Effective Hamiltonians for magnetic Bloch bands

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Zusammenfassung

In der vorliegenden Arbeit wird eine Einteilchen-Schrödinger-Gleichung mit einem periodischen Potential, einem starken, konstanten Magnetfeld und nicht-periodischen Störungen in zwei Dimensionen betrachtet. Unser Ziel ist die mathematisch rigorose Herleitung eines effektiven Modells, das die wesentlichen Eigenschaften der Gleichung wiedergibt. Dieses wird "effektiver Hamiltonian" genannt. Das Modell, das wir untersuchen möchten, beschreibt die Bewegung von Leitungselektronen in einem kristallinen Festkörper in dem von den Atomkernen erzeugten Potential. In der Festkörperphysik ist es eine Standardapproximation, bei vielen Fragestellungen die Coulomb-Abstoßung zwischen den Elektronen zu vernachlässigen. Deshalb reicht es aus, das Verhalten eines einzelnen Teilchens unter dem Einfluss eines periodischen Potentials anzuschauen. Sei nun $\Gamma \cong \mathbb{Z}^2$ das von den Atomkernen erzeugte Bravais-Gitter. Wir nehmen also an, dass sich die Kerne an festen Positionen befinden. Dann ist das mathematische Modell für diese Fragestellung (mit geeigneten physikalischen Einheiten) durch den folgenden Hamiltonian gegeben:

$$H^\varepsilon = \tfrac{1}{2}(-i\nabla_x - A_0(x) - A(\varepsilon x))^2 + V_\Gamma(x) + \Phi(\varepsilon x). \tag{1}$$

Unter geeigneten technischen Bedingungen ist dies ein selbstadjungierter Operator, dessen Definitionsbereich der magnetische Sobolev-Raum $\mathrm{D}(H^\varepsilon) = H^2_{A_0}(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$ ist. Hier ist V_Γ das kristalline Potential, welches periodisch bezüglich des Bravais-Gitters Γ ist, und A_0 das Vektorpotential des starken, konstanten Magnetfeldes $B = \mathrm{d}A_0$. Desweiteren betrachten wir auch nicht-periodische Störungen A und Φ. Die Potentiale $A = A(\varepsilon x)$ und $\Phi = \Phi(\varepsilon x)$ werden als langsam variierend auf der Skala des Gitters Γ angenommen. Hierbei ist ε ein dimensionsloser Parameter, von dem A und Φ unabhängig sind. Wir nehmen an, dass die Störungen glatt und einschließlich aller Ableitungen beschränkt sind. Die entsprechenden schwachen elektromagnetischen Felder sind $\mathcal{B}(x) = \mathrm{curl} A(x)$ beziehungsweise $\mathcal{E}(x) = -\nabla\Phi(x)$. Der ungestörte Operator ist dann definiert als

$$H_{\mathrm{MB}} = \tfrac{1}{2}(-i\nabla_x - A_0)^2 + V_\Gamma. \tag{2}$$

Nach einer magnetischen Bloch-Floquet-Transformation kann man diesen Hamiltonian als Weyl-Quantisierung eines operatorwertigen Symbols $H_{\mathrm{per}}(k)$ auffassen.

Für festes k ist dieses Symbol ein Operator auf einem Hilbertraum $\mathcal{H}_{\mathrm{f}}$ und hat diskretes Spektrum. Wir bezeichnen die Eigenwerte als $E_1(k) \leq E_2(k) \leq \ldots$. Die dadurch gegebenen Eigenwertfunktionen $(E_n)_{n\geq 1}$ des transformierten Hamiltonians werden magnetische Blochbänder genannt. Unser Ziel ist es, das Verhalten von Teilchen, deren Dynamik durch den Hamiltonian H^ε beschrieben wird, für $\varepsilon \ll 1$ genau zu verstehen. Dazu möchten wir effektive Modelle herleiten, die zu einem magnetischen Blochband gehören. Damit diese Einleitung so verständlich wie möglich bleibt, geben wir unsere Ergebnisse für den Fall $\Gamma = \mathbb{Z}^2$ an.
In [Teu03, PST03b] wird der nicht-magnetische Fall, also der Fall $A_0 \equiv 0$, genau untersucht. Wir möchten die Methoden der eben genannten Arbeiten auf den magnetischen Fall verallgemeinern. Im nicht-magnetischen Fall konnte zu einem isolierten, nicht degenerierten Blochband E ein sogenannter fast invarianter Unterraum definiert werden, sodass der Operator H^ε eingeschränkt auf diesen Unterraum durch das effektive Modell

$$\widehat{h_{\mathrm{eff}}} = E(k - A(i\varepsilon\nabla_k)) + \Phi(i\varepsilon\nabla_k) + \mathcal{O}(\varepsilon)$$

beschrieben werden kann, wobei

$$\widehat{h_{\mathrm{eff}}} \quad \text{auf} \quad L^2(\mathbb{T}^{2*})$$

operiert und $\widehat{h_{\mathrm{eff}}}$ die Weyl-Quantisierung des Symbols h_{eff} ist. Darüberhinaus ist $\mathbb{T}^{2*} = \mathbb{R}^2/\Gamma^*$, wobei Γ^* das duale Gitter zu Γ ist. Die Darstellung $E(k-A(i\varepsilon\nabla_k))+\Phi(i\varepsilon\nabla_k)$ wird Peierls Substitution genannt.
Im magnetischen Fall möchten wir ebenfalls mathematisch rigoros ein effektives Modell für H^ε herleiten, welches als Pseudodifferentialoperator gegeben ist und in führender Ordnung durch eine Peierls Substitution gegeben ist. Allerdings führt die Einbindung des Potentials A_0 im Hamiltonian H^ε auch zu einigen Unterschieden zwischen unseren Resultaten und denen aus dem nicht-magnetischen Fall. Das effektive Modell auf dem zu dem magnetischen Blochband E gehörigen Unterraum ist

$$\widehat{h_{\mathrm{eff}}}^{\mathrm{eff}} = E(k - A(i\varepsilon\nabla_k^{\mathrm{eff}})) + \Phi(i\varepsilon\nabla_k^{\mathrm{eff}}) + \mathcal{O}(\varepsilon),$$

wobei $\widehat{h_{\mathrm{eff}}}^{\mathrm{eff}}$ auf

$$\mathcal{H}_\theta = \{\psi \in L^2_{\mathrm{loc}}(\mathbb{R}^2) : \psi(k - \gamma^*) = e^{\frac{i\theta}{2\pi}\gamma_1^* k_2}\psi(k) \quad \forall \gamma^* \in \Gamma^*\}$$

operiert und

$$\nabla_k^{\mathrm{eff}} = \nabla_k + (0, \tfrac{i\theta}{2\pi}k_1)^{\mathrm{T}}.$$

Hier bezeichnet θ die Chernzahl eines bestimmten Linienbündels (des Blochbündels) und ist somit ganzzahlig. Wir erhalten also immer noch ein Operator vom

Typ einer Peierls Substitution, aber im Unterschied zum nicht-magnetischen Fall ist die Quantisierung, die benutzt werden muss, neu. Sie bildet das Symbol $f(k,r)$ auf den Operator $f(k, i\varepsilon\nabla_k^{\text{eff}})$ ab und setzt also einen nicht-trivialen Zusammenhang ein. Die Definition dieser Quantisierung ist ein wesentlicher Teil dieser Arbeit. Ein weiterer Unterschied ist, dass der effektive Operator nicht mehr auf Funktionen über dem Torus, sondern auf Schnitten eines möglicherweise nicht-trivialen Linienbündels operiert.

Der Grund dafür ist, dass der effektive Hamiltonian auf einem isolierten, nicht degenerierten Blochband E immer ein Operator zwischen Schnitten eines Linienbündels über dem Torus ist. Dieses Bündel heißt Blochbündel und wird mit der Spektralprojektion P (die Spektralprojektion, die zu dem Blochband E gehört) assoziiert. Im nicht-magnetischen Fall ist das Blochbündel stets trivial und daher isomorph zu $\mathbb{T}^{2*} \times \mathbb{C}$. Damit sind die Schnitte in diesem Bündel gerade die Funktionen von $\mathbb{T}^{2*}$ nach $\mathbb{C}$. In diesem Fall muss also nicht weiter beachtet werden, dass der effektive Hamiltonian ein Operator zwischen Schnitten eines Linienbündels ist. Im magnetischen Fall hebt jedoch das Auftreten von A_0 die Zeitumkehrsymmetrie des ungestörten Operators H_{MB} und damit die Trivialisierbarkeit des Blochbündels auf. Folglich können wir nicht mehr vernachlässigen, dass wir einen Operator zwischen Schnitten eines Linienbündels erhalten. Der einfachste Raum für den effektiven Operator ist der Raum $\mathcal{H}_\theta$, welcher Funktionen von $\mathbb{R}^2$ nach $\mathbb{C}$ beinhaltet, die bis auf eine Phase periodisch sind. Wendet man unsere Ergebnisse auf den nicht-magnetischen Fall an, erhält man $\theta = 0$ und $\mathcal{H}_{\theta=0} \cong L^2(\mathbb{T}^{2*})$ sowie $\nabla^{\text{eff}} = \nabla$. Somit reproduziert diese Arbeit auch den nicht-magnetischen Fall.

Wie in [PST03b, Teu03] möchten wir unsere Ergebnisse mittels raumadiabatischer Störungstheorie herleiten. Diese wurde in [PST03a] entwickelt und wir müssen die Methoden aus [PST03b, Teu03] auf den magnetischen Fall verallgemeinern. Das wichtigste mathematische Instrument, das in der raumadiabatischen Störungstheorie verwendet wird, sind Pseudodifferentialoperatoren mit operatorwertigen Symbolen. Ein Pseudodifferentialoperator H ist die Quantisierung eines Symbols $h \in C^\infty(\mathbb{R}^4, \mathcal{L}(\mathcal{H}))$ mit $H = h(k, -i\varepsilon\nabla_k) = \widehat{h}$. Ein Symbol ist immer eine glatte Funktion über deren Ableitungen man eine gewisse Kontrolle hat, wie zum Beispiel die Existenz einer (Ordnungs-)Funktion w, sodass für alle α, β eine Konstante $C_{\alpha,\beta}$ existiert, sodass für alle $k, r \in \mathbb{R}^2$ gilt

$$\left\|(\partial_k^\alpha \partial_r^\beta h)(k,r)\right\|_{\mathcal{L}(\mathcal{H})} \leq C_{\alpha,\beta} w(k,r).$$

Die wichtigste Eigenschaft von Pseudodifferentialoperatoren ist, dass man im Symbolraum ein Produkt $\sharp$ definieren kann, dass auf Operatorebene der Hintereinanderausführung von Abbildungen entspricht. Für zwei Pseudodifferentialoperatoren $A = \widehat{a}$ und $B = \widehat{b}$ gilt also $AB = \widehat{a \sharp b}$. Oft ist es einfacher, einen Operator erst auf der Symbolebene zu konstruieren und ihn danach zu quantisieren.

Um die raumadiabatische Störungstheorie anwenden zu können, muss der betreffende Hamiltonian als Pseudodifferentialoperator gegeben sein. Der Hamiltonian (1) kann durch eine magnetische Bloch-Floquet-Transformation $\mathcal{U}_{\mathrm{BF}}$, welche im Wesentlichen eine Fourier-Transformation auf dem Gitter Γ ist, in die gewünschte Form gebracht werden. Um die Transformation definieren zu können, brauchen wir die zusätzliche Bedingung $B|M| \in 2\pi\mathbb{Z}$, wobei M die Fundamentalzelle des Gitters Γ ist. (Man beachte, dass es ausreichend ist, $B|M| \in \pi\mathbb{Q}$ zu fordern und zu einem Untergitter $\widetilde{\Gamma} \subset \Gamma$ überzugehen, welches $B|\widetilde{M}| \in 2\pi\mathbb{Z}$ erfüllt.) Sei nun M^* die erste Brillouin-Zone des Gitters Γ, also die Fundamentalzelle des dualen Gitters Γ^*. Dann bildet $\mathcal{U}_{\mathrm{BF}}$ den Raum $L^2(\mathbb{R}^2)$ auf den Raum $\mathcal{H}_\tau \cong L^2(M^*) \otimes \mathcal{H}_{\mathrm{f}}$ ab, wobei $\mathcal{H}_{\mathrm{f}}$ ein seperabler Hilbertraum ist und

$$\mathcal{H}_\tau = \{\varphi \in L^2_{\mathrm{loc}}(\mathbb{R}^2, \mathcal{H}_{\mathrm{f}}) : \varphi(k - \gamma^*) = \tau(\gamma^*)\varphi(k) \quad \forall \gamma^* \in \Gamma^*\},$$

wobei τ eine unitäre Darstellung von Γ^* in $\mathcal{L}(\mathcal{H}_{\mathrm{f}})$ ist. Dies transformiert den ungestörten Operator (2) zu

$$\mathcal{U}_{\mathrm{BF}} H_{\mathrm{MB}} \mathcal{U}^*_{\mathrm{BF}} = \int_{M^*}^{\oplus} H_{\mathrm{per}}(k) \mathrm{d}k,$$

wobei

$$H_{\mathrm{per}}(k) := \tfrac{1}{2}(-i\nabla_y - A_0(y) + k)^2 + V_\Gamma(y).$$

Der volle Operator (1) wird transformiert zu

$$\mathcal{U}_{\mathrm{BF}} H^\varepsilon \mathcal{U}^*_{\mathrm{BF}} =: H^\varepsilon_{\mathrm{BF}} = \widehat{H_0(k,r)},$$

wobei

$$H_0(k,r) = H_{\mathrm{per}}(k - A(r)) + \Phi(r).$$

Eine weitere Voraussetzung dafür, die raumadiabatische Störungstheorie anwenden zu können, ist eine Lücke im Spektrum des Hauptsymbols des Hamiltonians $\widehat{H}$. In dieser Arbeit nehmen wir deshalb an, dass $E(k)$ ein nicht degenerierter, isolierter Eigenwert von $H_{\mathrm{per}}(k)$ mit zugehörgier Spektralprojektion $P(k)$ ist. Damit ist $E(k - A(r)) + \Phi(r)$ ein nicht degenerierter, isolierter Eigenwert von $H_0(k,r)$ mit Spektralprojektion $P(k - A(r))$.

Der erste Schritt in der raumadiabatischen Störungstheorie ist die Konstruktion der sogenannten fast invarianten Unterräume. Diese Konstruktion funktioniert abgesehen von kleinen, technischen Änderungen wie im nicht-magnetischen Fall. Das heißt, wir bekommen eine Projektion $\Pi^\varepsilon = \widehat{\pi} + \mathcal{O}(\varepsilon^\infty)$, die fast als Pseudodifferentialoperator mit Hauptsymbol $\pi_0(k,r) = P(k - A(r))$ gegeben ist. Der Raum $\Pi^\varepsilon \mathcal{H}_\tau$ heißt "fast invarianter Unterraum", da $[H^\varepsilon_{\mathrm{BF}}, \Pi^\varepsilon] = \mathcal{O}(\varepsilon^\infty)$ gilt.

Allerdings ist dieser Unterraum schlecht zugänglich und man kann die Funktionen, die er enthält, nicht explizit beschreiben. Desweiteren hängt er von ε ab und es ist klar, dass es keinen Limes für $\varepsilon \to 0$ gibt. Von daher ist der nächste Schritt in der raumadiabatischen Störungstheorie, diesen Unterraum unitär auf einen ε-unabhängigen und explizit gegebenen Raum $\mathcal{H}_{\mathrm{ref}}$ abzubilden. Dieser Schritt kann im Gegensatz zur Definition der fast invarianten Unterräume nicht aus dem nicht-magnetischen Fall übernommen werden. Die Definition der verknüpfenden Abbildung U^ε und des geeigenten Referenzraums $\mathcal{H}_{\mathrm{ref}}$ sind einige der Hauptziele und der Ausgangspunkt dieser Arbeit.

Dies wird nun genauer beschrieben. Wir betrachten das folgende Bündel über $\mathbb{T}^{2*}$:

$$E = (\mathbb{R}^2 \times \mathcal{H}_{\mathrm{f}})_\sim \quad \text{wobei} \quad (k,\varphi) \sim (k',\varphi') :\Leftrightarrow k' = k - \gamma^* \quad \text{und} \quad \varphi' = \tau(\gamma^*)\varphi$$

$$\text{mit Zusammenhang} \quad \nabla^{\mathrm{B}} = P(k)\nabla P(k) + P^\perp(k)\nabla P^\perp(k).$$

Dieser Zusammenhang wird Berry-Zusammenhang genannt und $\mathcal{H}_\tau$ ist der Raum der L^2-Schnitte in diesem Bündel. Das Blochbündel ist das zur Projektion $P(k)$ assoziierte Unterbündel von E, das heißt

$$E_{\mathrm{Bl}} = \{(k,\varphi) \in (\mathbb{R}^2, \mathcal{H}_{\mathrm{f}})_\sim : \varphi \in P(k)\mathcal{H}_{\mathrm{f}}\} \quad \text{mit Zusammenhang} \quad \nabla^{\mathrm{B}}_k = P(k)\nabla_k.$$

Die L^2-Schnitte im Blochbündel sind dann gegeben durch

$$\Pi^0\mathcal{H}_\tau := \{f \in \mathcal{H}_\tau : f(k) \in P(k)\mathcal{H}_{\mathrm{f}}\}.$$

Im nicht-magnetischen Fall ist die wesentliche Voraussetzung für die Konstruktion von U^ε, dass E_{Bl} trivial ist. Folglich gibt es eine Funktion φ mit $\varphi(k-\gamma^*) = \tau(\gamma^*)\varphi(k)$ für alle $k \in \mathbb{R}^2$ und $\varphi(k) \in P(k)\mathcal{H}_{\mathrm{f}}$ mit $\|\varphi(k)\|_{\mathcal{H}_{\mathrm{f}}} \equiv 1$. Mit Hilfe dieser Funktion kann man, ausgehend vom Hauptsymbol $u_0(k,r) := \langle\varphi(k - A(r))| + u_0^\perp$, die verknüpfende unitäre Abbildung fast als Pseudodifferentialoperator $U^\varepsilon = \widehat{u} + \mathcal{O}(\varepsilon^\infty)$ konstruieren. Der Referenzraum kann als $L^2(\mathbb{T}^{2*})$ gewählt werden, da dieser Raum isomorph zum Raum der L^2-Schnitte im Blochbündel E_{Bl} ist. Die Tatsache, dass Π^ε und U^ε fast Pseudodifferentialoperatoren sind, ist ausschlaggebend dafür, den effektiven Operator ebenfalls als Pseudodifferentialoperator zu erhalten, da man dann ausnutzen kann, dass das Produkt "$\sharp$" auf der Symbolebene mit der Hintereinanderausführung auf der Operatorebene übereinstimmt:

$$\begin{aligned} H_{\mathrm{eff}} &= U^\varepsilon\Pi^\varepsilon H^\varepsilon_{\mathrm{BF}}\Pi^\varepsilon U^{\varepsilon*} = \widehat{u}\widehat{\pi}\widehat{H}\widehat{\pi}\widehat{u^*} + \mathcal{O}(\varepsilon^\infty) = \mathrm{Op}(\underbrace{u\sharp\pi\sharp H\sharp\pi\sharp u^*}_{=:h}) + \mathcal{O}(\varepsilon^\infty) \\ &= \widehat{h} + \mathcal{O}(\varepsilon^\infty). \end{aligned}$$

Die Konstruktion liefert den effektiven Hamiltonian also direkt als Pseudodifferentialoperator.

Nun wenden wir uns wieder dem magnetischen Fall zu. Das Hauptproblem ist, dass das Blochbündel in diesem Fall nicht mehr trivialisierbar ist. Also gibt es keine Funktion φ mehr wie vorher - jeder globale Schnitt φ im Blochbündel muss Nullstellen haben. Wir werden daher wie folgt vorgehen: Wir betrachten das Bündel

$$E'_{\mathrm{Bl}} = \{(k, \varphi) \in (\mathbb{R}^2, \mathcal{H}_{\mathrm{f}}) : \varphi \in P(k)\mathcal{H}_{\mathrm{f}}\},$$

welches als Bündel über $\mathbb{R}^2$ trivial sein muss. Also muss es einen globalen Schnitt in diesem Bündel geben, der nirgends verschwindet. Einen solchen Schnitt φ konstruieren wir mit Hilfe des Paralleltransports bezüglich des Berry-Zusammenhangs ∇^{B}. Dies liefert (entspricht Lemma 3.3.2 im Haupttext)

Lemma. *Es gibt eine Funktion $\varphi \in C^\infty(\mathbb{R}^2, \mathcal{H}_{\mathrm{f}})$, sodass für alle $k \in \mathbb{R}^2$ gilt: $\varphi(k) \in P(k)\mathcal{H}_{\mathrm{f}}$, $\|\varphi(k)\|_{\mathcal{H}_{\mathrm{f}}} = 1$ und*

$$\varphi(k - \gamma^*) = e^{-\frac{i\theta}{2\pi}k_2\gamma_1^*}\tau(\gamma^*)\varphi(k) \quad \textit{für alle} \quad \gamma^* \in \Gamma^*,$$

wobei θ die Chernzahl des Blochbündels ist.

Man beachte, dass die Chernzahl eine ganze Zahl ist, die genau dann null ist, wenn das entsprechende Linienbündel trivial ist, siehe z.B. [BT82].
Mit Hilfe dieser Funktion kann man das zum Blochbündel unitär äquivalente Linienbündel E_θ definieren als

$$E_\theta := \{(k, \lambda)_\sim \in \mathbb{R}^2 \times \mathbb{C}\},$$

$$\text{wobei} \quad (k, \lambda) \sim (k', \lambda') \Leftrightarrow k' = k - \gamma^* \quad \text{und} \quad \lambda' = e^{\frac{i\theta}{2\pi}k_2\gamma_1^*}\lambda.$$

Der gewünschte Referenzraum ist dann der Raum der L^2-Schnitte in diesem Bündel, also der Raum

$$\mathcal{H}_\theta = \{\psi \in L^2_{\mathrm{loc}}(\mathbb{R}^2) : \psi(k - \gamma^*) = e^{\frac{i\theta}{2\pi}\gamma_1^* k_2}\psi(k) \quad \forall \gamma^* \in \Gamma^*\}.$$

Natürlich sind $\mathcal{H}_\theta$ und $\Pi^0\mathcal{H}_\tau$ unitär äquivalent.
Die erste Idee für die Konstruktion von U^ε könnte sein, als Hauptsymbol $u_0(k, r) := \langle\varphi(k - A(r))| + u_0^\perp$ zu wählen. Aber im Gegensatz zum nicht-magnetischen Fall ist diese Funktion in keiner geeigenten Symbolklasse, da sie keine beschränkten Ableitungen mehr hat. Deshalb vernachlässigen wir zunächst, dass der Referenzraum ein möglichst einfacher Raum sein soll. Stattdessen konzentrieren wir uns zunächst darauf, die ε-Abhängigkeit des fast invarianten Unterraums zu beseitigen. Mit Hilfe der Funktion φ definieren wir eine unitäre Abbildung $U_1^\varepsilon = \widehat{u} + \mathcal{O}(\varepsilon^\infty)$, welche $\Pi^\varepsilon\mathcal{H}_\tau$ auf $\Pi^0\mathcal{H}_\tau$ abbildet. Das Hauptsymbol von u ist $u_0(k, r) := |\varphi(k)\rangle\langle\varphi(k - A(r))| + u_0^\perp$. Diese Funktion ist (mit einer geeigneten Eichung für A

oder einer zusätzliche Phase) τ-äquivariant und hat daher beschränkte Ableitungen.
Die unitäre Abbildung $U^\theta : \Pi^0\mathcal{H}_\tau \to \mathcal{H}_\theta$ zwischen den Schnitten im Blochbündel und den Schnitten in E_θ ist dann durch $\langle\varphi(k)|$ gegeben. Also setzen wir als verknüpfende unitäre Abbildung

$$U^\varepsilon := U^\theta \circ U_1^\varepsilon.$$

Das Problem ist, dass U^θ kein Pseudodifferentialoperator ist. Es ist offensichtlich, dass das Symbol $u(k,r) = \langle\varphi(k)|$ sein müsste, man aber die Ableitungen dieser Funktion nicht kontrollieren kann. Bis hierher erhalten wir also als effektiven Hamiltonian

$$\begin{aligned} H_{\text{eff}} &= U^\varepsilon\Pi^\varepsilon H_{\text{BF}}^\varepsilon\Pi^\varepsilon U^{\varepsilon *} = U^\theta U_1^\varepsilon\Pi^\varepsilon H_{\text{BF}}^\varepsilon\Pi^\varepsilon U_1^{\varepsilon *}U^{\theta *} = U^\theta\widehat{u}\widehat{\pi}\widehat{H}\widehat{\pi}\widehat{u}^* U^{\theta *} + \mathcal{O}(\varepsilon^\infty) \\ &= U^\theta \mathrm{Op}(u\sharp\pi\sharp H\sharp\pi\sharp u^*)U^{\theta *} + \mathcal{O}(\varepsilon^\infty) \\ &= U^\theta\widehat{h}U^{\theta *} + \mathcal{O}(\varepsilon^\infty). \end{aligned}$$

Man kann hier also nicht wie im nicht-magnetischen Fall die unitäre Abbildung via Weyl-Produkt zum Symbol hinzuzufügen, da U^θ kein Symbol hat. Deshalb müssen wir eine andere Methode finden, die es ermöglicht $U^\theta\widehat{h}U^{\theta *}$ als Pseudodifferentialoperator darzustellen.
Dies ist der nächste wichtige Schritt dieser Arbeit. Wir müssen einen Weg finden, die nicht-triviale Geometrie des Blochbündels in den Quantisierungsvorgang einzubinden. Dazu ist zu beachten, dass der Zusammenhang des Bündels nicht der triviale Zusammenhang ∇_k, sondern der Berryzusammenhang $\nabla_k^{\text{B}} = P(k)\nabla_k$ ist. Deshalb werden wir nun Quantisierungen definieren, die für ein Symbol $f(k,r)$ zwar immer noch k auf k abbilden, aber r nicht mehr mit dem trivialen Zusammenhang $-i\varepsilon\nabla_k$, sondern mit anderen Zusammenhängen ersetzen, wie zum Beispiel mit $-i\varepsilon\nabla_k^{\text{B}}$. Desweiteren müssen diese neuen Quantisierungen Operatoren zwischen Schnitten von möglicherweise nicht-trivialen Bündeln erzeugen. Dies is nötig, da wir auf einen Pseudodifferentialoperator zwischen Schnitten eines Bündels mit einem nicht-trivialen Zusammenhang abzielen. Die benötigten Bündel und Zusammenhänge sind

- die Bündel E und E_{Bl} versehen mit dem Berry-Zusammenhang ∇^{B},
- das Bündel E_θ versehen mit dem θ-Zusammenhang $\nabla^\theta = U^\theta\nabla^{\text{B}}U^{\theta *}$ und
- das Bündel E_θ versehen mit dem effektiven Zusammenhang $\nabla^{\text{eff}} = (\nabla_k + (0, \frac{i\theta}{2\pi}k_1)^{\text{T}})$.

Also sind die drei Weyl-Quantisierungen

- die Berry-Quantisierung $\mathrm{Op}^{\mathrm{B}}(f) = f(k, -i\varepsilon\nabla_k^{\mathrm{B}})$, die Operatoren auf $\mathcal{H}_\tau$ beziehungsweise $\Pi^0\mathcal{H}_\tau$ erzeugt,
- die θ-Quantisierung $\mathrm{Op}^\theta(f) = f(k, -i\varepsilon\nabla_k^\theta)$, die Operatoren auf $\mathcal{H}_\theta$ erzeugt, und
- die effektive Quantisierung $\mathrm{Op}^{\mathrm{eff}}(f) = f(k, -i\varepsilon(\nabla_k + (0, \frac{i\theta}{2\pi}k_1)^{\mathrm{T}})$, die Operatoren auf $\mathcal{H}_\theta$ erzeugt.

Diese Quantisierungen werden definiert, indem man die entsprechenden Paralleltransporte in die Integralformeln der Quantisierungen einfügt.
Die Motivation für die ersten zwei Quantisierungen ist

$$U^\theta \Pi^0 \widehat{r_j}^\tau \Pi^0 U^{\theta *} = U^\theta \widehat{r_j}^{\mathrm{B}} U^{\theta *} = \widehat{r_j}^\theta.$$

Wir werden also zunächst von der τ- zur Berry-Quantisierung übergehen. Danach zeigen wir, wie es möglich ist, die unitäre Abbildung U^θ in die Berry-Quantisierung "hineinzuziehen", indem man zur θ-Quantisierung übergeht. Dies ist der wesentliche Schritt um einen Pseudodifferentialoperator auf $\mathcal{H}_\theta$ zu bekommen. Danach gehen wir außerdem noch zur effektiven Quantisierung über, weil der Zusammenhang ∇^{eff} unabhängig von φ ist und für $\theta = 0$ mit dem trivialen Zusammenhang ∇_k übereinstimmt. Dies ermöglicht es, unsere Ergebnisse mit dem nicht-magnetischen Fall $A_0 \equiv 0$ zu vergleichen beziehungsweise diesen Fall einzuschließen.
In Kapitel vier werden diese drei Quantisierungen mathematisch rigoros definiert und zueinander in Beziehung gesetzt. Letzteres bedeutet zu zeigen, wie man ein Symbol f zu einem Symbol f_{c} abändern muss, sodass

$$\widehat{f}^{\,\tau} = \widehat{f_{\mathrm{c}}}^{\mathrm{B}} + \mathcal{O}(\varepsilon^\infty)$$

gilt. Nachdem wir gezeigt haben, wie man die τ- in die Berry-Quantisierung umrechnet, nutzen wir als nächstes die unitäre Äquivalenz von ∇^{B} und ∇^θ aus, um einen Operator der Form $U^\theta \widehat{f}^{\,\mathrm{B}} U^{\theta *}$ in einen der Form $\widehat{f_\theta}^{\theta}$ umzuschreiben. Das Symbol dieses Operators kann wiederum mit dem gleichen Vorgehen wie vorher umgerechnet werden, sodass

$$\widehat{(f_\theta)_{\mathrm{c}}}^{\mathrm{eff}} = \widehat{f_\theta}^{\theta} + \mathcal{O}(\varepsilon^\infty)$$

gilt. Dieses Programm kann also folgendermaßen zusammengefasst werden: Wir beginnen mit

$$H_{\mathrm{eff}} = U^\varepsilon \Pi^\varepsilon \widehat{H}^\tau \Pi^\varepsilon U^{\varepsilon *} = U^\theta \Pi^0 \widehat{h}^{\,\tau} U^{\theta *} = \widehat{h_{\mathrm{eff}}}^{?}$$

und lösen dieses Problem durch

$$U^\theta \Pi^0 \widehat{h}^{\,\tau} U^{\theta *} = U^\theta \widehat{h_{\mathrm{c}}}^{\mathrm{B}} U^{\theta *} = \widehat{(h_{\mathrm{c}})_\theta}^{\theta} = \widehat{h_{\mathrm{eff}}}^{\mathrm{eff}}$$

(wobei Gleichheit bis auf $\mathcal{O}(\varepsilon^\infty)$ gilt).
Dies wenden wir nun auf unseren Hamiltonian $H^\varepsilon_{\mathrm{BF}}$ an und bekommen das folgende Ergebnis (hier lassen wir die genauen technischen Voraussetzungen weg - sie stehen in Theorem 5.1.1).

Theorem. Sei E ein nicht degeneriertes, isoliertes Eigenwertband. Dann gibt es

(i) eine orthogonale Projektion $\Pi^\varepsilon \in \mathcal{L}(\mathcal{H}_\tau)$,

(ii) eine unitäre Abbildung $U^\varepsilon \in \mathcal{L}(\Pi^\varepsilon \mathcal{H}_\tau, \mathcal{H}_\theta)$ und

(iii) einen selbstadjungierten Operator $\widehat{h_{\mathrm{eff}}}^{\mathrm{eff}} \in \mathcal{L}(\mathcal{H}_\theta)$,

sodass
$$\|[H^\varepsilon_{\mathrm{BF}}, \Pi^\varepsilon]\|_{\mathcal{L}(\mathcal{H}_\tau)} = \mathcal{O}(\varepsilon^\infty)$$
und
$$\left\|(e^{-iH^\varepsilon_{\mathrm{BF}}t} - U^{\varepsilon *} e^{-i\widehat{h_{\mathrm{eff}}}^{\mathrm{eff}} t} U^\varepsilon)\Pi^\varepsilon\right\|_{\mathcal{L}(\mathcal{H}_\tau)} = \mathcal{O}(\varepsilon^\infty(1+|t|)).$$
Der effektive Hamiltonian ist die effektive Quantisierung des Symbols $h_{\mathrm{eff}} \in S^1_{\tau\equiv 1}(\varepsilon, \mathbb{C})$, welches in jeder Ordnung berechnet werden kann.

Desweiteren berechnen wir (in Theorem 5.2.4) die führenden Ordnungen des Symbols explizit:

Theorem. Das Haupt- und Subhauptsymbol des Symbols $h_{\mathrm{eff}} = h_0 + \varepsilon h_1 + \mathcal{O}(\varepsilon^2)$ aus dem obigen Theorem sind

$$h_0(k,r) = E(k - A(r)) + \Phi(r)$$

und

$$h_1(k,r) = (\nabla\Phi(r) - \nabla E(\widetilde{k}) \times B(r)) \cdot (i\mathcal{A}_1(\widetilde{k}), i(\mathcal{A}_2(\widetilde{k}) - \tfrac{i\theta}{2\pi}k_1))^{\mathrm{T}} - B(r)\cdot\mathcal{M}(\widetilde{k}),$$

wobei $\mathcal{A}_j(k) = \langle\varphi(k), \partial_j\varphi(k)\rangle_{\mathcal{H}_{\mathrm{f}}}$, $\widetilde{k} = k - A(r)$, $B(r) = \partial_1 A_2 - \partial_2 A_1$ und $\mathcal{M}(\widetilde{k})$ der Rammal-Wilkinson-Term ist.

Am Ende der Arbeit bringen wir unsere Resultate mit dem Hofstadter-Modell in Verbindung.

Chapter 1

Introduction

In this work we consider a one-particle Schrödinger equation with a periodic potential, a strong constant magnetic field, and non-periodic perturbations in two dimensions. Our goal is a rigorous derivation of an effective model called effective Hamiltonian that captures the main features of the equation.
The model we want to analyse describes the movement of conduction electrons in a crystalline solid in the potential created by the atomic cores, for instance electrons in some metal or crystal. For a lot of questions, it is a standard approximation in solid state physics to neglect the Coulomb repulsion between the electrons. Hence, it suffices to look at the behaviour of a single particle under the influence of a periodic potential. Let $\Gamma \cong \mathbb{Z}^2$ be the Bravais lattice generated by the nuclei. This means that we assume the nuclei to be on fixed positions. Then the mathematical model for this problem is given (with suitable physical units) by the Hamiltonian

$$H^\varepsilon = \tfrac{1}{2}(-i\nabla_x - A_0(x) - A(\varepsilon x))^2 + V_\Gamma(x) + \Phi(\varepsilon x). \tag{1.1}$$

Under suitable technical conditions, this is a self-adjoint operator defined on the magnetic Sobolev space $\mathrm{D}(H^\varepsilon) = H^2_{A_0}(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$. Here, V_Γ is the crystal potential which is periodic with respect to the Bravais lattice Γ and A_0 is the vector potential of a strong constant magnetic field $B = \mathrm{d}A_0$. Moreover, we also take into account some non-periodic perturbations A and Φ. The potentials $A = A(\varepsilon x)$ and $\Phi = \Phi(\varepsilon x)$ are assumed to be slowly varying on the scale of the lattice Γ. Here ε is a dimensionless parameter and A and Φ are independent of ε. The perturbations are assumed to be smooth and bounded together with all their derivatives. The corresponding weak electromagnetic fields are $\mathcal{B}(x) = \mathrm{curl} A(x)$ respectively $\mathcal{E}(x) = -\nabla\Phi(x)$. The unperturbed operator is denoted by

$$H_{\mathrm{MB}} = \tfrac{1}{2}(-i\nabla_x - A_0)^2 + V_\Gamma. \tag{1.2}$$

After a magnetic Bloch-Floquet transformation, this operator is given as the Weyl quantisation of an operator-valued symbol $H_{\mathrm{per}}(k)$. For fixed k, this symbol acts on some Hilbert space $\mathcal{H}_{\mathrm{f}}$ and has discrete spectrum. The eigenvalues are denoted by $E_1(k) \leq E_2(k) \leq \ldots$ and the resulting eigenvalue functions $(E_n)_{n\geq 1}$ of the transformed Hamiltonian are called magnetic Bloch bands. Our aim is to understand in detail the behaviour of particles governed by the Hamiltonian H^ε for $\varepsilon \ll 1$. Thereto we want to get effective models associated to a magnetic Bloch band. To keep this introduction as clear as possible, we state our results for the case $\Gamma = \mathbb{Z}^2$. In [Teu03, PST03b] the non-magnetic case, that is to say the case where $A_0 \equiv 0$, is discussed in detail. Our goal is to generalise the methods of the aforementioned works to the magnetic case. The result in the non-magnetic case is that for an isolated, non-degenerate Bloch band E one can define a so-called almost invariant subspace associated to this Bloch band E so that restricted to this subspace, the operator H^ε can be described by the effective model

$$\widehat{h_{\mathrm{eff}}} = E(k - A(i\varepsilon\nabla_k)) + \Phi(i\varepsilon\nabla_k) + \mathcal{O}(\varepsilon),$$

where

$$\widehat{h_{\mathrm{eff}}} \quad \text{acts on} \quad L^2(\mathbb{T}^{2*})$$

and the " $\widehat{\ }$ " indicates that $\widehat{h_{\mathrm{eff}}}$ is the Weyl quantisation of the symbol h_{eff}. Moreover, $\mathbb{T}^{2*} = \mathbb{R}^2/\Gamma^*$, where Γ^* is the dual lattice of Γ. The representation $E(k - A(i\varepsilon\nabla_k)) + \Phi(i\varepsilon\nabla_k)$ is called Peierls substitution.

In the magnetic case, we again aim for a rigorous derivation of an effective model for H^ε given as a pseudodifferential operator with a Peierls substitution type operator in the leading order. However, the inclusion of the potential A_0 in the Hamiltonian H^ε causes some differences between our results and the results in the non-magnetic case. On the subspace corresponding to the magnetic Bloch band E, the effective model is

$$\widehat{h_{\mathrm{eff}}}^{\mathrm{eff}} = E(k - A(i\varepsilon\nabla_k^{\mathrm{eff}})) + \Phi(i\varepsilon\nabla_k^{\mathrm{eff}}) + \mathcal{O}(\varepsilon),$$

where $\widehat{h_{\mathrm{eff}}}^{\mathrm{eff}}$ acts on

$$\mathcal{H}_\theta = \{\psi \in L^2_{\mathrm{loc}}(\mathbb{R}^2) : \psi(k - \gamma^*) = e^{\frac{i\theta}{2\pi}\gamma_1^* k_2}\psi(k) \quad \forall \gamma^* \in \Gamma^*\}$$

and

$$\nabla_k^{\mathrm{eff}} = \nabla_k + (0, \tfrac{i\theta}{2\pi}k_1)^{\mathrm{T}}.$$

Here θ is the Chern number of a certain line bundle (the Bloch bundle) and hence an integer. So we still get a Peierls substitution type operator, but in contrast to the non-magnetic case, we need to use a different quantisation that maps a symbol $f(k,r)$ to the operator $f(k, i\varepsilon\nabla_k^{\mathrm{eff}})$ and thus inserts a non-trivial connection. It is a main part of this thesis to define this new quantisation. Moreover, the effective operator no longer acts on functions over the torus but on sections of a possibly non-trivial line bundle.

The main reason for this is that for an isolated, non-degenerate eigenvalue band E, the effective Hamiltonian is always an operator on sections of a line bundle over the torus called Bloch bundle. This vector bundle is associated to the spectral projection P (the spectral projection associated to the Bloch band E), and is always trivial in the non-magnetic case. Hence in the non-magnetic case, it is isomorphic to $\mathbb{T}^{2*} \times \mathbb{C}$ and the sections are just functions from $\mathbb{T}^{2*}$ to $\mathbb{C}$. Thus in this case, one can neglect that the effective Hamiltonian is an operator on sections of a line bundle and treat it like an operator between function spaces. However, in the magnetic case, the inclusion of A_0 breaks the time-reversal symmetry of the unperturbed operator H_{MB} and therewith the trivialisability of the Bloch bundle. Thus we cannot ignore that we get an operator on sections of a line bundle. The simplest space for the effective operator to act on is the space $\mathcal{H}_\theta$ that consists of functions from $\mathbb{R}^2$ to $\mathbb{C}$ that, however, must fulfil a twisted periodic boundary condition. Applying these results to the non-magnetic case yields $\theta = 0$ and $\mathcal{H}_{\theta=0} \cong L^2(\mathbb{T}^{2*})$ as well as $\nabla^{\mathrm{eff}} = \nabla$. So this thesis also reproduces the non-magnetic case.
As in [PST03b, Teu03], the method we want to apply and generalise is space-adiabatic perturbation theory - a method developed in [PST03a]. The main mathematical tool used in this theory are pseudodifferential operators with operator-valued symbols. A pseudodifferential operator H is the quantisation of a symbol $h \in C^\infty(\mathbb{R}^4, \mathcal{L}(\mathcal{H}))$ with $H = h(k, -i\varepsilon\nabla_k) = \widehat{h}$. Recall that a symbol is always a smooth function and one always has some kind of control over the derivatives, like the existence of a (order) function w so that for all α, β there is a constant $C_{\alpha,\beta}$ so that for all $k, r \in \mathbb{R}^2$

$$\left\|(\partial_k^\alpha \partial_r^\beta h)(k,r)\right\|_{\mathcal{L}(\mathcal{H})} \leq C_{\alpha,\beta} w(k,r).$$

The most important property of pseudodifferential operators is that one can define a product $\sharp$ on the space of symbols that corresponds to the composition of operators, i.e. for two pseudodifferential operators $A = \widehat{a}$ and $B = \widehat{b}$ we have $AB = \widehat{a \sharp b}$. Often it is easier to construct an operator on the level of symbols first and to quantise it afterwards.
To apply space-adiabatic perturbation theory, the Hamiltonian in question must be given as a pseudodifferential operator. The Hamiltonian (1.1) can be put into this framework using a magnetic Bloch-Floquet transform $\mathcal{U}_{\mathrm{BF}}$ which is basically a Fourier transformation on the lattice Γ. To define the transformation, we need the additional condition $B|M| \in 2\pi\mathbb{Z}$, where M is the fundamental cell of the lattice Γ. (Note that it is sufficient to take $B|M| \in \pi\mathbb{Q}$ and then pass to a sublattice $\widetilde{\Gamma} \subset \Gamma$ that fulfils $B|\widetilde{M}| \in 2\pi\mathbb{Z}$.) Let M^* be the first Brillouin zone of the lattice Γ that is to say the fundamental cell of the dual lattice Γ^*. Then, $\mathcal{U}_{\mathrm{BF}}$ maps the space $L^2(\mathbb{R}^2)$ to the space $\mathcal{H}_\tau \cong L^2(M^*) \otimes \mathcal{H}_{\mathrm{f}}$, where $\mathcal{H}_{\mathrm{f}}$ is a separable Hilbert

space, and

$$\mathcal{H}_\tau = \{\varphi \in L^2_{\mathrm{loc}}(\mathbb{R}^2, \mathcal{H}_\mathrm{f}) : \varphi(k-\gamma^*) = \tau(\gamma^*)\varphi(k) \quad \forall \gamma^* \in \Gamma^*\},$$

where τ is a unitary representation of Γ^* in $\mathcal{L}(\mathcal{H}_\mathrm{f})$. This transforms the unperturbed operator (1.2) into

$$\mathcal{U}_{\mathrm{BF}} H_{\mathrm{MB}} \mathcal{U}^*_{\mathrm{BF}} = \int^{\oplus}_{M^*} H_{\mathrm{per}}(k)\mathrm{d}k,$$

where

$$H_{\mathrm{per}}(k) := \tfrac{1}{2}(-i\nabla_y - A_0(y) + k)^2 + V_\Gamma(y).$$

The full operator (1.1) transforms as

$$\mathcal{U}_{\mathrm{BF}} H^\varepsilon \mathcal{U}^*_{\mathrm{BF}} =: H^\varepsilon_{\mathrm{BF}} = \widehat{H_0(k,r)},$$

where

$$H_0(k,r) = H_{\mathrm{per}}(k - A(r)) + \Phi(r).$$

Another requirement to apply space-adiabatic perturbation theory is that one needs some gap in the spectrum of the principal symbol of the Hamiltonian $\widehat{H}$. Throughout this work, we assume that $E(k)$ is a non-degenerate, isolated eigenvalue of $H_{\mathrm{per}}(k)$ with spectral projection $P(k)$. Hence $E(k-A(r))+\Phi(r)$ is a a non-degenerate, isolated eigenvalue of $H_0(k,r)$ with spectral projection $P(k-A(r))$.
The first step of space-adiabatic perturbation theory is the construction of the so-called almost invariant subspaces. This construction carries over from the non-magnetic case with only small technical modifications, which means that we get a projection $\Pi^\varepsilon = \widehat{\pi} + \mathcal{O}(\varepsilon^\infty)$ that is nearly a pseudodifferential operator with principal symbol $\pi_0(k,r) = P(k-A(r))$. The space $\Pi^\varepsilon \mathcal{H}_\tau$ is called "almost invariant subspace" since $[H^\varepsilon_{\mathrm{BF}}, \Pi^\varepsilon] = \mathcal{O}(\varepsilon^\infty)$ holds.
However, this subspace is not easily accessible at all and we cannot describe the functions it contains explicitly. Moreover, it depends on ε and it is clear that there is no limit for $\varepsilon \to 0$. Hence the next step of space-adiabatic perturbation theory is to unitarily map this space to a space $\mathcal{H}_{\mathrm{ref}}$ that does not depend on ε and additionally is given explicitly. In contrast to the definition of the almost invariant subspace, this step does not carry over from the non-magnetic case. The definition of the intertwining unitary U^ε and an appropriate reference space $\mathcal{H}_{\mathrm{ref}}$ are one of the main goals and the motivation of this work.
Let us explain this in more detail. Consider the following bundle over $\mathbb{T}^{2*}$:

$$E = (\mathbb{R}^2 \times \mathcal{H}_\mathrm{f})_\sim \quad \text{where} \quad (k,\varphi) \sim (k',\varphi') :\Leftrightarrow k' = k - \gamma^* \quad \text{and} \quad \varphi' = \tau(\gamma^*)\varphi$$

$$\text{with connection} \quad \nabla^{\mathrm{B}} = P(k)\nabla P(k) + P^\perp(k)\nabla P^\perp(k).$$

The connection is called "Berry connection" and $\mathcal{H}_\tau$ is the space of L^2-sections of this bundle. The Bloch bundle is a subbundle of E associated to the projection $P(k)$, that means

$$E_{\mathrm{Bl}} = \{(k,\varphi) \in (\mathbb{R}^2, \mathcal{H}_{\mathrm{f}})_\sim : \varphi \in P(k)\mathcal{H}_{\mathrm{f}}\} \quad \text{with connection} \quad \nabla_k^{\mathrm{B}} = P(k)\nabla_k.$$

The L^2-sections of the Bloch bundle are given by

$$\Pi^0\mathcal{H}_\tau := \{f \in \mathcal{H}_\tau : f(k) \in P(k)\mathcal{H}_{\mathrm{f}}\}.$$

In the non-magnetic case, the key ingredient for the construction of U^ε is that E_{Bl} is trivial and hence there is a function φ so that $\varphi(k-\gamma^*) = \tau(\gamma^*)\varphi(k)$ for all $k \in \mathbb{R}^2$ and $\varphi(k) \in P(k)\mathcal{H}_{\mathrm{f}}$ satisfying $\|\varphi(k)\|_{\mathcal{H}_{\mathrm{f}}} \equiv 1$. With the help of this function one can construct the intertwining unitary as $U^\varepsilon = \widehat{u} + \mathcal{O}(\varepsilon^\infty)$, that is to say nearly as pseudodifferential operator, with principal symbol $u_0(k,r) = \langle\varphi(k - A(r))| + u_0^\perp$. The reference space can be chosen as $L^2(\mathbb{T}^{2*})$ because this space is isomorphic to the space of L^2-sections of the Bloch bundle E_{Bl}. The fact that Π^ε and U^ε are nearly pseudodifferential operators is crucial to get the effective operator as a pseudodifferential operator since one now can exploit that the product "$\sharp$" on the level of symbols corresponds to the composition on the level of operators:

$$\begin{aligned} H_{\mathrm{eff}} &= U^\varepsilon \Pi^\varepsilon H_{\mathrm{BF}}^\varepsilon \Pi^\varepsilon U^{\varepsilon *} = \widehat{u}\widehat{\pi}\widehat{H}\widehat{\pi}\widehat{u^*} + \mathcal{O}(\varepsilon^\infty) = \mathrm{Op}(\underbrace{u\sharp\pi\sharp H\sharp\pi\sharp u^*}_{=:h}) + \mathcal{O}(\varepsilon^\infty) \\ &= \widehat{h} + \mathcal{O}(\varepsilon^\infty). \end{aligned}$$

So the construction directly gives the effective Hamiltonian as a pseudodifferential operator.

Let us go back to the magnetic case. The key point or better key problem is that we lose the trivialisability of the Bloch bundle. So we do not get a function φ as before - every global section φ of the Bloch bundle must have zeros. The approach in this work is to take a global non-zero section of the bundle

$$E'_{\mathrm{Bl}} = \{(k,\varphi) \in (\mathbb{R}^2, \mathcal{H}_{\mathrm{f}}) : \varphi \in P(k)\mathcal{H}_{\mathrm{f}}\},$$

which must be trivial because it is a bundle over $\mathbb{R}^2$. We construct such a section φ by using the parallel transport with respect to the Berry connection ∇^{B}. The result is (see Lemma 3.3.2 in the main text)

Lemma. *There is a function $\varphi \in C^\infty(\mathbb{R}^2, \mathcal{H}_{\mathrm{f}})$ so that for all $k \in \mathbb{R}^2$ we have $\varphi(k) \in P(k)\mathcal{H}_{\mathrm{f}}$, $\|\varphi(k)\|_{\mathcal{H}_{\mathrm{f}}} = 1$, and*

$$\varphi(k-\gamma^*) = e^{-\frac{i\theta}{2\pi}k_2\gamma_1^*}\tau(\gamma^*)\varphi(k) \quad \textit{for all} \quad \gamma^* \in \Gamma^*,$$

where θ is the Chern number of the Bloch bundle.

Recall that the Chern number is an integer that is zero if and only if the line bundle is trivial, see e.g. [BT82].
With this function at hand, we can define a line bundle E_θ that is unitarily equivalent to the Bloch bundle by

$$E_\theta := \{(k,\lambda)_\sim \in \mathbb{R}^2 \times \mathbb{C}\},$$

$$\text{where} \quad (k,\lambda) \sim (k',\lambda') \Leftrightarrow k' = k - \gamma^* \quad \text{and} \quad \lambda' = e^{\frac{i\theta}{2\pi} k_2 \gamma_1^*} \lambda.$$

The reference space we aim for is the space of L^2-sections of this line bundle, that is to say the space

$$\mathcal{H}_\theta = \{\psi \in L^2_{\text{loc}}(\mathbb{R}^2) : \psi(k-\gamma^*) = e^{\frac{i\theta}{2\pi}\gamma_1^* k_2}\psi(k) \quad \forall \gamma^* \in \Gamma^*\}.$$

Of course, $\mathcal{H}_\theta$ and $\Pi^0\mathcal{H}_\tau$ are unitarily equivalent.
For the construction of U^ε, the first idea is to define the principal symbol u_0 as $\langle\varphi(k - A(r))| + u_0^\perp$. But in contrast to the non-magnetic case, this function is in no suitable symbol class since it does not have bounded derivatives. Thus the next difference to the non-magnetic case is that we first neglect that we want the reference space to be the space that is as simple as possible and focus on getting rid of the ε in the almost invariant subspace. Thereto, we use the function φ to construct a unitary $U_1^\varepsilon = \widehat{u} + \mathcal{O}(\varepsilon^\infty)$ that maps $\Pi^\varepsilon\mathcal{H}_\tau$ to $\Pi^0\mathcal{H}_\tau$. The principal symbol of u is $u_0(k,r) := |\varphi(k)\rangle\langle\varphi(k-A(r))| + u_0^\perp$, which is (with an appropriate gauge for A or an additional phase) τ-equivariant and hence has bounded derivatives.
The unitary map between the sections of the Bloch bundle and the sections of the bundle E_θ is given by $U^\theta : \Pi^0\mathcal{H}_\tau \to \mathcal{H}_\theta$ defined as $\langle\varphi(k)|$. So we set

$$U^\varepsilon := U^\theta \circ U_1^\varepsilon$$

as the intertwining unitary. The problem is that the map U^θ is not a pseudo-differential operator. Obviously, the symbol would have to be $u(k,r) = \langle\varphi(k)|$, but the problem is that we cannot get any control over the derivatives of this function. Until now, we get as effective Hamiltonian

$$\begin{aligned} H_{\text{eff}} &= U^\varepsilon \Pi^\varepsilon H_{\text{BF}}^\varepsilon \Pi^\varepsilon U^{\varepsilon *} = U^\theta U_1^\varepsilon \Pi^\varepsilon H_{\text{BF}}^\varepsilon \Pi^\varepsilon U_1^{\varepsilon *} U^{\theta *} = U^\theta \widehat{u}\widehat{\pi}\widehat{H}\widehat{\pi}\widehat{u}^* U^{\theta *} + \mathcal{O}(\varepsilon^\infty) \\ &= U^\theta \mathrm{Op}(u \sharp \pi \sharp H \sharp \pi \sharp u^*) U^{\theta *} + \mathcal{O}(\varepsilon^\infty) \\ &= U^\theta \widehat{h} U^{\theta *} + \mathcal{O}(\varepsilon^\infty). \end{aligned}$$

Hence the procedure from the non-magnetic case to add the unitary map to the symbol via Weyl product does not work because there is no symbol for U^θ. Thus we need to think of other ways to make it possible to write $U^\theta \widehat{h} U^{\theta *}$ as a pseudo-differential operator.

This is the next main step in this thesis. We need to find a way to include the non-trivial geometry of the Bloch bundle into our quantisation procedure. Thereto note that the connection on the bundle is not the trivial connection ∇_k but the Berry connection $\nabla_k^{\mathrm{B}} = P(k)\nabla_k$. Our approach here is to define quantisations that, although they still map k to k for a symbol $f(k,r)$, no longer replace r with the trivial connection $-i\varepsilon\nabla_k$, but with other connections, for example $-i\varepsilon\nabla_k^{\mathrm{B}}$. Moreover, those new quantisations need to generate operators that act on sections of possibly non-trivial bundles. This is necessary since we aim for a pseudodifferential operator on sections of a bundle with a non-trivial connection. The bundles and connections we need are

- the bundles E and E_{Bl} with the Berry connection ∇^{B},
- the bundle E_θ with the θ-connection $\nabla^\theta = U^\theta \nabla^{\mathrm{B}} U^{\theta *}$, and
- the bundle E_θ with the effective connection $\nabla^{\mathrm{eff}} = (\nabla_k + (0, \frac{i\theta}{2\pi}k_1)^{\mathrm{T}})$.

Hence the three new Weyl quantisations are

- the Berry quantisation $\mathrm{Op}^{\mathrm{B}}(f) = f(k, -i\varepsilon\nabla_k^{\mathrm{B}})$ that generates operators on $\mathcal{H}_\tau$ respectively $\Pi^0\mathcal{H}_\tau$,
- the θ-quantisation $\mathrm{Op}^\theta(f) = f(k, -i\varepsilon\nabla_k^\theta)$ that generates operators on $\mathcal{H}_\theta$, and
- the effective quantisation $\mathrm{Op}^{\mathrm{eff}}(f) = f(k, -i\varepsilon(\nabla_k + (0, \frac{i\theta}{2\pi}k_1)^{\mathrm{T}})$ that generates operators on $\mathcal{H}_\theta$.

Those quantisations are defined by including the respective parallel transport maps in the integral formulas of the quantisations.
The motivation for the first two quantisations is

$$U^\theta \Pi^0 \widehat{r_j}^\tau \Pi^0 U^{\theta *} = U^\theta \widehat{r_j}^{\mathrm{B}} U^{\theta *} = \widehat{r_j}^\theta.$$

This means that we first pass from the τ-quantisation to the Berry quantisation. Then we show how it is possible to "absorb" the unitary map U^θ into the Berry-quantisation passing to the θ-quantisation. This is the main step to get a pseudodifferential operator on $\mathcal{H}_\theta$. After that, we even pass to the effective quantisation because the connection ∇^{eff} is independent of φ and coincides for $\theta = 0$ with the trivial connection ∇_k. This permits us to include respectively compare our results with the non-magnetic case $A_0 \equiv 0$.
The content of the fourth chapter is to rigorously define these quantisations and to link them together. This means that we show how we have to alter a symbol f to a symbol f_{c} so that

$$\widehat{f}^{\,\tau} = \widehat{f_{\mathrm{c}}}^{\mathrm{B}} + \mathcal{O}(\varepsilon^\infty)$$

holds. After we have linked the τ- to the Berry quantisation, we can exploit the unitary equivalence of ∇^{B} and ∇^{θ} to translate an operator $U^{\theta}\widehat{f}^{\,\mathrm{B}}U^{\theta *}$ into an operator $\widehat{f_\theta}^{\,\theta}$. Following the same ideas as before, the symbol of this operator can again be corrected so that

$$\widehat{(f_\theta)_{\mathrm{c}}}^{\,\mathrm{eff}} = \widehat{f_\theta}^{\,\theta} + \mathcal{O}(\varepsilon^\infty).$$

So one can summarise our program to get to the effective Hamiltonian as a pseudo-differential operator as follows: We started from

$$H_{\mathrm{eff}} = U^{\varepsilon}\Pi^{\varepsilon}\widehat{H}^{\tau}\Pi^{\varepsilon}U^{\varepsilon *} = U^{\theta}\Pi^{0}\widehat{h}^{\,\tau}U^{\theta *} = \widehat{h_{\mathrm{eff}}}^{\,?}$$

and solved this problem by

$$U^{\theta}\Pi^{0}\widehat{h}^{\,\tau}U^{\theta *} = U^{\theta}\widehat{h_{\mathrm{c}}}^{\,\mathrm{B}}U^{\theta *} = \widehat{(h_{\mathrm{c}})_\theta}^{\,\theta} = \widehat{h_{\mathrm{eff}}}^{\,\mathrm{eff}}$$

(where the equalities are up to $\mathcal{O}(\varepsilon^\infty)$).
Applying this program to our Hamiltonian $H^{\varepsilon}_{\mathrm{BF}}$ yields our result (here we do not state the exact technical conditions - they can be found in Theorem 5.1.1).

Theorem. Let E be a non-degenerate, isolated eigenvalue band. Then there exist

(i) an orthogonal projection $\Pi^{\varepsilon} \in \mathcal{L}(\mathcal{H}_\tau)$,

(ii) a unitary map $U^{\varepsilon} \in \mathcal{L}(\Pi^{\varepsilon}\mathcal{H}_\tau, \mathcal{H}_\theta)$, and

(iii) a self-adjoint operator $\widehat{h_{\mathrm{eff}}}^{\,\mathrm{eff}} \in \mathcal{L}(\mathcal{H}_\theta)$

such that

$$\|[H^{\varepsilon}_{\mathrm{BF}}, \Pi^{\varepsilon}]\|_{\mathcal{L}(\mathcal{H}_\tau)} = \mathcal{O}(\varepsilon^\infty)$$

and

$$\left\|(e^{-iH^{\varepsilon}_{\mathrm{BF}}t} - U^{\varepsilon *}e^{-i\widehat{h_{\mathrm{eff}}}^{\,\mathrm{eff}}t}U^{\varepsilon})\Pi^{\varepsilon}\right\|_{\mathcal{L}(\mathcal{H}_\tau)} = \mathcal{O}(\varepsilon^\infty(1+|t|)).$$

The effective Hamiltonian is the effective quantisation of the symbol $h_{\mathrm{eff}} \in S^1_{\tau\equiv 1}(\varepsilon, \mathbb{C})$ which can be computed to any order.

We also compute (in Theorem 5.2.4) the leading orders of the symbol:

Theorem. The principal and subprincipal symbol of the symbol $h_{\mathrm{eff}} = h_0 + \varepsilon h_1 + \mathcal{O}(\varepsilon^2)$ from the theorem above are

$$h_0(k,r) = E(k - A(r)) + \Phi(r)$$

and

$$h_1(k,r) = (\nabla\Phi(r) - \nabla E(\widetilde{k}) \times B(r)) \cdot (i\mathcal{A}_1(\widetilde{k}), i(\mathcal{A}_2(\widetilde{k}) - \tfrac{i\theta}{2\pi}k_1))^{\mathrm{T}} - B(r) \cdot \mathcal{M}(\widetilde{k}),$$

where $\mathcal{A}_j(k) = \langle \varphi(k), \partial_j \varphi(k) \rangle_{\mathcal{H}_\mathrm{f}}$, $\widetilde{k} = k - A(r)$, $B(r) = \partial_1 A_2 - \partial_2 A_1$, and $\mathcal{M}(\widetilde{k})$ is the Rammal-Wilkinson term.

We conclude this work with some comments on the connection of our results with the Hofstadter model.
This thesis is organised as follows: In Chapter 2 we introduce the setting and the strategy of space-adiabatic perturbation theory and summarise the construction of the almost invariant subspace. Moreover, we state conditions under which the subspace coincides with the corresponding spectral subspace. Chapter 3 is dedicated to the intertwining unitary. The new Weyl quantisations are the content of Chapter 4. Finally, the effective dynamics of our model are treated in Chapter 5, where we also connect our results with the Hofstadter model.

Chapter 2

Space-adiabatic perturbation theory

2.1 The model

In this work, we consider a one-particle Schrödinger equation with a periodic potential, a strong constant magnetic field, and non-periodic perturbations. Our goal is a rigorous derivation of an effective model called effective Hamiltonian that captures the main features of the equation. In a crystalline solid, we consider the problem of conduction electrons moving in the potential created by the atomic cores, for instance electrons in some metal or crystal. For many questions, it is a standard approximation in solid state physics to neglect the Coulomb repulsion between the electrons. This leads us to study one fundamental problem of solid state physics: We want to understand the behaviour of a single particle under the influence of a periodic potential. Taking into account the preceding considerations, it is clear that this includes a grasp of the behaviour of an ideal fermi gas. For a more detailed introduction we refer the reader to [AM76].
Let us look at this problem more closely. Let Γ_0 be the Bravais lattice generated by the nuclei. This means that we assume the nuclei to be on fixed positions. Then it is possible to chose a basis $\{\gamma^1, \gamma^2\}$ of $\mathbb{R}^2$ so that

$$\Gamma_0 = \{\gamma = \sum_{j=1}^{2} \lambda_j \gamma^j, \quad \text{where} \quad \lambda \in \mathbb{Z}^2\}.$$

The crystal potential V_{Γ_0} is then periodic with respect to Γ_0 which means that $V_{\Gamma_0} : \mathbb{R}^2 \to \mathbb{R}$ satisfies $V_{\Gamma_0}(x + \gamma) = V_{\Gamma_0}(x)$ for all $\gamma \in \Gamma_0$. If there are no other external forces involved, the one-particle Hamiltonian reads

$$H_{\mathrm{Bl}} = -\tfrac{1}{2}\Delta + V_{\Gamma_0} \tag{2.1}$$

and is defined on the Sobolev space $\mathrm{D}(H_{\mathrm{Bl}}) = H^2(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$. To simplify matters here and in the following, we have chosen physical units so that $m_e =$

$\hbar = c = 1$, where m_e denotes the electron mass, $\hbar$ the reduced Planck constant, and c the speed of light. The purely periodic operator (2.1) can be diagonalised via Bloch-Floquet transformation. The eigenvalue bands E of the transformed Hamiltonian are called Bloch bands. However, we want to study the dynamics of particles in solids under the influence of external non-periodic forces. Laboratory-generated magnetic fields can produce forces comparable with $-\nabla V_{\Gamma_0}$. Hence, the associated vector potential A is split up into $A = A_0 + A_{\text{ext}}$, where A_0 is the vector potential of a strong constant magnetic field $B = \mathrm{d}A_0$. This means that we regard $A_0 \in \Omega^1(\mathbb{R}^2)$ as a one-form and $B = \mathrm{d}A_0 \in \Omega^2(\mathbb{R}^2)$ as its exterior derivative. We choose the symmetric gauge $A_0(x) = (A_1(x), A_2(x)) = \frac{B}{2}(-x_2, x_1)$. Adding A_0 to our Hamiltonian (2.1), the magnetic Bloch Hamiltonian reads

$$H_{\text{MB}} = \tfrac{1}{2}(-i\nabla_x - A_0)^2 + V_{\Gamma_0}. \tag{2.2}$$

We take this as our unperturbed Hamiltonian again defined on a suitable domain $\mathrm{D}(H_{\text{MB}}) \subset L^2(\mathbb{R}^2)$. Due to the linearity of A_0, this domain is different from the domain $\mathrm{D}(H_{\text{Bl}})$ of the non-magnetic Bloch Hamiltonian (2.1). The Sobolev space has to be replaced by a magnetic Sobolev space $H^2_{A_0}(\mathbb{R}^2)$, see Appendix A.

We do not just want to study the Hamiltonian (2.2), but we also want to take into account some non-periodic perturbations A and Φ. The potential $A_{\text{ext}}(x) = A(\varepsilon x)$ is assumed to be slowly varying on the scale of the lattice Γ_0. Here ε is a dimensionless parameter and A is independent of ε. For this weak perturbation we choose the gauge $A(x) = (A_1(x), 0)$. Also the laboratory-generated electrostatic potentials vary slowly on the lattice scale of Γ_0. Thus, we set $\Phi_{\text{ext}}(x) = \Phi(\varepsilon x)$, where again ε is a dimensionless parameter and Φ is independent of ε. After including all electromagnetic potentials, the full Hamiltonian H^ε finally reads

$$H^\varepsilon = \tfrac{1}{2}(-i\nabla_x - A_0(x) - A(\varepsilon x))^2 + V_{\Gamma_0}(x) + \Phi(\varepsilon x) \tag{2.3}$$

with domain $H^2_{A_0}(\mathbb{R}^2)$. Our purpose is a detailed comprehension of the behaviour of particles governed by the Hamiltonian (2.3) for $\varepsilon \ll 1$.

The method we want to use is space-adiabatic perturbation theory, see [PST03a, PST03b, Teu03]. Thereto we must generalise the methods developed in [PST03b] and [Teu03] to the case of a strong magnetic field with potential $A_0 \neq 0$. We aim for a rigorous derivation of an effective Hamiltonian. Thus we first have to construct subspaces which are almost invariant under the time evolution generated by the Hamiltonian (2.3). Restricted to these subspaces, we want to show that the Hamiltonian is unitarily equivalent to suitable simpler effective operators (up to an error in powers of ε). The basic idea of this procedure is well known in physics under the name "Peierls substitution", see [Pei33].

The main obstruction we have to overcome is that the obtained effective operators are no longer operating on L^2-functions over the torus, but on sections of a non-trivial line bundle over the torus (in the case of an isolated non-degenerate eigenvalue band).

Let us introduce some technical assumptions. The perturbations are assumed to be smooth and bounded together with all their derivatives which means $A \in C_{\mathrm{b}}^{\infty}(\mathbb{R}^2, \mathbb{R}^2)$ and $\Phi \in C_{\mathrm{b}}^{\infty}(\mathbb{R}^2, \mathbb{R})$. The corresponding weak electromagnetic fields are $\mathcal{B}(x) = \mathrm{curl} A(x)$ respectively $\mathcal{E}(x) = -\nabla\Phi(x)$.

To ensure the self-adjointness of the Hamiltonian (2.3), we need V_{Γ_0} to be relatively $(-i\nabla + A_0)^2$-bounded with relative bound smaller than 1. This is shown in Appendix A, Proposition A.0.8. These technical assumptions are summarised as

Assumption 1. *Assume that $A \in C_{\mathrm{b}}^{\infty}(\mathbb{R}^2, \mathbb{R}^2)$, $\Phi \in C_{\mathrm{b}}^{\infty}(\mathbb{R}^2, \mathbb{R})$ and that V_{Γ_0} is relatively $(-i\nabla + A_0)^2$-bounded with relative bound smaller than* 1.

We will always make this assumption in the following if we do not explicitly state that we do not.

Remark 2.1.1. Note that if V_{Γ_0} is $-\Delta$-bounded with relative bound smaller than 1, this implies (see [AHS78], Theorem 2.4) that it is relatively $(-i\nabla + A_0)^2$-bounded with relative bound smaller than 1.

Let us take a closer look at the differences of the Hamiltonians (2.1) and (2.2). While the non-magnetic Hamiltonian (2.1) commutes with the ordinary lattice translations, this is not the case for the magnetic Hamiltonian (2.2). Yet for the magnetic Hamiltonian, there is an analogon of the lattice transformations due to [Zak68] called magnetic translations. Often they are defined as

$$\widetilde{T}_\gamma \psi(x) := e^{-i\langle A_0(x), \gamma\rangle} \psi(x - \gamma).$$

An easy calculation shows that for this translation, $[H_{\mathrm{MB}}, T_\gamma] = 0$ holds for all $\gamma \in \Gamma_0$. However, unlike the ordinary lattice translations, these magnetic translations do in general not commute with each other:

$$\widetilde{T}_\gamma \widetilde{T}_{\widetilde{\gamma}} \psi(y) = e^{i\langle A_0(\gamma), \widetilde{\gamma}\rangle} \widetilde{T}_{\gamma + \widetilde{\gamma}}.$$

They only commute if

$$B(-\gamma_2\widetilde{\gamma}_1 + \gamma_1\widetilde{\gamma}_2) =: B(\gamma \wedge \widetilde{\gamma}) \in 2\pi\mathbb{Z} \tag{2.4}$$

for all γ, $\widetilde{\gamma} \in \Gamma_0$. To get general commutativity, we have to make the assumption

$$B(\gamma^1 \wedge \gamma^2) \in \pi\mathbb{Q}$$

because then it is possible to choose a sublattice $\Gamma \subset \Gamma_0$ so that (2.4) holds for all γ, $\widetilde{\gamma} \in \Gamma$. This implies

$$\widetilde{T}_\gamma \widetilde{T}_{\widetilde{\gamma}} = \widetilde{T}_{\widetilde{\gamma}} \widetilde{T}_\gamma = \pm \widetilde{T}_{\gamma + \widetilde{\gamma}}.$$

So if we additionally wanted $\{\widetilde{T}_\gamma, \gamma \in \Gamma\}$ to form a group representation, we would have to demand $B(\gamma \wedge \widetilde{\gamma}) \in \mathbf{4}\pi\mathbb{Z}$ for all γ, $\widetilde{\gamma} \in \Gamma$. To avoid this, we choose a slightly different definition of the magnetic translations that form a group already if (2.4) holds for all γ, $\widetilde{\gamma} \in \Gamma$. For $\gamma = a\gamma^1 + b\gamma^2 \in \Gamma_0$ with a, $b \in \mathbb{Z}$, we set

$$T_\gamma := (\widetilde{T}_{\gamma^2})^b (\widetilde{T}_{\gamma^1})^a,$$

which yields

$$T_\gamma \psi(x) := e^{-iab\langle A_0(\gamma^1),\gamma^2\rangle} e^{-i\langle A_0(x),\gamma\rangle} \psi(x-\gamma) \left(= e^{-iab\langle A_0(\gamma^1),\gamma^2\rangle} \widetilde{T}_\gamma \psi(x)\right) \tag{2.5}$$

Hence, if (2.4) holds for all γ, $\widetilde{\gamma} \in \Gamma$, we immediately get for $\gamma = a\gamma^1 + b\gamma^2$ and $\widetilde{\gamma} = \widetilde{a}\gamma^1 + \widetilde{b}\gamma^2$

$$T_\gamma T_{\widetilde{\gamma}} = (\widetilde{T}_{\gamma^2})^b (\widetilde{T}_{\gamma^1})^a (\widetilde{T}_{\gamma^2})^{\widetilde{b}} (\widetilde{T}_{\gamma^1})^{\widetilde{a}} = (\widetilde{T}_{\gamma^2})^{b+\widetilde{b}} (\widetilde{T}_{\gamma^1})^{a+\widetilde{a}} = T_{\gamma+\widetilde{\gamma}}.$$

Of course, $[T_\gamma, H_{\mathrm{MB}}] = 0$ still holds for all $\gamma \in \Gamma$ since we have only altered $\widetilde{T}_\gamma$ by a constant factor (for constant γ). We will see why we need $\{T_\gamma, \gamma \in \Gamma\}$ to form a group representation when we later define the magnetic Bloch-Floquet transformation.
In the following, let Γ be the lattice generated by $\{\gamma^1, \gamma^2\}$, M the fundamental cell of Γ, and $\mathbb{T}^2 = \mathbb{R}^2/\Gamma$ the corresponding torus. Furthermore, let Γ^* be the dual lattice generated by $\{\gamma^{1*}, \gamma^{2*}\}$ with $\gamma^i\gamma^{j*} = \delta_{ij}$ and M^* the fundamental cell of Γ^* and hence the first Brillouin zone. In the following, M^* is always equipped with the normalised Lebesgue measure dk. Finally $\mathbb{T}^{2*} = \mathbb{R}^2/\Gamma^*$.
The presence of the magnetic field in the Hamiltonian (2.2) causes a splitting of the Bloch bands into magnetic sub-bands. To be more precise, if the magnetic flux per unit cell of the lattice Γ is a rational multiple $\frac{p}{q}$ of the flux quantum $\frac{h}{e}$, which in our units chosen above equals 2π, the original Bloch band splits into q magnetic subbands, see [Hof76]. Another difference already indicated is the domains of those Hamiltonians, which is treated in Appendix A.

2.2 A general overview of space-adiabatic perturbation theory

Since we want to study the operator (2.3) using space-adiabatic perturbation theory, we first give an outline of this theory. For a more detailed introduction we refer the reader to [PST03a] and [Teu03] respectively [PST03b] and [Teu03] for the case of the Bloch electron without strong magnetic field, that is to say $A_0 \equiv 0$. The key ingredient of space-adiabatic perturbation theory is a distinction of some

degrees of freedom of a physical system named slow respectively fast degrees of freedom. The theory shows that certain dynamic degrees of freedom are in some sense subordinate and no longer autonomous. The point is that the fast modes adjust quickly to the slow modes which in return are ruled by a suitable effective Hamiltonian H_{eff}. This phenomenon is called adiabatic decoupling.
The prime example for this is the Born-Oppenheimer approximation [BO27]. There, molecules are analysed. Due to the fact that the nuclei have a much larger mass than the electrons, they move considerably slower than the electrons. So the separation is to take the fast degrees of freedom as the movement of the electrons and the slow ones as the movement of the nuclei. Then, the fast modes adapt to the status of the lowest energy at the given position of the nuclei, while in turn the slow modes are governed by an effective potential which arises from the electronic energy band.
The philosophy of space-adiabatic perturbation theory is to use the separation of scales and some gap in the spectrum to split up the original Hamiltonian into a direct sum of simpler operators which are more accessible. The main mathematical tool used are pseudodifferential operators with operator-valued symbols. For more detailed information about this we refer the reader to the Appendix B and references therein, which contains all information about pseudodifferential calculus that we need throughout this work. In that appendix we also settle the notation as far as pseudodifferential calculus and associated symbol spaces are concerned.
To apply space-adiabatic perturbation theory to an operator $\widehat{H}$ acting on some Hilbert space $\mathcal{H}$, three basic ingredients are needed:

(i) The state space $\mathcal{H}$ decomposes as

$$\mathcal{H} = \mathcal{H}_{\text{s}} \otimes \mathcal{H}_{\text{f}} = L^2(\mathbb{R}^d) \otimes \mathcal{H}_{\text{f}} \cong L^2(\mathbb{R}^d, \mathcal{H}_{\text{f}}).$$

The Hilbert space $\mathcal{H}_{\text{s}}$ is called the state space of the slow degrees of freedom and must be of the form $L^2(\mathbb{R}^d)$. The space $\mathcal{H}_{\text{f}}$, the state space of the fast degrees of freedom, may be any separable Hilbert space.

(ii) The Hamiltonian $\widehat{H}$ which generates the time-evolution of states is given as Weyl quantisation of a semiclassical symbol

$$H(z,\varepsilon) \asymp \sum_{j=0}^{\infty} \varepsilon^j H_j(z)$$

in some suitable symbol space $S^m_\rho(\varepsilon, \mathcal{L}(\mathcal{H}_{\text{f}}))$ with values in the bounded self-adjoint operators on $\mathcal{H}_{\text{f}}$.

(iii) The principal symbol $H_0(z)$ of $H(z,\varepsilon)$ has a pointwise isolated part of the spectrum which means that it fulfils the Gap-condition

Condition 1. (Gap)$_\gamma$
For all $z \in \mathbb{R}^{2d}$ there is a relevant subset $\sigma_*(z) \subset \sigma(z)$ of the spectrum $\sigma(z)$ of $H_0(z)$ so that there exist $f_\pm \in C(\mathbb{R}^{2d}, \mathbb{R})$ with $f_- \leq f_+$ so that

- for all $z \in \mathbb{R}^{2d}$ the set $\sigma_*(z)$ is entirely contained in the interval $I(z) := [f_-(z), f_+(z)]$,
- $\operatorname{dist}(\sigma(z) \setminus \sigma_*(z), I(z)) \geq C_{\mathrm{g}} \langle p \rangle^\gamma$, and
- $\sup_{z \in \mathbb{R}^{2d}} |f_+(z) - f_-(z)| \leq C_{\mathrm{d}}$.

Note that for $\gamma = 0$ there is a constant gap.

However, in our special case of the Bloch electron, some modifications are necessary:

(i) The Hilbert space of the slow degrees of freedom will be $L^2(M^*)$, where M^* is the first Brillouin zone.

(ii) We will need to allow for more general symbol classes $S^w(\mathbb{R}^{2d}, \mathcal{L}(\mathcal{D}, \mathcal{H}_{\mathrm{f}}))$ because the operator H_0 will be unbounded as an operator on $\mathcal{H}_{\mathrm{f}}$. Thus, we will have to perceive it as an operator on its domain $\mathcal{D}$, equipped with the graph norm, to $\mathcal{H}_{\mathrm{f}}$ to get a bounded operator.

(iii) In our case, the spectrum of H_0 will consist only of eigenvalue bands, where the unperturbed band is periodic with respect to the lattice Γ^*. Because of the periodicity, the spectrum will not fulfil an increasing gap condition but a constant gap condition, which can be written down in a way which is adapted to the structure of the spectrum in question.

Now we give a quick outline of the strategy. There are three main steps:

(i) The first step is to construct an orthogonal projection $\Pi^\varepsilon \in \mathcal{L}(\mathcal{H})$ which is associated to an isolated part $\sigma_*(z)$ of the spectrum of $H_0(z)$. The accordant subspace $\Pi^\varepsilon \mathcal{H}$ of $\mathcal{H}$ is called almost invariant subspace because it is approximately invariant under the time-evolution generated by the Hamiltonian $\widehat{H}$. The usual construction procedure is to first construct a semiclassical symbol π which Weyl-commutes with the symbol H up to an error of $\mathcal{O}(\varepsilon^\infty)$. The prinicipal symbol π_0 of π is just the spectral projection associated to $\sigma_*(z)$. Then one takes the quantisation $\widehat{\pi}$ and slightly modifies it to turn it into a true projector $\Pi^\varepsilon = \widehat{\pi} + \mathcal{O}(\varepsilon^\infty)$.

(ii) We are interested in the dynamics inside the almost invariant subspace $\Pi^\varepsilon \mathcal{H}$. But this space is ε-dependent and we do not know how it looks like. Moreover, it is clear that there is no limit for $\varepsilon \to 0$. Therefore, the second step is to construct a unitary map U^ε which maps the subspace $\Pi^\varepsilon\mathcal{H}$ to a simpler, explicitly given, and ε-independent subspace $\mathcal{H}_{\text{ref}}$. The choice of this space and of the unitary is not unique but nevertheless natural and chosen in order to reflect the physics of the reduced system. As in the first step, the usual procedure to construct U^ε is to first define a symbol u, then to take its quantisation $\widehat{u}$, and afterwards modify the quantisation to turn it into a true unitary $U^\varepsilon = \widehat{u} + \mathcal{O}(\varepsilon^\infty)$.

(iii) The third step is dedicated to the effective dynamics inside the almost invariant subspace. Thereto the effective Hamiltonian is defined by first projecting it to the subspace $\Pi^\varepsilon\mathcal{H}$ and then rotating it to the simpler space $\mathcal{H}_{\text{ref}}$. Thus, we get

$$H_{\text{eff}} = U^\varepsilon\Pi^\varepsilon\widehat{H}\Pi^\varepsilon U^{\varepsilon*} = \widehat{h} + \mathcal{O}(\varepsilon^\infty)$$

where $h := u\sharp\pi\sharp H\sharp\pi\sharp u^*$. Although the effective Hamiltonian obtained this way is still quite abstract, we can exploit that it is given in powers of ε and compute its leading order terms. It turns out that they provide quite interesting information about the dynamics inside the subspace.

This is the general strategy of space-adiabatic perturbation theory. Here $\mathcal{O}(\varepsilon^\infty)$ means:

Definition 2.2.1. *Let R_ε and S_ε be two ε-dependent operators on $\mathcal{H}$. One says that $R_\varepsilon = S_\varepsilon + \mathcal{O}(\varepsilon^\infty)$ if for every $n \in \mathbb{N}$ there is a constant C_n such that*

$$\|R_\varepsilon - S_\varepsilon\|_{\mathcal{L}(\mathcal{H})} \leq C_n\varepsilon^n$$

for all $\varepsilon \in [0, \varepsilon_0)$. One says that R_ε is $\mathcal{O}(\varepsilon^\infty)$-close to S_ε.

The next goal is to show how the above defined Hamiltonian (2.3) can be put into a form that fits into the space-adiabatic framework with the three ingredients described above. This will be the content of the next section.

2.3 The magnetic Bloch-Floquet transformation

In this section, we want to show how the Hamiltonian (2.3) can be transformed into a more convenient form for the space-adiabatic framework. For this purpose we use the fact that the operator H^ε is invariant under magnetic translations with respect to the lattice Γ. This suggests a split-up of $\mathbb{R}^2$ into $\mathbb{R}^2 \cong \Gamma \times M$, where M is the fundamental cell of Γ, and hence a split-up of

$$L^2(\mathbb{R}^2) \cong L^2(\Gamma \times M) \cong l^2(\Gamma) \otimes L^2(M).$$

This method is a known technique called "Bloch theory" by physicists, see [AM76], or Floquet theory, see [Kuc93], where we also refer the reader to for general results about Floquet theory. The transformation we are going to use will be called "Bloch-Floquet transformation", although it is sometimes also called "Zak transformation" due to [Zak68]. So let us introduce the transformation.
Since we aim for a continuous L^2-space for the slow degrees of freedom, we basically do a Fourier transformation on the space $l^2(\Gamma)$ to transform it into $L^2(M^*)$. For $\psi \in S(\mathbb{R}^2)$, this is the Fourier transformation

$$(\mathcal{F} \otimes 1)\psi(k,y) = \sum_{\gamma\in\Gamma} e^{ik\gamma}\psi(y-\gamma).$$

The fact that the operator (2.3) is only invariant under the magnetic translations T_γ and not under ordinary translations reflects in the fact that we take a magnetic Bloch-Floquet transformation where the ordinary translations are replaced by the magnetic translations defined by (2.5):

$$(\mathcal{F}_{\mathrm{magn}} \otimes 1)\psi(k,y) = \sum_{\gamma\in\Gamma} e^{ik\gamma}T_\gamma\psi(y).$$

For technical reasons, we define the magnetic Bloch-Floquet transformation for functions $\psi \in S(\mathbb{R}^2)$ as

$$\begin{aligned} \mathcal{U}_{\mathrm{BF}}\psi(k,y) &:= e^{-iky}(\mathcal{F}_{\mathrm{magn}} \otimes 1)\psi(k,y) = \sum_{\gamma\in\Gamma} e^{-i(y-\gamma)k}T_\gamma\psi(y) \qquad (2.6)\\ &= \sum_{\gamma\in\Gamma} e^{-i(y-\gamma)k}e^{-iab\langle A_0(\gamma^1),\gamma^2\rangle}e^{-i\langle A_0(y),\gamma\rangle}\psi(y-\gamma). \end{aligned}$$

Lemma 2.3.1. *For $\psi \in S(\mathbb{R}^2)$ it holds*

- $(T_\gamma\mathcal{U}_{\mathrm{BF}}\psi)(k,y) := e^{-iab\langle A_0(\gamma^1),\gamma^2\rangle}e^{-i\langle A_0(y),\gamma\rangle}(\mathcal{U}_{\mathrm{BF}}\psi)(k,y-\gamma) = (\mathcal{U}_{\mathrm{BF}}\psi)(k,y)$ *for all* $\gamma \in \Gamma$ *and*
- $(\mathcal{U}_{\mathrm{BF}}\psi)(k-\gamma^*,y) = e^{iy\gamma^*}(\mathcal{U}_{\mathrm{BF}}\psi)(k,y)$ *for all* $\gamma^* \in \Gamma^*$.

Proof.
Let $\psi \in S(\mathbb{R}^2)$ and $\widetilde{\gamma} \in \Gamma$. Then

$$\begin{aligned} T_{\widetilde{\gamma}}\mathcal{U}_{\mathrm{BF}}\psi(k,y) &= T_{\widetilde{\gamma}}\left(\sum_{\gamma\in\Gamma} e^{-i(y-\gamma)k}T_\gamma\psi(y)\right) = \sum_{\gamma\in\Gamma} e^{-i(y-\widetilde{\gamma}-\gamma)k}T_{\widetilde{\gamma}}T_\gamma\psi(y)\\ &= \mathcal{U}_{\mathrm{BF}}\psi(k,y), \end{aligned}$$

where in the last equality we used that the set $\{T_\gamma, \gamma \in \Gamma\}$ is a group. The second claim is straight forward. □

Now we need to define a target space for the magnetic Bloch-Floquet transformation. Thereto and for later use we need the following definition.

Definition 2.3.2. *Let $m \in \mathbb{N}_0$. Then*

$$\mathcal{H}_{\mathrm{f}} := \{\psi \in L^2_{\mathrm{loc}}(\mathbb{R}^2) : T_\gamma \psi = \psi \quad \textit{for all} \quad \gamma \in \Gamma\}$$

is a Hilbert space with inner product

$$\langle f, g \rangle_{\mathcal{H}_{\mathrm{f}}} := \int_M \overline{f(y)} g(y) \mathrm{d}y$$

and

$$\mathcal{H}^m_{A_0}(\mathbb{R}^2) := \{f \in \mathcal{H}_{\mathrm{f}} : (-i\nabla - A_0)^\alpha f \in \mathcal{H}_{\mathrm{f}} \quad \textit{for all} \quad |\alpha| \leq m\}$$

is a Hilbert space with inner product

$$\langle f, g \rangle_{\mathcal{H}^m_{A_0}(\mathbb{R}^2)} := \sum_{|\alpha| \leq m} \langle (-i\nabla - A_0)^\alpha f, (-i\nabla - A_0)^\alpha g \rangle_{\mathcal{H}_{\mathrm{f}}}.$$

The target space for the magnetic Bloch-Floquet transformation is then defined as follows:

Definition 2.3.3. *Let $\tau(\gamma^*) \in \mathcal{U}(\mathcal{H}_{\mathrm{f}})$ be given by*

$$\tau(\gamma^*)\psi(y) := e^{iy\gamma^*}\psi(y) \quad \textit{for} \quad \gamma^* \in \Gamma^* \quad \textit{and} \quad \psi \in \mathcal{H}_{\mathrm{f}}.$$

Then

$$\mathcal{H}_\tau := \{f \in L^2_{\mathrm{loc}}(\mathbb{R}^2_k, (\mathcal{H}_{\mathrm{f}})_y) : f(k - \gamma^*) = \tau(\gamma^*) f(k) \quad \textit{for all} \quad \gamma^* \in \Gamma^*\}$$

equipped with the inner product

$$\langle f, g \rangle_{\mathcal{H}_\tau} = \int_{M^*} \langle f(k), g(k) \rangle_{\mathcal{H}_{\mathrm{f}}} \mathrm{d}k,$$

where $\mathrm{d}k$ *is the normalised Lebesgue measure, is a Hilbert space.*

Proposition 2.3.4. *$\mathcal{U}_{\mathrm{BF}}$ can be extended to a unitary map*

$$\mathcal{U}_{\mathrm{BF}} : L^2(\mathbb{R}^2) \to \mathcal{H}_\tau.$$

The inverse map is defined by

$$(\mathcal{U}^{-1}_{\mathrm{BF}}\phi)(x) := \int_{M^*} e^{ikx}\phi(k, x)\mathrm{d}k.$$

Proof.
It can be easily checked that $\mathcal{U}_{\mathrm{BF}}$ is an isometry. Thereto it suffices to consider functions in the dense subset $S(\mathbb{R}^2)$. Let $\psi \in S(\mathbb{R}^2)$. Then

$$\begin{aligned}
\|\mathcal{U}_{\mathrm{BF}}\psi\|^2_{\mathcal{H}_\tau} &= \langle \mathcal{U}_{\mathrm{BF}}\psi, \mathcal{U}_{\mathrm{BF}}\psi\rangle_{\mathcal{H}_\tau} \\
&= \int_{M^*}\int_M \sum_{\gamma\in\Gamma}\sum_{\gamma'\in\Gamma} e^{-i(\gamma-\gamma')k} e^{-i\langle A_0(y),\gamma'-\gamma\rangle}\overline{\psi(y-\gamma)}\psi(y-\gamma')\mathrm{d}y\mathrm{d}k \\
&= \sum_{\gamma\in\Gamma}\sum_{\gamma'\in\Gamma}\int_M e^{-i\langle A_0(y),\gamma'-\gamma\rangle}\overline{\psi(y-\gamma)}\psi(y-\gamma')\int_{M^*} e^{-i(\gamma-\gamma')k}\mathrm{d}k\mathrm{d}y \\
&= \sum_{\gamma\in\Gamma}\int_M |\psi(y-\gamma)|^2\mathrm{d}y \\
&= \|\psi\|^2_{L^2(\mathbb{R}^2)}
\end{aligned}$$

since $\int_{M^*} e^{-i(\gamma-\gamma')k}\mathrm{d}k = \delta_{\gamma,\gamma'}$.
A quick computation shows $\mathcal{U}_{\mathrm{BF}}^{-1}\mathcal{U}_{\mathrm{BF}}\psi = \psi$ for all $\psi \in S(\mathbb{R}^2)$:

$$\begin{aligned}
&\mathcal{U}_{\mathrm{BF}}^{-1}\mathcal{U}_{\mathrm{BF}}\psi(x) \\
&= \int_{M^*} e^{ikx}\left(\sum_{\gamma\in\Gamma} e^{-i(x-\gamma)k}e^{-iab\langle A_0(\gamma^1),\gamma^2\rangle}e^{-i\langle A_0(x),\gamma\rangle}\psi(x-\gamma)\right)\mathrm{d}k \\
&= \sum_{\gamma\in\Gamma} e^{-iab\langle A_0(\gamma^1),\gamma^2\rangle}e^{-i\langle A_0(x),\gamma\rangle}\psi(x-\gamma)\int_{M^*} e^{ik\gamma}\mathrm{d}k \\
&= \psi(x).
\end{aligned}$$

It is also checked easily that $\mathcal{U}_{\mathrm{BF}}^{-1}$ extends to an isometry from $\mathcal{H}_\tau$ to $L^2(\mathbb{R}^2)$ since for $\psi \in C^\infty_\tau$ it holds

$$\begin{aligned}
\left\|\mathcal{U}_{\mathrm{BF}}^{-1}\psi\right\|^2_{L^2(\mathbb{R}^2)} &= \int_{\mathbb{R}^2}|\int_{M^*} e^{ikx}\psi(k,x)\mathrm{d}k|^2\mathrm{d}x \\
&= \sum_{\gamma\in\Gamma}\int_M|\int_{M^*} e^{ikx}e^{-ik\gamma}\psi(k,x-\gamma)\mathrm{d}k|^2\mathrm{d}x \\
&= \sum_{\gamma\in\Gamma}\int_M|\int_{M^*} e^{ikx}e^{-ik\gamma}e^{-iab\langle A_0(\gamma^1),\gamma^2\rangle}e^{i\langle A_0(x),\gamma\rangle}\psi(k,x)\mathrm{d}k|^2\mathrm{d}x \quad (2.7) \\
&= \sum_{\gamma\in\Gamma}\int_M|\int_{M^*} e^{ikx}e^{-ik\gamma}\psi(k,x)\mathrm{d}k|^2\mathrm{d}x = \int_M\sum_{\gamma\in\Gamma}|a_\gamma|^2\mathrm{d}x \\
&= \int_M\int_{M*}|e^{ikx}\psi(k,x)|^2\mathrm{d}k\mathrm{d}x = \int_{M^*}\int_M|\psi(k,x)|^2\mathrm{d}x\mathrm{d}k \\
&= \|\psi\|^2_{\mathcal{H}_\tau},
\end{aligned}$$

where a_γ is the Fourier coefficient of the function $e^{i\langle\cdot,x\rangle}\psi(\cdot,x)|_{M^*} \in L^2(M^*)$ and, moreover, in (2.7) we exploited that ψ is in $\mathcal{H}_\tau$ which yields $\psi(k,x-\gamma) = T_{-\gamma}\psi(k,x-\gamma) = e^{-iab\langle A_0(\gamma^1),\gamma^2\rangle}e^{i\langle A_0(x),\gamma\rangle}\psi(k,x)$. Thus $\mathcal{U}_{\mathrm{BF}}^{-1}$ is injective and hence $\mathcal{U}_{\mathrm{BF}}$ is surjective and therefore a unitary. □

Since we want to transform the Hamiltonian (2.3) via magnetic Bloch-Floquet transformation, we first investigate how the operators Q = "multiplication with x" and $P = -i\nabla_x - A_0(x)$ and the multiplication with the periodic potential V_Γ transform under $\mathcal{U}_{\mathrm{BF}}$. We will keep things short.

Proposition 2.3.5. *Under the above defined magnetic Bloch-Floquet transformation, the operators*

- *Q as multiplication with x on its maximal domain* $\mathrm{D}(Q) = \{\psi \in L^2(\mathbb{R}^2) : x_j\psi \in L^2(\mathbb{R}^2)\forall j \in \{1,2\}\}$
- *$P := -i\nabla_x - A_0(x)$ with domain $H^1_{A_0}(\mathbb{R}^2)$*
- *V_Γ as multiplication with V_Γ*

transform as

- *$\mathcal{U}_{\mathrm{BF}}Q\mathcal{U}_{\mathrm{BF}}^* = i\nabla_k^\tau \otimes 1_{\mathcal{H}_\mathrm{f}}$ with domain $\mathcal{H}_\tau \cap H^1_{\mathrm{loc}}(\mathbb{R}^2,\mathcal{H}_\mathrm{f})$*
- *$\mathcal{U}_{\mathrm{BF}}P\mathcal{U}_{\mathrm{BF}}^* = 1_{L^2(M^*)}\otimes(-i\nabla_y - A_0(y)) + k\otimes 1_{\mathcal{H}_\mathrm{f}}$ with domain $\mathcal{H}_\tau \cap L^2_{\mathrm{loc}}(\mathbb{R}^2,\mathcal{H}^1_{A_0})$*
- *$\mathcal{U}_{\mathrm{BF}}V_\Gamma\mathcal{U}_{\mathrm{BF}}^* = 1_{L^2(M^*)} \otimes V_\Gamma$.*

Proof.
For the proof one just uses the definition of the magnetic Bloch-Floquet transformation and for the domains also the formula for the inverse of $\mathcal{U}_{\mathrm{BF}}$. □

Remark 2.3.6. Note that the domain of $i\nabla_k^\tau$ is independent of k because we have chosen the suitable definition for the magnetic Bloch-Floquet transformation in (2.6).

The next step is to use this knowledge to write the transformed operator $\mathcal{U}_{\mathrm{BF}}H^\varepsilon\mathcal{U}_{\mathrm{BF}}^*$ as a pseudodifferential operator. We will start with the transformation of the unperturbed Hamiltonian (2.2).

Proposition 2.3.7. *The Hamiltonian $H_{\mathrm{MB}} = \frac{1}{2}(-i\nabla_x - A_0(x))^2 + V_\Gamma(x)$ transforms under magnetic Bloch-Floquet transformation as*

$$H^0_{\mathrm{BF}} := \mathcal{U}_{\mathrm{BF}}H_{\mathrm{MB}}\mathcal{U}_{\mathrm{BF}}^* = \int_{M^*}^{\oplus} H_{\mathrm{per}}(k)\mathrm{d}k,$$

where

$$H_{\mathrm{per}}(k) := \tfrac{1}{2}(-i\nabla_y - A_0(y) + k)^2 + V_\Gamma(y). \tag{2.8}$$

For fixed k, the domain of $H_{\mathrm{per}}(k)$ is $\mathcal{H}^2_{A_0}(\mathbb{R}^2)$ and thus independent of k. The domain of H^0_{BF} is $L^2_\tau(\mathbb{R}^2, \mathcal{H}^2_{A_0}(\mathbb{R}^2))$.

We skip the proof since it directly follows from the above results.
Now we finally transform the full Hamiltonian (2.3).

Theorem 2.3.8. *The Hamiltonian (2.3) defined on $\mathrm{D}(H^\varepsilon) = H^2_{A_0}(\mathbb{R}^2)$ transforms under magnetic Bloch-Floquet transformation as*

$$H^\varepsilon_{\mathrm{BF}} = \mathcal{U}_{\mathrm{BF}} H^\varepsilon \mathcal{U}^*_{\mathrm{BF}} = \tfrac{1}{2}(-i\nabla_y - A_0(y) + k - A(i\varepsilon\nabla^\tau_k))^2 + V_\Gamma(y) + \Phi(i\varepsilon\nabla^\tau_k) \tag{2.9}$$

with domain $\mathrm{D}(H^\varepsilon_{\mathrm{BF}}) = L^2_\tau(\mathbb{R}^2, \mathcal{H}^2_{A_0}(\mathbb{R}^2))$.

We again skip the proof.
Let us now make the connection with the theory of pseudodifferential operators. Formally, $H^\varepsilon_{\mathrm{BF}}$ just looks like the quantisation of the symbol

$$H_0(k,r) = \tfrac{1}{2}(-i\nabla_y - A_0(y) + k - A(r))^2 + V_\Gamma(y) + \Phi(r). \tag{2.10}$$

To be more precise, this is a symbol in $S^{w=1+k^2}_{(\tau_1,\tau_2)}(\mathbb{R}^4, \mathcal{L}(\mathcal{H}^2_{A_0}, \mathcal{H}_{\mathrm{f}}))$ where $\tau_1 = \tau|_{\mathcal{H}^2_{A_0}}$ and $\tau_2 = \tau$ defined on $\mathcal{H}_{\mathrm{f}}$. Thus, its τ-quantisation $\widehat{H_0}^\tau : S'_{\tau_1}(\mathbb{R}^2, \mathcal{H}^2_{A_0}) \to S'_\tau(\mathbb{R}^2, \mathcal{H}_{\mathrm{f}})$ can be restricted to $L^2_\tau(\mathbb{R}^2, \mathcal{H}^2_{A_0}(\mathbb{R}^2))$. Hence, recalling that $i\nabla^\tau_k$ is just defined as the restriction of $i\nabla_k|_{H^1(\mathbb{R}^2,\mathcal{H}_{\mathrm{f}})\cap\mathcal{H}_\tau}$ and using spectral calculus, one can see that $\widehat{H_0}^\tau$ and $H^\varepsilon_{\mathrm{BF}}$ coincide on $L^2_\tau(\mathbb{R}^2, \mathcal{H}^2_{A_0}(\mathbb{R}^2))$. Thus we have proven

Theorem 2.3.9. *Let $H_0(k,r) = \frac{1}{2}(-i\nabla_y - A_0(y) + k - A(r))^2 + V_\Gamma(y) + \Phi(r)$. Then under the Assumption (1) it holds*

- $H_0 \in S^{w=1+k^2}_{(\tau_1,\tau_2)}(\mathbb{R}^4, \mathcal{L}(\mathcal{H}^2_{A_0}, \mathcal{H}_{\mathrm{f}}))$*, where $\tau_1 = \tau|_{\mathcal{H}^2_{A_0}}$ and $\tau_2 = \tau$ defined on $\mathcal{H}_{\mathrm{f}}$, and*
- $\widehat{H_0}^\tau|_{L^2_\tau(\mathbb{R}^2,\mathcal{H}^2_{A_0}(\mathbb{R}^2))} = H^\varepsilon_{\mathrm{BF}}$.

2.4 Space-adiabatic perturbation theory in the (magnetic) Bloch case

So far we have shown how the Hamiltonian (2.3) can be unitarily transformed into a pseudodifferential operator acting on a Hilbert space which is suited to fit into the framework of space-adiabatic perturbation theory.

Let us now go back to showing how the three ingredients for applying space-adiabatic perturbation theory are satisfied in the magnetic Bloch case. The results of the previous section provide a distinction of slow and fast degrees of freedom and also show how the Hamiltonian can be written as a pseudodifferential operator. So two of the ingredients for applying space-adiabatic perturbation theory are given by

$$\mathcal{H} = \mathcal{H}_\tau \cong L^2(M^*) \otimes \mathcal{H}_\mathrm{f}$$

and

$$H^\varepsilon_{\mathrm{BF}} = \widehat{H_0(k,r)}^{\tau} \quad \text{with} \quad H_0 \in S^{w=1+k^2}_\tau(\mathbb{R}^4, \mathcal{L}(\mathcal{H}^2_{A_0}, \mathcal{H}_\mathrm{f})).$$

The fact that the space $\mathcal{H}_\mathrm{s}$ of the slow degrees of freedom is not $L^2(\mathbb{R}^2)$ any more and the differences in the symbol spaces are solved by using the τ-quantisation from Appendix B. Let us take a closer look at the spectrum of the symbol (2.10). First, we analyse the spectrum of the unperturbed periodic operator (2.8) defined on $\mathcal{H}^2_{A_0}(\mathbb{R}^2)$. As in the case $A_0 \equiv 0$, one can show that $H_{\mathrm{per}}(k)$ has a compact resolvent for every $k \in M^*$ and hence has discret spectrum with eigenvalues of finite multiplicity that accumulate at infinity. So let

$$E_1(k) \leq E_2(k) \leq ...$$

be the eigenvalue bands $\{E_n(k), n \in \mathbb{N}\}$ repeated according to their multiplicity and let

$$\{\varphi_n(k), n \in \mathbb{N}\} \subset \mathcal{H}^2_{A_0}$$

be the corresponding eigenfunctions which thus form an orthonormal basis of $\mathcal{H}_\mathrm{f}$. In the following, E_n will be called the n^{th} band function or just the n^{th} band. Note that in the case of eigenvalue crossings they do not have to be smooth functions. However, this will not bother us in the following since we are going to work with an isolated non-degenerate band $E(k)$, where the meaning of "isolated" will be specified below. Note also that because of the τ-equivariance of $H_{\mathrm{per}}(k)$, the band functions $E_n(k)$ are periodic with respect to Γ^*. Hence we will not get an increasing gap condition but some constant gap condition. Using this knowledge about the spectrum of $H_{\mathrm{per}}(k)$, we can define a suitable gap condition as follows:

Definition 2.4.1. *A family of bands $\{E_n(k)\}_{n\in I}$ with $I = [I_-, I_+] \cap \mathbb{N}$ is called isolated respectively satisfies the gap condition if*

$$\inf_{k \in M^*} \mathrm{dist}\left(\cup_{n\in I}\{E_n(k)\}, \cup_{m \notin I}\{E_m(k)\}\right) =: C_\mathrm{g} > 0.$$

Remark 2.4.2. Often not the function $E_n(k)$ itself is called Bloch band but the set

$$\overline{E_n} := \overline{\cup_{k\in M^*} E_n(k)}.$$

In this setting, a band is called isolated if $\overline{E_n} \cap \overline{E_{n\pm 1}} = \emptyset$. Of course, this is a stronger request than our gap condition above. Since for our purposes we only need the weaker condition given in Definition 2.4.1, we stick to that terminology of isolated Bloch bands.

The connection with the perturbed Hamiltonian is $H_0(k,r) = H_{\mathrm{per}}(k - A(r)) + \Phi(r)$. So in our magnetic Bloch case, the three ingredients of space-adiabatic perturbation theory are fulfilled by

- the Hilbert space $\mathcal{H} = \mathcal{H}_\tau \cong L^2(M^*) \otimes \mathcal{H}_{\mathrm{f}}$,
- the operator $H^\varepsilon_{\mathrm{BF}} = \widehat{H_0(k,r)}^\tau$, where $H_0 \in S^{w=1+k^2}_\tau(\mathbb{R}^4, \mathcal{L}(\mathcal{H}^2_{A_0}, \mathcal{H}_{\mathrm{f}}))$, and
- we assume that $E(k)$ is an isolated, non-degenerate eigenvalue of $H_{\mathrm{per}}(k)$.

Now we are in position to apply space-adiabatic perturbation theory to our Hamiltonian $H^\varepsilon_{\mathrm{BF}}$.
The non-magnetic case, where H^ε does not include the potential of a strong constant magnetic field A_0, is done in Chapter 5 of [Teu03] respectively in [PST03b]. Our aim is to generalise the methods used there to the magnetic Bloch case $A_0 \neq 0$. Let us now go through the three main steps of space-adiabatic perturbation theory and illustrate where we can follow the line of the non-magnetic case and where we cannot and why. We also sketch our further proceeding and point out where we take care of which step.
For the construction of the almost invariant subspace, we can follow the line of the non-magnetic case with slight technical modifications. The construction of Π^ε was already done in [Sti11], but for the sake of completeness we will give a quick overview of the construction in the next section.
In contrast, the construction of the intertwining unitary U^ε fails. The main ingredient in the non-magnetic case is the trivialisability of a vector bundle called Bloch bundle. For $A_0 \equiv 0$ and $\dim P(k) = m$, it is shown in [Pan07], that the corresponding Bloch bundle E_{Bl} is trivial if $m = 1$ or $m \in \mathbb{N}$ and $d \leq 3$. The key ingredient in [Pan07] is the time-reversal symmetry of the Hamiltonian H_{Bl}. However, the inclusion of the strong constant magnetic field breaks the time-reversal symmetry and thus in the magnetic case, the results of [Pan07] do not hold. The fact that the Bloch bundle is not trivial in the case of magnetic Bloch bands is also studied elaborately in papers of Dubrovin and Novikov, see [DN80a], [DN80b] and [Nov81].
So our main aim is to define a reference Hilbert space $\mathcal{H}_{\mathrm{ref}}$ and construct the unitary map U^ε which maps $\Pi^\varepsilon \mathcal{H}_\tau$ to $\mathcal{H}_{\mathrm{ref}}$ as well as the derivation of an effective Hamiltonian acting on $\mathcal{H}_{\mathrm{ref}}$ which is given as a pseudodifferential operator. The construction of the intertwining unitary U^ε will be the content of Chapter

3. The main difference in the construction of U^ε is, as already indicated, the non-trivialisability of the Bloch bundle. Thus we will use local trivialisations to at least map the space $\Pi^\varepsilon \mathcal{H}_\tau$ to an ε-independent space denoted by $\Pi^0 \mathcal{H}_\tau$. This can be done using the pseudodifferential methods suggested above. Afterwards, the space $\Pi^0 \mathcal{H}_\tau$ has to be unitarily mapped to a simpler space $\mathcal{H}_{\mathrm{ref}}$ since we want the simplest space we can get for our reference space. The difficulty is that this map cannot be defined as a pseudodifferential operator. So these ideas only give a definition of U^ε as the combination of the two described unitaries, but do not immediately yield the symbol for the effective Hamiltionian.
This is the motivation for Chapter 4, where we show how we have to include the non-trivial geometry of the Bloch bundle into our quantisation procedures. Thereto we develop three new Weyl quantisations which we need to finally write the effective Hamiltonian $H_{\mathrm{eff}} = U^\varepsilon \Pi^\varepsilon \widehat{H}^\tau \Pi^\varepsilon U^{\varepsilon *}$ as a pseudodifferential operator. The point is that without a definition of U^ε as a pseudodifferential operator, we do not get the symbol of the effective operator by just Weyl-multiplying the symbols of $\widehat{H}$, Π^ε, and U^ε. Therefore, we are going to introduce pseudodifferential calculi for sections of non-trivial bundles. Those calculi will also be linked together in some way so that it will be possible to include the unitary map from Chapter 3 in the quantisation respectively the symbols.
In Chapter 5, we will use the results of the aforegoing chapters to finally derive the effective Hamiltonian as a pseudodifferential operator. Moreover, we will as well compute the principal and subprincipal symbol of it.
The definitions of the three new pseudodifferential calculi are new as well as the resulting effective Hamiltonian. Also the definition of the reference Hilbert space and the unitary map U^ε are new.
Let us give a quick overview of existing results about the Bloch electron case. The case without a strong magnetic field, that is to say $A_0 \equiv 0$, is done in [Teu03] and [PST03b] and it is slightly generalised in [DL11] to the case that not the potential A of the weak perturbation has to be in $C^\infty_{\mathrm{b}}(\mathbb{R}^d, \mathbb{R}^d)$ but only the magnetic field B. This is done by using a magnetic Weyl calculus. In [DP10] the authors derive an effective model for the case $A_0 \equiv 0$ and $A(x) = \frac{B}{2}(-x_2, x_1)^{\mathrm{T}}$, where the modification of [Teu03] and [PST03b] is mainly just the use of magnetic Sobolev spaces. In the papers mentioned so far, the authors always derive an effective Hamiltonian whose leading order is given by the Peierls Substitution. All these works share that they have the trivialisability of the Bloch bundle.
For the case $A_0 \neq 0$ there is no corresponding rigorous derivation of an effective model for H^ε in the literature. But in [Sti11] the author establishes a new method to derive semiclassical results for this case. More precisely, this work contains a rigorous justification of the equations proposed by Niu at al. in [SN99, SNSN05]. In [Sti11], the authors do not need the trivialisability of the Bloch bundle any

more because they just construct invariant subspaces as usual but then, instead of constructing an intertwining unitary U^ε and an effective Hamiltonian, they derive their semiclassical result with a different method, which can also be found in [ST11a] and for the Bloch case in [ST11b]. So it is not possible to derive effective Hamiltonians with their methods.
Let us also mention that in [DGR04] the authors claim to have already solved the problem. But they use the assumption that the Bloch bundle is trivial and hence their results do not hold in general. Moreover, because of the assumption that the Bloch bundle is trivial, the results of [DGR04] could have also been achieved using the methods of [Teu03] and [PST03b]. In fact, the non-trivialisability of the Bloch bundle is the main obstruction one has to overcome in the magnetic Bloch case in contrast to the non-magnetic case. All in all, the result of the paper coincides with our result for the (trivial) case $\theta = 0$, but in general it holds $\theta \neq 0$ - the case which is excluded in the [DGR04].
Now we quickly motivate why we expect a Peierls Substitution type operator for the effective Hamiltonian on an almost invariant subspace. Let $E(k)$ be a non-degenerate isolated Bloch band with associated projection $P(k)$. Then for $\psi \in \mathcal{U}_{\mathrm{BF}}^* P(k)\mathcal{U}_{\mathrm{BF}} L^2(\mathbb{R}^2)$ we get from Proposition 2.3.5(ii) that

$$\begin{aligned} H_{\mathrm{MB}}\psi(x) &= \mathcal{U}_{\mathrm{BF}}^* H_{\mathrm{per}}(k)\mathcal{U}_{\mathrm{BF}}\psi(x) = \mathcal{U}_{\mathrm{BF}}^* E(k)\mathcal{U}_{\mathrm{BF}}\psi(x) \\ &= E(-i\nabla_x - A_0(x))\psi(x). \end{aligned}$$

Thus the wave functions from the band subspace associated to E propagate with dispersion relation $E(p_{\mathrm{magn}})$ where p_{magn} is the magnetic momentum. However, when we include non-periodic potentials A and Φ, the subspace $P\mathcal{H}_\tau$ is no longer invariant since the perturbations A and Φ cause band transitions. But if we assume them to be slowly varying as we indeed do, those transitions will be small and we can still expect approximately invariant subspaces which are associated to the magnetic Bloch band E. Hence we expect the dynamics inside an almost invariant subspace to be given through a Peierls substitution type Hamiltonian.
We conclude this section with an outlook on how the effective Hamiltonian will look like compared to the non-magnetic case $A_0 \equiv 0$. For the sake of clarity let $\Gamma = \mathbb{Z}^2$. In the non-magnetic case we know from [Teu03, PST03b] that

$$\widehat{h_{\mathrm{eff}}}^{\tau} = E(k - A(i\varepsilon\nabla_k)) + \Phi(i\varepsilon\nabla_k) + \mathcal{O}(\varepsilon),$$

where

$$\widehat{h_{\mathrm{eff}}}^{\tau} \quad \text{operates on} \quad L^2(\mathbb{T}^{2*})$$

with $\mathbb{T}^{2*} = \mathbb{R}^2/(2\pi\mathbb{Z})^2$. In our more general, magnetic case we will get the same symbol for the leading order of the effective Hamiltonian but the fact that it no longer operates on L^2-functions on the torus but on L^2-sections of a non-trivial line bundle reflects both in the fact that we have to take a different quantisation procedure as well as a different Hilbert space which are both better adjusted to the non-trivial geometry of the Bloch bundle. So our effective Hamiltonian will read

$$\widehat{h_{\text{eff}}}^{\text{eff}} = E(k - A(i\varepsilon\nabla_k^{\text{eff}})) + \Phi(i\varepsilon\nabla_k^{\text{eff}}) + \mathcal{O}(\varepsilon),$$

where

$$\widehat{h_{\text{eff}}}^{\text{eff}} \quad \text{operates on} \quad \mathcal{H}_\theta$$

with

$$\nabla_k^{\text{eff}} = \nabla_k + (0, \tfrac{i\theta}{2\pi}k_1)^{\mathrm{T}}.$$

Here θ is supposed to be the Chern number of the Bloch bundle and thus an in general non-zero integer. Note that for the case $A_0 \equiv 0$, we get exactly the result of [Teu03, PST03b] since then the Bloch bundle is trivial which is equivalent to the condition $\theta = 0$, the vanishing of the Chern number, see for example [BT82]. We will see later that $\mathcal{H}_{\theta=0} \cong L^2(\mathbb{T}^{2*})$.

2.5 The almost invariant subspace

This section is dedicated to the construction of the almost invariant subspace $\Pi^\varepsilon\mathcal{H}_\tau$. This concept goes back to [Nen02] and a similar construction can also be found in [NeSo04] and [MaSo02]. We, however, will follow the lines of [PST03a, PST03b, Teu03]. The goal is to construct a subspace of the statespace $\mathcal{H}_\tau$ which is approximately invariant under the time-evolution of the Hamiltonian $\widehat{H}^\tau$. After that, we conclude this section with a new result. We state conditions under that the constructed subspace can be taken as the spectral subspace associated to the family of bands in question.

The construction is first done on the level of symbols. We start with the spectral projection $P_I(k)$ belonging to the isolated band family as principal symbol. Afterwards, one takes the quantisation and then turns that operator into a true projector. We will shortly sketch the way this is done in our case. A detailed description of the whole construction can be found in [PST03b, Teu03], since the methods there carry over to the magnetic case without difficulties, as it is done in [Sti11].

Note that if we take $P_I(k)$ as the spectral projection associated to an isolated band familiy $\{E_n(k)\}_{n\in I}$, we get a smooth function P_I because of the gap condition. Moreover, the τ-equivariance of $H_{\text{per}}(k)$ implies the τ-equivariance of $P_I(k)$. We present the results for semiclassical symbols $H \in S_\tau^{w=1+k^2}(\varepsilon, \mathcal{L}(\mathcal{H}^2_{A_0}, \mathcal{H}_{\text{f}}))$ with principal symbol H_0.

Lemma 2.5.1. *Let $\{E_n\}_{n\in I}$ be an isolated family of bands according to Defintion 2.4.1 and P_I the associated spectral projection. Then there exists a unique formal symbol*

$$\pi(k,r) = \sum_{j=0}^{\infty} \varepsilon^j \pi_j(k,r) \in M^1_\tau(\mathcal{L}(\mathcal{H}_\mathrm{f})) \cap M^{w=1+k^2}_\tau(\mathcal{L}(\mathcal{H}_\mathrm{f}, \mathcal{H}^2_{A_0}(\mathbb{R}^2)))$$

with principal symbol $\pi_0(k,r) = P_I(k - A(r))$ such that

- $\pi \sharp \pi = \pi$,
- $\pi^* = \pi$, *and*
- $[\pi, H]_\sharp = 0$.

Theorem 2.5.2. *Let the assumptions of Lemma 2.5.1 be satisfied. Then there exists an orthogonal projection $\Pi^\varepsilon \in \mathcal{L}(\mathcal{H}_\tau)$ such that*

$$[H^\varepsilon_{\mathrm{BF}}, \Pi^\varepsilon] = \mathcal{O}(\varepsilon^\infty)$$

and

$$\Pi^\varepsilon = \widehat{\pi}^\tau + \mathcal{O}(\varepsilon^\infty)$$

with $\pi \in S^1_\tau(\varepsilon, \mathcal{L}(\mathcal{H}_\mathrm{f}))$ with principal symbol $\pi_0(k,r) = P_I(k - A(r))$.

Corollary 2.5.3. *It holds*

$$[e^{-iH^\varepsilon_{\mathrm{BF}} s}, \Pi^\varepsilon] = \mathcal{O}(\varepsilon^\infty |s|).$$

For the proofs, we refer the reader to the beforehand mentioned sources [Teu03, PST03b, Sti11].
Note that, although the obtained almost invariant subspaces $\Pi^\varepsilon \mathcal{H}_\tau$ are associated to some isolated family of bands $\{E_n\}_{n\in I}$ of the spectrum of the principal symbol H_0 of H, they are in general not spectral. This is due to the fact that, although the family of bands is isolated from the other bands, the image of it may overlap with the image of other bands. Yet the gap condition can be sharpened, as already indicated in Remark (2.4.2), so that the projection Π^ε can be taken as the spectral projection P^ε of $H^\varepsilon_{\mathrm{BF}}$ associated to the gap corresponding to the family of bands. So in this case, the effective Hamiltonian $P^\varepsilon H^\varepsilon_{\mathrm{BF}} P^\varepsilon$ has exactly the same spectral properties as $H^\varepsilon_{\mathrm{BF}}$ on the spectral subspace $P^\varepsilon \mathcal{H}_\tau$. Let us formulate this more precisely.

Proposition 2.5.4. *Let the assumptions of Lemma 2.5.1 be satisfied so that even*

$$\mathrm{dist}(\overline{\cup_{n\in I}\cup_{k\in M^*} E_n(k)}, \overline{\cup_{n\notin I}\cup_{k\in M^*} E_n(k)}) := d_{\mathrm{g}} > 0.$$

Let $\|\phi\|_\infty < \frac{1}{2}d_{\mathrm{g}}$ *and moreover* P^ε *be the spectral projection associated to the subset* $\sigma(H^\varepsilon_{\mathrm{BF}}) \cap B_{\frac{d_{\mathrm{g}}}{2}}(\overline{\cup_{n\in I}\cup_{k\in M^*} E_n(k)})$ *of the spectrum of* $H^\varepsilon_{\mathrm{BF}}$. *Then for the projection constructed with respect to Lemma 2.5.1 it holds*

$$\widehat{\pi} = P^\varepsilon + \mathcal{O}(\varepsilon^\infty).$$

Proof.
To prove the above lemma, one has to look at the construction of the symbol π in Lemma 2.5.1. There, one starts constructing a local Moyal resolvent of H. We adopt the notation from the proof of Lemma 5.17 in [Teu03]. Under the above conditions, for any $z_0 = (k_0, r_0) \in \mathbb{R}^4$, the circle $\Lambda(z_0)$ can be chosen as the same circle Λ that encloses the set $\overline{\cup_{n\in I}\cup_{k\in M^*} E_n(k)}$ in a way that the circular line has the distance greater or equal than $\frac{1}{2}d_{\mathrm{g}}$ to the set $\overline{\cup_{n\in I}\cup_{k\in M^*} E_n(k)}$ and the area of the circle has the distance greater or equal than $\frac{1}{2}d_{\mathrm{g}}$ to the set $\overline{\cup_{n\notin I}\cup_{k\in M^*} E_n(k)}$. This circle Λ fulfils

$$\mathrm{dist}(\Lambda, \sigma(H_0(k,r))) \geq \tfrac{1}{2}d_{\mathrm{g}} - \|\phi\|_\infty > 0 \quad \text{for all} \quad (k,r) \in \mathbb{R}^4$$

and

$$\mathrm{radius}(\Lambda) \leq C_{\mathrm{r}}. \tag{2.11}$$

Now we show $\Lambda \subset \rho(H^\varepsilon_{\mathrm{BF}})$. To see this let $\zeta \in \Lambda$. Then one can construct the Moyal resolvent $R(\zeta)$ and it holds

$$\begin{aligned} (\widehat{H}^\tau - \zeta)\widehat{R(\zeta)}^\tau &= \mathrm{id}_{H_\tau} + U \\ \widehat{R(\zeta)}^\tau(\widehat{H}^\tau - \zeta) &= \mathrm{id}_{H_\tau} + V \end{aligned} \tag{2.12}$$

with $U, V = \mathcal{O}(\varepsilon^\infty)$. Hence one can use the Neumann series of $-U$ to define the operator $S := \widehat{R}^\tau(\mathrm{id} + U)^{-1}$. It is easy to see that S is the continuous inverse of $\widehat{H}^\tau - \zeta$ and hence ζ must be in the resolvent set of $\widehat{H}^\tau = H^\varepsilon_{\mathrm{BF}}$.
Moreover, for every $\zeta \in \Lambda$ the quantisation of $R(\zeta)$ is $\mathcal{O}(\varepsilon^\infty)$-close to the resolvent $T(\zeta)$ of $H^\varepsilon_{\mathrm{BF}}$ since (2.12) implies (multiplying with $T(\zeta)$ from the left)

$$\widehat{R(\zeta)}^\tau = T(\zeta) + \mathcal{O}(\varepsilon^\infty). \tag{2.13}$$

Then it holds for any $n \in \mathbb{N}$ (again adopting the notation from [Teu03]) that

$$\begin{aligned}
\Pi^\varepsilon &= \widehat{\pi^{(n)}}^{\tau} + \mathcal{O}(\varepsilon^{n+1}) = \mathrm{Op}^\tau\left(\tfrac{i}{2\pi}\int_\Lambda R^{(n)}(\zeta,k,r)\mathrm{d}\zeta\right) + \mathcal{O}(\varepsilon^{n+1}) && (2.14)\\
&= \tfrac{i}{2\pi}\int_\Lambda \mathrm{Op}^\tau(R^{(n)}(\zeta,k,r))(-T(\zeta)+T(\zeta))\mathrm{d}\zeta + \mathcal{O}(\varepsilon^{n+1}) && (2.15)\\
&= \tfrac{i}{2\pi}\int_\Lambda T(\zeta)\mathrm{d}\zeta + \mathcal{O}(\varepsilon^{n+1}) && (2.16)\\
&= P^\varepsilon + \mathcal{O}(\varepsilon^{n+1}).
\end{aligned}$$

Here equality (2.14) follows by construction and equation (2.16) follows using (2.13) and (2.11). For equation (2.15), note that for $T \in S'_\tau$ and $\varphi \in S(\mathbb{R}^2, \mathcal{H}_\mathrm{f})$ it holds

$$\begin{aligned}
&\mathrm{Op}^\tau\left(\tfrac{i}{2\pi}\int_\Lambda R^{(n)}(\zeta,k,r)\mathrm{d}\zeta\right)(T)(\varphi)\\
&= T\left(k \mapsto \mathrm{Op}^\tau\left(\tfrac{-i}{2\pi}\int_\Lambda R^{(n)*}(\zeta,k,r)\mathrm{d}\zeta\right)\varphi(k)\right)\\
&= T\left(k \mapsto \tfrac{1}{(2\pi\varepsilon)^2}\int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}} \tfrac{-i}{2\pi}\int_\Lambda R^{(n)*}(\zeta, \tfrac{k+y}{2}, r)\mathrm{d}\zeta\varphi(y)\mathrm{d}y\mathrm{d}r\right)\\
&= T\left(k \mapsto \tfrac{-i}{2\pi}\int_\Lambda \tfrac{1}{(2\pi\varepsilon)^2}\int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}} R^{(n)*}(\zeta, \tfrac{k+y}{2}, r)\varphi(y)\mathrm{d}y\mathrm{d}r\mathrm{d}\zeta\right) && (2.17)\\
&= T\left(k \mapsto \tfrac{-i}{2\pi}\int_\Lambda \mathrm{Op}^\tau(R^{(n)*}(\zeta))\mathrm{d}\zeta\varphi(k)\right)\\
&= T\left(k \mapsto \tfrac{-i}{2\pi}\int_\Lambda \mathrm{Op}^\tau(R^{(n)}(\zeta))^*\mathrm{d}\zeta\varphi(k)\right) && (2.18)\\
&= \tfrac{i}{2\pi}\int_\Lambda \mathrm{Op}^\tau(R^{(n)}(\zeta))\mathrm{d}\zeta(T)(\varphi).
\end{aligned}$$

For equation (2.17) we again used (2.11), while equation (2.18) follows by using Proposition B.3.8, $R \in S^1_\tau(\varepsilon)$, and Proposition B.3.6. □

Chapter 3

The intertwining unitary

3.1 The general construction of the intertwining unitary

In the previous chapter we showed how the almost invariant subspace $\Pi^\varepsilon\mathcal{H}_\tau$ associated to a family of bands $\{E_n\}_{n\in I}$ is constructed. Hence, so far we have a subspace which decouples from its complement up to a small error in ε. The question is how to describe the dynamics inside this subspace. As already indicated, the problem is that the subspace $\Pi^\varepsilon\mathcal{H}_\tau$ is ε-dependent and in general not even a limit $\varepsilon \to 0$ exists. Hence the space is not easily accessible at all. One way to deal with this complicacy is to find an explicit space $\mathcal{H}_{\mathrm{ref}}$ which is ε-independent and unitarily equivalent to $\Pi^\varepsilon\mathcal{H}_\tau$. This is the method we want to use to get into the dynamics inside the decoupled subspace.
We will define the unitary map to the reference space in two steps. The first step is to construct an almost unitary pseudodifferential operator which diagonalises the pseudodifferential operator $\mathrm{Op}^\tau(\pi\sharp H\sharp\pi)$. This approach has a long tradition, see [Nir73] and references therein, and has been used in different settings, see for example [Tay75, HeSj90]. The successive diagonalisation also appears in the physics literature as for example in [Bl62a], [Bl62b] and [LiWe93].
Before we present our construction, we want to give a quick outline how this is done in general and in [Teu03, PST03b] in the case $A_0 \equiv 0$. Afterwards, we will point out why and where this construction fails in our case. Recall that the main difference is the non-trivialisability of the Bloch bundle.
In the general case, one first fixes an adequate reference space. The strategy is the following: The smoothness of the prinicipal symbol $H_0(z)$ of the Hamiltonian and the gap condition imply the smoothness of the spectral projection $P(z) = \pi_0(z)$. This means that the subspaces $\pi_0(z)\mathcal{H}_{\mathrm{f}}$ are all of the same dimension and hence isomorphic to some subspace $\mathcal{K}_{\mathrm{f}} \subset \mathcal{H}_{\mathrm{f}}$ which does not depend on z; for example,

one could take $\mathcal{K}_\mathrm{f} = \pi_0(0)\mathcal{H}_\mathrm{f}$. Let π_r be the projection in $\mathcal{H}_\mathrm{f}$ onto $\mathcal{K}_\mathrm{f}$. With our above choice we get $\pi_\mathrm{r} = \pi_0(0)$. Then a projection $\Pi_\mathrm{r} = \widehat{\pi}_\mathrm{r} = 1 \otimes \pi_\mathrm{r} \in \mathcal{L}(\mathcal{H})$ can be defined since π_r is in $S^0_\rho(\mathcal{L}(\mathcal{H}_\mathrm{f}))$ and $\mathcal{H}_\mathrm{ref}$ is chosen as $\mathcal{H}_\mathrm{ref} = \Pi_\mathrm{r}\mathcal{H}$. The goal is to unitarily map $\Pi^\varepsilon\mathcal{H}$ to $\mathcal{H}_\mathrm{ref}$. To do this, one needs a symbol $u_0(z)$ which pointwise intertwines $\pi_0(z)$ and π_r, which means

$$u_0(z)\pi_0(z)u_0(z)^* = \pi_\mathrm{r}.$$

The existence of $u_0(z)$ and its smoothness follow from the trivialisability of the smooth vector bundle

$$E = \{(z, \psi) \in (\mathbb{R}^{2d}, \mathcal{H}_\mathrm{f}) : \psi \in \pi_0(z)\mathcal{H}_\mathrm{f}\}$$

because then the associated bundle of the orthonormal frames admits a global section which in turn implies the existence of $u_0(z)$. For example in the special case $\mathrm{ran}P(z) = 1$, E is a trivial line bundle and thus admits a global section φ without zeros and $\|\varphi(z)\|_{\mathcal{H}_\mathrm{f}} = 1$ for all $z \in \mathbb{R}^{2d}$. Then $\widetilde{u}_0(z) := |\varphi(0)\rangle\langle\varphi(z)|$ can be extended to a map $u_0(z) \in \mathcal{U}(\mathcal{H}_\mathrm{f})$. Note that u_0 is not unique. The problem is that, although from this construction we get the existence and smoothness of u_0, it cannot be proven that it is in some appropriate symbol class $S^m_\rho(\mathcal{L}(\mathcal{H}_\mathrm{f}))$ since we do not get any information about boundedness of its derivatives. Hence in the general setting, one has to make the assumption that $u_0 \in S^0_\rho(\mathcal{L}(\mathcal{H}_\mathrm{f}))$. Normally in physical examples for which intertwining unitaries have been constructed in the framework of space-adiabatic perturbation theory so far, like for example the Bloch electron in the non-magnetic case, u_0 can be chosen conveniently so that the boundary conditions for it and its derivatives are fulfilled. We emphasise that this will not be the case in the magnetic Bloch case and one of our goals is to show how an intertwining unitary can be constructed nevertheless.
After having defined $u_0(z)$, one has to extend it to a formal symbol $u(z) = u_0(z) + \mathcal{O}(\varepsilon)$ which satisfies $u\sharp u^* = 1 = u^*\sharp u$ and $u\sharp\pi\sharp u^* = \pi_\mathrm{r}$. Then one takes a resummation of it whose quantisation $\widehat{u}$ has to be modified to be turned into a true unitary U^ε that exactly intertwines Π_r and Π^ε.

3.2 The construction of the intertwining unitary in the non-magnetic Bloch case

In the case of the Bloch electron without strong magnetic field the just described strategy can be followed with some small modifications. To be as much comprehensible as possible we will stick to the case of an isolated Bloch band $E(k)$ which is non-degenerate. One can define π_r as $\pi_\mathrm{r} = \pi_0(0)$. But π_r is in the symbol class $S^1_{\tau\equiv 1}(\mathcal{L}(\mathcal{H}_\mathrm{f}))$ and not in $S^1_\tau(\mathcal{L}(\mathcal{H}_\mathrm{f}))$. So the projection $\Pi_\mathrm{r} = \widehat{\pi}_\mathrm{r}^{\tau\equiv 1} =$

$1 \otimes \pi_{\mathrm{r}}$ is in $\mathcal{L}(\mathcal{H}_{\tau\equiv 1})$ and not in $\mathcal{H}_\tau$ and hence the reference space is chosen to be $\Pi_{\mathrm{r}} H_{\tau\equiv 1} \cong L^2(\mathbb{T}^{2*})$. However, it is not hard to solve this problem. One just needs to find a symbol $u_0(k,r)$ which is right-τ-equivariant, that is to say $u_0(k-\gamma^*,r) = u_0(k,r)\tau(\gamma^*)^{-1}$ for all $\gamma^* \in \Gamma^*$. So again one aims for a function $\varphi \neq 0$ which fulfils $\varphi(k) \in P(k)\mathcal{H}_{\mathrm{f}}$. To get the right-$\tau$-equivariance, additionally we need $\varphi(k-\gamma^*) = \tau(\gamma^*)\varphi(k)$. This is were the Bloch bundle

$$E_{\mathrm{Bl}} = \{(k,\varphi) \in (\mathbb{R}^2, \mathcal{H}_{\mathrm{f}})_\sim : \varphi \in P(k)\mathcal{H}_{\mathrm{f}}\}, \tag{3.1}$$

$$\text{where} \quad (k,\varphi) \sim (k',\varphi') :\Leftrightarrow k' = k - \gamma^* \quad \text{and} \quad \varphi' = \tau(\gamma^*)\varphi,$$

comes into play. The point is that the existence of a τ-equivariant function φ without zeros is equivalent to the trivialisability of the Bloch bundle. It is shown in [Pan07] that this is the case when $A_0 \equiv 0$. So in this case, one sets $\widetilde{u}_0(k,r) := |\varphi(0)\rangle\langle\varphi(k - A(r))|$ and gets a symbol which pointwise intertwines $\pi_0(k,r)$ and π_{r}. Moreover, $u_0(k,r)$ is bounded in the $\mathcal{H}_{\mathrm{f}}$-norm together with all its derivatives because of the right τ-equivariance, which implies $u_0(k,r) \in S^1_{(\tau,\tau\equiv 1)}(\mathcal{L}(\mathcal{H}_{\mathrm{f}}))$. Then this symbol is extended as in the general case to a symbol $u \in S^1_{(\tau,\tau\equiv 1)}(\varepsilon, \mathcal{L}(\mathcal{H}_{\mathrm{f}}))$ and then modified to a true unitary $U^\varepsilon \in \mathcal{L}(\mathcal{H}_\tau, \mathcal{H}_{\tau\equiv 1})$ so that $U^\varepsilon \Pi^\varepsilon U^{\varepsilon *} = \Pi_{\mathrm{r}}$.

3.3 The construction of the intertwining unitary in the magnetic Bloch case

Now let us return to our general case $A_0 \neq 0$ for the Bloch electron. Let moreover from now on E be a non-degenerate isolated eigenvalue band. The above described approach does not work anymore since the Bloch bundle is no longer trivial. So if we defined $\pi_{\mathrm{r}} = \pi_0(0)$, $\mathcal{K}_{\mathrm{f}} = \pi_0(0)\mathcal{H}_{\mathrm{f}}$, and $\Pi_{\mathrm{r}} = \widehat{\pi}_{\mathrm{r}}{}^\tau$ as above, we would again need a function $\varphi \neq 0$ which is τ-equivariant and forms an orthonormal basis of $\mathrm{ran}P(k)$. But if we have a function φ which is τ-equivariant it must have zeros and otherwise if we have a function φ which forms for every k an orthonormal basis of $\mathrm{ran}P(k)$ it cannot be τ-equivariant, because in either case it would be a contradiction of the non-triviality of the line bundle E_{Bl}. So the best we can look for is a function φ with norm $\|\varphi(k)\|_{\mathcal{H}_{\mathrm{f}}} = 1$ and $\varphi(k) \in P(k)\mathcal{H}_{\mathrm{f}}$, which then must have some sort of twisted periodic boundary conditions like $\varphi(k-\gamma^*) = e^{ig(k,\gamma^*)}\tau(\gamma^*)\varphi(k)$, where g is a smooth, real function. Our strategy will be to construct such a function and analyse explicitly its twisted periodic boundary conditions. The obtained function φ will not have bounded derivatives. This is because the Bloch bundle now has a curvature. So we cannot define a symbol $\widetilde{u}_0 = |\varphi(0)\rangle\langle\varphi(k - A(r))|$ as in the non-magnetic case since we cannot control its derivatives in any way. To deal with this, we first focus on getting rid of the ε-dependence of the decoupled

subspace $\Pi^\varepsilon \mathcal{H}_\tau$ and neglect that we want to map $\Pi^\varepsilon \mathcal{H}_\tau$ to a subspace $\mathcal{H}_{\rm ref}$ which is as simple as possible. Thereto we take $\mathcal{K}_{\rm f} := \pi_0(k)\mathcal{H}_{\rm f}$ and thus $\pi_{\rm r} := \pi_0(k)$ is no longer independent of k. But at least it does not depend on r, which after all is the reason for the appearance of the ε in the quantisation procedure ($k \mapsto k$ and $r \mapsto i\varepsilon\nabla_k$). Then $\pi_{\rm r}(k)$ is in the suitable symbol class $S^1_\tau(\mathcal{L}(\mathcal{H}_{\rm f}))$ and thus $\Pi_{\rm r} = \widehat{\pi_{\rm r}}^\tau = 1 \otimes \pi_0(k) := \Pi^0$ is an operator in $\mathcal{L}(\mathcal{H}_\tau)$ whose range $\Pi^0\mathcal{H}_\tau$ is independent of ε. Then the symbol $\widetilde{u}_0$, which pointwise intertwines $\pi_0(k,r)$ and $\pi_{\rm r}(k)$, is basically given by $|\varphi(k)\rangle\langle\varphi(k-A(r))|$. With an appropriate gauge for A, which is adapted to the definition of φ, this is a τ-equivariant function. So we can define $u_0(k,r) \in S^1_\tau(\mathcal{L}(\mathcal{H}_{\rm f}))$ and proceed as usual, that is to say that one can define a semiclassical symbol $u \in S^1_\tau(\varepsilon, \mathcal{L}(\mathcal{H}_{\rm f}))$ with principal symbol u_0 and then modify its quantisation $\widehat{u}^\tau = U_1^\varepsilon + \mathcal{O}(\varepsilon^\infty)$ to turn $\widehat{u}^\tau$ into a true unitary satisfying $U_1^\varepsilon \Pi^\varepsilon U_1^\varepsilon = \Pi^0$.
Using the function φ, we can write down explicitly how the space $\Pi^0\mathcal{H}_\tau$ looks like:

$$\begin{aligned}
\Pi^0\mathcal{H}_\tau &= \{f \in \mathcal{H}_\tau : f(k) \in P(k)\mathcal{H}_{\rm f}\ \forall k\} \\
&= \{f(k) = \psi(k)\varphi(k) \text{ with } \psi \in L^2_{\rm loc}(\mathbb{R}^2) \text{ and } \psi(k-\gamma^*) = e^{-ig(k,\gamma^*)}\psi(k)\} \\
&=: \{f \in \mathcal{H}_\tau : f(k) = \psi(k)\varphi(k) \ \text{ with } \ \psi \in \mathcal{H}_{\rm ref}\}.
\end{aligned}$$

So there is our reference space $\mathcal{H}_{\rm ref}$ which will be named $\mathcal{H}_\theta$ as it will become clear after the construction and analysis of φ. Here we just give the hint that θ will be the Chern number of the Bloch bundle and thus in $\mathbb{Z}(\setminus\{0\})$. It is also clear how to unitarily map $\Pi^0\mathcal{H}_\tau$ to $\mathcal{H}_{\rm ref}$; this is just the map $\langle\varphi(k)|$. Now we can combine this map with U_1^ε to get the required unitary map from $\Pi^\varepsilon\mathcal{H}_\tau$ to $\mathcal{H}_{\rm ref}$. The problem is that the map $\langle\varphi(k)|$ cannot be written as a pseudodifferential operator. This problem will be treated in Chapter 4.
So now we start the explicit construction of the intertwining unitary U^ε. Before we start with the construction of the function φ, we need the following Lemma:

Lemma 3.3.1. *Let*

$$E'_{\rm Bl} := \{(k,\varphi) \in \mathbb{R}^2 \times \mathcal{H}_{\rm f} : \varphi \in P(k)\mathcal{H}_{\rm f}\}$$

and let $\nabla^{\rm B} = P(k)\nabla$ be the connection on the bundle. For arbitrary $x, y \in \mathbb{R}^2$ let $t^{\rm B}(x,y)$ be the parallel transport with respect to the Berry connection along the straight line from y to x. Then

$$t^{\rm B}(x-\gamma^*, y-\gamma^*) = \tau(\gamma^*)t^{\rm B}(x,y)\tau(\gamma^*)^{-1}. \tag{3.2}$$

Proof.
Let $x, y \in \mathbb{R}^2$, $\gamma^* \in \Gamma^*$, and $\alpha(s) = y + s(x-y)$. Then, for any $h_y \in P(y)\mathcal{H}_{\rm f}$, it must hold

$$\nabla^{\rm B}_{\dot\alpha} t^{\rm B}(\alpha(s), y)h_y = 0. \tag{3.3}$$

This determines the map t^{B} uniquely. Thus to verify (3.2), it suffices to show

$$\nabla^{\mathrm{B}}_{\dot{\widetilde{\alpha}}}\tau(\gamma^*)t^{\mathrm{B}}(\widetilde{\alpha}(s)+\gamma^*,y)\tau(\gamma^*)^{-1}h_{y-\gamma^*}=0,$$

where $\widetilde{\alpha}(s)=y-\gamma^*+s(x-y)$ and $h_{y-\gamma^*}\in P(y-\gamma^*)\mathcal{H}_{\mathrm{f}}$. Thereto note that $\widetilde{\alpha}(s)=\alpha(s)-\gamma^*$ and $\dot{\widetilde{\alpha}}=\dot{\alpha}$. Hence

$$\begin{aligned}
&\nabla^{\mathrm{B}}_{\dot{\widetilde{\alpha}}}\tau(\gamma^*)t^{\mathrm{B}}(\widetilde{\alpha}(s)+\gamma^*,y)\tau(\gamma^*)^{-1}h_{y-\gamma^*}\\
&\quad=P(\alpha(s)-\gamma^*)\nabla_{\dot{\alpha}}\tau(\gamma^*)t^{\mathrm{B}}(\alpha(s),y)\tau(\gamma^*)^{-1}h_{y-\gamma^*}\\
&\quad=\tau(\gamma^*)P(\alpha(s))\tau(\gamma^*)^{-1}\nabla_{\dot{\alpha}}\tau(\gamma^*)t^{\mathrm{B}}(\alpha(s),y)\tau(\gamma^*)^{-1}h_{y-\gamma^*}\\
&\quad=\tau(\gamma^*)P(\alpha(s))\nabla_{\dot{\alpha}}t^{\mathrm{B}}(\alpha(s),y)\underbrace{\tau(\gamma^*)^{-1}h_{y-\gamma^*}}_{\in P(y)\mathcal{H}_{\mathrm{f}}}\\
&\quad=0
\end{aligned}$$

because of (3.3). □

The content of the following lemma is the construction of the function φ. Shortly speaking, we first construct a function using the parallel transport with respect to the induced connection on the Bloch bundle, the so-called Berry connection, and then analyse its "quasi-periodicity". From now on, we will restrict ourselves to the case $\Gamma=\mathbb{Z}^2$, which implies $\Gamma^*=(2\pi\mathbb{Z})^2$, to keep everything as clear as possible. Nevertheless, our results can be generalised easily to an arbitrary Bravais lattice Γ. We will give the general results at the end of each section and comment shortly on the changes in the proofs.

Lemma 3.3.2. *There is a function $\varphi\in C^\infty(\mathbb{R}^2,\mathcal{H}_{\mathrm{f}})$ so that for all $k\in\mathbb{R}^2$ we have $\varphi(k)\in P(k)\mathcal{H}_{\mathrm{f}}$, $\|\varphi(k)\|_{\mathcal{H}_{\mathrm{f}}}=1$, and*

$$\varphi(k-\gamma^*)=e^{-\frac{i\theta}{2\pi}k_2\gamma_1^*}\tau(\gamma^*)\varphi(k)\quad\textit{for all}\quad\gamma^*\in\Gamma^*,\tag{3.4}$$

where θ is the Chern number of the Bloch bundle (3.1).

Proof.
First let us mention that if the Bloch bundle (3.1) was trivial, the statement of the lemma would follow directly. But the point is that it holds also in the non-trivial case. The idea is to consider the bundle

$$E'_{\mathrm{Bl}}:=\{(k,\varphi)\in\mathbb{R}^2\times\mathcal{H}_{\mathrm{f}}:\varphi\in P(k)\mathcal{H}_{\mathrm{f}}\}.$$

This is a line bundle with base space $\mathbb{R}^2$ and thus trivial. This is the difference from the Bloch bundle (3.1). Nevertheless, the Berry connection $\nabla^{\mathrm{B}}=P(k)\nabla$ is still a connection on this bundle. Now we construct a trivialisation $\widetilde{\varphi}$ of E'_{Bl} by using the parallel transport with respect to the Berry connection and then analyse

the relation between $\widetilde{\varphi}(k-\gamma^*)$ and $\widetilde{\varphi}(k)$ for $\gamma^* \in \Gamma^*$.
For arbitrary $x, y \in \mathbb{R}^2$ let $t^{\mathrm{B}}(x,y)$ be the parallel transport with respect to the Berry connection along the straight line from y to x. Then the τ-equivariance of P implies, as we have seen in Lemma 3.3.1,

$$t^{\mathrm{B}}(x-\gamma^*, y-\gamma^*) = \tau(\gamma^*)t^{\mathrm{B}}(x,y)\tau(\gamma^*)^{-1}$$

for all $\gamma^* \in \Gamma^*$. Now we define the function $\widetilde{\varphi}$. Let h_0 be an arbitrary element in $P(0)\mathcal{H}_{\mathrm{f}}$ with $\|h_0\|_{\mathcal{H}_{\mathrm{f}}} = 1$ and fix

$$\widetilde{\varphi}(0) := h_0.$$

Now we set

$$\widetilde{\varphi}(0,k_2) := t^{\mathrm{B}}((0,k_2),(0,0))h_0$$

and

$$\widetilde{\varphi}(k_1,k_2) := t^{\mathrm{B}}((k_1,k_2),(0,k_2))\widetilde{\varphi}(0,k_2).$$

By construction, $\widetilde{\varphi}$ is a smooth function that fulfils $\widetilde{\varphi}(k) \in P(k)\mathcal{H}_{\mathrm{f}}$ and $\|\widetilde{\varphi}(k)\|_{\mathcal{H}_{\mathrm{f}}} = 1$ for all $k \in \mathbb{R}^2$.
The next step is to analyse the relation of $\widetilde{\varphi}(k-\gamma^*)$ and $\widetilde{\varphi}(k)$ for $k \in \mathbb{R}^2$ and $\gamma^* \in \Gamma^*$. From the τ-equivariance of $P(k)$ and the fact that $\varphi(k)$ spans $P(k)\mathcal{H}_{\mathrm{f}}$ for every $k \in \mathbb{R}^2$, we get

$$\begin{aligned}
\widetilde{\varphi}(k-\gamma^*) &= \tau(\gamma^*)P(k)\tau^{-1}(\gamma^*)\widetilde{\varphi}(k-\gamma^*) \\
&= \tau(\gamma^*)\langle\widetilde{\varphi}(k), \tau^{-1}(\gamma^*)\widetilde{\varphi}(k-\gamma^*)\rangle_{\mathcal{H}_{\mathrm{f}}}\widetilde{\varphi}(k) \\
&=: \overline{\widetilde{\alpha}(k,\gamma^*)}\tau(\gamma^*)\widetilde{\varphi}(k).
\end{aligned} \tag{3.5}$$

Thus we have to identify $\widetilde{\alpha}(k,\gamma^*)$. It is already clear that $\widetilde{\alpha}$ must have absolut value 1 since τ is a unitary and $\|\widetilde{\varphi}(k)\|_{\mathcal{H}_{\mathrm{f}}} = 1$ for all $k \in \mathbb{R}^2$.
As we want to map the space $\Pi^0\mathcal{H}_\tau$ to a suitable subspace of $L^2_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{C})$, we regard the following: Let ψ be a function in $L^2_{\mathrm{loc}}(\mathbb{R}^2)$. Then, using (3.5), we get

$$\psi(k)\widetilde{\varphi}(k) \in \mathcal{H}_\tau \;\text{ iff }\; \psi(k-\gamma^*)\widetilde{\varphi}(k-\gamma^*) = \psi(k)\tau(\gamma^*)\widetilde{\varphi}(k) = \psi(k)\widetilde{\alpha}(k,\gamma^*)\widetilde{\varphi}(k-\gamma^*).$$

This is equivalent to the condition

$$\psi(k-\gamma^*) = \widetilde{\alpha}(k,\gamma^*)\psi(k), \tag{3.6}$$

where

$$\widetilde{\alpha}(k,\gamma^*) = \langle\widetilde{\varphi}(k-\gamma^*), \tau(\gamma^*)\widetilde{\varphi}(k)\rangle_{\mathcal{H}_{\mathrm{f}}}.$$

Now let $\widetilde{\mathcal{H}}_\theta := \{\psi \in L^2_{\mathrm{loc}}(\mathbb{R}^2) : \psi(k-\gamma^*) = \widetilde{\alpha}(k,\gamma^*)\psi(k)\}$ and let $\widetilde{U} : \Pi^0\mathcal{H}_\tau \to \widetilde{\mathcal{H}}_\theta$ be the map defined by $f \mapsto \langle\widetilde{\varphi}(k), f(k)\rangle_{\mathcal{H}_{\mathrm{f}}}$. Then $\widetilde{U}$ is a unitary map between the Hilbert spaces $\Pi^0\mathcal{H}_\tau$ and $\widetilde{\mathcal{H}}_\theta$. For $j \in \{1,2\}$, consider the maps

$$\partial^\tau_{k_j} : \mathcal{H}_\tau \cap H^1_{\mathrm{loc}}(\mathbb{R}^2, \mathcal{H}_{\mathrm{f}}) \to \mathcal{H}_\tau$$

and

$$\widetilde{\partial}^\theta_{k_j}\psi(k) := \widetilde{U}\partial^\tau_{k_j}\widetilde{U}^* : \widetilde{\mathcal{H}}_\theta \cap H^1_{\mathrm{loc}}(\mathbb{R}^2) \to \widetilde{\mathcal{H}}_\theta.$$

We have $\widetilde{\partial}^\theta_{k_j}\psi(k) = \partial_{k_j}\psi(k) + \left\langle \widetilde{\varphi}(k), \partial^\tau_{k_j}\widetilde{\varphi}(k)\right\rangle_{\mathcal{H}_\mathrm{f}} \psi(k)$ for all $\psi \in \widetilde{\mathcal{H}}_\theta \cap H^1_{\mathrm{loc}}(\mathbb{R}^2)$. Let $\widetilde{\mathcal{A}}_j(k) := \left\langle \widetilde{\varphi}(k), \partial^\tau_{k_j}\widetilde{\varphi}(k)\right\rangle_{\mathcal{H}_\mathrm{f}}$ for $j \in \{1,2\}$. By construction, we have $\widetilde{\mathcal{A}}_1 \equiv 0$ as $\widetilde{\varphi}$ is parallel along horizontal lines. Denote by $\Omega(k) = \partial_1\widetilde{\mathcal{A}}_2(k) - \partial_2\widetilde{\mathcal{A}}_1(k) = \partial_1\widetilde{\mathcal{A}}_2(k)$ the curvature of the Berry connection. To get defining equations for $\widetilde{\alpha}(k,\gamma^*)$, on the one hand we differentiate equation (3.6) for $j \in \{1,2\}$ and $\psi \in \widetilde{\mathcal{H}}_\theta \cap H^1_{\mathrm{loc}}(\mathbb{R}^2)$ and get

$$\partial_{k_j}\psi(k-\gamma^*) = \widetilde{\alpha}(k,\gamma^*)\partial_{k_j}\psi(k) + (\partial_{k_j}\widetilde{\alpha}(k,\gamma^*))\psi(k). \tag{3.7}$$

On the other hand, $\widetilde{\partial}^\theta_{k_j}\psi$ must be in $\widetilde{\mathcal{H}}_\theta$, so we get

$$\widetilde{\partial}^\theta_{k_j}\psi(k-\gamma^*) = \widetilde{\alpha}(k,\gamma^*)(\partial_{k_j}\psi(k) + \widetilde{\mathcal{A}}_j(k)\psi(k)) \tag{3.8}$$

and, because ψ is in $\widetilde{\mathcal{H}}_\theta$,

$$\begin{aligned}\widetilde{\partial}^\theta_{k_j}\psi(k-\gamma^*) &= \partial_{k_j}\psi(k-\gamma^*) + \widetilde{\mathcal{A}}_j(k-\gamma^*)\psi(k-\gamma^*)\\ &= \partial_{k_j}\psi(k-\gamma^*) + \widetilde{\mathcal{A}}_j(k-\gamma^*)\widetilde{\alpha}(k,\gamma^*)\psi(k). \end{aligned}\tag{3.9}$$

Combining equation (3.8) and (3.9) yields

$$\widetilde{\alpha}(k,\gamma^*)(\partial_{k_j}\psi(k) + \widetilde{\mathcal{A}}_j(k)\psi(k)) = \partial_{k_j}\psi(k-\gamma^*) + \widetilde{\mathcal{A}}_j(k-\gamma^*)\widetilde{\alpha}(k,\gamma^*)\psi(k). \tag{3.10}$$

Inserting equation (3.7) into equation (3.10) yields

$$\partial_{k_j}\widetilde{\alpha}(k,\gamma^*) = \widetilde{\alpha}(k,\gamma^*)(\widetilde{\mathcal{A}}_j(k) - \widetilde{\mathcal{A}}_j(k-\gamma^*)) \quad \text{for all} \quad j \in \{1,2\}.$$

Hence $\partial_{k_1}\widetilde{\alpha}(k,\gamma^*) = 0$, which means that $\widetilde{\alpha}$ does not depend on k_1, and thus

$$\widetilde{\alpha}(k,\gamma^*) = \widetilde{\alpha}(0,\gamma^*)e^{\int_0^{k_2}(\widetilde{\mathcal{A}}_2(k_1,\kappa) - \widetilde{\mathcal{A}}_2(k_1-\gamma_1^*,\kappa-\gamma_2^*))\mathrm{d}\kappa}.$$

Now we take a closer look at the obtained two factors of $\widetilde{\alpha}$. Thereto, first note that the curvature Ω must be periodic with respect to Γ^*. Let therefor be $k \in \mathbb{R}^2$ and $\gamma^* \in \Gamma^*$. Then

$$\begin{aligned}\Omega(k-\gamma^*) &= \partial_{k_1}\langle \widetilde{\varphi}(k-\gamma^*), \partial_{k_2}\widetilde{\varphi}(k-\gamma^*)\rangle_{\mathcal{H}_\mathrm{f}}\\ &= \partial_{k_1}(\widetilde{\alpha}(k,\gamma^*)\partial_{k_2}\overline{\widetilde{\alpha}(k_2,\gamma^*)}) + \partial_{k_1}\left(|\widetilde{\alpha}(k,\gamma^*)|^2\widetilde{\mathcal{A}}_2(k)\right) \qquad (3.11)\\ &= \Omega(k). \qquad (3.12)\end{aligned}$$

For equation (3.11), we used the identity (3.5). In equation (3.12), we used that $\widetilde{\alpha}$ is independent of k_1 and thus the first summand is 0, and the fact that $|\widetilde{\alpha}(k,\gamma^*)| \equiv 1$. This yields

$$\begin{aligned}
\int_0^{k_2} (\widetilde{\mathcal{A}}_2(k_1,\kappa) - \widetilde{\mathcal{A}}_2(k_1-\gamma_1^*,\kappa-\gamma_2^*))\mathrm{d}\kappa &= \int_0^{k_2} (\widetilde{\mathcal{A}}_2(k_1,\kappa) - \widetilde{\mathcal{A}}_2(k_1-\gamma_1^*,\kappa))\mathrm{d}\kappa \\
&= \int_0^{k_2}\int_{k_1-\gamma_1^*}^{k_1} \Omega(p,\kappa)\mathrm{d}p\mathrm{d}\kappa \\
&= \frac{\gamma_1^*}{2\pi}\int_0^{k_2}\int_0^{2\pi} \Omega(p,\kappa)\mathrm{d}p\mathrm{d}\kappa \\
&=: \frac{\gamma_1^*}{2\pi}\overline{\Omega}(k_2).
\end{aligned}$$

Now let

$$t^{\mathrm{B}}(-\gamma^{1*},0)h_0 = e^{-i\beta_1}\tau(\gamma^{1*})h_0$$

and

$$t^{\mathrm{B}}(0,-\gamma^{2*})h_0 = e^{-i\beta_2}\tau(\gamma^{2*})h_0.$$

With $\beta = (\beta_1,\beta_2)$ it holds $\widetilde{\alpha}(0,\gamma^*) = e^{\frac{i}{2\pi}\beta\cdot\gamma^*}$. This follows by using the definition (3.5) of $\widetilde{\alpha}$ and the property (3.2) of t^{B}.
In order to get a more convenient parameter α we slightly modify the function $\widetilde{\varphi}$ by putting

$$\varphi(k) := e^{\frac{i\theta}{2\pi}k_1k_2 - \frac{k_1}{2\pi}\overline{\Omega}(k_2) - \frac{i}{2\pi}\beta\cdot k}\widetilde{\varphi}(k),$$

where θ is the Chern number of the Bloch bundle (3.1). Using the definition of φ, $\overline{\Omega}(k_2-\gamma_2^*) = \overline{\Omega}(k_2) - \overline{\Omega}(\gamma_2^*)$, and $\overline{\Omega}(\gamma_2^*) = \gamma_2^* i\theta$, we then get

$$\begin{aligned}
\alpha(k_2,\gamma^*) &= \langle \varphi(k-\gamma^*), \tau(\gamma^*)\varphi(k)\rangle_{\mathcal{H}_{\mathrm{f}}} \\
&= e^{\frac{i\theta}{2\pi}k_2\gamma_1^*}.
\end{aligned}$$

□

Remark 3.3.3. In the following we will always work with

- the function φ, which fulfils $\varphi(k-\gamma^*) = \tau(\gamma^*)e^{-\frac{i\theta}{2\pi}k_2\gamma_1^*}\varphi(k)$ for all $k \in \mathbb{R}^2$ and $\gamma^* \in \Gamma^*$,
- the Hilbert space $\mathcal{H}_\theta = \{\psi \in L^2_{\mathrm{loc}}(\mathbb{R}^2) : \psi(k-\gamma^*) = e^{\frac{i\theta}{2\pi}k_2\gamma_1^*}\psi(k)$ for all $k \in \mathbb{R}^2$ and $\gamma^* \in \Gamma^*\}$ with inner product $\langle\psi,\phi\rangle_{\mathcal{H}_\theta} = \int_{M^*}\overline{\psi(k)}\phi(k)\mathrm{d}k$, where $\mathrm{d}k$ denotes the normalised Lebesgue measure,
- the unitary map $U^\theta : \Pi_0\mathcal{H}_\tau \to \mathcal{H}_\theta$ defined by $f \mapsto \langle\varphi(k), f\rangle_{\mathcal{H}_{\mathrm{f}}}$,

- the operators $\partial_j^\theta : H^1_{\mathrm{loc}}(\mathbb{R}^2) \cap \mathcal{H}_\theta \to \mathcal{H}_\theta$ defined by $\partial_j^\theta := U^\theta \partial_j^\tau U^{\theta *}$ for $j \in \{1,2\}$, and
- $\mathcal{A}_j := \langle \varphi(k), \partial_j \varphi(k) \rangle_{\mathcal{H}_\mathrm{f}}$ for $j \in \{1,2\}$. In particular, we have $\mathcal{A}_1(k-\gamma^*) = \mathcal{A}_1(k)$ and $\mathcal{A}_2(k-\gamma^*) = -\frac{i\theta}{2\pi}\gamma_1^* + \mathcal{A}_2(k)$ for all $k \in \mathbb{R}^2$ and $\gamma^* \in \Gamma^*$.

Now we have fixed the subspace $\mathcal{H}_\theta$ whose elements can be identified in a natural way with the L^2-sections of the non-trivial Bloch bundle. We take this Hilbert space $\mathcal{H}_\theta$ as the reference space on which the effective operator shall finally operate. The normal proceeding would now be to take $u_0(k,r) = \langle \varphi(k-A(r))| + u_0^\perp$ as the principal symbol of a pseudodifferential operator U^ε. But as it will be shown in the proof of the next theorem, this function does not have bounded derivatives and is hence in no suitable symbol class. So we cannot use the methods of [Teu03, PST03b] to construct a unitary between $\Pi^\varepsilon \mathcal{H}_\tau$ and $\mathcal{H}_\theta$.
Thus we take a different approach. We first construct a unitary U_1^ε from $\Pi^\varepsilon \mathcal{H}_\tau$ to $\Pi^0 \mathcal{H}_\tau$ and afterwards combine it with the map U^θ. For the construction of U_1^ε we can use the usual pseudodifferential methods, because with the help of the function φ from Lemma 3.3.2 we can define a suitable symbol $u_0(k,r)$ which is τ-equivariant and thus solves the problem of the unboundedness of the derivatives of u_0.

Theorem 3.3.4. *There exists a unitary operator $U_1^\varepsilon : \mathcal{H}_\tau \to \mathcal{H}_\tau$ such that*

$$U_1^\varepsilon \Pi^\varepsilon U_1^{\varepsilon *} = \Pi^0 \tag{3.13}$$

and $U_1^\varepsilon = \widehat{u} + \mathcal{O}_0(\varepsilon^\infty)$, where $u \asymp \sum_{j\geq 0} \varepsilon^j u_j$ belongs to $S^1_\tau(\varepsilon, \mathcal{L}(\mathcal{H}_\mathrm{f}))$ and has the principal symbol $u_0(k,r) = |\varphi(k)\rangle \langle \varphi(k-A(r))| \, e^{-\frac{i\theta}{2\pi} A_2(r) k_1} + u_0^\perp(k,r)$.

Proof.
First we show the existence of a symbol $u_0(k,r) \in S^1_\tau(\mathcal{L}(\mathcal{H}_\mathrm{f}))$ that is a (pointwise) unitary and fulfils

$$u_0(k,r)\pi_0(k,r)u_0^*(k,r) = \pi_0(k) \tag{3.14}$$

and

$$u_0(k,r)\varphi(k-A(r)) = e^{-\frac{i\theta}{2\pi}A_2(r)k_1}\varphi(k). \tag{3.15}$$

Recall $\pi_0(k,r) = \pi_0(k-A(r))$. Here we see why we choose the gauge $A_2 \equiv 0$. It eliminates the phase in (3.15) which will facilitate further calculations.
We want to define the unitary operator using a variant of the Nagy formula. To be able to apply the formula, we need that $|\langle \varphi(k), \varphi(k-A(r)) \rangle_{\mathcal{H}_\mathrm{f}}| > 0$ holds. If $A \equiv 0$ this is clear since $\langle \varphi(k), \varphi(k) \rangle_{\mathcal{H}_\mathrm{f}} \equiv 1$. So let us first assume that the perturbation potential A is small enough so that $|\langle \varphi(k), \varphi(k-A(r)) \rangle_{\mathcal{H}_\mathrm{f}}| \geq \frac{1}{2}$ holds for all $k, r \in \mathbb{R}^2$. Then we use a slightly modified version of the Nagy formula

to define the symbol u_0. Note thereto that $|\langle\varphi(k),\varphi(k-A(r))\rangle_{\mathcal{H}_\mathrm{f}}| > 0$ implies $\|\pi_0(k)-\pi_0(k,r)\|_{\mathcal{L}(\mathcal{H}_\mathrm{f})} < 1$. Let

$$\begin{aligned} u_0(k,r) \;&:=\; (1-(\pi_0(k)-\pi_0(k,r))^2)^{-\frac{1}{2}}\big(\tfrac{\langle\varphi(k-A(r)),\varphi(k)\rangle_{\mathcal{H}_\mathrm{f}}}{|\langle\varphi(k-A(r)),\varphi(k)\rangle_{\mathcal{H}_\mathrm{f}}|}e^{-\frac{i\theta}{2\pi}A_2(r)k_1}\pi_0(k)\pi_0(k,r) \\ &\quad +\pi_0^\perp(k)\pi_0^\perp(k,r)\big). \qquad (3.16)\end{aligned}$$

We immediately get that $u_0(k,r)$ is a unitary and that (3.14) holds. Moreover, a short calculation shows the correctness of (3.15): Note thereto that

$$(1-(\pi_0(k)-\pi_0(k,r))^2)\varphi(k) = |\langle\varphi(k-A(r)),\varphi(k)\rangle_{\mathcal{H}_\mathrm{f}}|^2\varphi(k)$$

and hence

$$\begin{aligned} & u_0(k,r)\varphi(k-A(r)) \\ &= (1-(\pi_0(k)-\pi_0(k,r))^2)^{-\frac{1}{2}}\frac{\langle\varphi(k-A(r)),\varphi(k)\rangle_{\mathcal{H}_\mathrm{f}}e^{-\frac{i\theta}{2\pi}A_2(r)k_1}\langle\varphi(k),\varphi(k-A(r))\rangle_{\mathcal{H}_\mathrm{f}}}{|\langle\varphi(k-A(r)),\varphi(k)\rangle_{\mathcal{H}_\mathrm{f}}|}\varphi(k) \\ &= \frac{\langle\varphi(k-A(r)),\varphi(k)\rangle_{\mathcal{H}_\mathrm{f}}\langle\varphi(k),\varphi(k-A(r))\rangle_{\mathcal{H}_\mathrm{f}}}{|\langle\varphi(k-A(r)),\varphi(k)\rangle_{\mathcal{H}_\mathrm{f}}|^2}e^{-\frac{i\theta}{2\pi}A_2(r)k_1}\varphi(k) \\ &= e^{-\frac{i\theta}{2\pi}A_2(r)k_1}\varphi(k).\end{aligned}$$

Now we check the τ-equivariance of the symbol. It suffices to show that the phase in (3.16) is periodic in k with respect to Γ^*, which follows from an elementary calculation. So u_0 is a combination of τ-equivariant functions and hence τ-equivariant. It remains to show the boundedness of the derivatives. Because of the τ-equivariance of u_0, we have to show boundedness only for $(k,r)\in M^*\times\mathbb{R}^2$. Thereto we take a closer look at the behaviour of the derivatives of φ: For the derivative in k_1-direction, it can be easily seen that $\|\partial_1\varphi(k)\|_{\mathcal{H}_\mathrm{f}}$ is periodic with respect to Γ^* and thus bounded:

$$\partial_1\varphi(k-\gamma^*) = \partial_1(\tau(\gamma^*)e^{-\frac{i\theta}{2\pi}k_2\gamma_1^*}\varphi(k)) = \tau(\gamma^*)e^{-\frac{i\theta}{2\pi}k_2\gamma_1^*}\partial_1\varphi(k).$$

For the derivative in the other direction we do not get a periodicity:

$$\begin{aligned}\partial_2\varphi(k-\gamma^*) &= \partial_2(\tau(\gamma^*)e^{-\frac{i\theta}{2\pi}k_2\gamma_1^*}\varphi(k)) \\ &= \tau(\gamma^*)(-\tfrac{i\theta}{2\pi}\gamma_1^*)e^{-\frac{i\theta}{2\pi}k_2\gamma_1^*}\varphi(k)+\tau(\gamma^*)e^{-\frac{i\theta}{2\pi}k_2\gamma_1^*}\partial_2\varphi(k).\end{aligned}$$

So we can state that $\partial_2\varphi(k)$ has a growth which can only be controlled if we have a bounded k_1 variable. Thus we need the boundedness of the potential $A_1(r)$. Note that this is one of the obstructions of taking A as a linear function (which means as a potential of a constant magnetic field). All in all, it is now clear that $\langle\varphi(k-A(r)),\varphi(k)\rangle_{\mathcal{H}_\mathrm{f}}$ is bounded with all its derivatives, e.g.

$$
\begin{aligned}
&\sup_{(k,r)\in\mathbb{R}^2\times\mathbb{R}^2} |\partial_{k_2}\langle\varphi(k-A(r)),\varphi(k)\rangle_{\mathcal{H}_\mathrm{f}}| \\
&\quad\le \sup_{(k,r)\in M^*\times\mathbb{R}^2} (\|\varphi(k)\|_{\mathcal{H}_\mathrm{f}} \|\partial_2\varphi(k-A(r))\|_{\mathcal{H}_\mathrm{f}} + \|\partial_2\varphi(k)\|_{\mathcal{H}_\mathrm{f}} \|\varphi(k-A(r))\|_{\mathcal{H}_\mathrm{f}}) \\
&\quad\le \sup_{k'\in B_R(M^*)} \|\partial_2\varphi(k')\|_{\mathcal{H}_\mathrm{f}} + \sup_{k\in M^*} \|\partial_2\varphi(k)\|_{\mathcal{H}_\mathrm{f}} < \infty,
\end{aligned}
$$

where $R=\sup_{r\in\mathbb{R}^2}|A_1(r)|$ and $B_R(M^*)=\{k\in\mathbb{R}^2 : \mathrm{dist}(k,M^*)\le R\}$. Here we want to emphasise that one might think that if we take the gauge $A_1=0$, we could also insert a linear potential A. But this is not possible because then, one gets the additional phase $e^{-\frac{i\theta}{2\pi}A_2(r)k_1}$ which leads for example in the derivative in k_1-direction to a summand $-\frac{i\theta}{2\pi}A_2(r)\widetilde{u}_0(k,r)$ which is unbounded if A_2 is unbounded. Together with the boundedness of $|\langle\varphi(k-A(r)),\varphi(k)\rangle_{\mathcal{H}_\mathrm{f}}|$ from below, we get that $\frac{\langle\varphi(k-A(r)),\varphi(k)\rangle_{\mathcal{H}_\mathrm{f}}}{|\langle\varphi(k-A(r)),\varphi(k)\rangle_{\mathcal{H}_\mathrm{f}}|}$ is bounded together with all its derivatives. Thus u_0 is a combination of symbols in $S^1_\tau(\mathcal{L}(\mathcal{H}_\mathrm{f}))$ and hence in $S^1_\tau(\mathcal{L}(\mathcal{H}_\mathrm{f}))$.
Now we generalise this method to an arbitrary potential $A\in C^\infty_\mathrm{b}(\mathbb{R}^2,\mathbb{R}^2)$. Our idea is to define the desired unitary gradually exploiting the boundedness of A. Thereto note that there is a $\delta>0$ so that for all $k\in\mathbb{R}^2$ and for all $x,y\in\mathbb{R}^2$ with $\|x\|,\|y\|\le\|A\|_\infty$ it holds if $\|x-y\|<\delta$ then $|\langle\varphi(k-x),\varphi(k-y)\rangle_{\mathcal{H}_\mathrm{f}}|>\frac{1}{2}$. The proof is simple: If we assume the contrary this means that for every $n\in\mathbb{N}$ there is $k_n\in M^*$ (because of the periodicity of $\langle\varphi(k-x),\varphi(k-y)\rangle_{\mathcal{H}_\mathrm{f}}$) and $x_n,y_n\in\mathbb{R}^2$ with $\|x_n\|,\|y_n\|\le\|A\|_\infty$, $\|x_n-y_n\|<\frac{1}{n}$, and $|\langle\varphi(k_n-x_n),\varphi(k_n-y_n)\rangle_{\mathcal{H}_\mathrm{f}}|\le\frac{1}{2}$. Applying the Theorem of Bolzano-Weierstraß we get convergent subsequences $(x'_n)_n\to x$, $(y'_n)_n\to y$, and $(k'_n)_n\to k_0$. It must hold $x=y$ and hence $\lim_{n\to\infty}|\langle\varphi(k_n-x_n),\varphi(k_n-y_n)\rangle_{\mathcal{H}_\mathrm{f}}|=|\langle\varphi(k-x),\varphi(k-x)\rangle_{\mathcal{H}_\mathrm{f}}|=1>\frac{1}{2}$. Note that this proof will not work if A is not bounded.
Now we choose $m\in\mathbb{N}$ with $m>\frac{\|A\|_\infty}{\delta}$. For $j\in\{1,2,...m\}$

$$|\langle\varphi(k-\tfrac{j}{m}A(r)),\varphi(k-\tfrac{j-1}{m}A(r))\rangle_{\mathcal{H}_\mathrm{f}}|>\tfrac{1}{2}$$

holds since $\left\|\frac{j}{m}A(r)-\frac{j-1}{m}A(r)\right\|\le\frac{1}{m}\|A\|_\infty<\delta$. Thus we can define

$$
\begin{aligned}
&u_j(k,r)\\
&:=(1-(\pi_0(k-\tfrac{j-1}{m}A(r))-\pi_0(k-\tfrac{j}{m}A(r)))^2)^{-\frac{1}{2}}\Big(\frac{\langle\varphi(k-\frac{j}{m}A(r)),\varphi(k-\frac{j-1}{m}A(r))\rangle_{\mathcal{H}_\mathrm{f}}}{|\langle\varphi(k-\frac{j}{m}A(r)),\varphi(k-\frac{j-1}{m}A(r))\rangle_{\mathcal{H}_\mathrm{f}}|}\times\\
&\quad e^{-\frac{i\theta A_2(r)}{2\pi m}k_1}\pi_0(k-\tfrac{j-1}{m}A(r))\pi_0(k-\tfrac{j}{m}A(r))+\pi_0^\perp(k-\tfrac{j-1}{m}A(r))\pi_0^\perp(k-\tfrac{j}{m}A(r))\Big).
\end{aligned}
$$

Then $u_j\in S^1_\tau(\mathcal{L}(\mathcal{H}_\mathrm{f}))$, is a unitary, and fulfils

$$u_j(k,r)\pi_0(k-\tfrac{j}{m}A(r))u_j^*(k,r)=\pi_0(k-\tfrac{j-1}{m}A(r))$$

and

$$u_j(k,r)\varphi(k-\tfrac{j}{m}A(r)) = e^{-\frac{i\theta A_2(r)}{2\pi m}k_1}\varphi(k-\tfrac{j-1}{m}A(r))$$

for $j \in \{1,2,...m\}$. Finally we set

$$u_0(k,r) := u_1 \circ u_2 \circ ... \circ u_m(k,r).$$

This is the desired symbol.

Let $\widetilde{u}_0(k,r) := |\varphi(k)\rangle\,\langle\varphi(k-A(r))|\, e^{-\frac{i\theta}{2\pi}A_2(r)k_1}$. Obviously, $\widetilde{u}_0$ is a partial isometry with initial subspace $\pi_0(k-A(r))\mathcal{H}_{\mathrm{f}}$ and final subspace $\pi_0(k)\mathcal{H}_{\mathrm{f}}$. Now we can proceed along the lines of the construction in [Teu03, PST03b]. Note that, although in our case $\pi_{\mathrm{r}} = \pi_0(k)$ is k-dependent, the proof works as well. The only difference is that the symbol u is in $S^1_\tau(\mathcal{L}(\mathcal{H}_{\mathrm{f}}))$.

This means that we have to show by induction the existence of $u_n \in S^1_\tau(\mathcal{L}(\mathcal{H}_{\mathrm{f}}))$ for $n \in \mathbb{N}$ so that with $u^{(n)} = \sum_{j=0}^n \varepsilon^j u_j$ it holds

$$u^{(n)}\sharp u^{(n)*} = 1 + \mathcal{O}(\varepsilon^{n+1}) = u^{(n)*}\sharp u^{(n)}$$

and

$$u^{(n)}\sharp\pi\sharp u^{(n)*} = \pi_{\mathrm{r}} + \mathcal{O}(\varepsilon^{n+1}).$$

The induction starts with $n = 0$, where u_0 fulfils obviously everything. For the induction step $n \to n+1$ one sets (adopting the notation used in [Teu03])

$$u_{n+1} := (a_{n+1} + b_{n+1})u_0.$$

It remains to show $u_j \in S^1_\tau(\mathcal{L}(\mathcal{H}_{\mathrm{f}}))$ for all $j \in \mathbb{N}$. It is again clear for $j = 0$ as shown above. Let us assume that it holds for all $j \le n$. Then recall

$$A_{n+1} = [u^{(n)}\sharp u^{(n)*} - 1]_{n+1} \quad \text{and} \quad a_{n+1} = -\tfrac{1}{2}A_{n+1}.$$

From the assumption it follows that $u^{(n)}$ and $u^{(n)*}$ are in $S^1_\tau(\mathcal{L}(\mathcal{H}_{\mathrm{f}}))$ and thus, using Proposition B.3.5, A_{n+1} and hence a_{n+1} are in $S^1_\tau(\mathcal{L}(\mathcal{H}_{\mathrm{f}}))$. Analogously recall

$$B_{n+1} = [w^{(n)}\sharp\pi\sharp w^{(n)*} - \pi_{\mathrm{r}}]_{n+1} \quad \text{with} \quad w^{(n)} = u^{(n)} + \varepsilon^{n+1}a_{n+1}u_0.$$

Again from the assumption and the fact that $a_{n+1} \in S^1_\tau(\mathcal{L}(\mathcal{H}_{\mathrm{f}}))$, we get from Proposition B.3.5 that $w^{(n)} \in S^1_\tau(\mathcal{L}(\mathcal{H}_{\mathrm{f}}))$ and hence $B_{n+1} \in S^1_\tau(\mathcal{L}(\mathcal{H}_{\mathrm{f}}))$ and thus $b_{n+1} = [\pi_{\mathrm{r}}, B_{n+1}] \in S^1_\tau(\mathcal{L}(\mathcal{H}_{\mathrm{f}}))$. All in all, we get

$$u_{n+1} = (a_{n+1} + b_{n+1})u_0 \in S^1_\tau(\mathcal{L}(\mathcal{H}_{\mathrm{f}}))$$

which concludes the proof. $\square$

Remark 3.3.5. One might think that if we neglect in the previous proof that we want (3.15) to hold, we do not have to fil in the phase in the Nagy formula. Then it seems that we could do the construction also for a linear A. But this is not true because the boundedness of A is an essential ingredient to be able to apply the Nagy formula in the first place.

From the construction we directly get

Theorem 3.3.6. *Let h be a resummation in $S^1_\tau(\varepsilon, \mathcal{L}(\mathcal{H}_{\mathrm{f}}))$ of the formal symbol*

$$u\sharp\pi\sharp H\sharp\pi\sharp u^* \in M^1_\tau(\varepsilon, \mathcal{L}(\mathcal{H}_{\mathrm{f}})).$$

Then $\widehat{h} \in \mathcal{L}(\mathcal{H}_\tau)$, $[\widehat{h}, \Pi^0] = \mathcal{O}(\varepsilon^\infty)$, and

$$(e^{-iH^\varepsilon_{\mathrm{BF}}t} - U^{\varepsilon*}_1 e^{-i\widehat{h}t} U^\varepsilon_1)\Pi^\varepsilon = \mathcal{O}(\varepsilon^\infty(1+|t|)).$$

Proof.
First, note that h is really in the stated symbol class $S^1_\tau(\varepsilon, \mathcal{L}(\mathcal{H}_{\mathrm{f}}))$ since we get from Lemma 2.5.1 that $\pi \in S^{w=1+k^2}_\tau(\varepsilon, \mathcal{L}(\mathcal{H}_{\mathrm{f}}, \mathcal{H}^2_{A_0}))$, which implies that $H\sharp\pi \in S^{w=(1+k^2)^2}_\tau(\varepsilon, \mathcal{L}(\mathcal{H}_{\mathrm{f}})) = S^1_\tau(\varepsilon, \mathcal{L}(\mathcal{H}_{\mathrm{f}}))$ because of the τ-equivariance of the symbol and the fact that $\tau(\gamma^*)$ as an operator on $\mathcal{H}_{\mathrm{f}}$ is a unitary. Then we immediately get $h \in S^1_\tau(\varepsilon, \mathcal{L}(\mathcal{H}_{\mathrm{f}}))$. Hence it follows from Proposition B.3.6 that $\widehat{h}^\tau \in \mathcal{L}(\mathcal{H}_\tau)$. By construction, we have $[\widehat{h}, \Pi^0] = \mathcal{O}(\varepsilon^\infty)$. It remains to show the last statement:

$$\begin{aligned}
(e^{-i\widehat{H}^\tau t} - U^{\varepsilon*}_1 e^{-i\widehat{h}^\tau t} U^\varepsilon_1)\Pi^\varepsilon &= (e^{-i\widehat{H}^\tau t} - e^{-iU^{\varepsilon*}_1\widehat{h}^\tau U^\varepsilon_1 t})\widehat{\pi}^\tau + \mathcal{O}(\varepsilon^\infty) \\
&= (e^{-i\widehat{\pi}^\tau\widehat{H}^\tau\widehat{\pi}^\tau t} - e^{-iU^{\varepsilon*}_1\widehat{h}^\tau U^\varepsilon_1 t})\widehat{\pi}^\tau + \mathcal{O}(\varepsilon^\infty) \\
&= \mathcal{O}(\varepsilon^\infty(1+|t|)),
\end{aligned}$$

where the last equality follows from the usual Duhammel argument using that the difference of the generators is of order $\mathcal{O}(\varepsilon^\infty)$. ☐

Remark 3.3.7. Note that we only get $[\widehat{h}, \Pi^0] = \mathcal{O}(\varepsilon^\infty)$ and not exactly 0. In the corresponding theorem for the case $A_0 \equiv 0$, the projection π_{r} does not depend on k and is constant. Thus there we get from the construction

$$h = u\sharp\pi\sharp H\sharp\pi\sharp u^* = \pi_{\mathrm{r}}\sharp u\sharp H\sharp u^*\sharp\pi_{\mathrm{r}} = \pi_{\mathrm{r}} \circ h \circ \pi_{\mathrm{r}},$$

which follows directly from the definition (B.3) of the Moyal product and the fact that π_{r} is independent of k and r. Then it follows from the integral formula for the τ-quantisation that $\widehat{h}^\tau = \Pi_{\mathrm{r}}\widehat{h}^\tau\Pi_{\mathrm{r}}$ and thus $[\Pi_{\mathrm{r}}, \widehat{h}^\tau] = 0$.
However, in our case the projection π_{r} depends on k. Thus we cannot conclude $[\Pi^0, \widehat{h}^\tau] = 0$ as before. Moreover, neither it holds $h = \pi_0(k)h(k,r)\pi_0(k)$ nor

$\mathrm{Op}^\tau(\pi_0(k)h(k,r)\pi_0(k)) = \Pi^0\widehat{h}^{\,\tau}\Pi^0$. Instead, we will later define a quantisation $\mathrm{Op}^{\mathrm{Berry}}$ which will satisfy $\mathrm{Op}^{\mathrm{Berry}}(\pi_0(k)h(k,r)\pi_0(k)) = \Pi^0\widehat{h}^{\,\mathrm{Berry}}\Pi^0$. So this will just cause some technical effort. And for the moment we just work with $\Pi^0\widehat{h}^{\,\tau}\Pi^0$ which is at least $\mathcal{O}(\varepsilon^\infty)$ close to $\widehat{h}^{\,\tau}$.

Remark 3.3.8. We get an operator which leaves the fibers of the Bloch bundle almost invariant, but the symbol does not have to commute almost with $\pi_0(k)$. Again this will be the case for the new quantisation $\mathrm{Op}^{\mathrm{Berry}}$ which will be introduced in Chapter 4.

Now we can take

$$U^\varepsilon := U_\theta \circ U_1^\varepsilon$$

as the interwining unitary between $\Pi^0\mathcal{H}_\tau$ and $\mathcal{H}_\theta$ and

$$H_{\mathrm{eff}} := U^\theta \Pi^0 \widehat{h} \Pi^0 U^{\theta *}$$

as the effective Hamiltonian operating on $\mathcal{H}_\theta$. But this is not yet the form of an effective operator we are aiming for, because we want to get a description of the effective Hamiltonian as the quantisation of a semiclassical symbol. Here we cannot proceed along the lines of the constructions in [Teu03, PST03b], since we are not able to construct a symbol whose quantisation is U^θ. This problem arises because the Bloch bundle is no longer trivial and thus the partial derivative with respect to k_2 of $u(k,r) = \langle\varphi(k - A(r))|$ has a growth in k_1 which cannot be controlled. How we can get H_{eff} as the quantisation of a semiclassical symbol will be the content of the next two chapters. We will have to define new quantisations and show methods how to translate one quantisation into the other one.

3.4 The corresponding results for an arbitrary Bravais lattice Γ

For a general Bravais lattice Γ, denote the components of the generating vectors γ^1 and γ^2 by $\gamma^1 = (\gamma^1_1, \gamma^1_2)$ and $\gamma^2 = (\gamma^2_1, \gamma^2_2)$ and analogously for the dual lattice $\gamma^{1*} = (\gamma^{1*}_1, \gamma^{1*}_2)$ and $\gamma^{2*} = (\gamma^{2*}_1, \gamma^{2*}_2)$. Then it holds for the function φ:

Lemma 3.4.1. *There is a function $\varphi \in C^\infty(\mathbb{R}^2, \mathcal{H}_{\mathrm{f}})$ so that for all $k \in \mathbb{R}^2$ we have $\varphi(k) \in P(k)\mathcal{H}_{\mathrm{f}}$, $\|\varphi(k)\|_{\mathcal{H}_{\mathrm{f}}} = 1$, and*

$$\varphi(k - \gamma^*) = e^{-\frac{i\theta}{2\pi}\langle\gamma^2,k\rangle\langle\gamma^1,\gamma^*\rangle}\tau(\gamma^*)\varphi(k) \quad \textit{for all} \quad \gamma^* \in \Gamma^*, \tag{3.17}$$

where θ is the Chern number of the Bloch bundle (3.1).

Proof.
The proof works as the proof of Lemma 3.3.2. The difference is that one has to do the construction of $\widetilde{\varphi}$ with respect to the basis $\{\gamma^{1*}, \gamma^{2*}\}$ of $\mathbb{Z}^2$. This means to take $k_j^* = \frac{1}{2\pi}\langle \gamma^j, k\rangle$ for $j \in \{1,2\}$ and set

$$\widetilde{\varphi}(0, k_2^*) := t^{\mathrm{B}}((0, k_2^*), (0,0))h_0$$

and

$$\widetilde{\varphi}(k_1^*, k_2^*) := t^{\mathrm{B}}((k_1^*, k_2^*), (0, k_2^*))\widetilde{\varphi}(0, k_2^*).$$

Also the derivative $\widetilde{\partial}_j$ and $\widetilde{\mathcal{A}}$ are with respect to this basis. This yields

$$\begin{aligned} \alpha(k^*, \gamma^*) &= e^{2\pi i\theta a k_2^*} \quad \text{where} \quad \gamma^* = a\gamma^{1*} + b\gamma^{2*} \\ &= e^{-\frac{i\theta}{2\pi}\langle \gamma^2, k\rangle\langle\gamma^1, \gamma^*\rangle}. \end{aligned}$$

□

This means that in analogy to Remark 3.3.3 one works with

- the function φ, which fulfils $\varphi(k-\gamma^*) = \tau(\gamma^*)e^{-\frac{i\theta}{2\pi}\langle\gamma^2,k\rangle\langle\gamma^1,\gamma^*\rangle}\varphi(k)$ for all $k \in \mathbb{R}^2$ and $\gamma^* \in \Gamma^*$,
- the Hilbert space $\mathcal{H}_\theta = \{\psi \in L^2_{\mathrm{loc}}(\mathbb{R}^2) : \psi(k-\gamma^*) = e^{\frac{i\theta}{2\pi}\langle\gamma^2,k\rangle\langle\gamma^1,\gamma^*\rangle}\psi(k)$ for all $k \in \mathbb{R}^2$ and $\gamma^* \in \Gamma^*\}$ with inner product $\langle \psi, \phi\rangle_{\mathcal{H}_\theta} = \int_{M^*} \overline{\psi(k)}\phi(k)\mathrm{d}k$, where $\mathrm{d}k$ denotes the normalised Lebesgue measure,
- the operators $\partial_j^\theta : H^1_{\mathrm{loc}}(\mathbb{R}^2) \cap \mathcal{H}_\theta \to \mathcal{H}_\theta$ defined by $\partial_j^\theta := U^\theta \partial_j^\tau U^{\theta *}$ for $j \in \{1,2\}$, and
- $\mathcal{A}_j := \langle \varphi(k), \partial_j\varphi(k)\rangle_{\mathcal{H}_{\mathrm{f}}}$ for $j \in \{1,2\}$. In particular, we have for $j \in \{1,2\}$ that $\mathcal{A}_j(k-\gamma^*) = -\frac{i\theta}{2\pi}\gamma_j^2\langle\gamma^1,\gamma^*\rangle + \mathcal{A}_j(k)$ for all $k \in \mathbb{R}^2$ and $\gamma^* \in \Gamma^*$.

Note that now the derivatives are in the ordinary directions $\partial_j = \partial_{e_j}$ for $j \in \{1,2\}$ and not any more in the directions of the generating vectors of the Bravais lattice Γ^*.
Also the definition of the first part of the intertwining unitary has to be adapted to the lattice:

Theorem 3.4.2. *There exists a unitary operator $U_1^\varepsilon : \mathcal{H}_\tau \to \mathcal{H}_\tau$ such that*

$$U_1^\varepsilon \Pi^\varepsilon U_1^{\varepsilon *} = \Pi^0$$

and $U_1^\varepsilon = \widehat{u} + \mathcal{O}_0(\varepsilon^\infty)$, where $u \asymp \sum_{j\geq 0}\varepsilon^j u_j$ belongs to $S^1_\tau(\varepsilon, \mathcal{L}(\mathcal{H}_{\mathrm{f}}))$ and has the principal symbol $u_0(k,r) = |\varphi(k)\rangle\,\langle\varphi(k-A(r))|\,e^{-\frac{i\theta}{2\pi}\langle\gamma^1,k\rangle\langle\gamma^2,A(r)\rangle} + u_0^\perp(k,r)$.

For the perturbation A the appropriate gauge is $\langle \gamma^2, A(r) \rangle \equiv 0$.

Proof.
The proof works as the proof of Theorem 3.3.4. The only difference is that the norm of $\partial_1 \varphi$ is no longer periodic, but as for $\partial_2 \varphi$ this is not a problem for a bounded perturbation A. $\square$

Chapter 4

The new Weyl quantisations

4.1 Motivation

The motivation for this chapter is to develop methods to express the effective Hamiltonian $U^\theta \Pi^0 \widehat{h} U^{\theta *}$ obtained in the previous chapter as the quantisation of a semiclassical symbol. To this purpose, we have to take a different approach than usually (for example in the case $A_0 \equiv 0$). It is clear that we are going to need a different pseudodifferential calculus than the one we have used so far. Our strategy is to define three new pseudodifferential calculi and to prove theorems which tell us how and for which symbols we can translate one calculus into the other. Here, translating one calculus into the other means to compute corrections f_c of a symbol f so that

$$\widehat{f(k,r)}^{\text{"old quantisation"}} \approx \widehat{f_\mathrm{c}(k,r)}^{\text{"new quantisation"}},$$

where the quantisations and the "$\approx$" will be specified below in terms of an exact definition respectively in terms of $\mathcal{O}(\varepsilon^n)$ with $n \in \mathbb{N} \cup \{\infty\}$.
The first quantisation we are going to introduce is called "Berry quantisation". The difference to the τ-quantisation is that for a symbol $f(k,r)$, the variable r gets replaced by $-i\varepsilon\nabla^\mathrm{B}$ which is supposed to be the Berry connection. The Berry quantisation has the advantage that the connection ∇^B restricted to sections on the Bloch bundle is unitarily equivalent to the connection ∇^θ on the bundle $E_\theta = \{(k,\lambda) \in (\mathbb{R}^2 \times \mathbb{C})_\sim\}$, where $(k,\lambda) \sim (k',\lambda') :\Leftrightarrow \exists \gamma^* \in \Gamma^* : k' = k - \gamma^*$ and $\lambda' = e^{\frac{i\theta}{2\pi} k_2 \gamma_1^*}\lambda$. The bundle E_θ, of course, is the bundle which can be identified with the Bloch bundle E_Bl in a natural way. A second advantage of the Berry quantisation is that its symbol commutes with $\pi_0(k)$ pointwise if and only if its Berry quantisation commutes with Π^0.
The next quantisation we need is the θ-quantisation. This is a quantisation which induces operators that act on $\mathcal{H}_\theta$ for suitable symbols. Roughly speaking it re-

places r by $-i\varepsilon\nabla^\theta$. After that, we show how to translate $U^\theta \widehat{f}^{\,\mathrm{Berry}} U^{\theta*}$ into $\widehat{f_\theta}^{\,\theta}$ exploiting the unitary equivalence of the connections ∇^{B} and ∇^θ.
After this, we introduce a last quantisation which we call effective quantisation because it is the final quantisation for our effective operator $H_{\mathrm{eff}} \overset{!}{=} \widehat{h_{\mathrm{eff}}}^{\,\mathrm{eff}}$. The difference to the θ-quantisation is that r is replaced by the connection $-i\varepsilon\nabla_k^{\mathrm{eff}} = -i\varepsilon(\nabla_k + (0, \frac{i\theta}{2\pi}k_1)^{\mathrm{T}})$, which is a canonical connection on the bundle E_θ because it is the one where the curvature is just $\frac{i\theta}{2\pi}$. If the curvature of the bundle E_θ was indeed $\frac{i\theta}{2\pi}$, it would follow from the construction of φ in Lemma 3.3.2 that the connections ∇^θ and ∇^{eff} are the same. Accessorily, this connection is independent of the function $\varphi(k)$. Moreover, the usage of this connection makes it possible to compare our case to the non-magnetic case $A_0 \equiv 0$ where the Bloch bundle is trivial. One just has to recall that a line bundle is trivial if and only if its Chern number θ is zero, see for example [BT82]. So in this case, we get with our approach $\mathcal{H}_{\theta=0} \cong L^2(\mathbb{T}^{2*})$ and $\nabla_k^{\mathrm{eff}} = \nabla_k$, so it just includes the case $A_0 \equiv 0$. We will see that we get the appropriate symbols for h_{eff} when we compute its principal and subprincipal symbol.
Our goal is to get quantisation formulas that map suitable symbols to pseudodifferential operators that act on sections of possibly non-trivial bundles. There are some similar works about such quantisation maps in the literature, as in [Pfl98a, Pfl98b, Saf98, Sha05a, Sha05b, Han10]. As opposed to the Euclidian case, the relation between a pseudodifferential operator on sections of vector bundles and its symbol becomes more subtle. If one just defines a corresponding pseudodifferential calculus in local coordinates, like this is for example done in [Hör85], one can associate a symbol to an operator which is unique only up to an error of ε. To define a full symbol, one has to take into account the geometry of the vector bundle. This means that instead of local coordinates, one must use a connection on the vector bundle and a connection on the base space. This idea goes back to Widom [Wid78, Wid80], who was the first to develop a complete isomorphism between such pseudodifferential operators and their symbols. However, he just showed how to recover the full symbol from a pseudodifferential operator and proved that this map is bijective, but he did not show how to get from a symbol to the corresponding pseudodifferential operator. His work was developed further by Pflaum [Pfl98b] and Safarov [Saf98]. In [Pfl98b], the author is the first to give a quantisation map which maps symbols that are sections of endomorphism bundles to operators between the sections of the corresponding bundles. In his quantisation formulas he uses a cutoff function so that he can use the exponential map corresponding to a given connection on the manifold that may not be defined globally. A geometric symbol calculus for pseudodifferential operators between sections of vector bundles can also be found in [Sha05a, Sha05b], where the author moreover introduces the notion of a geometric symbol in comparison to a coordinatewise

symbol. A semiclassical variant of this calculus can be found in [Han10]. When we compute the correction f_{c} of f so that $\widehat{f}^{\,\tau} = \widehat{f_{\mathrm{c}}}^{\mathrm{B}}$, one could say, using the language of [Sha05a, Sha05b], that f_{c} is the geometric symbol with respect to the Berry connection of the operator $\widehat{f}^{\,\tau}$.
But in all these works there is no Weyl calculus. In [Saf98] and [Pfl98a], the authors give formulas for Weyl quantisations but only for pseudodifferential operators on manifolds and not for operators between sections of vector bundles. Additionally, the authors only use Hörmander symbol classes, see [Hör85]. So what we are going to do is to define semiclassical Weyl calculi for more general symbol classes. Moreover, the Berry calculus is a calculus on a bundle whose fiber is an infinite dimensional Hilbert space. Although in our cases the exponential map will always be defined everywhere, we use cutoff functions since we need them to get control over the parallel transport maps corresponding to the connections on the bundles and their derivatives. Another difference is that we will transfer the phase space $\mathbb{R}^2 \times \mathbb{R}^2$ to $\mathbb{T}^{2*} \times \mathbb{R}^2$ by using periodic-like conditions for symbols and functions. This approach is also used in [GN98] and [Teu03, PST03b].

4.2 The Berry quantisation

Let us quickly sketch the following proceeding. We start with the bundle

$$E' = \mathbb{R}^2 \times \mathcal{H}_{\mathrm{f}} \quad \text{with connection} \quad \nabla^{\mathrm{B}} = P(k)\nabla P(k) + P^{\perp}(k)\nabla P^{\perp}(k).$$

Note that the property (3.2) of the parallel transport with respect to the Berry connection still holds since besides P also $P^{\perp}$ is τ-equivariant. For symbols $f : \mathbb{R}^4 \to \mathcal{L}(\mathcal{H}_{\mathrm{f}})$ we then define a quantisation $\widehat{f}^{\,\mathrm{B}}$ on the sections $\varphi : \mathbb{R}^2 \to \mathcal{H}_{\mathrm{f}}$. As already indicated, we transfer this results to the bundle

$$E = (\mathbb{R}^2 \times \mathcal{H}_{\mathrm{f}})_{\sim} \quad \text{with connection} \quad \nabla^{\mathrm{B}} = P(k)\nabla P(k) + P^{\perp}(k)\nabla P^{\perp}(k),$$

where

$$(k, \varphi) \sim (k', \varphi') :\Leftrightarrow k' = k - \gamma^* \quad \text{and} \quad \varphi' = \tau(\gamma^*)\varphi.$$

This is done by showing that for τ-equivariant symbols $f : \mathbb{R}^4 \to \mathcal{L}(\mathcal{H}_{\mathrm{f}})$ the quantisation $\widehat{f}^{\,\mathrm{B}}$ maps sections $\varphi : \mathbb{R}^2 \to \mathcal{H}_{\mathrm{f}}$ of the bundle E, that are the τ-equivariant functions, to sections of E.
Then we collect several important properties of this quantisation; among them is the fact that if and only if a τ-equivariant symbol commutes with $P(k)$ pointwise, the corresponding pseudodifferential operator commutes with Π^0. Hence we can say that for those symbols we have a quantisation that is an operator between sections of the Bloch bundle (3.1) with the connection $\nabla^{\mathrm{B}} = P(k)\nabla$.

At the end of this section we will show how we can correct a symbol so that $\widehat{f}^{\,\tau} \approx \widehat{f_{\mathrm{c}}}^{\mathrm{B}}$.
Let us now start the rigorous maths and introduce the Berry quantisation.

Definition 4.2.1. *A function $\chi \in C^\infty(\mathbb{R}^2)$ is called a smooth cutoff function if* $\mathrm{supp}\chi$ *is compact, $\chi \equiv 1$ in a neigbourhood of* 0*, and* $0 \leq \chi \leq 1$.

Throughout this chapter, we will always assume for τ-equivariant symbols that τ is a unitary representation and $\tau_1 = \tau_2 = \tau$.

Definition 4.2.2. *Let $f \in S^w(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_{\mathrm{f}})) \cup S^m_\rho(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_{\mathrm{f}}))$ and let $t^{\mathrm{B}}(x,y)$ be the parallel transport with respect to the Berry connection $\nabla^{\mathrm{B}} = \Pi^0 \nabla \Pi^0 + \Pi^{0^\perp} \nabla \Pi^{0^\perp}$ along the straight line from y to x. Let $\chi \in C_0^\infty(\mathbb{R}^2)$ be a smooth cutoff function. Then the Berry quantisation $\widehat{f}^{\,\mathrm{B},\chi}$ is defined by*

$$(\widehat{f}^{\,\mathrm{B},\chi}\psi)(k) = \frac{1}{(2\pi\varepsilon)^2} \int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}} \chi(k-y) t^{\mathrm{B}}\left(k, \tfrac{k+y}{2}\right) f\left(\tfrac{k+y}{2}, r\right) t^{\mathrm{B}}\left(\tfrac{k+y}{2}, y\right) \psi(y) \mathrm{d}r\mathrm{d}y$$

for $\psi \in S(\mathbb{R}^2, \mathcal{H}_{\mathrm{f}})$.

Remark 4.2.3. We will see that for suitable symbols the Berry quantisation does not depend on the cutoff up to an error of $\mathcal{O}(\varepsilon^\infty)$.

To show the well-definedness of this quantisation, we follow the usual routine: First, we show that the quantised symbol is a continuous map from the Schwartz space $S(\mathbb{R}^2, \mathcal{H}_{\mathrm{f}})$ to itself. Then we extend this mapping by duality to $S'(\mathbb{R}^2, \mathcal{H}_{\mathrm{f}})$. After that, we show that for τ-equivariant symbols the quantised symbol maps τ-equivariant functions to τ-equivariant functions.

Proposition 4.2.4. *For $f \in S^w(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_{\mathrm{f}})) \cup S^m_\rho(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_{\mathrm{f}}))$, χ a smooth cutoff function, and $\psi \in S(\mathbb{R}^2, \mathcal{H}_{\mathrm{f}})$ the integral*

$$(\widehat{f}^{\,\mathrm{B},\chi}\psi)(k) = \frac{1}{(2\pi\varepsilon)^2} \int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}} \chi(k-y) t^{\mathrm{B}}\left(k, \tfrac{k+y}{2}\right) f\left(\tfrac{k+y}{2}, r\right) t^{\mathrm{B}}\left(\tfrac{k+y}{2}, y\right) \psi(y) \mathrm{d}r\mathrm{d}y$$

defines a continuous mapping from $S(\mathbb{R}^2, \mathcal{H}_{\mathrm{f}})$ to $S(\mathbb{R}^2, \mathcal{H}_{\mathrm{f}})$.

Proof.
For the proof, we can proceed along the standard lines, like this is for example done in the appendix of [Teu03]. The only problem we have to overcome is that the derivatives of the parallel transport t^{B} are in general not bounded. To deal with this, we have introduced the cutoff function in the definition of the Berry quantisation. Then one can exploit the property (3.2) of t^{B}.

Let $f \in S^m_\rho(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathrm{f}))$ and $\delta > 0$ so that $\mathrm{supp}\chi \subset B_\delta(0)$. First, we show that $\sup_{k\in\mathbb{R}^2} \left\| (\widehat{f}^{\,\mathrm{B},\chi}\psi)(k) \right\|_{\mathcal{H}_\mathrm{f}} < \infty$. From

$$
\begin{aligned}
&(\widehat{f}^{\,\mathrm{B},\chi}\psi)(k) \\
&= \tfrac{1}{(2\pi\varepsilon)^2} \int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}} \chi(k-y) t^\mathrm{B}\left(k, \tfrac{k+y}{2}\right) f\left(\tfrac{k+y}{2}, r\right) t^\mathrm{B}\left(\tfrac{k+y}{2}, y\right) \psi(y) \mathrm{d}r\mathrm{d}y \\
&= \tfrac{1}{(2\pi\varepsilon)^2} \int_{\mathbb{R}^4} \left(\tfrac{1-\varepsilon^2\Delta_y}{\langle r\rangle^2}\right)^M e^{\frac{i(k-y)r}{\varepsilon}} \chi(k-y) t^\mathrm{B}\left(k, \tfrac{k+y}{2}\right) f\left(\tfrac{k+y}{2}, r\right) t^\mathrm{B}\left(\tfrac{k+y}{2}, y\right) \\
&\quad \psi(y) \mathrm{d}r\mathrm{d}y \\
&= \tfrac{1}{(2\pi\varepsilon)^2} \int_{\mathbb{R}^4} \tfrac{1}{\langle r\rangle^{2M}} e^{\frac{i(k-y)r}{\varepsilon}} \left(1-\varepsilon^2\Delta_y\right)^M \left(\chi(k-y) t^\mathrm{B}\left(k, \tfrac{k+y}{2}\right) f\left(\tfrac{k+y}{2}, r\right)\right. \\
&\quad \left. t^\mathrm{B}\left(\tfrac{k+y}{2}, y\right) \psi(y)\right) \mathrm{d}r\mathrm{d}y && (4.1) \\
&= \tfrac{1}{(2\pi\varepsilon)^2} \sum_{|\alpha_1+\ldots+\alpha_5|\leq 2M} c_{\alpha_1\ldots\alpha_5} \int_{\mathbb{R}^4} \tfrac{1}{\langle r\rangle^{2M}} e^{\frac{i(k-y)r}{\varepsilon}} \partial_y^{\alpha_1}\chi(k-y) \partial_y^{\alpha_2} t^\mathrm{B}\left(k, \tfrac{k+y}{2}\right) && (4.2) \\
&\quad \partial_y^{\alpha_3} f\left(\tfrac{k+y}{2}, r\right) \partial_y^{\alpha_4} t^\mathrm{B}\left(\tfrac{k+y}{2}, y\right) \partial_y^{\alpha_5}\psi(y) \mathrm{d}r\mathrm{d}y && (4.3)
\end{aligned}
$$

we get

$$
\begin{aligned}
&\left\| (\widehat{f}^{\,\mathrm{B},\chi}\psi)(k) \right\|_{\mathcal{H}_\mathrm{f}} \\
&\leq \tfrac{C_M}{(2\pi\varepsilon)^2} \sum_{|\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5|\leq 2M} \int_{\mathbb{R}^4} \tfrac{1}{\langle r\rangle^{2M}} |\partial_y^{\alpha_1}\chi(k-y)| \left\|\partial_y^{\alpha_2} t^\mathrm{B}\left(k, \tfrac{k+y}{2}\right)\right\|_{\mathcal{L}(\mathcal{H}_\mathrm{f})} \\
&\quad \left\|\partial_y^{\alpha_3} f\left(\tfrac{k+y}{2}, r\right)\right\|_{\mathcal{L}(\mathcal{H}_\mathrm{f})} \left\|\partial_y^{\alpha_4} t^\mathrm{B}\left(\tfrac{k+y}{2}, y\right)\right\|_{\mathcal{L}(\mathcal{H}_\mathrm{f})} \left\|\partial_y^{\alpha_5}\psi(y)\right\|_{\mathcal{H}_\mathrm{f}} \mathrm{d}r\mathrm{d}y \\
&\leq C'_M \underbrace{\sup_{|k-y|\leq\delta} \sup_{|\alpha|\leq 2M} \left\|\partial_y^{\alpha} t^\mathrm{B}\left(k, \tfrac{k+y}{2}\right)\right\|_{\mathcal{L}(\mathcal{H}_\mathrm{f})} \cdot \sup_{|k-y|\leq\delta} \sup_{|\alpha|\leq 2M} \left\|\partial_y^{\alpha} t^\mathrm{B}\left(\tfrac{k+y}{2}, y\right)\right\|_{\mathcal{L}(\mathcal{H}_\mathrm{f})}}_{:=C(k)} \cdot \\
&\quad \sum_{|\alpha|\leq 2M} \int_{\mathbb{R}^4} \tfrac{1}{\langle r\rangle^{2M-m}} \left\|\partial_y^{\alpha}\psi(y)\right\|_{\mathcal{H}_\mathrm{f}} \mathrm{d}r\mathrm{d}y && (4.4) \\
&\leq C'_M C(k) \sum_{|\alpha|\leq 2M} \int_{\mathbb{R}^2} \tfrac{1}{\langle r\rangle^{2M-m}} \mathrm{d}r \int_{\mathbb{R}^2} \left\|\partial_y^{\alpha}\psi(y)\right\|_{\mathcal{H}_\mathrm{f}} \mathrm{d}y \\
&\leq C''_M C(k) \sup_{|\alpha|\leq 2M} \left\|\partial^\alpha\psi\right\|_{L^1(\mathbb{R}^2, \mathcal{H}_\mathrm{f})}.
\end{aligned}
$$

So after the partial integration in (4.1), the occurrence of the parallel transport maps t^B and the cutoff function causes new factors of the form $\partial_y^\alpha t^\mathrm{B}(k, \frac{k+y}{2})$ and derivatives of the cutoff. For the cutoff, this is not a problem since the support stays in $B_\delta(0)$. But the derivatives of t^{B} induce the constant $C(k)$. We need to show $\sup_{k\in\mathbb{R}^2} C(k) < \infty$. Using $\frac{k+y}{2} = k + \frac{y-k}{2}$ and (3.2) we get

$$\begin{aligned}
&\sup_{k\in\mathbb{R}^2} \sup_{|k-y|\leq\delta} \sup_{|\alpha|\leq 2M} \left\|\partial_y^\alpha t^{\mathrm{B}}\left(k, \tfrac{k+y}{2}\right)\right\|_{\mathcal{L}(\mathcal{H}_{\mathrm{f}})} \\
&\quad\leq \sup_{\gamma^*\in\Gamma^*} \sup_{k\in M^*} \sup_{|v|\leq\delta} \sup_{|\alpha|\leq 2M} \left\|\partial_v^\alpha t^{\mathrm{B}}\left(k-\gamma^*, k-\gamma^* + \tfrac{v}{2}\right)\right\|_{\mathcal{L}(\mathcal{H}_{\mathrm{f}})} \\
&\quad= \sup_{k\in M^*} \sup_{|v|\leq\delta} \sup_{|\alpha|\leq 2M} \left\|\partial_v^\alpha t^{\mathrm{B}}\left(k, k + \tfrac{v}{2}\right)\right\|_{\mathcal{L}(\mathcal{H}_{\mathrm{f}})} \leq C. \qquad (4.5)
\end{aligned}$$

Analogously one can bound the second part of $C(k)$ using $y = k + (y-k)$. This yields the desired estimation.
Now we show for arbitrary $\alpha, \beta \in \mathbb{N}_0^2$ that $\sup_{k\in\mathbb{R}^2} \left\|k^\alpha \partial_k^\beta (\widehat{f}^{\,\mathrm{B},\chi}\psi)(k)\right\|_{\mathcal{H}_{\mathrm{f}}} < \infty$. From

$$\begin{aligned}
&k^\alpha \partial_k^\beta (\widehat{f}^{\,\mathrm{B},\chi}\psi)(k) \\
&= \frac{1}{(2\pi\varepsilon)^2} \int_{\mathbb{R}^4} \frac{k^\alpha}{\langle r\rangle^{2M}} e^{\frac{i(k-y)r}{\varepsilon}} \left(1-\varepsilon^2\Delta_y\right)^M \partial_k^\beta(\chi(k-y) t^{\mathrm{B}}\left(k, \tfrac{k+y}{2}\right) f\left(\tfrac{k+y}{2}, r\right) \\
&\quad t^{\mathrm{B}}\left(\tfrac{k+y}{2}, y\right)\psi(y))\mathrm{d}r\mathrm{d}y \\
&= \frac{1}{(2\pi\varepsilon)^2} \int_{\mathbb{R}^4} \frac{(y-i\varepsilon\partial_r)^\alpha}{\langle r\rangle^{2M}} e^{\frac{i(k-y)r}{\varepsilon}} \left(1-\varepsilon^2\Delta_y\right)^M \partial_k^\beta(\chi(k-y) t^{\mathrm{B}}\left(k, \tfrac{k+y}{2}\right) f\left(\tfrac{k+y}{2}, r\right) \\
&\quad t^{\mathrm{B}}\left(\tfrac{k+y}{2}, y\right)\psi(y))\mathrm{d}r\mathrm{d}y \qquad (4.6) \\
&= \frac{1}{(2\pi\varepsilon)^2} \int_{\mathbb{R}^4} \frac{1}{\langle r\rangle^{2M}} e^{\frac{i(k-y)r}{\varepsilon}} (y+i\varepsilon\partial_r)^\alpha \left(1-\varepsilon^2\Delta_y\right)^M \partial_k^\beta(\chi(k-y) t^{\mathrm{B}}\left(k, \tfrac{k+y}{2}\right) \\
&\quad f\left(\tfrac{k+y}{2}, r\right) t^{\mathrm{B}}\left(\tfrac{k+y}{2}, y\right)\psi(y))\mathrm{d}r\mathrm{d}y \\
&= \frac{1}{(2\pi\varepsilon)^2} \sum_{\alpha_6+\alpha_7=\alpha} \sum_{|\alpha_1+\ldots+\alpha_5|\leq 2M} \sum_{\alpha_8+\ldots+\alpha_{11}=\beta} c_{\alpha_6,\alpha_7} c_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5} c_{\alpha_8,\alpha_9,\alpha_{10},\alpha_{11}} \\
&\quad \int_{\mathbb{R}^4} \frac{1}{\langle r\rangle^{2M}} e^{\frac{i(k-y)r}{\varepsilon}} \partial_y^{\alpha_1}\partial_k^{\alpha_8}\chi(k-y) \partial_y^{\alpha_2}\partial_k^{\alpha_9} t^{\mathrm{B}}\left(k, \tfrac{k+y}{2}\right) \partial_r^{\alpha_7}\partial_y^{\alpha_3}\partial_k^{\alpha_{10}} f\left(\tfrac{k+y}{2}, r\right) \\
&\quad \partial_y^{\alpha_4}\partial_k^{\alpha_{11}} t^{\mathrm{B}}\left(\tfrac{k+y}{2}, y\right) y^{\alpha_6}\partial_y^{\alpha_5}\psi(y))\mathrm{d}r\mathrm{d}y,
\end{aligned}$$

where in (4.6) we used the equality

$$k^\alpha e^{\frac{i(k-y)r}{\varepsilon}} = (y-i\varepsilon\partial_r)^\alpha e^{\frac{i(k-y)r}{\varepsilon}},$$

we get

$$
\begin{aligned}
&\left\|k^\alpha \partial_k^\beta (\widehat{f}^{\,\mathrm{B},\chi}\psi)(k)\right\|_{\mathcal{H}_\mathrm{f}} \\
&\leq C_M \sum_{\alpha_6+\alpha_7=\alpha} \sum_{|\alpha_1+\ldots+\alpha_5|\leq 2M} \sum_{\alpha_8+\ldots+\alpha_{11}=\beta} |c_{\alpha_6,\alpha_7} c_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5} c_{\alpha_8,\alpha_9,\alpha_{10},\alpha_{11}}| \\
&\quad \int_{\mathbb{R}^4} \tfrac{1}{\langle r\rangle^{2M-m}} \left\|y^{\alpha_6}\partial_y^{\alpha_5}\psi(y)\right\|_{\mathcal{H}_\mathrm{f}} \mathrm{d}r\mathrm{d}y \qquad (4.7) \\
&\leq C'_M \sup_{\gamma\leq\alpha, |\beta|\leq 2M} \left\|y^\gamma \partial^\beta \psi\right\|_{L^1(\mathbb{R}^2,\mathcal{H}_\mathrm{f})}.
\end{aligned}
$$

All in all, this implies that $\widehat{f}^{\,\mathrm{B},\chi}$ maps S to S. Now we use the continuity of the embedding $S(\mathbb{R}^2,\mathcal{H}_\mathrm{f}) \hookrightarrow L^1(\mathbb{R}^2,\mathcal{H}_\mathrm{f})$. Let for $n, m \in \mathbb{N}_0$

$$
\|\psi\|_{n,m(S(\mathbb{R}^2,\mathcal{H}_\mathrm{f}))} := \sup_{|\alpha|\leq n, |\beta|\leq m} \sup_{k\in\mathbb{R}^2} \left\|k^\alpha \partial_k^\beta \psi(k)\right\|_{\mathcal{H}_\mathrm{f}}.
$$

Then we have shown that for arbitrary $n, m \in \mathbb{N}_0$ there exist $k, l \in \mathbb{N}_0$ such that $\left\|\widehat{f}^{\,\mathrm{B},\chi}\psi\right\|_{n,m(S(\mathbb{R}^2,\mathcal{H}_\mathrm{f}))} \leq C\,\|\psi\|_{k,l(S(\mathbb{R}^2,\mathcal{H}_\mathrm{f}))}$, which implies that $\widehat{f}^{\,\mathrm{B},\chi}$ is a continuous map from S to S.
For symbols $f \in S^w(\mathbb{R}^4,\mathcal{L}(\mathcal{H}_\mathrm{f}))$ the proof works as well, just note that in (4.4) (and analogously in (4.7)) instead of

$$
\left\|\partial_y^{\alpha_3} f(\tfrac{k+y}{2}, r)\right\|_{\mathcal{L}(\mathcal{H}_\mathrm{f})} \leq C\langle r\rangle^m
$$

we must use that w is an order function and hence there is an $N > 0$ so that

$$
\begin{aligned}
\left\|\partial_y^{\alpha_3} f(\tfrac{k+y}{2}, r)\right\|_{\mathcal{L}(\mathcal{H}_\mathrm{f})} &\leq Cw(\tfrac{k+y}{2}, r) \leq C'\langle r\rangle^N w((\tfrac{k+y}{2}, 0)) \\
&\leq C''\langle r\rangle^N \langle y\rangle^N w((\tfrac{k-y}{2}, 0)) \\
&\leq C'''\langle r\rangle^N \langle y\rangle^N \langle \tfrac{k-y}{2}\rangle^N w(0).
\end{aligned}
$$

This yields

$$
\left\|(\widehat{f}^{\,\mathrm{B},\chi}\psi)(k)\right\|_{\mathcal{H}_\mathrm{f}} \leq C \sup_{|\alpha|\leq 2M} \left\|\langle\cdot\rangle^N \partial^\alpha \psi(\cdot)\right\|_{L^1(\mathbb{R}^2,\mathcal{H}_\mathrm{f})}.
$$

□

Now for $f \in S^w(\mathbb{R}^4,\mathcal{L}(\mathcal{H}_\mathrm{f})) \cup S^m_\rho(\mathbb{R}^4,\mathcal{L}(\mathcal{H}_\mathrm{f}))$ the mapping $\widehat{f}^{\,\mathrm{B},\chi}$ can be extended in a natural way to a continuous mapping from $S'(\mathbb{R}^2,\mathcal{H}_\mathrm{f})$ to $S'(\mathbb{R}^2,\mathcal{H}_\mathrm{f})$ by putting

$$
\widehat{f}^{\,\mathrm{B},\chi}(T)(\psi) := T(\widehat{f^*}^{\,\mathrm{B},\chi}(\psi)) \quad \text{for} \quad T \in S' \quad \text{and} \quad \psi \in S.
$$

The next step is to assure that the Berry quantisation of a τ-equivariant symbol maps τ-equivariant functions to τ-equivariant functions.

Proposition 4.2.5. *For $f \in S^w_\tau(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathrm{f})) \cup S^m_{\rho,\tau}(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathrm{f}))$ we have*

$$\widehat{f}^{\,\mathrm{B},\chi} S'_\tau \subset S'_\tau.$$

Proof.
This can be seen following the lines of the proof of B.3 in [Teu03]. Additionally, we have to use (3.2). Let $T \in S'_\tau$ and $\psi \in S(\mathbb{R}^2, \mathcal{H}_\mathrm{f})$. Then

$$\begin{aligned}
&L_{\gamma^*}\widehat{f}^{\,\mathrm{B},\chi} T(\psi) = T(\widehat{f^*}^{\,\mathrm{B},\chi} L_{-\gamma^*}\psi) \\
&= \; T(k \mapsto \tfrac{1}{(2\pi\varepsilon)^2}\int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}}\chi(k-y)t^\mathrm{B}\left(k, \tfrac{k+y}{2}\right) f^*\left(\tfrac{k+y}{2}, r\right) t^\mathrm{B}\left(\tfrac{k+y}{2}, y\right) \\
&\quad\; \psi(y+\gamma^*)\mathrm{d}r\mathrm{d}y) \\
&= \; T(k \mapsto \tfrac{1}{(2\pi\varepsilon)^2}\int_{\mathbb{R}^4} e^{\frac{i(k+\gamma^*-y)r}{\varepsilon}}\chi(k+\gamma^*-y)t^\mathrm{B}\left(k, \tfrac{k-\gamma^*+y}{2}\right) f^*\left(\tfrac{k-\gamma^*+y}{2}, r\right) \\
&\quad\; t^\mathrm{B}\left(\tfrac{k-\gamma^*+y}{2}, y-\gamma^*\right)\psi(y)\mathrm{d}r\mathrm{d}y) \\
&= \; T(k \mapsto \tfrac{1}{(2\pi\varepsilon)^2}\int_{\mathbb{R}^4} e^{\frac{i(k+\gamma^*-y)r}{\varepsilon}}\chi(k+\gamma^*-y)\tau(\gamma^*)t^\mathrm{B}\left(k+\gamma^*, \tfrac{k+\gamma^*+y}{2}\right) \\
&\quad\; f^*\left(\tfrac{k+\gamma^*+y}{2}, r\right) t^\mathrm{B}\left(\tfrac{k+\gamma^*+y}{2}, y\right)\tau(\gamma^*)^{-1}\psi(y)\mathrm{d}r\mathrm{d}y) \\
&= \; T(L_{-\gamma^*}\tau(\gamma^*)\widehat{f^*}^{\,\mathrm{B},\chi}\tau(\gamma^*)^{-1}\psi) = T(\widehat{f^*}^{\,\mathrm{B},\chi}\tau(\gamma^*)^{-1}\psi) \\
&= \; \tau(\gamma^*)\widehat{f}^{\,\mathrm{B},\chi}T(\psi).
\end{aligned}$$

□

Now we identify the symbols for which the Berry quantisation does not depend on the cutoff up to a “small“ error:

Proposition 4.2.6. *Let $f \in S^w_\tau(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathrm{f}))$ fulfil*

($*$) $\exists \alpha_0 \in \mathbb{N}$ *so that for* $|\alpha| \geq \alpha_0$ *it holds* $\partial^\alpha_r f(k,r) \in L^1(\mathbb{R}^2_r, \mathcal{H}_\mathrm{f}) \cap L^2(\mathbb{R}^2_r, \mathcal{H}_\mathrm{f})$ *for all* $k \in \mathbb{R}^2$ *and* $\|\partial^\alpha_r f(k,r)\|_{L^1(\mathbb{R}^2_r, \mathcal{L}(\mathcal{H}_\mathrm{f}))} \leq h_\alpha(k)$ *with* $h_\alpha \in C(\mathbb{R}^2, \mathbb{R}_{\geq 0})$.

Then the Berry quantisation of this symbol does not depend on the cutoff χ up to an error of $\mathcal{O}(\varepsilon^\infty)$, which means that for two smooth cutoff functions χ and $\widetilde{\chi}$ it holds

$$\widehat{f}^{\,\mathrm{B},\chi} = \widehat{f}^{\,\mathrm{B},\widetilde{\chi}} + \mathcal{O}(\varepsilon^\infty).$$

Proof.
This proof works analogously to the proof of Lemma 4.2.14, so we do not give details here. One only has to note that in equation (4.8), instead of $1-\chi(v)$, one has to write $\chi - \widetilde{\chi}$. □

Remark 4.2.7. Note that Proposition 4.2.6 includes symbols in $S^m_{\rho,\tau}(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathrm{f}))$ with $\rho > 0$.

One of the advantages of the Berry quantisation is that if we have a symbol f satisfying $[\widehat{f}^{\,\mathrm{B}}, \Pi^0] = 0$, the symbol itself must commute pointwise with $\pi_0(k)$.

Proposition 4.2.8. *Let $f \in S^w_\tau(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathrm{f})) \cup S^m_{\rho,\tau}(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathrm{f}))$ and χ a smooth cutoff function. Then it holds*

$$[f(k,r), \pi_0(k)] = 0 \quad \forall k \in \mathbb{R}^2 \quad \textit{iff} \quad [\widehat{f}^{\,\mathrm{B},\chi}, \Pi^0] = 0.$$

Proof.
Let $T \in S'_\tau(\mathbb{R}^2, \mathcal{H}_\mathrm{f})$ and $\psi \in S(\mathbb{R}^2, \mathcal{H}_\mathrm{f})$. Note that $\Pi^0 = \widehat{\pi_0(k)}^{\tau} = \widehat{\pi_0(k)}^{\mathrm{B},\chi}$.
"$\Rightarrow$"

$$\begin{aligned}
&\widehat{f}^{\,\mathrm{B},\chi}\Pi^0 T(\psi) = T(k \mapsto \pi_0(k)\widehat{f^*}^{\,\mathrm{B},\chi}\psi(k)) \\
&= T(k \mapsto \pi_0(k)\tfrac{1}{(2\pi\varepsilon)^2}\int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}}\chi(k-y)t^\mathrm{B}\left(k, \tfrac{k+y}{2}\right) f^*\left(\tfrac{k+y}{2}, r\right) t^\mathrm{B}\left(\tfrac{k+y}{2}, y\right) \\
&\quad \psi(y)\mathrm{d}r\mathrm{d}y) \\
&= T(k \mapsto \tfrac{1}{(2\pi\varepsilon)^2}\int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}}\chi(k-y)t^\mathrm{B}\left(k, \tfrac{k+y}{2}\right) f^*\left(\tfrac{k+y}{2}, r\right) t^\mathrm{B}\left(\tfrac{k+y}{2}, y\right)\pi_0(y) \\
&\quad \psi(y)\mathrm{d}r\mathrm{d}y) = T(\widehat{f^*}^{\,\mathrm{B},\chi}\Pi^0\psi) \\
&= \Pi^0\widehat{f}^{\,\mathrm{B},\chi}(T)(\psi)
\end{aligned}$$

because of $t^\mathrm{B}(x,y)\pi_0(y) = \pi_0(x)t^\mathrm{B}(x,y)$ and the assumption that f and π_0 commute. So we have

$$\widehat{f}^{\,\mathrm{B},\chi}\Pi^0 = \Pi^0\widehat{f}^{\,\mathrm{B},\chi}.$$

"$\Leftarrow$"
Let $g(k,r) = [f(k,r), \pi_0(k)]$. Note that from Proposition B.3.5 it follows that g is in the same symbol class as f. Then

$$\begin{aligned}
&\Pi^0\widehat{f}^{\,\mathrm{B},\chi}(T)(\psi) = T(\widehat{f^*}^{\,\mathrm{B},\chi}\Pi^0\psi) \\
&= T(k \mapsto \tfrac{1}{(2\pi\varepsilon)^2}\int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}}\chi(k-y)t^\mathrm{B}\left(k, \tfrac{k+y}{2}\right) f^*\left(\tfrac{k+y}{2}, r\right) t^\mathrm{B}\left(\tfrac{k+y}{2}, y\right)\pi_0(y) \\
&\quad \psi(y)\mathrm{d}r\mathrm{d}y) \\
&= T(k \mapsto \pi_0(k)\tfrac{1}{(2\pi\varepsilon)^2}\int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}}\chi(k-y)t^\mathrm{B}\left(k, \tfrac{k+y}{2}\right) f^*\left(\tfrac{k+y}{2}, r\right) t^\mathrm{B}\left(\tfrac{k+y}{2}, y\right) \\
&\quad \psi(y)\mathrm{d}r\mathrm{d}y - \tfrac{1}{(2\pi\varepsilon)^2}\int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}}\chi(k-y)t^\mathrm{B}\left(k, \tfrac{k+y}{2}\right)\left[\pi_0\left(\tfrac{k+y}{2}\right), f^*\left(\tfrac{k+y}{2}, r\right)\right] \\
&\quad t^\mathrm{B}\left(\tfrac{k+y}{2}, y\right)\psi(y)\mathrm{d}r\mathrm{d}y) \\
&= \widehat{f}^{\,\mathrm{B},\chi}\Pi^0(T)(\psi) - \widehat{g}^{\,\mathrm{B},\chi}(T)(\psi),
\end{aligned}$$

which yields $\widehat{g}^{\,\mathrm{B},\chi} = 0$ and therefore $g(k,r) \equiv 0$. $\square$

Since we will have to deal with symbols that just satisfy $[\widehat{f}^{\,\mathrm{B}}, \Pi^0] = \mathcal{O}(\varepsilon^\infty)$, we also prove the following corollary:

Corollary 4.2.9. *Let $f \in S^w_\tau(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathrm{f})) \cup S^m_{\rho,\tau}(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathrm{f}))$, χ a smooth cutoff function, and $n \in \mathbb{N} \cup \{\infty\}$. Then we have*

$$\widehat{f_{\mathrm{diag}}}^{\mathrm{B},\chi} = \widehat{f}^{\,\mathrm{B},\chi} + \mathcal{O}(\varepsilon^n) \quad \textit{iff} \quad [\widehat{f}^{\,\mathrm{B},\chi}, \Pi^0] = \mathcal{O}(\varepsilon^n),$$

where $f_{\mathrm{diag}}(k,r) = \pi_0(k) f(k,r) \pi_0(k) + \pi_0(k)^\perp f(k,r) \pi_0(k)^\perp$.

Proof.
Let $T \in S'_\tau(\mathbb{R}^2, \mathcal{H}_\mathrm{f})$ and $\psi \in S(\mathbb{R}^2, \mathcal{H}_\mathrm{f})$.
"$\Rightarrow$"
Using Proposition 4.2.8, we get

$$\widehat{f}^{\,\mathrm{B},\chi}\Pi^0 = \widehat{f_{\mathrm{diag}}}^{\mathrm{B},\chi}\Pi^0 + \mathcal{O}(\varepsilon^n) = \Pi^0 \widehat{f_{\mathrm{diag}}}^{\mathrm{B},\chi} + \mathcal{O}(\varepsilon^n) = \Pi^0 \widehat{f}^{\,\mathrm{B},\chi} + \mathcal{O}(\varepsilon^n).$$

Note that the key point is that $[f_{\mathrm{diag}}(k,r), \pi_0(k)] = 0$.
"$\Leftarrow$"
Adopting the notation from the previous proof, we get from the subsequent Lemma 4.2.10

$$\widehat{g}^{\,\mathrm{B},\chi} = [\widehat{f}^{\,\mathrm{B},\chi}, \Pi^0] = \mathcal{O}(\varepsilon^n).$$

Since $f(k,r) - f_{\mathrm{diag}}(k,r) = [g(k,r), \pi_0(k)] := h(k,r)$, this yields, again using Lemma 4.2.10,

$$\widehat{f}^{\,\mathrm{B},\chi} = \widehat{f_{\mathrm{diag}}}^{\mathrm{B},\chi} + \widehat{h}^{\,B,\chi} = \widehat{f_{\mathrm{diag}}}^{\mathrm{B},\chi} + \widehat{g}^{\,\mathrm{B},\chi}\Pi^0 - \Pi^0 \widehat{g}^{\,\mathrm{B},\chi} = \widehat{f_{\mathrm{diag}}}^{\mathrm{B},\chi} + \mathcal{O}(\varepsilon^n).$$

$\square$

Lemma 4.2.10. *Let $f \in S^w_\tau(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathrm{f})) \cup S^m_{\rho,\tau}(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathrm{f}))$ and χ a smooth cutoff function. Then it holds*

$$\widehat{f\pi_0}^{\mathrm{B},\chi} = \widehat{f(k,r)}^{\mathrm{B},\chi}\Pi^0$$

and

$$\widehat{\pi_0 f}^{\mathrm{B},\chi} = \Pi^0 \widehat{f(k,r)}^{\mathrm{B},\chi},$$

where $(f\pi_0)(k,r) = f(k,r)\pi_0(k)$ and $(\pi_0 f)(k,r) = \pi_0(k) f(k,r)$.

Proof.
Let $T \in S'_\tau(\mathbb{R}^2, \mathcal{H}_\mathrm{f})$ and $\psi \in S(\mathbb{R}^2, \mathcal{H}_\mathrm{f})$. Then

$$\begin{aligned}
\widehat{f\pi_0}^{\mathrm{B},\chi}(T)(\psi) &= T(\widehat{\pi_0 f^*}^{\mathrm{B},\chi}\psi) \\
&= T(k \mapsto \tfrac{1}{(2\pi\varepsilon)^2} \int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}} \chi(k-y) t^\mathrm{B}\left(k, \tfrac{k+y}{2}\right) \pi_0\left(\tfrac{k+y}{2}\right) f^*\left(\tfrac{k+y}{2}, r\right) t^\mathrm{B}\left(\tfrac{k+y}{2}, y\right) \\
&\quad \psi(y)\mathrm{d}r\mathrm{d}y) \\
&= T(k \mapsto \pi_0(k) \tfrac{1}{(2\pi\varepsilon)^2} \int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}} \chi(k-y) t^\mathrm{B}\left(k, \tfrac{k+y}{2}\right) f^*\left(\tfrac{k+y}{2}, r\right) t^\mathrm{B}\left(\tfrac{k+y}{2}, y\right) \\
&\quad \psi(y)\mathrm{d}r\mathrm{d}y) = T(\Pi^0 \widehat{f^*}^{\mathrm{B},\chi}(\psi)) \\
&= \widehat{f}^{\,\mathrm{B},\chi} \Pi^0(T)(\psi).
\end{aligned}$$

Here we used $t^\mathrm{B}(k,z)\pi_0(z) = \pi_0(k) t^\mathrm{B}(k,z)$. The second statement follows quite analogously. ☐

There is also a version of the Calderon-Vaillancourt theorem:

Theorem 4.2.11. *Let $f \in S^1_\tau(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathrm{f}))$ fulfil*

(∗) $\exists \alpha_0 \in \mathbb{N}$ *so that for* $|\alpha| \geq \alpha_0$ *it holds* $\partial_r^\alpha f(k,r) \in L^1(\mathbb{R}^2_r, \mathcal{H}_\mathrm{f}) \cap L^2(\mathbb{R}^2_r, \mathcal{H}_\mathrm{f})$ *for all* $k \in \mathbb{R}^2$ *and* $\|\partial_r^\alpha f(k,r)\|_{L^1(\mathbb{R}^2_r, \mathcal{L}(\mathcal{H}_\mathrm{f}))} \leq h_\alpha(k)$ *with* $h_\alpha \in C(\mathbb{R}^2, \mathbb{R}_{\geq 0})$.

Then

$$\widehat{f}^{\,\mathrm{B}} \in \mathcal{L}(\mathcal{H}_\tau).$$

Proof.
For the proof see the proof of Theorem 4.2.16. ☐

Remark 4.2.12. The theorem above includes symbols $f \in S^{m=0}_{\rho,\tau}(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathrm{f}))$ with $\rho > 0$.

Yet another advantage of the quantisation is the usual property of the Weyl quantisation: For symbols f with $\widehat{f}^{\,\mathrm{B}} \in \mathcal{L}(\mathcal{H}_\tau)$ the adjoint of the quantised symbol should be the quantisation of the pointwise adjoint of the symbol f.

Proposition 4.2.13. *Let $f \in S^w_\tau(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathrm{f})) \cup S^m_{\rho,\tau}(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathrm{f}))$ with $\widehat{f}^{\,\mathrm{B},\chi} \in \mathcal{L}(\mathcal{H}_\tau)$. Then*

$$(\widehat{f}^{\,\mathrm{B},\chi})^* = \widehat{f^*}^{\mathrm{B},\chi}.$$

Proof.
For the proof we follow the line of the proof of B.7 in [Teu03]. Let $\psi \in \mathcal{H}_\tau$, $\phi \in C^\infty_\tau$, and $\widetilde{\phi} = 1_{M^*}\phi$. Denote by

$$K_f(k,y) := \tfrac{1}{(2\pi\varepsilon)^2} \int_{\mathbb{R}^2} e^{\frac{i(k-y)r}{\varepsilon}} \chi(k-y) t^\mathrm{B}\left(k, \tfrac{k+y}{2}\right) f\left(\tfrac{k+y}{2}, r\right) t^\mathrm{B}\left(\tfrac{k+y}{2}, y\right) \mathrm{d}r$$

the distributional integral kernel of $\widehat{f}^{\,\mathrm{B},\chi}$ regarded as an operator on $S'(\mathbb{R}^2.\mathcal{H}_\mathrm{f})$. From the τ-equivariance of the symbol f and property (3.2) it follows

$$K_f(k-\gamma^*, y-\gamma^*) = \tau(\gamma^*)K_f(k,y)\tau(\gamma^*)^{-1}.$$

Thus, using $\widetilde{\phi}$ as a test function, we get

$$\begin{aligned}
\langle \phi, \widehat{f}^{\,\mathrm{B},\chi}\psi\rangle_{\mathcal{H}_\tau} &= \int_{M^*}\langle \phi(k), \widehat{f}^{\,\mathrm{B},\chi}\psi(k)\rangle_{\mathcal{H}_\mathrm{f}}\mathrm{d}k = \int_{\mathbb{R}^2}\langle \widetilde{\phi}(k), \widehat{f}^{\,\mathrm{B},\chi}\psi(k)\rangle_{\mathcal{H}_\mathrm{f}}\mathrm{d}k \\
&= \overline{\widehat{f}^{\,\mathrm{B},\chi}\psi(\widetilde{\phi})} = \overline{\psi(\widehat{f^*}^{\,\mathrm{B},\chi}\widetilde{\phi})} = \int_{\mathbb{R}^2}\langle \widehat{f^*}^{\,\mathrm{B},\chi}\widetilde{\phi}(k), \psi(k)\rangle_{\mathcal{H}_\mathrm{f}}\mathrm{d}k \\
&= \int_{\mathbb{R}^2}\langle \int_{\mathbb{R}^2} K_{f^*}(k,y)\widetilde{\phi}(y)\mathrm{d}y, \psi(k)\rangle_{\mathcal{H}_\mathrm{f}}\mathrm{d}k = \int_{\mathbb{R}^2}\langle \int_{M^*} K_{f^*}(k,y)\phi(y)\mathrm{d}y, \psi(k)\rangle_{\mathcal{H}_\mathrm{f}}\mathrm{d}k \\
&= \int_{M^*}\sum_{\gamma^*\in\Gamma^*}\langle \int_{M^*} K_{f^*}(k+\gamma^*,y)\phi(y)\mathrm{d}y, \psi(k+\gamma^*)\rangle_{\mathcal{H}_\mathrm{f}}\mathrm{d}k \\
&= \int_{M^*}\sum_{\gamma^*\in\Gamma^*}\langle \int_{M^*} \tau(\gamma^*)^{-1}K_{f^*}(k,y-\gamma^*)\tau(\gamma^*)\phi(y)\mathrm{d}y, \tau(\gamma^*)^{-1}\psi(k)\rangle_{\mathcal{H}_\mathrm{f}}\mathrm{d}k \\
&= \int_{M^*}\sum_{\gamma^*\in\Gamma^*}\langle \int_{M^*} K_{f^*}(k,y-\gamma^*)\phi(y-\gamma^*)\mathrm{d}y, \psi(k)\rangle_{\mathcal{H}_\mathrm{f}}\mathrm{d}k \\
&= \int_{M^*}\langle \int_{\mathbb{R}^2} K_{f^*}(k,y)\phi(y)\mathrm{d}y, \psi(k)\rangle_{\mathcal{H}_\mathrm{f}}\mathrm{d}k = \int_{M^*}\langle (\widehat{f^*}^{\,\mathrm{B},\chi}\phi)(k), \psi(k)\rangle_{\mathcal{H}_\mathrm{f}}\mathrm{d}k \\
&= \langle \widehat{f^*}^{\,\mathrm{B},\chi}\phi, \psi\rangle_{\mathcal{H}_\tau}.
\end{aligned}$$

Since C^∞_τ is dense in $\mathcal{H}_\tau$ and $\widehat{f}^{\,\mathrm{B},\chi}$ is continuous, the claim follows. □

Now we want to compare this new quantisation to the τ-quantisation. The main idea to convert the symbols is to do a Taylor expansion of the parallel transport maps in the formula for $\widehat{f}^{\,\mathrm{B},\chi}$. The other difference between the formulas for $\widehat{f}^{\,\tau}$ and $\widehat{f}^{\,\mathrm{B},\chi}$ is the cutoff function. Hence, to get in a better position for comparing these quantisations, we first show that adding an arbitrary cutoff to the definition of $\widehat{f}^{\,\tau}$ just causes an error of order $\mathcal{O}(\varepsilon^\infty)$.

Lemma 4.2.14. *Let $f \in S^w_\tau(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathrm{f}))$ fulfil*

(∗) $\exists \alpha_0 \in \mathbb{N}$ *so that for all* $|\alpha| \geq \alpha_0$ *it holds* $\partial^\alpha_r f(k,r) \in L^1(\mathbb{R}^2_r, \mathcal{H}_\mathrm{f}) \cap L^2(\mathbb{R}^2_r, \mathcal{H}_\mathrm{f})$ *for all* $k \in \mathbb{R}^2$ *and* $\|\partial^\alpha_r f(k,r)\|_{L^1(\mathbb{R}^2_r,\mathcal{L}(\mathcal{H}_\mathrm{f}))} \leq h_\alpha(k)$ *with* $h_\alpha \in C(\mathbb{R}^2, \mathbb{R}_{\geq 0})$.

Then with

$$\widehat{f}^{\,\tau,\chi}\psi(k) := \tfrac{1}{(2\pi\varepsilon)^2}\int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}}\chi(k-y) f\left(\tfrac{k+y}{2}, r\right)\psi(y)\mathrm{d}r\mathrm{d}y$$

and χ a smooth cutoff function, it holds

$$\widehat{f}^{\,\tau} = \widehat{f}^{\,\tau,\chi} + \mathcal{O}(\varepsilon^\infty).$$

Proof.
Using

$$(1-\chi(v))e^{\frac{ivr}{\varepsilon}} = (1-\chi(v))\left(\tfrac{-\varepsilon^2\Delta_r}{|v|^2}\right)^M e^{\frac{ivr}{\varepsilon}} \quad \text{for} \quad M \in \mathbb{N} \tag{4.8}$$

and the fact that for M large enough we have $\Delta_r^M f(k,r) \in L^2(\mathbb{R}^2_r, \mathcal{L}(\mathcal{H}_\mathrm{f})) \cap L^1(\mathbb{R}^2_r, \mathcal{L}(\mathcal{H}_\mathrm{f}))$, performing a partial integration we get

$$\begin{aligned}
\widehat{f}^{\,\tau}\psi(k) - \widehat{f}^{\,\tau,\chi}\psi(k) &= \tfrac{1}{(2\pi\varepsilon)^2}\int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}}(1-\chi(k-y))f\left(\tfrac{k+y}{2}, r\right)\psi(y)\mathrm{d}r\mathrm{d}y \\
&= \tfrac{(-1)^M\varepsilon^{2M}}{(2\pi\varepsilon)^2}\int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}}\tfrac{(1-\chi(k-y))}{|k-y|^{2M}}\Delta_r^M f\left(\tfrac{k+y}{2}, r\right)\psi(y)\mathrm{d}r\mathrm{d}y \\
&= \tfrac{(-1)^M\varepsilon^{2M}2\pi}{(2\pi\varepsilon)^2}\int_{\mathbb{R}^2} \tfrac{(1-\chi(k-y))}{|k-y|^{2M}}\mathcal{F}(\Delta^M f_{\frac{k+y}{2}})\left(\tfrac{y-k}{\varepsilon}\right)\psi(y)\mathrm{d}y.
\end{aligned}$$

Hence with the same strategy as in the proof of Theorem 4.2.16,

$$\begin{aligned}
\left\|\widehat{f}^{\,\tau}\psi(k) - \widehat{f}^{\,\tau,\chi}\psi(k)\right\|_{\mathcal{H}_\mathrm{f}} &\le C\varepsilon^{2M-2}\int_{\mathbb{R}^2}\left\|\mathcal{F}(\Delta^M f_{\frac{k+y}{2}})\left(\tfrac{y-k}{\varepsilon}\right)\right\|_{\mathcal{L}(\mathcal{H}_\mathrm{f})}\|\psi(y)\|_{\mathcal{H}_\mathrm{f}}\,\mathrm{d}y \\
&\le C'\varepsilon^{2M-1}\|\psi\|_{\mathcal{H}_\tau}
\end{aligned}$$

holds and thus

$$\left\|\widehat{f}^{\,\tau} - \widehat{f}^{\,\tau,\chi}\right\|_{\mathcal{L}(\mathcal{H}_\tau)} \le C'\varepsilon^{2M-1}.$$

□

Remark 4.2.15. Note that Lemma 4.2.14 includes symbols in $S^m_{\rho,\tau}(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathrm{f}))$ with $\rho > 0$.

The next step is to analyse the difference between the operators $\widehat{f}^{\,\tau}$ and $\widehat{f}^{\,\mathrm{B}}$ and to show how we can connect them.

Theorem 4.2.16. *Let $f \in S^1_\tau(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathrm{f}))$ fulfil*

(∗) $\exists\alpha_0 \in \mathbb{N}$ *so that for* $|\alpha| \ge \alpha_0$ *we have* $\partial_r^\alpha f(k,r) \in L^1(\mathbb{R}^2_r, \mathcal{H}_\mathrm{f}) \cap L^2(\mathbb{R}^2_r, \mathcal{H}_\mathrm{f})$ *for all* $k \in \mathbb{R}^2$ *and* $\|\partial_r^\alpha f(k,r)\|_{L^1(\mathbb{R}^2_r, \mathcal{L}(\mathcal{H}_\mathrm{f}))} \le h_\alpha(k)$ *with* $h_\alpha \in C(\mathbb{R}^2, \mathbb{R}_{\ge 0})$,

or let $f \in S^m_{\rho,\tau}(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathrm{f}))$ with $\rho > 0$. Then *(i) For every $N \in \mathbb{N}_0$ there is a correction $f_{\mathrm{c}}^{(N)}$ in the same symbol class as f so that*

$$\widehat{f}^{\,\tau} = \widehat{f_{\mathrm{c}}^{(N)}}^{\mathrm{B}} + \mathcal{O}(\varepsilon^{N+1}).$$

For $N \in \mathbb{N}_0$, the correction $f_{\mathrm{c}}^{(N)}$ of the symbol f is given by

$$f_{\mathrm{c}}^{(N)}(k,r) = \sum_{n=0}^{N} \varepsilon^n f_{\mathrm{c}n}(k,r),$$

where

$$f_{\mathrm{c}n}(k,r) = \sum_{j=0}^{n} \left(\tfrac{i}{2}\right)^n \sum_{\alpha\in\{1,2\}^j} \sum_{\beta\in\{1,2\}^{n-j}} c_\alpha(k)\partial_{r_{\alpha_1}}...\partial_{r_{\alpha_j}}\partial_{r_{\beta_1}}...\partial_{r_{\beta_{n-j}}} f(k,r)\widetilde{c}_\beta(k),$$

with

$$\begin{aligned} c_\alpha(k) &= \tfrac{1}{j!}\partial_{y_{\alpha_1}}...\partial_{y_{\alpha_j}} t^{\mathrm{B}}(k,y)|_{y=k} \quad \textit{for} \quad \alpha\in\{1,2\}^j \quad \textit{and} \quad j\geq 1, \\ \widetilde{c}_\alpha(k) &= \tfrac{1}{j!}(-1)^j\partial_{y_{\alpha_1}}...\partial_{y_{\alpha_j}} t^{\mathrm{B}}(y,k)|_{y=k} \quad \textit{for} \quad \alpha\in\{1,2\}^j \quad \textit{and} \quad j\geq 1, \\ c_\alpha(k) &= \widetilde{c}_\alpha(k) = \mathrm{id}_{\mathcal{H}_{\mathrm{f}}} \quad \textit{for} \quad \alpha\in\{1,2\}^0. \end{aligned}$$

For $f \in S^1_\tau(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_{\mathrm{f}}))$, the correction $f_{\mathrm{c}}^{(N)}$ fulfils $(*)$.
For $f \in S^m_{\rho,\tau}(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_{\mathrm{f}}))$, it even holds $f_{\mathrm{c}n} \in S^{m-n\rho}_{\rho,\tau}(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_{\mathrm{f}}))$.
In particular, we have

$$f_{\mathrm{c}0}(k,r) = f(k,r)$$

and

$$f_{\mathrm{c}1}(k,r) = -\tfrac{i}{2}\left(D(k)\nabla_r f(k,r) + \nabla_r f(k,r) D(k)\right),$$

where $D(k) = \nabla_k - \nabla_k^{\mathrm{B}} = \pi_0^\perp(k)\nabla_k\pi_0(k) + \pi_0(k)\nabla_k\pi_0^\perp(k)$.

(ii) For every $N \in \mathbb{N}_0$ there is a correction $f_{\mathrm{c}}^{(N)}$ in the same symbol class as f so that

$$\widehat{f}^{\,\mathrm{B}} = \widehat{f_{\mathrm{c}}^{(N)}}^{\tau} + \mathcal{O}(\varepsilon^{N+1}).$$

For $N \in \mathbb{N}_0$, the correction $f_{\mathrm{c}}^{(N)}$ of the symbol f is given by

$$f_{\mathrm{c}}^{(N)}(k,r) = \sum_{n=0}^{N} \varepsilon^n f_{\mathrm{c}n}(k,r),$$

where

$$f_{\mathrm{c}n}(k,r) = \sum_{j=0}^{n} \left(\tfrac{i}{2}\right)^n \sum_{\alpha\in\{1,2\}^j} \sum_{\beta\in\{1,2\}^{n-j}} c_\alpha(k)\partial_{r_{\alpha_1}}...\partial_{r_{\alpha_j}}\partial_{r_{\beta_1}}...\partial_{r_{\beta_{n-j}}} f(k,r)\widetilde{c}_\beta(k),$$

with

$$
\begin{aligned}
c_\alpha(k) &= \tfrac{1}{j!}\partial_{y_{\alpha_1}}...\partial_{y_{\alpha_j}} t^{\mathrm{B}}(y,k)|_{y=k} \quad \textit{for} \quad \alpha \in \{1,2\}^j \quad \textit{and} \quad j \geq 1, \\
\widetilde{c}_\alpha(k) &= \tfrac{1}{j!}(-1)^j \partial_{y_{\alpha_1}}...\partial_{y_{\alpha_j}} t^{\mathrm{B}}(k,y)|_{y=k} \quad \textit{for} \quad \alpha \in \{1,2\}^j \quad \textit{and} \quad j \geq 1, \\
c_\alpha(k) &= \widetilde{c}_\alpha(k) = \mathrm{id}_{\mathcal{H}_\mathrm{f}} \quad \textit{for} \quad \alpha \in \{1,2\}^0.
\end{aligned}
$$

For $f \in S^1_\tau(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathrm{f}))$, the correction $f_\mathrm{c}^{(N)}$ fulfils $(*)$.
For $f \in S^m_{\rho,\tau}(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathrm{f}))$ with $\rho > 0$, it even holds $f_{\mathrm{c}n} \in S^{m-n\rho}_{\rho,\tau}(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathrm{f}))$.
In particular, we have

$$f_{\mathrm{c}0}(k,r) = f(k,r)$$

and

$$f_{\mathrm{c}1}(k,r) = \tfrac{i}{2}\left(D(k)\nabla_r f(k,r) + \nabla_r f(k,r) D(k)\right),$$

where $D(k) = \nabla_k - \nabla_k^{\mathrm{B}} = \pi_0^\perp(k)\nabla_k \pi_0(k) + \pi_0(k)\nabla_k\pi_0^\perp(k)$.

Proof.
The proof is divided into six parts. In parts 1-5, we prove the theorem for symbols $f \in S^1_\tau(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathrm{f}))$ that fulfil $(*)$.

- In the first part, we show that $f_\mathrm{c}^{(N)}$ is in the same symbol class as f and that it fulfils $(*)$.
- In the second part, we compute a Taylor expansion of the parallel transport $t^{\mathrm{B}}(x,y)$ which we need for the proof and put $t^{\mathrm{B}}(z+\delta,z)f(z,r)t^{\mathrm{B}}(z,z-\delta)$ into the form we need, which mainly means arranging it after powers of δ. Then we use this to show how the corrections $f_\mathrm{c}^{(N)}$ of the symbol f emerge.
- The third step is to show that $\widehat{f}^{\,\mathrm{B}} = \widehat{f_\mathrm{c}^{(N)}}^{\tau} + \mathcal{O}(\varepsilon^{N+1})$. Thereto, the main work will be required for the estimation of the term caused by the remainder term of the Taylor expansion. Furthermore, we will have to use the Calderon-Vaillancourt theorem for the τ-quantisation. Up to here, we have shown $\widehat{f}^{\,\mathrm{B}} = \widehat{f_\mathrm{c}^{(N)}}^{\tau} + \mathcal{O}(\varepsilon^{N+1})$.
- In the fourth part, we deduce from this result a Calderon-Vaillancourt theorem for the Berry quantisation, that is to say we prove Theorem 4.2.11.
- In the fifth step, we show $\widehat{f}^{\,\tau} = \widehat{f_\mathrm{c}^{(N)}}^{\mathrm{B}} + \mathcal{O}(\varepsilon^{N+1})$. The proof works as in part three, so we just point out the few modifications which have to be done.
- In the sixth step, we give the modifications of the proof for symbols $f \in S^m_{\rho,\tau}(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathrm{f}))$ with $\rho > 0$.

Part 1.
Let $f \in S^1_\tau(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathrm{f}))$ fulfil $(*)$. We want to verify that $f_\mathrm{c}^{(N)}$ is in the same symbol class as f and that it fulfils $(*)$. Thereto we show that c_α respectively $\widetilde{c}_\alpha$ are symbols in $S^1_\tau(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathrm{f}))$: Since

$$\begin{aligned} c_\alpha(k-\gamma^*) &= \tfrac{1}{j!}\partial_{y_{\alpha_1}}...\partial_{y_{\alpha_j}} t^\mathrm{B}(y, k-\gamma^*)|_{y=k-\gamma^*} \\ &= \tfrac{1}{j!}\partial_{y_{\alpha_1}}...\partial_{y_{\alpha_j}} \tau(\gamma^*) t^\mathrm{B}(y+\gamma^*, k)\tau(\gamma^*)^{-1}|_{y=k-\gamma^*} \\ &= \tfrac{1}{j!}\partial_{y_{\alpha_1}}...\partial_{y_{\alpha_j}} \tau(\gamma^*) t^\mathrm{B}(y, k)\tau(\gamma^*)^{-1}|_{y=k} \\ &= \tau(\gamma^*)c_\alpha(k)\tau(\gamma^*)^{-1}, \end{aligned}$$

c_α is τ-equivariant, which directly induces the boundedness of c_α together with all the derivatives. The same is true for $\widetilde{c}_\alpha$. Using Proposition B.3.5 then yields $f_\mathrm{c}^{(N)} \in S^1_\tau(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathrm{f}))$ as a sum of products of symbols in this symbol class. Furthermore, from the boundedness of the c_α we immediately get that $f_\mathrm{c}^{(N)}$ fulfils $(*)$.

Part 2.
The only differences between the integral formulas of the quantisations are the cutoff function χ and the twofold occurrence of the map t^B around the symbol which is quantised. However, the appearance of the cutoff χ is not really a problem because for the symbols considered here we can use Lemma 4.2.14. So the key idea for the proof is to somehow "mingle“ the map t^B with the symbol f. The first naive idea one might have is to just take $t^\mathrm{B}(k, \frac{k+y}{2}) f(\frac{k+y}{2}, r) t^\mathrm{B}(\frac{k+y}{2}, y)$ as a new symbol. But this is not possible because a symbol must be a function depending only on $\frac{k+y}{2}$ and r and not also on k. So our strategy to deal with this is to compute a Taylor expansion of the above expression and to show that the remainder term does only cause a small error. Let us do this more precisely:
Our idea is that $k-y$ is "small" because for the symbols in consideration the quantisation does (up to an error of $\mathcal{O}(\varepsilon^\infty)$) not depend on the cutoff, so the support can be made small. Hence with

$$\tfrac{k+y}{2} = z \quad \text{and} \quad \delta = \tfrac{k-y}{2}$$

we can write

$$k = z+\delta \quad \text{and} \quad y = z-\delta.$$

The next step is straightforward: We just compute the Taylor expansion of $t^\mathrm{B}(z+\delta, z)$ and $t^\mathrm{B}(z, z-\delta)$ to get a new symbol that just depends on $\frac{k+y}{2}$ and r:

$$t^\mathrm{B}(z+\delta, z) = \sum_{j=0}^{N+1} \tfrac{1}{j!} \sum_{\alpha\in\{1,2\}^j} \partial_{y_{\alpha_1}}...\partial_{y_{\alpha_j}} t^\mathrm{B}(y,z)|_{y=z}\delta_{\alpha_1}..\delta_{\alpha_j} + R_{N+2}(z,\xi,\delta)$$

with

$$R_{N+2}(z,\xi,\delta) = \sum_{\alpha\in\{1,2\}^{N+2}} \tfrac{1}{(N+2)!}\partial_{y_{\alpha_1}}...\partial_{y_{\alpha_{N+2}}} t^\mathrm{B}(y,z)|_{y=\xi}\delta_{\alpha_1}..\delta_{\alpha_{N+2}}$$

and $\xi = z + t\delta$ with $t \in [0,1]$. For later use we define

$$r_\alpha(z,\xi) := \tfrac{1}{(N+2)!}\partial_{y_{\alpha_1}}...\partial_{y_{\alpha_{N+2}}} t^{\mathrm{B}}(y,z)|_{y=\xi}$$

and get

$$R_{N+2}(z,\xi,\delta) = \sum_{\alpha\in\{1,2\}^{N+2}} r_\alpha(z,\xi)\delta_{\alpha_1}..\delta_{\alpha_{N+2}}$$

and

$$t^{\mathrm{B}}(z+\delta,z) = \sum_{j=0}^{N+1}\sum_{\alpha\in\{1,2\}^j} c_\alpha(z)\delta_{\alpha_1}..\delta_{\alpha_j} + R_{N+2}(z,\xi,\delta).$$

Analogously

$$t^{\mathrm{B}}(z,z-\delta) = \sum_{j=0}^{N+1}\tfrac{1}{j!}\sum_{\alpha\in\{1,2\}^j}(-1)^j\partial_{y_{\alpha_1}}...\partial_{y_{\alpha_j}} t^{\mathrm{B}}(z,y)|_{y=z}\delta_{\alpha_1}..\delta_{\alpha_j} + \widetilde{R}_{N+2}(z,\widetilde{\xi},\delta)$$

with

$$\widetilde{R}_{N+2}(z,\widetilde{\xi},\delta) = \sum_{\alpha\in\{1,2\}^{N+2}} \tfrac{(-1)^N}{(N+2)!}\partial_{y_{\alpha_1}...\partial_{y_{\alpha_{N+2}}}} t^{\mathrm{B}}(z,y)\big|_{y=\widetilde{\xi}}\delta_{\alpha_1}..\delta_{\alpha_{N+2}}$$

and $\widetilde{\xi} = z - t\delta$ with $t \in [0,1]$. Again we set

$$\widetilde{r}_\alpha(z,\widetilde{\xi}) := \tfrac{(-1)^{N+2}}{(N+2)!}\partial_{y_{\alpha_1}}...\partial_{y_{\alpha_{N+2}}} t^{\mathrm{B}}(z,y)\big|_{y=\widetilde{\xi}}$$

and get

$$\widetilde{R}_{N+2}(z,\widetilde{\xi},\delta) = \sum_{\alpha\in\{1,2\}^{N+2}} \widetilde{r}_\alpha(z,\widetilde{\xi})\delta_{\alpha_1}..\delta_{\alpha_{N+2}}$$

and

$$t^{\mathrm{B}}(z,z-\delta) = \sum_{j=0}^{N+1}\sum_{\alpha\in\{1,2\}^j} \widetilde{c}_\alpha(z)\delta_{\alpha_1}..\delta_{\alpha_j} + \widetilde{R}_{N+2}(z,\widetilde{\xi},\delta).$$

Though this can be done for every $N \in \mathbb{N}_0$, for the proof we choose $N \geq \alpha_0 - 2$, which we need when we estimate the remainder terms of the expansion.
Using the Taylor expansions above, we get after arranging the summands according to the powers of δ

$$\begin{aligned} & t^{\mathrm{B}}(z+\delta,z)f(z,r)t^{\mathrm{B}}(z,z-\delta) \\ = & \sum_{n=0}^{N}\sum_{j=0}^{n}\sum_{\alpha\in\{1,2\}^j}\sum_{\beta\in\{1,2\}^{n-j}} c_\alpha(z)f(z,r)\widetilde{c}_\beta(z)\delta_{\alpha_1}...\delta_{\alpha_j}\delta_{\beta_1}...\delta_{\beta_{n-j}} \\ & +\mathcal{R}_{N+1}(z,\xi,\widetilde{\xi},\delta,r), \end{aligned}$$

where $c_\alpha(k)$ and $\widetilde{c}_\alpha(k)$ are defined as above. The remainder term $\mathcal{R}_{N+1}$ can be described more explicitly as follows:

$$
\begin{aligned}
&\mathcal{R}_{N+1}(z,\xi,\widetilde{\xi},\delta,r)\\
&= \sum_{m=0}^{N+1}\sum_{j=m}^{N+1}\sum_{\alpha\in\{1,2\}^j}\sum_{\beta\in\{1,2\}^{N+1+m-j}} c_\alpha(z)f(z,r)\widetilde{c}_\beta(z)\delta_{\alpha_1}...\delta_{\alpha_j}\delta_{\beta_1}...\delta_{\beta_{N+1+m-j}}\\
&+ \sum_{m=0}^{N+1}\sum_{\alpha\in\{1,2\}^{N+2}}\sum_{\beta\in\{1,2\}^m}(r_\alpha(z,\xi)f(z,r)\widetilde{c}_\beta(z)\delta_{\alpha_1}...\delta_{\alpha_{N+2}}\delta_{\beta_1}...\delta_{\beta_m}\\
&\quad +c_\beta(z)f(z,r)\widetilde{r}_\alpha(z,\widetilde{\xi})\delta_{\alpha_1}...\delta_{\alpha_{N+2}}\delta_{\beta_1}...\delta_{\beta_m})\\
&+ \sum_{\alpha\in\{1,2\}^{N+2}}\sum_{\beta\in\{1,2\}^{N+2}} r_\alpha(z,\xi)f(z,r)\widetilde{r}_\beta(z,\widetilde{\xi})\delta_{\alpha_1}...\delta_{\alpha_{N+2}}\delta_{\beta_1}...\delta_{\beta_{N+2}}.
\end{aligned}
$$

Here we again have arranged the summands according to powers of δ, but we also have distinguished between summands which do not include r_α and $\widetilde{r}_\alpha$, summands which include one of them, and summands which include both (and thus are the only coefficients of δ^α with $|\alpha| = 2N+4$). We will use this representation when we estimate the remainder term in step three.
Now we show how the corrections $f_{\mathrm{c}}^{(N)}$ are derived. Let $\psi \in C_\tau^\infty(\mathbb{R}^2,\mathcal{H}_{\mathrm{f}})$, which is a dense subset of $\mathcal{H}_\tau$, and χ a smooth cutoff function. Then with $\delta = \frac{k-y}{2}$

$$
\begin{aligned}
&(\widehat{f}^{\,\mathrm{B},\chi}\psi)(k)\\
&= \tfrac{1}{(2\pi\varepsilon)^2}\int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}}\chi(k-y)t^{\mathrm{B}}\left(k,\tfrac{k+y}{2}\right)f\left(\tfrac{k+y}{2},r\right)t^{\mathrm{B}}\left(\tfrac{k+y}{2},y\right)\psi(y)\mathrm{d}r\mathrm{d}y\\
&= \sum_{n=0}^{N}\sum_{j=0}^{n}\sum_{\alpha\in\{1,2\}^j}\sum_{\beta\in\{1,2\}^{n-j}}\tfrac{1}{(2\pi\varepsilon)^2}\int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}}\chi(k-y)c_\alpha\left(\tfrac{k+y}{2}\right)f\left(\tfrac{k+y}{2},r\right)\\
&\quad \widetilde{c}_\beta\left(\tfrac{k+y}{2}\right)\delta_{\alpha_1}...\delta_{\alpha_j}\delta_{\beta_1}...\delta_{\beta_{n-j}}\psi(y)\mathrm{d}r\mathrm{d}y\\
&\quad +\tfrac{1}{(2\pi\varepsilon)^2}\int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}}\chi(k-y)\mathcal{R}_{N+1}\left(\tfrac{k+y}{2},\xi,\widetilde{\xi},\delta,r\right)\psi(y)\mathrm{d}r\mathrm{d}y
\end{aligned}
$$

if we can show that the remainder term in the last row,

$$
\mathcal{T}_{N+1}\psi(k) := \tfrac{1}{(2\pi\varepsilon)^2}\int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}}\chi(k-y)\mathcal{R}_{N+1}\left(\tfrac{k+y}{2},\xi,\widetilde{\xi},\delta,r\right)\psi(y)\mathrm{d}r\mathrm{d}y,
$$

defines a convergent integral. This will be done in step three. Here we just take care of the first sum without the remainder term and show that it equals $\widehat{f_{\mathrm{c}}^{(N)}}^{\tau,\chi}$. So let $n \in \{0,1,..,N\}$, $j \in \{0,1,..n\}$, $\alpha \in \{1,2\}^j$, and $\beta \in \{1,2\}^{n-j}$. We look at the accordant summand:

$$
\begin{aligned}
&\tfrac{1}{(2\pi\varepsilon)^2}\int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}}\chi(k-y)c_\alpha\left(\tfrac{k+y}{2}\right)f\left(\tfrac{k+y}{2},r\right)\widetilde{c}_\beta\left(\tfrac{k+y}{2}\right)\delta_{\alpha_1}...\delta_{\alpha_j}\delta_{\beta_1}...\delta_{\beta_{n-j}}\psi(y)\mathrm{d}r\mathrm{d}y\\
&= \tfrac{1}{(2\pi\varepsilon)^2}\left(\tfrac{-i\varepsilon}{2}\right)^n\int_{\mathbb{R}^4}\left(\partial_{r_{\alpha_1}}...\partial_{r_{\alpha_j}}\partial_{r_{\beta_1}}...\partial_{r_{\beta_{n-j}}}e^{\frac{i(k-y)r}{\varepsilon}}\right)\chi(k-y)c_\alpha\left(\tfrac{k+y}{2}\right)f\left(\tfrac{k+y}{2},r\right)\\
&\quad\ \widetilde{c}_\beta\left(\tfrac{k+y}{2}\right)\psi(y)\mathrm{d}r\mathrm{d}y \qquad (4.9)\\
&= \tfrac{1}{(2\pi\varepsilon)^2}\left(\tfrac{-i\varepsilon}{2}\right)^n\int_{\mathbb{R}^4}\left(\partial_{r_{\alpha_1}}...\partial_{r_{\alpha_j}}\partial_{r_{\beta_1}}...\partial_{r_{\beta_{n-j}}}\left(\tfrac{(1-\varepsilon^2\Delta_y)^M}{\langle r\rangle^{2M}}e^{\frac{i(k-y)r}{\varepsilon}}\right)\right)\chi(k-y)\\
&\quad\ c_\alpha\left(\tfrac{k+y}{2}\right)f\left(\tfrac{k+y}{2},y\right)\widetilde{c}_\beta\left(\tfrac{k+y}{2}\right)\psi(y)\mathrm{d}r\mathrm{d}y \qquad (4.10)\\
&= \tfrac{1}{(2\pi\varepsilon)^2}\left(\tfrac{-i\varepsilon}{2}\right)^n(-1)^n\int_{\mathbb{R}^4}\left(\tfrac{(1-\varepsilon^2\Delta_y)^M}{\langle r\rangle^{2M}}e^{\frac{i(k-y)r}{\varepsilon}}\right)\chi(k-y)c_\alpha\left(\tfrac{k+y}{2}\right)\\
&\quad\ \left(\partial_{r_{\alpha_1}}...\partial_{r_{\alpha_j}}\partial_{r_{\beta_1}}...\partial_{r_{\beta_{n-j}}}f\left(\tfrac{k+y}{2},r\right)\right)\widetilde{c}_\beta\left(\tfrac{k+y}{2}\right)\psi(y)\mathrm{d}r\mathrm{d}y \qquad (4.11)\\
&= \tfrac{1}{(2\pi\varepsilon)^2}\int_{\mathbb{R}^4}e^{\frac{i(k-y)r}{\varepsilon}}\chi(k-y)\left(\tfrac{i\varepsilon}{2}\right)^n c_\alpha\left(\tfrac{k+y}{2}\right)\left(\partial_{r_{\alpha_1}}...\partial_{r_{\alpha_j}}\partial_{r_{\beta_1}}...\partial_{r_{\beta_{n-j}}}f\left(\tfrac{k+y}{2},r\right)\right)\\
&\quad\ \widetilde{c}_\beta\left(\tfrac{k+y}{2}\right)\psi(y)\mathrm{d}r\mathrm{d}y.
\end{aligned}
$$

In equality (4.9) we used the identity

$$\delta_j e^{\frac{i(k-y)r}{\varepsilon}} = \tfrac{-i\varepsilon}{2}\partial_{r_j}e^{\frac{i(k-y)r}{\varepsilon}}, \tag{4.12}$$

and in equality (4.10) we used the identity

$$\tfrac{(1-\varepsilon^2\Delta_y)^M}{\langle r\rangle^{2M}}e^{\frac{i(k-y)r}{\varepsilon}} = e^{\frac{i(k-y)r}{\varepsilon}} \quad \text{for} \quad M\in\mathbb{N},$$

which is crucial for the definition of the integral as an elliptic integral. Since $f(k,r)$ is bounded, the integration by parts in equality (4.11) is possible if we take M large enough so that the boundary term of the integration by parts vanishes. This procedure, of course, works for every summand, meaning that so far we have

$$\widehat{f}^{\,\mathrm{B},\chi}\psi(k) = \widehat{f_{\mathrm{c}}^{(N)}}^{\tau,\chi}\psi(k) + \mathcal{T}_{N+1}\psi(k)$$

if we assume the convergence of $\mathcal{T}_{N+1}\psi(k)$. This leads us to part three, where we will not only prove the convergence of this integral, but moreover will show that $\mathcal{T}_{N+1}$ is of order $\mathcal{O}(\varepsilon^{N+1})$.

Part 3.

We are left to show that for $\psi\in C^\infty_\tau$

$$\mathcal{T}_{N+1}\psi(k) := \tfrac{1}{(2\pi\varepsilon)^2}\int_{\mathbb{R}^4}e^{\frac{i(k-y)r}{\varepsilon}}\chi(k-y)\mathcal{R}_{N+1}(z,\xi,\widetilde{\xi},\delta,r)\psi(y)\mathrm{d}r\mathrm{d}y$$

is convergent, defines a τ-equivariant function, and it holds

$$\|\mathcal{T}_{N+1}\psi\|_{\mathcal{H}_\tau} \leq C_{N+1}\varepsilon^{N+1}\|\psi\|_{\mathcal{H}_\tau}.$$

Thereto we have to take a closer look at the representation of the remainder term $\mathcal{R}_{N+1}$ from part two. The summands of the representation where r_α and $\widetilde{r}_\alpha$ do not appear can be treated as follows: As in part two, we can perform a partial integration using the identity (4.12) and add ε^{N+1+m} as well as the maps c_α and $\widetilde{c}_\alpha$ to $\partial_{r_{\alpha_1}}..\partial_{r_{\alpha_j}}\partial_{r_{\beta_1}}..\partial_{r_{\beta_{N+1+m-j}}}f(k,r)$. Then we use Propositions B.3.5 and B.3.6 and get that these summands are of order $\mathcal{O}(\varepsilon^{N+1})$.
The obstruction that we cannot estimate the remaining terms of $\mathcal{T}_{N+1}\psi(k)$ analogously is that the terms which include r_α or $\widetilde{r}_\alpha$ do not only depend on $\frac{k+y}{2}$ and r, but also depend on ξ respectively $\widetilde{\xi}$. Thus we cannot treat them as symbols and use Proposition B.3.6. Let us look at an arbitrary summand of $\mathcal{T}_{N+1}\psi(k)$ which contains r_α and which we name $\rho(r_\alpha,\widetilde{c}_\beta)\psi(k)$: For $m\in\{0,1,...N+1\}$, $\alpha\in\{1,2\}^{N+2}$, and $\beta\in\{1,2\}^m$ we get with $z=\frac{k+y}{2}$

$$\begin{aligned}
&\rho(r_\alpha,\widetilde{c}_\beta)\psi(k)\\
:=\ & \tfrac{1}{(2\pi\varepsilon)^2}\int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}}\chi(k-y)r_\alpha(z,\xi)f(z,r)\widetilde{c}_\beta(z)\delta_{\alpha_1}...\delta_{\alpha_{N+2}}\delta_{\beta_1}...\delta_{\beta_m}\psi(y)\mathrm{d}r\mathrm{d}y\\
=\ & (\tfrac{-i\varepsilon}{2})^{N+m+2}\tfrac{1}{(2\pi\varepsilon)^2}\int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}}\chi(k-y)r_\alpha(z,\xi)\partial_{r_{\alpha_1}}...\partial_{r_{\alpha_{N+2}}}\partial_{r_{\beta_1}}...\partial_{r_{\beta_m}}f(z,r)\\
& \widetilde{c}_\beta(z)\psi(y)\mathrm{d}r\mathrm{d}y \qquad (4.13)\\
=\ & (\tfrac{-i\varepsilon}{2})^{N+m+2}\tfrac{1}{2\pi\varepsilon^2}\int_{\mathbb{R}^2}\chi(k-y)r_\alpha(z,\xi)\mathcal{F}(\partial_{\alpha_1}...\partial_{\alpha_{N+2}}\partial_{\beta_1}...\partial_{\beta_m}f_z)\left(\tfrac{y-k}{\varepsilon}\right)\\
& \widetilde{c}_\beta(z)\psi(y)\mathrm{d}y.
\end{aligned}$$

In equality (4.13), we made the usual partial integration. $\mathcal{F}$ is the Fourier transform in $L^2(\mathbb{R}^2,\mathcal{L}(\mathcal{H}_\mathrm{f}))$. Note that the condition $N\geq\alpha_0-2$ assures that we have $\partial_{r_{\alpha_1}}...\partial_{r_{\alpha_{N+2}}}\partial_{r_{\beta_1}}...\partial_{r_{\beta_m}}f_z(r)\in L^2(\mathbb{R}^2_r,\mathcal{L}(\mathcal{H}_\mathrm{f}))$. Thus, the $\mathcal{H}_\mathrm{f}$-norm of the whole term can be estimated by

$$\begin{aligned}
&\|\rho(r_\alpha,\widetilde{c}_\beta)\psi(k)\|_{\mathcal{H}_\mathrm{f}}\\
\leq\ & C_1\varepsilon^{N+m}\int_{\mathbb{R}^2}\chi(k-y)\|r_\alpha(z,\xi)\|_{\mathcal{L}(\mathcal{H}_\mathrm{f})}\left\|\mathcal{F}(\partial_{\alpha_1}...\partial_{\alpha_{N+2}}\partial_{\beta_1}...\partial_{\beta_m}f_z)\left(\tfrac{y-k}{\varepsilon}\right)\right\|_{\mathcal{L}(\mathcal{H}_\mathrm{f})}\\
& \|\widetilde{c}_\beta(z)\|_{\mathcal{L}(\mathcal{H}_\mathrm{f})}\|\psi(y)\|_{\mathcal{H}_\mathrm{f}}\mathrm{d}y\\
\leq\ & C_2\varepsilon^{N+m}\int_{\mathbb{R}^2}\left\|\mathcal{F}(\partial_{\alpha_1}...\partial_{\alpha_{N+2}}\partial_{\beta_1}...\partial_{\beta_m}f_z)\left(\tfrac{y-k}{\varepsilon}\right)\right\|_{\mathcal{L}(\mathcal{H}_\mathrm{f})}\|\psi(y)\|_{\mathcal{H}_\mathrm{f}}\mathrm{d}y \qquad (4.14)
\end{aligned}$$

because $\widetilde{c}_\beta$ is bounded and, recalling $\xi = \frac{k+y}{2} + t\frac{k-y}{2}$ with $t \in [0,1]$,

$$
\begin{aligned}
&\chi(k-y)\left\|r_\alpha(\tfrac{k+y}{2},\xi)\right\|_{\mathcal{L}(\mathcal{H}_\mathrm{f})} \\
&\quad\leq\; c\sup_{k\in\mathbb{R}^2}\sup_{t\in[0,1]}\sup_{v,w\in\frac{1}{2}\mathrm{supp}\chi}\left\|\partial_{x_{\alpha_1}}...\partial_{x_{\alpha_{N+2}}}t^\mathrm{B}(x,k-w)|_{x=k-w+tv}\right\|_{\mathcal{L}(\mathcal{H}_\mathrm{f})} \\
&\quad\leq\; c\sup_{\gamma^*\in\Gamma^*}\sup_{k\in M^*}\sup_{t\in[0,1]}\sup_{v,w\in\frac{1}{2}\mathrm{supp}\chi}\left\|\partial_{x_{\alpha_1}}...\partial_{x_{\alpha_{N+2}}}t^\mathrm{B}(x,k-w-\gamma^*)|_{x=k-\gamma^*-w+tv}\right\|_{\mathcal{L}(\mathcal{H}_\mathrm{f})} \\
&\quad\leq\; c\sup_{k\in M^*}\sup_{t\in[0,1]}\sup_{v,w\in\frac{1}{2}\mathrm{supp}\chi}\left\|\tau(\gamma^*)\partial_{x_{\alpha_1}}...\partial_{x_{\alpha_{N+2}}}t^\mathrm{B}(x,k-w)|_{x=k-w+tv}\tau(\gamma^*)^{-1}\right\|_{\mathcal{L}(\mathcal{H}_\mathrm{f})} \\
&\quad\leq\; c'.
\end{aligned}
$$

Then we insert $\langle y-k\rangle^2\langle y-k\rangle^{-2}$ in the integral in (4.14) and use the Cauchy-Schwarz inequality in $L^2(\mathbb{R}^2)$ to get the further estimation

$$
\begin{aligned}
&\|\rho(r_\alpha,\widetilde{c}_\beta)\psi(k)\|_{\mathcal{H}_\mathrm{f}} \\
&\quad\leq\; C_2\varepsilon^{N+m}\left(\int_{\mathbb{R}^2}\langle y-k\rangle^{-4}\|\psi(y)\|^2_{\mathcal{H}_\mathrm{f}}\,\mathrm{d}y\right)^{\frac{1}{2}}\times \qquad (4.15)\\
&\qquad\left(\int_{\mathbb{R}^2}\left\|\mathcal{F}(\partial_{\alpha_1}...\partial_{\alpha_{N+2}}\partial_{\beta_1}...\partial_{\beta_m}f_z)\left(\tfrac{y-k}{\varepsilon}\right)\right\|^2_{\mathcal{L}(\mathcal{H}_\mathrm{f})}\langle y-k\rangle^4\mathrm{d}y\right)^{\frac{1}{2}}. \qquad (4.16)
\end{aligned}
$$

The integral in (4.16) is estimated as follows using the condition $N \geq \alpha_0 - 2$:

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\left\|\mathcal{F}(\partial_{\alpha_1}...\partial_{\alpha_{N+2}}\partial_{\beta_1}...\partial_{\beta_m}f_z)\left(\tfrac{y-k}{\varepsilon}\right)\right\|^2_{\mathcal{L}(\mathcal{H}_\mathrm{f})}\langle y-k\rangle^4\mathrm{d}y \\
&\quad=\; \varepsilon^2\int_{\mathbb{R}^2}\left\|\mathcal{F}(\partial_{\alpha_1}...\partial_{\alpha_{N+2}}\partial_{\beta_1}...\partial_{\beta_m}f_{k+\frac{\varepsilon y}{2}})(y)\right\|^2_{\mathcal{L}(\mathcal{H}_\mathrm{f})}\langle\varepsilon y\rangle^4\langle y\rangle^4\langle y\rangle^{-4}\mathrm{d}y \\
&\quad\leq\; \varepsilon^2\int_{\mathbb{R}^2}\left\|(1-\Delta_r)(1-\varepsilon^2\Delta_r)\partial_{r_{\alpha_1}}..\partial_{r_{\alpha_{N+2}}}\partial_{r_{\beta_1}}..\partial_{r_{\beta_m}}f(k+\tfrac{\varepsilon y}{2},r)\right\|^2_{L^1(\mathbb{R}^2_r,\mathcal{L}(\mathcal{H}_\mathrm{f}))} \\
&\qquad\langle y\rangle^{-4}\mathrm{d}y \\
&\quad\leq\; \varepsilon^2\sup_{k\in M^*}\left\|(1-\Delta_r)(1-\varepsilon^2\Delta_r)\partial_{r_{\alpha_1}}...\partial_{r_{\alpha_{N+2}}}\partial_{r_{\beta_1}}...\partial_{r_{\beta_m}}f(k,r)\right\|^2_{L^1(\mathbb{R}^2_r,\mathcal{L}(\mathcal{H}_\mathrm{f}))} \\
&\qquad\int_{\mathbb{R}^2}\langle y\rangle^{-4}\mathrm{d}y \\
&\quad=\; C_3\varepsilon^2.
\end{aligned}
$$

Note that in the last inequality, we used that τ is a unitary.
The integral in (4.15) is estimated as follows: Let $k = [k] - \gamma^*_k$ with $[k] \in M^*$ and

$\gamma_k^* \in \Gamma^*$. Then

$$
\begin{aligned}
\int_{\mathbb{R}^2} \langle y-k\rangle^{-4} \|\psi(y)\|^2_{\mathcal{H}_\mathrm{f}}\,\mathrm{d}y &= \sum_{\gamma^*\in\Gamma^*}\int_{M^*}\langle y+\gamma^*-k\rangle^{-4}\|\psi(y+\gamma^*)\|^2_{\mathcal{H}_\mathrm{f}}\,\mathrm{d}y \\
&\leq \int_{M^*}\|\psi(y)\|^2_{\mathcal{H}_\mathrm{f}}\,\mathrm{d}y \sum_{\gamma^*\in\Gamma^*}\sup_{y\in M^*}\langle y+\gamma^*-k\rangle^{-4} \\
&= \|\psi\|^2_{\mathcal{H}_\tau}\sum_{\gamma^*\in\Gamma^*}\sup_{y\in M^*}\langle y+\gamma^*-[k]+\gamma_k^*\rangle^{-4} \\
&= \|\psi\|^2_{\mathcal{H}_\tau}\sum_{\gamma^*\in\Gamma^*}\sup_{y\in M^*}\langle y+\gamma^*-[k]\rangle^{-4} \\
&\leq \|\psi\|^2_{\mathcal{H}_\tau}\sum_{\gamma^*\in\Gamma^*}\sup_{y\in 2M^*}\langle y+\gamma^*\rangle^{-4} \\
&\leq C_4\,\|\psi\|^2_{\mathcal{H}_\tau}.
\end{aligned}
$$

So far we have shown

$$\|\rho(r_\alpha,\widetilde{c}_\beta)\psi(k)\|_{\mathcal{H}_\mathrm{f}} \leq C\varepsilon^{N+m+1}\,\|\psi\|_{\mathcal{H}_\tau}$$

and hence $\mathcal{T}_{N+1}\psi(k)$ is a convergent integral.
A quick computation shows that the thereby given function is τ-equivariant; note thereto that (3.2) implies $r_\alpha(z-\gamma^*,\xi-\gamma^*)=\tau(\gamma^*)r_\alpha(z,\xi)\tau(\gamma^*)^{-1}$. Let $\psi\in C^\infty_\tau$. Then

$$
\begin{aligned}
&\rho(r_\alpha,\widetilde{c}_\beta)\psi(k-\gamma^*) \\
&= \left(\tfrac{-i\varepsilon}{2}\right)^{N+m+2}\tfrac{1}{(2\pi\varepsilon)^2}\int_{\mathbb{R}^4} e^{\frac{i(k-\gamma^*-y)r}{\varepsilon}}\chi(k-\gamma^*-y)r_\alpha(\tfrac{k-\gamma^*+y}{2},\xi) \\
&\quad \partial_{r_{\alpha_1}}...\partial_{r_{\alpha_{N+2}}}\partial_{r_{\beta_1}}...\partial_{r_{\beta_m}}f(\tfrac{k-\gamma^*+y}{2},r)\widetilde{c}_\beta(\tfrac{k-\gamma^*+y}{2})\psi(y)\mathrm{d}r\mathrm{d}y \\
&\quad \text{with}\quad \xi\in[\tfrac{k-\gamma^*+y}{2},k-\gamma^*] \\
&= \left(\tfrac{-i\varepsilon}{2}\right)^{N+m+2}\tfrac{1}{(2\pi\varepsilon)^2}\int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}}\chi(k-y)r_\alpha(\tfrac{k+y}{2}-\gamma^*,\xi) \\
&\quad \partial_{r_{\alpha_1}}...\partial_{r_{\alpha_{N+2}}}\partial_{r_{\beta_1}}...\partial_{r_{\beta_m}}f(\tfrac{k+y}{2}-\gamma^*,r)\widetilde{c}_\beta(\tfrac{k+y}{2}-\gamma^*)\psi(y-\gamma^*)\mathrm{d}r\mathrm{d}y \\
&\quad \text{with}\quad \xi\in[\tfrac{k+y}{2}-\gamma^*,k-\gamma^*] \\
&= \left(\tfrac{-i\varepsilon}{2}\right)^{N+m+2}\tfrac{1}{(2\pi\varepsilon)^2}\int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}}\chi(k-y)\tau(\gamma^*)r_\alpha(\tfrac{k+y}{2},\xi+\gamma^*)\tau(\gamma^*)^{-1}\tau(\gamma^*) \\
&\quad \partial_{r_{\alpha_1}}...\partial_{r_{\alpha_{N+2}}}\partial_{r_{\beta_1}}...\partial_{r_{\beta_m}}f(\tfrac{k+y}{2},r)\tau(\gamma^*)^{-1}\tau(\gamma^*)\widetilde{c}_\beta(\tfrac{k+y}{2})\tau(\gamma^*)^{-1}\tau(\gamma^*)\psi(y)\mathrm{d}r\mathrm{d}y \\
&\quad \text{with}\quad \xi\in[\tfrac{k+y}{2}-\gamma^*,k-\gamma^*] \\
&= \tau(\gamma^*)\rho(r_\alpha,\widetilde{c}_\beta)\psi(k).
\end{aligned}
$$

So now we know that $\rho(r_\alpha, \widetilde{c}_\beta)\psi \in \mathcal{H}_\tau$ and finally we get

$$\|\rho(r_\alpha, \widetilde{c}_\beta)\psi\|_{\mathcal{H}_\tau} = \left(\int_{M*} \|\rho(r_\alpha, \widetilde{c}_\beta)\psi(k)\|^2_{\mathcal{H}_\mathrm{f}}\, \mathrm{d}k\right)^{\frac{1}{2}} \leq C_5 \varepsilon^{N+m+1} \|\psi\|_{\mathcal{H}_\tau},$$

which suffices because C^∞_τ is dense in $\mathcal{H}_\tau$.
The other parts can be estimated in the same way, so we have proven

$$\left\| \widehat{f}^{\,\mathrm{B}} - \widehat{f_\mathrm{c}^{(N)}}^{\,\tau} \right\|_{\mathcal{L}(\mathcal{H}_\tau)} \leq C_{N+1}\varepsilon^{N+1}.$$

Part 4.
Now using Theorem B.3.6, the proof of Theorem 4.2.11, the Calderon-Vaillancourt version for the Berry quantisation, is short:

$$\begin{aligned} &\sup_{\psi \in C^\infty_\tau, \|\psi\|_{\mathcal{H}_\tau} \leq 1} \left\| \widehat{f}^{\,\mathrm{B}} \psi \right\|_{\mathcal{H}_\tau} \\ \leq\; & \sup_{\psi \in C^\infty_\tau, \|\psi\|_{\mathcal{H}_\tau} \leq 1} \left\| (\widehat{f}^{\,\mathrm{B}} - \widehat{f}^{\,\tau})\psi \right\|_{\mathcal{H}_\tau} + \sup_{\psi \in C^\infty_\tau, \|\psi\|_{\mathcal{H}_\tau} \leq 1} \left\| \widehat{f}^{\,\tau} \psi \right\|_{\mathcal{H}_\tau} \\ \leq\; & c_1 \varepsilon + c_2. \end{aligned}$$

Part 5.
With Theorem 4.2.11 at our disposal, we can do the proof the other way round and show

$$\widehat{f}^{\,\tau} = \widehat{f_\mathrm{c}^{(N)}}^{\,\mathrm{B}} + \mathcal{O}(\varepsilon^{N+1}).$$

The proof works as in step two and three, so we just point out briefly the two basic modifications.
In step two, we start with $\widehat{f}^{\,\tau}$. To compare it with $\widehat{f}^{\,\mathrm{B}}$, we insert the identity maps $t^\mathrm{B}(k, \frac{k+y}{2})t^\mathrm{B}(\frac{k+y}{2}, k)$ and $t^\mathrm{B}(y, \frac{k+y}{2})t^\mathrm{B}(\frac{k+y}{2}, y)$ around the symbol f in the formula for the τ-quantisation and take care of the term $t^\mathrm{B}(\frac{k+y}{2}, k)f(\frac{k+y}{2}, r)t^\mathrm{B}(y, \frac{k+y}{2})$ with the methods above. The difference is that the derivatives in the formulas for the c_α, r_α, etc. are now supposed to fall in each case onto the other components.
The second difference is the occurrence of the parallel transport maps $t^\mathrm{B}(k, \frac{k+y}{2})$ and $t^\mathrm{B}(k, \frac{k+y}{2})$ in the remainder terms which are estimated in step three. That is why in this case, we need the Calderon-Vaillancourt theorem for the Berry quantisation and not the one for the τ-quantisation. In the second estimation, the maps t^B do not cause problems because their norm equals one since they are unitary maps.

Part 6.
Now we briefly point out the changes of the proof if we take a symbol $f \in$

$S^m_{\rho,\tau}(\mathbb{R}^4,\mathcal{L}(\mathcal{H}_\mathrm{f}))$ with $\rho > 0$. In step one, we have already asserted that $c_\alpha \in S^1_\tau(\mathbb{R}^4,\mathcal{L}(\mathcal{H}_\mathrm{f}))$. Since it does not depend on r, it also holds $c_\alpha \in S^0_{\rho,\tau}(\mathbb{R}^4,\mathcal{L}(\mathcal{H}_\mathrm{f}))$ for every $\rho > 0$. Using $\partial^\alpha_r f \in S^{m-|\alpha|\rho}_{\rho,\tau}(\mathbb{R}^4,\mathcal{L}(\mathcal{H}_\mathrm{f}))$, we get from Proposition B.3.5 $f_{\mathrm{c}n} \in S^{m-n\rho}_{\rho,\tau}(\mathbb{R}^4,\mathcal{L}(\mathcal{H}_\mathrm{f}))$ and $f^{(N)}_\mathrm{c} \in S^m_{\rho,\tau}(\mathbb{R}^4,\mathcal{L}(\mathcal{H}_\mathrm{f}))$.
In part two, nothing changes but the fact that we must do the Taylor expansions for $N > \frac{m+4}{\rho} - 2$ to assure that the respective derivatives of f are in $L^1 \cap L^2$ with $\|\partial^\alpha_r f(k,r)\|_{L^1(\mathbb{R}^2_r,\mathcal{L}(\mathcal{H}_\mathrm{f}))} \leq c_\alpha$ and that they are in S^1_τ, which is both needed in the estimation for the remainder term in part three. □

Remark 4.2.17. The theorem also implies the existence of a semiclassical symbol $f_\mathrm{c} \asymp \sum_{n\geq 0} \varepsilon^n f_{\mathrm{c}n}$ in the accordant symbol class which satisfies $\widehat{f}^{\,\tau} = \widehat{f_\mathrm{c}}^{\,\mathrm{B}} + \mathcal{O}(\varepsilon^\infty)$ and the other way round. Moreover, we can also correct semiclassical symbols. This is the content of the following corollary.

Corollary 4.2.18. *Let $f \in S^1_\tau(\varepsilon,\mathcal{L}(\mathcal{H}_\mathrm{f}))$ with $f \asymp \sum_{n\geq 0} \varepsilon^n f_n$ fulfil*

(∗) *for every $n \in \mathbb{N}_0$ there is $\alpha_n \in \mathbb{N}$ so that for $|\alpha| \geq \alpha_n$ it holds $\partial^\alpha_r f_n(k,r) \in L^1(\mathbb{R}^2_r,\mathcal{H}_\mathrm{f}) \cap L^2(\mathbb{R}^2_r,\mathcal{H}_\mathrm{f})$ for all $k \in \mathbb{R}^2$ and $\|\partial^\alpha_r f_n(k,r)\|_{L^1(\mathbb{R}^2_r,\mathcal{L}(\mathcal{H}_\mathrm{f}))} \leq h_{n,\alpha}(k)$ with $h_{n,\alpha} \in C(\mathbb{R}^2,\mathbb{R}_{\geq 0})$,*

or let $f \in S^m_{\rho,\tau}(\varepsilon,\mathcal{L}(\mathcal{H}_\mathrm{f}))$ with $\rho > 0$.

(i) *There is a semiclassical symbol f_c in the same symbol class as f so that*

$$\widehat{f}^{\,\tau} = \widehat{f_\mathrm{c}}^{\,\mathrm{B}} + \mathcal{O}(\varepsilon^\infty).$$

It holds

$$f_\mathrm{c} \asymp \sum_{n\geq 0} \varepsilon^n g_n,$$

where

$$g_n = \sum_{j=0}^{n} (f_j)_{c(n-j)},$$

where we used the notation from Theorem 4.2.16, which means that $(f_j)_{c(n-j)}$ is the $(n-j)$th correction of the ordinary symbol f_j according to Theorem 4.2.16(i).
For $f \in S^1_\tau(\varepsilon,\mathcal{L}(\mathcal{H}_\mathrm{f}))$ the correction f_c fulfils (∗).
In particular, the principal and subprincipal symbol of f_c read

$$g_0(k,r) = (f_\mathrm{c})_0(k,r) = f_0(k,r)$$

and

$$g_1(k,r) = (f_\mathrm{c})_1(k,r) = f_1(k,r) - \tfrac{i}{2}\left(D(k)\nabla_r f_0(k,r) + \nabla_r f_0(k,r) D(k)\right),$$

where $D(k) = \nabla_k - \nabla^\mathrm{B}_k = \pi_0^\perp(k)\nabla_k \pi_0(k) + \pi_0(k)\nabla_k\pi_0^\perp(k)$.

(ii) There is a semiclassical symbol f_{c} in the same symbol class as f so that

$$\widehat{f}^{\,\mathrm{B}} = \widehat{f_{\mathrm{c}}}^{\,\tau} + \mathcal{O}(\varepsilon^\infty).$$

It holds

$$f_{\mathrm{c}} \asymp \sum_{n\geq 0} \varepsilon^n g_n,$$

where

$$g_n = \sum_{j=0}^{n} (f_j)_{c(n-j)},$$

where we used the notation from Theorem 4.2.16, which means that $(f_j)_{c(n-j)}$ is the $(n-j)$th correction of the ordinary symbol f_j according to Theorem 4.2.16(ii).
For $f \in S^1_\tau(\varepsilon, \mathcal{L}(\mathcal{H}_{\mathrm{f}}))$ the correction f_{c} fulfils $(*)$.
In particular, the principal and subprincipal symbol of f_{c} read

$$g_0(k,r) = (f_{\mathrm{c}})_0(k,r) = f_0(k,r)$$

and

$$g_1(k,r) = (f_{\mathrm{c}})_1(k,r) = f_1(k,r) + \tfrac{i}{2}\left(D(k)\nabla_r f_0(k,r) + \nabla_r f_0(k,r) D(k)\right),$$

where $D(k) = \nabla_k - \nabla_k^{\mathrm{B}} = \pi_0^\perp(k)\nabla_k \pi_0(k) + \pi_0(k)\nabla_k\pi_0^\perp(k)$.

Proposition 4.2.19. *Let f be a symbol which satisfies the assumptions of Theorem 4.2.16 and is self-adjoint. Then also its correction f_{c} is self-adjoint.*

Proof.
Let first f_{c} be the correction of f in the sense of Theorem 4.2.16 (i). We will just give the proof for this case since the other case can be proven analoguosly. First note that for $\alpha \in \{1,2\}^j$ and $j \geq 0$ it holds

$$(c_\alpha(k))^* = \tfrac{1}{j!}(\partial_{y_{\alpha_1}}...\partial_{y_{\alpha_j}} t^{\mathrm{B}}(k,y)|_{y=k})^* = \tfrac{1}{j!}\partial_{y_{\alpha_1}}...\partial_{y_{\alpha_j}} t^{\mathrm{B}}(y,k)|_{y=k} = (-1)^j \widetilde{c_\alpha}(k).$$

Thus we get

$$
\begin{aligned}
&(f_{\mathrm{c}}^{(N)}(k,r))^* \\
&= \sum_{n=0}^{N}\sum_{j=0}^{n}\left(\tfrac{-i\varepsilon}{2}\right)^n \sum_{\alpha\in\{1,2\}^j}\sum_{\beta\in\{1,2\}^{n-j}} (\widetilde{c}_\beta(k))^* \partial_{r_{\alpha_1}}..\partial_{r_{\alpha_j}}\partial_{r_{\beta_1}}..\partial_{r_{\beta_{n-j}}} f(k,r)(c_\alpha(k))^* \\
&= \sum_{n=0}^{N}\sum_{j=0}^{n}(-1)^n\left(\tfrac{i\varepsilon}{2}\right)^n \sum_{\alpha\in\{1,2\}^j}\sum_{\beta\in\{1,2\}^{n-j}} (-1)^{n-j} c_\beta(k) \partial_{r_{\alpha_1}}..\partial_{r_{\alpha_j}}\partial_{r_{\beta_1}}..\partial_{r_{\beta_{n-j}}} f(k,r) \\
&\quad (-1)^j \widetilde{c_\alpha}(k) \\
&= \sum_{n=0}^{N}\sum_{j=0}^{n}\left(\tfrac{i\varepsilon}{2}\right)^n \sum_{\alpha\in\{1,2\}^j}\sum_{\beta\in\{1,2\}^{n-j}} c_\beta(k) \partial_{r_{\alpha_1}}...\partial_{r_{\alpha_j}}\partial_{r_{\beta_1}}...\partial_{r_{\beta_{n-j}}} f(k,r)\widetilde{c_\alpha}(k) \\
&= f_{\mathrm{c}}^{(N)}(k,r).
\end{aligned}
$$

□

4.3 The θ-quantisation

Now we are in a position to compute a correction f_{c} of f so that the difference between the τ-quantisation of f and the Berry quantisation of f_{c} is of order $\mathcal{O}(\varepsilon^\infty)$. The next step is to exploit the unitary equivalence of the connections ∇^{Berry} and $\nabla^\theta = U^\theta \nabla^{\mathrm{B}} U^{\theta *}$ and to "absorb" the unitary map U^θ into a new quantisation, namely the θ-quantisation Op^θ. Thereto, we first introduce the θ-quantisation and the corresponding symbols. Note that the idea for the quantisation is exactly the same as the one for the Berry quantisation, which is to use the parallel transport of the desired connection in the definition of the quantisation. Then we show how the symbol has to be changed to get

$$U^\theta \widehat{f}^{\,\mathrm{B},\chi} U^{\theta *} = \widehat{f_\theta}^{\,\theta,\chi}.$$

The emerging pseudodifferential operator $\widehat{f_\theta}^{\,\theta,\chi}$ does not operate on $\mathcal{H}_\tau$ any more but on $\mathcal{H}_\theta$. Thus we also expect the accordant symbol to be in $C^\infty(\mathbb{R}^4,\mathbb{C})$ and not in $C^\infty(\mathbb{R}^4,\mathcal{L}(\mathcal{H}_{\mathrm{f}}))$.

Definition 4.3.1. *Let $f \in S^w(\mathbb{R}^4,\mathbb{C}) \cup S^m_\rho(\mathbb{R}^4,\mathbb{C})$ and χ be a smooth cutoff function. Then for $\psi \in S(\mathbb{R}^2,\mathbb{C})$ we define the θ-quantisation by*

$$\widehat{f}^{\,\theta,\chi}\psi(k) := \tfrac{1}{(2\pi\varepsilon)^2}\int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}} \chi(k-y) t^\theta(k,y)\, f\left(\tfrac{k+y}{2}, r\right)\psi(y)\mathrm{d}r\mathrm{d}y,$$

where $t^\theta(x,y)$ is the parallel transport along the straight line from y to x with respect to the connection $\nabla^\theta = U^\theta \nabla^{\mathrm{B}} U^{\theta^*}$.

Remark 4.3.2. In this case, the parallel transport map t^θ appears only once in the quantisation formula and not twice as in the Berry connection. This happens because the symbol and the parallel transport maps are each $\mathbb{C}$-valued and hence it holds

$$t^\theta(k, \tfrac{k+y}{2})f(\tfrac{k+y}{2}, r)t^\theta(\tfrac{k+y}{2}, y) = t^\theta(k, \tfrac{k+y}{2})t^\theta(\tfrac{k+y}{2}, y)f(\tfrac{k+y}{2}, r) = t^\theta(k, y)f(\tfrac{k+y}{2}, r).$$

Remark 4.3.3. We will see that for suitable symbols the θ-quantisation does not depend on the cutoff up to an error of $\mathcal{O}(\varepsilon^\infty)$.

To show the well-definedness of this quantisation, we again follow the usual routine as we did with the Berry quantisation: First, we show that the quantised symbol is a continuous map from the Schwartz space $S(\mathbb{R}^2)$ to itself. Then we extend this mapping by duality to $S'(\mathbb{R}^2)$. As counterpart to the τ-equivariance of functions in the context of the Berry connection, we need to define an operator V_{γ^*} for a multiplication with the phase $e^{\frac{i\theta}{2\pi}k_2\gamma_1^*}$ that enables us to define a space S_θ respectively S'_θ as the set of functions respectively distributions which satisfy $V_{\gamma^*}f = L_{\gamma^*}f$.

Then it can be shown that the quantisation of Γ^*-periodic symbols, which are the correspondents of the τ-equivariant symbols in the context of the Berry quantisation, leaves S'_θ invariant. The proofs are very similar to the ones for the Berry quantisation, so we keep everything short.

Proposition 4.3.4. *For $f \in S^w(\mathbb{R}^4, \mathbb{C}) \cup S^m_\rho(\mathbb{R}^4, \mathbb{C})$, χ a smooth cutoff function, and $\psi \in S(\mathbb{R}^2)$, the integral*

$$(\widehat{f}^{\,\theta,\chi}\psi)(k) = \frac{1}{(2\pi\varepsilon)^2}\int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}}\chi(k-y)t^\theta(k,y)\, f\left(\tfrac{k+y}{2}, r\right)\psi(y)\mathrm{d}r\mathrm{d}y$$

defines a continuous mapping from $S(\mathbb{R}^2)$ to $S(\mathbb{R}^2)$.

Proof.
The proof works as the proof of Proposition 4.2.4. One only has to consider that to show $\sup_{k\in\mathbb{R}^2} C(k) < \infty$ in estimation (4.5), one uses the property

$$t^\theta(k-\gamma^*, y-\gamma^*) = e^{\frac{i\theta}{2\pi}(k_2-y_2)\gamma_1^*}t^\theta(k,y), \tag{4.17}$$

which is the counterpart of the property (3.2). □

Again, this mapping can be extended to a map

$$\widehat{f}^{\,\theta,\chi} : S'(\mathbb{R}^2) \to S'(\mathbb{R}^2)$$

by putting

$$\widehat{f}^{\,\theta,\chi}(T)(\psi) := T(\widehat{\overline{f}}^{\,\theta,\chi}\psi)$$

for $\psi \in S(\mathbb{R}^2)$ and $T \in S'(\mathbb{R}^2)$.

Definition 4.3.5. *Let $\gamma \in \mathbb{R}^2$, ψ a function defined on $\mathbb{R}^2$, and T a distribution in $S'(\mathbb{R}^2)$. Then*

$$V_\gamma \psi(k) := e^{\frac{i\theta}{2\pi} k_2 \gamma_1} \psi(k)$$

and

$$V_\gamma(T)(\psi) := T(V_{-\gamma}\psi) \quad \textit{for} \quad \psi \in S(\mathbb{R}^2) \quad \textit{and} \quad T \in S'(\mathbb{R}^2)$$

as well as

$$S_\theta(\mathbb{R}^2) = \{f \in S(\mathbb{R}^2) : V_{\gamma^*} f = L_{\gamma^*} f \quad \forall \gamma^* \in \Gamma^*\}$$

and

$$S'_\theta(\mathbb{R}^2) = \{T \in S'(\mathbb{R}^2) : V_{\gamma^*} T = L_{\gamma^*} T \quad \forall \gamma^* \in \Gamma^*\}$$

and moreover

$$C^\infty_\theta(\mathbb{R}^2) := \{f \in C^\infty(\mathbb{R}^2) : f(k - \gamma^*) = V_{\gamma^*} f(k) \quad \forall \gamma^* \in \Gamma^*\}.$$

Proposition 4.3.6. *For $f \in S^w_{\tau\equiv 1}(\mathbb{R}^4, \mathbb{C}) \cup S^m_{\rho,\tau\equiv 1}(\mathbb{R}^4, \mathbb{C})$ we have*

$$\widehat{f}^{\,\theta,\chi} S'_\theta \subset S'_\theta.$$

Proof.
This can be seen as in the proof of Proposition 4.2.5. Let $T \in S'_\theta$ and $\psi \in S(\mathbb{R}^2)$. Then

$$\begin{aligned}
L_{\gamma^*} \widehat{f}^{\,\theta,\chi} T(\psi) &= T(\widehat{\overline{f}}^{\,\theta,\chi} L_{-\gamma^*}\psi) \\
&= T\Big(k \mapsto \tfrac{1}{(2\pi\varepsilon)^2} \int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}} \chi(k-y) t^\theta(k,y) \overline{f}\left(\tfrac{k+y}{2}, r\right) \psi(y+\gamma^*) \mathrm{d}r \mathrm{d}y\Big) \\
&= T\Big(k \mapsto \tfrac{1}{(2\pi\varepsilon)^2} \int_{\mathbb{R}^4} e^{\frac{i(k+\gamma^*-y)r}{\varepsilon}} \chi(k+\gamma^*-y) t^\theta(k, y-\gamma^*) \overline{f}\left(\tfrac{k-\gamma^*+y}{2}, r\right) \\
&\quad \psi(y) \mathrm{d}r \mathrm{d}y\Big) \\
&= T\Big(k \mapsto \tfrac{1}{(2\pi\varepsilon)^2} \int_{\mathbb{R}^4} e^{\frac{i(k+\gamma^*-y)r}{\varepsilon}} \chi(k+\gamma^*-y) e^{\frac{i\theta}{2\pi}(k_2-y_2)\gamma_1^*} t^\theta(k+\gamma^*, y) \\
&\quad \overline{f}\left(\tfrac{k+\gamma^*+y}{2}, r\right) \psi(y) \mathrm{d}r \mathrm{d}y\Big) \\
&= T(V_{\gamma^*} L_{-\gamma^*} \widehat{\overline{f}}^{\,\theta,\chi} V_{-\gamma^*}\psi) = T(\widehat{\overline{f}}^{\,\theta,\chi} V_{-\gamma^*}\psi) \\
&= V_{\gamma^*} \widehat{f}^{\,\theta,\chi} T(\psi).
\end{aligned}$$

Here we used (4.17). ☐

The symbols for which the θ-quantisation does not depend on the cutoff up to a “small“ error are again those with an improving behaviour for the derivatives with respect to r:

Proposition 4.3.7. *Let* $f \in S^w_{\tau\equiv 1}(\mathbb{R}^4, \mathbb{C})$ *fulfil*

(∗) $\exists \alpha_0 \in \mathbb{N}$ *so that for all* $|\alpha| \geq \alpha_0$ *it holds* $\partial_r^\alpha f(k,r) \in L^1(\mathbb{R}^2_r) \cap L^2(\mathbb{R}^2_r)$ *for all* $k \in \mathbb{R}^2$ *and* $\|\partial_r^\alpha f(k,r)\|_{L^1(\mathbb{R}^2_r)} \leq h_\alpha(k)$ *with* $h_\alpha \in C(\mathbb{R}^2, \mathbb{R}_{\geq 0})$.

Then the θ*-quantisation of this symbol does not depend on the cutoff* χ *up to an error of* $\mathcal{O}(\varepsilon^\infty)$*, which means that for two smooth cutoff functions* χ *and* $\widetilde{\chi}$ *it holds*

$$\widehat{f}^{\,\theta,\chi} = \widehat{f}^{\,\theta,\widetilde{\chi}} + \mathcal{O}(\varepsilon^\infty).$$

Proof.
This proof is analogous to the proof of Proposition 4.2.6, so we do not give details here. □

Remark 4.3.8. Note that Proposition 4.3.7 includes symbols $f \in S^m_{\rho,\tau}(\mathbb{R}^4, \mathbb{C})$ with $\rho > 0$.

We proceed by showing that also for the θ-quantisation it holds that for symbols f with $\widehat{f}^{\,\theta} \in \mathcal{L}(\mathcal{H}_\theta)$, the adjoint of the quantised symbol is the quantisation of the pointwise adjoint of the symbol f.

Proposition 4.3.9. *Let* $f \in S^w_{\tau\equiv 1}(\mathbb{R}^4, \mathbb{C}) \cup S^m_{\rho,\tau\equiv 1}(\mathbb{R}^4, \mathbb{C})$ *with* $\widehat{f}^{\,\theta,\chi} \in \mathcal{L}(\mathcal{H}_\theta)$*. Then*

$$(\widehat{f}^{\,\theta,\chi})^* = \widehat{\overline{f}}^{\,\theta,\chi}.$$

Proof.
The theorem can be proven following the line of the proof of Proposition 4.2.13. One difference is that the distributional integral kernel K_f now fulfils

$$K_f(k - \gamma^*, y - \gamma^*) = e^{\frac{i\theta}{2\pi}(k_2 - y_2)\gamma_1^*} K_f(k, y), \tag{4.18}$$

which follows from the Γ^*-periodicity of f and the property (4.17) of the parallel transport with respect to ∇^θ. The other difference is, of course, that we have to take $\psi \in \mathcal{H}_\theta$ and $\phi \in C^\infty_\theta$. Let again $\widetilde{\phi} = 1_{M^*}\phi$. Then, using $\widetilde{\phi}$ as a test function, we get

$$
\begin{aligned}
\langle \phi, \widehat{f}^{\,\theta,\chi}\psi\rangle_{\mathcal{H}_\theta} &= \int_{M^*} \overline{\phi(k)}\widehat{f}^{\,\theta,\chi}\psi(k)\mathrm{d}k = \int_{\mathbb{R}^2} \overline{\widetilde{\phi}(k)}\widehat{f}^{\,\theta,\chi}\psi(k)\mathrm{d}k = \overline{\widehat{f}^{\,\theta,\chi}(\psi)(\widetilde{\phi})} \\
&= \overline{\psi(\widehat{\overline{f}}^{\,\theta,\chi}\widetilde{\phi})} = \int_{\mathbb{R}^2} \overline{\int_{\mathbb{R}^2} K_{\overline{f}}(k,y)\widetilde{\phi}(y)\mathrm{d}y}\psi(k)\mathrm{d}k \\
&= \int_{\mathbb{R}^2} \overline{\int_{M^*} K_{\overline{f}}(k,y)\phi(y)\mathrm{d}y}\psi(k)\mathrm{d}k \\
&= \int_{M^*} \sum_{\gamma^*\in\Gamma^*} \overline{\int_{M^*} K_{\overline{f}}(k+\gamma^*,y)\phi(y)\mathrm{d}y}\psi(k+\gamma^*)\mathrm{d}k \\
&= \int_{M^*} \sum_{\gamma^*\in\Gamma^*} \overline{\int_{M^*} e^{\frac{i\theta}{2\pi}(y_2-k_2)\gamma_1^*} K_{\overline{f}}(k,y-\gamma^*)\phi(y)\mathrm{d}y e^{-\frac{i\theta}{2\pi}k_2\gamma_1^*}}\psi(k)\mathrm{d}k \\
&= \int_{M^*} \sum_{\gamma^*\in\Gamma^*} \overline{\int_{M^*} K_{\overline{f}}(k,y-\gamma^*)e^{\frac{i\theta}{2\pi}y_2\gamma_1^*}\phi(y)\mathrm{d}y}\psi(k)\mathrm{d}k \\
&= \int_{M^*} \sum_{\gamma^*\in\Gamma^*} \overline{\int_{M^*} K_{\overline{f}}(k,y-\gamma^*)\phi(y-\gamma^*)\mathrm{d}y}\psi(k)\mathrm{d}k \\
&= \int_{M^*} \overline{\int_{\mathbb{R}^2} K_{\overline{f}}(k,y)\phi(y)\mathrm{d}y}\psi(k)\mathrm{d}k = \int_{M^*} \overline{(\widehat{\overline{f}}^{\,\theta,\chi}\phi)(k)}\psi(k)\mathrm{d}k \\
&= \langle \widehat{\overline{f}}^{\,\theta,\chi}\phi, \psi\rangle_{\mathcal{H}_\theta}.
\end{aligned}
$$

Since C_θ^∞ is dense in $\mathcal{H}_\theta$ and $\widehat{f}^{\,\theta,\chi}$ is continuous, the claim follows. $\square$

Now that we have introduced the θ-quantisation, we show how it is connected to the Berry quantisation. It is clear that we are only interested in symbols for which $\widehat{f}^{\,\mathrm{B},\chi}$ commutes with Π^0 because we want to exploit the unitary equivalence of $\Pi^0\mathcal{H}_\tau$ and $\mathcal{H}_\theta$.

Theorem 4.3.10. *Let $f \in S^w_\tau(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathrm{f})) \cup S^m_{\rho,\tau}(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathrm{f}))$ with $[\pi_0(k), f(k,r)] = 0$. Then it holds*

(i) $f_\theta(k,r) := \langle\varphi(k), f(k,r)\varphi(k)\rangle_{\mathcal{H}_\mathrm{f}} \in S^w_{\tau\equiv 1}(\mathbb{R}^4,\mathbb{C}) \cup S^m_{\rho,\tau\equiv 1}(\mathbb{R}^4,\mathbb{C})$ *and*

(ii) $U^\theta \widehat{f}^{\,\mathrm{B},\chi} U^{\theta *} = \widehat{f_\theta}^{\,\theta,\chi}$.

Proof.
The first statement that f_θ is in the accordant symbol class is a simple calculation. The periodicity follows directly since τ is a unitary representation, f is τ-equivariant, and φ has the property (3.4). From the periodicity of f_θ in k, we

get the boundedness of the derivative with respect to k. The properties of the derivatives with respect to r also follow immediately.
For the second statement, note first that Proposition 4.2.8 assures that $\widehat{f}^{\,\mathrm{B},\chi}$ can be perceived as a map from $\Pi^0\mathcal{H}_\tau$ to $\Pi^0\mathcal{H}_\tau$. Second, note that the unitary U^θ can be perceived both as a map between functions and a map between distributions in the usual way setting $U^\theta(T)(\psi) = T(U^{\theta*}\psi)$. The same holds true for the map $U^{\theta*}$. Hence for $T \in S'_\theta(\mathbb{R}^2)$ and $\psi \in S(\mathbb{R}^2)$ we get

$$\begin{aligned}
& U^\theta \widehat{f}^{\,\mathrm{B},\chi} U^{\theta*}(T)(\psi) = T(U^\theta \widehat{f^*}^{\,\mathrm{B},\chi} U^{\theta*}\psi) = T(k \mapsto \langle \varphi(k), \widehat{f^*}^{\,\mathrm{B},\chi}(\varphi\psi)(k)\rangle_{\mathcal{H}_\mathrm{f}}) \\
= \;& T(k \mapsto \langle \varphi(k), \tfrac{1}{(2\pi\varepsilon)^2} \int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}} \chi(k-y) t^\mathrm{B}\left(k, \tfrac{k+y}{2}\right) f^*\left(\tfrac{k+y}{2}, r\right) t^\mathrm{B}\left(\tfrac{k+y}{2}, y\right) \\
& \varphi(y)\psi(y)\mathrm{d}r\mathrm{d}y\rangle_{\mathcal{H}_\mathrm{f}}) \\
= \;& T(k \mapsto \langle \varphi(k), \tfrac{1}{(2\pi\varepsilon)^2} \int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}} \chi(k-y)\varphi(k) t^\theta\left(k, \tfrac{k+y}{2}\right) \langle \varphi(\tfrac{k+y}{2}), f^*\left(\tfrac{k+y}{2}, r\right) \\
& \varphi(\tfrac{k+y}{2})\rangle_{\mathcal{H}_\mathrm{f}} t^\theta\left(\tfrac{k+y}{2}, y\right)\psi(y)\mathrm{d}r\mathrm{d}y\rangle_{\mathcal{H}_\mathrm{f}}) = T(\widehat{\overline{f_\theta}}^{\,\theta}\psi) \\
= \;& \widehat{f_\theta}^{\,\theta}(T)(\psi).
\end{aligned}$$

Note that here we used the identities

$$t^\mathrm{B}(z,y)\varphi(y) = \varphi(z)t^\theta(z,y),$$

$$\langle \varphi(z), f^*(z,r)\,\varphi(z)\rangle_{\mathcal{H}_\mathrm{f}} = \langle f(z,r)\,\varphi(z), \varphi(z)\rangle_{\mathcal{H}_\mathrm{f}} = \overline{f_\theta}(z),$$

and the fact that f and $\pi_0(k)$ commute. □

This theorem can be generalised to semiclassical symbols.

Corollary 4.3.11. *Let $f \in S^w_\tau(\varepsilon, \mathcal{L}(\mathcal{H}_\mathrm{f})) \cup S^m_{\rho,\tau}(\varepsilon, \mathcal{L}(\mathcal{H}_\mathrm{f}))$ be a semiclassical symbol satisfying $f \asymp \sum_{j\geq 0} \varepsilon^j f_j$ and $[\pi_0(k), f(k,r)] = 0$. Then*

- *$f_\theta(k,r) = \langle \varphi(k), f(k,r)\varphi(k)\rangle_{\mathcal{H}_\mathrm{f}} \in S^w_{\tau\equiv 1}(\varepsilon, \mathbb{C})$ respectively $S^m_{\rho,\tau\equiv 1}(\varepsilon, \mathbb{C})$ with $f_\theta \asymp \sum_{j\geq 0} \varepsilon^j (f_j)_\theta$ and*
- *$U^\theta \widehat{f}^{\,\mathrm{B}} U^{\theta*} = \widehat{f_\theta}^{\,\theta}$.*

Proof.
For the proof, one just has to check that $f_\theta \asymp \sum_{j\geq 0} \varepsilon^j (f_j)_\theta$. Let $f \in S^w_\tau(\varepsilon)$, $n \in \mathbb{N}$, and $l \in \mathbb{N}_0$. Then it follows from

$$f_\theta(k,r) - \sum_{j=0}^{n-1} \varepsilon^j f_{j\theta}(k,r) = \langle \varphi(k), (f(k,r) - \sum_{j=0}^{n-1} \varepsilon^j f_j(k,r))\varphi(k)\rangle_{\mathcal{H}_\mathrm{f}}$$

that

$$\begin{aligned}
&\sup_{|\alpha+\beta|\leq l}\sup_{(k,r)\in\mathbb{R}^4} w(k,r)^{-1}|\partial_k^\alpha\partial_r^\beta \tfrac{1}{\varepsilon^n}(f_\theta(k,r)-\sum_{j=0}^{n-1}\varepsilon^j f_{j\theta}(k,r))| \\
&= \sup_{|\alpha+\beta|\leq l}\sup_{k\in M^*, r\in\mathbb{R}^2}\sup_{\gamma^*\in\Gamma^*} w(k-\gamma^*,r)^{-1}\times \\
&\quad |\sum_{\alpha_1+\alpha_2+\alpha_3\leq\alpha} c_{\alpha_1\alpha_2\alpha_3}\langle\partial_k^{\alpha_1}\varphi(k),\partial_k^{\alpha_2}\partial_r^\beta\tfrac{1}{\varepsilon^n}(f(k,r)-\sum_{j=0}^{n-1}\varepsilon^j f_j(k,r))\partial_k^{\alpha_3}\varphi(k)\rangle_{\mathcal{H}_\mathrm{f}}| \\
&\leq C\sup_{|\alpha+\beta|\leq l}\sup_{k\in M^*, r\in\mathbb{R}^2}\sup_{\gamma^*\in\Gamma^*} w(k-\gamma^*,r)^{-1}\times \\
&\quad \left\|\partial_k^\alpha\partial_r^\beta\tfrac{1}{\varepsilon^n}(f(k-\gamma^*,r)-\sum_{j=0}^{n-1}\varepsilon^j f_j(k-\gamma^*,r))\right\|_{\mathcal{L}(\mathcal{H}_\mathrm{f})} \\
&= C\sup_{|\alpha+\beta|\leq l}\sup_{k\in\mathbb{R}^2, r\in\mathbb{R}^2} w(k,r)^{-1}\left\|\partial_k^\alpha\partial_r^\beta\tfrac{1}{\varepsilon^n}(f(k,r)-\sum_{j=0}^{n-1}\varepsilon^j f_j(k,r))\right\|_{\mathcal{L}(\mathcal{H}_\mathrm{f})} \\
&\leq C'',
\end{aligned}$$

where we exploited that $\langle\varphi(k),(f(k,r)-\sum_{j=0}^{n-1}\varepsilon^j f_j(k,r))\varphi(k)\rangle_{\mathcal{H}_\mathrm{f}}$ is Γ^*-periodic in k and $f\asymp\sum_{j\geq 0}\varepsilon^j f_j$ in S^w_τ. The analogous statement for $f\in S^m_{\rho,\tau}$ can be proven in the same way. $\square$

Remark 4.3.12. Obviously, if $f(k,r)$ is self-adjoint, also f_θ is self-adjoint since

$$\begin{aligned}
\overline{f_\theta(k,r)} &= \overline{\langle\varphi(k),f(k,r)\varphi(k)\rangle_{\mathcal{H}_\mathrm{f}}} = \langle f(k,r)\varphi(k),\varphi(k)\rangle_{\mathcal{H}_\mathrm{f}} = \langle\varphi(k),f(k,r)\varphi(k)\rangle_{\mathcal{H}_\mathrm{f}} \\
&= f_\theta(k,r).
\end{aligned}$$

Theorem 4.3.13. *Let $f\in S^1_{\tau\equiv 1}(\mathbb{R}^4,\mathbb{C})$ fulfil*

(∗) *$\exists\alpha_0\in\mathbb{N}$ so that for $|\alpha|\geq\alpha_0$ it holds $\partial_r^\alpha f(k,r)\in L^1(\mathbb{R}^2_r)\cap L^2(\mathbb{R}^2_r)$ for all $k\in\mathbb{R}^2$ and $\|\partial_r^\alpha f(k,r)\|_{L^1(\mathbb{R}^2_r)}\leq h_\alpha(k)$ with $h_\alpha\in C(\mathbb{R}^2,\mathbb{R}_{\geq 0})$.*

Then

$$\widehat{f}^{\,\theta}\in\mathcal{L}(\mathcal{H}_\theta).$$

Proof.
Let $\psi\in\mathcal{H}_\theta$ and note that we can perceive f as a symbol for the Berry quantisation which fulfils the requirements of Theorem 4.3.10 and 4.2.11. Hence, we get from Theorem 4.2.11

$$\left\|\widehat{f}^{\,\mathrm{B}}U^{\theta *}\psi\right\|_{\mathcal{H}_\tau}\leq C\left\|U^{\theta *}\psi\right\|_{\mathcal{H}_\tau}.$$

Then
$$\left\|\widehat{f}^{\,\mathrm{B}} U^{\theta *}\psi\right\|_{\mathcal{H}_\tau} = \left\|U^\theta \widehat{f}^{\,\mathrm{B}} U^{\theta *}\psi\right\|_{\mathcal{H}_\theta} = \left\|\widehat{f_\theta}^{\,\theta}\psi\right\|_{\mathcal{H}_\theta} = \left\|\widehat{f}^{\,\theta}\psi\right\|_{\mathcal{H}_\theta}$$
and
$$\left\|U^{\theta *}\psi\right\|_{\mathcal{H}_\tau} = \|\psi\|_{\mathcal{H}_\theta}$$
provide the claim. □

Remark 4.3.14. The theorem above includes symbols $f \in S^{m=0}_{\rho,\tau\equiv 1}(\mathbb{R}^4, \mathbb{C})$ with $\rho > 0$.

Note that the key point that made it possible to "translate" $U^\theta \widehat{f}^{\,\mathrm{B}} U^{\theta *}$ into $\widehat{f_\theta}^{\,\theta}$ was that the operator $\widehat{f}^{\,\mathrm{B}}$ is an operator on sections of the hermitian line bundle E_{Bl} with connection ∇^{Berry} (the Bloch bundle) and that the quantisation replaces r by the connection $-i\varepsilon\nabla^{\mathrm{Berry}}_k$. So from a geometrical point of view, we are looking at an operator which operates on sections of an in general non-trivial line bundle over the torus $\mathbb{T}^{2*}$ with connection ∇^{Berry} and curvature Ω. In [AOS94], the authors consider the geometrical construction for a system under the influence of a magnetic field with integral flux. The state space there consists of the L^2-sections of a hermitian line bundle with connection which has the magnetic field as curvature. The Hamiltonian for the system is just the Bochner Laplacian.
In the second chapter of the paper (Prop.4), they show how for a hermitian line bundle over the two-torus $\mathbb{T}^2$ with connection and curvature b (which they define by $b(X,Y)s := i(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})s$ for X, Y smooth vectorfields and s a smooth section of the line bundle) the Bochner Laplacian is unitarily equivalent to a self-adjoint realisation of $-(\partial_{k_1} - ia_{k_1})^2 - (\partial_{k_2} - ia_{k_2})^2$ in L^2(unit cell) with suitable boundary conditions. This is similar to what we were looking for when we translated $U^\theta \widehat{f}^{\,\mathrm{B}} U^{\theta *}$ into $\widehat{f_\theta}^{\,\theta}$, although we are interested in a much larger class of operators and not only in the translation of the Bochner-Laplacian $\widehat{r^2}^{\,\mathrm{Berry}}$ with $\varepsilon = 1$. But the idea is more or less the same. The difference in the definition of the unitary is that we just took, for lack of a global trivialisation, a function $\varphi : \mathbb{R}^2 \to E_{\mathrm{Bl}}$ for which $\varphi(k) \in P(k)\mathcal{H}_{\mathrm{f}}$ for every $k \in \mathbb{R}^2$ but $\varphi(k-\gamma^*) \neq \tau(\gamma^*)\varphi(k)$ while the authors there took lokal trivialisations and get their boundary conditions from the corresponding transition functions. This also reflects in the fact that we have chosen to transfer the torus $\mathbb{T}^{2*}$ to $\mathbb{R}^2$ using (quasi)-perodicities, while the authors of the paper do not do that. So the connection between these two works can be made in the following way:
Using the notation from [AOS94], in our case the line bundle is $E_{\mathrm{Bl}} \xrightarrow{\pi} \mathbb{T}^{2*}$ with connection ∇^{Berry} and curvature $i\Omega$. Possible trivialisations are (with φ the func-

tion from Lemma 3.3.2)

$$\begin{aligned}
\psi_1 &: V_1 \times \mathbb{C} \to \pi^{-1}(V_1) \text{ defined by } (k,\lambda) \mapsto \lambda\varphi(k), \\
\psi_2 &: V_2 \times \mathbb{C} \to \pi^{-1}(V_2) \text{ defined by } (k,\lambda) \mapsto \lambda e^{-\frac{i\theta}{2\pi}k_1k_2}\varphi(k), \\
\psi_3 &: V_3 \times \mathbb{C} \to \pi^{-1}(V_3) \text{ defined by } (k,\lambda) \mapsto \lambda\varphi(k), \quad \text{and} \\
\psi_4 &: V_4 \times \mathbb{C} \to \pi^{-1}(V_4) \text{ defined by } (k,\lambda) \mapsto \lambda\tfrac{\chi(k)}{|\chi(k)|}\varphi(k),
\end{aligned}$$

where $\chi(k) = (1 - \frac{k_1}{2\pi}) + \frac{k_1}{2\pi}e^{-i\theta k_2}$, which can have zeros only at $k_1 = \pi$. Then we get for the transition functions

$$c_{12}(k) = e^{-\frac{i\theta}{2\pi}k_1k_2}$$

and

$$c_{13}(k) \equiv 1$$

and for the connection forms we get

$$a_1(k) = i\mathcal{A}_1(k)\mathrm{d}k^1 + i\mathcal{A}_2(k)\mathrm{d}k^2 \tag{4.19}$$

on V_1 and

$$\begin{aligned}
a_2(k) &= i\langle e^{-\frac{i\theta}{2\pi}k_1k_2}\varphi(k), \partial_{k_1}(e^{-\frac{i\theta}{2\pi}k_1k_2}\varphi(k))\rangle_{\mathcal{H}_\mathrm{f}}\mathrm{d}k^1 \\
&\quad + i\langle e^{-\frac{i\theta}{2\pi}k_1k_2}\varphi(k), \partial_{k_2}(e^{-\frac{i\theta}{2\pi}k_1k_2}\varphi(k))\rangle_{\mathcal{H}_\mathrm{f}}\mathrm{d}k^2 \\
&= i\mathcal{A}_1(k)\mathrm{d}k^1 + i\mathcal{A}_2(k)\mathrm{d}k^2 + \tfrac{\theta}{2\pi}k_2\mathrm{d}k^1 + \tfrac{\theta}{2\pi}k_1\mathrm{d}k^2
\end{aligned}$$

on V_2. In [AOS94], Proposition 4(ii), this yields the boundary conditions

$$\psi(2\pi, k_2) = e^{-i\theta k_2}\psi(0, k_2) \quad \text{and} \quad \psi(k_1, 2\pi) = \psi(k_1, 0).$$

Following their arguments, for the connection form we get from $a_1 - a_2 = \mathrm{d}f_{12}$ with $f_{12}(k) = -\frac{\theta}{2\pi}k_1k_2$ that

$$\begin{aligned}
a_1(2\pi, k_2) &:= \mathrm{d}f_{12}(2\pi, k_2) + a_2(2\pi, k_2) = \mathrm{d}f_{12}(2\pi, k_2) + a_2(0, k_2) \\
&=: \mathrm{d}f_{12}(2\pi, k_2) + \mathrm{d}f_{21}(0, k_2) + a_1(0, k_2)
\end{aligned} \tag{4.20}$$

and

$$a_1(k_1, 2\pi) := a_3(k_1, 2\pi) = a_3(k_1, 0) =: a_1(k_1, 0). \tag{4.21}$$

This yields (inserting (4.19) in (4.20))

$$i\mathcal{A}_1(2\pi, k_2) = -\tfrac{\theta}{2\pi}k_2 + \tfrac{\theta}{2\pi}k_2 + i\mathcal{A}_1(0, k_2) = i\mathcal{A}_1(0, k_2)$$

$$i\mathcal{A}_2(2\pi, k_2) = -\theta + i\mathcal{A}_2(0, k_2)$$

and analogously from (4.21)

$$i\mathcal{A}_1(k_1, 2\pi) = i\mathcal{A}_1(k_1, 0)$$

$$i\mathcal{A}_2(k_1, 2\pi) = i\mathcal{A}_2(k_1, 0).$$

This all fits with the fact that the reference space is isomorphic to

$$\mathcal{H}_\theta \cong \{\psi \in L^2(M^*) : \psi(2\pi, k_2) = e^{-i\theta k_2}\psi(0, k_2) \ \text{ and } \ \psi(k_1, 2\pi) = \psi(k_1, 0)\}$$

endowed with the connection

$$\widetilde{\nabla}_k^\theta = \nabla_k + \mathcal{A},$$

where

$$\mathcal{A}_1(2\pi, k_2) = \mathcal{A}_1(0, k_2)$$
$$\mathcal{A}_2(2\pi, k_2) = i\theta + \mathcal{A}_2(0, k_2)$$

and

$$\mathcal{A}_1(k_1, 2\pi) = \mathcal{A}_1(k_1, 0)$$
$$\mathcal{A}_2(k_1, 2\pi) = \mathcal{A}_2(k_1, 0).$$

So in our construction, the connection forms $\mathcal{A}_j(k)$ are directly defined on M^*, while in [AOS94] the boundary conditions again follow from the transition functions. Note that $\Pi^0\mathcal{H}_\tau$ is isomorphic to the space of the L^2-sections of the line bundle E_{Bl} as well as $\mathcal{H}_\theta$ is isomorphic to the space of the L^2-sections of the line bundle E_θ.

In Chapter 3 of the paper, the authors do a magnetic Bloch-Floquet transformation on the Bochner Laplacian H of the trivial line bundle $\mathbb{R}^2 \times \mathbb{C}$ with curvature $b = B(x)\mathrm{d}x^1 \wedge \mathrm{d}x^2$, where B is a Γ-periodic function with constant $B_{\mathrm{c}} \in 2\pi\mathbb{Z}$. So this is similar to our case, where we do the same with the original unperturbed Hamiltonian H_{MB}, because we could perceive B as curvature of a connection ∇^{MB} on the trivial line bundle $\mathbb{R}^2 \times \mathbb{C}$ with vector potential $a = \frac{B}{2}(-x_2, x_1)$ and $H_{\mathrm{MB}} - V_\Gamma(x)$ as the corresponding Bochner Laplacian. But they use a different Bloch-Floquet transformation than we. Roughly speaking, their transformation is $(\mathcal{F}_{\mathrm{magn}} \otimes 1)$, while ours is $e^{-iky}(\mathcal{F}_{\mathrm{magn}} \otimes 1)$. As already mentioned, this is the reason that in the paper, the obtained operators $H(k)$ have a domain which is dependent of k, while in our case the domain of the operators stays independent of k. Therefore in our case, the k-dependence of $H(k)$ reflects in the representation of $H_{\mathrm{per}}(k)$, while in the paper the representation of $H(k)$ is independent of k. So the characterisation in the paper of the operator $\int_{M^*}^{\oplus} H(k)\mathrm{d}k$ is not useful for our

problem.
Moreover, there is another difference. The authors of [AOS94] transform the Bochner Laplacian H via Bloch-Floquet transformation to get a decomposition

$$UHU^* = \int_{\mathbb{R}^2/\Gamma}^{\oplus} H(k)\mathrm{d}k.$$

Then they show that for every k, the operator $H(k)$ is a self-adjoint realisation of $-(\partial_{k_1} - ia_{k_1})^2 - (\partial_{k_2} - ia_{k_2})^2$ in L^2(unit cell) with suitable boundary conditions. So using Proposition 4 from [AOS94], they conclude that there must be a hermitian line bundle with connection so that the corresponding Bochner Laplacian of this bundle is unitarily equivalent to $H(k)$. They then use this knowledge to prove their result that the direct integral of the Bochner Laplacians over all non-equivalent hermitian line bundles with connection over the torus with curvature b is unitarily equivalent to the unique Bochner Laplacian on the hermitian line bundle with curvature b on its universal cover. We, in contrast, use the Bloch-Floquet transformation to get the operator H^ε as a quantised symbol. Then we derive the effective model H_{eff} acting on sections of E_{Bl} endowed with the Berry connection. But then we use the characterisation given in Proposition 4 of [AOS94] to get from this operator to an operator acting on $L^2(M^*)$ with suitable boundary conditions. So we use this Proposition the other way round and, moreover, apply it to the "full" operator $H_{\text{eff}} = \widehat{h_{\text{eff}}}^{\text{eff}}$.

4.4 The effective quantisation

The last quantisation we want to introduce is the effective quantisation. It is the quantisation we want to use for the effective Hamiltonian we want to compute. Roughly speaking, this quantisation maps $r \mapsto -i\varepsilon\nabla_k^{\text{eff}} = -i\varepsilon(\nabla_k + (0, \frac{i\theta}{2\pi}k_1)^{\mathrm{T}})$, which is also a connection on the line bundle E_θ. Moreover, the connection form $\mathcal{A}^{\text{eff}} = (0, \frac{i\theta}{2\pi}k_1)^{\mathrm{T}}$ is the connection form that would be achieved in the construction in the proof of Lemma 3.3.2 if the curvature form was constant and thus $\Omega(k) = \frac{i\theta}{2\pi}$. The advantage over the θ-quantisation is that we can write this connection independent from φ and π_0. Moreover, with this connection it is easy to compare the case without the strong magnetic field A_0 with our general case $A_0 \neq 0$. In the case $A_0 \equiv 0$, the Bloch bundle is trivial and therefore $\theta = 0$ since it is the Chern number of this line bundle. Thus, our result includes the case $A_0 \equiv 0$ and it should be simple to read off the according result, for example when we compute the symbols of the effective Hamiltonian. It also holds true that $\mathcal{H}_{\theta=0} \cong L^2(\mathbb{T}^{2*})$.

Definition 4.4.1. *Let $f \in S^w(\mathbb{R}^4, \mathbb{C}) \cup S^m_\rho(\mathbb{R}^4, \mathbb{C})$ and χ a smooth cutoff function.*

Then for $\psi \in S(\mathbb{R}^2, \mathbb{C})$ we define the effective quantisation by

$$\widehat{f}^{\,\mathrm{eff},\chi}\psi(k) := \tfrac{1}{(2\pi\varepsilon)^2} \int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}} \chi(k-y) t^{\mathrm{eff}}(k,y)\, f\left(\tfrac{k+y}{2}, r\right) \psi(y)\mathrm{d}r\mathrm{d}y,$$

where $t^{\mathrm{eff}}(x,y)$ is the parallel transport along the straight line from y to x with respect to the connection $\nabla^{\mathrm{eff}}_k = \nabla_k + (0, \frac{i\theta}{2\pi}k_1)^{\mathrm{T}}$.

Remark 4.4.2. We will see that for suitable symbols the effective quantisation does not depend on the cutoff up to an error of $\mathcal{O}(\varepsilon^\infty)$.

To show the well-definedness of this quantisation, we again follow the usual routine as we did with the θ-quantisation: First, we show that the quantised symbol is a continuous map from the Schwartz space $S(\mathbb{R}^2)$ to itself and extend this mapping by duality to $S'(\mathbb{R}^2)$. Then it can be shown that for Γ^*-periodic symbols the quantisation leaves S'_θ invariant. The proofs are very similar to the ones for the θ-quantisation, so we do not give details.

Proposition 4.4.3. *For $f \in S^w(\mathbb{R}^4, \mathbb{C}) \cup S^m_\rho(\mathbb{R}^4, \mathbb{C})$, χ a smooth cutoff function, and $\psi \in S(\mathbb{R}^2)$, the integral*

$$(\widehat{f}^{\,\mathrm{eff},\chi}\psi)(k) = \tfrac{1}{(2\pi\varepsilon)^2} \int_{\mathbb{R}^4} e^{\frac{i(k-y)r}{\varepsilon}} \chi(k-y) t^{\mathrm{eff}}(k,y)\, f\left(\tfrac{k+y}{2}, r\right) \psi(y)\mathrm{d}r\mathrm{d}y$$

defines a continuous mapping from $S(\mathbb{R}^2)$ to $S(\mathbb{R}^2)$.

Proof.
The proof works exactly as the proof of Proposition 4.3.4. □

Again, this mapping can be extended to a map

$$\widehat{f}^{\,\mathrm{eff},\chi} : S'(\mathbb{R}^2) \to S'(\mathbb{R}^2)$$

by putting

$$\widehat{f}^{\,\mathrm{eff},\chi}(T)(\psi) := T(\widehat{\overline{f}}^{\,\mathrm{eff},\chi}\psi)$$

for $\psi \in S(\mathbb{R}^2)$.

Next, we show that the effective quantisation of symbols which are Γ^*-periodic in k maps S'_θ to S'_θ.

Proposition 4.4.4. *For $f \in S^w_{\tau\equiv 1}(\mathbb{R}^4, \mathbb{C}) \cup S^m_{\rho,\tau\equiv 1}(\mathbb{R}^4, \mathbb{C})$ we have*

$$\widehat{f}^{\,\mathrm{eff},\chi} S'_\theta \subset S'_\theta.$$

Proof.
Since

$$t^{\mathrm{eff}}(k-\gamma^*, y-\gamma^*) = e^{\frac{i\theta}{2\pi}\gamma_1^*(k_2-y_2)} t^{\mathrm{eff}}(k,y)$$

holds also for the connection ∇^{eff}, the proof works exactly as the one of Proposition 4.3.6. □

The symbols for which the effective quantisation does not depend on the cutoff up to a "small" error are again those with an improving behaviour for the derivatives with respect to r:

Proposition 4.4.5. *Let $f \in S^w_{\tau\equiv 1}(\mathbb{R}^4, \mathbb{C})$ fulfil*

(∗) *$\exists \alpha_0 \in \mathbb{N}$ so that for $|\alpha| \geq \alpha_0$ it holds $\partial_r^\alpha f(k,r) \in L^1(\mathbb{R}^2_r) \cap L^2(\mathbb{R}^2_r)$ for all $k \in \mathbb{R}^2$ and $\|\partial_r^\alpha f(k,r)\|_{L^1(\mathbb{R}^2_r)} \leq h_\alpha(k)$ with $h_\alpha \in C(\mathbb{R}^2, \mathbb{R}_{\geq 0})$.*

Then the effective quantisation of this symbol does not depend on the cutoff χ up to an error of $\mathcal{O}(\varepsilon^\infty)$, which means that for two smooth cutoff functions χ and $\widetilde{\chi}$ it holds

$$\widehat{f}^{\,\mathrm{eff},\chi} = \widehat{f}^{\,\mathrm{eff},\widetilde{\chi}} + \mathcal{O}(\varepsilon^\infty).$$

Proof.
The proof works exactly as the proof of Proposition 4.3.7. □

Remark 4.4.6. Note that Proposition 4.4.5 includes symbols in $S^m_{\rho,\tau\equiv 1}(\mathbb{R}^4, \mathbb{C})$ with $\rho > 0$.

Now we again show that for symbols whose quantisation is in $\mathcal{L}(\mathcal{H}_\theta)$, the adjoint of this operator is given through the quantisation of the pointwise adjoint of the symbol.

Proposition 4.4.7. *Let $f \in S^w_{\tau\equiv 1}(\mathbb{R}^4, \mathbb{C}) \cup S^m_{\rho,\tau\equiv 1}(\mathbb{R}^4, \mathbb{C})$ with $\widehat{f}^{\,\mathrm{eff}} \in \mathcal{L}(\mathcal{H}_\theta)$. Then*

$$(\widehat{f}^{\,\mathrm{eff}})^* = \widehat{\overline{f}}^{\,\mathrm{eff}}.$$

Proof.
The proof works exactly as the proof of Proposition 4.3.9. □

The quantisation we have just introduced is, finally, the one for our effective Hamiltonian. Thus, we need to show how we can compute corrections f_{c} of a symbol f so that it holds

$$\widehat{f_{\mathrm{c}}}^{\,\mathrm{eff}} = \widehat{f}^{\,\theta} + \mathcal{O}(\varepsilon^\infty).$$

The method is the same as in Theorem 4.2.16 about the τ- and the Berry quantisation. Therefore, again we will keep the proofs short.

Theorem 4.4.8. *Let $f \in S^1_{\tau\equiv 1}(\mathbb{R}^4, \mathbb{C})$ fulfil*

$(*)$ $\exists \alpha_0 \in \mathbb{N}$ *so that for* $|\alpha| \geq \alpha_0$ *it holds* $\partial_r^\alpha f(k,r) \in L^1(\mathbb{R}^2_r) \cap L^2(\mathbb{R}^2_r)$ *for all* $k \in \mathbb{R}^2$ *and* $\|\partial_r^\alpha f(k,r)\|_{L^1(\mathbb{R}^2_r)} \leq h_\alpha(k)$ *with* $h_\alpha \in C(\mathbb{R}^2, \mathbb{R}_{\geq 0})$,

or let $f \in S^m_{\rho,\tau}(\mathbb{R}^4, \mathbb{C})$ *with* $\rho > 0$.

(i) For every $N \in \mathbb{N}_0$ *there is a correction* $f_{\mathrm{c}}^{(N)}$ *in the same symbol class as* f *so that*

$$\widehat{f}^{\,\mathrm{eff}} = \widehat{f_{\mathrm{c}}^{(N)}}^{\,\theta} + \mathcal{O}(\varepsilon^{N+1}).$$

For $N \in \mathbb{N}_0$ *the correction* $f_{\mathrm{c}}^{(N)}$ *of the symbol* f *is given by*

$$f_{\mathrm{c}}^{(N)} = \sum_{n=0}^{N} \varepsilon^n f_{\mathrm{c}n}(k,r),$$

where

$$f_{\mathrm{c}n}(k,r) = \sum_{\alpha \in \mathbb{N}_0^4,\ |\alpha|=n} \left(\tfrac{i}{2}\right)^n (-1)^{\alpha_3+\alpha_4} \tfrac{1}{\alpha!} \partial^\alpha \overline{t(k,k)} \partial_{r_1}^{\alpha_1+\alpha_3} \partial_{r_2}^{\alpha_2+\alpha_4} f(k,r)$$

with

$$t(k,y) = t^{\mathrm{eff}}(y,k) t^\theta(k,y).$$

For $f \in S^1_{\tau\equiv 1}(\mathbb{R}^4, \mathbb{C})$, $f_{\mathrm{c}}^{(N)}$ *fulfils* $(*)$.
For $f \in S^m_{\rho,\tau}(\mathbb{R}^4, \mathbb{C})$, *it even holds* $f_{\mathrm{c}n} \in S^{m-n\rho}_{\rho,\tau}(\mathbb{R}^4, \mathbb{C})$.
In particular, we have

$$f_{\mathrm{c}0}(k,r) = f(k,r)$$

and

$$\begin{aligned} f_{\mathrm{c}1}(k,r) &= i\left(\mathcal{A}_1(k)\partial_{r_1} f(k,r) + (\mathcal{A}_2(k) - \tfrac{i\theta}{2\pi}k_1)\partial_{r_2} f(k,r)\right) \\ &= iD(k) \cdot \nabla_r f(k,r), \end{aligned}$$

where $D(k) = \mathcal{A}(k) - (0, \frac{i\theta}{2\pi}k_1)^{\mathrm{T}}$.

(ii) For every $N \in \mathbb{N}_0$ *there is a correction* $f_{\mathrm{c}}^{(N)}$ *in the same symbol class as* f *so that*

$$\widehat{f}^{\,\theta} = \widehat{f_{\mathrm{c}}^{(N)}}^{\,\mathrm{eff}} + \mathcal{O}(\varepsilon^{N+1}).$$

The corrected symbol can be computed by

$$f_{\mathrm{c}}^{(N)} = \sum_{n=0}^{N} \varepsilon^n f_{\mathrm{c}n},$$

where

$$f_{\mathrm{c}n}(k,r) = \sum_{\alpha\in\mathbb{N}_0^4,\ |\alpha|=n} \left(\tfrac{i}{2}\right)^n (-1)^{\alpha_3+\alpha_4}\tfrac{1}{\alpha!}\partial^\alpha t(k,k)\partial_{r_1}^{\alpha_1+\alpha_3}\partial_{r_2}^{\alpha_2+\alpha_4} f(k,r)$$

with

$$t(k,y) = t^{\mathrm{eff}}(y,k)t^\theta(k,y).$$

For $f \in S^1_{\tau\equiv 1}(\mathbb{R}^4,\mathbb{C})$, $f_{\mathrm{c}}^{(N)}$ *fulfils* $(*)$.
For $f \in S^m_{\rho,\tau}(\mathbb{R}^4,\mathbb{C})$, *it even holds* $f_{\mathrm{c}n} \in S^{m-n\rho}_{\rho,\tau}(\mathbb{R}^4,\mathbb{C})$.
In particular, we have

$$f_{\mathrm{c}0}(k,r) = f(k,r)$$

and

$$\begin{aligned} f_{\mathrm{c}1}(k,r) &= -i\left(\mathcal{A}_1(k)\partial_{r_1} f(k,r) + (\mathcal{A}_2(k) - \tfrac{i\theta}{2\pi}k_1)\partial_{r_2} f(k,r)\right) \\ &= -iD(k)\cdot\nabla_r f(k,r), \end{aligned}$$

where $D(k) = \mathcal{A}(k) - (0, \frac{i\theta}{2\pi}k_1)^{\mathrm{T}}$.

Proof.
The idea for the above theorem is the same as in Theorem 4.2.16 and hence we follow the line of this proof. Moreover, the case here is easier because here the parallel transport is only multiplication with a phase. More precisely, we have

$$t^\theta(k,y) = e^{-\int_0^1 \mathcal{A}_1(y+t(k-y))\mathrm{d}t(k_1-y_1) - \int_0^1 \mathcal{A}_2(y+t(k-y))\mathrm{d}t(k_2-y_2)}$$

and

$$t^{\mathrm{eff}}(k,y) = e^{\frac{i\theta}{4\pi}(y_1+k_1)(y_2-k_2)}.$$

The idea is again to insert the identity $t^{\mathrm{eff}}(k,y)t^{\mathrm{eff}}(y,k)$ and use the Taylor expansion of $t(k,y) = t^{\mathrm{eff}}(y,k)t^\theta(k,y)$. Let us quickly comment on the changes in each of the parts of the proof of Theorem 4.2.16. In part one, note that for arbitrary $N \in \mathbb{N}$ the periodicity of the corrected symbol $f_{\mathrm{c}}^{(N)}$ with respect to Γ^* follows from the periodicity of $t(k,y)$ and f. Simple calculations show that $f_{\mathrm{c}}^{(N)} \in S^1_{\tau\equiv 1}(\mathbb{R}^4,\mathbb{C})$ and that it fulfils $(*)$.
In part two, the Taylor expansion gets simpler: With $\delta = \frac{k-y}{2}$, the n^{th} order Taylor polynomial of $t : \mathbb{R}^4 \to \mathbb{C}$ is

$$t(z+\delta, z-\delta) = \sum_{\alpha\in\mathbb{N}_0^4,\ |\alpha|=0}^{n} \tfrac{1}{\alpha!}\partial^\alpha t(k,k)\delta_1^{\alpha_1+\alpha_3}\delta_2^{\alpha_2+\alpha_4}(-1)^{\alpha_3+\alpha_4}.$$

Partial integration turns δ_j in $\frac{i\varepsilon}{2}\partial_{r_j}$ for $j \in \{1,2\}$, which leads to the given formula in the theorem.

For the estimation of the remainder term in part three we can use Theorem 4.3.13 and prove (i). Thus in part 4, a Calderon-Vaillancourt theorem for the effective quantisation has to be shown using (i). The rest is clear. □

The theorem can be generalised to semiclassical symbols.

Corollary 4.4.9. *Let $f \in S^1_{\tau\equiv 1}(\varepsilon, \mathbb{C})$ with $f \asymp \sum_{n\geq 0} \varepsilon^n f_n$ fulfil*

(∗) *for every $n \in \mathbb{N}_0$ there is $\alpha_n \in \mathbb{N}$ so that for $|\alpha| \geq \alpha_n$ it holds $\partial_r^\alpha f_n(k,r) \in L^1(\mathbb{R}^2_r) \cap L^2(\mathbb{R}^2_r)$ for all $k \in \mathbb{R}^2$ and $\|\partial_r^\alpha f_n(k,r)\|_{L^1(\mathbb{R}^2_r)} \leq h_{n,\alpha}(k)$ with $h_{n,\alpha} \in C(\mathbb{R}^2, \mathbb{R}_{\geq 0})$,*

or let $f \in S^m_{\rho,\tau\equiv 1}(\varepsilon, \mathbb{C})$ with $\rho > 0$.

(i) There exists a semiclassical symbol f_{c} in the same symbol class as f so that

$$\widehat{f}^{\,\mathrm{eff}} = \widehat{f_{\mathrm{c}}}^{\,\theta} + \mathcal{O}(\varepsilon^\infty).$$

It holds

$$f_{\mathrm{c}} \asymp \sum_{n\geq 0} \varepsilon^n g_n,$$

where

$$g_n = \sum_{j=0}^{n} (f_j)_{c(n-j)},$$

where we used the notation from Theorem 4.4.8, which means that $(f_j)_{c(n-j)}$ is the $(n-j)$th correction of the ordinary symbol f_j according to Theorem 4.4.8(i).
For $f \in S^1_{\tau\equiv 1}(\varepsilon, \mathbb{C})$, the correction f_{c} fulfils (∗).
The principal and subprincipal symbol of f_{c} read

$$(f_{\mathrm{c}})_0(k,r) = f_0(k,r)$$

and

$$(f_{\mathrm{c}})_1(k,r) = f_1(k,r) + i\left(\mathcal{A}_1(k)\partial_{r_1} f_0(k,r) + (\mathcal{A}_2(k) - \tfrac{i\theta}{2\pi}k_1)\partial_{r_2} f_0(k,r)\right).$$

(ii) There exists a semiclassical symbol $f_{\mathrm{c}} \in S^1_{\tau\equiv 1}(\varepsilon, \mathbb{C})$ respectively $S^m_{\rho,\tau\equiv 1}(\varepsilon, \mathbb{C})$ which has the same properties as f so that

$$\widehat{f}^{\,\theta} = \widehat{f_{\mathrm{c}}}^{\,\mathrm{eff}} + \mathcal{O}(\varepsilon^\infty).$$

It holds

$$f_{\mathrm{c}} \asymp \sum_{n\geq 0} \varepsilon^n g_n,$$

where

$$g_n = \sum_{j=0}^{n} (f_j)_{c(n-j)},$$

where we used the notation from Theorem 4.4.8, which means that $(f_j)_{c(n-j)}$ *is the* $(n-j)$*th correction of the ordinary symbol* f_j *according to Theorem 4.4.8(ii).*
For $f \in S^1_{\tau\equiv 1}(\varepsilon, \mathbb{C})$*, the correction* f_{c} *fulfils* $(*)$*.*
The principal and subprincipal symbol of f_{c} *read*

$$(f_{\mathrm{c}})_0(k,r) = f_0(k,r)$$

and

$$(f_{\mathrm{c}})_1(k,r) = f_1(k,r) - i\left(\mathcal{A}_1(k)\partial_{r_1} f_0(k,r) + (\mathcal{A}_2(k) - \tfrac{i\theta}{2\pi}k_1)\partial_{r_2} f_0(k,r)\right).$$

Proposition 4.4.10. *Let* f *satisfy the assumptions of Theorem 4.4.8 and let* $f(k,r) = \overline{f(k,r)}$*. Then* $f_{\mathrm{c}}(k,r) = \overline{f_{\mathrm{c}}(k,r)}$ *holds.*

Proof.
Simple calculations show for $j \in \{1,2\}$

$$\partial_{k_j} t^\theta(k,y)|_{k=y} = -\partial_{y_j} t^\theta(k,y)|_{k=y} \quad \text{and} \quad \partial_{k_j} t^{\mathrm{eff}}(k,y)|_{k=y} = -\partial_{y_j} t^{\mathrm{eff}}(k,y)|_{k=y}.$$

Now for $\alpha = (\alpha_1, ...\alpha_4) \in \mathbb{N}_0^4$, let $\widetilde{\alpha}_j := \alpha_{j+2}$ if $j \in \{1,2\}$ and $\widetilde{\alpha}_j := \alpha_{j-2}$ if $j \in \{3,4\}$ respectively $\widetilde{\alpha} := (\alpha_3, \alpha_4, \alpha_1, \alpha_2)$. Then

$$\begin{aligned}
\overline{\partial^\alpha t(k,k)} &= \overline{\partial^\alpha (t^{\mathrm{eff}}(x_3,x_4,x_1,x_2) t^\theta(x_1,x_2,x_3,x_4))}|_{(x_1,x_2,x_3,x_4)=(k,k)} \\
&= \partial^\alpha (t^{\mathrm{eff}}(x_1,x_2,x_3,x_4) t^\theta(x_3,x_4,x_1,x_2))|_{(x_1,x_2,x_3,x_4)=(k,k)} \\
&= \partial^{\widetilde{\alpha}} (t^{\mathrm{eff}}(x_3,x_4,x_1,x_2) t^\theta(x_1,x_2,x_3,x_4))|_{(x_1,x_2,x_3,x_4)=(k,k)} \\
&= \sum_{\beta \le \widetilde{\alpha}} \binom{\widetilde{\alpha}}{\beta} \partial^{\widetilde{\alpha}-\beta} t^{\mathrm{eff}}(x_3,x_4,x_1,x_2) \partial^\beta t^\theta(x_1,x_2,x_3,x_4)|_{(x_1,x_2,x_3,x_4)=(k,k)} \\
&= \sum_{\beta \le \widetilde{\alpha}} \binom{\alpha}{\widetilde{\beta}} (-1)^{|\alpha|} \partial^{\alpha-\widetilde{\beta}} t^{\mathrm{eff}}(x_3,x_4,x_1,x_2) \partial^{\widetilde{\beta}} t^\theta(x_1,x_2,x_3,x_4)|_{(x_1,x_2,x_3,x_4)=(k,k)} \\
&= (-1)^{|\alpha|} \partial^\alpha (t^{\mathrm{eff}}(x_3,x_4,x_1,x_2) t^\theta(x_1,x_2,x_3,x_4))|_{(x_1,x_2,x_3,x_4)=(k,k)} \\
&= (-1)^{|\alpha|} \partial^\alpha t(k,k)
\end{aligned}$$

and hence

$$
\begin{aligned}
\overline{f_{\mathrm{c}}^{(N)}(k,r)} &= \sum_{\alpha\in\mathbb{N}_0^4,\ |\alpha|=0}^{N} (-1)^{|\alpha|}\left(\tfrac{i\varepsilon}{2}\right)^{|\alpha|}(-1)^{\alpha_3+\alpha_4}\tfrac{1}{\alpha!}\partial^\alpha\overline{t(k,k)}\partial_{r_1}^{\alpha_1+\alpha_3}\partial_{r_2}^{\alpha_2+\alpha_4}f(k,r)\\
&= \sum_{\alpha\in\mathbb{N}_0^4,\ |\alpha|=0}^{N} \left(\tfrac{i\varepsilon}{2}\right)^{|\alpha|}(-1)^{\alpha_3+\alpha_4}\tfrac{1}{\alpha!}\partial^\alpha t(k,k)\partial_{r_1}^{\alpha_1+\alpha_3}\partial_{r_2}^{\alpha_2+\alpha_4}f(k,r)\\
&= f_{\mathrm{c}}^{(N)}(k,r).
\end{aligned}
$$

□

There is also a Calderon-Vaillancourt theorem for the effective quantisation.

Theorem 4.4.11. *Let $f\in S^1_{\tau\equiv 1}(\mathbb{R}^4,\mathbb{C})$ fulfil*

(∗) *$\exists\alpha_0\in\mathbb{N}$ so that for $|\alpha|\geq\alpha_0$ it holds $\partial_r^\alpha f(k,r)\in L^1(\mathbb{R}_r^2)\cap L^2(\mathbb{R}_r^2)$ for all $k\in\mathbb{R}^2$ and $\|\partial_r^\alpha f(k,r)\|_{L^1(\mathbb{R}_r^2)}\leq h_\alpha(k)$ with $h_\alpha\in C(\mathbb{R}^2,\mathbb{R}_{\geq 0})$.*

Then

$$\widehat{f}^{\,\mathrm{eff}}\in\mathcal{L}(\mathcal{H}_\theta).$$

Remark 4.4.12. The above theorem includes symbols $f\in S_\rho^{m=0}(\mathbb{R}^4,\mathbb{C})$ with $\rho>0$.

4.5 The corresponding results for an arbitrary Bravais lattice Γ

In this section we again comment shortly on the changes that have to be made in case of an arbitrarily chosen Bravais lattice Γ generated by $\{\gamma^1,\gamma^2\}$. We again denote the components of the generating vectors by $\gamma^1=(\gamma_1^1,\gamma_2^1)$ and $\gamma^2=(\gamma_1^2,\gamma_2^2)$ and analogously for the dual lattice $\gamma^{1*}=(\gamma_1^{1*},\gamma_2^{1*})$ and $\gamma^{2*}=(\gamma_1^{2*},\gamma_2^{2*})$, as we did in Section 3.4. There are no modifications necessary for the Berry quantisation and corresponding theorems. This is because the Bloch bundle only depends on the projections $P(k)$ and not on the particular choice of the function φ from Lemma 3.3.2 respectively 3.4.1.
For the θ-quantisation the situation changes. The bundle in question is now the bundle $E_\theta=\{(k,\lambda)\in(\mathbb{R}^2,\mathbb{C})_\sim\}$, where the equivalence relation depends on the phase of φ. This means that for our choice of φ with

$$\varphi(k-\gamma^*)=e^{-\frac{i\theta}{2\pi}\langle\gamma^2,k\rangle\langle\gamma^1,\gamma^*\rangle}\tau(\gamma^*)\varphi(k)\quad\text{for all}\quad\gamma^*\in\Gamma^*,$$

the equivalence relation is

$$(k,\lambda)\sim(k',\lambda') \quad \text{iff} \quad k'=k-\gamma^* \quad \text{and} \quad \lambda'=e^{\frac{i\theta}{2\pi}\langle\gamma^2,k\rangle\langle\gamma^1,\gamma^*\rangle}\lambda.$$

Therefore, also Definition 4.3.5 has to be adapted: Now

$$V_{\gamma^*}\psi(k):=e^{\frac{i\theta}{2\pi}\langle\gamma^2,k\rangle\langle\gamma^1,\gamma^*\rangle}\psi(k).$$

The conformance of the bundle E_θ to the phase of φ also induces that the property (4.17) of the parallel transport with respect to the θ-connection is now

$$t^\theta(k-\gamma^*,y-\gamma^*)=e^{\frac{i\theta}{2\pi}\langle\gamma^2,k-y\rangle\langle\gamma^1,\gamma^*\rangle}t^\theta(k,y). \tag{4.22}$$

This has to be considered in the proof of Proposition 4.3.6 and in the proof of Proposition 4.3.9, where the property (4.22) implies that the property (4.18) of the distributional integral kernel K_f now is

$$K_f(k-\gamma^*,y-\gamma^*)=e^{\frac{i\theta}{2\pi}\langle\gamma^2,k-y\rangle\langle\gamma^1,\gamma^*\rangle}K_f(k,y). \tag{4.23}$$

As the θ-quantisation, also the effective quantisation depends on the bundle E_θ and thus we again get the corresponding modifications. The connection form is

$$\mathcal{A}(k)=\tfrac{i\theta}{2\pi}\langle\gamma^1,k\rangle\gamma^2.$$

Hence the parallel transport is

$$t^{\mathrm{eff}}(k,y)=e^{\frac{i\theta}{4\pi}\langle\gamma^1,k+y\rangle\langle\gamma^2,y-k\rangle}.$$

Of course, the properties (4.22) and (4.23) hold for t^{eff}. Finally, we have to take care of the modified connection ∇_k^{eff} in the theorems for the corrections of symbols, that is to say Theorem 4.4.8 and Corollary 4.4.9. In Theorem 4.4.8(i), we now get

$$\begin{aligned} f_{\mathrm{c}1}(k,r) &= i((\mathcal{A}_1(k)-\tfrac{i\theta}{2\pi}\langle\gamma^1,k\rangle\gamma_1^2)\partial_{r_1}f(k,r)+(\mathcal{A}_2(k)-\tfrac{i\theta}{2\pi}\langle\gamma^1,k\rangle\gamma_2^2)\partial_{r_2}f(k,r)) \\ &= iD(k)\cdot\nabla_r f(k,r) \end{aligned}$$

where $D(k)=\mathcal{A}(k)-\frac{i\theta}{2\pi}\langle\gamma^1,k\rangle\gamma^2$. Analogous modifications have to be done in Theorem 4.4.8(ii) and Corollary 4.4.9.

Chapter 5

The effective dynamics

5.1 The effective Hamiltonian as the quantisation of a semiclassical symbol

Now we use our new pseudodifferential calculi and their properties and relations among each other to, finally, define our effective Hamiltonian as the quantisation of a semiclassical symbol and thus as an operator acting on $\mathcal{H}_\theta$. Before we start the rigorous maths, we give an informal overview.
The question is how to write the effective Hamiltonian H_{eff} as a pseudodifferential operator

$$H_{\text{eff}} = U^\varepsilon \Pi^\varepsilon \widehat{H}^\tau \Pi^\varepsilon U^{\varepsilon *} = U^\theta \Pi^0 \widehat{h}^\tau \Pi^0 U^{\theta *} + \mathcal{O}(\varepsilon^\infty) = \widehat{h_{\text{eff}}}^? + \mathcal{O}(\varepsilon^\infty).$$

The idea we are going to follow is

$$U^\theta \Pi^0 \widehat{h}^\tau \Pi^0 U^{\theta *} = U^\theta \Pi^0 \widehat{h_{\text{c}}}^{\text{B}} \Pi^0 U^{\theta *} + \mathcal{O}(\varepsilon^\infty) = \widehat{(h_{\text{c}})_\theta}^\theta + \mathcal{O}(\varepsilon^\infty) = \widehat{h_{\text{eff}}}^{\text{eff}} + \mathcal{O}(\varepsilon^\infty),$$

where the quantisations are

- $\mathrm{Op}^\tau : k \mapsto k, r \mapsto -i\varepsilon \nabla_k^\tau$ and $\widehat{h}^\tau \in \mathcal{L}(\mathcal{H}_\tau)$
- $\mathrm{Op}^{\text{B}} : k \mapsto k, r \mapsto -i\varepsilon \nabla_k^{\text{Berry}}$ and $\widehat{h}^{\text{B}} \in \mathcal{L}(\mathcal{H}_\tau)$
- $\mathrm{Op}^\theta : k \mapsto k, r \mapsto -i\varepsilon \nabla_k^\theta$ and $\widehat{h}^\theta \in \mathcal{L}(\mathcal{H}_\theta)$
- $\mathrm{Op}^{\text{eff}} : k \mapsto k, r \mapsto -i\varepsilon(\nabla_k + (0, \frac{i\theta}{2\pi} k_1)^{\text{T}})$ and $\widehat{h}^{\text{eff}} \in \mathcal{L}(\mathcal{H}_\theta)$.

So let us make the described procedure rigorous.

Theorem 5.1.1. *Let* $H_0(k,r) = \frac{1}{2}(-i\nabla_y + k - A_0(y) - A(r))^2 + V_\Gamma(y) + \Phi(r) \in S^{w=1+k^2}_\tau(\mathbb{R}^4, \mathcal{L}(\mathcal{H}^2_{A_0}, \mathcal{H}_\mathrm{f}))$*, let* E *be an isolated band according to Definition 2.4.1, and let Assumption 1 hold. Moreover, let the semiclassical symbol* $h \asymp \sum_{j\geq 0} \varepsilon^j h_j$ *from Theorem 3.3.6 fulfil*

(∗) *for all* $j \geq 1$ *there is* $\alpha_j \in \mathbb{N}$ *so that for all* $|\alpha| \geq \alpha_j$ *we have* $\partial^\alpha_r h_j(k,r) \in L^1(\mathbb{R}^2_r, \mathcal{H}_\mathrm{f}) \cap L^2(\mathbb{R}^2_r, \mathcal{H}_\mathrm{f})$ *for all* $k \in \mathbb{R}^2$ *and* $\|\partial^\alpha_r h_j(k,r)\|_{L^1(\mathbb{R}^2_r, \mathcal{L}(\mathcal{H}_\mathrm{f}))} \leq g_{j,\alpha}(k)$ *with* $g_{j,\alpha} \in C(\mathbb{R}^2)$*.*

Then there exist

(i) an orthogonal projection $\Pi^\varepsilon \in \mathcal{L}(\mathcal{H}_\tau)$*,*

(ii) a unitary map $U^\varepsilon \in \mathcal{L}(\Pi^\varepsilon \mathcal{H}_\tau, \mathcal{H}_\theta)$*, and*

(iii) a self-adjoint operator $\widehat{h_\mathrm{eff}}^{\mathrm{eff}} \in \mathcal{L}(\mathcal{H}_\theta)$

such that

$$\|[H^\varepsilon_\mathrm{BF}, \Pi^\varepsilon]\|_{\mathcal{L}(\mathcal{H}_\tau)} = \mathcal{O}(\varepsilon^\infty)$$

and

$$\left\|(e^{-iH^\varepsilon_\mathrm{BF}t} - U^{\varepsilon *} e^{-i\widehat{h_\mathrm{eff}}^{\mathrm{eff}}t} U^\varepsilon)\Pi^\varepsilon\right\|_{\mathcal{L}(\mathcal{H}_\tau)} = \mathcal{O}(\varepsilon^\infty(1+|t|)).$$

The effective Hamiltonian is the effective quantisation of the symbol $h_\mathrm{eff} \in S^1_{\tau\equiv 1}(\varepsilon, \mathbb{C})$ *which can be computed to any order.*

Proof.
First recall $H^\varepsilon_\mathrm{BF} = \widehat{H}^\tau$ with $H \in S^{w=1+k^2}_\tau(\varepsilon, \mathcal{L}(\mathcal{H}^2_{A_0}, \mathcal{H}_\mathrm{f}))$. Let Π^ε be the projection from Theorem 2.5.2. Moreover, let $U^\varepsilon := U^\theta \circ U^\varepsilon_1$ with U^ε_1 the unitary map from Theorem 3.3.4 and U^θ the map defined in Remark 3.3.3. Then let

$$h = u \sharp \pi \sharp H \sharp \pi \sharp u^* \in S^1_\tau(\varepsilon, \mathcal{L}(\mathcal{H}_\mathrm{f}))$$

be the symbol from Theorem 3.3.6. Now take

$$h_\mathrm{c} := (u \sharp \pi \sharp H \sharp \pi \sharp u^*)_\mathrm{c} \in S^1_\tau(\varepsilon, \mathcal{L}(\mathcal{H}_\mathrm{f}))$$

as the corrected symbol according to Corollary 4.2.18(i) and let

$$h_\theta := (h_\mathrm{c})_\theta = \langle \varphi(k), h_\mathrm{c}(k,r)\varphi(k)\rangle_{\mathcal{H}_\mathrm{f}} \in S^1_\tau(\varepsilon, \mathbb{C}).$$

Then we take

$$h_\mathrm{eff} := (h_\theta)_\mathrm{c} \in S^1_{\tau\equiv 1}(\varepsilon, \mathbb{C})$$

as the correction of h_θ with respect to Corollary 4.4.9(ii). By construction, we get using Proposition 4.2.19, Remark 4.3.12, and Proposition 4.4.10 that $h_\mathrm{eff}(k,r) \in \mathbb{R}$

and thus from Proposition 4.4.7 that $\widehat{h_{\text{eff}}}^{\text{eff}}$ is a self-adjoint operator in $\mathcal{L}(\mathcal{H}_\theta)$. So we are left to show that $\left\|(e^{-iH^\varepsilon_{\text{BF}}t} - U^{\varepsilon *}e^{-i\widehat{h_{\text{eff}}}^{\text{eff}}t}U^\varepsilon)\Pi^\varepsilon\right\|_{\mathcal{L}(\mathcal{H}_\tau)} = \mathcal{O}(\varepsilon^\infty(1+|t|))$. It holds

$$
\begin{aligned}
\widehat{h_{\text{eff}}}^{\text{eff}} &= \widehat{h_\theta}^{\theta} + \mathcal{O}(\varepsilon^\infty) = \mathrm{Op}^\theta\left(((h_{\mathrm{c}})_{\text{diag}})_\theta\right) + \mathcal{O}(\varepsilon^\infty) &(5.1)\\
&= U^\theta \widehat{(h_{\mathrm{c}})_{\text{diag}}}^{\mathrm{B}} U^{\theta *} + \mathcal{O}(\varepsilon^\infty) = U^\theta \Pi^0 \widehat{(h_{\mathrm{c}})_{\text{diag}}}^{\mathrm{B}} U^{\theta *} + \mathcal{O}(\varepsilon^\infty) &(5.2)\\
&= U^\theta \Pi^0 \widehat{h_{\mathrm{c}}}^{\mathrm{B}} U^{\theta *} + \mathcal{O}(\varepsilon^\infty) &(5.3)\\
&= U^\theta \Pi^0 \widehat{h}^{\tau} U^{\theta *} + \mathcal{O}(\varepsilon^\infty). &(5.4)
\end{aligned}
$$

In the second transformation of equation (5.1), we used that $h_\theta = ((h_{\mathrm{c}})_{\text{diag}})_\theta$ $(:= \langle \varphi(k), (h_{\mathrm{c}})_{\text{diag}}\varphi(k)\rangle_{\mathcal{H}_{\mathrm{f}}})$, in the first transformation of (5.2) we used Theorem 4.3.10, and in (5.3) we used $[\widehat{h_{\mathrm{c}}}^{\mathrm{B}}, \Pi^0] = \mathcal{O}(\varepsilon^\infty)$ and Corollary 4.2.9. Hence, (5.4) implies

$$
\begin{aligned}
U^{\varepsilon *}\widehat{h_{\text{eff}}}^{\text{eff}}U^\varepsilon &= U_1^{\varepsilon *}\Pi^0\widehat{h}^{\tau}U_1^\varepsilon + \mathcal{O}(\varepsilon^\infty) = U_1^{\varepsilon *}\Pi^0 U_1^\varepsilon \Pi^\varepsilon \widehat{H}^{\tau}\Pi^\varepsilon U_1^{\varepsilon *}U_1^\varepsilon + \mathcal{O}(\varepsilon^\infty)\\
&= \widehat{\pi}^\tau \widehat{H}^{\tau}\widehat{\pi}^\tau + \mathcal{O}(\varepsilon^\infty),
\end{aligned}
$$

where we used (3.13), and thus

$$
\begin{aligned}
(e^{-iH^\varepsilon_{\text{BF}}t} - U^{\varepsilon *}e^{-i\widehat{h_{\text{eff}}}^{\text{eff}}t}U^\varepsilon)\Pi^\varepsilon &= (e^{-i\widehat{H}^{\tau}t} - e^{-iU^{\varepsilon *}\widehat{h_{\text{eff}}}^{\text{eff}}U^\varepsilon t})\widehat{\pi}^\tau + \mathcal{O}(\varepsilon^\infty)\\
&= (e^{-i\widehat{\pi}^\tau\widehat{H}^{\tau}\widehat{\pi}^\tau t} - e^{-iU^{\varepsilon *}\widehat{h_{\text{eff}}}^{\text{eff}}U^\varepsilon t})\widehat{\pi}^\tau + \mathcal{O}(\varepsilon^\infty)\\
&= \mathcal{O}(\varepsilon^\infty(1+|t|)).
\end{aligned}
$$

The last equality follows by the usual Duhammel argument exploiting that the difference of the generators is, according to the previous discussion, of order $\mathcal{O}(\varepsilon^\infty)$:

$$
\begin{aligned}
&e^{-i\widehat{\pi}^\tau\widehat{H}^{\tau}\widehat{\pi}^\tau t} - e^{-iU^{\varepsilon *}\widehat{h_{\text{eff}}}^{\text{eff}}U^\varepsilon t} = e^{-i\widehat{\pi}^\tau\widehat{H}^{\tau}\widehat{\pi}^\tau t}(1 - e^{i\widehat{\pi}^\tau\widehat{H}^{\tau}\widehat{\pi}^\tau t}e^{-iU^{\varepsilon *}\widehat{h_{\text{eff}}}^{\text{eff}}U^\varepsilon t})\\
&= e^{-i\widehat{\pi}^\tau\widehat{H}^{\tau}\widehat{\pi}^\tau t}(-i)\int_0^t e^{i\widehat{\pi}^\tau\widehat{H}^{\tau}\widehat{\pi}^\tau s}(\widehat{\pi}^\tau\widehat{H}^{\tau}\widehat{\pi}^\tau - U^{\varepsilon *}\widehat{h_{\text{eff}}}^{\text{eff}}U^\varepsilon)e^{-iU^{\varepsilon *}\widehat{h_{\text{eff}}}^{\text{eff}}U^\varepsilon s}\mathrm{d}s
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\left\|(e^{-iH^\varepsilon_{\text{BF}}t} - U^{\varepsilon *}e^{-i\widehat{h_{\text{eff}}}^{\text{eff}}t}U^\varepsilon)\Pi^\varepsilon\right\|_{\mathcal{L}(\mathcal{H}_\tau)}\\
&\leq C|t|\left\|\widehat{\pi}^\tau\widehat{H}^{\tau}\widehat{\pi}^\tau - U^{\varepsilon *}\widehat{h_{\text{eff}}}^{\text{eff}}U^\varepsilon\right\|_{\mathcal{L}(\mathcal{H}_\tau)} + \mathcal{O}(\varepsilon^\infty)\\
&= \mathcal{O}((1+|t|)\varepsilon^\infty).
\end{aligned}
$$

$\square$

So we have seen how we can apply our framework developed in Chapter 4 to define an effective Hamiltonian as a pseudodifferential operator. However, the condition for the derivatives of the symbol $h = u\sharp\pi\sharp H\sharp\pi\sharp u^*$ is rather impractical and unhandy. Therefore, we prove the following Lemmata to show that we can break this assumption down to some assumptions on the potentials A and Φ of the weakly varying perturbations. Note that since they are already smooth, bounded together with all their derivatives, and non-periodic, the additional assumption that the derivatives should be in L^1 is not a big confinement but only a technical detail.
So the following Lemmata will assure that $h = u\sharp\pi\sharp H\sharp\pi\sharp u^*$ satisfies the condition for the derivatives with respect to r if we just put a technical condition on A and Φ. The proofs will be more or less straightforward; one only has to keep track of the constructions of the symbols π and u in Theorem 2.5.1 respectively Theorem 3.3.4 and use the formula (B.3) of the Moyal product.

Lemma 5.1.2. *Let* $H_0(k,r) = \frac{1}{2}(-i\nabla_y + k - A_0(y) - A(r))^2 + V_\Gamma(y) + \Phi(r) = H_{\mathrm{per}}(k - A(r)) + \Phi(r) \in S^{w=1+k^2}_\tau(\mathbb{R}^4, \mathcal{L}(\mathcal{H}^2_{A_0}, \mathcal{H}_\mathrm{f}))$ *with*

$(*)$ $\quad A \in C^\infty_\mathrm{b}(\mathbb{R}^2,\mathbb{R}^2)$ *so that* $|\partial^\alpha_r A(r)| \in L^1(\mathbb{R}^2)$ *for all* $|\alpha| \geq 1$ *and*
$\quad\quad \Phi \in C^\infty_\mathrm{b}(\mathbb{R}^2,\mathbb{R})$ *so that* $\partial^\alpha_r \Phi(r) \in L^1(\mathbb{R}^2)$ *for all* $|\alpha| \geq \alpha_0$*, where* $\alpha_0 \geq 1$.

Then for every $\alpha, \beta \in \mathbb{N}^2_0$, $|\beta| \geq \alpha_0$*, and* $p \in \{1,2\}$ *there is a constant* $c_{\alpha,\beta}$ *so that*

$$\left\|\partial^\alpha_k \partial^\beta_r H_0(k,r)\right\|_{L^p(\mathbb{R}^2_r, \mathcal{L}(\mathcal{H}^2_{A_0}, \mathcal{H}_\mathrm{f}))} \leq c_{\alpha,\beta}(1+k^2).$$

Proof.
For the proof, we notice that for $|\beta| \geq \alpha_0$, $\partial^\alpha_k \partial^\beta_r H_{\mathrm{per}}(k - A(r))$ is a sum which can be estimated (using $H_{\mathrm{per}}(k - A(r)) \in S^{w=1+k^2}_\tau(\mathbb{R}^4, \mathcal{L}(\mathcal{H}^2_{A_0}, \mathcal{H}_\mathrm{f}))$ and $A \in C^\infty_\mathrm{b}(\mathbb{R}^2,\mathbb{R}^2)$) in the $\mathcal{L}(\mathcal{H}^2_{A_0}, \mathcal{H}_\mathrm{f})$-norm by a sum of the form $(1+k^2)c\sum_{i=1}^2 \sum_{|\gamma|=1}^{|\beta|} |\partial^\gamma_r A_i(r)|$. Thus, $(*)$ yields $\left\|\partial^\alpha_k \partial^\beta_r H_0(k,r)\right\|_{L^p(\mathbb{R}^2_r, \mathcal{L}(\mathcal{H}^2_{A_0}, \mathcal{H}_\mathrm{f}))} \leq c_{\alpha,\beta}(1+k^2)$. ∎

Lemma 5.1.3. *Let* $H_0(k,r) = \frac{1}{2}(-i\nabla_y + k - A_0(y) - A(r))^2 + V_\Gamma(y) + \Phi(r) = H_{\mathrm{per}}(k - A(r)) + \Phi(r) \in S^{w=1+k^2}_\tau(\mathbb{R}^4, \mathcal{L}(\mathcal{H}^2_{A_0}, \mathcal{H}_\mathrm{f}))$ *with*

$(*)$ $\quad A \in C^\infty_\mathrm{b}(\mathbb{R}^2,\mathbb{R}^2)$ *so that* $|\partial^\alpha_r A(r)| \in L^1(\mathbb{R}^2)$ *for all* $|\alpha| \geq 1$ *and*
$\quad\quad \Phi \in C^\infty_\mathrm{b}(\mathbb{R}^2,\mathbb{R})$ *so that* $\partial^\alpha_r \Phi(r) \in L^1(\mathbb{R}^2)$ *for all* $|\alpha| \geq 1$.

Moreover, let $\pi \asymp \sum_{j\geq 0} \varepsilon^j \pi_j$ *be the symbol constructed according to Theorem 2.5.1. Then for all* $j \geq 0$ *and for every* $\alpha, \beta \in \mathbb{N}^2_0$ *with* $|\beta| \geq 1$*, there are even polynomials* $q_{j,\alpha,\beta}$ *and* $\widetilde{q}_{j,\alpha,\beta}$ *so that*

$$\left\|\partial^\alpha_k \partial^\beta_r \pi_j(k,r)\right\|_{L^p(\mathbb{R}^2_r, \mathcal{L}(\mathcal{H}_\mathrm{f}, \mathcal{H}^2_{A_0})} \leq q_{j,\alpha,\beta}(k) \tag{5.5}$$

and

$$\left\|\partial_k^\alpha \partial_r^\beta \pi_j(k,r)\right\|_{L^p(\mathbb{R}_r^2,\mathcal{L}(\mathcal{H}_{\mathrm{f}}))} \leq \widetilde{q}_{j,\alpha,\beta}(k) \tag{5.6}$$

for $p \in \{1,2\}$.

Proof.
With very similar arguments as in the proof of Lemma 5.1.2, using $\pi_0(k) \in S_\tau^{w=1+k^2}(\mathbb{R}^4,\mathcal{L}(\mathcal{H}_{A_0}^2,\mathcal{H}_{\mathrm{f}})) \cap S_\tau^{w\equiv 1}(\mathbb{R}^4,\mathcal{L}(\mathcal{H}_{\mathrm{f}}))$, one gets that $\pi_0(k,r)$ fulfils the above conditions.
For π_j with $j \geq 1$ we have to take into account its construction from Lemma 2.5.1. Since π_j is defined as an integral over R_j, we first show that the Moyal resolvent constructed in the proof of Theorem 2.5.1 fulfils that for all $j \geq 0$ and $\alpha,\beta \in \mathbb{N}_0^2$ with $|\beta| \geq 1$, there are even polynomials $q_{j,\alpha,\beta}$ and $\widetilde{q}_{j,\alpha,\beta}$ so that

$$\left\|\partial_k^\alpha \partial_r^\beta R_j(k,r)\right\|_{L^p(\mathbb{R}_r^2,\mathcal{L}(\mathcal{H}_{\mathrm{f}},\mathcal{H}_{A_0}^2)} \leq q_{j,\alpha,\beta}(k)$$

and

$$\left\|\partial_k^\alpha \partial_r^\beta R_j(k,r)\right\|_{L^p(\mathbb{R}_r^2,\mathcal{L}(\mathcal{H}_{\mathrm{f}}))} \leq \widetilde{q}_{j,\alpha,\beta}(k)$$

for $p \in \{1,2\}$. We proceed by induction over j.
The induction starts at $j=0$. Let $R_0(\zeta,k,r) = (H_0(k,r)-\zeta)^{-1}$. Using

$$R_0(\zeta,z) - R_0(\zeta,\widetilde{z}) = R(\zeta,z)(H_0(z) - H_0(\widetilde{z}))R(\zeta,\widetilde{z}),$$

we get

$$\partial_{z_j} R_0(\zeta,z) = R_0(\zeta,z)\partial_{z_j} H_0(z) R_0(\zeta,z).$$

Hence, we can conclude that $\partial_k^\alpha \partial_r^\beta R_0(\zeta,k,r)$ must be a sum with summands of the form $R_0(k,r,\zeta)\partial_k^{\gamma^1}\partial_r^{\delta^1} H_0(k,r) R_0 .. \partial_k^{\gamma^m}\partial_r^{\delta^m} H_0(k,r) R_0(k,r,\zeta)$ with $|\gamma^1 + .. + \gamma^m| = |\alpha|$, $|\delta^1 + .. + \delta^m| = |\beta|$, and $m \leq \max\{|\alpha|,|\beta|\}$. This yields, using $R_0(k,r,\zeta) \in S_\tau^{w=1+k^2}(\mathbb{R}^4,\mathcal{L}(\mathcal{H}_{A_0}^2,\mathcal{H}_{\mathrm{f}})) \cap S_\tau^{w=1}(\mathbb{R}^4,\mathcal{L}(\mathcal{H}_{\mathrm{f}}))$ uniformly in ζ (which is shown in the proof of Lemma 2.5.1), that

$$\left\|\partial_k^\alpha \partial_r^\beta R_0(k,r)\right\|_{L^p(\mathbb{R}_r^2,\mathcal{L}(\mathcal{H}_{\mathrm{f}},\mathcal{H}_{A_0}^2)} \leq c_{\alpha,\beta}(1+k^2)^{2|\alpha+\beta|+1} =: q_{0,\alpha,\beta}(k)$$

and

$$\left\|\partial_k^\alpha \partial_r^\beta R_0(k,r)\right\|_{L^p(\mathbb{R}_r^2,\mathcal{L}(\mathcal{H}_{\mathrm{f}}))} \leq c_{\alpha,\beta}(1+k^2)^{2|\alpha+\beta|} =: \widetilde{q}_{0,\alpha,\beta}(k)$$

for $\alpha,\beta \in \mathbb{N}_0^2$, $|\beta| \geq 1$, and $p \in \{1,2\}$.
Now we assume that the claim holds for all R_j with $j \leq n$ and prove it for R_{n+1}. In the proof of Theorem 2.5.1, R_{n+1} is constructed as

$$R_{n+1}(\zeta,k,r) = -R_0(\zeta,k,r)E_{n+1}(\zeta,k,r)$$

with

$$E_{n+1}(\zeta,k,r) = [(H_0(\zeta,k,r)-\zeta)\sharp R^{(n)}(\zeta,k,r)]_{n+1}.$$

Now we use the formula (B.3) for the Moyal product and get

$$\begin{aligned} &E_{n+1}(\zeta,k,r) \\ &\quad= \sum_{|\alpha|+|\beta|+l=n+1,\ l\le n} c_{\alpha,\beta}(\partial_k^\alpha\partial_r^\beta(H_0(k,r)-\zeta))(\partial_r^\alpha\partial_k^\beta R_l(\zeta,k,r)) \\ &\quad= \sum_{|\alpha|+|\beta|+l=n+1,\ l\le n} c_{\alpha,\beta}(\partial_k^\alpha\partial_r^\beta H_0(k,r))(\partial_r^\alpha\partial_k^\beta R_l(\zeta,k,r)), \end{aligned}$$

which yields

$$\begin{aligned} &R_{n+1}(k,r,\zeta) \\ &\quad= -\sum_{|\alpha|+|\beta|+l=n+1,\ l\le n} c_{\alpha,\beta}R_0(k,r,\zeta)(\partial_k^\alpha\partial_r^\beta H_0(k,r))(\partial_r^\alpha\partial_k^\beta R_l(\zeta,k,r)). \end{aligned}$$

Hence we can use the induction hypothesis and Lemma 5.1.2 to conclude that $R_{n+1}(\zeta,k,r)$ has the required property.
Finally, recall the definition of π_n for $n\in\mathbb{N}_0$ in the proof of Theorem 2.5.1 as

$$\pi_n(k,r) := \tfrac{i}{2\pi}\oint_{\Lambda(z_0)} R_n(\zeta,k,r)\mathrm{d}\zeta \quad \text{on } \mathcal{U}_{z_0}.$$

This directly yields that the symbols $\pi_n(k,r)$ satisfy the conditions (5.5) and (5.6). □

Lemma 5.1.4. *Let* $H_0(k,r) = \frac{1}{2}(-i\nabla_y + k - A_0(y) - A(r))^2 + V_\Gamma(y) + \Phi(r) = H_{\mathrm{per}}(k-A(r)) + \Phi(r) \in S_\tau^{w=1+k^2}(\mathbb{R}^4, \mathcal{L}(\mathcal{H}^2_{A_0}, \mathcal{H}_{\mathrm{f}}))$ *with*

(∗) $A\in C_{\mathrm{b}}^\infty(\mathbb{R}^2,\mathbb{R}^2)$ *so that* $|\partial_r^\alpha A(r)|\in L^1(\mathbb{R}^2)$ *for all* $|\alpha|\ge 1$ *and*
$\Phi\in C_{\mathrm{b}}^\infty(\mathbb{R}^2,\mathbb{R})$ *so that* $\partial_r^\alpha\Phi(r)\in L^1(\mathbb{R}^2)$ *for all* $|\alpha|\ge 1$.

Moreover, let $u\asymp\sum_{j\ge 0}\varepsilon^j u_j$ *be the symbol constructed according to Theorem 3.3.4. Then for all* $j\ge 0$ *and for every* $\alpha,\beta\in\mathbb{N}_0^2$ *with* $|\beta|\ge 1$, *there is an even polynomial* $q_{j,\alpha,\beta}$ *so that*

$$\left\|\partial_k^\alpha\partial_r^\beta u_j(k,r)\right\|_{L^p(\mathbb{R}_r^2,\mathcal{L}(\mathcal{H}_{\mathrm{f}}))} \le q_{j,\alpha,\beta}(k) \tag{5.7}$$

for $p\in\{1,2\}$.

Proof.
We show this by induction. For $n=0$ the claim follows from the definition of $u_0(k,r)$, see the proof of Thereom 3.3.4, and the assumptions (∗). Similar to the proof of Lemma 5.1.2, we notice that $\left\|\partial_k^\alpha\partial_r^\beta u_0(k,r)\right\|_{\mathcal{L}(\mathcal{H}_{\mathrm{f}})}$ can be estimated by a

finite sum with summands looking like $c\sum_{|\gamma|=1}^{|\beta|}|\partial_r^\gamma A_1(r)|$. This yields the claim for $n=0$.
If we assume

$$\left\|\partial_k^\alpha\partial_r^\beta u_j(k,r)\right\|_{L^p(\mathbb{R}^2_r,\mathcal{L}(\mathcal{H}_\mathrm{f}))}\leq q_{j,\alpha,\beta}(k)\quad\text{for}\quad j\leq n,$$

it is quite easy to see that it also holds for $u_{n+1}(k,r)$. We just have to recall the definition of $u_{n+1}(k,r)$ in the proof of Theorem 3.3.4 and verify (adopting the notation used there) step by step that A_{n+1} and thus a_{n+1} and thus $w^{(n)}$ and $w^{(n)*}$ and, because of Lemma 5.1.3, B_{n+1} and hence b_{n+1} satisfy (5.7) and thus so does u_{n+1}. We always use the formula (B.3) of the Moyal product, the boundedness of the symbols, and the fact that they fulfil (5.7). □

Lemma 5.1.5. *Let $H_0(k,r)=\frac{1}{2}(-i\nabla_y+k-A_0(y)-A(r))^2+V_\Gamma(y)+\Phi(r)=H_{\mathrm{per}}(k-A(r))+\Phi(r)\in S_\tau^{w=1+k^2}(\mathbb{R}^4,\mathcal{L}(\mathcal{H}^2_{A_0},\mathcal{H}_\mathrm{f}))$ with*

(∗) $A\in C_\mathrm{b}^\infty(\mathbb{R}^2,\mathbb{R}^2)$ *so that* $|\partial_r^\alpha A(r)|\in L^1(\mathbb{R}^2)$ *for all* $|\alpha|\geq 1$ *and*
$\Phi\in C_\mathrm{b}^\infty(\mathbb{R}^2,\mathbb{R})$ *so that* $\partial_r^\alpha\Phi(r)\in L^1(\mathbb{R}^2)$ *for all* $|\alpha|\geq 1$.

Moreover, let $h\asymp\sum_{j\geq 0}\varepsilon^j h_j$ be the symbol constructed according to Theorem 3.3.6. Then for all $j\geq 0$ and for every $\alpha,\beta\in\mathbb{N}_0^2$ with $|\beta|\geq 1$, there is an even polynomial $q_{j,\alpha,\beta}$ so that

$$\left\|\partial_k^\alpha\partial_r^\beta h_j(k,r)\right\|_{L^p(\mathbb{R}^2_r,\mathcal{L}(\mathcal{H}_\mathrm{f}))}\leq q_{j,\alpha,\beta}(k)$$

for $p\in\{1,2\}$.

Proof.
This follows by using the Lemmata 5.1.2, 5.1.3, 5.1.4, and the formula (B.3) of the Moyal product. □

Now we can reformulate Theorem 5.1.1 in a more convenient form.

Corollary 5.1.6. *Let $H_0(k,r)=\frac{1}{2}(-i\nabla_y+k-A_0(y)-A(r))^2+V_\Gamma(y)+\Phi(r)\in S_\tau^{w=1+k^2}(\mathbb{R}^4,\mathcal{L}(\mathcal{H}^2_{A_0},\mathcal{H}_\mathrm{f}))$, let E be an isolated band according to Definition 2.4.1, and let Assumption 1 hold. Moreover, let $|\partial_r^\alpha A(r)|\in L^1(\mathbb{R}^2)$ for all $|\alpha|\geq 1$ and $\partial_r^\alpha\Phi(r)\in L^1(\mathbb{R}^2)$ for all $|\alpha|\geq 1$. Then there exist*

(i) an orthogonal projection $\Pi^\varepsilon\in\mathcal{L}(\mathcal{H}_\tau)$,

(ii) a unitary map $U^\varepsilon\in\mathcal{L}(\Pi^\varepsilon\mathcal{H}_\tau,\mathcal{H}_\theta)$, and

(iii) a self-adjoint operator $\widehat{h_{\mathrm{eff}}}^{\mathrm{eff}}\in\mathcal{L}(\mathcal{H}_\theta)$

such that

$$\|[H^\varepsilon_{\mathrm{BF}}, \Pi^\varepsilon]\|_{\mathcal{L}(\mathcal{H}_\tau)} = \mathcal{O}(\varepsilon^\infty)$$

and

$$\left\|(e^{-iH^\varepsilon_{\mathrm{BF}}t} - U^{\varepsilon *} e^{-i\widehat{h_{\mathrm{eff}}}^{\mathrm{eff}}t} U^\varepsilon)\Pi^\varepsilon\right\|_{\mathcal{L}(\mathcal{H}_\tau)} = \mathcal{O}(\varepsilon^\infty(1+|t|)).$$

The effective Hamiltonian is the effective quantisation of the symbol $h_{\mathrm{eff}} \in S^1_{\tau\equiv 1}(\varepsilon, \mathbb{C})$ *which can be computed to any order.*

Proof.
By Lemma 5.1.5, we get that the assumptions for Theorem 5.1.1 are satisfied with $\alpha_j = 1$ for all $j \geq 0$. □

Remark 5.1.7. Corollary 5.1.6 can be generalised to semiclassical symbols $H \asymp \sum_{j\geq 0} \varepsilon^j H_j \in S^1_\tau(\varepsilon, \mathcal{L}(\mathcal{H}^2_{A_0}, \mathcal{H}_{\mathrm{f}}))$ with principal symbol H_0 defined as above that satisfy that for all $j \geq 1$ and for every $\alpha, \beta \in \mathbb{N}_0^2$ with $|\beta| \geq 1$, there is a function $g_{j,\alpha,\beta} \in C(\mathbb{R}^2)$ so that

$$\left\|\partial^\alpha_k \partial^\beta_r H_j(k,r)\right\|_{L^p(\mathbb{R}^2_r, \mathcal{L}(\mathcal{H}_{\mathrm{f}}))} \leq g_{j,\alpha,\beta}(k)$$

for $p \in \{1, 2\}$.

5.2 The leading orders of the symbol

The next goal is to compute the principal and subprincipal symbol of the effective Hamiltonian from Theorem 5.1.1 for our case of a non-degenerate isolated eigenvalue band E. Thereto, we look at its construction in the proof of Theorem 5.1.1. Hence, we first must compute the principal and subprincipal symbol of $h = u\sharp\pi\sharp H\sharp\pi\sharp u^*$. After that, we compute the principal and subprincipal symbol of h_{c} according to Corollary 4.2.18(i). By construction, these symbols commute with $\pi_0(k)$. Then the principal and subprincipal symbol of h_θ are given by $h_{\theta 0} = U^\theta h_{c0} U^{\theta^*} = \langle\varphi(k), h_{c0}\varphi(k)\rangle_{\mathcal{H}_{\mathrm{f}}}$ and $h_{\theta 1} = U^\theta h_{c1} U^{\theta^*} = \langle\varphi(k), h_{c1}\varphi(k)\rangle_{\mathcal{H}_{\mathrm{f}}}$. The last step is to compute the principal and subprincipal symbol of h_{eff} as the corrections of h_θ according to Corollary 4.4.9(ii).
Note that we started from the pseudodifferential operator $H^\varepsilon_{\mathrm{BF}} = H_0(k, i\varepsilon\nabla^\tau_k)$ and not $H_0(k, \text{-}i\varepsilon\nabla^\tau_k)$. For the constructions made so far, we could neglect this. But now, when we explicitly compute symbols, we need to pay attention that in the Weyl product and in the corrections there are some changes of signs.

Proposition 5.2.1. *The subprincipal symbol of* h_θ *from the construction in the proof of Theorem 5.1.1 can be computed by*

$$\begin{aligned} h_{\theta 1} &= \tfrac{i}{2}\langle\varphi(k), \{\widetilde{u}_0(k,r), H_{\mathrm{per}}(k,r) - E(k,r)\} e^{\frac{i\theta}{2\pi}k_1 A_2(r)}\varphi(k,r)\rangle_{\mathcal{H}_{\mathrm{f}}} \\ &\quad + i\langle\varphi(k), \{\widetilde{u}_0(k,r), E(k,r) + \Phi(r)\} e^{\frac{i\theta}{2\pi}k_1 A_2(r)}\varphi(k,r)\rangle_{\mathcal{H}_{\mathrm{f}}}. \end{aligned}$$

Proof.
For this proof, we can first proceed along the ideas of the computation in paragraph 3.3.1 of [Teu03]. Note that we will get additional terms because the projection $\pi_0(k)$, which corresponds to the projection π_{r} in [Teu03, PST03b], is not constant, but depends on k, and also the symbol u_0 is a bit more complex. For the following considerations keep in mind that $u_0(k,r)$ can be split up into

$$\begin{aligned} u_0(k,r) &= \pi_0(k)u_0(k,r)\pi_0(k,r) + \pi_0^\perp(k)u_0(k,r)\pi_0^\perp(k,r) && (5.8)\\ &= |\varphi(k)\rangle\,\langle\varphi(k-A(r))|\,e^{-\frac{i\theta}{2\pi}A_2(r)k_1} + u_0^\perp(k,r) && (5.9)\\ &:= \widetilde{u}_0(k,r) + u_0^\perp(k,r). && (5.10)\end{aligned}$$

First, note that by construction of u we have $h = u\sharp\pi\sharp H\sharp\pi\sharp u^* = \pi_{\mathrm{r}}\sharp u\sharp H\sharp u^*\sharp\pi_{\mathrm{r}}$ where $\pi_{\mathrm{r}}(k,r) = \pi_0(k)$. So with $\widetilde{h} := u\sharp H\sharp u^*$ we get

$$\begin{aligned} \widetilde{h}_0(k,r) &= u_0(k,r)H_0(k-A(r))u_0^*(k,r)\\ &= |\varphi(k)\rangle\langle\varphi(k-A(r))|\,(H_{\mathrm{per}}(k-A(r))+\Phi(r))\,|\varphi(k-A(r))\rangle\langle\varphi(k)|\\ &\quad + u_0^\perp(k,r)H_0(k,r)u_0^{\perp *}(k,r)\\ &= (E(k-A(r))+\Phi(r))\pi_0(k) + \pi_0^\perp(k)u_0^\perp(k,r)H_0(k,r)u_0^{\perp *}(k,r)\pi_0^\perp(k)\end{aligned}$$

and therefore

$$h_0(k,r) = (E(k-A(r))+\Phi(r))\pi_0(k)$$

and

$$\begin{aligned} h_1(k,r) &= (\pi_{\mathrm{r}}\sharp\widetilde{h}\sharp\pi_{\mathrm{r}})_1(k,r) = (\pi_{\mathrm{r}}\sharp\widetilde{h})_1(k,r)\pi_0(k) + \tfrac{i}{2}\{\pi_0(k)\widetilde{h}_0(k,r),\pi_0(k)\}\\ &= \pi_0(k)\widetilde{h}_1(k,r)\pi_0(k) + \tfrac{i}{2}\{\pi_0(k),\widetilde{h}_0(k,r)\}\pi_0(k) + \tfrac{i}{2}\{h_0(k,r),\pi_0(k)\}\\ &= \pi_0(k)\widetilde{h}_1(k,r)\pi_0(k) + \tfrac{i}{2}(-\nabla_k\pi_0(k)\nabla_r(E(k-A(r))+\Phi(r))\pi_0(k)\\ &\quad +\nabla_r(E(k-A(r))+\Phi(r))\pi_0(k)\nabla_k\pi_0(k))\\ &= \pi_0(k)\widetilde{h}_1(k,r)\pi_0(k) + \tfrac{i}{2}\nabla_r\,(E(k-A(r))+\Phi(r))\,[\pi_0(k),\nabla_k\pi_0(k)].\end{aligned}$$

Hence, for the corrections of the symbols we get from corollary 4.2.18(i)

$$h_{c0} = (E(k-A(r))+\Phi(r))\pi_0(k)$$

and

$$\begin{aligned} h_{c1} &= h_1(k,r) + \tfrac{i}{2}(D(k)\nabla_r(E(k-A(r))+\Phi(r))\pi_0(k) + \nabla_r(E(k-A(r))\\ &\quad +\Phi(r))\pi_0(k)D(k))\\ &= h_1(k,r) + \tfrac{i}{2}\nabla_r(E(k-A(r))+\Phi(r))(D(k)\pi_0(k)+\pi_0(k)D(k))\\ &= h_1(k,r) + \tfrac{i}{2}\nabla_r(E(k-A(r))+\Phi(r))D(k)\\ &= h_1(k,r) + \tfrac{i}{2}\nabla_r(E(k-A(r))+\Phi(r))[\nabla_k\pi_0(k),\pi_0(k)]\\ &= \pi_0(k)\widetilde{h}_1(k,r)\pi_0(k).\end{aligned}$$

Here we used

$$
\begin{aligned}
D(k) &= \pi_0(k)\nabla_k(\pi_0^\perp(k)\pi_0^\perp(k)) + \pi_0^\perp(k)\nabla_k(\pi_0(k)\pi_0(k)) \\
&= \pi_0(k)\left(\nabla_k\pi_0^\perp(k)\right)\pi_0^\perp(k) + \pi_0^\perp(k)\left(\nabla_k\pi_0(k)\right)\pi_0(k) \\
&= -\pi_0(k)\left(\nabla_k\pi_0(k)\right) + \pi_0(k)\left(\nabla_k\pi_0(k)\right)\pi_0(k) + \left(\nabla_k\pi_0(k)\right)\pi_0(k) \\
&\quad -\pi_0(k)\left(\nabla_k\pi_0(k)\right)\pi_0(k) \\
&= [\nabla_k\pi_0(k), \pi_0(k)]
\end{aligned}
$$

exploiting

$$\nabla_k \circ \pi_0(k) = (\nabla_k\pi_0(k)) + \pi_0(k)\nabla_k.$$

It remains to compute $\pi_0(k)\widetilde{h}_1(k,r)\pi_0(k)$.
To avoid having to compute $u_1(k,r)$, which would be quite complicated, we furthermore proceed along the lines of the computation in [Teu03, PST03b]. On the one hand, we get, after Moyal-multiplying $\widetilde{h} = u\sharp H\sharp u^*$ with u from the right, $u\sharp H = \widetilde{h}\sharp u$ and thus

$$u\sharp H - \widetilde{h}_0\sharp u = \varepsilon\widetilde{h}_1 u_0 + \mathcal{O}(\varepsilon^2). \tag{5.11}$$

On the other hand, we can compute the subprincipal symbol by

$$(u\sharp H - \widetilde{h}_0\sharp u)_1 = u_1 H_0 + u_0 H_1 - \widetilde{h}_0 u_1 + (u_0\sharp H_0)_1 - (\widetilde{h}_0\sharp u_0)_1. \tag{5.12}$$

Combining equations (5.11) and(5.12) yields

$$\widetilde{h}_1 = (u\sharp H - \widetilde{h}_0\sharp u)_1 u_0^* = (u_1 H_0 + u_0 H_1 - \widetilde{h}_0 u_1 + (u_0\sharp H_0)_1 - (\widetilde{h}_0\sharp u_0)_1)u_0^*.$$

Having in mind the construction of $u_1 = (a+b)u_0$ and $\widetilde{h}_0$, we get

$$\pi_0(k)(u_1 H_0 u_0^* - \widetilde{h}_0 u_1 u_0^*)\pi_0(k) = 0.$$

Since in our case also $H_1 \equiv 0$, we just must take care of the term

$$\pi_0(k)\Big((u_0\sharp H_0)_1 - (\widetilde{h}_0\sharp u_0)_1\Big)\underbrace{u_0^*(k,r)\pi_0(k)}_{=\pi_0(k,r)u_0^*(k,r)}.$$

Now we want to show that for the calculation of h_1, only the part $\widetilde{u}_0$ of u_0 is relevant. Recall $\widetilde{h}_0 = u_0 H_0 u_0^*$ and (5.8). We first split up u_0 to get

$$\pi_0(k)\{u_0, H_0\}\pi_0(k,r) = \pi_0(k)\{\widetilde{u}_0, H_0\}\pi_0(k,r) + \pi_0(k)\{u_0^\perp, H_0\}\pi_0(k,r) \tag{5.13}$$

and calculate

$$
\begin{aligned}
\pi_0(k)\{u_0^\perp, H_0\}\pi_0(k,r) &= -\pi_0(k)(\nabla_k\pi_0^\perp(k))u_0^\perp(\nabla_r H_0)\pi_0(k,r) \\
&= \pi_0(k)(\nabla_k\pi_0(k))u_0^\perp(\nabla_r\pi_0(k,r))H_0\pi_0(k,r) \\
&\quad -\pi_0(k)(\nabla_k\pi_0(k))u_0^\perp H_0(\nabla_r\pi_0(k,r))\pi_0(k,r),
\end{aligned}
$$

where in the last step we exploited $H_0 = \pi_0(k,r)H_0\pi_0(k,r) + \pi_0^\perp(k,r)H_0\pi_0^\perp(k,r)$. Next, we look at

$$\pi_0(k)\{\widetilde{h}_0, u_0\}\pi_0(k,r) = \pi_0(k)\{\widetilde{h}_0, \widetilde{u}_0\}\pi_0(k,r) + \pi_0(k)\{\widetilde{h}_0, u_0^\perp\}\pi_0(k,r) \tag{5.14}$$

and calculate

$$\begin{aligned}
\pi_0(k)\{\widetilde{h}_0, u_0^\perp\}\pi_0(k,r) &= \pi_0(k)(\nabla_r\widetilde{h}_0)u_0^\perp(\nabla_k\pi_0^\perp(k,r))\pi_0(k,r) & (5.15)\\
&\quad -\pi_0(k)(\nabla_k\widetilde{h}_0)u_0^\perp(\nabla_r\pi_0^\perp(k,r))\pi_0(k,r) \\
&= \pi_0(k)\widetilde{h}_0\nabla_k(\pi_0(k))u_0^\perp(\nabla_r\pi_0^\perp(k,r))\pi_0(k,r) & (5.16)\\
&\quad -\pi_0(k)(\nabla_k\pi_0(k))\widetilde{h}_0u_0^\perp(\nabla_r\pi_0^\perp(k,r))\pi_0(k,r) & (5.17)\\
&= \pi_0(k)(\nabla_k\pi_0(k))u_0^\perp(\nabla_r\pi_0(k,r))H_0\pi_0(k,r) & (5.18)\\
&\quad -\pi_0(k)(\nabla_k\pi_0(k))u_0^\perp H_0(\nabla_r\pi_0(k,r))\pi_0(k,r) & (5.19)\\
&= \pi_0(k)\{u_0^\perp, H_0\}\pi_0(k,r). & (5.20)
\end{aligned}$$

In (5.15), the right hand side vanishes since $\pi_0(k)(\nabla_r\widetilde{h}_0)\pi_0^\perp(k,r) = 0$. Then we use that the off-diagonal part of $\widetilde{h}_0$ vanishes and treat the part $\pi_0(k)\widetilde{h}_0\pi_0(k)$ in (5.16) and $\pi_0^\perp(k)\widetilde{h}_0\pi_0^\perp(k)$ in (5.17). Moreover, in (5.18) we used $\pi_0(k)\widetilde{h}_0 = (E(k,r)+\Phi(r))\pi_0(k)$, moved the scalar part $(E(k,r)+\Phi(r))$ to the end of the expression, and then in turn exploited $(E(k,r)+\Phi(r))\pi_0(k,r) = H_0(k,r)\pi_0(k,r)$. Finally, in (5.19) note that $\widetilde{h}_0u_0^\perp = \widetilde{h}_0\pi_0^\perp(k,r)u_0(k,r) = \pi_0^\perp(k,r)\widetilde{h}_0u_0(k,r) = \pi_0^\perp(k,r)u_0(k,r)H_0(k,r) = u_0^\perp(k,r)H_0(k,r)$.
All in all, we get from (5.13),(5.14), and (5.20) that

$$\pi_0(k)\left(\{u_0, H_0\} - \{\widetilde{h}_0, u_0\}\right)\pi_0(k,r) \quad = \quad \pi_0(k)\left(\{\widetilde{u}_0, H_0\} - \{\widetilde{h}_0, \widetilde{u}_0\}\right)\pi_0(k,r),$$

Finally note that $\pi_0(k)\{\pi_0^\perp(k)\widetilde{h}_0\pi_0^\perp(k), \widetilde{u}_0\}\pi_0(k,r) = 0$ implies

$$\pi_0(k)\left(\{u_0, H_0\} - \{\widetilde{h}_0, u_0\}\right)\pi_0(k,r) \quad = \quad \pi_0(k)\left(\{\widetilde{u}_0, H_0\} - \{h_0, \widetilde{u}_0\}\right)\pi_0(k,r)$$

and hence, as for u_0, only the part $\pi_0(k)\widetilde{h}_0\pi_0(k) = h_0$ of $\widetilde{h}_0$ is relevant for the following calculations.
Thus, we just must compute the Poisson brackets

$$\{\widetilde{u}_0(k,r), H_{\mathrm{per}}(k - A(r)) + \Phi(r)\}$$

and

$$\begin{aligned}
\{(E+\Phi)\pi_0(k), \widetilde{u}_0\} &= \pi_0(k)\{E+\Phi, \widetilde{u}_0\} - (E+\Phi)\nabla_k\pi_0(k)\nabla_r\widetilde{u}_0\\
&= -\pi_0(k)\{\widetilde{u}_0, E+\Phi\} - (E+\Phi)(\nabla_k\pi_0(k))\pi_0(k)\nabla_r\widetilde{u}_0,
\end{aligned}$$

where the last equality holds because of $\widetilde{u}_0(k,r) = \pi_0(k)\widetilde{u}_0(k,r)$. So we get

$$\begin{aligned}
h_{c1} &= \pi_0(k)\widetilde{h}_1\pi_0(k) \\
&= +\tfrac{i}{2}\langle\varphi(k), \{\widetilde{u}_0, H_{\mathrm{per}} + \Phi\} + \{\widetilde{u}_0, E + \Phi\}\varphi(k - A(r))\rangle_{\mathcal{H}_\mathrm{f}}\pi_0(k) \\
&\quad +\tfrac{i}{2}\langle\varphi(k), (E + \Phi)\pi_0(k)(\nabla_k\pi_0(k))\pi_0(k)\nabla_r\widetilde{u}_0\varphi(k - A(r))\rangle_{\mathcal{H}_\mathrm{f}}\pi_0(k) \\
&= +\tfrac{i}{2}\langle\varphi(k), \{\widetilde{u}_0, H_{\mathrm{per}} - E\}\varphi(k - A(r))\rangle_{\mathcal{H}_\mathrm{f}}\pi_0(k) \\
&\quad +i\langle\varphi(k), \{\widetilde{u}_0, E + \Phi\}\varphi(k - A(r))\rangle_{\mathcal{H}_\mathrm{f}}\pi_0(k)
\end{aligned}$$

because of $\pi_0(k)(\nabla_k\pi_0(k))\pi_0(k) = 0$.
With $h_{\theta 0} = \langle\varphi(k), h_{c0}\varphi(k)\rangle_{\mathcal{H}_\mathrm{f}}$ and $h_{\theta 1} = \langle\varphi(k), h_{c1}\varphi(k)\rangle_{\mathcal{H}_\mathrm{f}}$, the claim follows. $\square$

Proposition 5.2.2. *The symbol $h_\theta(k,r) = h_{\theta 0}(k,r) + \varepsilon h_{\theta 1}(k,r) + \mathcal{O}(\varepsilon^2)$ from the construction in the proof of Theorem 5.1.1 is given by*

$$h_{\theta 0}(k,r) = E(k - A(r)) + \Phi(r)$$

and

$$\begin{aligned}
&h_{\theta 1}(k,r) \\
&= i\mathcal{A}_1(k - A(r))(\partial_1\Phi(r) + \partial_2 E(k - A(r))\partial_2 A_1(r)) \\
&\quad -i(\mathcal{A}_2(k) - \mathcal{A}_2(k - A(r)))(\partial_2\Phi(r) - \partial_1 E(k - A(r))\partial_2 A_1(r)) \\
&\quad -i\mathcal{A}_1(k)\partial_{r_1}(E(k - A(r)) + \Phi(r)) \\
&\quad +\partial_2 A_1(r)\mathrm{Re}\left(\tfrac{i}{2}\langle\partial_1\varphi(k - A(r)), (H_{\mathrm{per}} - E)(k - A(r))\partial_2\varphi(k - A(r))\rangle_{\mathcal{H}_\mathrm{f}}\right) \\
&\quad -\partial_2 A_1(r)\mathrm{Re}\left(\tfrac{i}{2}\langle\partial_2\varphi(k - A(r)), (H_{\mathrm{per}} - E)(k - A(r))\partial_1\varphi(k - A(r))\rangle_{\mathcal{H}_\mathrm{f}}\right).
\end{aligned}$$

Proof.
Bearing in mind that we have chosen the gauge $A_2(r) \equiv 0$, we can calculate using

Proposition 5.2.1

$$
\begin{aligned}
&\langle \varphi(k), \{\widetilde{u}_0(k,r), (H_{\mathrm{per}}-E)(k,r)\}\varphi(k,r)\rangle_{\mathcal{H}_{\mathrm{f}}} \\
&\quad = \langle \varphi(k), \nabla_r \widetilde{u}_0(k,r)\nabla_k (H_{\mathrm{per}}-E)(k-A(r)) \\
&\qquad -\nabla_k \widetilde{u}_0(k,r)\nabla_r (H_{\mathrm{per}}-E)(k-A(r))\varphi(k,r)\rangle_{\mathcal{H}_{\mathrm{f}}} \\
&\quad = \sum_{j=1}^{2}(\langle \partial_{r_j}\varphi(k-A(r)), \partial_{k_j}(H_{\mathrm{per}}-E)(k-A(r))\varphi(k-A(r))\rangle_{\mathcal{H}_{\mathrm{f}}} \\
&\qquad -\mathcal{A}_j(k)\langle \varphi(k-A(r)), \partial_{r_j}(H_{\mathrm{per}}-E)(k-A(r))\varphi(k-A(r))\rangle_{\mathcal{H}_{\mathrm{f}}} \\
&\qquad -\langle \partial_{k_j}\varphi(k-A(r)), \partial_{r_j}(H_{\mathrm{per}}-E)(k-A(r))\varphi(k-A(r))\rangle_{\mathcal{H}_{\mathrm{f}}}) \\
&\quad = \sum_{j=1}^{2}(-\langle -\partial_{k_1}\varphi(k-A(r))\partial_j A_1(r), (H_{\mathrm{per}}-E)(k-A(r))\partial_{k_j}\varphi(k-A(r))\rangle_{\mathcal{H}_{\mathrm{f}}} \\
&\qquad +\langle \partial_{k_j}\varphi(k-A(r)), (H_{\mathrm{per}}-E)(k-A(r))\partial_{r_j}\varphi(k-A(r))\rangle_{\mathcal{H}_{\mathrm{f}}}) \\
&\quad = \partial_2 A_1(r)\langle \partial_1\varphi(k-A(r)), (H_{\mathrm{per}}-E)(k-A(r))\partial_2\varphi(k-A(r))\rangle_{\mathcal{H}_{\mathrm{f}}} \\
&\qquad -\partial_2 A_1(r)\langle \partial_2\varphi(k-A(r)), (H_{\mathrm{per}}-E)(k-A(r))\partial_1\varphi(k-A(r))\rangle_{\mathcal{H}_{\mathrm{f}}}.
\end{aligned}
$$

Here we used the equality

$$
\langle \Phi, \partial_j (H_{\mathrm{per}}-E)\varphi\rangle = -\langle \Phi, (H_{\mathrm{per}}-E)\partial_j\varphi\rangle,
$$

which comes from the fact that φ is an eigenvector of H_{per} with eigenvalue E and so it holds for arbitrary Φ that $\langle \Phi, (H_{\mathrm{per}}-E)\varphi\rangle = 0 = \partial_j\langle \Phi, (H_{\mathrm{per}}-E)\varphi\rangle$. Moreover, note that the imaginary part of

$$
\begin{aligned}
&i(\partial_2 A_1(r)\langle \partial_1\varphi(k-A(r)), (H_{\mathrm{per}}-E)(k-A(r))\partial_2\varphi(k-A(r))\rangle_{\mathcal{H}_{\mathrm{f}}} \\
&-\partial_2 A_1(r)\langle \partial_2\varphi(k-A(r)), (H_{\mathrm{per}}-E)(k-A(r))\partial_1\varphi(k-A(r))\rangle_{\mathcal{H}_{\mathrm{f}}})
\end{aligned}
$$

vanishes. Furthermore, it holds

$$
\begin{aligned}
&\langle \varphi(k), \{\widetilde{u}_0, E+\Phi\}\varphi(k-A(r))\rangle_{\mathcal{H}_{\mathrm{f}}} \\
&\quad = \sum_{j=1}^{2}\langle \partial_{r_j}\varphi(k-A(r)), \partial_{k_j}E(k-A(r))\varphi(k-A(r))\rangle_{\mathcal{H}_{\mathrm{f}}} \\
&\qquad -\sum_{j=1}^{2}(\mathcal{A}_j(k)\langle \varphi(k-A(r)), \partial_{r_j}(E(k-A(r))+\Phi(r))\varphi(k-A(r))\rangle_{\mathcal{H}_{\mathrm{f}}} \\
&\qquad -\langle \partial_{k_j}\varphi(k-A(r)), \partial_{r_j}(E(k-A(r))+\Phi(r))\varphi(k-A(r))\rangle_{\mathcal{H}_{\mathrm{f}}}) \\
&\quad = \mathcal{A}_1(k-A(r))(\partial_2 E(k-A(r))\partial_2 A_1(r)+\partial_1\Phi(r)) - \mathcal{A}_1(k)\partial_{r_1}(E(k-A(r)) \\
&\qquad +\Phi(r)) + (\mathcal{A}_2(k-A(r))-\mathcal{A}_2(k))(\partial_{r_2}(E(k-A(r))+\Phi(r)).
\end{aligned}
$$

□

Remark 5.2.3. A formal Taylor expansion yields

$$\Phi(i\varepsilon\nabla_k + i\varepsilon\mathcal{A}(k)) = \Phi(i\varepsilon\nabla_k) + \varepsilon\nabla\Phi(i\varepsilon\nabla_k)i\mathcal{A}(k) + \varepsilon^2...$$

So one could expect that the Berry connection term in the subprincipal symbol of h_θ from Theorem 5.2.2 should vanish since it is already included in the quantisation. But as we see, this only is the case if the weak perturbation $A(r) \equiv 0$.

Theorem 5.2.4. *The principal and subprincipal symbol of* $h_{\mathrm{eff}} = h_0 + \varepsilon h_1 + \mathcal{O}(\varepsilon^2)$ *from Theorem 5.1.1 are given by*

$$h_0(k,r) = E(k - A(r)) + \Phi(r)$$

and

$$\begin{aligned} &h_1(k,r) \\ &\quad= \; i\mathcal{A}_1(k - A(r))(\partial_1\Phi(r) + \partial_2 E(k - A(r))\partial_2 A_1(r)) \\ &\qquad + i(\mathcal{A}_2(k - A(r)) - \tfrac{i\theta}{2\pi}k_1)(\partial_2\Phi(r) - \partial_1 E(k - A(r))\partial_2 A_1(r)) \\ &\qquad + \partial_2 A_1(r)\mathrm{Re}\left(\tfrac{i}{2}\langle\partial_1\varphi(k - A(r)), (H_{\mathrm{per}} - E)(k - A(r))\partial_2\varphi(k - A(r))\rangle_{\mathcal{H}_{\mathrm{f}}}\right) \\ &\qquad - \partial_2 A_1(r)\mathrm{Re}\left(\tfrac{i}{2}\langle\partial_2\varphi(k - A(r)), (H_{\mathrm{per}} - E)(k - A(r))\partial_1\varphi(k - A(r))\rangle_{\mathcal{H}_{\mathrm{f}}}\right) \\ &\quad =: \; (\nabla\Phi(r) - \nabla E(\widetilde{k}) \times B(r)) \cdot (i\mathcal{A}_1(\widetilde{k}), i(\mathcal{A}_2(\widetilde{k}) - \tfrac{i\theta}{2\pi}k_1))^{\mathrm{T}} - B(r) \cdot \mathcal{M}(\widetilde{k}), \end{aligned}$$

where $\widetilde{k} = k - A(r)$ *and* $B(r) = \partial_2 A_1$.

Proof.
From Theorem 4.4.8(ii), we get

$$\begin{aligned} (h_\theta)_{\mathrm{c}1}(k,r) &= h_{\theta 1}(k,r) + i(\mathcal{A}_1(k)\partial_{r_1}(E(k - A(r)) + \Phi(r)) + (\mathcal{A}_2(k) - \tfrac{i\theta}{2\pi}k_1) \times \\ &\quad \partial_{r_2}(E(k - A(r)) + \Phi(r))) \\ &= i\mathcal{A}_1(k - A(r))(\partial_1\Phi(r) + \partial_2 E(k - A(r))\partial_2 A_1(r)) \\ &\quad + i(\mathcal{A}_2(k - A(r)) - \tfrac{i\theta}{2\pi}k_1)(\partial_2\Phi(r) - \partial_1 E(k - A(r))\partial_2 A_1(r)). \end{aligned}$$

□

5.3 The corresponding results for an arbitrary Bravais lattice Γ

In this section, we give the corresponding results for an arbitrary Bravais lattice Γ generated by the vectors γ^1 and γ^2. As always, we denote the components of the generating vectors by $\gamma^1 = (\gamma^1_1, \gamma^1_2)$ and $\gamma^2 = (\gamma^2_1, \gamma^2_2)$.

In Proposition 5.2.1, one has to keep in mind the altered phase in the definition of u_0, which yields

$$\begin{aligned} h_{\theta 1} &= \tfrac{i}{2}\langle\varphi(k),\{u_0(k,r),H_{\mathrm{per}}(k,r)-E(k,r)\}e^{\frac{i\theta}{2\pi}\langle\gamma^1,k\rangle\langle\gamma^2,A(r)\rangle}\varphi(k,r)\rangle_{\mathcal{H}_{\mathrm{f}}} \\ &\quad +i\langle\varphi(k),\{u_0(k,r),E(k,r)+\Phi(r)\}e^{\frac{i\theta}{2\pi}\langle\gamma^1,k\rangle\langle\gamma^2,A(r)\rangle}\varphi(k,r)\rangle_{\mathcal{H}_{\mathrm{f}}}. \end{aligned}$$

However, this modification only has to be noted if one does not take our gauge $\langle\gamma^2,A(r)\rangle\equiv 0$.

The subprincipal symbol of h_θ computed in Proposition 5.2.2 is then given by

$$\begin{aligned} h_{\theta 1}(k,r) &= i\mathcal{A}_1(\widetilde{k})(\partial_1\Phi(r)+\partial_2E(\widetilde{k})\partial_2A_1(r)-\partial_2E(\widetilde{k})\partial_1A_2(r)) \\ &\quad +i\mathcal{A}_2(\widetilde{k})(\partial_2\Phi(r)-\partial_1E(\widetilde{k})\partial_2A_1(r)+\partial_1E(\widetilde{k})\partial_1A_2(r)) \\ &\quad -i\mathcal{A}_1(k)\partial_{r_1}(E(\widetilde{k})+\Phi(r))-\mathcal{A}_2(k)\partial_{r_2}(E(\widetilde{k})+\Phi(r)) \\ &\quad +(\partial_2A_1(r)-\partial_1A_2(r))\mathrm{Re}\left(\tfrac{i}{2}\langle\partial_1\varphi(\widetilde{k}),(H_{\mathrm{per}}-E)(\widetilde{k})\partial_2\varphi(\widetilde{k})\rangle_{\mathcal{H}_{\mathrm{f}}}\right) \\ &\quad +(\partial_1A_2(r)-\partial_2A_1(r))\mathrm{Re}\left(\tfrac{i}{2}\langle\partial_2\varphi(\widetilde{k}),(H_{\mathrm{per}}-E)(\widetilde{k})\partial_1\varphi(\widetilde{k})\rangle_{\mathcal{H}_{\mathrm{f}}}\right), \end{aligned}$$

where $\widetilde{k}=k-A(r)$.
Hence taking into account the modifications of Corollary 4.4.9, the subprincipal symbol of h_{eff} is

$$\begin{aligned} h_{\mathrm{eff}1}(k,r) &= i\mathcal{A}_1(\widetilde{k})(\partial_1\Phi(r)+\partial_2E(\widetilde{k})\partial_2A_1(r)-\partial_2E(\widetilde{k})\partial_1A_2(r)) \\ &\quad +i\mathcal{A}_2(\widetilde{k})(\partial_2\Phi(r)-\partial_1E(\widetilde{k})\partial_2A_1(r)+\partial_1E(\widetilde{k})\partial_1A_2(r)) \\ &\quad +\tfrac{\theta}{2\pi}\langle\gamma^1,k\rangle\left(\gamma_1^2\partial_{r_1}(E(\widetilde{k})+\Phi(r))+\gamma_2^2\partial_{r_2}(E(\widetilde{k})+\Phi(r))\right) \\ &\quad +(\partial_2A_1(r)-\partial_1A_2(r))\mathrm{Re}\left(\tfrac{i}{2}\langle\partial_1\varphi(\widetilde{k}),(H_{\mathrm{per}}-E)(\widetilde{k})\partial_2\varphi(\widetilde{k})\rangle_{\mathcal{H}_{\mathrm{f}}}\right) \\ &\quad +(\partial_1A_2(r)-\partial_2A_1(r))\mathrm{Re}\left(\tfrac{i}{2}\langle\partial_2\varphi(\widetilde{k}),(H_{\mathrm{per}}-E)(\widetilde{k})\partial_1\varphi(\widetilde{k})\rangle_{\mathcal{H}_{\mathrm{f}}}\right). \end{aligned}$$

Here we need to note that from the gauge $\langle A,\gamma^2\rangle=0$ we get

$$\partial_jA_1\gamma_1^2+\partial_jA_2\gamma_2^2=0 \quad \text{for} \quad j=1,2.$$

This yields

$$\begin{aligned}
&\gamma_1^2\partial_{r_1}E(k-A(r))+\gamma_2^2\partial_{r_2}E(k-A(r))\\
&= \quad -\partial_1E(k-A(r))\underbrace{\gamma_1^2\partial_1A_1(r)}_{=-\gamma_2^2\partial_1A_2(r)}-\partial_2E(k-A(r))\gamma_1^2\partial_1A_2(r)\\
&\qquad -\partial_1E(k-A(r))\gamma_2^2\partial_2A_1(r)-\partial_2E(k-A(r))\underbrace{\gamma_2^2\partial_2A_2(r)}_{=-\gamma_1^2\partial_2A_1(r)}\\
&= \quad \gamma_1^2(-\partial_2E(k-A(r))\partial_1A_2(r)+\partial_2E(k-A(r))\partial_2A_1(r))\\
&\qquad +\gamma_2^2(\partial_1E(k-A(r))\partial_1A_2(r)-\partial_1E(k-A(r))\partial_2A_1(r)).
\end{aligned}$$

Thus, we get for the leading orders of the symbol of the effective Hamiltonian:

Theorem 5.3.1. *The principal and subprincipal symbol of the semiclassical symbol $h_{\text{eff}}=h_0+\varepsilon h_1+\mathcal{O}(\varepsilon^2)$ from Theorem 5.1.1 are*

$$h_0(k,r)=E(k-A(r))+\Phi(r)$$

and

$$\begin{aligned}
h_1(k,r) &= i(\mathcal{A}_1(\widetilde{k})-\tfrac{i\theta}{2\pi}\langle\gamma^1,k\rangle\gamma_1^2)(\partial_1\Phi(r)+\partial_2E(\widetilde{k}))(\partial_2A_1(r)-\partial_1A_2(r))\\
&\quad +i(\mathcal{A}_2(\widetilde{k})-\tfrac{i\theta}{2\pi}\langle\gamma^1,k\rangle\gamma_2^2)(\partial_2\Phi(r)-\partial_1E(\widetilde{k}))(\partial_2A_1(r)-\partial_1A_2(r))\\
&\quad +(\partial_2A_1(r)-\partial_1A_2(r))\mathrm{Re}\left(\tfrac{i}{2}\langle\partial_1\varphi(\widetilde{k}),(H_{\text{per}}-E)(\widetilde{k})\partial_2\varphi(\widetilde{k})\rangle_{\mathcal{H}_{\text{f}}}\right)\\
&\quad +(\partial_1A_2(r)-\partial_2A_1(r))\mathrm{Re}\left(\tfrac{i}{2}\langle\partial_2\varphi(\widetilde{k}),(H_{\text{per}}-E)(\widetilde{k})\partial_1\varphi(\widetilde{k})\rangle_{\mathcal{H}_{\text{f}}}\right)\\
&=: (\nabla\Phi(r)-\nabla E(\widetilde{k})\times B(r))\cdot i(\mathcal{A}(\widetilde{k})-\tfrac{i\theta}{2\pi}\langle\gamma^1,k\rangle\gamma^2)-B(r)\cdot\mathcal{M}(\widetilde{k}),
\end{aligned}$$

where $\widetilde{k}=k-A(r)$ and $B(r)=\partial_1A_2-\partial_2A_1$.

5.4 The Hofstadter model

In this section, we want to connect our results with the Hofstadter model, which was, as the name suggests, first explored by Douglas R. Hofstadter in [Hof76]. Until today, the Hofstadter model is an object of interest for mathematical and physical research, see for example [AJ06] and [GA03] and references therein. It is given by the discrete magnetic Laplacian

$$H^B=\sum_{|a|=1}T_a^B$$

acting on $l^2(\mathbb{Z}^2)$. Here for $j, a \in \mathbb{Z}^2$, the magnetic translation with respect to the constant magnetic field B is defined by $(T_a^B x)_j = e^{\frac{i}{2}B(-a_1 j_2 + a_2 j_1)} x_{j-a}$. Setting $u := T^B_{(1,0)}$ and $v := T^B_{(0,1)}$, we see that the Hamiltonian is of the form $H^B = u + u^* + v + v^*$ and, moreover, $uv = e^{-iB} vu$ holds. This motivates the following, more general definition: Let $\mathcal{H}$ be a Hilbert space and let U and V be some unitary operators in $\mathcal{L}(\mathcal{H})$ that fulfil $UV = e^{-2\pi i \rho} VU$ with $\rho \in \mathbb{R}$. Then we call the operator

$$H = U + V + U^* + V^*$$

Hofstadter-Hamiltonian. Moreover, any operator of the form

$$H = \sum_{n,m \in \mathbb{Z}^2} c_{nm} U^n V^m,$$

with complex c_{nm}, is called Hofstadter-like Hamiltonian. Before we link this to our results, we want to imbed it into a more general structure. For this whole section, compare also Chapter 2.3 and 3.3.7 of [DeN10].

The main tool we need is the notion of the "Non-Commutative Torus", short NCT, or "Rotation C^*-Algebra". The NCT was first introduced by Connes [Con80] and an detailed monograph can be found in [Boc01]. Let u and v be two abstract elements satisfying $u^* = u^{-1}$ and $v^* = v^{-1}$ with respect to an involution $*$ and $uv = e^{-2\pi i \rho} vu$ with $\rho \in \mathbb{R}$. Then $\mathfrak{L}_\rho$, which consists of all finite, complex linear combinations of $u^n v^m$ with $n, m \in \mathbb{Z}$, can be endowed with the norm $\|a\| := \sup_\pi \{ \|\pi(a)\|_{\mathcal{L}(\mathcal{H})} : \pi : \mathfrak{L}_\rho \to \mathcal{L}(\mathcal{H}) \text{ is a } *\text{-representation}\}$. The unital C^*-algebra $\mathfrak{U}_\rho$ is then defined as the completion of $\mathfrak{L}_\rho$ with respect to this norm and ρ is called deformation parameter. One can show that the spectrum of the element $u + u^* + v + v^*$ is the Hofstadter butterfly.

An important property of the NCT $\mathfrak{U}_\rho$ is the so-called "surjective representation property": Let U and V be two unitary operators acting on a Hilbert space $\mathcal{H}$ that fulfil $UV = e^{-2\pi i \rho} VU$. Let moreover $C^*(U, V)$ be the C^*-algebra generated by U and V in $\mathcal{L}(\mathcal{H})$. Then the surjective representation property assures that the map π with $\pi(u) = U$ and $\pi(v) = V$ extends algebraically to a representation (a $*$-morphism) $\pi : \mathfrak{U}_\rho \to C^*(U, V)$ which is surjective.

So note that if we have an arbitrary representation like this, it is enough to show that it is injective to conclude that it is a $*$-isomorphism. This will get important for us because it implies that $\sigma(\pi(a)) = \sigma(a)$ for all $a \in \mathfrak{U}_\rho$.

Now we show how the effective Hamiltonian H_{eff} that we have derived can be perceived as Hofstadter-like Hamiltonian. For the weak perturbation A we take the linear potential of a constant magnetic field with our gauge $A(y) = (-by_2, 0)^{\mathrm{T}}$, which is adapted to the choice of φ from Lemma 3.3.2. Moreover, we set $\Phi \equiv 0$ so that the full Hamiltonian now reads

$$H^\varepsilon = \tfrac{1}{2}(-i\nabla_x - A_0(x) - A(\varepsilon x))^2 + V_\Gamma(x)$$

respectively after magnetic Bloch-Floquet transformation

$$H_{\mathrm{BF}}^{\varepsilon} = \widehat{H_0}^{\tau} \quad \text{with} \quad H_0(k,r) = \tfrac{1}{2}(-i\nabla_y - A_0(y) + k - A(r))^2 + V_\Gamma(y).$$

Now we assume that Theorem 5.1.1 holds also for this case and hence the leading order of the effective Hamiltonian is given by the Peierls substitution, so we formally get as effective Hamiltonian with accurancy $N_0 = 1$ for $H_{\mathrm{BF}}^{\varepsilon}$

$$H_{\mathrm{eff}} = E(k - A(i\varepsilon\nabla_k^{\mathrm{eff}})) \in \mathcal{L}(\mathcal{H}_\theta) \quad \text{with} \quad \nabla_k^{\mathrm{eff}} = (\partial_{k_1}, \partial_{k_2} + \tfrac{i\theta}{2\pi}k_1)^{\mathrm{T}}.$$

We want to show how we can perceive this effective operator as Hofstadter-like Hamiltonian. Thereto, we first must define appropriate operators U and V in $\mathcal{L}(\mathcal{H}_\theta)$. We can formally write

$$H_{\mathrm{eff}} = E(\mathcal{K}_1, \mathcal{K}_2)$$

where

$$\mathcal{K}_1 = k_1 + ib\varepsilon\partial_{k_2} - \tfrac{\theta}{2\pi}b\varepsilon k_1$$

and

$$\mathcal{K}_2 = k_2$$

with domains $\mathrm{D}(\mathcal{K}_1) = H^1_{\mathrm{loc}}(\mathbb{R}^2)$ and $\mathrm{D}(\mathcal{K}_2) = L^2_{\mathrm{loc}}(\mathbb{R}^2)$. Note that it holds

$$[\mathcal{K}_1, \mathcal{K}_2] = i\varepsilon b. \tag{5.21}$$

Using those operators, we introduce the Hofstadter unitaries $U := e^{i\mathcal{K}_1}$ and $V := e^{i\mathcal{K}_2}$. They act on $\mathcal{H}_\theta$ as

$$U\psi(k) = e^{i(1-\frac{\theta}{2\pi}b\varepsilon)k_1}\psi(k_1, k_2 - \varepsilon b)$$

and

$$V\psi(k) = e^{ik_2}\psi(k).$$

Moreover, a short calculation shows

$$UV = e^{-i\varepsilon b}VU = e^{-2\pi i\left(\frac{\varepsilon b}{2\pi}\right)}VU. \tag{5.22}$$

To get the operator as a power series, let $E(k) = \sum_{n,m\in\mathbb{Z}} c_{n,m}e^{ik_1 n}e^{ik_2 m}$ be the Fourier expansion of the $(2\pi\mathbb{Z})^2$-periodic band function E. Then, bearing in mind (5.21), we define the "Peierls quantisation" of the symbol E as

$$H_{\mathrm{eff}}^{\varepsilon} = E(\mathcal{K}_1, \mathcal{K}_2) := \sum_{n,m\in\mathbb{Z}} c_{n,m}e^{\frac{nmi\varepsilon b}{2}}U^n V^m. \tag{5.23}$$

Any operator of this form is a Hofstadter-like Hamiltonian and well-defined. If the band function is of the form $E(k) = 2\cos(k_1) + 2\cos(k_2)$, the corresponding effective operator $H_{\text{eff}}^{\text{Hof}} = U + U^* + V + V^*$ is a Hofstadter Hamiltonian.
Our next goal is to show that the spectrum of the operator $H_{\text{eff}}^{\text{Hof}}$ is the Hofstadter butterfly. In the non-magnetic case case $A_0 \equiv 0$ this is described at length in [DeN10]. We will just sketch the relevant details for our purpose and point out the small modification of the proofs. It is clear that the elements of the NCT $\mathfrak{U}_\rho$ are connected with the operators on $\mathcal{H}_\theta$ by the Hofstadter representation π_{Hof}. This is the representation generated by $\pi_{\text{Hof}}(u) = U$ and $\pi_{\text{Hof}}(v) = V$, where U and V are the above defined Hofstadter unitaries acting on $\mathcal{H}_\theta$. The surjective representation property and equation (5.22) assure that $\pi_{\text{Hof}} : \mathfrak{U}_\rho \to C^*(U, V)$ is a surjective $*$-morphism. We need to show that the Hofstadter representation π_{Hof} is a $*$-isomorphism. Hence we have to prove the following Lemma.

Lemma 5.4.1. *For any $\rho \in \mathbb{R}$, the Hofstadter representation $\pi_{\text{Hof}} : \mathfrak{U}_\rho \to C^*(U, V)$ is injective.*

Proof.
The proof follows the line of the proof of Lemma 2.3.2 in [DeN10]. The first step is to define the GNS representation π_{GNS} of $\mathfrak{U}_\rho$ relative to a faithful state $\oint$. The faithfulness of the state $\oint$ implies the faithfulness of the representation π_{GNS}. The second step is to show that π_{GNS} and π_{Hof} are unitarily equivalent, which implies that π_{Hof} is injective.
We just sketch the first part: the state $\oint$ is the linear extension of the map defined by $\oint u^n v^m := \delta_{n,o}\delta_{m,o}$ and is faithful. Hence $\langle a, b\rangle_{\text{GNS}} := \oint a^* b$ defines a scalar product on $\mathfrak{U}_\rho$ and turns it into a pre-Hilbert space whose completion is called $\mathcal{H}_{\text{GNS}}$. The representation $\pi_{\text{GNS}} : \mathfrak{U}_\rho \to \mathcal{L}(\mathcal{H}_{\text{GNS}})$ is defined as follows: For $a \in \mathfrak{U}_\rho$, $\pi_{\text{GNS}}(a)$ is defined on the dense subset $\mathfrak{U}_\rho$ by $\pi_{\text{GNS}}(a)b := ab$.
Now we go on with the second part of the proof. Let $\xi_{n,m} := e^{2\pi i \rho n m} u^n v^m \in \mathcal{H}_{\text{GNS}}$ for all $n, m \in \mathbb{Z}$. Then the set $\{\xi_{n,m} : n, m \in \mathbb{Z}\}$ forms an orthonormal basis of $\mathcal{H}_{\text{GNS}}$ with

$$\begin{aligned} \pi_{\text{GNS}}(u)\xi_{n,m} &= e^{-2\pi i \rho m}\xi_{n+1,m} \\ \pi_{\text{GNS}}(v)\xi_{n,m} &= \xi_{n,m+1}. \end{aligned}$$

Now let $\rho := \frac{\varepsilon b}{2\pi}$ and $\varphi_{n,m} := \frac{1}{2\pi} e^{-\frac{i\theta}{2\pi}k_1 k_2} e^{ik_1 n + ik_2 m}$. Then the set $\{\varphi_{n,m} : n, m \in \mathbb{Z}\}$ forms an orthonormal basis of $\mathcal{H}_\theta$ so that

$$\begin{aligned} U\varphi_{n,m} &= e^{-2\pi i \rho m}\varphi_{n+1,m} \\ V\varphi_{n,m} &= \varphi_{n,m+1}. \end{aligned}$$

Hence the unitary map $W : \mathcal{H}_{\text{GNS}} \to \mathcal{H}_\theta$ defined by $W\xi_{n,m} = \varphi_{n,m}$ intertwines the representations π_{GNS} and π_{Hof}. So the proof works almost analogously as the one in [DeN10], the only difference being the different choice of the basis $\{\varphi_{n,m} : n, m \in \mathbb{Z}\}$ of $\mathcal{H}_\theta$. This choice is necessary because of the additional phase $e^{-\frac{i\theta}{2\pi} b\varepsilon k_1}$ in the definition of the Hofstadter unitary U. □

We get that any Hofstadter-like operator (5.23) is realized as $\pi_{\mathrm{Hof}}(a)$ with a self-adjoint element a in $\mathfrak{U}_\rho$ and for the spectra we have $\sigma(\pi_{\mathrm{Hof}}(a)) = \sigma(a)$. In particular, we see that the spectrum of $H^{\mathrm{Hof}}_{\mathrm{eff}}$ is the Hofstadter butterfly (without colours).

So as in the non-magnetic case $A_0 \equiv 0$, we get an effective Hamiltonian H_{eff} given as Peierls substitution with accurancy $N_0 = 1$, which can be written as $H_{\mathrm{eff}} = \sum_{n,m\in\mathbb{Z}} c_{n,m} e^{\frac{nmi\varepsilon b}{2}} U^n V^m$ and particularly for the form $E = 2\cos(k_1) + 2\cos(k_2)$ of the band function as $H^{\mathrm{Hof}}_{\mathrm{eff}} = U + U^* + V + V^*$, where the Hofstadter unitaries U and V fulfil $UV = e^{-2\pi i\rho} VU$. The differences to the non-magnetic case $A_0 \equiv 0$ are of course the reference space which is now $\mathcal{H}_\theta$ and not $L^2(\mathbb{T}^{2*})$ any more and that in the Peierls substitution we have to insert the non-trivial connection ∇^{eff}_k. This implies that the Hofstadter unitaries look differently. Now we want to relate our results and the Hofstadter model to the Quantum Hall effect.

Remark 5.4.2. The connection with the Quantum Hall effect

The Quantum Hall effect was first discovered in 1980 by von Klitzing [vKDP80], who was awarded the nobel price for it. Two years later, the authors of [TKNN82] were the first to realise the coherency between the quantised values of the Hall conductance and topological quantum numbers. A nice overview about the history of the Quantum Hall effect and the development of its mathematical interpretation can be found in [AOS03] and [AO08].

The mathematical model is given by the Hamiltonian

$$H^{\mathrm{QHE}} = \tfrac{1}{2}(-i\nabla_x - \tfrac{B}{2}(-x_2, x_1)^{\mathrm{T}})^2 + V_\Gamma(x),$$

where B is a uniform magnetic field and V_Γ is periodic with respect to the lattice $\Gamma \cong \mathbb{Z}^2$.

The procedure suggested in [TKNN82] is to study the Quantum Hall effect by studying simpler, effective models. In [TKNN82], the Hamiltonian $H^{\mathrm{QHE}}_{A_0}$ is analysed for two different limits: the limit $\varepsilon \to 0$, which means that the magnetic field B is weak compared to V_Γ (the so-called Hofstadter-regime), and in the limit $\frac{1}{\varepsilon} \to 0$, which means that the magnetic field dominates all other interactions (the so-called Harper regime). Then one uses the Kubo-formula to compute the Hall conductance associated to spectral gaps as the Chern-number of underlying vector bundles which are related to spectral projections of the accordant effective Hamiltonians. For suitable values of the adiabatic parameters these effective models share the same spectral structure and the corresponding Chern numbers are related by a Diophantine equation sometimes referred to as ”TKNN-equations“.

However, the paper does not provide rigorous mathematical justifications for everything, and some essential mathematical gaps have been filled only recently in [DeN10], among them are the limit of the weak magnetic field using space-adiabatic perturbation theory and a rigorous justification of the TKNN-equations. A nice introduction to the coloured quantum butterflies seen as thermodynamic phase diagrams can also be found in [Avr03].
It should now be clear that our results fit into this framework: they can be perceived as a derivation of a unitarily effective model for the Hamiltonian H^{QHE} with a perturbed magnetic field $B + \delta B$ in the limit $\delta B \to 0$.

Appendix A

Magnetic Sobolev spaces

In this paragraph, we quickly recall the definition and some properties of magnetic Sobolev spaces as Hilbert spaces in order to establish the self-adjointness of our Hamiltonian (2.3) on the domain $H^2_{A_0}(\mathbb{R}^d)$. Proofs can be found in [Sti11] and for a more detailed presentation of magnetic Sobolev spaces we refer the reader to [GMS91] and [IMP07].

Definition A.0.3. *Let $A \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ satisfy $\sup_{x\in\mathbb{R}^d} |\partial_x^\alpha A(x)| \leq c_\alpha$ for all $\alpha \in \mathbb{N}^d \setminus \{0\}$. Then for $m \in \mathbb{N}_0$*

$$H^m_A(\mathbb{R}^d) := \{f \in L^2(\mathbb{R}^d) : (-i\partial_x - A)^\alpha f \in L^2(\mathbb{R}^d) \ \text{ for all } \ |\alpha| \leq m\}$$

is called the m^{th} magnetic Sobolev space. The inner product is defined by

$$\langle f, g\rangle_{H^m_A} := \sum_{|\alpha|\leq m} \langle (-i\partial_x - A)^\alpha f, (-i\partial_x - A)^\alpha g\rangle_{L^2(\mathbb{R}^d)}.$$

Proposition A.0.4. *Under the above assumptions, $H^m_A(\mathbb{R}^d)$ is a Hilbert space with dense subset $S(\mathbb{R}^d)$.*

Lemma A.0.5. *Let $L = \sum_{|\alpha|\leq 2m} a_\alpha(x)(-i\partial_x - A)^\alpha$ with $a_\alpha \in C^\infty_{\mathrm{b}}(\mathbb{R}^d, \mathbb{R})$ an elliptic differential operator, that is to say $\sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha \geq c|\xi|^{2m}$ for all $x \in \mathbb{R}^d$ and $\xi \neq 0$.*
Then the $(2m)^{th}$ Sobolev norm $\| \ \|_{H^{2m}_A}$ is equivalent to the graph norm

$$\| \ \|_L := (\| \ \|^2_{L^2(\mathbb{R}^d)} + \|L(\cdot)\|^2_{L^2(\mathbb{R}^d)})^{\frac{1}{2}}.$$

Corollary A.0.6. *Under the assumptions from Lemma A.0.5, we have $L \in \mathcal{L}(H^{2m}_A(\mathbb{R}^d), L^2(\mathbb{R}^d))$.*

Proposition A.0.7. *Under the above assumptions, it holds that if L is symmetric, the operator $L : H^{2m}_A(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is self-adjoint.*

Proposition A.0.8. *Let* $H^\varepsilon = \frac{1}{2}(-i\nabla_x - A_0(x) - A(\varepsilon x))^2 + V_\Gamma(x) + \Phi(\varepsilon x)$ *and let Assumption 1 hold. Then* $H^\varepsilon : H^2_{A_0}(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ *is self-adjoint.*

Proof.
Let $H := \frac{1}{2}(-i\nabla_x - A_0)^2 + V_\Gamma$, $T_1 := \frac{1}{2}A^2(\varepsilon x) + \Phi(\varepsilon x)$ and $T_2 := A(\varepsilon x)(i\nabla_x + A_0(x)) + \frac{i}{2}(\nabla_x \cdot A(\varepsilon x))$. Then

$$H^\varepsilon = H + T_1 + T_2$$

holds. So standard arguments show the claim, see for example [Kat66]. □

Appendix B

Weyl calculus

In this appendix, we want to give a short derivation of the τ-quantisation following the line of [Teu03, PST03b]. Thereto, we first sketch the operator-valued Weyl calculus and the Weyl product. Then we show how the τ-quantisation can be obtained by using suitable symbols and restricting to suitable functions. Another aim of this appendix is to fix notation and state important formulas, for example the Weyl product. For a more detailed presentation we refer the reader to [Teu03], Appendix A and B.

B.1 Operator-valued Weyl calculus

The theory of pseudodifferential operators can be generalised to operator-valued symbols. This is pointed out in many textbooks about the subject, for example in [Hör85, Fol89, DS99]. We want to state the formulas and results that are important for us. When it comes to symbol classes, we do not only use the usual Hörmander symbol classes, see [Fol89, Hör85]. We also need the symbol classes related to order functions that can be found for example in [DS99, Mar02].
Let in the following be

- $\langle p \rangle := (1 + |p|^2)^{\frac{1}{2}}$,
- $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2$, and $\mathcal{H}_3$ separable Hilbert spaces,
- $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ the space of continuous linear maps from $\mathcal{H}_1$ to $\mathcal{H}_2$ and $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$.

We first introduce the notion of an order function:

Definition B.1.1. *A function $w : \mathbb{R}^{2d} \to [0, \infty)$ is called "order function" if there exist constants $C > 0$ and $N > 0$ such that for every $x, y \in \mathbb{R}^{2d}$ it holds*

$$w(x) \leq C \langle x - y \rangle^N w(y).$$

Now we define the spaces of symbols:

Definition B.1.2. *Let $m \in \mathbb{R}$ and $0 \leq \rho \leq 1$. Then we define the symbol class $S^m_\rho(\mathbb{R}^{2d}, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$ by*

$$\begin{aligned} S^m_\rho(\mathbb{R}^{2d}, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)) \;:=\; & \{f \in C^\infty(\mathbb{R}^{2d}, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)) : \forall \alpha, \beta \in \mathbb{N}^d \;\; \exists C_{\alpha,\beta} > 0 : \\ & \sup_{q \in \mathbb{R}^d} \left\| (\partial^\alpha_q \partial^\beta_p f)(q,p) \right\|_{\mathcal{L}(\mathcal{H}_1 \mathcal{H}_2)} \leq C_{\alpha,\beta} \langle p \rangle^{m - \rho|\beta|} \; \forall p \in \mathbb{R}^d \;\}. \end{aligned}$$

Definition B.1.3. *Let $w : \mathbb{R}^{2d} \to [0, \infty)$ be an order function. Then the symbol class $S^w(\mathbb{R}^{2d}, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$ is defined by*

$$\begin{aligned} S^w(\mathbb{R}^{2d}, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)) \;:=\; & \{f \in C^\infty(\mathbb{R}^{2d}, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)) : \forall \alpha, \beta \in \mathbb{N}^d \;\; \exists C_{\alpha,\beta} > 0 : \\ & \left\| (\partial^\alpha_q \partial^\beta_p f)(q,p) \right\|_{\mathcal{L}(\mathcal{H}_1 \mathcal{H}_2)} \leq C_{\alpha,\beta} w(q,p) \;\; \forall q, p \in \mathbb{R}^d \;\}. \end{aligned}$$

Remark B.1.4. The just introduced symbol classes are Fréchet spaces with respect to the families of seminorms given through the respective minimal constants $C_{\alpha,\beta}$. Note also that $S^m_{\rho=0} = S^{w=\langle p \rangle^m}$.

Now we give the formula for the quantisation of the symbols.

Definition B.1.5. *Let $f \in S^m_\rho(\mathbb{R}^{2d}, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)) \cup S^w(\mathbb{R}^{2d}, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$ and $\psi \in S(\mathbb{R}^d, \mathcal{H}_1)$. Then the Weyl quantisation $\widehat{f}$ of f is given by*

$$(\widehat{f}\, \psi)(k) = \tfrac{1}{(2\pi\varepsilon)^d} \int_{\mathbb{R}^d} e^{\frac{i(k-y)r}{\varepsilon}} f\left(\tfrac{k+y}{2}, r\right) \psi(y) \mathrm{d}r \mathrm{d}y.$$

Proposition B.1.6. *Let $f \in S^m_\rho(\mathbb{R}^{2d}, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)) \cup S^w(\mathbb{R}^{2d}, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$. Then $\widehat{f}$ is a continuous mapping from $S(\mathbb{R}^d, \mathcal{H}_1)$ to $S(\mathbb{R}^d, \mathcal{H}_2)$.*

This mapping can be extended by duality to a continuous mapping between the respective dual spaces. Thereto we use the anti-linear inclusion $S(\mathbb{R}^d, \mathcal{H}) \hookrightarrow S'(\mathbb{R}^d, \mathcal{H})$ given by

$$S(\mathbb{R}^d, \mathcal{H}) \ni \psi \mapsto T_\psi \in S'(\mathbb{R}^d, \mathcal{H}) \quad \text{with} \quad T_\psi(\varphi) = \int_{\mathbb{R}^d} \langle \psi(x), \varphi(x) \rangle_{\mathcal{H}} \mathrm{d}x.$$

The extension of the map $\widehat{f}$ is, for $T \in S'(\mathbb{R}^d, \mathcal{H}_1)$ and $\varphi \in S(\mathbb{R}^d, \mathcal{H}_2)$, given by

$$\widehat{f}\,(T)(\varphi) := T(\widehat{f^*}\varphi),$$

where f^* denotes the pointwise adjoint of f.
We conclude this section with two important properties of the Weyl quantisation. The first one identifies symbols whose quantisation is a bounded operator:

Theorem B.1.7. *Let $f \in S^{w=1}(\mathbb{R}^{2d}, \mathcal{L}(\mathcal{H}))$. Then*

$$\widehat{f} \in \mathcal{L}(L^2(\mathbb{R}^d, \mathcal{H})).$$

Remark B.1.8. Note that $S_\rho^{m=0}(\mathbb{R}^{2d}, \mathcal{L}(\mathcal{H})) \subset S^{w=1}(\mathbb{R}^{2d}, \mathcal{L}(\mathcal{H}))$.

The second property is the usual property of the Weyl quantisation: The adjoint of the quantised symbol should be the quantisation of the pointwise adjoint of the symbol.

Proposition B.1.9. *Let $f \in S_\rho^m(\mathbb{R}^{2d}, \mathcal{L}(\mathcal{H})) \cup S^w(\mathbb{R}^{2d}, \mathcal{L}(\mathcal{H}))$ and let moreover $\widehat{f} \in \mathcal{L}(L^2(\mathbb{R}^d, \mathcal{H}))$. Then we have*

$$\left(\widehat{f}\,\right)^* = \widehat{f^*}.$$

B.2 Weyl product and semiclassical symbols

The most important observation about pseudodifferential operator theory is that one can define an associative product in the space of symbols that corresponds to the product of operators. We are going to call it the Weyl or the Moyal product. Nevertheless, we first regard the pointwise product.

Proposition B.2.1. *(i) For two symbols $f \in S_\rho^{m_1}(\mathbb{R}^{2d}, \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3))$ and $g \in S_\rho^{m_2}(\mathbb{R}^{2d}, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$, $f \cdot g$ is a symbol in $S_\rho^{m_1+m_2}(\mathbb{R}^{2d}, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_3))$.*

(ii) Let $f \in S^{w_1}(\mathbb{R}^{2d}, \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3))$, $g \in S^{w_2}(\mathbb{R}^{2d}, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$. Then $f \cdot g$ is a symbol in $S^{w_1 \cdot w_2}(\mathbb{R}^{2d}, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_3))$.

Now we turn to the Weyl product. The following theorem states that for suitable symbols, the product of to pseudodifferential operators is again a pseudodifferential operator whose symbol is given by the Weyl product of the symbols of the two pseudodifferential operators.

Proposition B.2.2. *Let $f \in S_\rho^{m_1}(\mathbb{R}^{2d}, \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3))$ and $g \in S_\rho^{m_2}(\mathbb{R}^{2d}, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$ or let $f \in S^{w_1}(\mathbb{R}^{2d}, \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3))$and $g \in S^{w_2}(\mathbb{R}^{2d}, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$. Then $\widehat{f} \circ \widehat{g} = \widehat{h}$ with $h \in S_\rho^{m_1+m_2}(\mathbb{R}^{2d}, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_3))$ respectively $h \in S^{w_1 w_2}(\mathbb{R}^{2d}, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_3))$ given by*

$$h(q,p) = \exp\left(\tfrac{i\varepsilon}{2}(\nabla_p \cdot \nabla_x - \nabla_\xi \cdot \nabla_q)\right) f(q,p) g(x,\xi)|_{x=q,\xi=p} =: f \sharp g.$$

The symbol $f \sharp g$ is called the "Weyl product of the symbols f and g".

We see from the formula that the Weyl product can be extended in orders of ε. Hence it is convenient to introduce also semiclassical symbols which are classes of ε-dependent symbols that are close to a power series in ε of classical symbols in a way which will be specified below.

Definition B.2.3. *A map $f : [0, \varepsilon_0) \to S^m_\rho(\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$ with $\varepsilon \mapsto f_\varepsilon$ is called semiclassical symbol of order m and weight ρ if there exists a sequence $\{f_j\}_{j\in\mathbb{N}}$ with $f_j \in S^{m-j\rho}_\rho(\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$ such that for every $n \in \mathbb{N}$ we have*

$$\varepsilon^{-n}\left(f_\varepsilon - \sum_{j=0}^{n-1} \varepsilon^j f_j\right) \in S^{m-n\rho}_\rho(\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)) \tag{B.1}$$

uniformly in ε in the following sense: For any $k \in \mathbb{N}$ there exists a constant $C_{n,k}$ such that for every $\varepsilon \in [0, \varepsilon_0)$ one has

$$\left\| f_\varepsilon - \sum_{J=0}^{n-1} \varepsilon^j f_j \right\|_k^{(m-n\rho)} \leq C_{n,k}\varepsilon^n,$$

where $\|\cdot\|_k^m$ is the k^{th} Fréchet semi-norm in $S^m_\rho(\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$. The space of these semiclassical symbols is denoted by $S^m_\rho(\varepsilon, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$ or simply by $S^m_\rho(\varepsilon)$.

Analogously we define

Definition B.2.4. *A map $f : [0, \varepsilon_0) \to S^w(\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$ with $\varepsilon \mapsto f_\varepsilon$ is called semiclassical symbol if there exists a sequence $\{f_j\}_{j\in\mathbb{N}}$ with $f_j \in S^w(\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$ such that for every $n \in \mathbb{N}$ we have*

$$\varepsilon^{-n}\left(f_\varepsilon - \sum_{j=0}^{n-1} \varepsilon^j f_j\right) \in S^w(\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$$

uniformly in ε in the following sense: For any $k \in \mathbb{N}$ there exists a constant $C_{n,k}$ such that for every $\varepsilon \in [0, \varepsilon_0)$ one has

$$\left\| f_\varepsilon - \sum_{j=0}^{n-1} \varepsilon^j f_j \right\|_k \leq C_{n,k}\varepsilon^n, \tag{B.2}$$

where $\|\cdot\|_k$ is the k^{th} Fréchet semi-norm in $S^w(\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$. The space of these semiclassical symbols is denoted by $S^w(\varepsilon, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$ or simply by $S^w(\varepsilon)$.

f_0 is called the "principal symbol" and f_1 is called the "subprincipal symbol" of f_ε. If the condition (B.1) respectively (B.2) is fulfilled, one writes

$$f_\varepsilon \asymp \sum_{j\geq 0} \varepsilon^j f_j \quad \text{in} \quad S^m_\rho(\varepsilon) \quad \text{respectively} \quad S^w(\varepsilon)$$

and says that f_ε is asymptotically equivalent to the series $\sum_{j\geq 0} \varepsilon^j f_j$ in $S^m_\rho(\varepsilon)$ respectively $S^w(\varepsilon)$.

A formal power series $\sum_{j\geq 0}\varepsilon^j f_j$ is in general not convergent. Yet it is always the expansion of a (non-unique) symbol f_ε. So one can construct a formal power series and knows a priori that there is a semiclassical symbol that is asymptotically equivalent to it.

Proposition B.2.5. *Let $\{f_j\}_{j\in\mathbb{N}}$ be a sequence such that $f_j \in S^{m-j\rho}_\rho$ respectively $f_j \in S^w$. Then there exists $f_\varepsilon \in S^m_\rho(\varepsilon)$ respectively $S^w(\varepsilon)$ such that $f \asymp \sum_{j\geq 0}\varepsilon^j f_j$ in S^m_ρ respectively S^w and f_ε is unique up to $\mathcal{O}(\varepsilon^\infty)$. The symbol f_ε is called a "resummation" of the formal symbol $\sum_{j\geq 0}\varepsilon^j f_j$.*

The Weyl product of two semiclassical symbols is again a semiclassical symbol with an explicit asymptotic expansion:

Proposition B.2.6. *Let*

$$f_\varepsilon \asymp \sum_{j\geq 0}\varepsilon^j f_j \quad \text{in } S^{m_1}_\rho(\varepsilon,\mathcal{L}(\mathcal{H}_3,\mathcal{H}_2)) \text{ respectively } S^{w_1}(\varepsilon,\mathcal{L}(\mathcal{H}_3,\mathcal{H}_2))$$

and

$$g_\varepsilon \asymp \sum_{j\geq 0}\varepsilon^j g_j \quad \text{in } S^{m_2}_\rho(\varepsilon,\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)) \text{ respectively } S^{w_2}(\varepsilon,\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)).$$

Then $f_\varepsilon \sharp g_\varepsilon \in S^{m_1+m_2}_\rho(\varepsilon,\mathcal{L}(\mathcal{H}_1,\mathcal{H}_3))$ respectively $S^w(\varepsilon,\mathcal{L}(\mathcal{H}_1,\mathcal{H}_3))$ has the asymptotic expansion

$$(f_\varepsilon \sharp g_\varepsilon)_k(q,p) = \sum_{|\alpha|+|\beta|+j+l=k} \tfrac{(-1)^\alpha}{(2i)^{|\alpha|+|\beta|}|\alpha|!|\beta|!}\left((\partial^\alpha_q\partial^\beta_p f_j)(\partial^\alpha_p\partial^\beta_q g_l)\right)(q,p). \tag{B.3}$$

The leading orders are

$$(f_\varepsilon \sharp g_\varepsilon)_0 = f_0 g_0$$

and

$$(f_\varepsilon \sharp g_\varepsilon)_1 = f_0 g_1 + f_1 g_0 - \tfrac{i}{2}\{f_0,g_0\},$$

where $\{\cdot,\cdot\}$ denotes the Poisson bracket.

For convenience, we also introduce spaces of formal power series.

Definition B.2.7. *Let*

$$M^m_\rho(\varepsilon,\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)) := \{\sum_{j\geq 0}\varepsilon^j f_j : f_j \in S^{m-j\rho}_\rho(\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2))\}$$

and

$$M^w(\varepsilon,\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)) := \{\sum_{j\geq 0}\varepsilon^j f_j : f_j \in S^w(\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2))\}.$$

In the context of formal power series, the product defined by (B.3) is called "Moyal product" and also denoted by $\sharp$.

We conclude this section with

Definition B.2.8. *Let R_ε and S_ε be two ε-dependent operators on $\mathcal{H}$. One says that $R_\varepsilon = S_\varepsilon + \mathcal{O}(\varepsilon^\infty)$ if for every $n \in \mathbb{N}$ there is a constant C_n such that*

$$\|R_\varepsilon - S_\varepsilon\|_{\mathcal{L}(\mathcal{H})} \leq C_n \varepsilon^n$$

for all $\varepsilon \in [0, \varepsilon_0)$. One says that R_ε is $\mathcal{O}(\varepsilon^\infty)$-close to S_ε.

Remark B.2.9. Often, the subscript ε of f_ε is dropped.

B.3 The τ-quantisation

In this section, we finally introduce the τ-quantisation. Our approach is to pass from the phase space $T^*\mathbb{R}^d = \mathbb{R}^{2d}$ to the phase space $T^*\mathbb{T}^d = \mathbb{T}^d \times \mathbb{R}^d$ by restricting the calculus to (more or less) periodic symbols. This approach has also been used in [GN98]. In our case, we deal with symbols and functions that are not exactly periodic, but τ-equivariant with respect to a representation τ of the group of lattice translations. This way, we get a pseudodifferential calculus for τ-equivariant symbols and we will see that the properties of the Weyl calculus described in the previous section carry over to the τ-calculus obtained that way.
In the following, let $\{\gamma^1, ..., \gamma^d\} \subset \mathbb{R}^d$ generate the Bravais lattice

$$\Gamma := \{\gamma \in \mathbb{R}^d : \gamma = \sum_{j=1}^d \lambda_j \gamma^j, \lambda_j \in \mathbb{Z} \quad \forall j = 1, ...d\}.$$

The centered fundamental cell of Γ is

$$M = \{x \in \mathbb{R}^d : x = \sum_{j=1}^d \alpha_j \gamma^j \quad \text{for} \quad \alpha_j \in [-\tfrac{1}{2}, \tfrac{1}{2}]\}.$$

τ is supposed to be a representation of Γ in $\mathcal{L}^*(\mathcal{H})$, the group of invertible elements of $\mathcal{L}(\mathcal{H})$, that means that it is a group homomorphism

$$\tau : \Gamma \to \mathcal{L}^*(\mathcal{H}), \quad \gamma \mapsto \tau(\gamma).$$

When we deal with two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we denote by $\tau = (\tau_1, \tau_2)$ a collection of such representations for the respective Hilbert spaces. Let moreover L_γ be the translation by the lattice vector γ; this means that for $\psi \in S(\mathbb{R}^d, \mathcal{H})$ we have $L_\gamma \psi(x) = \psi(x - \gamma)$ and for a distribution $T \in S'(\mathbb{R}^d, \mathcal{H})$ we have $L_\gamma(T)(\psi) = T(L_{-\gamma}\psi)$.

Definition B.3.1. *A distribution $T \in S'(\mathbb{R}^d, \mathcal{H})$ is called τ-equivariant if*

$$L_\gamma T = \tau(\gamma) T \quad \text{for all} \quad \gamma \in \Gamma,$$

where for $\varphi \in S(\mathbb{R}^d, \mathcal{H})$ we have $\tau(\gamma)T(\varphi) = T(\tau(\gamma)^{-1}\varphi)$. Denote by S'_τ the space of τ-equivariant distributions. Moreover, we define the Hilbert space

$$L^2_\tau(\mathbb{R}^d, \mathcal{H}) := \mathcal{H}_\tau := \{\psi \in L^2_{\mathrm{loc}}(\mathbb{R}^d, \mathcal{H}) : L_\gamma \psi = \tau(\gamma)\psi \quad \text{for all} \quad \gamma \in \Gamma\}$$

with inner product

$$\langle \varphi, \psi \rangle_{\mathcal{H}_\tau} = \int_M \langle \varphi(x), \psi(x) \rangle_{\mathcal{H}} \mathrm{d}x,$$

and the space

$$C^\infty_\tau := \{\psi \in C^\infty(\mathbb{R}^d, \mathcal{H}) : L_\gamma \psi = \tau(\gamma)\psi \quad \text{for all} \quad \gamma \in \Gamma\},$$

that is a dense subset of $\mathcal{H}_\tau$.

As mentioned before, we also need the notion of τ-equivariant symbols:

Definition B.3.2. *A symbol $f_\varepsilon \in S^w(\varepsilon, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)) \cup S^m_\rho(\varepsilon, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$ is called τ-equivariant (more precisely (τ_1, τ_2)-equivariant) if*

$$f_\varepsilon(q - \gamma, p) = \tau_2(\gamma) f_\varepsilon(q, p) \tau_1(\gamma)^{-1} \quad \text{for all} \quad \gamma \in \Gamma.$$

The symbol spaces of τ-equivariant symbols are denoted by $S^w_\tau(\varepsilon, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$ respectively $S^m_{\rho,\tau}(\varepsilon, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$.

Remark B.3.3. Note that the coefficients in the asymptotic expansion of a τ-equivariant symbol must be as well τ-equivariant.

Now we define the quantisation of a τ-equivariant symbol. The corresponding pseudodifferential operator should be an operator from $S'_{\tau_1}(\mathbb{R}^d, \mathcal{H}_1)$ to $S'_{\tau_2}(\mathbb{R}^d, \mathcal{H}_2)$. As indicated above, we just take the usual Weyl quantisation of the symbol and restrict it to τ-equivariant distributions. The only thing we need to show is that the Weyl quantisation maps τ-equivariant distributions to τ-equivariant distributions.

Proposition B.3.4. *Let $f \in S^w_\tau(\varepsilon, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)) \cup S^m_{\rho,\tau}(\varepsilon, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$. Then*

$$\widehat{f}\, S'_{\tau_1}(\mathbb{R}^d, \mathcal{H}_1) \subset S'_{\tau_2}(\mathbb{R}^d, \mathcal{H}_2).$$

To emphasise the fact that we quantise τ-equivariant symbols we also write $\widehat{f}^{\,\tau}$ instead of $\widehat{f}$.
So the formula for the τ-quantisation is given by Definition B.1.5. The next observation is that the pointwise, Weyl, and Moyal products of τ-equivariant symbols are again τ-equivariant.

Proposition B.3.5. *Let $f_\varepsilon \in S_\tau^{w_1}(\varepsilon, \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3))$ and $g \in S_\tau^{w_2}(\varepsilon, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$. Then $f_\varepsilon \cdot g_\varepsilon$ and $f_\varepsilon \sharp g_\varepsilon$ are in $S_\tau^{w_1 w_2}(\varepsilon, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_3))$.*
Analogously it holds for $f_\varepsilon \in S_{\rho,\tau}^{m_1}(\varepsilon, \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3))$ and $g \in S_{\rho,\tau}^{m_2}(\varepsilon, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$ that $f_\varepsilon \cdot g_\varepsilon$ and $f_\varepsilon \sharp g_\varepsilon$ are in $S_{\rho,\tau}^{m_1+m_2}(\varepsilon, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_3))$.

Note that the Weyl product is just the Weyl product introduced in Proposition B.2.2. An analogous statement holds for the Moyal product of formal symbols. Now we present a variant of the Calderon-Vaillancourt theorem for the Hilbert space $\mathcal{H}_\tau$.

Proposition B.3.6. *Let $f \in S_\tau^{w=1}(\mathcal{L}(\mathcal{H}))$ and τ_1 and τ_2 be unitary representations of Γ in $\mathcal{L}(\mathcal{H})$. Then*

$$\widehat{f}^{\,\tau} \in \mathcal{L}(\mathcal{H}_{\tau_1}, \mathcal{H}_{\tau_2})$$

and for $f_\varepsilon \in S_\tau^{w=1}(\varepsilon, \mathcal{L}(\mathcal{H}))$ we have that $\sup_{\varepsilon\in[0,\varepsilon_0)} \left\| \widehat{f_\varepsilon}^{\,\tau} \right\|_{\mathcal{L}(\mathcal{H}_{\tau_1}, \mathcal{H}_{\tau_2})} < \infty$.

Remark B.3.7. Note that $S_{\rho,\tau}^{m=0}(\mathcal{L}(\mathcal{H})) \subset S_\tau^{w=1}(\mathbb{R}^{2d}, \mathcal{L}(\mathcal{H}))$.

The last result presented is that for a symbol f, the adjoint of $\widehat{f}^{\,\tau}$ seen as an operator in $\mathcal{L}(\mathcal{H}_\tau)$ is the τ-quantisation of the pointwise adjoint of the symbol.

Proposition B.3.8. *Let $f \in S_\tau^{w}(\varepsilon, \mathcal{L}(\mathcal{H})) \cup S_{\rho,\tau}^{m}(\varepsilon, \mathcal{L}(\mathcal{H}))$ with a unitary representation τ (with $\tau_1 = \tau_2 = \tau$) and let $\widehat{f}^{\,\tau} \in \mathcal{L}(\mathcal{H}_\tau)$. Then*

$$\left(\widehat{f}^{\,\tau}\right)^* = \widehat{f^*}^{\,\tau}.$$

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