Y}}(\mathcal{V})$  and  $g \neq 0$  on  $\mathcal{V}$ . Since  $\mathcal{O}_{\mathcal{Y}}(\mathcal{V})$  is a subring of the valuation ring  $\mathcal{O}_{\mathcal{Y},\eta_Z}$ , with valuation  $v_Z$  extending  $v$ , see Proposition 3.9, we define

$$\text{ord}(E, Z) := v_Z(f) = v_Z(h) - v_Z(g).$$

Clearly it is independent of the representatives of the Cartier divisor and the open subset  $\mathcal{V}$ . Therefore the vertical component of  $\text{cyc}(E)$  is defined as

$$\sum_Z \text{ord}(E, Z)Z,$$

where  $Z$  runs through the irreducible components of the special fiber  $\mathcal{Y}_s$ . By Proposition 3.10  $\mathcal{Y}$  is a noetherian topological space, then this sum is finite. In particular, given a principal Cartier divisor  $\text{div}(f)$  on  $\mathcal{Y}$  corresponding to a rational function  $f$ , the associated cycle is denoted by  $\text{cyc}(\text{div}(f))$ .

If the variety  $\mathcal{Y}$  is not normal, we still can associate to every Cartier divisor a Weil divisor. However, the construction is more involved and the theory of admissible formal schemes over a valuation ring is needed, for details see [18, §2.9].

Let  $\text{CaDiv}(\mathcal{Y})$  be the group of Cartier divisors on  $\mathcal{Y}$ , we have shown that there exists a map from this group to the group of Weil divisors  $\text{Div}(\mathcal{Y})$  on  $\mathcal{Y}$ .

**Proposition 3.13.** *Suppose that  $\mathcal{Y}$  is normal, then the map  $\text{CaDiv}(\mathcal{Y}) \rightarrow \text{Div}(\mathcal{Y})$  is injective.*

*Proof.* Let  $D \in \text{CaDiv}(\mathcal{Y})$  be such that  $\text{cyc}(D) = 0$ . Locally, on  $\mathcal{U} = \text{Spec}(A) \subset \mathcal{Y}$ ,  $D$  is defined by a rational function  $f$ . Since its associated principal Weil divisor is zero, then  $f \in A_p$  for every height one prime ideal  $p$  of  $A$ . By Proposition 3.8 this implies that  $f \in A$ . Since  $v(1/f) = -v(f) = 0$ , then by the same argument  $1/f \in A$ , which means that  $f$  is invertible. Therefore the Cartier divisor is trivial.  $\square$

The map in Proposition 3.13 is compatible with flat morphisms in the sense of the following proposition.

**Proposition 3.14.** *Let  $\varphi : \mathcal{Y}' \rightarrow \mathcal{Y}$  be a flat morphism of varieties over  $K^\circ$ . Then the following diagram commutes.*

$$\begin{array}{ccc} \text{CaDiv}(\mathcal{Y}) & \longrightarrow & \text{Div}(\mathcal{Y}) \\ \downarrow \varphi^* & & \downarrow \varphi^* \\ \text{CaDiv}(\mathcal{Y}') & \longrightarrow & \text{Div}(\mathcal{Y}') \end{array}$$

*That is, given a Cartier divisor  $D$  on  $\mathcal{Y}$ , we have  $\varphi^*(\text{cyc}(D)) = \text{cyc}(\varphi^*(D))$ .*

*Proof.* This follows from [18, Proposition 2.15].  $\square$

Two Cartier divisors  $D, D'$  are called rationally equivalent if there exists a principal Cartier divisor  $\text{div}(f)$  such that  $D - D' = \text{div}(f)$ . We define analogously the notion of rational equivalence for Weil divisors. The group of rational equivalence classes of Weil divisors is the *first Chow group*  $CH^1(\mathcal{Y})$  of  $\mathcal{Y}$ . The equivalence class of a Weil divisor  $\mathcal{D}$  is denoted by  $[\mathcal{D}]$ . For Cartier divisors the group of equivalence classes is isomorphic to the Picard group  $\text{Pic}(\mathcal{Y})$  of  $\mathcal{Y}$ .

The cycle associated to a given Cartier divisor satisfies the following properties.

**Proposition 3.15.** *Let  $D$  be a Cartier divisor on  $\mathcal{Y}$ , then the following properties hold*

(a)  $\text{supp}(D) = \text{supp}(\text{cyc}(D))$ .

(b) *The Cartier divisor  $D$  is effective iff  $\text{cyc}(D)$  is an effective cycle.*

*Proof.* Clearly  $\text{supp}(\text{cyc}(D)) \subset \text{supp}(D)$ . To show the other inclusion take  $y \in \text{supp}(D)$  and suppose that  $y \notin \text{supp}(\text{cyc}(D))$ . Let  $\mathcal{V} = \text{Spec}(A)$  be an open affine subset of  $y$  such that the local equation for  $D$  on  $\mathcal{V}$  is given by a rational function  $f$ . Since  $A = \bigcap A_p$  and

$y$  is not in the support of  $\text{cyc}(D)$  hence  $v_p(f) = 0$  for all  $p$ , then  $f \in A$ . This means that  $y$  is not in the support of  $D$ , which contradicts the assumption. This proves (a). For (b) we just note that  $\text{cyc}(D)$  is effective iff  $v_p(f) \geq 0$  for all height one prime ideal  $p$ , then  $f \in A_p$  for all  $p$  as before, which implies that  $f \in A$ .  $\square$

With the previous discussion between Cartier and Weil divisors, we are ready to define the intersection product between a Cartier divisor and a cycle.

Let  $D$  be a Cartier divisor and  $W$  a prime cycle on  $\mathcal{Y}$  which intersect properly. To define the intersection product  $D.W$  we consider the following cases. First if  $W$  is vertical it is defined as  $\text{cyc}(D|_W)$ . If  $W$  is horizontal and the closure  $\overline{W} \subset \mathcal{Y}$  is normal it is defined as  $\text{cyc}(D|_{\overline{W}})$ .

If  $\overline{W} \subset \mathcal{Y}$  is not normal, it is still possible to associate to the Cartier divisor  $D|_{\overline{W}}$  a Weil divisor, see [18, §2.9–§2.9]. We define  $D.W$  as the associated Weil divisor of the restriction of  $D$  to the closure  $\overline{W}$  in  $\mathcal{Y}$ . By linearity we get the definition of the intersection product of  $D$  with an arbitrary cycle  $\mathcal{Z}$ . We also can define the product between two Cartier divisors as follows.

**Definition 3.16.** Let  $D, E$  be two Cartier divisors on  $\mathcal{Y}$ . The product of  $D$  and  $E$  is defined as  $D.E := D.\text{cyc}(E)$ .

The following proposition shows that this product is independent of the order, i.e. it is commutative, see [18, Proposition 2.14].

**Proposition 3.17.** *Given two Cartier divisors  $D$  and  $E$  intersecting properly on  $\mathcal{Y}$ , that is  $\text{codim}(\text{supp}(D) \cap \text{supp}(E), \mathcal{Y}) \geq 2$ , we have  $D.\text{cyc}(E) = E.\text{cyc}(D)$ .*

*Proof.* Since we have to show this equality for horizontal and vertical components, this is reduced to the case of fields. For the horizontal components it is a standard result in intersection theory, see [14, Theorem 2.4]. For the the vertical components it was proved by Gubler, see [19, Theorem 5.9].  $\square$

### 3.3 $\mathbb{T}$ -invariant neighborhoods and Cartier divisors

In order to prove Sumihiro's theorem, given a point  $y \in \mathcal{Y}$  we take an open affine neighborhood  $\mathcal{U}_0$  of it and construct the smallest  $\mathbb{T}$ -invariant neighborhood  $\mathcal{U}$  which contains  $\mathcal{U}_0$ . It would be very important for the proof to construct a Cartier divisor in  $\mathcal{U}$  with support  $\mathcal{U} \setminus \mathcal{U}_0$ . For this, the following result is needed.

**Proposition 3.18.** *Consider a non-empty affine open subset  $\mathcal{U}_0$  of a normal variety  $\mathcal{Y}$  over the valuation ring  $K^\circ$ . Then the components of  $\mathcal{Y} \setminus \mathcal{U}_0$  have codimension one in  $\mathcal{Y}$ .*

*Proof.* Replacing  $\mathcal{Y}$  by  $\mathcal{Y} \setminus W$  successively for every component  $W \subset \mathcal{Y} \setminus \mathcal{U}_0$  of codimension one, we may assume that  $\text{codim}(\mathcal{U}_0, \mathcal{Y}) \geq 2$ . We need to prove that  $\mathcal{Y} = \mathcal{U}_0$ . Given an affine open covering  $\{\mathcal{U}_i\}_{i \in I}$  of  $\mathcal{Y}$ , we note that any pair  $\mathcal{U}_i, \mathcal{U}_j$  has non-empty intersection because  $\mathcal{Y}$  is irreducible. Furthermore since  $\mathcal{Y}$  is separated, for all  $i \in I$

we have that  $\mathcal{U}_0 \cap \mathcal{U}_i$  is a non-empty affine subset. Therefore it is enough to show that  $\mathcal{U}_i = \mathcal{U}_i \cap \mathcal{U}_0$  for every  $i \in I$ . This shows that we can restrict to the affine case.

We assume that  $\mathcal{Y} = \text{Spec}(A)$  and let  $\mathcal{U}_0 = \text{Spec}(B)$ . Since  $\text{codim}(\mathcal{U}_0, \mathcal{Y}) \geq 2$ , by Proposition 3.8 we have

$$A = \bigcap_{\text{ht}(p)=1} A_p = \bigcap_{\text{ht}(q)=1} B_q = B.$$

Then  $\mathcal{Y} = \mathcal{U}_0$ , which proves the claim.  $\square$

In order to study the  $\mathbb{T}$ -action on Weil divisors, we need to restrict to the elements of the torus whose reduction is well defined. This is given by  $T^\circ$ , the affinoid torus in  $T$  defined as  $\{x \in T \mid |x_1(x)| = 1, \dots, |x_n(x)| = 1\}$  where  $x_1, \dots, x_n$  are the coordinates of  $\mathbb{T}$ .

**Definition 3.19.** We said that a cycle  $\mathcal{Z}$  in  $\mathbb{T} \times_{K^\circ} \mathcal{Y}$  satisfies the *flatness condition* if every component of the horizontal (resp. vertical) part of  $\mathcal{Z}$  is flat over  $T$  (resp.  $\mathbb{T}_s$ ).

Let  $\mathcal{Y}$  be a variety over  $K^\circ$  and let  $i_t : \mathcal{Y} \rightarrow \mathbb{T} \times_{K^\circ} \mathcal{Y}$  be the embedding over  $\mathcal{Y}$  induced by the integral point of  $\mathbb{T}$  corresponding to  $t$ . We are going to define the pull-back  $i_t^*(\mathcal{Z})$  for those cycles  $\mathcal{Z}$  in  $\mathbb{T} \times_{K^\circ} \mathcal{Y}$  satisfying the flatness condition.

The induced maps  $i_t : Y \rightarrow T \times_K Y$  and  $i_t : \mathcal{Y}_s \rightarrow \mathbb{T}_s \times_{\tilde{K}} \mathcal{Y}_s$  are clearly regular embeddings. Then it is possible to define the pull-back of the horizontal and vertical part of  $\mathcal{Z}$  on  $Y$  and  $\mathcal{Y}_s$  respectively, see [14, Chapter 6]. We define  $i_t^*(\mathcal{Z})$  as the sum of these two pull-backs. Clearly, this pull-back keeps the codimension and is linear in  $\mathcal{Z}$ .

The following proposition relates the cycles of codimension one in  $\mathbb{T} \times_{K^\circ} \mathcal{Y}$  with those in  $\mathcal{Y}$ . Recall that  $p_2$  is the canonical projection  $\mathbb{T} \times_{K^\circ} \mathcal{Y} \rightarrow \mathcal{Y}$  over  $K^\circ$ .

**Proposition 3.20.** *Given a cycle  $\mathcal{D}$  of codimension one in  $\mathbb{T} \times_{K^\circ} \mathcal{Y}$ , there exists a cycle  $\mathcal{D}'$  on  $\mathcal{Y}$  of codimension one such that  $p_2^*(\mathcal{D}')$  is rationally equivalent to  $\mathcal{D}$ .*

*Proof.* The statement is true over the generic fibers, see [14, Proposition 1.9]. For the vertical parts, we proceed as follows. First note that the special fiber of  $\mathbb{T} \times_{K^\circ} \mathcal{Y}$  is given by

$$\bigcup \mathbb{T}_s \times_{\tilde{K}} V,$$

where  $V$  runs through the irreducible components of  $\mathcal{Y}_s$ . Therefore every irreducible component of  $(\mathbb{T} \times_{K^\circ} \mathcal{Y})_s$  is of the form  $\mathbb{T}_s \times_{\tilde{K}} V = p_2^*(V)$ , for some irreducible component  $V$  of  $\mathcal{Y}_s$ . Then every vertical cycle of codimension 1 in  $\mathbb{T} \times_{K^\circ} \mathcal{Y}$  is given by the pullback of one in  $\mathcal{Y}$ . This proves the claim.  $\square$

Let  $x_1, \dots, x_n$  be a coordinate system of the split torus  $\mathbb{T} \simeq (\mathbb{G}_m)^n_{K^\circ}$  and  $p_j : \mathbb{T} \times_{K^\circ} \mathcal{Y} \rightarrow \mathbb{G}_m$  be the canonical projection onto the  $j$ -th factor of  $\mathbb{T}$ . Given a point  $t \in T^\circ(K)$  with coordinates  $t_1 = x_1(t), \dots, t_n = x_n(t)$ , we denote by  $D_{t_j}$  the Cartier divisor  $p_j^*(\text{div}(x_j - t_j))$  on  $\mathbb{T} \times_{K^\circ} \mathcal{Y}$  given by the pullback of the divisor  $\text{div}(x_j - t_j)$  on  $\mathbb{G}_m$ .

Let  $\mathcal{Z}$  be a cycle on  $\mathbb{T} \times_{K^\circ} \mathcal{Y}$  satisfying the flatness condition previously established in Definition 3.19. Using the proper intersection product for Cartier divisor from 3.2, we have

$$(i_t)_*(i_t^*(\mathcal{Z})) = D_{t_1} \dots D_{t_n} \cdot \mathcal{Z}.$$

To see that such a product makes sense, we note that by the flatness condition  $D_{t_1} \dots D_{t_n} \cdot \mathcal{Z}$  is a proper intersection product and hence it is well defined, see [14, Example 6.5.1]. By Proposition 3.17, the intersection product of two Cartier divisors is commutative, therefore the right hand side is symmetric respect to the Cartier divisors.

Given two varieties  $\mathcal{Y}, \mathcal{Y}'$  over  $K^\circ$  we consider a flat morphism  $\varphi : \mathbb{T} \times_{K^\circ} \mathcal{Y} \rightarrow \mathbb{T} \times_{K^\circ} \mathcal{Y}'$  over  $\mathbb{T}$ . Let  $\varphi_t : \mathcal{Y} \rightarrow \mathcal{Y}'$  be the flat morphism induced by an element  $t \in T^\circ(K)$  by base change. Note that in this situation, given a cycle  $\mathcal{Z}'$  in  $\mathcal{Y}'$  there are different ways to get a cycle in  $\mathcal{Y}$ , namely  $i_t^*(\varphi^*(p_2^*(\mathcal{Z}')))$  and  $\varphi_t^*(\mathcal{Z}')$ . The following result shows that both constructions agree. We denote by  $p_2 : \mathbb{T} \times_{K^\circ} \mathcal{Y} \rightarrow \mathcal{Y}$  the canonical projection.

**Proposition 3.21.** *Let  $\mathcal{Z}'$  be a cycle of  $\mathcal{Y}'$ ,  $\varphi : \mathbb{T} \times_{K^\circ} \mathcal{Y} \rightarrow \mathbb{T} \times_{K^\circ} \mathcal{Y}'$  be a flat morphism over  $\mathbb{T}$  and  $\varphi_t : \mathcal{Y} \rightarrow \mathcal{Y}'$  be the induced morphism by a point  $t \in T^\circ(K)$ . Then the cycle  $\varphi^*(p_2^*(\mathcal{Z}'))$  satisfies the flatness condition and the equality  $i_t^*(\varphi^*(p_2^*(\mathcal{Z}'))) = \varphi_t^*(\mathcal{Z}')$  holds.*

*Proof.* The cycle  $p_2^*(\mathcal{Z}') = \mathbb{T} \times_{K^\circ} \mathcal{Z}'$  clearly satisfies the flatness condition. Since  $\varphi$  is a flat morphism over  $\mathbb{T}$  and flat cycles are preserved by flat morphisms, we have that  $\varphi^*(p_2^*(\mathcal{Z}'))$  also fulfills the flatness condition. By the definition of pull-backs of cycles of varieties over  $K^\circ$ , it is enough to prove the claim for varieties defined over fields. Since in this case the statement holds, see [14, Proposition 6.5], we get the proof of the proposition.  $\square$

Now, we explore the relation between the principal cycles given by rational functions  $g$  on  $\mathbb{T} \times_{K^\circ} \mathcal{Y}$  and those in  $\mathcal{Y}$  given by evaluating the rational function  $g$  on an element  $t \in T^\circ(K)$ . Precisely, we have the following result.

**Lemma 3.22.** *Consider a variety  $\mathcal{Y}$  over the valuation ring  $K^\circ$  and let  $t \in T^\circ(K)$ . Given a rational function  $g$  on  $\mathbb{T} \times_{K^\circ} \mathcal{Y}$  such that the cycle  $\text{cyc}(\text{div}(g))$  satisfies the flatness condition, we have that  $g(t, \cdot)$  is a rational function on  $\mathcal{Y}$  and  $i_t^*(\text{cyc}(\text{div}(g))) = \text{cyc}(\text{div}(g(t, \cdot)))$ .*

*Proof.* Since the cycle associated to  $\text{div}(g)$  satisfies the flatness condition and flat morphisms are open, the support of  $\text{cyc}(\text{div}(g))$  can not be contained in  $i_t(\mathcal{Y})$ , where  $i_t$  is the regular embedding induced by  $t \in T^\circ(K)$ . Then the first claim follows from the flatness assumption. The last part of the statement follows from the fact that we may write  $i_t^*$  as an  $n$ -fold proper intersection product with Cartier divisors and from Proposition 3.17.  $\square$

**Proposition 3.23.** *Consider a variety  $\mathcal{Y}$  over the valuation ring  $K^\circ$ . Then there is a correspondence of cycles of codimension one in  $\mathbb{T} \times_{K^\circ} \mathcal{Y}$  and  $\mathcal{Y}$  up to rational equivalence.*

That is the pull-back with respect to the canonical projection  $p_2 : \mathbb{T} \times_{K^\circ} \mathcal{Y} \rightarrow \mathcal{Y}$  induces an isomorphism  $p_2^* : CH^1(\mathcal{Y}) \rightarrow CH^1(\mathbb{T} \times_{K^\circ} \mathcal{Y})$ .

*Proof.* Given two cycles  $\mathcal{Z}_1, \mathcal{Z}_2$  of codimension one on  $\mathcal{Y}$  which are rationally equivalent, by proposition 3.14, we get that  $p_2^*(\mathcal{Z}_1)$  and  $p_2^*(\mathcal{Z}_2)$  are rational equivalent as well. Then  $p_2^*$  is compatible with rational equivalence and hence it is well-defined on the Chow groups. Surjectivity follows from Proposition 3.20. To prove injectivity, suppose that  $\mathcal{D}$  is a cycle of codimension one on  $\mathcal{Y}$  such that  $p_2^*(\mathcal{D})$  is rationally equivalent to 0 on  $\mathbb{T} \times_{K^\circ} \mathcal{Y}$ . Using Lemma 3.22 for the unit element in  $T^\circ(K)$ , we deduce that  $\mathcal{D}$  is rationally equivalent to 0. This proves injectivity.  $\square$

If we consider the Picard groups instead of the Chow groups in the statement of the previous proposition for normal varieties, we get also an isomorphism.

**Proposition 3.24.** *Given a normal variety  $\mathcal{Y}$  over  $K^\circ$ , the pull-back with respect to  $p_2$  induces an isomorphism  $\text{Pic}(\mathcal{Y}) \rightarrow \text{Pic}(\mathbb{T} \times_{K^\circ} \mathcal{Y})$ .*

*Proof.* The claim follows from [17, Remark 9.6].  $\square$

With the previous results, we can understand the behavior of cycles under the action of the torus.

**Proposition 3.25.** *Let  $\mathcal{Y}$  be a normal  $\mathbb{T}$ -toric variety over  $K^\circ$  and let  $\mathcal{D}$  be a cycle of codimension one in  $\mathcal{Y}$ . We denote by  $\mathcal{D}^t$  the pull-back of  $\mathcal{D}$  with respect to the flat morphism corresponding to the action of an element  $t \in T^\circ(K)$  on  $\mathcal{Y}$ . Then  $\mathcal{D}^t$  is rationally equivalent to  $\mathcal{D}$ .*

*Proof.* From Propositions 3.20, we know that there exists a cycle  $\mathcal{D}'$  of codimension one in  $\mathcal{Y}$  such that  $\sigma^*(\mathcal{D})$  is rationally equivalent to  $p_2^*(\mathcal{D}')$ , where  $\sigma : \mathbb{T} \times_{K^\circ} \mathcal{Y} \rightarrow \mathcal{Y}$  is the action of the torus on  $\mathcal{Y}$ . By Proposition 3.21 applied to the unit element  $e$  of  $\mathbb{T}$  and by Lemma 3.22 it follows that  $\mathcal{D}$  is rationally equivalent to  $\mathcal{D}'$ . This implies that  $\sigma^*(\mathcal{D})$  is rationally equivalent to  $p_2^*(\mathcal{D})$ . If we proceed in the same way as before, applying Proposition 3.21 and Lemma 3.22 again, but now to  $t$  instead of the unit element  $e$ , we get the proof of the Proposition.  $\square$

As we have said at the beginning of the chapter, given an open affine subset  $\mathcal{U}_0 \subset \mathcal{Y}$  we are going to construct a  $\mathbb{T}$ -invariant open subset  $\mathcal{U}$  such that it contains  $\mathcal{U}_0$  and it has a Cartier divisor with support  $\mathcal{U} \setminus \mathcal{U}_0$ . The construction of this opens subset is given in the following lemma.

**Lemma 3.26.** *Given a non-empty open subset  $\mathcal{U}_0$  of the  $\mathbb{T}$ -toric variety  $\mathcal{Y}$  over  $K^\circ$ , the open subset  $\mathcal{U} := \bigcup_{t \in T^\circ(K)} t\mathcal{U}_0$  is the smallest  $\mathbb{T}$ -invariant subset containing  $\mathcal{U}_0$ .*

*Proof.* Consider the subset  $S$  of  $\mathbb{T}$  such that translation with its elements leaves  $\mathcal{U}$  invariant. The points in the special fibre of  $S$  are equal to the stabilizer of the boundary of  $\mathcal{U}_s$  and hence an algebraic subgroup of  $\mathbb{T}_s$ . By construction, it contains all rational points over the residue field of the torus and hence this algebraic subgroup is equal to



$\mathbb{T}_s$ . We use the same argument for the points of  $S$  contained in the generic fibre. Again, it is an algebraic subgroup now contained in  $\mathbb{T}_\eta$ . All points of  $T^\circ(K)$  are contained in this algebraic subgroup. Since  $T^\circ$  is an  $n$ -dimensional affinoid torus, we conclude that  $T^\circ(K)$  is Zariski dense in  $\mathbb{T}_\eta$  and hence the algebraic subgroup is the torus  $\mathbb{T}_\eta$  over  $K$ . We conclude that  $\mathcal{U}$  is  $\mathbb{T}$ -invariant. This proves the claim.  $\square$

Note that the action of  $T^\circ(K)$  on  $\mathcal{U}$  leave invariant the vertical part of  $\mathcal{U}_0$ , the reason is that the set of generic points of the irreducible components of  $(\mathcal{U}_0)_s$  is discrete and the action is continuous. Therefore by Proposition 3.18 the components of  $\mathcal{U} \setminus \mathcal{U}_0$  are of codimension one, furthermore they are all horizontal. Let  $\mathcal{D}$  be the horizontal cycle on  $\mathcal{U}$  given by the sum of those irreducible components.

**Proposition 3.27.** *Let  $\mathcal{D}$  be the cycle of codimension one in  $\mathcal{U}$  defined above. Then there is a unique Cartier divisor  $D$  on  $\mathcal{U}$  such that  $\mathcal{D} = \text{cyc}(D)$ .*

*Proof.* By Proposition 3.25, there is a non-zero rational function  $f_t$  on  $\mathcal{U}$  such that  $\mathcal{D} - \mathcal{D}^t = \text{cyc}(\text{div}(f_t))$ . Since  $\mathcal{U} \setminus t^{-1}\mathcal{U}_0$  is equal to the support of  $\mathcal{D}^t$ , we deduce that the restriction of  $\mathcal{D}$  to  $t^{-1}\mathcal{U}_0$  is the Weil divisor given by the rational function  $f_t$  on  $t^{-1}\mathcal{U}_0$ . By Proposition 3.13 and Proposition 3.15, the Cartier divisor on a normal variety is uniquely determined by its associated Weil divisor. This yields immediately that  $\{(t^{-1}\mathcal{U}_0, f_t)\}$  is a Cartier divisor on  $\mathcal{U}$  with associated Weil divisor  $\mathcal{D}$ . Uniqueness follows as well.  $\square$

*Remark 3.28.* Note that since the cycle  $\mathcal{D}$  is effective, then by Proposition 3.15 the associated Cartier divisor  $D$  is effective as well.

It is natural to ask if this Cartier divisor is well behaved respect to the  $\mathbb{T}$ -action. For this we need first the following property. Recall that  $\sigma : \mathbb{T} \times_{K^\circ} \mathcal{Y} \rightarrow \mathcal{Y}$  is the torus action of the normal  $\mathbb{T}$ -toric variety  $\mathcal{Y}$  over the valuation ring  $K^\circ$

**Proposition 3.29.** *Given the Cartier divisor  $D$  from Proposition 3.27, we have that  $\sigma^*(D)$  is linearly equivalent to  $p_2^*(D)$ .*

*Proof.* From Proposition 3.24, there is a Cartier divisor  $D'$  on  $\mathcal{Y}$  such that  $p_2^*(D')$  is rational equivalent to  $\sigma^*(D)$ . Since the unit element  $e$  in  $T^\circ(K)$  induces the section  $i_e$  of  $\sigma$  and  $p_2$ , we obtain that the divisor  $D$  is rational equivalent to  $D'$  which proves the claim.  $\square$

With this proposition we can obtain the following result.

**Corollary 3.30.** *Given the Cartier divisor  $D$  from Proposition 3.27, we denote by  $D^t$  the pull-back with respect the torus action by  $t$ , where  $t$  is an element in  $T^\circ(K)$ . Then the line bundles  $\mathcal{O}(D^t)$  and  $\mathcal{O}(D)$  on  $\mathcal{U}$  are isomorphic. Furthermore  $\mathcal{O}(D)$  is base point free, i.e. it is generated by global sections.*

*Proof.* By applying  $i_t^*$  to the Proposition 3.29, we get that the Cartier divisor  $D$  is linear equivalent to  $D^t$ , therefore the line bundle  $\mathcal{O}(D)$  is isomorphic to  $\mathcal{O}(D^t)$ . From proposition 3.27, we know that the Cartier divisor  $D^t$  is effective, hence it has a canonical section  $s_{D^t}$  whose support is  $\mathcal{U} \setminus t^{-1}\mathcal{U}_0$ , then the second claim follows from the first one.  $\square$

Note that this shows that the Cartier divisors  $D^t$  and  $D$  are linear equivalent for every  $t \in T^\circ(K)$ .

## 4 | $\mathbb{T}$ -Linearization

In chapter 2 we have proved Theorem 1, which is the first step toward the classification of normal  $\mathbb{T}$ -toric varieties over the valuation ring  $K^\circ$  stated in Theorem 3. In order to obtain such a result Sumihiro's theorem for normal  $\mathbb{T}$ -toric varieties over the valuation ring  $K^\circ$  is crucial. To prove it, we need to study invertible sheaves  $\mathcal{L}$  on normal  $\mathbb{T}$ -toric varieties over  $K^\circ$ . This is done in order to get embeddings into projective space and simplify the problem.

In the previous chapter, we consider a non-empty affine open subset  $\mathcal{U}_0$  of  $\mathcal{Y}$  and the smallest  $\mathbb{T}$ -invariant open subset  $\mathcal{U}$  of  $\mathcal{Y}$  containing  $\mathcal{U}_0$ . In Proposition 3.27, we have constructed an effective Cartier divisor  $D$  on  $\mathcal{U}$  with  $\text{supp}(D) = \mathcal{U} \setminus \mathcal{U}_0$ . We saw that  $\text{cyc}(D)$  is a horizontal cycle with all multiplicities equal to 1. In this chapter, we will see that  $\mathcal{O}(D)$  is ample and has a  $\mathbb{T}$ -linearization leading to a  $\mathbb{T}$ -equivariant immersion into a projective space.

### 4.1 $\mathbb{T}$ -Linearization of a line bundle

Before introducing the notion of a  $\mathbb{T}$ -linearization let us give some intuition behind the definition. When we have a line bundle  $L$  (or equivalently an invertible sheaf  $\mathcal{L}$ ) on a  $\mathbb{T}$ -toric variety  $\mathcal{Y}$ , it is useful to know when it is possible to lift the torus action on  $\mathcal{Y}$  to an action on  $L$ . When this is possible in an appropriated way, we say that  $\mathcal{L}$  has a  $\mathbb{T}$ -linearization. One of the advantages of this fact is that if the invertible sheaf is ample it is possible to get a  $\mathbb{T}$ -equivariant embedding into projective space which allow us, in principle, to simplify problems.

In order to make this concept precise, we denote by  $\mu : \mathbb{T} \times_{K^\circ} \mathbb{T} \rightarrow \mathbb{T}$ ,  $\sigma : \mathbb{T} \times_{K^\circ} \mathcal{Y} \rightarrow \mathcal{Y}$ ,  $p_2 : \mathbb{T} \times_{K^\circ} \mathcal{Y} \rightarrow \mathcal{Y}$  the group action, the torus action and the canonical projection into the second factor respectively, and by  $p_{23} : \mathbb{T} \times_{K^\circ} \mathbb{T} \times_{K^\circ} \mathcal{Y} \rightarrow \mathbb{T} \times_{K^\circ} \mathcal{Y}$  the projection to the last two factors.

**Definition 4.1.** Let  $\mathcal{L}$  be an invertible sheaf on a  $\mathbb{T}$ -toric variety  $\mathcal{Y}$ . A  $\mathbb{T}$ -linearization of  $\mathcal{L}$  consists of an isomorphism of sheaves on  $\mathbb{T} \times_{K^\circ} \mathcal{Y}$

$$\phi : \sigma^* \mathcal{L} \rightarrow p_2^* \mathcal{L},$$

satisfying the cocycle condition:

$$p_{23}^* \phi \circ (\text{id}_{\mathbb{T}} \times \sigma)^* \phi = (\mu \times \text{id}_{\mathcal{Y}})^* \phi. \quad (4.1)$$

Geometrically this can be seen clearly if we consider the line bundle  $L$  instead of the invertible sheaf  $\mathcal{L}$ . In this case a  $\mathbb{T}$ -linearization is a lift of the torus action on  $\mathcal{Y}$  to an action on  $L$  such that the zero section is  $\mathbb{T}$ -invariant, see [26] for details.

Given a  $\mathbb{T}$ -toric variety  $\mathcal{Y}$  and an invertible sheaf  $\mathcal{L}$  with a  $\mathbb{T}$ -linearization, we can define the action of  $\mathbb{T}$  on the sections of  $\mathcal{L}$ . In order to do this, we need to define a dual action of the split torus  $\mathbb{T}$  on a  $K^\circ$ -module. Recall that  $\hat{\mu} : K^\circ[M] \rightarrow K^\circ[M] \otimes_{K^\circ} K^\circ[M]$  and  $\hat{\varphi}_e : K^\circ[M] \rightarrow K^\circ$  are the morphisms defining the product and the unit element in  $\mathbb{T}$ .

**Definition 4.2.** Let  $V$  be a  $K^\circ$ -module and  $A = H^0(\mathbb{T}, \mathcal{O}_{\mathbb{T}}) = K^\circ[M]$ . A dual action of  $\mathbb{T}$  on  $V$  is given by a morphism over  $K^\circ$

$$\hat{\sigma} : V \rightarrow A \otimes_{K^\circ} V$$

which satisfies  $(\hat{\mu} \otimes \text{id}_V) \circ \hat{\sigma} = (\text{id}_A \otimes \hat{\sigma}) \circ \hat{\sigma}$  and  $(\hat{\varphi}_e \otimes \text{id}_V) \circ \hat{\sigma} = \text{id}_V$ .

A submodule  $W \subset V$  is called invariant under the action of  $\mathbb{T}$  if  $\hat{\sigma}(W) \subset A \otimes_{K^\circ} W$ . An element  $v \in V$  is called invariant if  $\hat{\sigma}(v) = 1 \otimes v$ . If it satisfies  $\hat{\sigma}(v) = \chi \otimes v$ , for some character  $\chi$  of  $\mathbb{T}$ , it is called semi  $\mathbb{T}$ -invariant.

The action of the torus on an element of a  $K^\circ$ -module will be very important for the proof of Sumihiro's theorem. We will consider the module of global sections of some line bundle and look for the semi  $\mathbb{T}$ -invariant sections which satisfy certain conditions, see Lemma 5.2 for a precise statement.

From now on  $\mathcal{Y}$  will always be a normal  $\mathbb{T}$ -toric variety over the valuation ring  $K^\circ$ . We have the following lemma which was proved by Rosenlicht in the case of toric varieties over a field, see [32]. We keep the same notation as in chapter 3, where  $\mathcal{U}_0 \subset \mathcal{Y}$  is an affine open subset and  $\mathcal{U}$  is the smallest  $\mathbb{T}$ -invariant open subset which contains it.

**Lemma 4.3.** For any  $f \in H^0(\mathbb{T} \times_{K^\circ} \mathbb{T} \times_{K^\circ} \mathcal{U}, \mathcal{O}_{\mathbb{T} \times_{K^\circ} \mathbb{T} \times_{K^\circ} \mathcal{U}}^*)$  there exist characters  $\chi_1, \chi_2$  of  $\mathbb{T}$  and  $g \in H^0(\mathcal{U}, \mathcal{O}_{\mathcal{U}}^*)$  such that

$$f = \chi_1 \chi_2 g$$

*Proof.* By the lemma of Rosenlicht [32], the statement is true in the case of fields. Then restricting  $f$  to the generic fiber we get  $f = \chi_1 \chi_2 g$ , where  $g \in H^0(U, \mathcal{O}_U^*)$ , with  $U := \mathcal{U}_\eta$ , and  $\chi_1, \chi_2$  are characters of  $T$ . Obviously  $\chi_1, \chi_2$  are characters of  $\mathbb{T}$  as well. Since

$$U \subset \mathcal{U}$$

is an open dense subset, we can extend  $g$  to  $\mathcal{U}$ . Composing  $f$  with the section  $\varphi_e$  induced by the unit element  $e$  of  $\mathbb{T}$ , we get  $g = f \circ \varphi_e \in H^0(\mathcal{U}, \mathcal{O}_{\mathcal{U}})$ . Now  $f$  is invertible, then  $g$  is invertible as well, actually we have  $g^{-1} = \chi_1 \chi_2 f^{-1}$  and since  $g^{-1} = f^{-1} \circ \varphi_e \in H^0(\mathcal{U}, \mathcal{O}_{\mathcal{U}})$  we conclude that  $g \in H^0(\mathcal{U}, \mathcal{O}_{\mathcal{U}}^*)$ .  $\square$

## 4.2 Existence of the $\mathbb{T}$ -linearization

Let  $D$  be the Cartier divisor on  $\mathcal{U}$  with  $\text{supp}(D) = \mathcal{U} \setminus \mathcal{U}_0$ . In order to show that  $\mathcal{L} := \mathcal{O}(D)$  has a  $\mathbb{T}$ -linearization, we need to show that the isomorphism

$$\phi : \sigma^* \mathcal{L} \rightarrow p_2^* \mathcal{L}$$

satisfies the cocycle condition (4.1).

**Proposition 4.4.** *With the previous notation,  $\mathcal{L}$  has a  $\mathbb{T}$ -linearization. Furthermore this induces a  $\mathbb{T}$ -action on the global sections.*

*Proof.* From Proposition 3.24, we know that the sheaves  $\sigma^* \mathcal{L}$  and  $p_2^* \mathcal{L}$  on  $\mathbb{T} \times_{K^\circ} \mathcal{U}$  are isomorphic. Then both sides of (4.1) give rise to isomorphic sheaves on  $\mathbb{T} \times_{K^\circ} \mathbb{T} \times_{K^\circ} \mathcal{U}$ . Therefore there exists an element  $f \in H^0(\mathbb{T} \times_{K^\circ} \mathbb{T} \times_{K^\circ} \mathcal{U}, \mathcal{O}_{\mathbb{T} \times_{K^\circ} \mathbb{T} \times_{K^\circ} \mathcal{U}}^*)$  such that

$$p_{23}^* \phi \circ (\text{id}_{\mathbb{T}} \times \sigma)^* \phi = f(\mu \times \text{id}_{\mathcal{U}})^* \phi.$$

By Lemma 4.3 there are characters  $\chi_1, \chi_2$  and a regular invertible function  $g$  on  $\mathcal{U}$  such that

$$f(t_1, t_2, u) = \chi_1(t_1) \chi_2(t_2) g(u),$$

for all  $t_1, t_2 \in T(\overline{K})$  and  $u \in \mathcal{U}(\overline{K})$ . Evaluating in the points  $t_1 = t_2 = e \in T(\overline{K})$ ,  $u \in \mathcal{U}(\overline{K})$  we get that  $f(e, e, u) = 1$ . Now, by the construction of  $\phi$  we also have

$$\begin{aligned} f(e, t_2, u) &= 1 \\ f(t_1, e, u) &= 1, \end{aligned}$$

then  $g = 1$ . In general, we have

$$f(t_1, t_2, u) = \chi_1(t_1) \chi_2(t_2) = (\chi_1(t_1) \chi_2(e)) (\chi_1(e) \chi_2(t_2)) = 1.$$

Since the  $\overline{K}$ -rational points are dense, this shows that the isomorphism  $\phi$  is such that  $f = 1$  and therefore (4.1) is satisfied. We conclude that  $\mathcal{L}$  has a  $\mathbb{T}$ -linearization.

To prove the second claim, we note that from Definition 4.2 the composition of the morphisms

$$H^0(\mathcal{U}, \mathcal{L}) \rightarrow H^0(\mathbb{T} \times_{K^\circ} \mathcal{U}, \sigma^* \mathcal{L}) \rightarrow H^0(\mathbb{T} \times_{K^\circ} \mathcal{U}, p_2^* \mathcal{L}) \rightarrow H^0(\mathbb{T}, \mathcal{O}_{\mathbb{T}}) \otimes_{K^\circ} H^0(\mathcal{U}, \mathcal{L})$$

defines a dual action of  $\mathbb{T}$  on the global sections.

The last map follows from the Künneth formula, which also holds in this more general setting, see [22]. Concretely given a section  $s \in H^0(\mathcal{U}, \mathcal{L})$ , by the commutativity of the diagram

$$\begin{array}{ccc} p_2^* \mathcal{L} & \longrightarrow & \mathcal{L} \\ \downarrow & & \downarrow \\ \mathbb{T} \times_{K^\circ} \mathcal{U} & \longrightarrow & \mathcal{U} \end{array} \quad (4.2)$$

we have that  $(t \cdot s)(u) = t^{-1}(s(tu))$ .

□

### 4.3 Ampleness

To show that  $\mathcal{L} = \mathcal{O}(D)$  is ample, we consider first the following fact.

**Lemma 4.5.** *Let  $x_1, \dots, x_k$  be the affine coordinates of  $\mathcal{U}_0$ . Since  $\mathcal{U}_0 \subset \mathcal{U}$  is an open dense subset, we consider them as rational functions on  $\mathcal{U}$ . Then there exists  $\ell \in \mathbb{Z}_+$  such that the sections  $s_i := x_i s_{\ell D}$  are global sections of the line bundle  $\mathcal{O}(\ell D)$ , for  $i = 1, \dots, k$ .*

*Proof.* For every principal divisor  $\text{div}(x_i)$  the associated cycle is given by

$$\text{cyc}(\text{div}(x_i)) = \sum_j m_{ij} Z_j + \mathcal{V},$$

with  $Z_j$  the boundary divisors of  $\mathcal{U} \setminus \mathcal{U}_0$  and  $\mathcal{V}$  an effective Weil divisor of  $\mathcal{U}_0$  which meets  $\mathcal{U}_0$ . Recall that by construction  $\mathcal{D} = \text{cyc}(D) = \sum_j Z_j$ . Let  $\ell := -\min\{m_{ij}, 0\}$ . Then the following divisor

$$\text{cyc}(\text{div}(x_i)) + \ell \mathcal{D} = \sum_j m_{ij} Z_j + \ell \mathcal{D} + \mathcal{V} \geq \sum_j (m_{ij} + \ell) Z_j + \mathcal{V} \geq 0$$

is effective. In general, it could happen that the local functions for the Cartier divisor  $\text{div}(x_i) + \ell D$  are not regular, this is because the scheme  $\mathcal{U}$  is not necessarily regular. Fortunately it doesn't happen in our situation, actually by Proposition 3.15 the Cartier divisor  $\text{div}(x_i) + \ell D$  is effective and we conclude that  $x_i s_{\ell D}$  is a global section of  $\mathcal{O}(\ell D)$ .  $\square$

Recall that  $\mathcal{Y}$  is a Noetherian topological space, see Proposition 3.10. As a consequence, we obtain that  $\mathcal{U}$  is quasicompact. Then by Lemma 3.26 there is a finite subset  $S \subset T^\circ(K)$ , which contains  $e$ , such that

$$\mathcal{U} = \bigcup_{t \in S} t^{-1} \cdot \mathcal{U}_0.$$

Since the local coordinates  $x_1, \dots, x_k$  of  $\mathcal{U}_0$  induces the global sections  $s_1, \dots, s_k$  of  $\mathcal{O}(\ell D)$ , then  $t \cdot s_1, \dots, t \cdot s_k$  are global sections of  $\mathcal{O}(\ell D^t) \simeq \mathcal{O}(\ell D)$  and local coordinates of  $t^{-1} \mathcal{U}_0$ . Hence, it is clear that the sections  $\{t \cdot s_j\}_{t \in S, j=1, \dots, k}$  generate  $\mathcal{O}(\ell D)$ . Note that the section  $t \cdot s_D$  is supported in  $\mathcal{U} \setminus t^{-1} \mathcal{U}_0$ .

**Lemma 4.6.** *The map  $\psi : \mathcal{U} \rightarrow \mathbb{P}_{K^\circ}^{R'}$ , defined by*

$$u \mapsto (\cdots : t \cdot s_1(u) : \cdots : t \cdot s_k(u) : t \cdot s_D^\ell(u) : \cdots)_{t \in S}$$

*gives an embedding into  $\mathbb{P}_{K^\circ}^{R'}$ , with  $R' = |S|(k+1) - 1$ . This proves that  $\mathcal{L}$  is ample.*

*Proof.* Since  $\mathcal{L}^\ell$  is generated by these global sections, this map is well defined. To show that it is an open immersion it is enough to show that there is an open covering  $\{\mathcal{V}_i\}$  of  $\psi(\mathcal{U}) \subset \mathbb{P}^{R'}$  such that the restriction of  $\psi$  to  $\psi^{-1}(\mathcal{V}_i)$  is an immersion on  $\mathcal{V}_i$ , see ÉGA I [16, Corollaire 4.2.4]. Since this map is defined by the coordinates of  $\mathbb{P}_{K^\circ}^{R'}$ , it is enough to

take the covering of  $\psi(\mathcal{U})$  induced by the open covering of  $\mathbb{P}_{K^\circ}^R$  given by  $\{\mathcal{W}_t\}_{t \in S}$ , where  $\mathcal{W}_t$  is given by the complement of the set where the projective coordinates corresponding to  $\{t \cdot s_i, t \cdot s_D^\ell\}$  are zero. We have that  $t^{-1}\mathcal{U}_0 \subset \psi^{-1}(\mathcal{W}_t)$ . Clearly it is an immersion on each  $t^{-1}\mathcal{U}_0$  because  $\{t \cdot s_i\}$  are local coordinates. Furthermore  $\psi$  is injective. Suppose that  $\psi(u) = \psi(v)$ , looking at the coordinates given by the sections  $t \cdot s_D^\ell$  we have that  $u, v \in t^{-1}\mathcal{U}_0$  for some  $t \in S$ . Since  $t \cdot s_1, \dots, t \cdot s_k$  is a coordinate system of  $t^{-1}\mathcal{U}$  we have  $u = v$ . Therefore we conclude that  $\psi$  is an immersion.  $\square$

Although  $\psi$  is an immersion, it is not necessarily compatible with the torus action. The reason is that in general the submodule  $V_{\ell_0}$  of  $V_\ell := H^0(\mathcal{U}, \mathcal{L}^\ell)$  generated by the global sections  $\{t \cdot s_j\}_{t \in S, j=1, \dots, k}$  and  $\{t \cdot s_D\}_{t \in S}$  used in the definition of  $\psi$  in Lemma 4.6 is not  $\mathbb{T}$ -invariant. To solve this problem, since  $\mathcal{L}^\ell$  has a  $\mathbb{T}$ -linearization, we need to consider the dual action  $\hat{\sigma}$  of  $\mathbb{T}$  on  $V_\ell$ , see Proposition 4.4.

Recall that given a  $K^\circ$ -submodule  $W$  of  $V_\ell$ , it is called *invariant* under the action of  $\mathbb{T}$  if  $\hat{\sigma}(W) \subset A \otimes_{K^\circ} W$ , for  $A := H^0(\mathbb{T}, \mathcal{O}_{\mathbb{T}}) = K^\circ[M]$ . Since the same proof of [26, lemma\* on p. 25] works for modules over the valuation ring  $K^\circ$ , there is a finitely generated submodule  $W$  of  $V_\ell$  which is invariant under the action of  $\mathbb{T}$  and contains  $V_{\ell_0}$ . Since  $W$  is  $K^\circ$ -torsion free it is flat over  $K^\circ$ , therefore we conclude that  $W$  is a free  $K^\circ$ -module of finite rank. Let  $R + 1$  be the rank of  $W$ .

Using this  $\mathbb{T}$ -invariant  $K^\circ$ -module  $W$ , we get a morphism  $i : \mathcal{U} \rightarrow \mathbb{P}(W)$  with  $i^*(\mathcal{O}_{\mathbb{P}(W)}(1)) \cong \mathcal{L}^\ell$ . The dual action of  $\mathbb{T}$  on  $W$  induces a linear action of  $\mathbb{T}$  on the projective space  $\mathbb{P}(W)$ . By construction,  $i$  is  $\mathbb{T}$ -equivariant.

$$\begin{aligned} i(tu) &= (\cdots : w(tu) : \cdots)_{w \in W} \\ &= (\cdots : t(t \cdot w)(u) : \cdots)_{w \in W} \\ &= (\cdots : t \cdot w(u) : \cdots)_{w \in W} \\ &= t \cdot i(u). \end{aligned}$$

Since  $i$  factorizes through  $\psi$ , we deduce from Proposition 4.6 that  $i$  is an immersion.

We conclude that the line bundle  $\mathcal{L}$  is ample. Furthermore since the embedding is  $\mathbb{T}$ -equivariant, it allows us to consider the closure of an embedded  $\mathbb{T}$ -invariant subset  $\mathcal{U}$  in projective space as a projective  $\mathbb{T}$ -toric variety with a linear action of the torus. Explicitly we have the following result.

**Proposition 4.7.** *Given a non-empty open affine subset  $\mathcal{U}_0$  of a normal  $\mathbb{T}$ -toric variety  $\mathcal{Y}$  over  $K^\circ$  and the smallest  $\mathbb{T}$ -invariant open subset  $\mathcal{U}$  of  $\mathcal{Y}$  containing  $\mathcal{U}_0$ . We have a  $\mathbb{T}$ -equivariant open immersion of  $\mathcal{U}$  into a projective  $\mathbb{T}$ -toric variety  $\mathcal{Y}_{A,a}$ , with  $A \in M^{R+1}$  and a the height function as in 1.4.*

*Proof.* Let  $i : \mathcal{U} \rightarrow \mathbb{P}(W)$  be the  $\mathbb{T}$ -equivariant immersion constructed above. We take the closure  $\mathcal{Y}$  of  $i(\mathcal{U})$  in  $\mathbb{P}(W)$ . By construction it is  $\mathbb{T}$ -invariant and contains a dense  $\mathbb{T}$ -orbit. Furthermore since the line bundle  $\mathcal{O}(D)$  has a  $\mathbb{T}$ -linearization the  $\mathbb{T}$ -action on  $\mathcal{Y}$  is linear. In other words,  $\mathcal{Y}$  is a projective  $\mathbb{T}$ -toric variety over  $K^\circ$  on which  $\mathbb{T}$  acts linearly. Since  $i(\mathcal{U})(K) \neq \emptyset$  it is possible to choose a  $K$ -rational point  $\mathbf{y}$  in the open

dense orbit of  $i(\mathcal{U})$ . By [17, Proposition 9.8] it is possible to find coordinates in  $\mathbb{P}(W)$  such that for some  $A \in M^{R+1}$  we have  $\mathcal{Y} = \mathcal{Y}_{A,a}$  for the height function  $a$  of  $\mathbf{y}$  defined in 1.4.  $\square$



## 5 | Proof of Sumihiro's theorem

With the theory developed so far, we are ready to prove Theorem 2. By the Proposition 4.7 it is enough to prove it for projective normal  $\mathbb{T}$ -toric varieties over  $K^\circ$  with a linear action of  $\mathbb{T}$ . From [17, Proposition 9.8] follows that this varieties are of the form  $\mathcal{Y}_{A,a}$ , where  $A \in M^{R+1}$  and  $a$  is a height function defined as in 1.4. The general case will follow from this one.

### 5.1 Construction of the $\mathbb{T}$ -invariant neighbourhood

Let  $z \in \mathcal{Y}_{A,a}$  and let  $Y$  be a closed  $\mathbb{T}$ -invariant subset which does not contain  $z$ . Then we have the following result.

**Lemma 5.1.** *There exists a positive integer  $k$  and a section  $s_0$  of  $H^0(\mathbb{P}_{K^\circ}^R, \mathcal{O}(k))$  such that  $s_0|_Y = 0$  and  $s_0(z) \neq 0$ .*

*Proof.* Suppose that  $\mathcal{Y}_{A,a} = V(\mathfrak{a}) \subset \mathbb{P}_{K^\circ}^R$  for some homogenous ideal  $\mathfrak{a} \subset K^\circ[x_0, \dots, x_R]$ . Since  $Y \subset \mathcal{Y}_{A,a}$  is closed, it is closed as well in  $\mathbb{P}_{K^\circ}^R$ . Then there is an homogenous ideal  $\mathfrak{b}$  such that  $Y = V(\mathfrak{b})$ . Since  $z \in \mathcal{Y}_{A,a} \setminus Y$ , hence  $\mathfrak{a} \subsetneq \mathfrak{b}$  then we can choose an element  $f \in \mathfrak{b} \setminus \mathfrak{a}$  such that  $f|_Y = 0$  but  $f(z) \neq 0$ . Then with  $s_0 = f \in H^0(\mathbb{P}_{K^\circ}^R, \mathcal{O}(k))$  and  $k = \deg(f)$ , we obtain the result.  $\square$

Consider the  $K^\circ$ -module  $V := H^0(\mathbb{P}_{K^\circ}^R, \mathcal{O}(k))$  of global sections in projective space. It is clear that  $V$  is a free  $K^\circ$ -module of finite rank. Since the action of  $\mathbb{T}$  in  $\mathbb{P}_{K^\circ}^R$  is linear and it induces an action on the  $K^\circ$ -module  $V$ , we have a linear representation of  $\mathbb{T}$  on  $V$ , i.e. a homomorphism  $S : \mathbb{T} \rightarrow GL(V)$  of group schemes over  $K^\circ$ . A section  $s \in V$  is called *semi  $\mathbb{T}$ -invariant* if there is  $u \in M$  such that  $S_t(s) = \chi^u(t)s$  for every  $t \in \mathbb{T}$  and for the character  $\chi^u$  of  $\mathbb{T}$  associated to  $u$ . Now consider the following  $K^\circ$ -submodule

$$W := \{s \in H^0(\mathbb{P}_{K^\circ}^R, \mathcal{O}(k)) \mid \exists \lambda \in K^\circ \setminus \{0\} \text{ s.t. } \lambda s|_Y = 0\}$$

of  $V$ . Since  $Y$  is  $\mathbb{T}$ -invariant, it is clear that  $W$  is invariant under the action of  $\mathbb{T}$ . Note that  $W$  is equal to the set of global sections of  $\mathcal{O}(k)$  which vanishes on  $Y_\eta$ .

**Lemma 5.2.**  *$W$  is a free  $K^\circ$ -module of finite rank, besides its basis can be chosen semi  $\mathbb{T}$ -invariant.*

*Proof.* Let  $\{x_i | i = 1, \dots, l\}$  be a basis of  $H^0(\mathbb{P}_{K^\circ}^R, \mathcal{O}(k))$ . It is also a basis of the vector space  $H^0(\mathbb{P}_{K^\circ}^R, \mathcal{O}(k)) \otimes K \simeq K^l$ . Defining

$$\left| \sum \lambda_i x_i \right| = \max |\lambda_i| |x_i|,$$

we see that it is a Cartesian vector space as well as its subspace  $W_K := W \otimes_{K^\circ} K \subset K^l$ . Let  $r = \dim(W_K)$ , by [6, Proposition 2.4.1/5 ] there is an orthogonal basis  $\{s_1, \dots, s_r\}$  of  $W_K$  which can be extended to an orthogonal basis of  $K^l$ , furthermore we can choose this basis such that  $|s_i| = 1$ , i.e. “orthonormal”. Note that the elements of this basis are in  $(K^\circ)^l$ , which we may identify with  $H^0(\mathbb{P}_{K^\circ}^R, \mathcal{O}(k))$ . Since  $W$  is saturated and has an integral basis, we have that  $W = W_K \cap (K^\circ)^l$ . Now let  $v \in W$ , then  $v = \sum \lambda_i s_i$  with  $\lambda_i \in K$ . Since  $v \in (K^\circ)^l$  then

$$1 \geq |v| = \max |\lambda_i|,$$

which implies that  $\lambda_i \in K^\circ$ , hence  $\{s_i\}$  is a basis of  $W$ . Now since  $W$  is a free  $K^\circ$ -module of finite rank, as in the proof of [17, Proposition 9.8] we conclude that there is a basis given by eigenvectors of the  $\mathbb{T}$ -action, i.e. semi  $\mathbb{T}$ -invariant sections.  $\square$

Now, we can prove Sumihiro’s theorem for normal projective  $\mathbb{T}$ -toric varieties with a linear action of the torus.

**Proposition 5.3.** *Let  $\mathcal{U}$  be a  $\mathbb{T}$ -invariant open subset of  $\mathcal{Y}_{A,a}$ . Then every point of  $\mathcal{U}$  has a  $\mathbb{T}$ -invariant open affine neighbourhood in  $\mathcal{U}$ .*

*Proof.* Let  $z \in \mathcal{U}$  and let  $Y := \mathcal{Y}_{A,a} \setminus \mathcal{U}$ . Since  $Y$  is  $\mathbb{T}$ -invariant, we can use the previous results as well as the notation from there. In particular, we have  $s_0 \in H^0(\mathbb{P}_{K^\circ}^{N+1}, \mathcal{O}(k))$  such that  $s_0|_Y = 0$  and  $s_0(z) \neq 0$ . Since  $s_0 \in W$ , by Lemma 5.2, we see that there is a semi  $\mathbb{T}$ -invariant section  $s_1 \in H^0(\mathbb{P}_{K^\circ}^{N+1}, \mathcal{O}(k))$  with  $s_1(z) \neq 0$  and  $\lambda s_1|_Y = 0$  for some  $\lambda \in K^\circ \setminus \{0\}$ .

To construct the affine invariant neighborhood of  $z$ , we consider two cases: when  $z$  is contained in the generic fiber and when  $z$  is contained in the special fiber. Suppose that  $z$  is contained in the generic fibre of  $\mathcal{U}$  over  $K^\circ$ . Then  $\mathcal{U}_1 := \{x \in \mathcal{Y}_{A,a} | \lambda s_1(x) \neq 0\}$  is an affine open subset of  $\mathcal{U}$  that contains  $z$ . Since  $s_1$  is semi  $\mathbb{T}$ -invariant, it follows that  $\mathcal{U}_1$  is  $\mathbb{T}$ -invariant: this can be seen from the equality  $\lambda s_i(tx) = t(t \cdot \lambda s_i)(x) = t\chi^u(t)\lambda s_i(u)$ . This proves the claim for this case.

Now suppose that  $z \in \mathcal{U}_s$ . Let  $Z$  be a  $\mathbb{T}$ -orbit containing  $z$ , by Proposition 1.15 there is a face  $Q_z$  of  $\text{Wt}(\mathbf{y}, a)$  corresponding to it. Since  $Y$  is  $\mathbb{T}$ -invariant,  $Y_s$  contains at least one dense  $\mathbb{T}$ -orbit. Suppose it is irreducible and let  $Q_Y$  be the face corresponding to the dense  $\mathbb{T}$ -orbit contained in  $Y_s$ . Since  $z \notin Y$  there exist  $u_i \in Q_z \cap A(\mathbf{y})$  such that  $u_i \notin Q_Y \cap A(\mathbf{y})$ , which means that for a closed point  $\mathbf{x}$  of  $Y_s$ ,  $x_i = 0$  and for a closed point of  $\bar{z}$  with coordinates  $(z_0 : \dots : z_n)$  we have that  $z_i \neq 0$ . Let  $s_i$  be the corresponding global section of  $\mathcal{O}(1)$ . Let  $s_1$  be one generator of  $V$  as above which is semi  $\mathbb{T}$ -invariant and is non-zero on  $z$ . For  $s := s_1 \otimes s_i$ , we define

$$\mathcal{U}_1 := \{x \in \mathcal{Y}_{A,a} | s(x) \neq 0\}.$$

By definition it is an affine  $\mathbb{T}$ -invariant subset and since  $s_1$  and  $s_i$  vanish on the horizontal and vertical components of  $Y$  respectively, it is contained in  $\mathcal{Y}_{A,a} \setminus Y$ .

If  $Y_s$  is not irreducible, then for each component  $Y_s^{(k)}$  we proceed as before to get the sections  $s_{i(k)}$ . With the product  $s := s_1 \otimes (\otimes_k s_{i(k)})$ , we obtain a global section of some tensor power of  $\mathcal{O}(1)$  which vanishes on  $Y$  but not on  $z$  and which is semi  $\mathbb{T}$ -invariant. Then  $\mathcal{U}_1 := \{x \in \mathcal{Y}_{A,a} \mid s(x) \neq 0\}$  defines an open affine  $\mathbb{T}$ -invariant neighborhood of  $z$  contained in  $\mathcal{Y}_{A,a} \setminus Y$ . This proves the claim.  $\square$

With this result, we can conclude the proof of the Theorem 2.

*Proof of Theorem 2.* Given a point  $z \in \mathcal{Y}$  in a normal  $\mathbb{T}$ -toric variety  $\mathcal{Y}$  over  $K^\circ$ , we take  $\mathcal{U}_0$  as an affine open neighbourhood of it and  $\mathcal{U}$  as the smallest  $\mathbb{T}$ -invariant open subset of  $\mathcal{Y}$  which contains  $\mathcal{U}_0$ . Let  $i : \mathcal{U} \rightarrow \mathbb{P}_{K^\circ}^R$  be the  $\mathbb{T}$ -equivariant embedding constructed in chapter 4. By Proposition 4.7 we know that there exists an element  $A \in M^{R+1}$  and a height function  $a$  such that the closure of  $i(\mathcal{U})$  in  $\mathbb{P}_{K^\circ}^R$  is a projective  $\mathbb{T}$ -toric variety  $\mathcal{Y}_{A,a}$  with a linear action of the torus. From Proposition 5.3 there is an affine  $\mathbb{T}$ -invariant open neighbourhood  $\mathcal{U}_1$  of  $i(z)$  which is contained in  $i(\mathcal{U})$ . The preimage  $i^{-1}(\mathcal{U}_1)$  of this neighbourhood is an affine  $\mathbb{T}$ -invariant open neighbourhood of  $\mathcal{U}$ , and therefore of  $\mathcal{Y}$ , which contains  $z$ . This proves Sumihiro's theorem.  $\square$

## 5.2 Classification

Finally in order to complete the picture which give rise to the interplay between toric geometry and convex geometry, we prove Theorem 3 which give us a bijective correspondence between normal  $\mathbb{T}$ -toric varieties and  $\Gamma$ -admissible fans.

*Proof of Theorem 3.* Let  $\mathcal{Y}$  be a normal  $\mathbb{T}$ -toric variety over the valuation ring  $K^\circ$ . We assume that the valuation is neither discrete nor trivial, since for those cases the result is known. By Theorem 1 and Theorem 2 there is open covering of  $\mathcal{Y}$  of the form  $\{\mathcal{V}_{\sigma_i}\}$ , with  $\mathcal{V}_{\sigma_i}$  normal affine  $\mathbb{T}$ -toric varieties constructed from the  $\Gamma$ -admissible cones  $\sigma_i$  for which the vertices of  $\sigma_i \cap (N_{\mathbb{R}} \times \{1\})$  are contained in  $N_{\Gamma} \times \{1\}$ .

We want to show that the fan  $\Sigma$  made from the cones  $\{\sigma_i\}$  is  $\Gamma$ -admissible and that  $\mathcal{Y}_{\Sigma} \simeq \mathcal{Y}$ . For this, we have to show that  $\mathcal{V}_{\sigma_i} \cap \mathcal{V}_{\sigma_j}$  is an affine toric scheme and that the  $\mathbb{T}$ -action is compatible with the corresponding action in  $\mathcal{V}_{\sigma_i}$  and  $\mathcal{V}_{\sigma_j}$ . Since  $\mathcal{Y}$  is separated the intersection of two affine charts  $\mathcal{V}_{ij} = \mathcal{V}_{\sigma_i} \cap \mathcal{V}_{\sigma_j}$  is affine and  $\mathbb{T}$ -invariant. Then by Theorem 1 there is a  $\Gamma$ -admissible cone  $\sigma_{ij} \subset N_{\mathbb{R}} \times \mathbb{R}_+$  such that  $\mathcal{V}_{ij} \simeq \mathcal{V}_{\sigma_{ij}}$ . Applying the orbit-face correspondence [17, Proposition 8.8] to the open immersions  $\mathcal{V}_{\sigma_{ij}} \rightarrow \mathcal{V}_{\sigma_i}$  and  $\mathcal{V}_{\sigma_{ij}} \rightarrow \mathcal{V}_{\sigma_j}$  we have that  $\sigma_{ij}$  is a closed face of  $\sigma_i$  and  $\sigma_j$ . By the definition of  $\mathcal{V}_{ij}$  the same argument shows that  $\sigma_{ij} = \sigma_i \cap \sigma_j$ . Therefore the cones  $\{\sigma_i\}$  and their closed faces form a  $\Gamma$ -admissible fan  $\Sigma$  in  $N_{\mathbb{R}} \times \mathbb{R}_+$  such that  $\mathcal{Y} \simeq \mathcal{Y}_{\Sigma}$ . This gives the result.  $\square$

These results allow us to say that the description of normal  $\mathbb{T}$ -toric varieties over an arbitrary valuation ring of rank one can be understood completely. This follows from the

theory available for toric varieties over a field and from tropical geometry, as described in chapter 2. Furthermore, although we have non-noetherian schemes, we still have a classification of  $\mathbb{T}$ -toric varieties in terms of fans. Since there are plenty of valuation rings of rank one, the theory have been extended in a non-trivial way. The good understanding of toric geometry over valuation rings allows us to have very solid foundations for the theories which are based on those, for instance the theory of tropical compactifications.

# Appendices



# A | Convex Geometry

In this appendix we summarize the notions of convex geometry which are used throughout this paper. The main references are Rockafellar [30], Ziegler [36] and Barvinok [4].

Let  $W$  be a real vector space and  $\Gamma \subset \mathbb{R}$  an additive subgroup. Let  $M$  be a free abelian group of rank  $n$  and  $N = \text{Hom}(M, \mathbb{Z})$  its dual. We denote by  $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$  and  $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ . A polyhedron  $\Delta$  in  $W$  is a finite intersection of closed half spaces

$$\bigcap_{i=1}^k \{\omega \in W \mid \langle u_i, \omega \rangle \geq c_i\}.$$

We say that the polyhedron  $\Delta$  is  $\Gamma$ -rational if  $u_i \in M$  and  $c_i \in \Gamma$ . It is called *pointed* if it is  $\Gamma$ -rational and do not contains affine subspaces of dimension  $> 0$ . When  $\Gamma = \mathbb{Q}$ , this coincides with the notion of rational polyhedra used by Fulton in [13]. The dimension of a polyhedron  $\Delta$  is the dimension of the subspace  $\text{aff}(\Delta)$  spanned by it.

A cone is a polyhedron with all  $c_i = 0$ , i.e. a polyhedron of the form

$$\bigcap_{i=1}^k \{\omega \in W \mid \langle u_i, \omega \rangle \geq 0\}.$$

A *closed face* of a polyhedron  $\Delta$  is  $\Delta$  itself or it is of the form  $H \cap \Delta$  where  $H$  is the boundary of a half space. An *open face* is a closed face without its properly contained closed faces.

Let  $\sigma$  be a face of a polyhedron  $\Delta$ . An interior point of  $\sigma$  is an interior point relative to  $\text{aff}(\sigma)$ . The set of the interior points of  $\sigma$  is called the *relative interior* of  $\sigma$ . It is denoted by  $\text{relint}(\sigma)$ . The relative interior of  $\Delta$  is called the *interior* of  $\Delta$ .

A cone  $\sigma$  in  $N_{\mathbb{R}} \times \mathbb{R}_+$  is called  $\Gamma$ -admissible if it can be written as

$$\sigma = \bigcap_{i=1}^k \{(\omega, s) \in N_{\mathbb{R}} \times \mathbb{R}_+ \mid \langle u_i, \omega \rangle + sc_i \geq 0\}, \quad u_1, \dots, u_k \in M, c_1, \dots, c_k \in \Gamma,$$

and does not contain a line.

A bounded polyhedron is called a *polytope*. This objects are easier to describe because of the Minkowski-Weyl theorem, or the Krein-Milman theorem in the infinite dimensional case, see [4, Theorem 3.3]. It says that every polytope is the convex hull of a finite

number of points. Using the addition in  $W$  we can add polyhedra. This makes the set of polyhedra into a monoid. Furthermore every polyhedron can be written as the sum of a polytope and a cone.

Let  $\Delta$  be a polyhedron. The recession cone or the cone of unbounded directions of  $\Delta$  is defined as

$$\text{rec}(\Delta) := \{\omega \in W \mid \Delta + \omega \subset \Delta\}.$$

The closure of the cone generated by  $\Delta \times \{1\}$  in  $W \times \mathbb{R}_+$  is denoted by  $c(\Delta)$ . For a cone  $\sigma$  in  $W \times \mathbb{R}_+$ , we define  $\sigma_s := \{w \in W \mid (w, s) \in \sigma\}$ . With  $\sigma = c(\Delta)$ , we have  $\sigma_0 = \text{rec}(\Delta)$  and  $\sigma_1 = \Delta$ . A polyhedron is called pointed iff its recession cone contains no lines which is equivalent to say that  $0 \in W$  is a vertex of  $\sigma_0$ .

A fan  $\Sigma$  is a collection of cones such that every face of a cone is in  $\Sigma$  and the intersection of any two cones is a face of each. A polyhedral complex  $\mathcal{C}$  is a set of polyhedra which satisfies that for any  $\Delta \in \mathcal{C}$  every closed face of  $\Delta$  is in  $\mathcal{C}$  and for any  $\Delta, \Delta' \in \mathcal{C}$  their intersection is in  $\mathcal{C}$  or is empty. The support of a polyhedral complex  $\mathcal{C}$  is given by  $|\mathcal{C}| = \cup |\Delta| \subset W$  for every  $\Delta \in \mathcal{C}$ , where  $|\Delta|$  denotes the support of the polyhedron  $\Delta$ . If  $|\mathcal{C}| = W$  the polyhedral complex  $\mathcal{C}$  is called complete.

Let  $f$  be a real function on  $W$ . The *epigraph* of  $f$ , denoted by  $\text{epi}(f)$ , is defined as the convex hull of  $\{(w, \lambda) \in W \times \mathbb{R} \mid \lambda \geq f(w)\}$ . A real function  $f$  convex iff  $\text{epi}(f)$  is a convex set in  $W \times \mathbb{R}$ . For instance the affine functions on  $W$  are convex. On  $W = \mathbb{R}^n$  a twice continuously differentiable function is convex iff its Hessian is positive semi-definite, see [30, Theorem 4.5].



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# Zusammenfassung

In der Mathematik wird die Konstruktion konkreter Beispiele nicht nur zur Illustration der Theorie genutzt, sondern ebenfalls zum Auffinden von Gegenbeispielen für offene Vermutungen. Konkrete Konstruktionen sind in der algebraischen Geometrie im Allgemeinen schwierig durchzuführen. Diese Tatsache verdeckt manchmal die, der Theorie zugrunde liegende, Geometrie. Jedoch sind solche Konstruktionen im Fall einer torischen Varietät über einem Körper einfacher durchzuführen. Dies ist mit der bekannten Übersetzung zwischen torischer und konvexer Geometrie begründet. Dieses erlaubt uns konkrete Beispiele durch die Implementierung der kombinatorischen Beschreibung in Computeralgebrasysteme zu berechnen. Dadurch werden die Anwendungsmöglichkeiten und das Verständnis der algebraischen Geometrie verbessert. Der Grundpfeiler dieser Resultate stellt ein Satz von Sumihiro dar, welcher besagt, dass jede torische Varietät eine Überdeckung aus affinen offenen Torus invarianten Teilmengen besitzt. Eines der Ziele dieser Arbeit ist es, dieses Resultat für torische Varietäten, die über einem beliebigen Bewertungsring vom Rang eins definiert sind, zu verallgemeinern.

Die Theorie torischer Varietäten über einem Körper ist sehr gut verstanden und es gibt viele gute Referenzen hierfür, vgl. etwa Cox–Little–Schenk [11], Ewald [12], Fulton [13], Kempf–Knudsen–Mumford–Saint-Donat [21] und Oda [27]. In [21] erweiterte Mumford die torische Geometrie auf normale Varietäten, welche über einem diskreten Bewertungsring definiert sind. Seither wurden nur sehr wenige Versuche unternommen torische Geometrie auf allgemeineren Basen zu erweitern. Das Hauptproblem ist, dass für Bewertungsringe, deren Bewertungen weder trivial noch diskret sind, die noetherschen Methoden der algebraischen Geometrie nicht mehr zur Verfügung stehen.

In [17] führte Walter Gubler torische Varietäten über einem beliebigen Bewertungsring vom Rang 1 ein. Eine  $\mathbb{T}$ -torische Varietät  $\mathcal{Y}$  über einem Bewertungsring  $K^\circ$  ist ein integrales separiertes flaches Schema vom endlichen Typ über  $K^\circ$ , sodass die allgemeine Faser  $\mathcal{Y}_\eta$  von  $\mathcal{Y}$  den Torus  $T := (\mathbb{G}_m^n)_K$  enthält und sich die natürliche Wirkung von  $T$  auf sich selbst zu einer Wirkung von  $\mathbb{T} := (\mathbb{G}_m^n)_{K^\circ}$  auf  $\mathcal{Y}$  fortsetzen lässt. Für jeden Kegel  $\sigma \in \Sigma$  eines zulässigen Fächers  $\Sigma \subset \mathbb{R}^n \times \mathbb{R}_+$  ist es möglich, ein affines  $\mathbb{T}$ -torisches Schema  $\mathcal{Y}_\sigma$  zu konstruieren. Durch Verkleben dieser affinen Schemata erhalten wir ein  $\mathbb{T}$ -torisches Schema  $\mathcal{Y}_\Sigma$ . Zu beachten ist, dass die kombinatorische Beschreibung, ähnlich wie im Fall einer torischen Varietät über einem diskreten Bewertungsring, von Kegeln und Fächern in  $\mathbb{R}^n \times \mathbb{R}_+$  herrührt. Eine Besonderheit in diesem Fall ist, dass die tropische Geometrie eine sehr gute Beschreibung der Kegel-Bahn-Beziehung liefert (für Details vgl. [17, §8]

und §1.3).

Torische Varietäten über einem beliebigen Grundring  $R$  wurden von Fred Rohrer in [31] untersucht. Er startet mit einem Fächer  $\Pi \subset \mathbb{R}^n$  und geht wie üblich vor, d.h. zu jedem Kegel  $\sigma \in \Pi$  ordnet er eine  $R$ -Algebra  $A_\sigma$  zu. Die Schemata  $\text{Spec}(A_\sigma)$  können verklebt werden, wodurch man das Schema  $X_\Pi$  erhält. Nach Konstruktion besitzt dieses Schema eine algebraische Wirkung durch den spaltenden Torus  $(\mathbb{G}_m^n)_R$ . Falls wir für  $R$  den Bewertungsring  $K^\circ$  wählen, liefert diese Konstruktion einen Spezialfall von Walter Gublers Konstruktion. In der Tat haben wir  $X_\Pi = \mathcal{Y}_{\Pi \times \mathbb{R}_+}$  für einen Fächer  $\Pi$ . Darüberhinaus liefern diese beiden Ansätze dasselbe Resultat, wenn man für  $R$  einen Körper  $K$  einsetzt und sich auf normale Varietäten beschränkt.

Der Unterschied zwischen diesen beiden Ansätzen ist, dass man beim ersten verlangt, dass die Toruswirkung sich auf das ganze Schema fortsetzen lässt, wohingegen man im zweiten mit der kombinatorischen Beschreibung startet und man daher, nach Definition, automatisch eine Toruswirkung auf dem ganzen Schema gegeben hat.

Die Konstruktion von Walter Gubler hängt stark vom Grundring, insbesondere von dem Fakt, dass dieser Rang 1 hat, ab. Dadurch wird der Einsatz der Theorie analytischer Räume nach Berkovich sowie der Einsatz der tropischen Geometrie möglich.

In derselben Weise hängen die Ergebnisse dieser Arbeit von diesen Voraussetzungen ab. Die Frage, ob sich die oben genannten Konstruktionen für Bewertungsringe von höherem Rang verallgemeinern lassen, bleibt weiterhin unbeantwortet.

## Hauptresultate

Das Ziel dieser Arbeit ist es torische Varietäten über einem Bewertungsring  $K^\circ$ , wie von Walter Gubler in [17] eingeführt, zu klassifizieren. Um dies zu erreichen werden drei Sätze bewiesen. Die dadurch erreichte Klassifikation wird die übliche Beziehung zwischen normalen torischen Varietäten und Kegeln, im affinen Fall, und Fächern, im Allgemeinen, verallgemeinern.

Sei  $K$  ein bewerteter Körper vom Rang 1,  $K^\circ$  sein Bewertungsring und  $\Gamma$  seine Wertegruppe. Sei  $\mathbb{T} = \text{Spec}(K^\circ[M])$  der spaltende Torus über  $K^\circ$ . Mit  $M$  bezeichnen wir das Charaktergitter von  $\mathbb{T}$ . In Kapitel 1 wiederholen wir die Theorie torischer Varietäten über dem Bewertungsring  $K^\circ$ . Wir zeigen wie man zu einem  $\Gamma$ -zulässigen Kegel  $\sigma \subset \mathbb{R}^n \times \mathbb{R}_+$  eine  $K^\circ$ -Algebra  $K[M]^\sigma$  konstruiert, die das  $\mathbb{T}$ -torische Schema  $\mathcal{V}_\sigma = \text{Spec}(K[M]^\sigma)$  liefert. Falls die Bewertung trivial oder diskret ist oder, falls die Bewertung weder trivial noch diskret ist aber die Ecken von  $\sigma \cap (\mathbb{R}^n \times \{1\})$  in  $\Gamma^n \times \{1\}$  enthalten sind, dann ist die Algebra  $K[M]^\sigma$  vom endlichen Typ und das Schema  $\mathcal{V}_\sigma$  ist eine normale  $\mathbb{T}$ -torische Varietät über  $K^\circ$ . Auf diese Weise erhalten wir viele Beispiele von  $\mathbb{T}$ -torischen Varietäten über einem Bewertungsring vom Rang 1. Eine natürliche Frage ist es, ob alle affinen normalen  $\mathbb{T}$ -torische Varietäten über  $K^\circ$  von dieser Form sind. Das erste hier vorgestellte Resultat bejaht diese Frage. Im Folgendem nehmen wir stets an, dass die Bewertung  $v$  nicht trivial ist. Dann haben wir folgendes Resultat.

**Satz 1.** *Falls die Bewertung  $v$  nicht diskret ist, definiert die Zuordnung  $\sigma \mapsto \mathcal{V}_\sigma$  eine Bijektion zwischen der Menge der  $\Gamma$ -zulässigen Kegel in  $\mathbb{R}^n \times \mathbb{R}_+$ , für die die Eckpunkte*

von  $\sigma \cap (\mathbb{R}^n \times \{1\})$  in  $\Gamma^n \times \{1\}$  enthalten sind, und der Menge der Isomorphieklassen normaler affiner  $\mathbb{T}$ -torischer Varietäten über dem Bewertungsring  $K^\circ$ .

Mit Hilfe dieses Satzes erhalten wir eine Klassifikation normaler affiner  $\mathbb{T}$ -torischer Varietäten über einem Bewertungsring von Rang 1, welche die bekannten Standardresultate von torischen Varietäten über einem Körper bzw. über einem diskreten Bewertungsring erweitert. Zu beachten ist, dass im Fall eines diskreten oder trivialen Bewertungsringes, die Zusatzbedingung für die Kegel entfällt.

Der Beweis von Satz 1 wird in Kapitel 2 geführt. Wir zeigen, dass man zu einer affinen normalen  $\mathbb{T}$ -torischen Varietät  $\mathcal{Y} = \text{Spec}(A)$  einen  $\Gamma$ -zulässigen Kegel  $\sigma$  konstruieren kann, sodass  $K[M]^\sigma = A$ . Zunächst konstruieren wir mit Hilfe der  $K^\circ$ -Algebra  $A$  eine Halbgruppe  $S$  und nehmen den davon erzeugten Kegel  $\text{cone}(S)$ . Der gesuchte Kegel ist dann der duale Kegel  $\sigma := \text{cone}(S)^\vee$ .

Darüber hinaus lässt sich  $\sigma$  durch die Tropikalisierung von  $\mathcal{Y}_\eta \cap T^\circ$  rekonstruieren (vgl. Proposition 1.11). Damit ist der Kegel  $\sigma$  eindeutig durch  $\mathcal{Y}$  gegeben.

Um normale, nicht notwendig affine,  $\mathbb{T}$ -torische Varietäten über dem Bewertungsring  $K^\circ$  zu klassifizieren, ist es notwendig den, aus der torischen Geometrie bekannten, Satz von Sumihiro zu verallgemeinern.

**Satz 2.** *Sei  $v$  eine Bewertung mit Werten in  $\mathbb{R}$  und Bewertungsring  $K^\circ$ . Dann besitzt jeder Punkt von  $\mathcal{Y}$  eine affine offene  $\mathbb{T}$ -invariante Umgebung.*

Dieses Resultat verallgemeinert den Satz von Sumihiro für normale torische Varietäten über einen Körper (vgl. [34]) auf normale  $\mathbb{T}$ -torische Varietäten über einem Bewertungsring vom Rang 1. Der Beweis ist deutlich schwerer als im klassischen Fall, da hier keine noetherschen Methoden zur Verfügung stehen. Stattdessen wenden wir die Schnitttheorie mit Cartierdivisoren auf Varietäten über einem Bewertungsring vom Rang 1 an. Dies wird in Kapitel 3 getan. Die Ergebnisse folgen aus der Schnitttheorie mit Cartierdivisoren auf zulässigen formalen Schemata über  $K^\circ$ , welche von Walter Gubler in [19] entwickelt wurde. Wir benutzen die Notation der PvM-Ringe um Weildivisoren auf normalen Varietäten über einem Bewertungsring zu untersuchen und um jedem Cartierdivisor einen Zykel der Kodimension 1 zuzuordnen. Wir nutzen dies um in Kapitel 4 zu zeigen, dass man zu jeder offenen affinen Teilmenge  $\mathcal{U}_0$  einer normalen  $\mathbb{T}$ -torischen Varietät  $\mathcal{Y}$  eine  $\mathbb{T}$ -invariante offene Teilmenge  $\mathcal{U}$  und einen Cartierdivisor  $D$  konstruieren kann, sodass  $\mathcal{U}$  die Menge  $\mathcal{U}_0$  enthält und  $D$  den Träger  $\mathcal{U} \setminus \mathcal{U}_0$  hat. Daraufhin zeigen wir, dass das Geradenbündel  $\mathcal{O}(D)$  eine  $\mathbb{T}$ -Linearisierung besitzt, was zu einer  $\mathbb{T}$ -äquivarianten Einbettung von  $\mathcal{U}$  in eine projektive  $\mathbb{T}$ -torische Varietät mit einer linearen Wirkung des Torus führt. Schließlich können wir durch Anwendung dieser Fakten Satz 2 in Kapitel 5 beweisen.

Als Folgerung aus Satz 1 und Satz 2 erhalten wir unser Hauptklassifikationsresultat.

**Satz 3.** *Falls  $v$  keine diskrete Bewertung ist, liefert die Zuordnung  $\Sigma \mapsto \mathcal{Y}_\Sigma$  eine Bijektion zwischen der Menge der Fächer in  $\mathbb{R}^n \times \mathbb{R}_+$ , deren Kegel wie in Satz 1 beschrieben sind, und der Menge der Isomorphieklassen normaler  $\mathbb{T}$ -torischer Varietäten über  $K^\circ$ .*

Dieser Satz verallgemeinert die Klassifikation normaler torischer Varietäten über einem Körper bzw. einem diskreten Bewertungsring auf normale  $\mathbb{T}$ -torische Varietäten über einem Bewertungsring vom Rang 1. Zu beachten ist, dass im Fall einer trivialen oder einer diskreten Bewertung die Zusatzbedingung für die Kegel entfällt.

Dieses Resultat ermöglicht uns ein besseres Verständnis der torischen Geometrie über einem Bewertungsring vom Rang 1 zu erhalten. Es lohnt sich zu betonen, dass diese Objekte in enger Beziehung mit tropischer Geometrie stehen: z.B. kann man den Kegel einer normalen affinen  $\mathbb{T}$ -torischen Varietät  $\mathcal{V}_\sigma$  aus der Tropikalisierung der Teilmenge der potentiell integren Punkte von  $T^\circ \cap (\mathcal{V}_\sigma)_\eta$ , wobei  $(\mathcal{V}_\sigma)_\eta$  die generische Faser bezeichnet, gewinnen. Darüberhinaus werden in [17] diese torischen Schematas zur Verallgemeinerung einiger Resultate über tropische Kompaktifizierungen abgeschlossener Unterschematas eines Torus  $T$  auf beliebige bewertete Körper von Rang 1 verwendet (vgl. [17, §12]).