Localization, Diffusivity and Transience in Random Media

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Zusammenfassung

Die vorliegende Dissertation befaßt sich mit der Asymptotik der beiden Modelle „Random Walk in Random Environment“ (RWRE) und „Branching Random Walks in Random Environment“ (BRWRE), was mit „Irrfahrt in zufälliger Umgebung“ und „Verzweigende Irrfahrten in zufälliger Umgebung“ übersetzt werden kann.

Beim RWRE wird die Bewegung eines Teilchens modelliert, dessen zufällige Bewegung von der am jeweiligen Ort gegebenen Übergangswahrscheinlichkeit abhängt. Diese ist zufällig und wird vor dem Start des Teilchens festgelegt. Eine interessante Frage in diesem Modell ist, ob die Projektion des Ortes des Teilchens auf einen eindimensionalen Unterraum rekurrent oder transient ist, und falls transient, in welche Richtung. Ist es möglich oder unmöglich, daß das Teilchen manchmal nach rechts verschwindet und manchmal nach links? Unter welchen Voraussetzungen?

BRWRE ist ein Modell für eine Population von mobilen, reproduzierenden Teilchen, deren Nachkommenanzahl vom Ort und von der Zeit ihrer Geburt beeinflußt wird. Die Bewegung der Teilchen ist von der Nachkommenentstehung unabhängig und entspricht der einer einfachen Irrfahrt. In diesem Modell gibt es einen Phasenübergang zwischen der „regular growth phase“ (Phase regulären Wachstums) und der „slow growth phase“ (Phase langsamen Wachstums). Der Begriff „Phase“ ist hier zu verstehen als Bereich von Parametern, in dem bestimmte Eigenschaften vorherrschen. Dies kommt der Verwendung im thermodynamischen Kontext nahe. In der regular growth phase wächst die Gesamtpopulation ähnlich schnell wie ihr Erwartungswert, während sie in der slow growth phase langsamer wächst.

Drei Artikel bilden den Hauptteil der vorliegenden Arbeit. Die ersten beiden betreffen BRWRE, der letzte RWRE. Der erste Artikel befaßt sich mit der Entwicklung der Teilchenverteilung in der slow groth phase und beweist ein immer wiederkehrendes Zusammenballen der Teilchen, während im zweiten Artikel in der regular growth phase ein zentraler Grenzwertsatz für die Teilchenverteilung bewiesen wird. Im dritten Artikel wird ein zweidimensionales Beispiel eines random environment konstruiert, das die Eigenschaft hat, daß die darauf laufende Irrfahrt mit nichttrivialer (nicht null oder eins) Wahrscheinlichkeit in positive Richtung transient ist. Die konstruierte Umgebung ist stationär und ergodisch, aber nicht i.i.d.; in der Tat ist bekannt, daß im i.i.d.–Fall ein solches Verhalten nicht auftreten kann.

Introduction

In order to model certain phenomena in nature, it is useful to separate the immobile conditions of the system from the mobile objects present therein; these two parts then are linked by the description of the rules by which the so called *environment* has an influence on the *particles*.

The mathematical models in this dissertation have in common that simple microscopic rules lead to non–trivial macroscopic effects.

We proceed by first describing them heuristically.

Branching Random Walks in Random Environment

The first model, also known under its acronym "BRWRE", describes a population of reproducing particles in a nonhomogeneous environment that changes over time. Both time and space are discrete in this setting.

At time zero, there is one particle at the origin. It moves randomly to a neighbouring point, which is chosen uniformly. There, in the next time step, the particle is replaced by its children. These perform independently the same sort of random movement, are replaced by their children, and so on.

Now, we need to say how many children each particle gets. This is where the random environment comes into play. It is fixed beforehand (but random) and sets for each point in space and each timestep the offspring distribution for all particles originating there at that time.

Various questions are of interest in this model, some of which are the following:

- Will the population survive?
- If it survives, how fast will the overall population count evolve?
- How evolves the probability of two randomly chosen particles being at the same point in space?
- How is the distribution of particles in space evoluting over time?

Random Walk in Random Environment

The second model, abbreviated "RWRE", describes the movement of a single particle. One similarity to BRWRE is the proceeding of drawing a random environment in a first step, which then is fixed and has lasting influence. As there is no branching present in this case, the influence of the environment is on the local transition probabilities of the random walk rather than on the offspring distribution. This means the neighbouring site to which the particle moves is not chosen uniformly, but according to the transition probabilities at the point where the particle is. The transition probabilities do not depend on time in this model.

The main question in this setting is quite similar to the last one presented above: How is the distribution of the particle's position evoluting over time? As this is in general a too broad question, often subquestions are considered:

- Is the Random Walk in Random Environment transient?
- Is the random walk's projection on a one–dimensional subspace transient?
- Is the random walk's projection on a one–dimensional subspace transient in one direction only?
- Do the above mentioned possibilities hold with non–trivial probability (i.e. not zero or one)?
- Under which assumptions on the environment are the answers to the above questions positive or negative?

Common points between the two models

Despite their obvious differences (many particles vs. one particle, influence on the offspring vs. influence on the movement etc.), the two BRWRE and RWRE share quite many similarities.

On the side of the described phenomenon,

- both model the random movement of particles or of a particle,
- this movement takes place in Euklidean space,
- it is under some random influence from the environment in which it is happening.

On the side of the mathematical models,

- they share the two–step approach of drawing a random environment and fixing it, then letting run the particle(s),
- the modeling is performed in both cases in the discrete space $\mathbb{Z}^d$.

As for the questions of interest, they can in both cases be subsumed under the main question: "What happens with the particle(s)"?

Composition of this dissertation

The present work consists of three articles, the first two of which are on BRWRE, while the third one is on RWRE.

The first article treats the behaviour of the population in the so called "slow growth phase" in which the total population count grows slower than its expectation. This regime prevails in dimensions one and two, and in higher dimensions if the environment is "too random", featuring large fluctuations in the offspring distributions. Under weak moment conditions, it is proven that in the entire slow growth phase, localization occurs, which means that again and again, more than a fixed percentage of the particles will mass at one point. This article is joint work with Makoto Nakashima of Kyoto University.

The second article is concerned with the "regular growth phase", in which the total population and its expectation are of the same order. Here, a central limit theorem and an invariance principle are proven, again under weak moment conditions. This part is joint work with Makoto Nakashima and Professor Nobuo Yoshida of Kyoto University.

Note that in both slow and regular growth phase, the population grows exponentially. The difference lies in the comparison of the exponent with the expected offspring rate.

One can say that the two articles on BRWRE complement each other as both regular and slow growth phase are covered. They give qualitative answers to the last question concerning BRWRE stated above. The results differ from previously available ones inasmuch the necessary moment conditions are weakened, and it is not assumed anymore that the particles have a minimum of one child. This leads to the possibility of the entire population becoming extinct. However, it can be shown that if the probability of having no children is not too high at too many places, the process has positive probability to survive. As a consequence, all theorems hold "on the event of survival".

The last part of this dissertation is devoted to the last two questions stated above concerning RWRE. For RWRE with independent and identically distributed environment in dimension $d = 2$, it is known that a so called 0–1–law holds. It states that the random walk's projection on a one–dimensional subspace has a probability of either zero or one to be transient to the right. This is also conjectured to hold true for higher dimensions, but the proof remains wide open for now. There is a counterexample to this assertion in dimensions three and higher for an environment that is not i.i.d., but still stationary and mixing. The present work gives such a counterexample for dimension two, meaning that for a 0–1–law to hold, one cannot distance oneself too much from the i.i.d.–assumption.

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A REMARK ON LOCALIZATION FOR BRANCHING RANDOM WALKS IN RANDOM ENVIRONMENT

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Abstract
We prove a localization-result for branching random walks in random environment, namely that if the process does not die out, the most populated site will infinitely often contain more than a fixed percentage of the population. This had been proven already before by Hu and Yoshida, but it is possible to drop their assumption that particles may not die.

1 Branching Random Walks in Random Environment

1.1 Informal descripion

Branching Random Walks in Random Environment (BRWRE) are a model for the spread of particles on an inhomogeneous media, such as bacteria that move around and encounter food supply or environmental conditions variable in time and space. These environmental conditions have an impact on the reproduction rate of the particles.
The randomness of the model occurs in two steps. The first step is the setting of the environment, which determines the offspring distribution at different times and places. In our case, these offspring distributions are to be i.i.d..

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The second step is the development of the population given the environment randomly generated in the first step. Starting with one particle at the origin, each particle generates offspring according to the offspring distribution associated with the time-space-location where it is born. It carries this offspring to adjacent sites in the manner of a simple random walk, and dies, leaving the new particles to start over, independently of each other.
As it is possible that particles die without leaving any offspring, the whole population might die out. This phenomenon is described in the event of "extinction". In the present article, however, we are more interested in the long-term-behaviour of the population, and usually work on the complementary event, called "survival". All the notions will be thoroughly defined in Subsection 1.3.

1.2 Brief history

Branching random walks in random environment have been introduced in [Birk], and Birkner, Geiger and Kersting [BGK05] revealed a phase change of the model which was subsequently characterized as a dichotomy: [Nak11] revealed that this model exhibits a phase transition beween what is called slow and regular growth, respectively.
The question of localization in this model, that is whether or not it is possible that in the long term, many particles may become concentrated on few sites, was answered positively for the slow growth phase by Hu and Yoshida [HY09] for environments that do not allow for extinction. A similar answer is given for the more general model of Linear Stochastic Evolution (LSE) in [Yos10]. BRWRE's survival, together with growth rates for the population, are studied by Comets and Yoshida [CY].
Uniting tools from the last three articles is what allows us to prove a localization result in a setting where extinction is possible.
A central limit theorem for BRWRE in the regular growth phase is proved in [HNY]. In that article, a more complete outline of the history of CLTs for BRW, BRWRE and related models can be found, and pictures of the BRWRE are given.

1.3 Thorough definition of the model

We define the random environment as i.i.d. offspring distributions $(q_{t,x})_{t\in\mathbb{N}_0,x\in\mathbb{Z}^d}$ under some (product-)measure Q on $\Omega_q := \mathscr{P}(\mathbb{N}_0)^{\mathbb{N}_0\times\mathbb{Z}^d}$, where $\mathscr{P}(\mathbb{N}_0)$ is the set of probability measures on $\mathbb{N}_0$, and may be equipped with the natural Borel-σ-field induced from that of $[0,1]^{\mathbb{N}_0}$. We call this product-σ-field $\mathscr{F}_q$.

$$q_{t,x} = (q_{t,x}(k))_{k\in\mathbb{N}_0} \in [0,1]^{\mathbb{N}_0}, \ \sum_{k\in\mathbb{N}_0} q_{t,x}(k) = 1.$$

On a measurable space $(\Omega_K, \mathscr{F}_K)$, to each fixed environment $q = (q_{t,x})_{t\in\mathbb{N}_0,x\in\mathbb{Z}^d}$ we associate a probability measure P_K^q such that the random variables $K := (K_{t,x}^\nu)_{t\in\mathbb{N}_0,x\in\mathbb{Z}^d,\nu\in\mathbb{N}}$ are independent in the number ν of the particle and the space-time point (t,x) while being distributed according to $q_{t,x}$:

$$P_K^q(K_{t,x}^\nu = k) = q_{t,x}(k), \ k \in \mathbb{N}_0. \tag{1.1}$$

These random variables $K_{t,x}^\nu$ describe the number of children born to the ν-th particle at time-space-location (t,x).
The ν-th particle ($\nu \in \mathbb{N}$) at time $t \in \mathbb{N}_0 := \{0,1,\dots\} =: \mathbb{N}\cup\{0\}$ and site $x \in \mathbb{Z}^d$ moves (together with all of his offspring) to some site adjacent to his birthplace, determined by the $\mathbb{Z}^d$-valued

random variable $X_{t,x}^{\nu}$. The $X := (X_{t,x}^{\nu})_{t\in\mathbb{N}_0,x\in\mathbb{Z}^d,\nu\in\mathbb{N}}$, defined on a probability space $(\Omega_X, \mathscr{F}_X, P_X)$, are defined to be the one-step transitions of a simple random walk, and i.i.d. in all three time, space, and particles:

$$P_X(X_{t,x}^{\nu} = y) = p(x,y) := \begin{cases} 1/2d & \text{if } |x-y| = 1 \\ 0 & \text{if } |x-y| \neq 1; \end{cases} \tag{1.2}$$

$|\cdot|$ designates the one-norm.

At its time-space destination $(t+1, X_{t,x}^{\nu})$, the said ν-th particle from (t,x) dies and leaves place to its children, and the procedure starts over for every child.

Of course, we can combine the realization of X and K on one probability space

$$(\Omega_X \times \Omega_K, \mathscr{F}_X \otimes \mathscr{F}_K, P^q), \text{ where } P^q := P_X \otimes P_K^q \tag{1.3}$$

and finally merge all our construction to

$$\begin{aligned} \Omega &:= \Omega_X \times \Omega_K \times \Omega_q,\ \mathscr{F} := \mathscr{F}_X \otimes \mathscr{F}_K \otimes \mathscr{F}_q, \\ P(A) &:= \int_A Q(dq) P^q(d\omega), \quad A \in \mathscr{F}. \end{aligned} \tag{1.4}$$

P^q can be seen as the quenched measure and P as the annealed one of the model.

Now we come to the population at time t and site x. We start at time 0 with one particle at the origin, and define inductively

$$N_{0,x} := \mathbb{1}_{x=0}, \quad N_{t,x} = \sum_{y\in\mathbb{Z}^d} \sum_{\nu=1}^{N_{t-1,y}} \mathbb{1}_{X_{t,y}^{\nu}=x} K_{t-1,y}^{\nu},\ t \geq 1. \tag{1.5}$$

The filtration

$$\mathscr{F}_0 := \{\emptyset, \Omega\}, \quad \mathscr{F}_t := \sigma\big(X_{s,\cdot}^{\cdot}, K_{s,\cdot}^{\cdot}, q_{s,\cdot};\ s \leq t-1\big), \quad t \geq 1, \tag{1.6}$$

makes the process $t \mapsto (N_{t,x})_{x\in\mathbb{Z}^d}$ adapted. The total population at time t can now be obtained by summation over all sites:

$$N_t := \sum_{y\in\mathbb{Z}^d} N_{t,y} = \sum_{y\in\mathbb{Z}^d} \sum_{\nu=1}^{N_{t-1,y}} K_{t-1,y}^{\nu}\ \ t \geq 1. \tag{1.7}$$

Important quantities of this model are the averaged and local moments of the offpring distributions

$$m^{(p)} := Q\big(m_{t,x}^{(p)}\big),\ m_{t,x}^{(p)} := \sum_{k\in\mathbb{N}_0} k^p q_{t,x}(k),\ p \in \mathbb{N}. \tag{1.8}$$

We also write $\mathbf{m} := m^{(1)}$.

1.4 The phase transition of the normalized population

It has been proven in [Nak11] that the total population exhibits a phase transition, where the one phase amounts to population growing as fast as its expectation, while the other phase means slower-than-the-expectation growth.

Proposition 1.4.1. *The normalized population* $\overline{N}_t := N_t/\mathbf{m}^t$ *is a martingale, and hence its limit exists:*

$$\overline{N}_\infty := \lim_{t\to\infty} \frac{N_t}{\mathbf{m}^t},\ P\text{-a.s.}. \tag{1.9}$$

Further,

$$P(\overline{N}_\infty) = \begin{cases} 1 & \text{“regular growth phase”, or} \\ 0 & \text{“slow growth phase”.} \end{cases}$$

Sufficient conditions for both phases are given by the following two Propositions 1.4.3 and 1.4.4, which necessitate a bit of

Notation 1.4.2. Given the simple symmetric random walk S_t on $\mathbb{Z}^d$, we call π_d the probability of the return event $\bigcup_{t\geq 1}\{S_t = 0\}$. Furthermore, we write

$$\alpha := \frac{Q(m_{t,x}^2)}{\mathbf{m}^2}.$$

Proposition 1.4.3. *There exists a constant* $\alpha^* > 1/\pi_d$ *such that, if*

$$\mathbf{m} > 1,\ m^{(2)} < \infty,\ d \geq 3, \text{ and } \alpha < \alpha^*, \tag{1.10}$$

then $P(\overline{N}_\infty > 0) > 0$.

Proposition 1.4.4. *On the other hand,* $P(\overline{N}_\infty = 0) = 1$ *is provided by any of the following three conditions:*

(a1) $d = 1;\ Q(m_{t,x} = \mathbf{m}) \neq 1.$
(a2) $d = 2;\ Q(m_{t,x} = \mathbf{m}) \neq 1.$
(a3) $d \geq 3;\ Q\Big(\frac{m_{t,x}}{\mathbf{m}} \ln \frac{m_{t,x}}{\mathbf{m}}\Big) > \ln(2d).$

Propositions 1.4.3 and 1.4.4 were obtained first in [BGK05, Theorem 4]. Proposition 1.4.3 plays a crucial role in our proof as it allows us in the slow growth phase to conclude $\alpha > \alpha^* > 1/\pi_d$.

Remark 1.4.5. We would at this point recall the non-random environment case [AN72, Theorem 1, page 24], where

$$P(\overline{N}_\infty = 0) = 1 \text{ if and only if } P(K_{t,x}^\nu \ln K_{t,x}^\nu) = \infty \text{ or } \mathbf{m} \leq 1. \tag{1.11}$$

In our case here, with the additional randomness of the environment, $P(\overline{N}_\infty = 0) = 1$ can happen even if the $K_{t,x}^\nu$ are bounded (see Remark 1.6.3 b) below).

1.5 Survival and the global growth estimate

Another dichotomy of this model is the one of survival and extinction. We define

$$\{survival\} := \{\forall t \in \mathbb{N}_0, N_t > 0\}. \tag{1.12}$$

The event of extinction is defined as the complement.
The following global growth estimate obtained in [CY, Theorem 2.1.1] characterizes the event of survival:

Lemma 1.5.1. *Suppose* $Q(m_{t,x} + m_{t,x}^{-1}) < \infty$ *and let* $\varepsilon > 0$. *Then, for large* t,

$$N_t \le e^{(\Psi+\varepsilon)t}, \ P\text{-a.s.},$$

where the limit

$$\Psi := \lim_{t\to\infty} \frac{1}{t} Q\big(\ln P^q(N_{t,0})\big)$$

exists.
If $\Psi > 0$ *and* $m^{(2)} < \infty$, *then*

$$\{survival\} = \{N_t \ge e^{(\Psi-\varepsilon)t} \text{ for all large } t\}, \ P\text{-a.s.}. \tag{1.13}$$

Remark 1.5.2. a) Actually, the hypotheses given in [CY] are somewhat weaker, and are implied by our assumption $m^{(2)} < \infty$. See the Remark 2) right after [CY, Theorem 2.1.1].
b) It is proved in [CY] as well that "$\Psi > 0$" is implied by

$$Q(m_{t,x} = \mathbf{m}) \neq 1, \ Q(\ln m_{t,x}) \ge 0. \tag{1.14}$$

The object we investigate is the population density

$$\rho_{t,x} = \rho_t(x) := \frac{N_{t,x}}{N_t} \mathbb{1}_{N_t>0}, \ t \in \mathbb{N}_0, \ x \in \mathbb{Z}^d. \tag{1.15}$$

It describes the distribution of the population in space.
Related important objects are

$$\rho_t^* := \max_{x\in\mathbb{Z}^d} \rho_{t,x} \quad \text{and} \quad \mathscr{R}_t := \sum_{x\in\mathbb{Z}^d} \rho_{t,x}^2. \tag{1.16}$$

They are, respectively, the density at the most populated site and the probability that two particles picked randomly from the total population are at the same site at time t. We will call this latter value the "replica overlap".
It is possible to relate the event of survival to this replica overlap.

Theorem 1.5.3. *Suppose* $m^{(2)} < \infty$. *Then, if* $P(\overline{N}_\infty = 0) = 1$,

$$\{survival\} \subseteq \Big\{ \sum_{t=1}^{\infty} \mathscr{R}_t = \infty \Big\}. \tag{1.17}$$

The proof of this Theorem can be found in Section 2.2. While it is true that the opposite inclusion does hold under the stronger assumption $m^{(3)} < \infty$, we do not state this formally here. The proof can be found in [HNY].

1.6 The main result

Hu and Yoshida, using the assumption that particles may not die, proved in [HY09, Theorem 1.3.2] the following

Theorem 1.6.1. *Suppose* $P(\overline{N}_\infty = 0) = 1$ *and*

$$m^{(3)} < \infty, \ Q(m_{t,x} = \mathbf{m}) \neq 1, \ Q\big(q_{t,x}(0) = 0\big) = 1. \qquad ([\text{HY09}, (1.18)])$$

Then, there exists a non-random number $c \in (0,1)$ *such that,*

$$\limsup_{t\to\infty} \rho_t^* \ge \limsup_{t\to\infty} \mathscr{R}_t \ge c, \ P\text{-a.s.}. \tag{6.19}$$

In this setting, extinction (i.e. the event that at some time, the total population becomes 0) cannot occur. However, it is possible to drop this assumption with the help of a few additional tools.
Our main result is indeed that the last two hypotheses can be replaced by weaker ones.

Theorem 1.6.2. *Suppose* $P(\overline{N}_\infty = 0) = 1$ *and*

$$m^{(3)} < \infty, \ \Psi > 0, \ Q(m_{t,x}^{-1}) < \infty. \tag{1.20}$$

Then, there exists a non-random number $c \in (0,1)$ *such that*

$$\limsup_{t\to\infty} \rho_t^* \geq \limsup_{t\to\infty} \mathscr{R}_t \geq c, \ \textit{P-a.s. on the event of survival.} \tag{1.21}$$

The proof of the Theorem is postponed to its own Section 2.4.

Remark 1.6.3. a) The fact that Theorem 1.6.1 does not allow for dying particles has two implications, namely $\Psi > 0$ (rather trivially by (1.14)) and $Q(m_{t,x}^{-1}) < \infty$. Our theorem shows that we can indeed content ourselves with these two weaker conditions themselves.
b) The hypotheses $P(\overline{N}_\infty = 0) = 1$ and $\Psi > 0$ are difficult to check in practice. Yet, it is possible to give an example that satisfies the easier **(a1)** $-$ **(a3)** of Proposition 1.4.4 and (1.14), but not the hypotheses of Theorem 1.6.1. It is given by the following class of environments constituted only of two states: for $n \in \mathbb{N}$,

$$q_{\cdot,\cdot}(0) = q_{\cdot,\cdot}(n^2) = \frac{1}{2} \text{ with probability } \frac{1}{n}, \tag{1.22}$$

$$q_{\cdot,\cdot}(1) = 1 \text{ with probability } 1 - \frac{1}{n}. \tag{1.23}$$

In this case, $Q\left(\frac{m_{t,x}}{\mathbf{m}} \ln \frac{m_{t,x}}{\mathbf{m}}\right) \sim \ln n$, and hence any dimension can be covered by n large enough.

2 Proofs

2.1 Tools for the proof of Theorem 1.5.3

The following Definition will be useful at several points. It provides notation for the thorough calculus of the fluctuation of the normalized population.

Definition 2.1.1. *Let*

$$U_{s+1,x} := \frac{\mathbb{1}_{N_s>0}}{\mathbf{m}N_s} \sum_{\nu=1}^{N_{s,x}} K_{s,x}^{\nu} \geq 0, \quad U_{s+1} := \sum_{x\in\mathbb{Z}^d} U_{s+1,x} = \frac{N_{s+1}}{\mathbf{m}N_s} \mathbb{1}_{N_s>0} = \frac{\overline{N}_{s+1}}{\overline{N}_s} \mathbb{1}_{\overline{N}_s>0}.$$

The $(U_{s+1,x})_{x\in\mathbb{Z}^d}$ *are independent under* $P(\cdot|\mathscr{F}_s)$*. It is not difficult to see that, on the event* $\{N_s > 0\}$*,*

$$P(U_{s+1,x}|\mathscr{F}_s) = \rho_s(x), \textit{ and hence } P(U_{s+1}|\mathscr{F}_s) = 1.$$

Also, with $\widetilde{c}_i = \frac{m^{(i)}}{\mathbf{m}^i}$, $i = 2, 3$,

$$\alpha\rho(x)^2 = \frac{1}{\mathbf{m}^2 N_s^2} N_{s,x}^2 Q\big(m_{s,x}^2\big) \le P\big(U_{s+1,x}^2 \big| \mathscr{F}_s\big) \tag{2.1}$$

$$= \frac{1}{\mathbf{m}^2 N_s^2} P\Big(\big(\sum_{\nu=1}^{N_{s,x}} K_{s,x}^{\nu}\big)^2 \Big| \mathscr{F}_s\Big) \le \frac{N_{s,x}^2 m^{(2)}}{\mathbf{m}^2 N_s^2} = \widetilde{c}_2 \rho_s(x)^2, \tag{2.2}$$

$$P\big(U_{s+1,x}^3 \big| \mathscr{F}_s\big) \le \frac{m^{(3)}}{\mathbf{m}^3} \rho_s(x)^3 = \widetilde{c}_3 \rho_s(x)^3, \text{ again on the event } \{N_s > 0\}. \tag{2.3}$$

Theorem 1.5.3 is a consequence of the following Proposition. It can be found in [Yos10, Proposition 2.1.2] and relates survival and boundedness of the predictable quadratic variation for some abstract martingale.

Proposition 2.1.2. *Let* $(Y_t)_{t\in\mathbb{N}_0}$ *be a mean-zero square-integrable martingale on a probability space with measure* $\mathbb{E}$ *and filtration* $(\mathscr{F}_t)_{t\in\mathbb{N}_0}$. *Suppose* $-1 \le \Delta Y_t := Y_t - Y_{t-1}$ *for all* $t \in \mathbb{N}$, *and let*

$$X_t := \prod_{s=1}^{t} (1 + \Delta Y_s). \tag{2.4}$$

If $P\big((\Delta Y_t)^2 \big| \mathscr{F}_{t-1}\big)$ *is uniformly bounded in* t, *then*

$$\{X_\infty = 0\} \subseteq \{Extinction\} \cup \left\{ \sum_{s=1}^{\infty} P\big((\Delta Y_s)^2 \big| \mathscr{F}_{s-1}\big) = \infty \right\}, \tag{2.5}$$

where $\{Extinction\} := \{\exists t > 0 : X_t = 0\}$.

2.2 Proof of Theorem 1.5.3

We want to apply the abstract result that is Proposition 2.1.2 to our setting. To get the notation right, we take $X_t := \overline{N}_t$, and remark that the definition

$$\Delta Y_t := \frac{\overline{N}_t}{\overline{N}_{t-1}} \mathbb{1}_{N_{t-1}>0} - \mathbb{1}_{N_{t-1}>0} = \sum_x \big[U_{t,x} - \rho_{t,x}\big] \ge -1 \tag{2.6}$$

verifies (2.4); the $U_{t,x}$ are taken from Definition 2.1.1. As for the other hypothesis of the Proposition, we need not even to check it in order to find $\sum_{s=1}^{\infty} P\big((\Delta Y_s)^2 \big| \mathscr{F}_{s-1}\big) = \infty$: if uniform boundedness does not hold, it is true anyway, and if uniform boundedness holds, we derive it from Proposition 2.1.2 on the event $\{survival\} \cap \{\overline{N}_\infty = 0\}$.
Now, with (2.1), we see that $\sum_{s=1}^{t} P\big((\Delta Y_s)^2 \big| \mathscr{F}_{s-1}\big)$ shares its asymptotic behaviour with $\sum_{s=1}^{t} \mathscr{R}_s$, so we conclude (1.17). □

2.3 Tools for the proof of Theorem 1.6.2

One result that has not been taken into account in [HY09] and that helps us making the slight improvement of the hypotheses is the following improved version of the Borel-Cantelli-lemma, stated in [Yos10, Lemma 2.2.1]:

Lemma 2.3.1. *Let $(R_t)_{t\in\mathbb{N}}$ be an integrable, adapted process defined on a probability space with measure $\mathbb{E}$ and a filtration $(\mathscr{F}_t)_{t\in\mathbb{N}_0}$. Define $V_0 := 0 =: T_0$ and*

$$V_t := \sum_{s=1}^{t} R_s, \qquad T_t := \sum_{s=1}^{t} \mathbb{E}[R_s|\mathscr{F}_{s-1}],\ t \in \mathbb{N}.$$

a) *Suppose there is a constant $C_1 \in (0,\infty)$ such that*

$$R_t - \mathbb{E}[R_t|\mathscr{F}_{t-1}] \geq -C_1,\ \mathbb{E}\text{-a.s. for all } t \in \mathbb{N}. \tag{2.7}$$

Then,

$$\Big\{\lim_{t\to\infty} V_t = \infty\Big\} = \Big\{\lim_{t\to\infty} V_t = \infty,\ \limsup_{t\to\infty} \frac{T_t}{V_t} \geq 1\Big\} \subseteq \Big\{\sup_{t\geq 1} T_t = \infty\Big\}.$$

b) *Suppose that $(R_t)_{t\in\mathbb{N}}$ is in $L^2(\mathbb{E})$, and that there exists a constant $C_2 \in (0,\infty)$ such that*

$$\mathbb{V}\mathrm{ar}[R_t|\mathscr{F}_{t-1}] \leq C_2 \mathbb{E}[R_t|\mathscr{F}_{t-1}]\ \mathbb{E}\text{-a.s. for all } t \in \mathbb{N},$$

where $\mathbb{V}\mathrm{ar}[R_t|\mathscr{F}_{t-1}] := \mathbb{E}[R_t^2|\mathscr{F}_{t-1}] - \mathbb{E}[R_t|\mathscr{F}_{t-1}]^2$. Then, $\mathbb{E}$-a.s.,

$$\Big\{\lim_{t\to\infty} T_t = \infty\Big\} = \Big\{\lim_{t\to\infty} T_t = \infty,\ \limsup_{t\to\infty} \frac{V_t}{T_t} = 1\Big\} \subseteq \Big\{\sup_{t\geq 1} V_t = \infty\Big\}.$$

This Lemma admits in our setting, with a slight abuse of notation, for the following

Corollary 2.3.2. *On the event $\{\lim_{t\to\infty} V_t = \infty\}$, there exists a constant $c_0 \in [1,\infty)$ such that*

$$T_t := \sum_{s=1}^{t} P(\mathscr{R}_s|\mathscr{F}_{s-1}) \leq c_0 \sum_{s=1}^{t} \mathscr{R}_s =: c_0 V_t \tag{2.8}$$

holds for large t.

Proof. In fact, the hypotheses of both **a)** and **b)** of Lemma 2.3.1 are satisfied. Indeed, $0 \leq \mathscr{R}_t = \sum_x \rho_{t,x}^2 \leq 1$ is square-integrable and adapted, and (2.7) is satisfied with $C_1 = 2$. Also,

$$\mathbb{V}\mathrm{ar}(\mathscr{R}_t|\mathscr{F}_{t-1}) \leq P(\mathscr{R}_t^2|\mathscr{F}_{t-1}) \leq P(\mathscr{R}_t|\mathscr{F}_{t-1}).$$

Hence, with **a)**, $\{\lim_{t\to\infty} V_t = \infty\}$ implies $\{\sup_t T_t = \infty\}$. But T_t is a sum over positive terms, so its supremum is equal to its limes, and we can readily apply part **b)**. The statement is then trivial. □

The following Lemma is an extension to [Yos10, Lemma 3.2.1] and replaces [HY09, Lemma 3.1.1].

Lemma 2.3.3. *Let $(U_i)_{1\leq i\leq n}$, $n \geq 2$, be non-negative, independent and cube-integrable random variables on our general probability space with probability measure $\mathbb{E}$ such that for*

$$U = \sum_{i=1}^{n} U_i, \qquad \mathbb{E}[U] = 1. \tag{2.9}$$

Let furthermore X be a random variable such that $0 \leq X \leq U_1^2$ a.s.. Then,

$$\mathbb{E}\Big[\frac{U_1 U_2}{U^2} : U > 0\Big] \geq \mathbb{E}[U_1]\mathbb{E}[U_2] - 2\mathbb{E}[U_2]\mathbb{V}\mathrm{ar}[U_1] - 2\mathbb{E}[U_1]\mathbb{V}\mathrm{ar}[U_2], \tag{2.10}$$

$$\mathbb{E}\Big[\frac{X}{U^2} : U > 0\Big] \geq \mathbb{E}[U_1^2](1 + 2\mathbb{E}[U_1]) - 2\mathbb{E}[U_1^3] - 3\mathbb{E}[U_1^2 - X]. \tag{2.11}$$

Proof. The first inequality is proved in [Yos10]. We will prove the second one.
Note that $u^{-2} \geq 3 - 2u$ for $u \in (0, \infty)$. Thus, we have that

$$\begin{aligned}
\mathbb{E}\Big[\frac{X}{U^2} : U > 0\Big] &\geq \mathbb{E}[X(3-2U) : U > 0] = \mathbb{E}[X(3-2U)] \\
&= \mathbb{E}[(X - U_1^2)(3-2U)] + \mathbb{E}[U_1^2(3-2U)] \\
&= \mathbb{E}[(X - U_1^2)(3-2U)] + 3\mathbb{E}[U_1^2] - 2\mathbb{E}[U_1^3] - 2\mathbb{E}[U_1^2]\mathbb{E}[\textstyle\sum_{i\neq 1} U_i] \\
&= 2\mathbb{E}[(U_1^2 - X)U] - 3\mathbb{E}[U_1^2 - X] + 3\mathbb{E}[U_1^2] - 2\mathbb{E}[U_1^3] - 2\mathbb{E}[U_1^2](1 - \mathbb{E}[U_1]) \\
&\geq -3\mathbb{E}[U_1^2 - X] + \mathbb{E}[U_1^2](1 + 2\mathbb{E}[U_1]) - 2\mathbb{E}[U_1^3]. \qquad \square
\end{aligned}$$

At this point, we need some further notations. We denote by $\mathscr{P}_s(x,y)$ the probability that the simple random walk starting in $x \in Z^d$ goes to $y \in Z^d$ in exactly $s \in \mathbb{N}$ steps. We write $r_j := \mathscr{P}_{2j}(x,x)$. Also, we can define the semigroup of the simple random walk by $\mathscr{P}_s f(x) := \sum_y \mathscr{P}_s(x,y) f(y)$. We write $\mathscr{P} := \mathscr{P}_1$.

Remark 2.3.4. With the Cauchy-Schwarz-inequality, we have

$$\max_x (\mathscr{P}_j \rho_t(x))^2 \leq \sum_x (\mathscr{P}_j \rho_t(x))^2 \leq \sum_x \mathscr{P}_j \rho_t^2(x) = \mathscr{R}_t = \sum_x \rho_t^2(x) \leq 1. \tag{2.12}$$

We now start estimates on the population density. The following result corresponds to the inequality (3.7) in [HY09, Lemma 3.1.4].

Lemma 2.3.5. *Suppose* (1.20). *On the event of survival up to time $s \in \mathbb{N}$, for any $y_1, y_2 \in \mathbb{Z}^d$, we have*

$$\begin{aligned}
&P\big(\rho_{s+1}(y_1)\rho_{s+1}(y_2)\big|\mathscr{F}_s\big) \\
&\quad\geq \mathscr{P}\rho_s(y_1)\mathscr{P}\rho_s(y_2) + (\alpha-1)\sum_z \rho_s(z)^2 p(z,y_1)p(z,y_2) \qquad (2.13)\\
&\quad\quad - 2\widetilde{c}_2\big[\mathscr{P}\rho_s(y_2)\mathscr{P}(\rho_s^2)(y_1) + \mathscr{P}\rho_s(y_1)\mathscr{P}(\rho_s^2)(y_2)\big] \\
&\quad\quad - 2\widetilde{c}_3 \sum_z \rho_s(z)^3 p(z,y_1)p(z,y_2) - 3\widetilde{c}_2 \frac{1}{N_s}\sum_z \rho_s(z)p(z,y_1)p(z,y_2).
\end{aligned}$$

where the $\widetilde{c}_\cdot$ are the same as in Definition 2.1.1.

Proof. We have

$$\begin{aligned}
P\big(\rho_{s+1}(y_1)\rho_{s+1}(y_2)|\mathscr{F}_s\big) &= \sum_{z_1,z_2}\sum_{\nu_1=1}^{N_{s,z_1}}\sum_{\nu_2=1}^{N_{s,z_2}} P\left(\frac{\mathbb{1}_{X_{s,z_1}^{\nu_1}=y_1}\mathbb{1}_{X_{s,z_2}^{\nu_2}=y_2} K_{s,z_1}^{\nu_1}K_{s,z_2}^{\nu_2}}{N_{s+1}^2}\mathbb{1}_{N_{s+1}>0}\middle|\mathscr{F}_s\right) \\
&\geq \sum_{z_1\neq z_2} p(z_1,y_1)p(z_2,y_2)P\left(\frac{\sum_{\nu_1=1}^{N_{s,z_1}}K_{s,z_1}^{\nu_1}\sum_{\nu_2=1}^{N_{s,z_2}}K_{s,z_2}^{\nu_2}}{N_{s+1}^2}\mathbb{1}_{N_{s+1}>0}\middle|\mathscr{F}_s\right) \qquad (2.14)\\
&\quad + \sum_z p(z,y_1)p(z,y_2)P\left(\frac{\sum_{\nu_1\neq\nu_2=1}^{N_{s,z}}K_{s,z}^{\nu_1}K_{s,z}^{\nu_2}}{N_{s+1}^2}\mathbb{1}_{N_{s+1}>0}\middle|\mathscr{F}_s\right). \qquad (2.15)
\end{aligned}$$

Now, we would like to estimate (2.14) and (2.15). We can rewrite these lines with the processes from Definition 2.1.1. These verify the hypotheses of Lemma 2.3.3. The estimates obtained by the application of this Lemma comprise second and third moments which we cannot provide explicitly. We therefore replace them by the estimates obtained in Definition 2.1.1; note that survival up to time $s+1$ implies survival up to time s.

Since $\{N_{s+1}>0\}\subseteq\{U_{s+1}>0\}$, by (2.10), we have

$$\begin{aligned}(2.14) &= \sum_{z_1\neq z_2} p(z_1,y_1)p(z_2,y_2)P\Big(\frac{U_{s+1,z_1}U_{s+1,z_2}}{U_{s+1}^2}\mathbb{1}_{U_{s+1}>0}\Big|\mathscr{F}_s\Big)\\ &\geq \sum_{z_1\neq z_2} p(z_1,y_1)p(z_2,y_2)\Big(\rho_s(z_1)\rho_s(z_2)-2\widetilde{c}_2\left[\rho_s(z_2)\rho_s(z_1)^2+\rho_s(z_1)\rho_s(z_2)^2\right]\Big).\end{aligned}$$

Also, with $X(z)=\Big(\big(\sum_{\nu=1}^{N_{s,z}}K_{s,z}^{\nu}\big)^2-\sum_{\nu=1}^{N_{s,z}}(K_{s,z}^{\nu})^2\Big)/\mathbf{m}^2N_s^2$ and (2.11),

$$\begin{aligned}(2.15) &= \sum_z p(z,y_1)p(z,y_2)P\Big(\frac{X(z)}{U_{s+1}^2}\mathbb{1}_{U_{s+1}>0}\Big|\mathscr{F}_s\Big)\\ &\geq \sum_z p(z,y_1)p(z,y_2)\\ &\qquad\Big[P\big(U_{s+1,z}^2\big|\mathscr{F}_s\big)\left[1+2\rho_s(z)\right]-2P\big(U_{s+1,z}^3\big|\mathscr{F}_s\big)-3\sum_{\nu=1}^{N_{s,z}}P\Big(\big(\frac{K_{s,z}^{\nu}}{\mathbf{m}N_s}\big)^2\Big|\mathscr{F}_s\Big)\Big]\\ &\geq \sum_z p(z,y_1)p(z,y_2)\Big[\alpha\rho_s(z)^2-2\widetilde{c}_3\rho_s(z)^3-3\widetilde{c}_2\frac{\rho_s(z)}{N_s}\Big].\end{aligned}$$

These estimates imply the statement. □

Lemma 2.3.6. *Suppose* (1.20). *For all* $1\leq j\leq t-1$,

$$\begin{aligned}P\Big(\sum_x\big(\mathscr{P}_{j-1}\rho_{t-j+1}(x)\big)^2\Big|\mathscr{F}_{t-j}\Big) &\geq \sum_x(\mathscr{P}_j\rho_{t-j}(x))^2+(\alpha-1)r_j\mathscr{R}_{t-j}\\ &\quad-(4\widetilde{c}_2+2\widetilde{c}_3)\mathscr{R}_{t-j}^{3/2}-\frac{3\widetilde{c}_2}{N_{t-j}}.\end{aligned}$$

Proof. If we apply the definition of the semigroup operator $\mathscr{P}$, we get

$$\sum_x\big(\mathscr{P}_{j-1}\rho_{t-j+1}(x)\big)^2=\sum_x\sum_{y_1,y_2}\mathscr{P}_{j-1}(x,y_1)\mathscr{P}_{j-1}(x,y_2)\rho_{t-j+1}(y_1)\rho_{t-j+1}(y_2).$$

Applying (2.13) gives

$$P\Big(\sum_x\big(\mathscr{P}_{j-1}\rho_{t-j+1}(x)\big)^2\Big|\mathscr{F}_{t-j}\Big)\geq\Big[I+(\alpha-1)II-2\widetilde{c}_2III-2\widetilde{c}_3IV-3\widetilde{c}_2\frac{1}{N_{t-j}}V\Big],$$

where

$$\begin{aligned}
I &:= \sum_x \sum_{y_1,y_2} \mathscr{P}_{j-1}(x,y_1)\mathscr{P}_{j-1}(x,y_2)\mathscr{P}\rho_{t-j}(y_1)\mathscr{P}\rho_{t-j}(y_2) \\
&= \sum_x \left(\mathscr{P}_j\rho_{t-j}(x)\right)^2 \text{ by definition of the semigroup-operator;} \\
II &:= \sum_x \sum_{y_1,y_2} \mathscr{P}_{j-1}(x,y_1)\mathscr{P}_{j-1}(x,y_2)\sum_z \rho_{t-j}(z)^2 p(z,y_1)p(z,y_2) \\
&= \sum_x\sum_z \left(\mathscr{P}_j(x,z)\right)^2\rho^2_{t-j}(z) = r_j\sum_z \rho^2_{t-j}(z) \text{ because } \sum_x \left(\mathscr{P}_j(x,z)\right)^2 = r_j; \\
III &:= \sum_x \sum_{y_1,y_2} \mathscr{P}_{j-1}(x,y_1)\mathscr{P}_{j-1}(x,y_2)\left(\mathscr{P}\rho_{t-j}(y_2)\mathscr{P}\rho^2_{t-j}(y_1) + \mathscr{P}\rho_{t-j}(y_1)\mathscr{P}\rho^2_{t-j}(y_2)\right) \\
&= 2\sum_x \mathscr{P}_j\rho_{t-j}(x)\mathscr{P}_j(\rho^2_{t-j})(x) \\
&\le 2\max_x \mathscr{P}_j\rho_{t-j}(x)\sum_x \mathscr{P}_j(\rho^2_{t-j})(x) \le 2\mathscr{R}^{1/2}_{t-j}\mathscr{R}_{t-j} \text{ by Remark 2.3.4;} \\
IV &:= \sum_x \sum_{y_1,y_2} \mathscr{P}_{j-1}(x,y_1)\mathscr{P}_{j-1}(x,y_2)\sum_z \rho^3_{t-j}(z)p(z,y_1)p(z,y_2) \\
&\le \sum_x \sum_{y_1,y_2} \mathscr{P}_{j-1}(x,y_1)\mathscr{P}_{j-1}(x,y_2)\sum_{z_1,z_2} \rho_{t-j}(z_1)p(y_1,z_1)\rho^2_{t-j}(z_2)p(y_2,z_2) \\
&= \sum_x \sum_{y_1,y_2} \mathscr{P}_{j-1}(x,y_1)\mathscr{P}_{j-1}(x,y_2)\mathscr{P}\rho_{t-j}(y_1)\mathscr{P}\rho^2_{t-j}(y_2) \le III; \\
V &:= \sum_x \sum_{y_1,y_2} \mathscr{P}_{j-1}(x,y_1)\mathscr{P}_{j-1}(x,y_2)\sum_z \rho_{t-j}(z)p(z,y_1)p(z,y_2) \\
&= \sum_x\sum_z \left(\mathscr{P}_j(x,z)\right)^2\rho_{t-j}(z) = \sum_z \rho_{t-j}(z)r_j = r_j.
\end{aligned}$$

In these computations, the symmetry of $p(\cdot,\cdot)$ has been used at appropriate places. If we put together the pieces, we obtain the statement of the Lemma. □

Later, in the proof of the main theorem, we are going to perform a division by $\sum_{s=1}^t \mathscr{R}_s$ at some point. The following Lemma helps showing that a certain term then vanishes asymptotically. We recall the definition of $V_t := \sum_{s=1}^t \mathscr{R}_s$ from Corollary 2.3.2 and write $V_\infty := \lim_{t\to\infty} V_t$.

Lemma 2.3.7. *Assume* (1.20)*, and fix some* $j \ge 1$*. The martingale* $Z_j(\cdot)$ *defined by*

$$Z_j(t) := \sum_{s=1}^t \sum_x \left[\left(\mathscr{P}_j\rho_s(x)\right)^2 - P\left((\mathscr{P}_j\rho_s(x))^2 \middle| \mathscr{F}_{s-1}\right)\right],\ t \ge 1,$$

satisfies the following law of large numbers:

$$\{V_\infty = \infty\} \subseteq \left\{\frac{Z_j(t)}{V_t} \xrightarrow[t\to\infty]{} 0\right\},\ P\text{-a.s.}.$$

Remark 2.3.8. The increments of $Z_j(t)$ will be used later, and in squared form in the proof of the Lemma. They are given by

$$Z_j(t+1) - Z_j(t) \quad = \sum_x \left[\left(\mathscr{P}_j\rho_{t+1}(x)\right)^2 - P\left((\mathscr{P}_j\rho_{t+1}(x))^2 \middle| \mathscr{F}_t\right)\right];$$

recalling Remark 2.3.4, we can further estimate

$$\begin{aligned}\big(Z_j(t+1)-Z_j(t)\big)^2 &\le \Big(\sum_x \big(\mathscr{P}_j\rho_{t+1}(x)\big)^2\Big)^2 + \Big(\sum_x P\big((\mathscr{P}_j\rho_{t+1}(x))^2\big|\mathscr{F}_t\big)\Big)^2\\ &\le \mathscr{R}_{t+1}^2 + P\big(\mathscr{R}_{t+1}\big|\mathscr{F}_t\big)^2 \le \mathscr{R}_{t+1} + P\big(\mathscr{R}_{t+1}\big|\mathscr{F}_t\big).\end{aligned}$$

Proof of Lemma 2.3.7. The idea of the proof is to make use of the increasing process $\langle Z_j\rangle_t$ associated with $Z_j(t)$ in order to monitor the growth of $Z_j(t)$ itself.
With the previous Remark, it is indeed possible to estimate $\langle Z_j\rangle_t$ by the sum of the conditional replica-overlap:

$$\begin{aligned}\langle Z_j\rangle_t &= \sum_{s=0}^{t-1}\langle Z_j\rangle_{s+1} - \langle Z_j\rangle_s = \sum_{s=0}^{t-1} P\Big(\big(Z_j(s+1)-Z_j(s)\big)^2\Big|\mathscr{F}_s\Big) \qquad (2.16)\\ &\le 2\sum_{s=0}^{t-1} P\big(\mathscr{R}_s\big|\mathscr{F}_{s-1}\big) = \dots,\end{aligned}$$

but this, by Corollary 2.3.2, is in turn related to the replica overlap itself:

$$\dots \le 2c_0 V_t,\ t \ge 1. \qquad (2.17)$$

The rest is easy. Either $\langle Z_j\rangle_\infty < \infty$, in which case $Z_j(t)$ converges and the statement is trivial anyway, or $\langle Z_j\rangle_\infty = \infty$, in which case we can apply the law of large numbers for square-integrable martingales, see [Dur91, p. 253], which gives us

$$\left|\frac{Z_j(t)}{V_t}\right| \le \frac{1}{2c_0}\left|\frac{Z_j(t)}{\langle Z_j\rangle_t}\right| \xrightarrow[t\to\infty]{} 0. \qquad \square$$

As a final ingredient, we give a statement that compares parameters of the simple random walk with ones of the BRWRE-model.

Lemma 2.3.9. *Suppose* (1.20) *and* $P(\overline{N}_\infty = 0) = 1$. *There exist* $\varepsilon > 0$ *and* $t_0 \in \mathbb{N}$ *such that*

$$\sum_{s=1}^{t_0} r_s \ge \frac{1+\varepsilon}{\alpha-1}. \qquad (2.18)$$

Furthermore, with $T > t_0$ *and* $c_4 := (\alpha-1)t_0\sum_{j=1}^{t_0} r_j$,

$$\sum_{t=t_0+1}^{T}\Big[(\alpha-1)\sum_{j=1}^{t_0} r_j\mathscr{R}_{t-j} - \mathscr{R}_t\Big] \ge \varepsilon V_T - c_4. \qquad (2.19)$$

Proof. In dimensions $d = 1, 2$, the first statement is trivial as $\sum_{s=1}^\infty r_s = \infty$.
In dimensions $d \ge 3$, the assumption $P(\overline{N}_\infty = 0) = 1$ in conjunction with Proposition 1.4.3 gives us $\alpha \ge \alpha^* > 1/\pi_d > 0$, which implies

$$(\alpha-1)\sum_{s=1}^\infty r_s = (\alpha-1)\frac{\pi_d}{1-\pi_d} > \frac{1}{1-\pi_d} > 1, \qquad (2.20)$$

so that (2.18) follows.

As for the second statement, we compute

$$\sum_{t=t_0+1}^{T}\Big[(\alpha-1)\sum_{j=1}^{t_0} r_j\mathscr{R}_{t-j}-\mathscr{R}_t\Big]=(\alpha-1)\sum_{j=1}^{t_0} r_j\sum_{t=t_0+1}^{T}\mathscr{R}_{t-j}-\sum_{t=t_0+1}^{T}\mathscr{R}_t$$
$$=(\alpha-1)\sum_{j=1}^{t_0} r_j(V_{T-j}-V_{t_0-j})-(V_T-V_{t_0})\geq(\alpha-1)\big(V_T-t_0)\big)\sum_{j=1}^{t_0} r_j-V_T$$
$$\geq(1+\varepsilon)V_T-(\alpha-1)t_0\sum_{j=1}^{t_0} r_j-V_T=\varepsilon V_T-c_4,$$

where for the last but one inequality, we used that $V_{t_0-j}\leq t_0-j$ and $V_{T-j}+j\geq V_T$ for all $1\leq j\leq t_0$. □

2.4 Proof of the main theorem

Proof of Theorem 1.6.2. The idea of the proof is to obtain some estimate of the form

$$\liminf_{T\to\infty}\frac{\sum_{t=1}^{T}\mathscr{R}_t^{3/2}}{\sum_{t=1}^{T}\mathscr{R}_t}\geq C \text{ some constant, } P\text{-a.s..}\tag{2.21}$$

This then implies

$$\limsup_{t\to\infty}\mathscr{R}_t\geq C^2,\ P\text{-a.s.,}\tag{2.22}$$

as can easily be verified by contradiction.
However, the only tool we have at hand to estimate $\mathscr{R}_t^{3/2}$ is Lemma 2.3.6, and we need to carry out several operations before arriving at (2.21).
First, we apply Lemma 2.3.6 to $j=1,\dots t_0$, with t_0 from (2.18), and take the sum:

$$\sum_{j=1}^{t_0}\left[(4\widetilde{c}_2+2\widetilde{c}_3)\mathscr{R}_{t-j}^{3/2}+\frac{3\widetilde{c}_2}{N_{t-j}}\right]$$
$$\geq\sum_{j=1}^{t_0}\sum_x\left[\big(\mathscr{P}_j\rho_{t-j}(x)\big)^2-P\Big(\big(\mathscr{P}_{j-1}\rho_{t-j+1}(x)\big)^2\Big|\mathscr{F}_{t-j}\Big)\right]+(\alpha-1)\sum_{j=1}^{t_0} r_j\mathscr{R}_{t-j}$$
$$=\sum_{j=1}^{t_0}\sum_x\left[\big(\mathscr{P}_{j-1}\rho_{t-j+1}(x)\big)^2-P\Big(\big(\mathscr{P}_{j-1}\rho_{t-j+1}(x)\big)^2\Big|\mathscr{F}_{t-j}\Big)\right]$$
$$+\sum_{j=1}^{t_0}\sum_x\left[\big(\mathscr{P}_j\rho_{t-j}(x)\big)^2-\big(\mathscr{P}_{j-1}\rho_{t-j+1}(x)\big)^2\right]+(\alpha-1)\sum_{j=1}^{t_0} r_j\mathscr{R}_{t-j}$$
$$=\sum_{j=1}^{t_0}\Big[Z_{j-1}(t-j+1)-Z_{j-1}(t-j)\Big]+\sum_x\big(\mathscr{P}_{t_0}\rho_{t-t_0}(x)\big)^2-\mathscr{R}_t+(\alpha-1)\sum_{j=1}^{t_0} r_j\mathscr{R}_{t-j}$$
$$\geq\sum_{j=1}^{t_0}\Big[Z_{j-1}(t-j+1)-Z_{j-1}(t-j)\Big]+(\alpha-1)\sum_{j=1}^{t_0} r_j\mathscr{R}_{t-j}-\mathscr{R}_t.$$

In the last equality, we made use of Remark 2.3.8.

Another summation, over $t = t_0 + 1, \dots, T$, makes appear $V_T = \sum_{t=1}^{T} \mathscr{R}_t$ on the right hand side (we immediately replace a telescopic sum by the end terms and apply (2.19)):

$$\sum_{t_0+1}^{T} \sum_{j=1}^{t_0} \left[(4\widetilde{c}_2 + 2\widetilde{c}_3)\mathscr{R}_{t-j}^{3/2} + \frac{3\widetilde{c}_2}{N_{t-j}} \right] \geq \sum_{j=1}^{t_0} \left[Z_{j-1}(T - j + 1) - Z_{j-1}(t_0 - j + 1) \right] + \varepsilon V_T - c_4 \quad (2.23)$$

Now, if we divide by V_T and let T tend to infinity, the fact that by Theorem 1.5.3, $V_\infty = \infty$ makes disappear several terms: the sum over $3\widetilde{c}_2/N_{t-j}$ is finite by (1.13), and the one with the martingales $Z_{\cdot}(\cdot)$ vanishes by Lemma 2.3.7. This leads directly to (2.21) and concludes the proof. □

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Branching random walks in random environment are diffusive in the regular growth phase

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Abstract

We treat branching random walks in random environment using the framework of Linear Stochastic Evolution. In spatial dimensions three or larger, we establish diffusive behaviour in the entire growth phase. This can be seen through a Central Limit Theorem with respect to the population density as well as through an invariance principle for a path measure we introduce.

Key words: branching random walk, random environment, central limit theorem, invariance principle, diffusivity.

AMS 2000 Subject Classification: Primary 60J80, 60K37, 60F17, 60K35, 82D30.

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1 Introduction

1.1 Background

Branching random walks (and their time–continuous counterpart branching Brownian motion) are treated, with the result of a central limit theorem (CLT), by Watanabe in [Wat67] and [Wat68]. Smith and Wilkinson introduce the notion of random (in time) environment to branching processes [SW69], and in 1972, the book by Athreya and Ney [AN72] appears and gives an excellent overview of the knowledge of the time.

A closely related model, the directed polymers in random environment (DPRE), is studied since the eighties, when the question of diffusivity is treated by Imbrie and Spencer [IS88] as well as Bolthausen [Bol89]. A review can be found in [CSY04].

It took until the new millenium for the time–space random environment known from DPRE to get applied to branching random walks by Birkner, Geiger and Kersting [BGK05]. A CLT in probability is proven in [Yos08a], and improved to an almost sure sense in [Nak11] with the help of Linear Stochastic Evolutions (LSE), which were introduced in [Yos08b] and [Yos10]. Linear stochastic evolutions build a frame to a variety of models, including DPRE. For LSE, the CLT was proven in [Nak09]. Shiozawa treats the time–continuous counterpart, namely branching Brownian motions in random environment [Shi09a, Shi09b].

The present article uses as a blueprint [CY06], which proves a CLT for DPRE, and the larger angle of view allowed by the LSE gives the crucial ingredients to conclude our result, which is a CLT on the event of survival on the entire regular growth phase, but under integrability conditions slightly more restrictive than those from [Nak11]. Compared to the case of DPRE, the necessary notational overhead is unfortunately significantly bigger. Speaking of DPRE, it is possible to extend the results of [CY06] to the case where completely repulsive sites are allowed, using the same conditioning–techniques as here.

A localization–result in the slow growth phase is proven by two of the authors of the present work in [HN11].

1.2 Branching random walks in random environment

We denote the natural numbers by $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ and $\mathbb{N} = \{1, 2, \dots\}$. We will need at various places sets of probability measures, which we write as $\mathscr{P}(\cdot)$; for instance,

$$\mathscr{P}(\mathbb{N}_0) := \Big\{ q = (q(k))_{k\in\mathbb{N}_0} \in [0,1]^{\mathbb{N}_0} : \sum_{k\in\mathbb{N}_0} q(k) = 1 \Big\}$$

stands for the set of probability measures on $\mathbb{N}_0$.

We consider particles in $\mathbb{Z}^d$, $d \geq 1$, each performing a simple random walk and branching into independent copies at each time–step.

i) At time $n = 0$, there is one particle born at the origin $x = 0$.

ii) A particle born at site $x \in \mathbb{Z}^d$ at time $n \in \mathbb{N}_0$ is equipped with k eggs with probability $q_{n,x}(k)$, $k \in \mathbb{N}_0$, independently from other particles.

iii) In the next time step, it takes its k eggs to a uniformly chosen nearest–neighbour site and dies. The eggs then are hatched.

The offspring distributions $q_{n,x} = (q_{n,x}(k))_{k\in\mathbb{N}_0}$ are assumed to be i.i.d. in time–space (n,x). This model is called Branching Random Walks in Random Environment (BRWRE). Let $N_{n,y}$ be the number of the particles which occupy the site $y \in \mathbb{Z}^d$ at time n.

For the proofs in this article, a modeling down to the level of individual particles is needed. First, we define namespaces $\mathcal{V}_n$, $n \in \mathbb{N}_0$ for the n–th generation particles and $\mathcal{V}_{\mathbb{N}_0}$ for the particles of all generations together:

$$\mathcal{V}_0 = \{\mathbf{1}\} = \{(1)\},\ \mathcal{V}_{n+1} = \mathcal{V}_n \times \mathbb{N},\ \text{for } n \geq 0,$$
$$\mathcal{V}_{\mathbb{N}_0} = \bigcup_{n\in\mathbb{N}_0} \mathcal{V}_n.$$

Then, we label all particles as follows:

i) At time $n = 0$, there is just one particle which we call $\mathbf{1} = (1) \in \mathcal{V}_0$.

ii) A particle at time n is identified with its genealogical chart $\mathbf{y} = (1, \mathrm{y}_1, \ldots, \mathrm{y}_n) \in \mathcal{V}_n$. If the particle $\mathbf{y}$ gives birth to $k_{\mathbf{y}}$ particles at time n, then the children are labeled by $(1, \mathrm{y}_1, \ldots, \mathrm{y}_n, 1), \ldots, (1, \mathrm{y}_1, \ldots, \mathrm{y}_n, k_{\mathbf{y}}) \in \mathcal{V}_{n+1}$.

By using this naming procedure, we define the branching of the particles rigorously. This definition is based on the one in [Yos08a].

Note that the particle with name $\mathbf{x}$ can be located at x anywhere in $\mathbb{Z}^d$. As both informations genealogy and place are usually necessary together, it is convenient to combine them to $\mathbb{x} = (x, \mathbf{x})$; think of x and $\mathbf{x}$ written very closely together.

• *Random environment of offspring distibutions:* Set $\Omega_q = \mathscr{P}(\mathbb{N}_0)^{\mathbb{N}_0\times\mathbb{Z}^d}$. The set $\mathscr{P}(\mathbb{N}_0)$ is equipped with the natural Borel σ–field induced by the one of $[0,1]^{\mathbb{N}_0}$. We denote by $\mathscr{G}_q$ the product σ–field on Ω_q.

We fix a product measure $Q \in \mathscr{P}(\Omega_q, \mathscr{G}_q)$ which describes the i.i.d. offspring distributions assigned to each time–space location.

Each environment $q \in \Omega_q$ is a function $(n,x) \mapsto q_{n,x} = (q_{n,x}(k))_{k\in\mathbb{N}_0}$ from $\mathbb{N}_0 \times \mathbb{Z}^d$ to $\mathscr{P}(\mathbb{N}_0)$. We interpret $q_{n,x}$ as the offspring distribution for each particle which occupies the time–space location (n,x).

• *Spatial motion:* A particle at time–space location (n,x) jumps to some neighbouring location $(n+1, y)$ before it is replaced by its children there. Therefore, the spatial motion should be described by assigning a destination to each particle at each time–space location (n,x). We define the measurable space $(\Omega_X, \mathscr{G}_X)$ as the set $(\mathbb{Z}^d)^{\mathbb{N}_0\times\mathbb{Z}^d\times\mathcal{V}_{\mathbb{N}_0}}$ with the product σ–field, and $\Omega_X \ni X \mapsto X_{n,\mathbb{x}}$ for each $(n,\mathbb{x}) \in \mathbb{N}_0 \times (\mathbb{Z}^d \times \mathcal{V}_{\mathbb{N}_0})$ as the projection. We define $P_X \in \mathscr{P}(\Omega_X, \mathscr{G}_X)$ as the product measure such that

$$P_X(X_{n,\mathbb{x}} = e) = \begin{cases} \frac{1}{2d} & \text{if } |e| = 1, \\ 0 & \text{if } |e| \neq 1 \end{cases}$$

for $e \in \mathbb{Z}^d$ and $(n,\mathbb{x}) \in \mathbb{N}_0 \times (\mathbb{Z}^d \times \mathcal{V}_{\mathbb{N}_0})$. Here, we interpret $X_{n,\mathbb{x}}$ as the step at time $n+1$ if the particle $\mathbf{x}$ is located space location x.

• *Offspring realization*: We define the measurable space $(\Omega_K, \mathscr{G}_K)$ as the set ${\mathbb{N}_0}^{\mathbb{N}_0 \times \mathbb{Z}^d \times \mathscr{V}_{\mathbb{N}_0}}$ with the product σ–field, and $\Omega_K \ni K \mapsto K_{n,\mathbb{x}}$ for each $(n,\mathbb{x}) \in \mathbb{N}_0 \times (\mathbb{Z}^d \times \mathscr{V}_{\mathbb{N}_0})$ as the projection. For each fixed $q \in \Omega_q$, we define $P_K^q \in \mathscr{P}(\Omega_K, \mathscr{G}_K)$ as the product measure such that

$$P_K^q(K_{n,\mathbb{x}} = k) = q_{n,x}(k) \quad \text{for all } (n,\mathbb{x}) = (n,x,\mathbf{x}) \in \mathbb{N}_0 \times \mathbb{Z}^d \times \mathscr{V}_{\mathbb{N}_0} \text{ and } k \in \mathbb{N}_0.$$

We interpret $K_{n,\mathbb{x}}$ as the number of eggs of the particle $\mathbf{x}$ if it is located at time–space location (n,x). One could directly speak of its children as well.

The first steps of such a BRWRE are shown in Figure 1.

Putting everything together, we arrive at the

• *Overall construction:* We define $(\Omega, \mathscr{G})$ by

$$\Omega = \Omega_X \times \Omega_K \times \Omega_q, \quad \mathscr{G} = \mathscr{G}_X \otimes \mathscr{G}_K \otimes \mathscr{G}_q,$$

and with $q \in \Omega_q$,

$$P^q = P_X \otimes P_K^q \otimes \delta_q, \quad P = \int Q(dq) P^q.$$

Now that the BRWRE is completely modeled, we can have a look at where the particles are: for $(n,\mathbb{y}) \in \mathbb{N}_0 \times (\mathbb{Z}^d \times \mathscr{V}_{\mathbb{N}_0})$, we define

$$N_{n,\mathbb{y}} = \mathbb{1}_{\{\text{the particle } \mathbf{y} \text{ is located at time–space location } (n,y)\}}.$$

This enables the

• *Placement of BRWRE into the framework of Linear Stochastic Evolutions:* We set the starting condition $N_{0,\mathbb{y}} = \mathbb{1}_{\mathbb{y}=(0,\mathbf{1})}$. Then, defining the matrices $(A_n)_n$ via their entries in the manner indicated below, we can describe $N_{n,\mathbb{y}}$ inductively by

$$\begin{aligned} N_{n,\mathbb{y}} &= \sum_{\mathbb{x} \in \mathbb{Z}^d \times \mathscr{V}_{\mathbb{N}_0}} N_{n-1,\mathbb{x}} \mathbb{1}_{\{y-x=X_{n-1,\mathbb{x}},\ 1 \le \mathbf{y}/\mathbf{x} \le K_{n-1,\mathbb{x}}\}}, \\ &= \sum_{\mathbb{x} \in \mathbb{Z}^d \times \mathscr{V}_{\mathbb{N}_0}} N_{n-1,\mathbb{x}} A_{n,\mathbb{x}}^{\mathbb{y}} \\ &= (N_0 A_1 \cdots A_n)_{\mathbb{y}},\ \mathbb{y} \in \mathbb{Z}^d \times \mathscr{V}_{\mathbb{N}_0}, \end{aligned}$$

where $\mathbf{y}/\mathbf{x}$ is given for $\mathbf{x},\mathbf{y} \in \mathscr{V}_{\mathbb{N}_0}$ as

$$\mathbf{y}/\mathbf{x} = \begin{cases} k & \text{if } \begin{array}{l} \mathbf{x} = (1, \mathrm{x}_1, \ldots, \mathrm{x}_n) \in \mathscr{V}_n, \\ \mathbf{y} = (1, \mathrm{x}_1, \ldots, \mathrm{x}_n, k) \in \mathscr{V}_{n+1} \end{array} \text{ for some } n \in \mathbb{N}_0, \\ \infty & \text{otherwise,} \end{cases}$$

and where

$$A_{n,\mathbb{x}}^{\mathbb{y}} := \mathbb{1}_{\{y-x=X_{n-1,\mathbb{x}},\ 1 \le \mathbf{y}/\mathbf{x} \le K_{n-1,\mathbb{x}}\}},\ \mathbb{x},\mathbb{y} \in \mathbb{Z}^d \times \mathscr{V}_{\mathbb{N}_0}.$$

One–site- and overall population can be defined respectively as

$$N_{n,y} = \sum_{\mathbf{y} \in \mathscr{V}_{\mathbb{N}_0}} N_{n,(y,\mathbf{y})}, \text{ and } N_n = \sum_{\mathbb{y} \in \mathbb{Z}^d \times \mathscr{V}_{\mathbb{N}_0}} N_{n,\mathbb{y}},$$

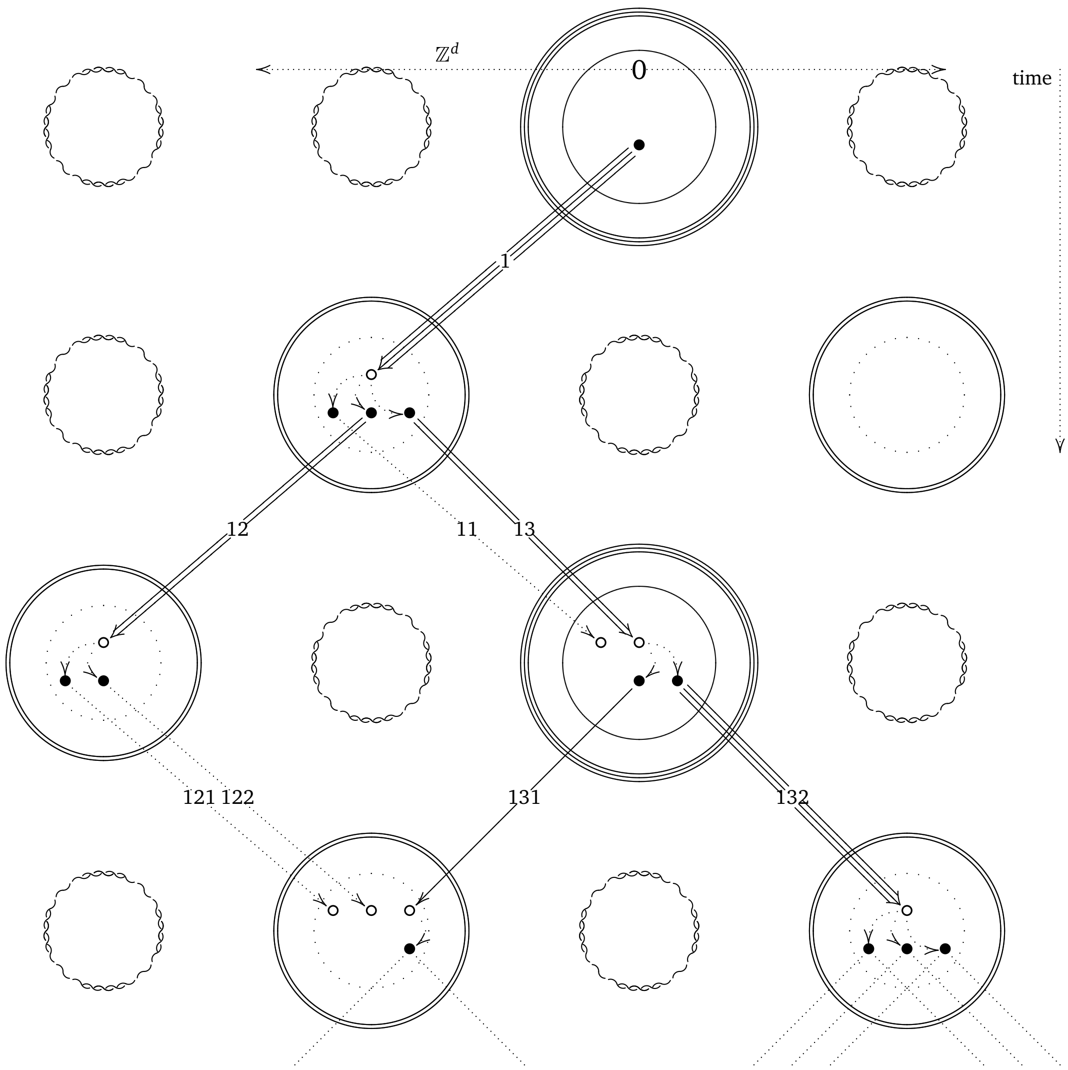


Figure 1: One realization of the first steps and branchings. In this particular example, there are only two types of offspring distibutions, one allowing for one or three eggs, the other one for two or none. This is indicated by the concentrical circles. The curly circles indicate points where the realization of the environment has no influence on the outcome of the random walk. The arrows indicate the movement of the particles, the number of strokes indicating the number of eggs carried. The cones in the lower part of the picture get their full meaning in Remark 2.1.2.

for $n \in \mathbb{N}_0,\ y \in \mathbb{Z}^d$. Other quantities needed later are the moments of the local offspring distributions for $n \in \mathbb{N}_0$ and $x \in \mathbb{Z}^d$,

$$m_{n,x}^{(p)} = \sum_{k \in \mathbb{N}_0} k^p q_{n,x}(k), \quad m^{(p)} = Q(m_{n,x}^{(p)}),\ p \in \mathbb{N}_0, \quad \mathbf{m} = m^{(1)},$$

and the normalized one–site and overall populations

$$\overline{N}_{n,y} = N_{n,y}/\mathbf{m}^n \text{ and } \overline{N}_n = N_n/\mathbf{m}^n, n \in \mathbb{N}_0, y \in \mathbb{Z}^d.$$

It is easy to see that the expectation of the matrix entries, which is an important parameter in the setting of LSE, for $\mathbb{x}, \mathbb{y} \in \mathbb{Z}^d \times \mathcal{V}_{\mathbb{N}_0}$computes as

$$a_{\mathbb{x}}^{\mathbb{y}} := P[A_{1,\mathbb{x}}^{\mathbb{y}}] = \begin{cases} \frac{1}{2d} \sum_{j \geq k} q(j) & \text{if } |x-y| = 1,\ \mathbf{y}/\mathbf{x} = k,\ k \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$$

with

$$q(j) := Q\big(q_{0,0}(j)\big), j \in \mathbb{N}_0.$$

Taking sums, we obtain

$$\sum_{\mathbb{y} \in \mathbb{Z}^d \times \mathcal{V}_{\mathbb{N}_0}} a_{\mathbb{x}}^{\mathbb{y}} = \mathbf{m}, \quad \text{for } \mathbb{x} \in \mathbb{Z}^d \times \mathcal{V}_{\mathbb{N}_0}.$$

1.3 Preliminaries

In this and the following subsection, we gather properties of BRWRE that are already known. First, we introduce the Markov chain $(\mathbb{S}, P_{\mathbb{S}}^{\mathbb{x}}) = ((S, \mathbf{S}), P_{(S,\mathbf{S})}^{(x,\mathbf{x})})$ on $\mathbb{Z}^d \times \mathcal{V}_{\mathbb{N}_0}$ for $\mathbb{x} = (x, \mathbf{x}) \in \mathbb{Z}^d \times \mathcal{V}_{\mathbb{N}_0}$, independent of $(A_n)_{n\geq 1}$, by

$$P_{\mathbb{S}}^{\mathbb{x}}(\mathbb{S}_0 = \mathbb{x}) = 1,$$

$$P_{\mathbb{S}}\big(\mathbb{S}_{n+1} = \mathbb{y} |\ \mathbb{S}_n = \mathbb{x}\big) = \frac{a_{\mathbb{x}}^{\mathbb{y}}}{\mathbf{m}} = \begin{cases} \frac{\sum_{j\geq k} q(j)}{2d\ \mathbf{m}} & \text{if } |x-y| = 1, \text{ and } \mathbf{y}/\mathbf{x} = k \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

where $\mathbb{x}, \mathbb{y} \in \mathbb{Z}^d \times \mathcal{V}_{\mathbb{N}_0}$. The filtration of this random walk will be called $\mathscr{F}_n = \sigma(\mathscr{F}_n^1 \times \mathscr{F}_n^2)$, with $\mathscr{F}_n^1 := \sigma(S_1, \ldots, S_n)$, $\mathscr{F}_n^2 := \sigma(\mathbf{S}_1, \ldots, \mathbf{S}_n)$, $n \in \mathbb{N}_0$, and the corresponding sample space $\Omega^1 \times \Omega^2$.

Note that we can regard S and $\mathbf{S}$ as independent Markov chains on $\mathbb{Z}^d$ and $\mathcal{V}_{\mathbb{N}_0}$, respectively, with S the simple random walk on $\mathbb{Z}^d$.

Next, we introduce a process which is essential to the proof of our results:

$$\zeta_0 = 1 \text{ and for } n \geq 1,\ \zeta_n = \zeta_n(\mathbb{S}) = \prod_{m=1}^{n} \frac{A_{m,\mathbb{S}_{m-1}}^{\mathbb{S}_m}}{a_{\mathbb{S}_{m-1}}^{\mathbb{S}_m}}. \tag{1.2}$$

Lemma 1.3.1. *ζ_n is a martingale with respect to the filtration given by*

$$\mathscr{H}_0 := \sigma(\mathbb{S}_0), \quad \mathscr{H}_n := \sigma(A_m, \mathbb{S}_m;\ m \le n),\ n \ge 1.$$

Moreover, we have that

$$N_{n,\mathbb{y}} = \mathbf{m}^n P_{\mathbb{S}}^{(0,1)}(\zeta_n : \mathbb{S}_n = \mathbb{y})\ P\text{–a.s. for } n \in \mathbb{N}_0,\ \mathbb{y} \in \mathbb{Z}^d \times \mathscr{V}_{\mathbb{N}_0}.$$

Remark 1.3.2. Summation over all possible sequences of names yields

$$N_{n,y} = \mathbf{m}^n P_{\mathbb{S}}^{(0,1)}(\zeta_n : S_n = y).$$

From this Lemma follows an important result: the next Lemma shows that a phase transition occurs for the growth rate of the total population.

Lemma 1.3.3. *$\overline{N}_n$ is a martingale with respect to $\mathscr{G}_n := \sigma(A_m : m \le n)$. Hence, the limit*

$$\overline{N}_\infty = \lim_{n\to\infty} \overline{N}_n,\ \text{exists } P\text{–a.s.} \tag{1.3}$$

and

$$P(\overline{N}_\infty) \in \{0,1\}.$$

Moreover, $P(\overline{N}_\infty) = 1$ if and only if the limit (1.3) *is convergent in $L^1(P)$.*

The proof of Lemmas 1.3.1 and 1.3.3 can be found in [Nak11].

We refer to the case $P(\overline{N}_\infty) = 1$ as *regular growth phase* and to the other one, $P[\overline{N}_\infty] = 0$, as *slow growth phase*. The regular growth phase means that the growth rate of the total population has the same order as the growth rate of the expectation of the total population $\mathbf{m}^n$; on the other hand, the slow growth phase means that, almost surely, the growth rate of the population is lower than the growth rate of its expectation.

One can also introduce the notions of 'survival' and 'extinction'.

Definition 1.3.4. *The event of survival is the existence of particles at all times:*

$$\{\text{survival}\} := \{\forall\ n \in \mathbb{N}_0,\ N_n > 0\}.$$

The extinction event is the complement of survival.

1.4 The result

Definition 1.4.1. *An important quantity of the model is the population density, which can be seen as a probability measure with support on $\mathbb{Z}^d$,*

$$\rho_{n,x} = \rho_n(x) := \frac{N_{n,x}}{N_n} \mathbb{1}_{N_n>0},\ n \in \mathbb{N}_0,\ x \in \mathbb{Z}^d.$$

Our main result is the following CLT, proven as Corollary 2.2.4 of the invariance principle Theorem 2.2.2.

Theorem 1.4.2. *Assume $d \geq 3$ and regular growth, and the moment conditions $m^{(3)} < \infty$ and $Q\big((m^{(2)}_{n,x})^2\big) < \infty$. Then, for all bounded continuous function $F \in \mathscr{C}_b(\mathbb{R}^d)$, in $P(\,\cdot\,|survival)$–probability,*

$$\lim_{n\to\infty}\sum_{x\in\mathbb{Z}^d}\rho_n(x)F\Big(\frac{x}{\sqrt{n}}\Big) = \int_{\mathbb{R}^d} F(x)\nu(dx),$$

where ν stands for the Gaussian measure with mean 0 and covariance matrix $\frac{1}{d}I$.

Remark 1.4.3. The hypothesis $d \geq 3$ is in fact not necessary because in dimensions one and two, regular growth cannot occur. Instead of a CLT, localized behaviour can be observed, see [HY09, HN11].

It is the following equivalence, recently proven as [CY, Proposition 2.2.2], that enables us to speak easily of $P(\,\cdot\,|survival)$–probability:

Lemma 1.4.4. *If $P(\overline{N}_\infty > 0) > 0$ and $\mathbf{m} < \infty$, then*

$$\{regular\ growth\} := \{\overline{N}_\infty > 0\} = \{survival\},\ P\text{–a.s.}.$$

[CY] handles also the case of slow growth.

2 Proofs

2.1 The path measure

Definition 2.1.1. *We set, on $\mathscr{F}_\infty$,*

$$\mu_n(d\mathbb{S}) := \frac{1}{\overline{N}_n}P_{\mathbb{S}}(\zeta_n d\mathbb{S})\mathbb{1}_{\overline{N}_\infty>0},\ n\in\mathbb{N}_0,$$

where ζ is defined in (1.2).

Additional notations and definitions comprise the shifted processes: for $m \in \mathbb{N}_0$, $\mathbb{z} \in \mathbb{Z}^d \times \mathscr{V}_{\mathbb{N}_0}$, we define $N^{m,\mathbb{z}}_n = (N^{m,\mathbb{z}}_{n,\mathbb{y}})_{\mathbb{y}\in\mathbb{Z}^d\times\mathscr{V}_{\mathbb{N}_0}}$ and $\overline{N}^{m,\mathbb{z}}_n = (\overline{N}^{m,\mathbb{z}}_{n,\mathbb{y}})_{\mathbb{y}\in\mathbb{Z}^d\times\mathscr{V}_{\mathbb{N}_0}}$, $n \in \mathbb{N}_0$, respectively by

$$N^{m,\mathbb{z}}_{0,\mathbb{y}} = \mathbb{1}_{\mathbb{y}=\mathbb{z}}, \qquad N^{m,\mathbb{z}}_{n+1,\mathbb{y}} = \sum_{\mathbb{x}\in\mathbb{Z}^d\times\mathscr{V}_{\mathbb{N}_0}} N^{m,\mathbb{z}}_{n,\mathbb{x}}A^{\mathbb{y}}_{m+n+1,\mathbb{x}}, \text{ and}$$
$$\overline{N}^{m,\mathbb{z}}_{n,\mathbb{y}} = N^{m,\mathbb{z}}_{n,\mathbb{y}}/\mathbf{m}^n.$$

Using this, we can, with $m \leq n$, express μ_n on a finite time–horizon as

$$\mu_n(\mathbb{S}_{[0,m]} = \mathbb{x}_{[0,m]}) = \zeta_m(\mathbb{x}_{[0,m]})\frac{\overline{N}^{m,\mathbb{x}_m}_{n-m}}{\overline{N}_n}P_{\mathbb{S}}(\mathbb{S}_{[0,m]} = \mathbb{x}_{[0,m]})\mathbb{1}_{\overline{N}_\infty>0}; \tag{2.1}$$

in particular,

$$\frac{N_{n,x}}{N_n}\mathbb{1}_{\overline{N}_\infty>0} = \sum_{\mathbb{x}_{[0,n]}:x_n=x}\mu_n(\mathbb{S}_{[0,n]} = \mathbb{x}_{[0,n]}).$$

Note that for $B \in \mathscr{F}_\infty$, the limit

$$\mu_\infty(B) = \lim_{n\to\infty} \mu_n(B)$$

exists P–a.s. because of the martingale limit theorem for $P_{\mathbb{S}}(\zeta_n : B)$, which is indeed a positive martingale with respect to the filtration $(\mathscr{G}_n)_n$, as can be easily checked, and for $\overline{N}_n$, see Lemma 1.3.3.

Remark 2.1.2. We can write, for $B \in \mathscr{F}_n^1$,

$$\mu_\infty(B \times \Omega^2) = \frac{1}{\overline{N}_\infty} \sum_{\mathbb{x}_n} P_{\mathbb{S}}(\zeta_n : (B \times \Omega^2) \cap \{\mathbb{S}_n = \mathbb{x}_n\}) \overline{N}_\infty^{n,\mathbb{x}_n} \mathbb{1}_{\overline{N}_\infty > 0}.$$

The reader who cares to return to the lower part of Figure 1 will be rewarded with an intuitive picture of how we can let run our BRW up to time $n = 3$ and plug in there the shifted processes, indicated by the dotted cones.

Definition 2.1.3. *We define the environmental measure conditional on survival, or under the assumptions of Lemma* 1.4.4 *equivalently, regular growth, by*

$$\widetilde{P}(\cdot) = P\left(\,\cdot\, \middle| \overline{N}_\infty > 0\right) = \frac{P(\cdot \cap \overline{N}_\infty > 0)}{P(\overline{N}_\infty > 0)}.$$

Lemma 2.1.4. *Assume regular growth. Then,*

$$\widetilde{P}\mu_\infty(\,\cdot \times \Omega^2) \text{ is a probability measure on } \mathscr{F}_\infty^1, \tag{2.2}$$

and

$$\widetilde{P}\mu_\infty(\,\cdot \times \Omega^2) \ll P_S \text{ on } \mathscr{F}_\infty^1, \tag{2.3}$$

where P_S denotes the measure of a simple random walk.

In order to prove this Lemma, we need the following observation:

Lemma 2.1.5. *Suppose $(B_m)_{m\geq 1} \subset \mathscr{F}_\infty^1$ are such that $\lim_{m\to\infty} P_{\mathbb{S}}(B_m \times \Omega^2) = 0$. Then*

$$0 = \lim_{m\to\infty} \sup_n \widetilde{P}\mu_n(B_m \times \Omega^2) = \lim_{m\to\infty} \widetilde{P}\mu_\infty(B_m \times \Omega^2).$$

Proof. We first prove the first equality. For $\delta > 0$,

$$P\big(\mu_n(B_m \times \Omega^2)\big) \leq P\big(\mu_n(B_m \times \Omega^2) : \overline{N}_n \geq \delta\big) + P\big(\mathbb{1}_{\overline{N}_\infty > 0} : \overline{N}_n \leq \delta\big).$$

We can estimate

$$\begin{aligned}
\sup_n P\big(\mu_n(B_m \times \Omega^2) : \overline{N}_n \geq \delta\big) &\leq \delta^{-1} \sup_n P\big(\overline{N}_n \mu_n(B_m \times \Omega^2)\big) \\
&= \delta^{-1} \sup_n P\Big(\overline{N}_n \frac{P_{\mathbb{S}}(\zeta_n : B_m \times \Omega^2)}{\overline{N}_n} \mathbb{1}_{\overline{N}_\infty > 0}\Big) \\
&\leq \delta^{-1} \sup_n P_{\mathbb{S}}\big(P(\zeta_n) : B_m \times \Omega^2\big) \\
&= \delta^{-1} P_{\mathbb{S}}(B_m \times \Omega^2) \xrightarrow[m\to\infty]{} 0.
\end{aligned}$$

On the other hand, as $\overline{N}_n^{-1}$ converges $\widetilde{P}$–a.s., their distributions are tight, and

$$\lim_{\delta\to 0}\sup_n \widetilde{P}(\overline{N}_n \le \delta) = 0.$$

The second equality follows directly by an application of dominated convergence. □

Proof of Lemma 2.1.4. The statement (2.2) is in some sense an affirmation of well–definiteness. The proof consists in verifying that $\widetilde{P}\mu_\infty$ is finitely additive, that $\widetilde{P}\mu_\infty(\Omega^1 \times \Omega^2) = 1$, and that $\mathscr{F}_\infty \ni B_n \times \Omega^2 \searrow \emptyset$ implies $\widetilde{P}\mu_\infty(B_n \times \Omega^2) \to 0$. The first two are quite obvious and the third one is a trivial application of the preceding Lemma 2.1.5, as is the absolute continuity (2.3). □

In the following Proposition, we introduce the variational norm

$$\|\nu - \nu'\|_{\mathscr{E}} := \sup\{\nu(B) - \nu'(B),\ B \in \mathscr{E}\},$$

where ν and ν' are probability measures on $\mathscr{E}$. This norm will be applied to $\mu_{n+m}(\cdot \times \Omega^2)$ and $\mu_\infty(\cdot \times \Omega^2)$, which are indeed, $\widetilde{P}$–a.s., probability measures on $\mathscr{F}_r^1$ because of the finiteness of $\mathscr{F}_r^1$, for all $r, m, n \in \mathbb{N}_0$.

Proposition 2.1.6. *In the regular growth phase,*

$$\lim_{m\to\infty}\sup_n \widetilde{P}\left(\|\mu_{m+n}(\cdot \times \Omega^2) - \mu_\infty(\cdot \times \Omega^2)\|_{\mathscr{F}_n^1}\right) = 0.$$

Proof. From (2.1) and its analogue for μ_∞, for $n, m \ge 0$,

$$\begin{aligned}
&\overline{N}_\infty \|\mu_{m+n}(\cdot \times \Omega^2) - \mu_\infty(\cdot \times \Omega^2)\|_{\mathscr{F}_n^1} \\
&= \overline{N}_\infty \sup_{B = B^1 \times \Omega^2 \in \mathscr{F}_n^1 \otimes \mathscr{F}_n^2} \left\{ P_{\mathbb{S}}\left(\zeta_n \frac{\overline{N}_m^{n,\mathbb{S}_n}}{\overline{N}_{n+m}} \mathbb{1}_B - \zeta_n \frac{\overline{N}_\infty^{n,\mathbb{S}_n}}{\overline{N}_\infty} \mathbb{1}_B\right) \mathbb{1}_{\overline{N}_\infty > 0}\right\} \\
&\le \overline{N}_\infty P_{\mathbb{S}}\left(\zeta_n \left| \frac{\overline{N}_m^{n,\mathbb{S}_n}}{\overline{N}_{n+m}} - \frac{\overline{N}_\infty^{n,\mathbb{S}_n}}{\overline{N}_\infty} \right| \right) \mathbb{1}_{\overline{N}_\infty > 0} \\
&= \mathbb{1}_{\overline{N}_\infty > 0} \overline{N}_{n+m}^{-1} P_{\mathbb{S}}\left(\zeta_n |\overline{N}_\infty \overline{N}_m^{n,\mathbb{S}_n} - \overline{N}_{n+m} \overline{N}_\infty^{n,\mathbb{S}_n}|\right) \\
&\le \mathbb{1}_{\overline{N}_\infty > 0} \overline{N}_{n+m}^{-1} P_{\mathbb{S}}\left(\zeta_n \left[|\overline{N}_\infty \overline{N}_m^{n,\mathbb{S}_n} - \overline{N}_{n+m} \overline{N}_m^{n,\mathbb{S}_n}| + |\overline{N}_{n+m} \overline{N}_m^{n,\mathbb{S}_n} - \overline{N}_{n+m} \overline{N}_\infty^{n,\mathbb{S}_n}| \right]\right) \\
&\le \frac{|\overline{N}_\infty - \overline{N}_{n+m}|}{\overline{N}_{n+m}} P_{\mathbb{S}}\left(\zeta_n \overline{N}_m^{n,\mathbb{S}_n}\right) + P_{\mathbb{S}}\left(\zeta_n |\overline{N}_m^{n,\mathbb{S}_n} - \overline{N}_\infty^{n,\mathbb{S}_n}|\right).
\end{aligned}$$

Note that in the first of the right–hand terms, the denominator is cancelled out with $P_{\mathbb{S}}(\zeta_n \overline{N}_m^{n,\mathbb{S}_n})$; so, as $\overline{N}_n$ converges in $L^1(P)$, the P–expectation of the first term vanishes as $m \to \infty$, and the second one yields

$$\begin{aligned}
PP_{\mathbb{S}}\left(\zeta_n \left|\overline{N}_m^{n,\mathbb{S}_n} - \overline{N}_\infty^{n,\mathbb{S}_n}\right|\right) &= PP_{\mathbb{S}}\left(\zeta_n P\left(\left|\overline{N}_m^{n,\mathbb{S}_n} - \overline{N}_\infty^{n,\mathbb{S}_n}\right| \,\middle|\, \mathscr{G}_n\right)\right) \\
&= PP_{\mathbb{S}}\left(\zeta_n \left\|\overline{N}_m - \overline{N}_\infty\right\|_{L^1(P)}\right) = \|\overline{N}_m - \overline{N}_\infty\|_{L^1(P)} \xrightarrow[m\to\infty]{} 0.
\end{aligned}$$

This proves

$$\sup_n P\big(\overline{N}_\infty \|\mu_{m+n} - \mu_\infty\|_{\mathscr{F}_n^1}\big) \xrightarrow[m\to\infty]{} 0.$$

Now, we use the same trick with the Chebychev–inequality that gives us a $\overline{N}_\infty$ in front of the norm as in Lemma 2.1.4:

$$\begin{aligned}\widetilde{P}\big(\|\mu_{m+n} - \mu_\infty\|_{\mathscr{F}_n^1}\big) &= \widetilde{P}\big(\|\mu_{m+n} - \mu\|_{\mathscr{F}_n^1}(\mathbb{1}_{\overline{N}_\infty > \delta} + \mathbb{1}_{\overline{N}_\infty \le \delta})\big) \\ &\le \delta^{-1}\widetilde{P}\big(\overline{N}_\infty \|\mu_{m+n} - \mu_\infty\|_{\mathscr{F}_n^1}\big) + 2\widetilde{P}\big(\overline{N}_\infty \le \delta\big)\end{aligned}$$

tends to 0 with $\delta \to 0,\ m \to \infty$ if we control δ and m appropriately, independently of n. □

2.2 The main statements

Definition 2.2.1. *For $n \ge 1$, the rescaling of the path S is defined by*

$$S_t^{(n)} = S_{nt}/\sqrt{n}, \quad 0 \le t \le 1,$$

with $(S_t)_{t\ge 0}$ the linear interpolation of $(S_n)_{n\in\mathbb{N}}$. We write $S^{(n)}$ for $(S_t^{(n)})_{t\ge 0}$.
Furthermore, we will denote by $\mathbb{W} = \{w \in \mathscr{C}([0,1] \to \mathbb{R}^d); w(0) = 0\}$ the d–dimensional Wiener–space, equipped with the topology induced by the supremum–norm. The probability space $(\mathbb{W}, \mathscr{F}^{\mathbb{W}}, P^{\mathbb{W}})$ features the Borel–σ–algebra $F^{\mathbb{W}}$ and $P^{\mathbb{W}}$ the Wiener–measure. We will be using $W = (W_t)_{t\ge 0}$ a Wiener-process on this probability-space.

Theorem 2.2.2. *Assume $d \ge 3$ and regular growth, and the technical assumptions $m^{(3)} < \infty$, $P\big((m_{0,0}^{(2)})^2\big) < \infty$. Then, for all $F \in \mathscr{C}_b(\mathbb{W})$,*

$$\lim_{n\to\infty} \mu_n\big(F(S^{(n)})\big) = P^{\mathbb{W}}\big(F(W/\sqrt{d})\big), \tag{2.4}$$

$$\lim_{n\to\infty} \mu_\infty\big(F(S^{(n)})\big) = P^{\mathbb{W}}\big(F(W/\sqrt{d})\big), \tag{2.5}$$

in $\widetilde{P}$–probability.

Remark 2.2.3. This is equivalent to $L^p(\widetilde{P})$–convergence for any finite p.

This Theorem implies the following CLT:

Corollary 2.2.4. *Under the same assumptions as in the Theorem, for all $F \in \mathscr{C}_b(\mathbb{R}^d)$,*

$$\lim_{n\to\infty} \sum_{x\in\mathbb{Z}^d} F\Big(\frac{x}{\sqrt{n}}\Big)\frac{\overline{N}_{n,x}}{\overline{N}_n} = \int_{\mathbb{R}^d} F(x)d\nu(x), \text{ in } \widetilde{P}\text{–probability,}$$

where ν designs the Gaussian measure with mean 0 and covariance matrix $\frac{1}{d}I$.

2.3 Some easier analogue of the main Theorem

The following Proposition is not needed for the proof of our result. We literally propose it nevertheless to the reader's attention because the proof is much easier than the one of Theorem 2.2.2, while the proceeding is the same. Basically, it can be done with the one–dimensional tools we have at hand from subsection 2.1 and without the technical hassles in Lemmas 2.4.2, 2.4.8, and 2.4.13. We will try to break it down to small parts as much as we can, and refer to these parts in the proof of Theorem 2.2.2.

Proposition 2.3.1. *Assume regular growth. Then,*

$$\lim_{n\to\infty} \widetilde{P}\mu_n(S^{(n)} \in \cdot) = P^{\mathbb{W}}(W/\sqrt{d} \in \cdot), \text{ weakly,} \tag{2.6}$$

$$\lim_{n\to\infty} \widetilde{P}\mu_\infty(S^{(n)} \in \cdot) = P^{\mathbb{W}}(W/\sqrt{d} \in \cdot), \text{ weakly.} \tag{2.7}$$

The following notation will prove useful.

Definition 2.3.2. *We define, for $w \in \mathbb{W}$,*

$$\overline{F}(w) = F(w) - P^{\mathbb{W}}\big(F(\frac{W}{\sqrt{d}})\big),\ F \in \mathscr{C}_b(\mathbb{W})$$

and

$$BL(\mathbb{W}) = \{F : \mathbb{W} \to \mathbb{R};\ \|F\|_{BL} := \|F\| + \|F\|_L < \infty\}$$

the set of bounded Lipschitz–functionals on $\mathbb{W}$. The two norms are defined respectively by

$$\|F\| := \sup_{w\in\mathbb{W}} |F(w)|,$$
$$\|F\|_L := \sup\left\{ \frac{F(w) - F(\widetilde{w})}{\|w - \widetilde{w}\|} :\ w \neq \widetilde{w} \in \mathbb{W} \right\}.$$

Proof of Proposition 2.3.1. The second statement is easier to prove. We attack it first, and use it later to manage the first one.

Two ingredients from outside this article will help us to prove (2.7). First, (2.7) is equivalent to

$$\lim_{m\to\infty} \widetilde{P}\mu_\infty(\overline{F}(S^{(m)})) = 0 \text{ for all } F \in BL(\mathbb{W}), \tag{2.8}$$

e.g., [Dud89, Theorem 11.3.3].

To prove (2.8), we make use of the following result for the simple random walk (S, P_S), see [AW00]: If $(n_k)_{k\geq 1} \subset \mathbb{Z}_+$ is an increasing sequence such that $\inf_{k\geq 1} n_{k+1}/n_k > 1$, then for any $F \in BL(\mathbb{W})$,

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} \overline{F}(S^{(n_k)}) = 0,\ P_S\text{–a.s..} \tag{2.9}$$

One of the key ideas of the proof is that in the last line, due to (2.3), we can replace 'P_S–a.s.' by '$\widetilde{P}\mu(\cdot \times \Omega^2)$–a.s.', and the statement still holds.

This enables us to prove (2.8) by contradiction. Assume that (2.8) does not hold. Then there is some subsequence $a_{m_l} = \widetilde{P}\mu_\infty(\overline{F}(S^{(m_l)})) > c > 0$ (or $< c < 0$). It has bounded domain, so has a convergent subsequence $a_{m_{l_k}}$ which can be chosen such that $n_k := m_{l_k}$ satisfies the above $\inf_{k\geq 1} n_{k+1}/n_k > 1$. To this n_k, we apply (2.9) and integrate with respect to $\widetilde{P}\mu_\infty$. By dominated convergence, we can switch integration and limit and get

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} \widetilde{P}\mu_\infty \big(\overline{F}(S^{(n_k)})\big) = 0.$$

But this is a contradition to the assumption that all the $\widetilde{P}\mu_\infty(\overline{F}(S^{(n_k)})) = \widetilde{P}\mu_\infty(\overline{F}(S^{(m_{l_k})})) > c$ (or $< c$). So we conclude that (2.8) does hold, indeed.

Now, it remains to prove (2.6) with the help of (2.7). We need to show the analogue of (2.8):

$$\lim_{n\to\infty} \widetilde{P}\mu_n \big(\overline{F}(S^{(n)})\big) = 0 \text{ for all } F \in BL(\mathbb{W}). \tag{2.10}$$

For $0 \leq k \leq n$, we add some telescopic terms:

$$\begin{aligned} \widetilde{P}\mu_n \big(\overline{F}(S^{(n)})\big) &= \widetilde{P}\mu_n \big(\overline{F}(S^{(n)}) - \overline{F}(S^{(n-k)})\big) \\ &\quad + \widetilde{P}\mu_n \big(\overline{F}(S^{(n-k)})\big) - \widetilde{P}\mu_\infty \big(\overline{F}(S^{(n-k)})\big) \\ &\quad + \widetilde{P}\mu_\infty \big(\overline{F}(S^{(n-k)})\big) \end{aligned} \tag{2.11}$$

We apply what we just proved, i.e. (2.8), and conclude that the last line vanishes for fixed k and $n \to \infty$. The middle one does the same due to Proposition 2.1.6. As for the first line, we note that $\overline{F}$ is uniformly continuous and that

$$\sup_{S\in\Omega^1} \max_{0\leq t\leq 1} \left| S_t^{(n)} - S_t^{(n-k)} \right| = O(k/\sqrt{n}).$$

Hence, (2.10) holds, so that we conclude (2.6) and thus the Proposition. □

2.4 The real work

In order to prove the statement of Theorem 2.2.2 'in probability', we take the path via 'L^2'. While the proceeding is basically the same as in the last section, the notation becomes much more complicated. As a start, we take a copy of our path $\mathbb{S}$:

Definition 2.4.1. *Let $(\widetilde{\mathbb{S}}, P_{\widetilde{\mathbb{S}}})$ be an independent copy of $(\mathbb{S}, P_{\mathbb{S}})$ defined on the probability space $(\widetilde{\Omega} = \widetilde{\Omega}^1 \times \widetilde{\Omega}^2, \widetilde{\mathscr{F}})$ for $i = 1, 2, 3, 4$. Similarily, we write $\widetilde{\zeta} = \zeta(\widetilde{\mathbb{S}})$, $P_{\mathbb{S}\widetilde{\mathbb{S}}}$, and $P_{S\widetilde{S}}$ for the simultaneous product measures and so on.*

Lemma 2.4.2. *For all $B \in \mathscr{F}_\infty^1 \otimes \widetilde{\mathscr{F}}_\infty^1$, with the notation $\overline{B} = B \times \Omega^2 \times \widetilde{\Omega}^2$, the following limit exists P–a.s. in the regular growth phase:*

$$\mu_\infty^{(2)}(B) = \lim_{n\to\infty} \mu_n^{\otimes 2}(\overline{B}), \tag{2.12}$$

where we define

$$\mu_n^{\otimes 2}(\overline{B}) = \frac{1}{\overline{N}_n^2} P_{\mathbb{S}\widetilde{\mathbb{S}}}\big(\zeta_n \widetilde{\zeta}_n \mathbb{1}_{\overline{B}}\big) \mathbb{1}_{\overline{N}_\infty > 0},$$

Moreover, we have that for all $n \in \mathbb{N}$, P–a.s. on $\{\overline{N}_\infty > 0\}$,

$$\begin{aligned} &\mu^{(2)}_\infty\Big((S,\widetilde{S})_{[0,n]} = \{(x_k,\widetilde{x}_k)\}_{k=1}^n\Big) \\ &= \frac{1}{\overline{N}^2_\infty} \sum_{\mathbf{x}_n,\widetilde{\mathbf{x}}_n \in \mathscr{V}_{\mathbb{N}_0}} \overline{N}^{n,(x_n,\mathbf{x}_n)}_\infty \overline{N}^{n,(\widetilde{x}_n,\widetilde{\mathbf{x}}_n)}_\infty \cdot \\ &\qquad \cdot P_{\mathbb{S}\widetilde{\mathbb{S}}}\Big(\zeta_n\widetilde{\zeta}_n : (S,\widetilde{S})_{[0,n]} = \{(x_k,\widetilde{x}_k)\}_{k=1}^n, (\mathbf{S}_n,\widetilde{\mathbf{S}}_n) = (\mathbf{x}_n,\widetilde{\mathbf{x}}_n)\Big). \end{aligned} \tag{2.13}$$

For the proof, we need a few Definitions and Lemmas.

Definition 2.4.3. *For $B \in \mathscr{F}^1_\infty \otimes \widetilde{\mathscr{F}}^1_\infty$, define the processes $(X_n)_{n\in\mathbb{N}_0}$ and $(Y_n)_{n\in\mathbb{N}_0}$ which depend on B as*

$$X_0 = X_1 := 0,\ X_n = X_n(B) := P_{\mathbb{S}\widetilde{\mathbb{S}}}\big(\zeta_n\widetilde{\zeta}_n \mathbb{1}_{\overline{B}\cap\{\mathbb{S}_{n-1}\neq\widetilde{\mathbb{S}}_{n-1}\}}\big);$$

$$Y_0 := P_{\mathbb{S}\widetilde{\mathbb{S}}}(\overline{B}),\ Y_1 = Y_1(B) := P_{\mathbb{S}\widetilde{\mathbb{S}}}\big(\zeta_1\widetilde{\zeta}_1\mathbb{1}_{\overline{B}}\big),\ Y_n = Y_n(B) := P_{\mathbb{S}\widetilde{\mathbb{S}}}\big(\zeta_n\widetilde{\zeta}_n\mathbb{1}_{\overline{B}\cap\{\mathbb{S}_{n-1}=\widetilde{\mathbb{S}}_{n-1}\}}\big).$$

Lemma 2.4.4. *Y_n converges to 0 P–almost surely, independently of B.*

Proof. A consequence of the construction of the BRWRE is that $\zeta_n\widetilde{\zeta}_n\mathbb{1}_{\{\mathbb{S}_{n-1}\neq\widetilde{\mathbb{S}}_{n-1},\mathbb{S}_n=\widetilde{\mathbb{S}}_n\}} = 0$, $P \otimes P_{\mathbb{S}\widetilde{\mathbb{S}}}$–a.s., so that we have

$$\begin{aligned} 0 \le P\big(Y_n\big) &\le PP_{\mathbb{S}\widetilde{\mathbb{S}}}\big(\zeta_n\widetilde{\zeta}_n\mathbb{1}_{\mathbb{S}_{n-1}=\widetilde{\mathbb{S}}_{n-1}}\big) \\ &= PP_{\mathbb{S}\widetilde{\mathbb{S}}}\big(\zeta_n\widetilde{\zeta}_n\mathbb{1}_{\mathbb{S}_{[0,n-1]}=\widetilde{\mathbb{S}}_{[0,n-1]}}\big) \\ &= PP_{\mathbb{S}\widetilde{\mathbb{S}}}\left(\prod_{k=1}^{n-1}\frac{(A^{\mathbb{S}_k}_{k,\mathbb{S}_{k-1}})^2}{(a^{\mathbb{S}_k}_{\mathbb{S}_{k-1}})^2}\mathbb{1}_{\mathbb{S}_{[0,n-1]}=\widetilde{\mathbb{S}}_{[0,n-1]}} P_{\mathbb{S}\widetilde{\mathbb{S}}}\Big(\frac{P(A^{\mathbb{S}_n}_{n,\mathbb{S}_{n-1}}A^{\widetilde{\mathbb{S}}_n}_{n,\widetilde{\mathbb{S}}_{n-1}}|\mathscr{G}_{n-1})}{a^{\mathbb{S}_n}_{\mathbb{S}_{n-1}}a^{\widetilde{\mathbb{S}}_n}_{\widetilde{\mathbb{S}}_{n-1}}}\Big|\mathscr{F}_{n-1},\widetilde{\mathscr{F}}_{n-1}\Big)\right) \\ &= P_{\mathbb{S}\widetilde{\mathbb{S}}}\Big(\prod_{k=1}^{n-1}\frac{1}{a^{\mathbb{S}_k}_{\mathbb{S}_{k-1}}} : \mathbb{S}_{[0,n-1]} = \widetilde{\mathbb{S}}_{[0,n-1]}\Big)\sum_{\mathbb{x},\,\mathbb{y}}\frac{P\big(A^{\mathbb{x}}_{1,(0,\mathbf{1})}A^{\mathbb{y}}_{1,(0,\mathbf{1})}\big)}{a^{\mathbb{y}}_{\mathbb{x}}}\frac{a^{\mathbb{y}}_{\mathbb{x}}}{\mathbf{m}^2} \\ &= \frac{m^{(2)}}{\mathbf{m}^2}\frac{1}{\mathbf{m}^{n-1}} \end{aligned} \tag{2.14}$$

We made use of the fact that in the third line, because the $A^{\mathbb{S}_k}_{k,\mathbb{S}_{k-1}}$'s are indicators, we can erase the square. Also erasable is the condition in the inner P–expectation. After that, the outmost P–expectation can be taken into the first fraction, cancelling out one of the $a^{\mathbb{S}_k}_{\mathbb{S}_{k-1}}$'s. To what remains, we apply the definition of the expectation, using (1.1). This technique is hinted in the second part of the fifth line, and applied similarly to the first part.

The assertion now follows from the Borel–Cantelli lemma. □

Lemma 2.4.5. *X_n is a submartingale with respect to $\mathscr{G}_n$.*

Proof. We start calculating

$$\begin{aligned} P(X_n|\mathscr{G}_{n-1}) &= P\big(P_{\mathbb{S}\widetilde{\mathbb{S}}}(\zeta_n\widetilde{\zeta}_n\mathbb{1}_{\overline{B}\cap\{\mathbb{S}_{n-1}\neq\widetilde{\mathbb{S}}_{n-1}\}})\big|\mathscr{G}_{n-1}\big) \\ &= P_{\mathbb{S}\widetilde{\mathbb{S}}}\left(\zeta_{n-1}\widetilde{\zeta}_{n-1}\mathbb{1}_{\overline{B}\cap\{\mathbb{S}_{n-1}\neq\widetilde{\mathbb{S}}_{n-1}\}}P\Big(\frac{A^{\mathbb{S}_n}_{n,\mathbb{S}_{n-1}}}{a^{\mathbb{S}_n}_{\mathbb{S}_{n-1}}}\frac{A^{\widetilde{\mathbb{S}}_n}_{n,\widetilde{\mathbb{S}}_{n-1}}}{a^{\widetilde{\mathbb{S}}_n}_{\widetilde{\mathbb{S}}_{n-1}}}\Big)\right). \end{aligned} \tag{2.15}$$

We do not use the following definition again, but we should like to point out its similarity to W to be defined later. The inner P–expectation computes as

$$w(\mathbb{x},\widetilde{\mathbb{x}},\mathbb{y},\widetilde{\mathbb{y}}) := \frac{P(A_{1,\mathbb{x}}^{\mathbb{y}}A_{1,\widetilde{\mathbb{x}}}^{\widetilde{\mathbb{y}}})}{a_{\mathbb{x}}^{\mathbb{y}}a_{\widetilde{\mathbb{x}}}^{\widetilde{\mathbb{y}}}} = \begin{cases} 0 & \text{if } a_{\mathbb{x}}^{\mathbb{y}}a_{\widetilde{\mathbb{x}}}^{\widetilde{\mathbb{y}}} = 0, \\ 1 & \text{if } x \neq \widetilde{x},\ a_{\mathbb{x}}^{\mathbb{y}}a_{\widetilde{\mathbb{x}}}^{\widetilde{\mathbb{y}}} \neq 0, \\ \frac{P(\sum_{i\geq k} q_{0,0}(i)\sum_{j\geq l} q_{0,0}(j))}{\sum_{i\geq k} q(i)\sum_{j\geq l} q(j)} & \text{if } \begin{array}{l} x=\widetilde{x},\ \mathbf{x}\neq\widetilde{\mathbf{x}}, \\ \mathbf{y}/\mathbf{x}=k,\ \widetilde{\mathbf{y}}/\widetilde{\mathbf{x}}=l, \end{array} a_{\mathbb{x}}^{\mathbb{y}}a_{\widetilde{\mathbb{x}}}^{\widetilde{\mathbb{y}}} \neq 0, \\ 0 & \text{if } \mathbb{x}=\widetilde{\mathbb{x}},\ y\neq\widetilde{y},\ a_{\mathbb{x}}^{\mathbb{y}}a_{\widetilde{\mathbb{x}}}^{\widetilde{\mathbb{y}}} \neq 0, \\ \left[\frac{1}{2d}\sum_{j\geq\min\{k,l\}} q(j)\right]^{-1} & \text{if } \begin{array}{l} \mathbb{x}=\widetilde{\mathbb{x}},\ y=\widetilde{y}, \\ \mathbf{y}/\mathbf{x}=k,\ \widetilde{\mathbf{y}}/\widetilde{\mathbf{x}}=l, \end{array} a_{\mathbb{x}}^{\mathbb{y}}a_{\widetilde{\mathbb{x}}}^{\widetilde{\mathbb{y}}} \neq 0. \end{cases}$$

Using this, we note that, under the condition $\{\mathbf{S}_{n-1}\neq\widetilde{\mathbf{S}}_{n-1}\}$, $w(\mathbb{S}_{n-1},\widetilde{\mathbb{S}}_{n-1},\mathbb{S}_n,\widetilde{\mathbb{S}}_n)$ depends only on $S_{n-1}-\widetilde{S}_{n-1}$, $\mathbf{S}_n/\mathbf{S}_{n-1}$ and $\widetilde{\mathbf{S}}_n/\widetilde{\mathbf{S}}_{n-1}$. Thus, we pursue

$$(2.15) = P_{\mathbb{S}\widetilde{\mathbb{S}}}\big(\zeta_{n-1}\widetilde{\zeta}_{n-1}\mathbb{1}_{\overline{B}\cap\{\mathbb{S}_{n-1}\neq\widetilde{\mathbb{S}}_{n-1}\}}(\mathbb{1}_{S_{n-1}\neq\widetilde{S}_{n-1}}+\alpha\mathbb{1}_{S_{n-1}=\widetilde{S}_{n-1}})\big), \tag{2.16}$$

where $\alpha = P(m_{0,0}^2)/\mathbf{m}^2 > 1$. This last equality is obtained by introducing a $P_{\mathbb{S}\widetilde{\mathbb{S}}}(\,\cdot\,|\mathscr{F}_{n-1},\widetilde{\mathscr{F}}_{n-1})$–conditional expectation, and remarking that the event $\overline{B}$ depends only on the random walk–part while the corresponding above fraction depends only on the children–part, and the two are thus independent. The calculus reads as follows:

$$\begin{aligned} &P_{\mathbb{S}\widetilde{\mathbb{S}}}\Bigg(\zeta_{n-1}\widetilde{\zeta}_{n-1}\mathbb{1}_{\{S_{n-1}=\widetilde{S}_{n-1},\mathbf{S}_{n-1}\neq\widetilde{\mathbf{S}}_{n-1}\}} \\ &\qquad\qquad P_{\mathbb{S}\widetilde{\mathbb{S}}}\Big(\frac{P(\sum_{i\geq\mathbf{S}_n/\mathbf{S}_{n-1}} q_{0,0}(i)\sum_{j\geq\widetilde{\mathbf{S}}_n/\widetilde{\mathbf{S}}_{n-1}} q_{0,0}(j))}{\sum_{i\geq\mathbf{S}_n/\mathbf{S}_{n-1}} q(i)\sum_{j\geq\widetilde{\mathbf{S}}_n/\widetilde{\mathbf{S}}_{n-1}} q(j)} : \overline{B}\Big|\mathscr{F}_{n-1},\widetilde{\mathscr{F}}_{n-1}\Big)\Bigg) \\ &= P_{\mathbb{S}\widetilde{\mathbb{S}}}\Big(\zeta_{n-1}\widetilde{\zeta}_{n-1}\mathbb{1}_{\{S_{n-1}=\widetilde{S}_{n-1},\mathbf{S}_{n-1}\neq\widetilde{\mathbf{S}}_{n-1}\}} \\ &\qquad\qquad \sum_{\mathbf{x},\mathbf{y}}\frac{P(\sum_{i\geq\mathbf{x}/\mathbf{S}_{n-1}} q_{0,0}(i)\sum_{j\geq\mathbf{y}/\widetilde{\mathbf{S}}_{n-1}} q_{0,0}(j))}{\mathbf{m}^2}P_{\mathbb{S}\widetilde{\mathbb{S}}}(\overline{B}|\mathscr{F}_{n-1},\widetilde{\mathscr{F}}_{n-1})\Big) \\ &= P_{\mathbb{S}\widetilde{\mathbb{S}}}\big(P_{\mathbb{S}\widetilde{\mathbb{S}}}(\zeta_{n-1}\widetilde{\zeta}_{n-1}\mathbb{1}_{\{S_{n-1}=\widetilde{S}_{n-1},\mathbf{S}_{n-1}\neq\widetilde{\mathbf{S}}_{n-1},\overline{B}\}}\frac{P(m_{0,0}^2)}{\mathbf{m}^2}|\mathscr{F}_{n-1},\widetilde{\mathscr{F}}_{n-1})\big) \end{aligned}$$

The BRWRE has, due to the strict construction of the ancestry, the feature that

$$\zeta_{n-1}\widetilde{\zeta}_{n-1}\mathbb{1}_{\overline{B}\cap\{\mathbb{S}_{n-1}\neq\widetilde{\mathbb{S}}_{n-1}\}} \geq \zeta_{n-1}\widetilde{\zeta}_{n-1}\mathbb{1}_{\overline{B}\cap\{\mathbb{S}_{n-2}\neq\widetilde{\mathbb{S}}_{n-2}\}}.$$

So, we continue (2.16) and finish the proof of the submartingale property by

$$(2.16) \geq P_{\mathbb{S}\widetilde{\mathbb{S}}}\Big(\zeta_{n-1}\widetilde{\zeta}_{n-1}\mathbb{1}_{\overline{B}\cap\{\mathbb{S}_{n-2}\neq\widetilde{\mathbb{S}}_{n-2}\}}\Big) = X_{n-1}.$$

□

Notation 2.4.6. For some sequence $(a_n)_{n\geq0}$, we set $\Delta a_n := a_n - a_{n-1}$ for $n\geq1$.

This notation is convenient when we treat the Doob–decomposition of the process X_n from Definition 2.4.3, i.e.

Definition 2.4.7.

$$X_n = X_n(B) =: M_n + \widehat{A}_n, \tag{2.17}$$

with M_n a martingale, $M_0 = \widehat{A}_0 = 0$, and $\widehat{A}_n$ the increasing process defined by its increments $\Delta\widehat{A}_n := P(\Delta X_n|\mathscr{G}_{n-1})$. By $\langle M\rangle_n$, we denote the quadratic variation of $(M_n)_n$, defined by $\Delta\langle M\rangle_n := P\big((\Delta M_n)^2|\mathscr{G}_{n-1}\big)$. Passing to the limit, we define

$$\widehat{A}_\infty := \lim_{n\to\infty} \widehat{A}_n, \quad \langle M\rangle_\infty := \lim_{n\to\infty} \langle M\rangle_n. \tag{2.18}$$

The next Lemma deals with the two processes $\widehat{A}_n$ and M_n:

Lemma 2.4.8.

$$\widehat{A}_\infty < \infty \text{ and } \langle M\rangle_\infty < \infty, \quad P\text{–a.s..}$$

Now take a sequence of events $(B_m)_{m\in\mathbb{N}_0}$ verifying $P^{\otimes 2}_{\widetilde{\mathbb{S}}\widetilde{\mathbb{S}}}(\overline{B}_m) \searrow 0$ with $m \to \infty$. If we replace B by B_m and define $X^m_n := X_n(B_m)$ together with its Doob–decomposition $M^m_n + \widehat{A}^m_n$, $m, n \in \mathbb{N}_0$, and the corresponding limits as in (2.18)*, we have*

$$\widehat{A}^m_\infty \xrightarrow[m\to\infty]{} 0 \text{ and } \langle M^m\rangle_\infty \xrightarrow[m\to\infty]{} 0, \quad P\text{–a.s..} \tag{2.19}$$

The proof is lengthy and will be postponed a little bit. But with this Lemma at hand, we can catch up on the

Proof of Lemma 2.4.2. Applying the 'B'–version of the last Lemma, we get that X_n converges, and by the convergence of $\overline{N}_n^{-2}$, $\mu_n^{\otimes 2} = \overline{N}_n^{-2}(X_n + Y_n)\mathbb{1}_{\overline{N}_\infty > 0}$ as well, $\widetilde{P}$–a.s. On the event of extinction, the statement is trivial, and we conclude (2.12).

The second statement (2.13) follows immediately from the definition. □

In order to prove Lemma 2.4.8, we also need the so called *replica overlap*, which is the probability of two particles to meet at the same place:

$$\mathscr{R}_n := \mathbb{1}_{N_n > 0} \sum_x \frac{N_{n,x}}{N_n}.$$

This replica overlap can be related to the event of survival via a Corollary of the following general result for martingales [Yos10, Proposition 2.1.2].

Proposition 2.4.9. *Let $(\mathsf{Y}_n)_{n\in\mathbb{N}_0}$ be a mean–zero martingale on a probability space with measure $\mathbb{E}$ and filtration $(\mathscr{I}_n)_{n\in\mathbb{N}_0}$ such that $-1 \le \Delta\mathsf{Y}_n$, $\mathbb{E}$–a.s. and*

$$\mathsf{X}_n := \prod_{m=1}^n (1 + \Delta\mathsf{Y}_m).$$

Then,

$$\{\mathsf{X}_\infty > 0\} \supseteq \{\mathsf{X}_n > 0 \text{ for all } n \ge 0\} \cap \left\{ \sum_{N=1}^\infty \mathbb{E}\left[(\Delta\mathsf{Y}_n)^2 \middle| \mathscr{I}_{n-1}\right] < \infty \right\}, \ \mathbb{E}\text{–a.s.,} \tag{2.20}$$

holds if Y_n is square–integrable and $\mathbb{E}\left[(\Delta Y_n)^2\middle|\mathscr{I}_{n-1}\right]$ is uniformly bounded.
The opposite inclusion is provided by Y_n being cube–integrable and

$$\mathbb{E}\left[(\Delta \mathsf{Y}_n)^3\middle|\mathscr{I}_{n-1}\right] \leq const \cdot \mathbb{E}\left[(\Delta \mathsf{Y}_n)^2\middle|\mathscr{I}_{n-1}\right].$$

Corollary 2.4.10. *Suppose $P(\overline{N}_\infty > 0) > 0$ and $m^{(3)} < \infty$. Then*

$$\{\overline{N}_\infty > 0\} = \{survival\} \cap \Big\{\sum_{n\geq 0} \mathscr{R}_n < \infty\Big\},\ P\text{–a.s.}.$$

For proving this Corollary, we start with some notation.
Notation 2.4.11. Define

$$U_{n+1,x} := \frac{\mathbb{1}_{N_n>0}}{\mathbf{m}N_n} \sum_{\substack{\mathbf{x}\in\mathscr{V}_{\mathbb{N}_0}:\\ N_{n,(x,\mathbf{x})}=1}} K_{n,(x,\mathbf{x})} \geq 0.$$

It is imporant to note that the sum in this definition is taken over exactly $N_{n,x}$ random variables. Also define

$$U_{n+1} := \sum_{x\in\mathbb{Z}^d} U_{n+1,x} = \frac{N_{n+1}}{\mathbf{m}N_n}\mathbb{1}_{N_n>0} = \frac{\overline{N}_{n+1}}{\overline{N}_n}\mathbb{1}_{\overline{N}_n>0}.$$

The $(U_{n+1,x})_{x\in\mathbb{Z}^d}$ are independent under $P(\cdot|\mathscr{G}_n)$. It is not difficult to see that, on the event $\{N_n > 0\}$,

$$P(U_{n+1,x}|\mathscr{G}_n) = \rho_n(x), \text{ and hence } P(U_{n+1}|\mathscr{G}_n) = 1.$$

Also, with $\widetilde{c}_i = \frac{m^{(i)}}{\mathbf{m}^i}$, $i = 2, 3$,

$$\alpha\rho(x)^2 = \frac{1}{\mathbf{m}^2 N_n^2} N_{n,x}^2 Q(m_{n,x}^2) \leq P\left(U_{n+1,x}^2\middle|\mathscr{G}_n\right)$$
$$= \frac{1}{\mathbf{m}^2 N_n^2} P\left(\Big(\sum_{\substack{\mathbf{x}\in\mathscr{V}_{\mathbb{N}_0}:\\ N_{n,(x,\mathbf{x})}=1}} K_{n,(x,\mathbf{x})}\Big)^2\middle|\ \mathscr{G}_n\right) \leq \frac{N_{n,x}^2 m^{(2)}}{\mathbf{m}^2 N_n^2} = \widetilde{c}_2\rho_n(x)^2,$$

$$P\left(U_{n+1,x}^3\middle|\mathscr{G}_n\right) \leq \frac{m^{(3)}}{\mathbf{m}^3}\rho_n(x)^3 = \widetilde{c}_3\rho_n(x)^3, \text{ again on the event } \{N_n > 0\}.$$

Proof of Corollary 2.4.10. We need to verify the prerequisites of Proposition 2.4.9 which we apply to $\mathsf{X}_n := \overline{N}_n$ and

$$\Delta\mathsf{Y}_n := \frac{\overline{N}_n}{\overline{N}_{n-1}}\mathbb{1}_{\overline{N}_{n-1}>0} - \mathbb{1}_{\overline{N}_{n-1}>0} = \sum_x [U_{n,x} - \rho_{n,x}] \geq 1.$$

The second moments compute as

$$P\left((\Delta\mathsf{Y}_n)^2\middle|\mathscr{G}_n\right) = P\left((\sum_x [U_{n,x} - \rho_{n-1,x}])^2\middle|\mathscr{G}_n\right)$$
$$= \sum_{x,y} P\left((U_{n,x} - \rho_{n-1,x})(U_{n,y} - \rho_{n-1,y})\middle|\mathscr{G}_{n-1}\right) = \sum_x P\left((U_{n,x} - \rho_{n-1,x})^2\middle|\mathscr{G}_{n-1}\right)$$
$$= \sum_x \left[P\left(U_{n,x}^2\middle|\mathscr{G}_{n-1}\right) - \rho_{n-1,x}^2\right].$$

Using the observations after Notation 2.4.11, we hence get

$$\Big(\frac{Q(m_{n,x}^2)}{\mathbf{m}^2}-1\Big)\sum_x \rho_{n-1,x}^2 \le P\Big(\big(\Delta \mathrm{Y}_n\big)^2\Big|\mathscr{G}_n\Big) \le \Big(\frac{m^{(2)}}{\mathbf{m}^2}-1\Big)\sum_x \rho_{n-1,x}^2.$$

Similar observations lead to estimate for the third moment:

$$\begin{aligned}
P\big((\Delta \mathrm{Y}_n)^3\big|\mathscr{G}_{n-1}\big) &= P\Big(\big(\sum_x [U_{n,x}-\rho_{n-1,x}]\big)^3\Big|\mathscr{G}_{n-1}\Big)\\
&= \sum_x P\Big((U_{n,x}-\rho_{n-1,x})^3\Big|\mathscr{G}_{n-1}\Big)\\
&\le 3\sum_x P\Big(U_{n,x}^3+\rho_{n-1,x}^3\Big|\mathscr{G}_{n-1}\Big) \le \Big(\frac{m^{(3)}}{\mathbf{m}^3}-1\Big)\sum_x \rho_{n-1,x}^3.
\end{aligned}$$

This proves that all hypotheses of Proposition 2.4.9 are fulfilled and in fact equality holds for (2.20). □

Proof of Lemma 2.4.8. We make a slight abuse of notation writing $B_{(m)}$ as templates for both the cases B and B_m, and so on for similar cases of notation. We can make use of (2.16) and, splitting two times 1 into complementary indicators, get

$$\begin{aligned}
\Delta\widehat{A}_n^{(m)} &= P\big(\Delta X_n^{(m)}\big|\mathscr{G}_{n-1}\big)\\
&= P_{\mathbb{S}\widetilde{\mathbb{S}}}\Big(\zeta_{n-1}\widetilde{\zeta}_{n-1}\mathbb{1}_{\overline{B}_{(m)}}\Big[\big(\mathbb{1}_{S_{n-1}\neq\widetilde{S}_{n-1}}+\alpha\mathbb{1}_{S_{n-1}=\widetilde{S}_{n-1}}\big)\mathbb{1}_{\mathbb{S}_{n-1}\neq\widetilde{\mathbb{S}}_{n-1}}\mathbb{1}_{\mathbb{S}_{n-2}\neq\widetilde{\mathbb{S}}_{n-2}}\\
&\qquad +\big(\mathbb{1}_{S_{n-1}\neq\widetilde{S}_{n-1}}+\alpha\mathbb{1}_{S_{n-1}=\widetilde{S}_{n-1}}\big)\mathbb{1}_{\mathbb{S}_{n-1}\neq\widetilde{\mathbb{S}}_{n-1}}\mathbb{1}_{\mathbb{S}_{n-2}=\widetilde{\mathbb{S}}_{n-2}}\\
&\qquad -\big(\mathbb{1}_{S_{n-1}\neq\widetilde{S}_{n-1}}+\mathbb{1}_{S_{n-1}=\widetilde{S}_{n-1}}\big)\mathbb{1}_{\mathbb{S}_{n-2}\neq\widetilde{\mathbb{S}}_{n-2}}\Big]\Big).
\end{aligned}\tag{2.21}$$

In the last term, $\mathbb{1}_{\mathbb{S}_{n-1}\neq\widetilde{\mathbb{S}}_{n-1}}$ is implied by the following indicator, while in the second term, $\mathbb{1}_{S_{n-1}\neq\widetilde{S}_{n-1}}$ is 0 due to the fact that $\zeta_{n-1}\widetilde{\zeta}_{n-1}\mathbb{1}_{S_{n-1}\neq\widetilde{S}_{n-1}}\mathbb{1}_{\mathbb{S}_{n-2}=\widetilde{\mathbb{S}}_{n-2}}=0$, $P\otimes P_{\mathbb{S}\widetilde{\mathbb{S}}}$–a.s.. Thus, we can continue

$$\begin{aligned}
(2.21) &= P_{\mathbb{S}\widetilde{\mathbb{S}}}\Big(\zeta_{n-1}\widetilde{\zeta}_{n-1}\mathbb{1}_{\overline{B}_{(m)}}\Big[(\alpha-1)\mathbb{1}_{S_{n-1}=\widetilde{S}_{n-1}}\mathbb{1}_{\mathbb{S}_{n-2}\neq\widetilde{\mathbb{S}}_{n-2}}\\
&\qquad +\alpha\mathbb{1}_{S_{n-1}=\widetilde{S}_{n-1}}\mathbb{1}_{\mathbb{S}_{n-1}\neq\widetilde{\mathbb{S}}_{n-1}}\mathbb{1}_{\mathbb{S}_{n-2}=\widetilde{\mathbb{S}}_{n-2}}\Big]\Big)\\
&\le \alpha P_{\mathbb{S}\widetilde{\mathbb{S}}}\Big(\zeta_{n-1}\widetilde{\zeta}_{n-1}\mathbb{1}_{S_{n-1}=\widetilde{S}_{n-1}}\mathbb{1}_{\overline{B}_{(m)}}\Big).
\end{aligned}$$

The sum

$$Z^{\widehat{A}} := \sum_n \zeta_{n-1}\widetilde{\zeta}_{n-1}\mathbb{1}_{S_{n-1}=\widetilde{S}_{n-1}}$$

is $P_{\mathbb{S}\widetilde{\mathbb{S}}}$–integrable, thanks to Corollary 2.4.10 together with Lemma 1.3.1 and Lemma 1.3.3. So, summation over all $n\in\mathbb{N}$ yields

$$\widehat{A}_n^{(m)} \nearrow \widehat{A}_\infty^{(m)} \le P_{\mathbb{S}\widetilde{\mathbb{S}}}\big(Z^{\widehat{A}}:B_{(m)}\big)\begin{cases} <\infty & \text{for } B_{(m)}=B\\ \xrightarrow[m\to\infty]{} 0 & \text{for } B_{(m)}=B_m,\end{cases}\tag{2.22}$$

P–almost surely.

Now, the same sort of estimates will be carried out for M_n, but involves much more work.

First, we note that $\Delta M_n^{(m)}$ can be written as

$$\begin{aligned}\Delta M_n^{(m)} &= X_n^{(m)} - P\big(X_n^{(m)}\big|\mathcal{G}_{n-1}\big)\\ &= P_{\mathbb{S}\widetilde{\mathbb{S}}}\Bigg(\zeta_{n-1}\widetilde{\zeta}_{n-1}\mathbb{1}_{\overline{B}_{(m)}}\Bigg[\frac{A_{n,\mathbb{S}_{n-1}}^{\mathbb{S}_n}}{a_{\mathbb{S}_{n-1}}^{\mathbb{S}_n}}\frac{A_{n,\widetilde{\mathbb{S}}_{n-1}}^{\widetilde{\mathbb{S}}_n}}{a_{\widetilde{\mathbb{S}}_{n-1}}^{\widetilde{\mathbb{S}}_n}}\mathbb{1}_{\mathbb{S}_{n-1}\neq\widetilde{\mathbb{S}}_{n-1}}\\ &\qquad - \mathbb{1}_{\mathbb{S}_{n-1}\neq\widetilde{\mathbb{S}}_{n-1}}\big(\alpha\mathbb{1}_{S_{n-1}=\widetilde{S}_{n-1}} + \mathbb{1}_{S_{n-1}\neq\widetilde{S}_{n-1}}\big)\Bigg]\Bigg).\end{aligned}$$

Definition 2.4.12. *For convenience, we define*

$$\varphi_n(\mathbb{S},\widetilde{\mathbb{S}}) := \frac{A_{n,\mathbb{S}_{n-1}}^{\mathbb{S}_n}}{a_{\mathbb{S}_{n-1}}^{\mathbb{S}_n}}\frac{A_{n,\widetilde{\mathbb{S}}_{n-1}}^{\widetilde{\mathbb{S}}_n}}{a_{\widetilde{\mathbb{S}}_{n-1}}^{\widetilde{\mathbb{S}}_n}} - \alpha\mathbb{1}_{S_{n-1}=\widetilde{S}_{n-1}} - \mathbb{1}_{S_{n-1}\neq\widetilde{S}_{n-1}}.$$

This is the point where we cannot maintain our easy notation of $\mathbb{S}$ and $\widetilde{\mathbb{S}}$, for we need four independent random walks $\mathbb{S}^{[1]}$, $\mathbb{S}^{[2]}$, $\mathbb{S}^{[3]}$, $\mathbb{S}^{[4]}$. The probability spaces and other notations are adjusted accordingly, refer to Definition 2.4.1. We compute

$$\begin{aligned}\Delta\langle M^{(m)}\rangle_n &= P\big((\Delta M_n^{(m)})^2\big|\mathcal{G}_{n-1}\big)\\ &= P_{\mathbb{S}}^{\otimes 4}\Big(\zeta_{n-1}^{[1]}\zeta_{n-1}^{[2]}\zeta_{n-1}^{[3]}\zeta_{n-1}^{[4]}\mathbb{1}_{\overline{B}_{(m)}\times\overline{B}_{(m)}}\mathbb{1}_{\mathbb{S}_{n-1}^{[1]}\neq\mathbb{S}_{n-1}^{[2]}}\mathbb{1}_{\mathbb{S}_{n-1}^{[3]}\neq\mathbb{S}_{n-1}^{[4]}}\\ &\qquad P\big(\varphi_n(\mathbb{S}^{[1]},\mathbb{S}^{[2]})\varphi_n(\mathbb{S}^{[3]},\mathbb{S}^{[4]})\big)\Big).\end{aligned} \tag{2.23}$$

We note that if $S_{n-1}^{[i]}\neq S_{n-1}^{[j]}$ for $i=1,2$ and $j=3,4$, then $\varphi_n(\mathbb{S}^{[1]},\mathbb{S}^{[2]})$ and $\varphi_n(\mathbb{S}^{[3]},\mathbb{S}^{[4]})$ are independent, and that under $\{\mathbb{S}_{n-1}^{[1]}\neq\mathbb{S}_{n-1}^{[2]}\}$, it holds that $P_{\mathbf{S}^{[1]},\mathbf{S}^{[2]}}\big(P(\varphi_n(\mathbb{S}^{[1]},\mathbb{S}^{[2]}))\big)=0$, where $P_{\mathbf{S}^{[1]},\mathbf{S}^{[2]}}$ is the probability measure with respect to $(\mathbf{S}^{[1]},\mathbf{S}^{[2]})$. From these observations, we get

$$\begin{aligned}(2.23) \leq \sum_{i=1,2;j=3,4} P_{\mathbb{S}}^{\otimes 4}\Big(&\zeta_{n-1}^{[1]}\zeta_{n-1}^{[2]}\zeta_{n-1}^{[3]}\zeta_{n-1}^{[4]}\mathbb{1}_{\overline{B}_{(m)}\times\overline{B}_{(m)}}\mathbb{1}_{\mathbb{S}_{n-1}^{[1]}\neq\mathbb{S}_{n-1}^{[2]}}\mathbb{1}_{\mathbb{S}_{n-1}^{[3]}\neq\mathbb{S}_{n-1}^{[4]}}\\ &P\big(\varphi_n(\mathbb{S}^{[1]},\mathbb{S}^{[2]})\varphi_n(\mathbb{S}^{[3]},\mathbb{S}^{[4]})\big) : S_{n-1}^{i} = S_{n-1}^{j}\Big).\end{aligned} \tag{2.24}$$

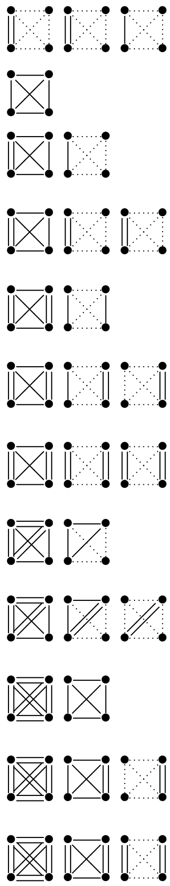
It is clear that

$$P\big(\varphi_n(\mathbb{S}^{[1]},\mathbb{S}^{[2]})\varphi_n(\mathbb{S}^{[3]},\mathbb{S}^{[4]})\big) \leq P\left(\frac{A_{n,\mathbb{S}_{n-1}^{[1]}}^{\mathbb{S}_n^{[1]}}A_{n,\mathbb{S}_{n-1}^{[2]}}^{\mathbb{S}_n^{[2]}}A_{n,\mathbb{S}_{n-1}^{[3]}}^{\mathbb{S}_n^{[3]}}A_{n,\mathbb{S}_{n-1}^{[4]}}^{\mathbb{S}_n^{[4]}}}{a_{\mathbb{S}_{n-1}^{[1]}}^{\mathbb{S}_n^{[1]}}a_{\mathbb{S}_{n-1}^{[2]}}^{\mathbb{S}_n^{[2]}}a_{\mathbb{S}_{n-1}^{[3]}}^{\mathbb{S}_n^{[3]}}a_{\mathbb{S}_{n-1}^{[4]}}^{\mathbb{S}_n^{[4]}}}\right).$$

We define $W(X,Y)$ for $X=\big(\mathbb{x}^{[1]},\mathbb{x}^{[2]},\mathbb{x}^{[3]},\mathbb{x}^{[4]}\big)$, $Y=\big(\mathbb{y}^{[1]},\mathbb{y}^{[2]},\mathbb{y}^{[3]},\mathbb{y}^{[4]}\big)$ by

$$W(X,Y) = P\left(A_{1,\mathbb{x}^{[1]}}^{\mathbb{y}^{[1]}}A_{1,\mathbb{x}^{[2]}}^{\mathbb{y}^{[2]}}A_{1,\mathbb{x}^{[3]}}^{\mathbb{y}^{[3]}}A_{1,\mathbb{x}^{[4]}}^{\mathbb{y}^{[4]}}\right).$$

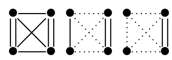
$W(X,Y)$ is zero whenever $\{a4\} := a_{\mathbb{S}_{n-1}^{[1]}}^{\mathbb{S}_n^{[1]}} a_{\mathbb{S}_{n-1}^{[2]}}^{\mathbb{S}_n^{[2]}} a_{\mathbb{S}_{n-1}^{[3]}}^{\mathbb{S}_n^{[3]}} a_{\mathbb{S}_{n-1}^{[4]}}^{\mathbb{S}_n^{[4]}}$ is zero; we hence care only about cases where $\{a4\} \neq 0$. Also, remember that (2.23) restricts to the event $\{\mathbb{S}_{n-1}^{[1]} \neq \mathbb{S}_{n-1}^{[2]},\ \mathbb{S}_{n-1}^{[3]} \neq \mathbb{S}_{n-1}^{[4]}\}$. Such cases can be separated as follows, with the definition $k^{[j]} := \mathbf{y}^{[j]}/\mathbf{x}^{[j]}$ for $j = 1,2,3,4$:



$$\begin{cases}
0 & x^{[1]} = x^{[3]},\ \mathbf{x}^{[1]} = \mathbf{x}^{[3]},\ y^{[1]} \neq y^{[3]},\ \{a4\} \neq 0 \\
1 & x^{[j]} \neq x^{(\ell)}\ \forall\, j,\ell \in \{1,2,3,4\} :\ j \neq \ell,\ \{a4\} \neq 0 \\
2 & x^{[1]} = x^{[3]} \neq x^{[2]} \neq x^{[4]} \neq x^{[1]},\ \mathbf{x}^{[1]} \neq \mathbf{x}^{[3]},\ \{a4\} \neq 0 \\
3 & x^{[1]} = x^{[3]} \neq x^{[2]} \neq x^{[4]} \neq x^{[1]},\ \mathbf{x}^{[1]} = \mathbf{x}^{[3]},\ y^{[1]} = y^{[3]},\ \{a4\} \neq 0 \\
4 & x^{[1]} = x^{[3]} \neq x^{[2]} = x^{[4]},\ \mathbf{x}^{[1]} \neq \mathbf{x}^{[3]},\ \mathbf{x}^{[2]} \neq \mathbf{x}^{[4]},\ \{a4\} \neq 0 \\
5 & x^{[1]} = x^{[3]} \neq x^{[2]} = x^{[4]},\ \mathbf{x}^{[1]} \neq \mathbf{x}^{[3]},\ \mathbf{x}^{[2]} = \mathbf{x}^{[4]},\ y^{[2]} = y^{[4]},\ \{a4\} \neq 0 \\
6 & x^{[1]} = x^{[3]} \neq x^{[2]} = x^{[4]},\ \mathbf{x}^{[1]} = \mathbf{x}^{[3]},\ \mathbf{x}^{[2]} = \mathbf{x}^{[4]},\ y^{[1]} = y^{[3]},\ y^{[2]} = y^{[4]},\ \{a4\} \neq 0 \\
7 & x^{[1]} = x^{[3]} = x^{[2]} \neq x^{[4]},\ \mathbf{x}^{[1]} \neq \mathbf{x}^{[3]} \neq \mathbf{x}^{[2]},\ \{a4\} \neq 0 \\
8 & x^{[1]} = x^{[3]} = x^{[2]} \neq x^{[4]},\ \mathbf{x}^{[1]} \neq \mathbf{x}^{[3]} = \mathbf{x}^{[2]},\ y^{[3]} = y^{[2]},\ \{a4\} \neq 0 \\
9 & x^{[1]} = x^{[3]} = x^{[2]} = x^{[4]},\ \mathbf{x}^{[j]} \neq \mathbf{x}^{(\ell)}\ \forall\, j,\ell \in \{1,2,3,4\} :\ j \neq \ell,\ \{a4\} \neq 0 \\
10 & x^{[1]} = x^{[3]} = x^{[2]} = x^{[4]},\ \mathbf{x}^{[2]} \neq \mathbf{x}^{[3]} \neq \mathbf{x}^{[1]} \neq \mathbf{x}^{[2]} = \mathbf{x}^{[4]}\ y^{[2]} = y^{[4]},\ \{a4\} \neq 0 \\
11 & x^{[1]} = x^{[3]} = x^{[2]} = x^{[4]},\ \mathbf{x}^{[1]} = \mathbf{x}^{[3]} \neq \mathbf{x}^{[2]} = \mathbf{x}^{[4]},\ y^{[1]} = y^{[3]},\ y^{[2]} = y^{[4]},\ \{a4\} \neq 0
\end{cases}$$

Cases that can be obtained by symmetry are not listed here. Case 0 yields $W(X,Y) = 0$ for it is impossible in the BRWRE–Model: particles with the same name at the same place are blown by the wind to the same site, so their children cannot be born at different sites.

The notation with the small squares is solely for the ease of understanding; all information is fully contained in the written part. For how to read it, let us take as an example case number 5:



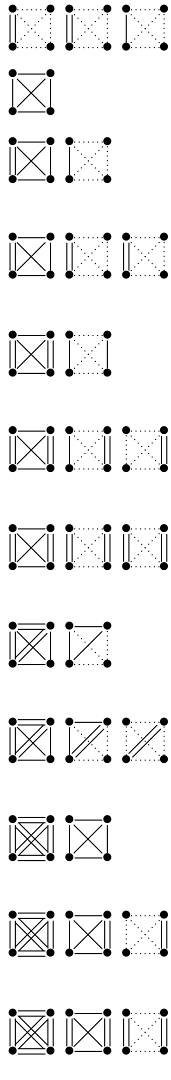
$$5 \qquad x^{[1]} = x^{[3]} \neq x^{[2]} = x^{[4]},\ \mathbf{x}^{[1]} \neq \mathbf{x}^{[3]},\ \mathbf{x}^{[2]} = \mathbf{x}^{[4]},\ y^{[2]} = y^{[4]},\ \{a4\} \neq 0$$

The first square corresponds to the 'x'–part, the second one to the '$\mathbf{x}$'–part, and the last one to the 'y'–part of the restriction. Each • corresponds to an index $j = 1,\ldots,4$, read left–right, top–down. The two left bullets of the first square are connected with a double stroke, read: equality sign, just as the two left ones. Indeed, $x^{[1]} = x^{[3]}$ and $x^{[2]} = x^{[4]}$. All other connections are single–stroked, and are supposed to be read as inequalties. The second square conveys hence the information that $\mathbf{x}^{[1]} \neq \mathbf{x}^{[3]}$ and $\mathbf{x}^{[2]} = \mathbf{x}^{[4]}$. The other dotted connections indicate that both the cases of equality and inequality are comprised. Lastly, the third square stands for all $y^{[j]},\ j \in \{1,\ldots,4\}$ with $y^{[2]} = y^{[4]}$.

If one changes the mapping of bullet–position and index, one gets all the symmetries immediately. A missing square has the same meaning as a square with only dotted lines would have.

Now, we can compute $W(X,Y)$, which equals in the respective cases to:

$$\begin{cases}
0 & 0 \\
a_{\mathbb{x}^{[1]}}^{\mathbb{y}^{[1]}} a_{\mathbb{x}^{[3]}}^{\mathbb{y}^{[3]}} a_{\mathbb{x}^{[2]}}^{\mathbb{y}^{[2]}} a_{\mathbb{x}^{[4]}}^{\mathbb{y}^{[4]}} & 1 \\
\left(\frac{1}{2d}\right)^2 P\Big(\sum_{i\geq k^{[1]}} q_{00}(i) \sum_{i\geq k^{[3]}} q_{00}(i)\Big) a_{\mathbb{x}^{[2]}}^{\mathbb{y}^{[2]}} a_{\mathbb{x}^{[4]}}^{\mathbb{y}^{[4]}} & 2 \\
\left(\frac{1}{2d}\right) P\Big(\sum_{i\geq \max\{k^{[1]},k^{[3]}\}} q_{00}(i)\Big) a_{\mathbb{x}^{[2]}}^{\mathbb{y}^{[2]}} a_{\mathbb{x}^{[4]}}^{\mathbb{y}^{[4]}} & 3 \\
\left(\frac{1}{2d}\right)^4 P\Big(\sum_{i\geq k^{[1]}} q_{00}(i) \sum_{i\geq k^{[3]}} q_{00}(i)\Big) P\Big(\sum_{i\geq k^{[2]}} q_{00}(i) \sum_{i\geq k^{[4]}} q_{00}(i)\Big) & 4 \\
\left(\frac{1}{2d}\right)^3 P\Big(\sum_{i\geq k^{[1]}} q_{00}(i) \sum_{i\geq k^{[3]}} q_{00}(i)\Big) P\Big(\sum_{i\geq \max\{k^{[2]},k^{[4]}\}} q_{00}(i)\Big) & 5 \\
\left(\frac{1}{2d}\right)^2 P\Big(\sum_{i\geq \max\{k^{[1]},k^{[3]}\}} q_{00}(i)\Big) P\Big(\sum_{i\geq \max\{k^{[2]},k^{[4]}\}} q_{00}(i)\Big) & 6 \\
\left(\frac{1}{2d}\right)^3 P\Big(\sum_{i\geq k^{[1]}} q_{00}(i) \sum_{i\geq k^{[3]}} q_{00}(i) \sum_{i\geq k^{[2]}} q_{00}(i)\Big) a_{\mathbb{x}^{[4]}}^{\mathbb{y}^{[4]}} & 7 \\
\left(\frac{1}{2d}\right)^2 P\Big(\sum_{i\geq k^{[1]}} q_{00}(i) \sum_{i\geq \max\{k^{[3]},k^{[2]}\}} q_{00}(i)\Big) a_{\mathbb{x}^{[4]}}^{\mathbb{y}^{[4]}} & 8 \\
\left(\frac{1}{2d}\right)^4 P\Big(\sum_{i\geq k^{[1]}} q_{00}(i) \sum_{i\geq k^{[3]}} q_{00}(i) \sum_{i\geq k^{[2]}} q_{00}(i) \sum_{i\geq k^{[4]}} q_{00}(i)\Big) & 9 \\
\left(\frac{1}{2d}\right)^3 P\Big(\sum_{i\geq k^{[1]}} q_{00}(i) \sum_{i\geq k^{[3]}} q_{00}(i) \sum_{i\geq \max\{k^{[2]},k^{[4]}\}} q_{00}(i)\Big) & 10 \\
\left(\frac{1}{2d}\right)^2 P\Big(\sum_{i\geq \max\{k^{[1]},k^{[3]}\}} q_{00}(i) \sum_{i\geq \max\{k^{[2]},k^{[4]}\}} q_{00}(i)\Big) & 11
\end{cases}$$



The number of *different* points in the first square corresponds to the number of separate expectations (there are expectations hidden in the $a_{\mathbb{x}^{[j]}}^{\mathbb{y}^{[j]}}$'s). The equalities in the second square that are written down are important inasmuch as they decide about which sums become united to one sum running over $i \geq \max\{\cdots\}$. The third square decides if in fact the case is at all possible. The exponent of the fraction corresponds to the number of summation marks (there are summation marks hidden in the $a_{\mathbb{x}^{[j]}}^{\mathbb{y}^{[j]}}$'s, but fractions, as well, so these $a_{\mathbb{x}^{[j]}}^{\mathbb{y}^{[j]}}$'s do not contribute to the exponent of the fraction).

Now, we can continue with $\Delta\langle M^{(m)}\rangle_n$. To get from (2.24) to the following line, one can apply the same trick with insterted conditional expectations as in the succession of equalities (2.14), and pick the worst case, which is case 11. We continue (2.24) and find

$$\Delta\langle M^{(m)}\rangle_n \le \sum_{i=1,2;j=3,4} C P_{\mathbb{S}}^{\otimes 4}\Big(\zeta_{n-1}^{[1]}\zeta_{n-1}^{[2]}\zeta_{n-1}^{[3]}\zeta_{n-1}^{[4]}\mathbb{1}_{S_{n-1}^{[i]}=S_{n-1}^{[j]}} : \overline{B}_{(m)}\times\overline{B}_{(m)}\Big),$$

where $C = c\, Q\big((m_{0,0}^{(2)})^2\big)/\mathbf{m}^4 < \infty$ and c is a constant depending only on d.

$$Z_{i,j}^M := \sum_n \zeta_{n-1}^{[1]}\zeta_{n-1}^{[2]}\zeta_{n-1}^{[3]}\zeta_{n-1}^{[4]}\mathbb{1}_{S_{n-1}^{[i]}=S_{n-1}^{[j]}}$$

serves the same aim as $Z^{\widehat{A}}$ in (2.22), and is $P_{\mathbb{S}}^{\otimes 4}$–integrable for the same reasons as for $Z^{\widehat{A}}$. So, in the same manner, we conclude

$$\langle M^{(m)}\rangle_\infty = \sum_n \Delta\langle M^{(m)}\rangle_n \le \sum_{i=1,2;j=3,4} P_{\mathbb{S}}^{\otimes 4}(Z_{i,j}^M : \overline{B}_{(m)}\times\overline{B}_{(m)}) \begin{cases} <\infty & \text{for } B_{(m)} = B \\ \xrightarrow[m\to\infty]{} 0 & \text{for } B_{(m)} = B_m, \end{cases}$$

P–almost surely. This finishes the proof of Lemma 2.4.8. □

Lemma 2.4.13.

$$\widetilde{P}\mu_\infty^{(2)} \text{ is a probability measure on } \mathscr{F}_\infty^1 \otimes \widetilde{\mathscr{F}}_\infty^1. \tag{2.25}$$

$$\widetilde{P}\mu_\infty^{(2)} \ll P_{S\widetilde{S}}(\cdot\times(\Omega^2\times\widetilde{\Omega}^2)) \text{ on } \mathscr{F}_\infty^1\otimes\widetilde{\mathscr{F}}_\infty^1. \tag{2.26}$$

Proof. As in the proof of (2.2), (2.25) and (2.26) boil down to proving that

$$\lim_{m\to\infty} \widetilde{P}\mu_\infty^{(2)}(\overline{B}_m) = 0,$$

for $\{B_m\}\subset(\mathscr{F}^1)^{\otimes 2}$ with $\lim_{m\to\infty} P_{S\widetilde{S}}(\overline{B}_m) = 0$. We show, in a way similar to the very end of the proof of Lemma 2.4.2,

$$\lim_{m\to\infty}\mu_\infty^{(2)}(B_m) = \lim_{m\to\infty}\lim_{n\to\infty}\mu_n^{\otimes 2}(\overline{B}_m) = 0 \text{ in } \widetilde{P}\text{–probability},$$

by proving that

$$\lim_{m\to\infty}\sup_n X_n^m = 0 \text{ in } \widetilde{P}\text{–probability}, \tag{2.27}$$

where $X_n^m = X_n(B_m)$ defined for B_m. Let also

$$X_n^m =: M_n^m + \widehat{A}_n^m$$

be the submartingale decomposition as in (2.17) and as hinted in Lemma 2.4.8. Now, we can apply the 'B_m'–version of Lemma 2.4.8. $\widehat{A}_n^m$ is taken care of by the first statement of (2.19), and for M_n^m, the second statement and a little calculus will yield

$$\lim_{m\nearrow\infty}\sup_n |M_n^m| = 0 \text{ in } \widetilde{P}\text{–probability}. \tag{2.28}$$

In fact, for $\ell\in\mathbb{R}$, let $\tau_\ell^m = \inf\{n\ge 0 : \langle M^m\rangle_{n+1} > \ell\}$. Then,

$$P\Big(\sup_n\big|M_n^m\big|\ge\varepsilon,\ \overline{N}_\infty>0\Big) \le P\Big(\langle M^m\rangle_\infty>\ell,\ \overline{N}_\infty>0\Big) + P\Big(\sup_n\big|M_n^m\big|\ge\varepsilon,\ \tau_\ell^m=\infty\Big).$$

Clearly, the first term on the right–hand–side vanishes as $m \nearrow \infty$ because of (2.19), and so does the second term as can be seen from the following application of Doob's inequality (for instance [Dur91, p.248]):

$$P\Big(\sup_n \big|M_n^m\big| \geq \varepsilon,\ \tau_\ell^m = \infty\Big) \leq P\Big(\sup_n \Big|M_{n\wedge\tau_\ell^m}^m\Big| \geq \varepsilon\Big)$$
$$\leq 4\varepsilon^{-2} P\Big(\langle M^m\rangle_{\tau_\ell^m}\Big) \leq 4\varepsilon^{-2} P\Big(\langle M^m\rangle_\infty \wedge \ell\Big).$$

Since ℓ is arbitrary, (2.28) follows and hence we conclude (2.27). □

Proof of Theorem 2.2.2. We are going to make use of the experience gathered in proving Proposition 2.3.1. In a manner very similar to the proof of (2.7), for (2.5), we need to show an analogue of (2.8) with the help of an analogue of (2.9). To be more concrete, we show

$$\lim_{n\to\infty} \widetilde{P}\Big(\big[\mu_\infty(\overline{F}(S^{(n)}))\big]^2\Big) = 0, \tag{2.29}$$

which implies

$$\widetilde{P}\big(|\mu_\infty(\overline{F}(S^{(n)}))|\big) \xrightarrow[n\to\infty]{} 0, \tag{2.30}$$

and hence the convergence in probability. Indeed, using the same replacement argument, but with (2.26) instead of (2.3), we get

$$\lim_{n\to\infty} \widetilde{P}\mu_\infty^{(2)}\Big(G\big(S^{(n)}, \widetilde{S}^{(n)}\big)\Big) = (P^{\mathbb{W}})^{\otimes 2}\Big(G\big(\,\cdot\,/\sqrt{d},\ \widetilde{\cdot}/\sqrt{d}\big)\Big)$$

for any $G \in \mathscr{C}_b(\mathbb{W}\times\mathbb{W})$. In particular, we can take $G(w,\widetilde{w}) = \overline{F}(w)\overline{F}(\widetilde{w})$, and get (2.29), and hence (2.5). The proof of (2.4) works with the same telescopic technique seen in (2.11) used in the proof of (2.6):

$$\widetilde{P}\big(|\mu_n(\overline{F}(S^{(n)}))|\big) = \widetilde{P}\big(|\mu_n(\overline{F}(S^{(n)} - \overline{F}(S^{(n-k)}))\big)$$
$$+ \widetilde{P}\big(|\mu_n(\overline{F}(S^{(n-k)})) - \mu_\infty(\overline{F}(S^{(n-k)}))|\big)$$
$$+ \widetilde{P}\big(|\mu_\infty(\overline{F}(S^{(n-k)}))|\big).$$

Note that the L^2–techniques in this paragraph that lead to (2.30) are needed only for the treatment of the last line; the other two can be dealt with with the same arguments than after (2.11). □

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A Stationary, Mixing and Perturbative Counterexample to the 0-1-law for Random Walk in Random Environment in Two Dimensions

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Abstract

We construct a two-dimensional counterexample of a random walk in random environment (RWRE). The environment is stationary, mixing and ε–perturbative, and the corresponding RWRE has non-trivial probability to wander off to the upper right. This is in contrast to the 0-1-law that holds for i.i.d. environments.

1 Random walk in random environment

We start by fixing the notation and the basic notions of the model.

We work in the d-dimensional space $\mathbb{Z}^d$, $d \geq 1$. $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ and $\mathbb{N} := \{1, 2, \dots\}$ stand for the natural numbers.

We will count dimensions from 0 to $d-1$; so, we write $u = (u_0, u_1, \dots, u_{d-1}) \in \mathbb{Z}^d$, and denote by $e_0, \dots e_{d-1}$ the canonical unit vectors in $\mathbb{Z}^d$. This nonstandard–notation will simplify things later. For two vectors $v, w \in \mathbb{Z}^d$, $v \cdot w$ denotes the scalar product.

For any real number $r \in \mathbb{R}$, we will be using the floor function $\lfloor r \rfloor := \max\{m \in \mathbb{N}_0 : m \leq r\}$ and for any natural number $l \in \mathbb{N}_0$ the modulo operation $l \bmod 2 := \mathbb{1}_{l \text{ is impair}} \in \{0, 1\}$.

If $\mathbb{P}$ is a probability measure, with the convenient notational abuse common in mathematical physics, we write "$\mathbb{P}$" for the expectation operator as well.

Define

$$S^d := \Big\{ \varpi \in [0,1]^{\{\pm e_j, 0 \leq j < d\}} : \sum_{e \in \{\pm e_j, 0 \leq j < d\}} \varpi(e) = 1 \Big\}, \ d \in \mathbb{N},$$

the set of nearest neighbour transition probabilities on $\mathbb{Z}^d$. We call a family $\omega = (\omega_u)_{u \in \mathbb{Z}^d}$ of S^d-valued random variables on an appropriate probability space $(\Omega, \mathcal{A}, P)$ a *random environment* on $\mathbb{Z}^d$.

One might ask for a random environment to satisfy, with $0 \leq \kappa < 1/2$ some *ellipticity constant*, the condition

$$P\big(\omega_u(e) \in (\kappa, 1-\kappa)\big) = 1 \text{ for all } u \in \mathbb{Z}^d, e \in \{\pm e_j, 0 \leq j < d\}. \tag{1.1}$$

If (1.1) is satisfied with $\kappa = 0$, the environment is called *elliptic*, and if it is even satisfied with some $\kappa > 0$, *uniformly elliptic*.

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A morally even stronger notion of homogeneity is reached when one pushes κ towards $\frac{1}{2d}$. For $\varepsilon > 0$, ω is called *ε–perturbative* if

$$P\big(\omega_u(e) \in [1/2d - \varepsilon, 1/2d + \varepsilon]\big) = 1 \text{ for all } u \in \mathbb{Z}^d,\ e \in \{\pm e_j, 0 \le j < d\}.$$

We use the term *totally ergodic* for "ergodic with respect to any shift".

Take a starting point $v \in \mathbb{Z}^d$. To a random environment ω on $(\Omega, \mathcal{A}, P)$, we associate the random probability measure P_v^ω, which, together with the $\mathbb{Z}^d$-valued random variables $(X_t)_{t\in\mathbb{N}_0}$, establishes the *random walk in random environment* $(P, P_v^\omega, (X_t)_{t\in\mathbb{N}_0})$. It is defined to satisfy the Markov-property and

$$P_v^\omega(X_0 = v) = 1, \tag{1.2}$$
$$P_v^\omega(X_{t+1} = X_t + e | X_t = u) = \omega_u(e),\ e \in \{\pm e_j, 0 \le j \le d-1\},\ u \in \mathbb{Z}^d.$$

In [Kal81], Kalikow considered questions of recurrence and transience of this model, and proved that for uniformly elliptic i.i.d.–environments,

$$PP_0^\omega(X_t \cdot v \text{ changes sign infinitely often}) \in \{0,1\},\ v \in \mathbb{Z}^d. \tag{1.3}$$

He also raised the question whether in $d = 2$, it holds that

$$PP_0^\omega(X_t \cdot v \xrightarrow[t\to\infty]{} \infty) \in \{0,1\},\ v \in \mathbb{R}^d \setminus \{0\}. \tag{1.4}$$

Sznitman and Zerner highlighted in [SZ99] that Kalikow's question (1.4) is valid in any dimension $d \ge 2$. They also pointed out that (1.3) implies

$$P\big(P_0^\omega(X_t \cdot v \text{ is transient})\big) \in \{0,1\},\ v \in \mathbb{Z}^d.$$

The term *Kalikow's* 0–1–*law* has since been established for this assertion.

For $d = 2$, Zerner and Merkl answer Kalikow's question (positively) for elliptic i.i.d.–environments in [ZM01]; an improved version of the proof is given in [Zer07]. Holmes and Salisbury treat the same questions without the assumption of ellipticity in [HS].

The necessity of the i.i.d.–assumption is assessed in [ZM01] by means of an example for $d = 2$ of an elliptic, ergodic and stationary environment that features

$$PP_0^\omega(X_t \cdot v \xrightarrow[t\to\infty]{} \infty) \notin \{0,1\} \text{ for some } v \in \mathbb{Z}^d. \tag{1.5}$$

[Zer07] gives a similar example with an even totally ergodic environment.

As for $d \ge 3$, Bramson, Zeitouni and Zerner [BZZ06] have a uniformly elliptic, stationary, totally ergodic, and even mixing example of an environment satisfying (1.5).

In the present article, we construct an environment with similar properties for dimension $d = 2$. Our main theorem is indeed:

Theorem 1.0.1. *For any $\varepsilon > 0$, there is an ε–perturbative, stationary, mixing random environment $\omega = (\omega_u)_{u\in\mathbb{Z}^2}$ with associated probability measure P such that for the associated random walk $((X_t), P_0^\omega)$, it holds that*

$$PP_0^\omega\big(X_t \cdot \vec{1} \xrightarrow[t\to\infty]{} \infty\big) > 0 \text{ as well as } PP_0^\omega\big(X_t \cdot \vec{1} \xrightarrow[t\to\infty]{} -\infty\big) > 0.$$

Here, $\vec{1}$ denotes the vector $(1,1)$.

A preprint by Guo [Guo] is concerned with the limiting velocity of the random walk in random environment on the events $\{X_t \cdot v \xrightarrow[t\to\infty]{} \dagger\infty\}$, $\dagger \in \{+,-\}$, $v \in \mathbb{R}^d$, in dimensions $d \geq 2$, in the case where the random environment satisfies uniform ellipticity and a certain strong mixing condition which holds in Gibbsian environments, for instance.

Proof of Theorem 1.0.1*, and organisation of the article.* In Section 2, we construct an object called streetgrid which we use to define the actual random environment in Subsection 2.3. We prove the streetgrid to be stationary and mixing in the Subsections 3.3 and 3.4. These properties are inherited in the definition of the random environment.

In Subsection 3.2, we show that there are areas growing in the direction of $\vec{1}$ that are in some sense large. This has the consequence, via the placement of the transition probabilities, that the random walk has positive probability of never leaving these areas, while wandering off to infinity in the direction of $\vec{1}$. This is shown in Subsection 4. The same arguments could be repeated for $-\vec{1}$, which finishes the proof. □

We should want to indicate some of the sources of inspiration that contributed to this article. The ideas of conducting the random walk to infinity on a "treelike structure" of "not too slowly growing roads leading to infinity" has been applied in [BZZ06]. As for how to construct such a structure in dimension $d = 2$, Häggström and Mester [HM09] had the idea of ever larger, ever rarer streets joining each other. By using Poisson processes of different intensities as the underlying structure instead of their"windows" of fixed length, we were able to avoid some of the rigidity of their model and to make assertions on mixing, at the price of developing a completely new construction.

2 Construction of a random environment

2.1 Notation

2.1.1 Boxes

Recall the convention to write $u = (u_0, u_1) \in \mathbb{Z}^2$. We call a *box* any subset B of $\mathbb{Z}^2$ that can be expressed as

$$B = \{b_0, \dots, b'_0\} \times \{b_1, \dots, b'_1\} \text{ for some } b_j, b'_j \in \mathbb{Z} \text{ with } b_j \leq b'_j,\ j \in \{0,1\}. \tag{2.1}$$

For a box B, we define the *emplacement of the faces of* B as

$$b_j(B) := b_j, \quad b'_j(B) := b'_j, \quad j \in \{0,1\}, \tag{2.2}$$

where b_j, b'_j, $j \in \{0,1\}$, are taken from (2.1).

For $v, w \in \mathbb{Z}^2$ we define the *box between* v *and* w as

$$\text{B'twn}(v,w) := \Big\{ \min\{v_0,w_0\}, \dots, \max\{v_0,w_0\} \Big\} \times \Big\{ \min\{v_1,w_1\}, \dots, \max\{v_1,w_1\} \Big\}.$$

The (outer) *boundary of a box* B may be defined as

$$\partial B := \{u \in \mathbb{Z}^2 : d(u,B) = 1\};$$

here, $d(\cdot,\cdot)$ means the 1-metric. It is convenient to define as well the *closure of* B, which is

$$\overline{B} := B \cup \partial B;$$

the *upper right corner* $\Rsh B$ of a box B is

$$\Rsh B := (b'_0(B), b'_1(B)).$$

2.1.2 Streets and streetgrid, and blocks

We call a number $m \in \mathbb{N}_0$ a *superlevel*, and $k \in \{0,1\}$ a *sublevel*. The mapping $(m,k) \mapsto 2m+k : \mathbb{N}_0 \times \{0,1\} \to \mathbb{N}_0$ is bijective, and this number is called the corresponding *level*. Given any level $l \in \mathbb{N}_0$, we can obviously reconstitute superlevel and sublevel using the inverse function, $(\lfloor \frac{l}{2} \rfloor, l \mod 2)$.

If a level has somehow been assigned to some object, we will speak of the superlevel and the sublevel of that object as well.

Given a level $l \in \mathbb{N}_0$ and a function $F \in {\mathbb{N}_0}^{\mathbb{D}}$, $\mathbb{D} \subseteq \mathbb{Z}^2$, a box $B \subseteq \mathbb{D}$ is called a *street of level l w.r.t. F* if

$$F_u = l \text{ for all } u \in B, \text{ and } F_u \neq l \text{ for all } u \in \partial B \cap \mathbb{D}.$$

We call it a *field w.r.t. F* if it is a street of level 0 w.r.t. F. When it is obvious or not important which level and function are meant, we will simply speak of "street" and "field".

We say F is a *streetgrid* if $\mathbb{D}$ is the union of streets and fields with respect to F, i.e.

$$\mathbb{D} = \dot{\bigcup_{l \in \mathbb{N}_0}} \dot{\bigcup_{\substack{B \text{ street of} \\ \text{level } l \text{ w.r.t. } F}}} B. \tag{2.3}$$

Given a box B contained in the domain of a streetgrid F, we define the *level of the box B w.r.t. F* as

$$\ell(B) = \ell^F(B) := \max_{u \in B} F_u.$$

Note that if the box B is a street, the two definitions of "level of the box B" and "level of the street B" coincide.

For $B \subseteq \mathbb{Z}^2$ a box such that $\overline{B} \subseteq \mathbb{D}$ the domain of F, we say B *is a block w.r.t. F* if all points $u \in \partial B$ are elements of exactly four different streets w.r.t. F, which are all of level greater than $\ell^F(B)$.

The *upper and lower levels of the block B* are defined respectively as

$$\overline{\ell}^F(B) := \max_{u \in \partial B} F_u \quad \text{and} \quad \underline{\ell}^F(B) := \min_{u \in \partial B} F_u.$$

$\underline{\ell}^{\cdot}(B)$ will be crucial in determining streets of which levels might be present if we have only information about ∂B the boundary of B, and $\overline{\ell}^{\cdot}(B)$ will constitute a lower bound to all levels that are not present in $\overline{B}$.

Given a streetgrid F and $u \in \mathbb{Z}^2$, we define S'rnd_u^F to be the *street or field around u*; to be precise,

$$\text{S'rnd}_u^F \text{ is defined to be the unique street or field } B \text{ w.r.t. } F \text{ such that } u \in B.$$

2.2 Construction of the streetgrid

2.2.1 Parameters and randomness used in the construction

$$\lambda_m := (m+1)!^{-2} \text{ and } \beta_m := m!^2, \; m \in \mathbb{N}_0, \tag{2.4}$$

are called the *rate of occurrence of streets at superlevel* m and the *planned widths of the streets at superlevel* m, respectively.

We define

$$\mathcal{Z} := \mathbb{Z} \times \mathbb{N}_0 \times \mathbb{Z}^2,$$

and, on some appropriate probability space $(\Omega, \mathcal{F}, P)$, a family of independent random variables

$$\mathbf{X} := \big(\mathbf{X}(x, l, w)\big)_{(x,l,w)\in\mathcal{Z}},$$

which are to be Bernoulli-distributed with parameters $\lambda_{\lfloor \frac{l}{2} \rfloor}$.

To understand the meaning of the index-set $\mathcal{Z}$, we need to read it backwards. Every point in $\mathbb{Z}^2$ gets for every level in $\mathbb{N}_0$ a Bernoulli-process $\{0,1\}^{\mathbb{Z}}$.

Having the necessary terms and definitions as well as the random ingredients at hand, we can start constructing the environment, beginning with a streetgrid. This will be done in two steps. Starting at the origin, we begin with narrow streets and make our way towards infinity by ever wider ones. This leaves wide areas of fields that will then be filled in the opposite direction with ever narrowing streets.

2.2.2 The initial grid

We could put the random ingredient $\mathbf{X}$ directly into our construction, which will be built gradually in several definitions. We prefer however to write down these definitions as functions on $\{0,1\}^{\mathcal{Z}}$, and to finally evaluate them at the random place $\mathbf{X}$. Notationwise, we will drop the dependence on $\mathbf{x} \in \{0,1\}^{\mathcal{Z}}$ after the first appearence, though. Please note that the definitions may, for some $\mathbf{x} \in \{0,1\}^{\mathcal{Z}}$, not make any sense; whenever there is some doubt on how $\mathbf{x}$ should look like, any typical realization $\mathbf{x}$ of $\mathbf{X}$ will do.

In a first step, we define processes that, roughly speaking, show where streets would be if each coordinate existed on its own. For each coordinate direction $j \in \{0,1\}$ and every superlevel $m \in \mathbb{N}_0$, we attach to the left of every point highlighted as 1 by the process $\mathbf{x}(\cdot, 2m+j, 0)$ an interval with the width β_m of the respective superlevel m. Then, for any point, we take the maximum level of all streets the point lies in; that is, in the case of overlapping intervals of different levels, the higher level prevails:

$$W_x^j(\mathbf{x}) := 2\max\big\{m \in \mathbb{N}_0 : \exists y \in \mathbb{Z} : x \le y < x + \beta_m,\ \mathbf{x}(y, 2m+j, 0) = 1\big\} + j, \qquad j \in \{0,1\},\ x \in \mathbb{Z},\ \mathbf{x} \in \{0,1\}^{\mathcal{Z}}. \tag{2.5}$$

We need to make sure $W_x^j(\mathbf{X})$ is P–a.s. finite for all $j \in \{0,1\}$ and all $x \in \mathbb{Z}$. For $m \in \mathbb{N}$, $x \in \mathbb{Z}$, it holds that

$$\begin{aligned}
&P\big(\exists\, y \in \mathbb{Z} :\ 0 \le y - x < \beta_m,\ \mathbf{X}(y, 2m+j, 0) = 1\big)\\
&\quad = P\big(\#\{y \in \mathbb{Z} :\ 0 \le y - x < \beta_m,\ \mathbf{X}(y, 2m+j, 0) = 1\} \ge 1\big)\\
&\quad \le P\big(\#\{y \in \mathbb{Z} :\ 0 \le y - x < \beta_m,\ \mathbf{X}(y, 2m+j, 0) = 1\}\big)\\
&\quad = \sum_{y=x}^{x+\beta_m-1} P\big(\mathbf{X}(y, 2m+j, 0) = 1\big) = \beta_m\lambda_m = \frac{m!^2}{(m+1)!^2} = \frac{1}{(m+1)^2}.
\end{aligned}$$

With the Borel–Cantelli–lemma, we conclude that there are P–a.s. only finitely many $m \in \mathbb{N}$ satisfying the condition of the maximum in (2.5), which hence is P–a.s. finite.

The dependence on $\mathbf{x}$ will de dropped for the next few definitions, even though it of course persists.

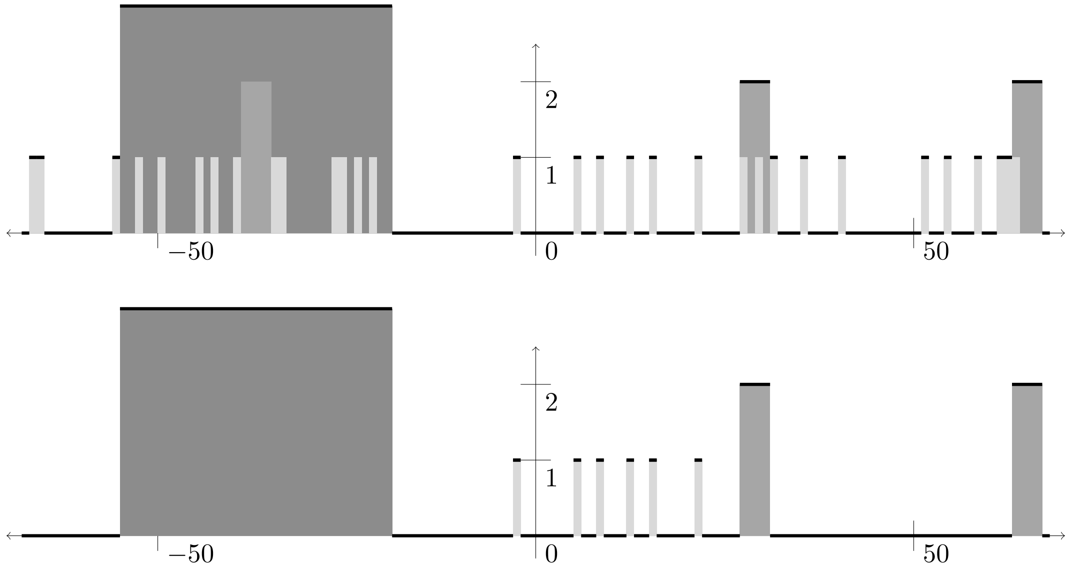

Figure 1: Simulation of a realization of $W_x^0(\mathbf{X})$ and V_x^0, $x \in \mathbb{Z}$, represented by the thick line. The rectangles indicate the intervals attached to the points highlighted by the Bernoulli-processes of different intensities. Although in this picture the domain of the two functions looks continuous, they are defined to have domain $\mathbb{Z}$.

The function W_x^j will be further transformed by removing the outer intervals of smaller value in

$$V_x^j := W_x^j \mathbb{1}_{W_x^j = (\max_{0 \le y \le x} W_y^j \vee \max_{x \le y \le 0} W_y^j)}, \ j \in \{0,1\}, \ x \in \mathbb{Z}.$$

Note that the maximum over an empty set is to be read as $-\infty$.

The transition from $W_\cdot^0$ to $V_\cdot^0$ is visualized in Figure 1.

Remark 2.2.1. *A monotonically increasing function* $f : \mathbb{N}_0 \to \mathbb{R}$ *satisfies* $f_x = \max_{0 \le y \le x} f_y$, $x \in \mathbb{N}_0$. $(V_x^j)_{x \in \mathbb{N}_0}$, $j \in \{0,1\}$ *are not monotonically increasing, but "weakly monotonically increasing, seen from* 0*" in the sense that they still satisfy*

$$V_x^j \in \left\{0, \max_{0 \le y \le x} V_y^j\right\}, \ x \in \mathbb{N}_0, \ j \in \{0,1\},$$

and a similar assertion for negative x.

With the following definition, we begin our two–dimensional construction. Any point $u = (u_0, u_1) \in \mathbb{Z}^2$ gets assigned a level by

$$\text{InitGrid}^{\mathbf{x}}(u) := \left(V_{u_0}^0 \vee V_{u_1}^1\right) \mathbb{1}_{V_{u_0}^0 \vee V_{u_1}^1 \ge \max_{j \in \{0,1\}} \left(\max_{0 \le x < u_j} V_x^j \vee \max_{u_j < x \le 0} V_x^j\right)}.$$

In words, the point u gets assigned the maximum of the two $V_{u_0}^0$ and $V_{u_1}^1$ provided this maximum is larger than any of the V_x^j for x between 0 and u_j, with $j \in \{0,1\}$. Thus, InitGrid satisfies a two-dimensional analogue of the heuristical notion of "weakly monotonically increasing seen from 0" mentioned in Remark 2.2.1.

Note that InitGrid is only the *initial streetgrid*, and w.r.t. this InitGrid, large fields remain.

We write $\text{InitGrid}(u) := \text{InitGrid}^{\mathbf{X}}(u)$; a simulation of InitGrid is shown in Figure 2.

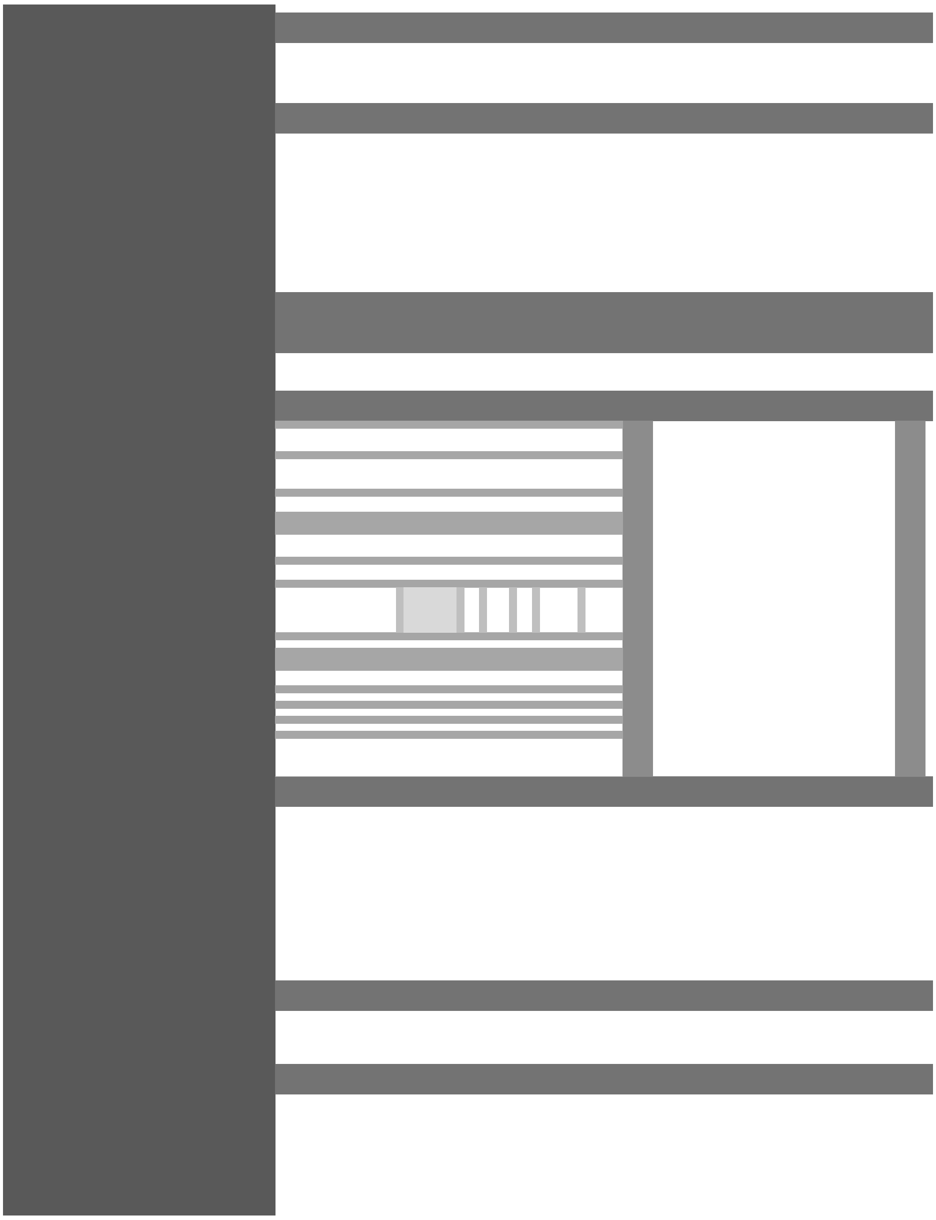

Figure 2: Simulation of $\text{InitGrid} = \text{InitGrid}^{\mathbf{X}}$. Again, the domain of InitGrid is not continuous, but $\mathbb{Z}^2$. $V_{\cdot}^{0}$ is the same as in Figure 1.

Lemma 2.2.2. $\mathrm{InitGrid}^{\mathbf{X}}(\cdot)$ *is P–a.s. a streetgrid.*

Proof. We need to show that $\mathbb{Z}^2$ is a patchwork of streets and fields w.r.t. $\mathrm{InitGrid}^{\mathbf{X}}(\cdot)$ as in (2.3). We will concentrate on the first quadrant, referring to analogy for the other ones.

Define

$$Y_m^j := \min\{x \in \mathbb{N}_0 | V_x^j > m\},\ m \in \mathbb{N}_0,\ j \in \{0,1\}.$$

On the coordinate axes, we have that

$$\begin{aligned}
\mathrm{InitGrid}(0) &= \max_{j\in\{0,1\}} V_0^j,\\
\mathrm{InitGrid}(xe_0) &= \mathrm{InitGrid}(0) \text{ for all } 0 \le x < Y^0_{\lfloor \frac{\mathrm{InitGrid}(0)}{2} \rfloor},\\
\mathrm{InitGrid}(ye_1) &= \mathrm{InitGrid}(0) \text{ for all } 0 \le y < Y^1_{\lfloor \frac{\mathrm{InitGrid}(0)}{2} \rfloor},\\
\mathrm{InitGrid}(xe_0) &= V_x^0 \text{ for all } x \ge Y^0_{\lfloor \frac{\mathrm{InitGrid}(0)}{2} \rfloor},\\
\mathrm{InitGrid}(ye_1) &= V_y^1 \text{ for all } y \ge Y^1_{\lfloor \frac{\mathrm{InitGrid}(0)}{2} \rfloor}.
\end{aligned}$$

On the first quadrant, it holds that

$$\mathrm{InitGrid}\big((x,y)\big) = \mathrm{InitGrid}(0) \text{ for all } 0 \le x < Y^0_{\lfloor \frac{\mathrm{InitGrid}(0)}{2} \rfloor},\ 0 \le y < Y^1_{\lfloor \frac{\mathrm{InitGrid}(0)}{2} \rfloor},$$

and more generally,

$$\mathrm{InitGrid}\big((x,y)\big) = \begin{cases} V_x^0 & \text{if } V_x^0 > V_z^1 \text{ for all } 0 \le z \le y,\\ V_y^1 & \text{if } V_y^1 > V_z^0 \text{ for all } 0 \le z \le x,\\ 0 & \text{else.} \end{cases}$$

If one takes this equation for fixed, say, x with $V_x^0 \neq 0$ and lets run y from 0 to infinity, one gets the value $\mathrm{InitGrid}((x,y)) = V_x^0 = \mathrm{InitGrid}(xe_0)$ for all $y < \min\{y \in \mathbb{N}_0 | V_y^1 > V_x^0\}$; in other words, until from the other coordinate, one gets blocked. Because $V_\cdot^0$ and $V_\cdot^1$ have disjoint codomains (except for 0, which they have in common), these blockings are sharp in the sense that one can always tell whether a point has got its value (different from 0) from $V_\cdot^0$ or $V_\cdot^1$.

Also, the other way around, if some point $(x,y) \in \mathbb{Z}^2$ has got its initial–grid–value from, say, V_x^0, then fixing x and letting z run from y to 0 yields

$$\mathrm{InitGrid}\big((x,z)\big) = \mathrm{InitGrid}\big((x,y)\big) \text{ for all } y \ge z \ge 0.$$

Combining the arguments of the last two paragraphs, one can see that all points $u \in \mathbb{Z}^2$ satisfying $\mathrm{InitGrid}(u) = 2m + j \neq 0$ lie in areas of constant InitGrid-value outgoing perpendicularly from the j-th coordinate axis. Each such area continues until it gets blocked by some area coming from the other coordinate axis. The areas are of rectangular shape, and P–a.s. finite.

This applies as well to the areas where the initial grid equals 0. These are indeed surrounded by four streets of different levels, so that they are fields.

Finally, we need not only to pay attention at the the four quadrants individually, but at the transition between them as well. Indeed, the streetgrid–property holds because between adjacent quadrants, the same $V_\cdot^j$, $j \in \{0,1\}$ influences the construction of the streets. □

Remark 2.2.3. *We will be saying "0 is responsible in* InitGrid *for the emplacement of streets of level l on D" for any block D w.r.t.* InitGrid *containing the origin and any level* $\ell^{\mathrm{InitGrid}}(D) \le l < \underline{\ell}^{\mathrm{InitGrid}}(D)$.

$\ell^{\text{InitGrid}}(D)$ is the highest level of any streets placed on D. The emplacement of these streets has been provided by the random ingredient $\mathbf{X}$ evaluated at points $(\cdot, l, 0)$, so it is sound to say 0 is responsible.

How about the levels $\ell^{\text{InitGrid}}(D) < l < \underline{\ell}^{\text{InitGrid}}(D)$? No street of these levels exists in D. But this absence of streets was stipulated by an absence of 1s in the random ingredient $\mathbf{X}$ at points $(\cdot, l, 0)$, $\ell^{\text{InitGrid}}(D) < l < \underline{\ell}^{\text{InitGrid}}(D)$. So it is legitimate to say 0 is responsible for those levels as well.

We will extend the notion of responsibility in Definition 2.2.5.

2.2.3 Asphalting of the remaining fields

After constructing $\text{InitGrid}^{\mathbf{x}}(u)$, we continue by iteratively putting the missing streets on the remaining fields. Let us describe informally how we proceed.

The streets that are not fields w.r.t. $\text{InitGrid}^{\mathbf{x}}$ are to remain untouched. We want to work exclusively on the fields.

By Lemma 2.2.2, any field B w.r.t. $\text{InitGrid}^{\mathbf{x}}$ is surrounded by four streets. The minimum of their level minus one is the level of the first streets that should be put on B. Determining the level of the streets to put is hence the first step.

Then, we need to know the place where we put these streets. To each field B will be assigned an own process resembling the one in (2.5); this time however, only one level at a time is taken into account. The random ingredient of this process will be the Bernoulli process associated to the upper right corner of B and the respective level.

Now, when the streets are put on the fields, smaller fields are created; on these, we put streets of the next lower level, and so on.

Now, back to rigid definitions. First, we define a dummy and the starting point of the iteration,

$$L_u^0 :\equiv 0, \quad L_u^1 := \text{InitGrid}^{\mathbf{x}}(u),\ u \in \mathbb{Z}^2.$$

For $i \geq 1$, and B a field with respect to the i-th iteration step $L_\cdot^i$, we associate a level to B by

$$l^i(B) := \begin{cases} \min_{v \in \partial B} L_v^i - 1 & \text{if } B \text{ is not a field with respect to } L_\cdot^{i-1} \\ l^{i-1}(B) - 1 & \text{if it is.} \end{cases}$$

This is the level of the streets that are going to be placed on B. The first line of the definition is used at the first iteration step, and also the default for the following steps; only if there has no street been put on a field in the last step, the second line makes sure that in the current step, the same level is not used again.

We provide the emplacement in B for the new streets of level l (we exceptionally remind the dependence on $\mathbf{x}$) by

$$W_x^{l,B}(\mathbf{x}) := l\mathbb{1}_{\exists y \in \mathbb{Z}:\ x \leq y < x + \beta_{\lfloor \frac{l}{2} \rfloor}, \mathbf{x}(y,l,\Rsh B)=1},\, l \in \mathbb{N}_0,\ x \in \mathbb{Z}.$$

Given l, the indicator function checks whether at the point x, there is a street of level l induced by the Bernoulli process at the upper right corner of the field.

The streets are placed on the field B using

$$L_u^{l,B} := W_{u_{l \bmod 2}}^{l,B} \mathbb{1}_{u \in B},\ u \in \mathbb{Z}^2.$$

The sublevel $l \bmod 2$ is taking care of the (vertical or horizontal) orientation of the streets.

We need to do this setting of streets in every field, and set the whole iteration step as

$$L_u^i := L_u^{i-1} + \sum_{\substack{B \text{ field} \\ \text{w.r.t. } L_\cdot^{i-1}}} L_u^{l^i(B),B}.$$

We have put, on every field w.r.t. $L_\cdot^{i-1}$, streets of "one level lower".

The process $L_\cdot^i = L_\cdot^i(\mathbf{x})$ converges pointwise with $i \to \infty$ for P–almost any realization $\mathbf{x}$ of $\mathbf{X}$: for any field w.r.t. $\text{InitGrid}^{\mathbf{x}}(\cdot)$, at some iteration, the level 1 (with superlevel 0) is reached and the remaining sub-fields are entirely filled with streets of level 1. Another way of seeing the convergence is by remarking that for every point $u \in \mathbb{Z}^2$, the sequence $(L_u^i)_{i\in\mathbb{N}}$ is monotonically increasing and bounded. The limes will be called the *final streetgrid* $\text{SG}(\mathbf{x}) = (\text{SG}(\mathbf{x})_u)_{u\in\mathbb{Z}^2}$ and we write $\text{SG} = (\text{SG}_u)_{u\in\mathbb{Z}^2} := (\text{SG}(\mathbf{X})_u)_{u\in\mathbb{Z}^2}$.

Based on the earlier simulation of the initial grid, a simulation of the final streetgrid can be found in Figure 3.

Lemma 2.2.4. $\text{SG}(\mathbf{X})$ *is P–a.s. a streetgrid.*

Proof. Each iteration step L^i is: to obtain L^i, only the fields of L^{i-1} are changed, and on these fields are placed streets extending in one coordinate direction up to the boundary of the field they are placed on. These streets are of strictly lower level than all surrounding streets.

Because the passage to the limit is of the type where for any finite region, the sequence is from some point on constant and equal to the limiting object, SG is a streetgrid as well. □

We turn again towards the concept of responsibility. This time, we give a precise definition, and then explain how it relates to our construction of the streetgrid.

Definition 2.2.5. *Take a streetgrid g. For any block D w.r.t. g and any $\ell^g(D) \le l < \underline{\ell}^g(D)$, there is a unique $w \in \mathbb{Z}^2$ of which we say that it is* in g responsible for the emplacement of streets of level l in D. *It is given by $w = 0$ if $0 \in D$, and $w = \urcorner D$ if $0 \notin D$.*

For $g = \text{InitGrid}$, this definition exactly reflects Remark 2.2.3. The streets already present in InitGrid are carried over to SG, so it is reasonable to say 0 is responsible for these in SG as well.

The responsibility of points $w \neq 0$ can be understood as follows: Any field D w.r.t. InitGrid does not contain the origin. It is also a block and will remain a block in the course of the construction.

The first iteration step is about placing streets of level $l = \underline{\ell}^{\text{InitGrid}}(D) - 1$ on D. The randomness for their emplacement comes from the Bernoulli process $\mathbf{X}((\cdot, l, \urcorner D))$. This is why $\urcorner D$ should be considered responsible for this block and level.

If no streets of level l are placed (because the Bernoulli process is 0 in the relevant range), $\urcorner D$ is responsible for the subsequent lower levels as well, until streets is placed. The level of these streets will later turn out to be the level $\ell^{\text{SG}}(D)$ of the block D.

By the placement of these streets, smaller fields are created, and it is *their* upper right corner that provides the randomness via $\mathbf{X}$. These upper right corners are hence the places that are responsible for the streets of these lower levels, on these smaller fields (which again are and remain blocks).

The next level shows that there is no conflict of responsibility.

Lemma 2.2.6. *Take a streetgrid g. If $w \in \mathbb{Z}^2$ is responsible in g for the emplacement of streets of level l in D, where $l \in \mathbb{N}$ is a level and D some block w.r.t. g, then w is* not *responsible in g for the emplacement of streets of level l in B, where $B \neq D$ is some other block w.r.t. g.*

Figure 3: Simulation of the final street grid SG. Where was the origin again?

Proof. Suppose w is responsible in g for the emplacement of streets of level l in both D and B, where both D and B are blocks w.r.t. g, but $D \neq B$. A first deduction is that either both B and D must contain the origin, or share the same upper right corner $w = \Rsh B = \Rsh D$. In either case, $B \cap D \neq \emptyset$.

As $B \neq D$, this implies that, without loss of generality, $\partial B \cap D \neq \emptyset$. Hence, $\ell^g(D) \geq \underline{\ell}^g(B)$. This is a contradiction to that w was to be responsible for the same level in B and D. $\square$

2.3 Transition probabilities for the random environment

In order to determine where what transition kernels will be placed, we cut down the streets of the streetgrid to lanes using the following definition:

Definition 2.3.1. *For $\diamondsuit, \heartsuit \in \{+,-\}$, B a street w.r.t.* $\mathrm{SG}(\mathbf{X})$ *of superlevel $m := \lfloor \frac{\ell^{\mathrm{SG}}(B)}{2} \rfloor \geq 2$ and sublevel $k := \ell^{\mathrm{SG}}(B) \bmod 2$, we define the* lanes

$$\mathrm{Lane}^{\mathrm{SG}}_{\diamondsuit,\heartsuit}(B) := \begin{cases} \{u \in B : b_k(B) \leq u_k < b_k(B) + \frac{\beta_m}{4}\} & \text{if } \diamondsuit = +, \heartsuit = +; \\ \{u \in B : b_k(B) + \frac{\beta_m}{4} \leq u_k < b_k(B) + \frac{\beta_m}{2}\} & \text{if } \diamondsuit = +, \heartsuit = -; \\ \{u \in B : b'_k(B) - \frac{\beta_m}{2} < u_k \leq b'_k(B) - \frac{\beta_m}{4}\} & \text{if } \diamondsuit = -, \heartsuit = -; \\ \{u \in B : b'_k(B) - \frac{\beta_m}{4} < u_k \leq b'_k(B)\} & \text{if } \diamondsuit = -, \heartsuit = +. \end{cases}$$

The definition of $b_\cdot(B)$ and $b'_\cdot(B)$ was given in (2.2).

Note that there might be some non–empty space between the two middle lanes $\mathrm{Lane}^{\mathrm{SG}}_{+,-}(B)$ and $\mathrm{Lane}^{\mathrm{SG}}_{-,-}(B)$.

We want to place the transition probabilities in a way that on the lanes with "+" as first index, the random walk feels a drift northwards or eastwards (if the sublevel of the street is 0 or 1, respectively), and on the lanes with "−" as first index, it feels a drift to the south or the west. The distinction between + and − in the second index is then used to provide a drift to the area where two lanes of the same street with the same first index meet.

Definition 2.3.2. *With $\diamondsuit, \heartsuit \in \{+,-\}$, we define*

$$\begin{aligned} \omega_{\diamondsuit,\heartsuit} &: \{\pm e_0, \pm e_1\} \to [\frac{1}{4} - \varepsilon, \frac{1}{4} + \varepsilon], \\ \omega_{\diamondsuit,\heartsuit}(\dagger e_0) &:= \frac{1}{4} + (\dagger(\diamondsuit(\heartsuit\varepsilon))), \\ \omega_{\diamondsuit,\heartsuit}(\dagger e_1) &:= \frac{1}{4} + (\dagger(\diamondsuit\varepsilon)),\ \dagger \in \{+,-\}, \end{aligned}$$

and

$$\omega_{\frac{1}{4}}(e) := \frac{1}{4},\ e \in \{\pm e_0, \pm e_1\}.$$

We will also be using the notation $\omega_{\nearrow} = \omega_{+,+}$ and visualize this local transition probability either by ⇹ or $\nearrow$. See also Figure 4.

We will need a reflection matrix, namely

$$R := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

to place the transition probabilities we just defined on the streets.

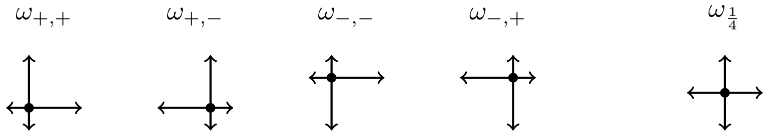

Figure 4: The transition probability kernels $\omega^{j}_{\cdot,\cdot}$. The lengths of the arrows are not to scale.

Definition 2.3.3. *Given the streetgrid* $\mathrm{SG}(\mathbf{X})$*, the transition probability kernels of the environment at place* $u \in \mathbb{Z}^2$ *will be defined as follows. If* $u \in B$ *a street w.r.t.* $\mathrm{SG}(\mathbf{X})$ *such that* $\lfloor \frac{\ell(B)}{2} \rfloor \geq 2$ *and* $b'_{\ell(B) \bmod 2}(B) - b_{\ell(B) \bmod 2}(B) + 1 \geq \beta_{\lfloor \frac{\ell(B)}{2} \rfloor}$*, set*

$$\omega_u = \omega_u(\mathbf{X}) := \begin{cases} \omega_{\diamondsuit,\heartsuit} & \text{if } u \in \mathrm{Lane}^{\mathrm{SG}}_{\diamondsuit,\heartsuit}(B),\ \diamondsuit,\heartsuit \in \{+,-\},\ \ell(B) \bmod 2 = 0, \\ \omega_{\diamondsuit,\heartsuit} \circ R & \text{if } u \in \mathrm{Lane}^{\mathrm{SG}}_{\diamondsuit,\heartsuit}(B),\ \diamondsuit,\heartsuit \in \{+,-\},\ \ell(B) \bmod 2 = 1, \\ \omega_{\frac{1}{4}}, & \text{else.} \end{cases}$$

If $u \in B$ *any other street, set* $\omega_u := \omega_{\frac{1}{4}}$.

Here, $\omega_{\diamondsuit,\heartsuit} \circ R(e) = \omega_{\diamondsuit,\heartsuit}(Re)$*,* $e \in \{\pm e_0, \pm e_1\}$*.*

A visualization of the lanes and the different corresponding transition probabilities can be found in Figure 5; the bigger picture can be seen in Figure 6.

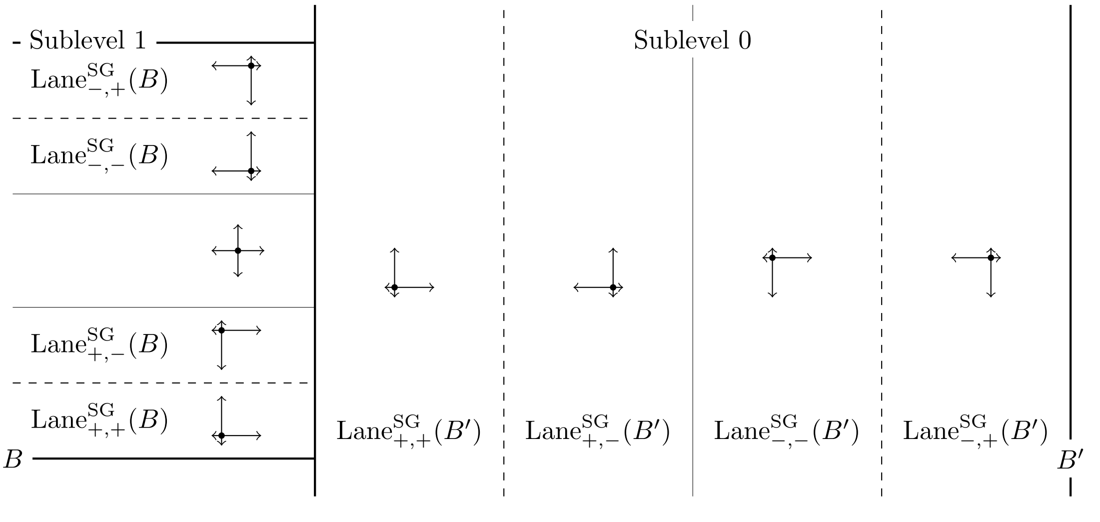

Figure 5: The horizontal street B (of sublevel 1) joining the vertical street B' (of sublevel 0). The street B to the left is wider than its planned width, so that there is some space between the lanes $\mathrm{Lane}^{\mathrm{SG}}_{+,-}(B)$ and $\mathrm{Lane}^{\mathrm{SG}}_{-,-}(B)$. The width of the streets is not to scale: B' ought to be much wider.

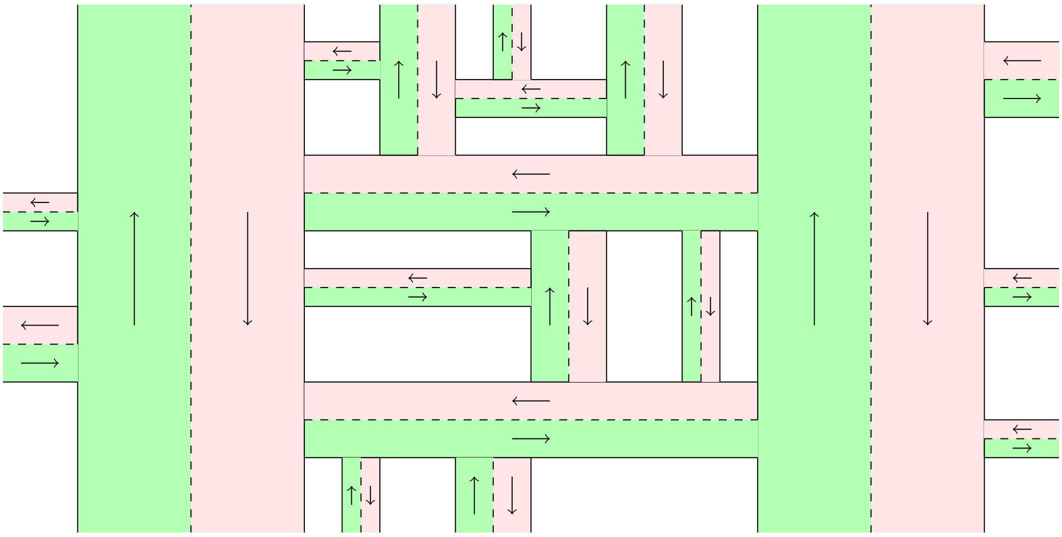

Figure 6: Artists rendering of the environment, and drift a particle would feel. The drift pushing it "towards the middle" of each tinted part of the street is not shown.

3 Properties of InitGrid and SG

3.1 Heuristial approach

Let us describe a very simple model of a random walk in a non–random environment. Define the environment ϖ by setting

$$\varpi_u \begin{cases} \omega_{\searrow} & \text{for all } u \in \mathbb{Z}^2 \text{ such that } u_1 \geq 0, \\ \omega_{\nearrow} & \text{for all } u \in \mathbb{Z}^2 \text{ such that } u_1 < 0. \end{cases}$$

That is, the random walk is subject to a uniform drift in direction of e_0 and towards the zeroth coordinate axis. It is easy to prove by standard martingale methods and the Borel–Cantelli–Lemma that the associated random walk in random environment $(X_n)_n$ starting at 0 has positive probability never to leave the set $\{x \in \mathbb{Z}^2 | x_0 \geq 0, |x_1| \leq \sqrt{x_0}\}$, while following the first coordinate axis to infinity.

The morality of this example is that a random walk with uniform drift along a line and with a drift pushing it back towards that line has positive probability to never be further away from the line than the square root of the travelled length.

We will prove that P–almost surely, somewhere, there is a street w.r.t. $\text{InitGrid}^{\mathbf{X}}$ on the first coordinate axis satisfying the following: if one walks down that street (northwards) until one hits a perpendicular street, walks eastwards on that new one until the next perpendicular street, starts walking northwards again, and so on; if one does so, then:

- at the end of one street, one always encounters one of the next higher level;
- the width of these streets grows nicely,
- the streets are not too long.

Also, there will always be a drift pushing forward and to the middle of the two lanes with first index "+" in these streets.

The idea is that, when walking like described above, the width of the street the walker is in as a function of the distance travelled is larger than the square root $(\cdot)^{1/2}$; this is in analogy to the above example.

An average–case–analysis shows heuristically why this is the case.

The streets of superlevel m have a planned width of β_m and, on average, a length of less than $\frac{1}{\lambda_{m+1}}$. The somewhat worst case for the random walk is if it has to go through the whole length of every street. The width of the n–th street the random walk visits is $\beta_n = n!^2$. The distance travelled is of the order of

$$\sum_{i=1}^{n} \frac{1}{\lambda_{i+1}} = \sum_{i=1}^{n} (i+2)!^2 \leq 2(n+2)!^2.$$

This shows that the square root of the travelled distance is of slower growth than the width of the streets, leaving enough room to the random walk for fluctuations without leaving the sequence of streets.

The exact proof stretches over the whole subsection, but the most pertinent statements can be found in Corollarys 3.2.3, 3.2.7, and 3.2.8.

Remark 3.1.1. *The statements above are even true with* any *root* $(\cdot)^{1/\alpha}$*,* $\alpha > 1$*, instead of the square root* $(\cdot)^{1/2}$*. Hence we need to fix the exponent.*

Definition 3.1.2.

Set $\alpha > 1$ *for the rest of the article.*

3.2 The way to infinity is eventually large

We start the proof of the above claims with a seemingly technical Definition and Lemma.

Definition 3.2.1. *The point inducing the (lowest part of the) first street of superlevel* $m \in \mathbb{N}_0$ *on the* j*-th coordinate axis is defined as*

$$G_m^j := \min\{x \in \mathbb{N}_0 : \mathbf{X}(x, 2m+j, 0) = 1\}, \ j \in \{0,1\}.$$

In view of the following Lemma, let us recall from (2.4) the parameters $\lambda_m := (m+1)!^{-2}$ and $\beta_m := m!^2, \ m \in \mathbb{N}_0$.

Lemma 3.2.2. *The following events all happen* P*–almost surely only finitely often (in* m*):*

$$\{G_{m-1}^0 \geq G_m^0 - \beta_m + 1\}; \quad \{G_{m-1}^1 - \beta_{m-1} + \beta_m \geq G_m^1 - \beta_m + 1\};$$
$$\{G_m^j > (\lambda_m)^{-\alpha}\}, \ j \in \{0,1\}.$$

Proof. The G_m^j, $m \in \mathbb{N}_0, j \in \{0,1\}$, are geometrically distributed, independent random variables with success probability $\lambda_m = \frac{1}{(m+1)!^2}$. We can calculate, for the first event,

$$\begin{aligned} &P(G_{m-1}^0 \geq G_m^0 - \beta_m + 1) \\ &\quad = \sum_{x \in \mathbb{N}_0} P(G_{m-1}^0 \geq x - \beta_m + 1) P(G_m^0 = x) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{x\in\mathbb{N}_0} (1-\lambda_{m-1})^{x-\beta_m+1}(1-\lambda_m)^x\lambda_m \\
&= (1-\lambda_{m-1})^{-\beta_m+1}\lambda_m \sum_{x\in\mathbb{N}_0} [(1-\lambda_{m-1})(1-\lambda_m)]^x \\
&= (1-\lambda_{m-1})^{-\beta_m+1}\frac{\lambda_m}{\lambda_{m-1}+\lambda_m-\lambda_{m-1}\lambda_m} \\
&= (1-\lambda_{m-1})^{-\beta_m+1}\Big(\frac{\lambda_{m-1}}{\lambda_m}+1-\lambda_{m-1}\Big)^{-1} \\
&= (1-\lambda_{m-1})^{-\beta_m+1}\big((m+1)^2+1-\lambda_{m-1}\big)^{-1},\ m\in\mathbb{N}.
\end{aligned}
$$

We see that the first term converges, while the second one is summable, so that we can conclude using the Borel-Cantelli-Lemma.

The probability of the second event computes just the same way, only the limit of the leading term is some other constant.

For the last event, we observe that

$$
P\big(G^j_m > (\lambda_m)^{-\alpha}\big) = (1-\lambda_m)^{\lfloor\lambda_m^{-\alpha}+1\rfloor} = \Big[\Big(1-\frac{1}{(m+1)!^2}\Big)^{(m+1)!^2}\Big]^{\frac{\lfloor(m+1)!^{2\alpha}\rfloor+1}{(m+1)!^2}} \sim e^{-(m+1)!^{2\alpha-2}}
$$

is indeed summable as well. □

Corollary 3.2.3. *The event*

$$
\big\{G^0_{m-1}+\beta_m \le G^0_m \le (\lambda_m)^{-\alpha}\big\} \cap \big\{G^1_{m-1}+2\beta_m-\beta_{m-1} \le G^1_m \le (\lambda_m)^{-\alpha}\big\} \tag{3.1}
$$

holds P–a.s. eventually. Hence, P–a.s.,

$$
M(\mathbf{X}) := \min\left\{m' \ge 5 \middle| \omega \in \bigcap_{m=m'}^{\infty} \begin{array}{c}\{G^0_{m-1}+\beta_m \le G^0_m \le \lambda_m^{-\alpha}\} \\ \cap\,\{G^1_{m-1}+2\beta_m-\beta_{m-1} \le G^1_m \le \lambda_m^{-\alpha}\}\end{array}\right\} < \infty. \tag{3.2}
$$

$M = M(\mathbf{X})$ is the superlevel from which the event defined in (3.1) always holds. The restriction to $m' \ge 5$ is made so that we do not have to worry about whether we can divide streets into four lanes, and subdivide lanes in four equal parts: already $\beta_4 = 576 = 36 * 16$.

There is a picture relating the terms of the event (3.1) in Figure 7.

Lemma 3.2.4. *It holds P–a.s. that for all $x \in \mathbb{N}$ and all $m' > m \ge M$,*

$$
\mathrm{InitGrid}^{\mathbf{X}}(xe_0) \ne 2m+1 \text{ and } \mathrm{InitGrid}^{\mathbf{X}}(G^0_m e_0) \ne 2m'.
$$

Proof. Take any $m \ge M$. We have

$$
G^1_m \ge G^1_{m-1} + 2\beta_m - \beta_{m-1} \ge 2\beta_m - \beta_{m-1} \ge \beta_m.
$$

$\mathbf{X}(G^1_m, 2m+1, 0) = 1$ induces a (part of a) street of superlevel m, with planned width β_m. Thus, this street of sublevel 1 does not reach the zeroth axis.

Now take $m' > m$. We know that $G^0_{m'} \ge G^0_{m'-1} + \beta_{m'} \ge G^0_m + \beta_{m'}$. This shows that any vertical street of higher level does not reach $G^0_m e_0$. □

Lemma 3.2.5.

$$
\mathrm{SG}_{G^j_m e_j} = 2m + j,\ m \ge M,\ j \in \{0,1\}.
$$

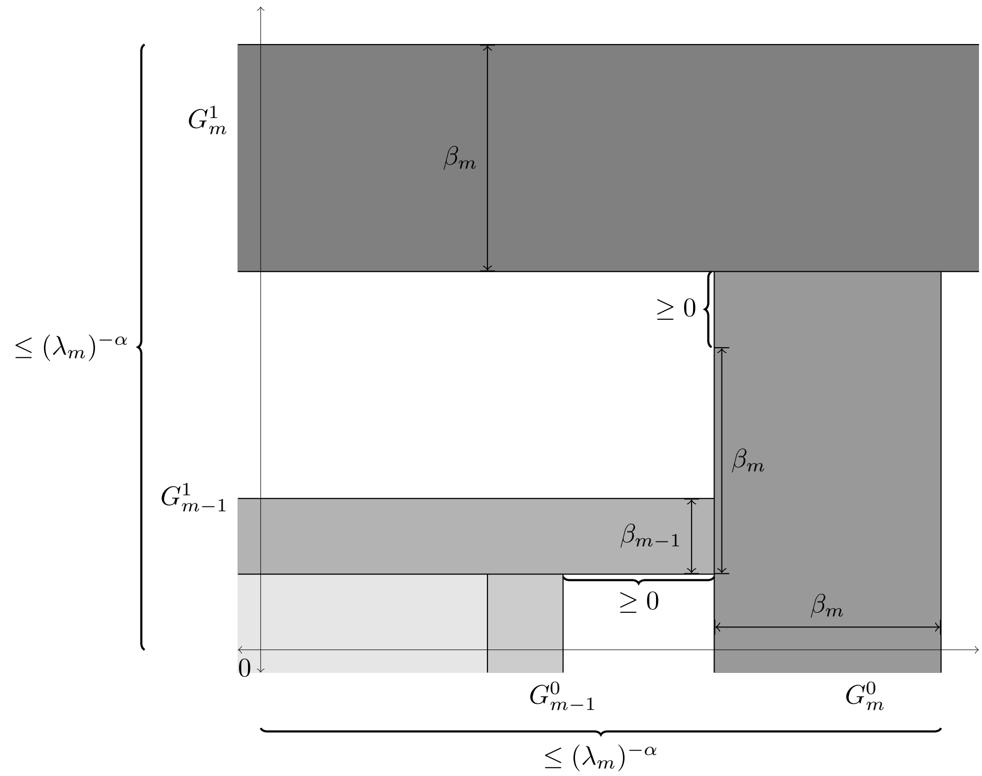

Figure 7: The implications of the event in (3.1).

Proof. We prove the case $j = 0$.

Take $m \geq M$. As G^0_m is a natural number such that $\mathbf{X}\big(G^0_m, 2m, 0\big) = 1$, we have $W^0_{G^0_m}(\mathbf{X}) \geq 2m$. $(G^0_n)_{n \geq M}$ is an increasing sequence. Thus, it holds that $V^0_{G^0_m} \geq 2m$. This implies that

$$\text{InitGrid}^{\mathbf{X}}(G^0_m e_0) \geq 2m. \tag{3.3}$$

The case ">" subdivides into the two

- $\text{InitGrid}^{\mathbf{X}}(G^0_m e_0) = 2m' + 1$ for some $m' \geq m$,
- $\text{InitGrid}^{\mathbf{X}}(G^0_m e_0) = 2m'$ for some $m' > m$,

which both are excluded by Lemma 3.2.4. Hence, equality holds in (3.3). G^j_m depends only on $\mathbf{X}(\cdot, \cdot, 0)$. As all streets that are not fields w.r.t. InitGrid remain untouched in the construction of the final streetgrid, the equality holds for $\text{SG}_{G^0_m e_0}$ as well.

Similar observations can be made for points of the form $G^1_m e_1$, $m \geq M$, using an adapted version of Lemma 3.2.4. □

Definition 3.2.6. *Set*

$$B^j_m := \text{S'rnd}^{\text{InitGrid}}(G^j_m e_j) = \text{S'rnd}^{\text{SG}}(G^j_m e_j),\ m \geq M,\ j \in \{0, 1\}.$$

Corollary 3.2.7. *For all $m \geq M$, the width $b_j'(B_m^j) - b_j(B_m^j) + 1$ of B_m^j is larger than or equal to β_m, while the length of the intersection of B_m^j and the first quadrant, $(\Rsh B_m^j)_i$, $i \neq j$, $i, j \in \{0, 1\}$, satisfies*

$$\lambda_{m+1}^{-\alpha} \geq \begin{cases} (\Rsh B_m^0)_1 \\ (\Rsh B_m^1)_0. \end{cases}$$

Proof. Both assertions follow from the same type of arguments as in the proof of the Lemmas 3.2.4 and 3.2.5; the second one makes also use of the upper bounds provided by (3.1). □

Corollary 3.2.8. *It holds for all $m \geq M$ that*

$$\text{S'rnd}(\Rsh B_m^0 + e_1) = B_m^1 \text{ and } \text{S'rnd}(\Rsh B_m^1 + e_0) = B_{m+1}^0.$$

Proof. We prove only the first assertion.

Let $m \geq M$. As we have seen in Lemma 3.2.5, $\ell(B_m^0) = 2m$.

By definition, the street B_m^0 extends vertically until it is blocked by some horizontal higher–level–street. The superlevel of this street is greater as or equal to m, otherwise there would be no blocking. Any horizontal street $B_{m'}^1$ of level $m' > m$ does not interfere with B_m^1, because the $G_\cdot^1$ all keep their distance from each other (see (3.1)). So, the blocking indeed happens by B_m^1. □

3.3 Stationarity

Notation 3.3.1. *Take $F : \{0,1\}^{\mathcal{Z}} \to {\mathbb{N}_0}^{\mathbb{Z}^2}$ a function. Note that the values $F(\mathbf{x}) : \mathbb{Z}^2 \to \mathbb{N}_0$ of this function are themselves functions $u \mapsto F(\mathbf{x})_u$. Let $I \subseteq \mathcal{Z}$, $D \subseteq \mathbb{Z}^2$. For $\bar{\mathbf{x}} \in \{0,1\}^I$, $g \in {\mathbb{N}_0}^D$, by the notation*

$$F(\bar{\mathbf{x}})|_D = g, \tag{3.4}$$

we shall express that

for all $\mathbf{x} \in \{0,1\}^{\mathcal{Z}}$ such that $\mathbf{x}|_I = \bar{\mathbf{x}}$, it holds that $F(\mathbf{x})_u = g_u$ for all $u \in D$.

Here, $\mathbf{x}|_I : I \to \{0,1\}$ denotes the usual restriction of the function $\mathbf{x} : \mathcal{Z} \to \{0,1\}$ on I. Notation (3.4) however is more restrictive than a mere restriction, because it is understood that on D, $F(\cdot)$ does not depend on the values at places in $\mathcal{Z} \setminus I$.

Also define $0|_I$ to be the constant mapping that assigns 0 to any element in I.

Lemma 3.3.2. *For any box $B \ni 0$, there is P–a.s. a block w.r.t. $\text{SG}(\mathbf{X})$ containing B:*

$$P\Big(\bigcup_{\substack{D \subseteq \mathbb{Z}^2: \\ B \subseteq D}} \{D \text{ is block w.r.t. } \text{SG}(\mathbf{X})\} \Big) = 1.$$

Proof. Take B a box containing the origin. For $j \in \{0,1\}$, define the random variables

$$d_j := \max\{x \leq b_j(B) | V_{x-1}^j > \ell^{\text{SG}}(B)\} \text{ and } d_j' := \min\{x \geq b_j'(B) | V_{x+1}^j > \ell^{\text{SG}}(B)\}.$$

These are P–almost surely finite, and $B \subseteq D := \{d_0, \dots, d_0'\} \times \{d_1, \dots, d_1'\}$. D is a random set and a block w.r.t. InitGrid($\cdot$). The streets placed on D by the iterative construction leading to SG are all of lower level than the minimum of the levels present in ∂D, so that the block-property is preserved. □

Definition 3.3.3. *Let D be a block w.r.t. $g \in \mathbb{N}_0{}^{\overline{D}}$ such that $0 \in D$. Note that this is more a condition on g than on D. Define*

$$J_g := \Big\{(y, 2m+j, 0)\big|\ m > \lfloor \frac{\overline{\ell}^g(D)}{2} \rfloor,\ j \in \{0,1\},\ b_j(D) - 1 \leq y \leq b_j'(D) + \beta_m\Big\} \subseteq \mathcal{Z}$$

and

$$I_g := \Big\{(y, 2m+j, u)\big|\ j \in \{0,1\},\ 0 \leq m \leq \lfloor \frac{\overline{\ell}^g(D)}{2} \rfloor,\ b_j(D) - 1 \leq y \leq b_j'(D) + \beta_m,\ u \in D\Big\} \subseteq \mathcal{Z}.$$

The dependence of I_g and J_g on D is omitted because it can be considered implicit via g.

To explain the meaning of these two sets, we need to go into greater detail.

Take a realization of $\mathbf{X}$. It is an element of $\{0,1\}^{\mathcal{Z}}$, and leads to $\mathrm{SG} = \mathrm{SG}(\mathbf{X})$. One can ask at which points in $\mathcal{Z}$ the values of $\mathbf{X}$ may be changed without changing the outcome of SG, or $\mathrm{SG}\,|_D$ for some fixed $D \subseteq \mathbb{Z}$.

The other way around, given a certain realization $g \in \mathbb{N}_0{}^{\overline{D}}$ of $\mathrm{SG}(\mathbf{X})|_{\overline{D}}$, where $D \subseteq \mathbb{Z}^2$ is a box, one can ask about the set of realizations of $\mathbf{X}$ such that

$$\mathrm{SG}(\mathbf{X})|_{\overline{D}} = g.$$

It turns out it is enough to look at the outcome of $\mathbf{X}$ on the two subsets I_g and J_g of $\mathcal{Z}$ in order to decide whether the last equation is true or not. All points that are responsible in the sense of Definition 2.2.5 are contained in I_g, and $\mathbf{X}(\cdot)$ being equal to 0 at all points in J_g stipulates the absence of big streets that are not supposed to be on $\overline{D}$.

Lemma 3.3.4. *Under the hypotheses of Definition* 3.3.3*, I_g is finite, and $I_g \cap J_g = \emptyset$. Let $\mathbf{x} \in \{0,1\}^{\mathcal{Z}}$. If $\mathrm{SG}(\mathbf{x})|_{\overline{D}} = g$, then it holds that $\mathrm{SG}(\mathbf{x}|_{I_g \cup J_g})|_{\overline{D}} = g$, in the notation of* (3.4)*. In other words, $\mathrm{SG}(\mathbf{x})|_{\overline{D}} = g$ does not depend on $\mathbf{x}|_{\mathcal{Z} \setminus (I_g \cup J_g)}$. Also, $\mathrm{SG}(\mathbf{x})|_{\overline{D}} = g$ implies $\mathbf{x}|_{J_g} = 0|_{J_g}$. Finally, $P(\mathbf{X}|_{J_g} \equiv 0) > 0$.*

Proof of Lemma 3.3.4. The first two assertions are obvious. $\mathrm{SG}(\mathbf{x})|_{\overline{D}} = g$ does hold or not no matter what the values of $\mathbf{x}$ at the points $(y, 2m+j, u)$ with

- $u \in \mathbb{Z}^2 \setminus D$, $m \in \mathbb{N}$, $y \in \mathbb{Z}$, $j \in \{0,1\}$,
- $u \in D \setminus \{0\}$, $m \geq \lfloor \frac{\ell^g(D)}{2} \rfloor$, $y \in \mathbb{Z}$, $j \in \{0,1\}$,
- $u \in D$, $m < \lfloor \frac{\ell^g(D)}{2} \rfloor$, $y \leq b_j(D) - 1$ or $y \geq b_j'(D) + \beta_m$, $j \in \{0,1\}$,
- $u = 0$, $m \geq \lfloor \frac{\ell^g(D)}{2} \rfloor$, $y \leq b_j(D) - 2$ or $y \geq b_j'(D) + \beta_m + 1$, $j \in \{0,1\}$.

Let us look at the lines one at a time.

As D is a block w.r.t. g, and $0 \in D$, all four streets in ∂D are already present in $\mathrm{InitGrid}^{\mathbf{x}}$. $\mathrm{InitGrid}^{\mathbf{x}}$ is only influenced by the values of $\mathbf{x}$ at points $(\cdot, \cdot, 0)$. The streets w.r.t. g in D are either streets w.r.t. $\mathrm{InitGrid}^{\mathbf{x}}$ or are influenced by the values of $\mathbf{x}$ at the upper right corners of fields w.r.t. $\mathrm{InitGrid}^{\mathbf{x}}$ or the subsequent iteration steps in the construction. These fields are entirely contained in D, again because D is a block w.r.t. g. This is why points $(\cdot, \cdot, u)$ with $u \notin D$ have no influence.

We just looked at the influence of points in the upper right corners of fields lying entirely in D. The streets they induce are all of lower level than the minimum level present in ∂D; higher levels are not even considered, and thus the values of $\mathbf{x}$ at the points in the second line have no influence on the equation.

The values of points with lower level do have an influence, but only if the index of the Bernoulli–process is not too far from D; to be precice, neither left to the lower end in the j–th coordinate–direction of D, nor farther than one street–width to the right of the upper end of D.

Similarily, the values at the origin do not have any influence if the index of the Bernoulli–process is too far from $\overline{D}$; this translates as slightly loosened boundaries in the last line.

All remaining points are contained in I_g and J_g, which contain however some of the cases above as well. This proves that $\mathrm{SG}(\mathbf{x})|_{\overline{D}} = g$ does not depend on $\mathbf{x}|_{\mathcal{Z}\setminus(I_g\cup J_g)}$.

The superlevels of the streets in $\overline{D}$ are *per definitionem* bounded by $\lfloor\frac{\overline{\ell}^g(D)}{2}\rfloor$. If the equation $\mathrm{SG}(\mathbf{x})|_{\overline{D}} = g$ is to hold, it is trivially true that

$$\text{there is no street w.r.t. } \mathrm{SG}(\mathbf{x}) \text{ of higher superlevel than } \lfloor\frac{\overline{\ell}^g(D)}{2}\rfloor \text{ in } \overline{D}. \tag{3.5}$$

This condition (3.5) is equivalent to

$$\mathbf{x}\big(y, 2m+j, 0\big) = 0 \text{ for all } b_j(D) - 1 \le y \le b'_j(D) + \beta_m, m > \lfloor\frac{\overline{\ell}^g(D)}{2}\rfloor, j \in \{0,1\}. \tag{3.6}$$

(3.6) can be written as $\mathbf{x}|_{J_g} \equiv 0$, which can hence be seen as an equivalent to (3.5).

Finally, we have, with some non-trivial, non-random constant c,

$$P(\mathbf{X}|_{J_g} \equiv 0) = \prod_{m>\lfloor\frac{\overline{\ell}^g(B)}{2}\rfloor} \prod_{j\in\{0,1\}} (1-\lambda_m)^{b'_j(D)-b_j(D)+\beta_m+2} \ge c \prod_{m\ge1}\prod_{j\in\{0,1\}}(1-\lambda_m)^{\beta_m} = c\prod_{m\ge1}(1-\lambda_m)^{2\beta_m}.$$

This value to be larger than zero is equivalent to

$$\sum_{m\ge1} \beta_m \ln(1-\lambda_m) > -\infty.$$

But

$$\beta_m \ln(1-\lambda_m) \sim \beta_m(-\lambda_m) = -\frac{m!^2}{(m+1)!^2} = -\frac{1}{(m+1)^2},$$

and we can, by the finiteness of the sum, confirm positive $P_{\mathbf{X}|_{J_g}}$-measure for $0|_{J_g}$. □

Definition 3.3.5. *We need to define some shift operators and related notations. Let $v \in \mathbb{Z}^2$ be the vector we want to shift by.*

For $D \subseteq \mathbb{Z}^2$, we write $D + v := \{u + v \mid u \in D\}$.

For $D \subseteq \mathbb{Z}^2$, $f \in {\mathbb{N}_0}^D$, we define the shifted $\theta_v f \in {\mathbb{N}_0}^{D+v}$ by

$$(\theta_v f)_u := f_{u-v} \text{ for all } u \in D + v.$$

We also can shift elements $(x, l, u) \in \mathcal{Z}$ by

$$\theta_v(x, l, u) := (x + v_{l \bmod 2}, l, u + v).$$

A slightly different shift will sometimes be needed for elements of the form $(x, l, 0) \in \mathcal{Z}$, namely one that preserves the special role of the origin:

$$\vartheta_v(x, l, 0) := (x + v_{l \bmod 2}, l, 0).$$

With these last two definitions at hand, we can shift the two I_g and J_g from Definition 3.3.3 *in the standard way by*

$$\theta_v I_g := \{\theta_v(x, l, u) \mid (x, l, u) \in I_g\},$$

$$\vartheta_v J_g := \{\vartheta_v(x,l,0) \mid (x,l,0) \in J_g\}.$$

Finally, we shift whole configurations $\bar{\mathbf{x}} \in \{0,1\}^I$, $I \subseteq \mathcal{Z}$ *by defining*

$$\theta_v \bar{\mathbf{x}}\big((x,l,u)\big) := \bar{\mathbf{x}}\big(\theta_{-v}(x,l,u)\big),\ (x,l,u) \in \theta_v I.$$

Lemma 3.3.6. *Let* $D \ni 0$ *be a block w.r.t.* $g \in \mathbb{N}_0{}^{\overline{D}}$, I_g, J_g *from Definition* 3.3.3. *Also take any* v *such that* $-v \in D$. *Then,* $I_{\theta_v g} = \theta_v I_g$ *and* $J_{\theta_v g} = \vartheta_v J_g$, *and for any* $\mathbf{x} \in \{0,1\}^{\mathcal{Z}}$, $\mathrm{SG}(\mathbf{x})|_{\overline{D}+v} = \theta_v g$ *implies* $\mathrm{SG}(\mathbf{x}|_{\theta_v I_g \cup \vartheta_v J_g})|_{\overline{D}+v} = \theta_v g$.

Proof. The first two equalities are easy exercises; an important point is how $\vartheta_\cdot$ preserves the special role of the origin, but at a different position relative to the shifted box.

The second assertion then follows directly frome Lemma 3.3.4, which tells us that $\mathrm{SG}(\mathbf{x})|_{\overline{D}+v} = \theta_v g$ implies $\mathrm{SG}(\mathbf{x}|_{I_{\theta_v g} \cup J_{\theta_v g}})|_{\overline{D}+v} = \theta_v g$. □

Figure 8 gives an idea of how the responsibility changes when the point of reference (the origin) is changed. This sort of changing will be employed in Definition 3.3.7 in order to create a configuration of $\{0,1\}^{I_g}$ that yields the same outcome of the final streetgrid's construction, only shifted.

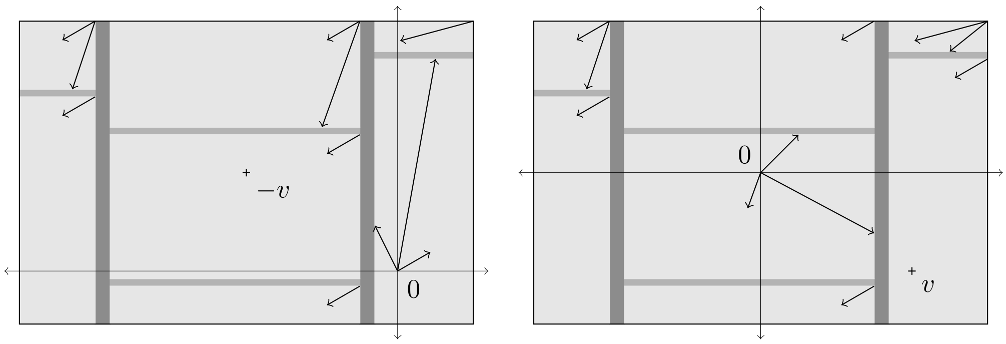

Figure 8: Responsibility. If the base of some arrow is at w and the tip points to some street of level l, then w is responsible for the emplacement of the streets of level l in D, where D is the smallest block containing the street.

Definition 3.3.7. *Take the hypotheses of Lemma* 3.3.6. *We define yet another operator on configurations on* $\{0,1\}^{I_g}$,

$$\bar{\mathbf{x}} \mapsto \swarrow\!\!\!\nearrow \bar{\mathbf{x}}:\ \big\{\bar{\mathbf{y}} \in \{0,1\}^{I_g} \big| \,\mathrm{SG}(\bar{\mathbf{y}}, 0|_{J_g})|_{\overline{D}} = g\big\} \to \big\{\bar{\mathbf{y}} \in \{0,1\}^{\theta_v I_g} \big| \,\mathrm{SG}(\bar{\mathbf{y}}, 0|_{\vartheta_v J_g})|_{\overline{D}+v} = \theta_v g\big\}.$$

So, we need to define the object $\big(\swarrow\!\!\!\nearrow \bar{\mathbf{x}}\big)(\cdot)$ *for all* $(y,l,w) \in I_{\theta_v g}$. *We do this first for a special case of pairs* (l,w), *and then for the rest.*

Take any block B *w.r.t.* g, *and* $\ell^g(B) \leq l < \underline{\ell}^g(B)$. *Recall that* $B+v$ *is a block w.r.t.* $\theta_v g$, *and that* $\ell^g(B) = \ell^{\theta_v g}(B+v)$ *and* $\underline{\ell}^g(B) = \underline{\ell}^{\theta_v g}(B+v)$. *So, we can apply Definition* 2.2.5 *and obtain* $w \in D$ *and* $\widetilde{w} \in D+v$ *such that*

- $\widetilde{w}$ *responsible in* $\theta_v g$ *for the emplacement of streets of level* l *in* $B+v$, *and*
- w *responsible in* g *for the emplacement of the streets of level* l *in* B,

which are both the only points to satisfy these conditions.

Write $m := \lfloor \frac{l}{2} \rfloor$ *and* $j := l \bmod 2$, *and define, for* $b_j(D+v) - 1 \le y \le b'_j(D+v) + \beta_m$,

$$\big(\swarrow\!\!\!\!\nearrow \bar{\mathbf{x}}\big)\big((y,l,\widetilde{w})\big) := \bar{\mathbf{x}}\big((y - v_j, l, w)\big),\ and\ \big(\swarrow\!\!\!\!\nearrow \bar{\mathbf{x}}\big)\big((y,l,w+v)\big) := \bar{\mathbf{x}}\big((y - v_j, l, \widetilde{w} - v)\big). \quad (3.7)$$

For any other case that has not yet been covered, take $l < \underline{\ell}^g(D)$ *and* $\widetilde{w} \in D + v$ *such that*

- $\widetilde{w}$ *is* not *in* $\theta_v g$ *responsible for the emplacement of the streets of level* l *in* $\widetilde{B}$ *for any block* $\widetilde{B}$ *w.r.t.* $\theta_v g$, *and*
- $\widetilde{w} - v$ *is* not *in* g *responsible for the emplacement of the streets of level* l *in* B *for any block* B *w.r.t.* g;

then, with $m := \lfloor \frac{l}{2} \rfloor$, $j := l \bmod 2$, *we define, for* $b_j(D+v) - 1 \le y \le b'_j(D+v) + \beta_m$,

$$\big(\swarrow\!\!\!\!\nearrow \bar{\mathbf{x}}\big)\big((y,l,\widetilde{w})\big) := \bar{\mathbf{x}}\big((y - v_j, l, \widetilde{w} - v)\big).$$

This last definition shows that the operator $\swarrow\!\!\!\!\nearrow$ is for most of the points really just the shift operator applied to the function $\bar{\mathbf{x}}$; only at the few points that are responsible, and at their counterparts in the shifted set, the special definition takes effect, and so to say, the responsibility is switched.

Note that $\swarrow\!\!\!\!\nearrow$ depends strongly on v and g, which gives again an implicit dependence on D. It also depends on our choice of I_g and J_g.

Lemma 3.3.8. $\swarrow\!\!\!\!\nearrow$ *is well-defined, and takes indeed values in the specified codomain. It is bijective, and probability–preserving in the sense that*

$$P(\mathbf{X}|_{I_g} = \bar{\mathbf{x}}) = P(\mathbf{X}|_{\theta_v I_g} = \swarrow\!\!\!\!\nearrow \bar{\mathbf{x}})\ \textit{for all}\ \bar{\mathbf{x}} \in \{0,1\}^{I_g}.$$

Also, the following equivalence holds:

$$\mathrm{SG}(\bar{\mathbf{x}}, 0|_{J_g})|_{\overline{D}} = g \Longleftrightarrow \mathrm{SG}(\swarrow\!\!\!\!\nearrow \bar{\mathbf{x}}, 0|_{\vartheta_v J_g})|_{\overline{D+v}} = \theta_v g.$$

Proof. For the first part of the definition, we remark that if $\widetilde{w} = w + v$, the two definitions in (3.7) coincide: $w = \widetilde{w} - v$. So, the two do not contradict each other immediately. Also, given any l, w and $\widetilde{w}$ are, respectively, responsible for l only in B and $B + v$. This was shown in Lemma 2.2.6. In the second part, the two bullets make sure that only cases not yet covered by the first part are defined. So, we indeed did not commit the error of multiply defining things.

The verification of $\swarrow\!\!\!\!\nearrow \bar{\mathbf{x}} \in \{0,1\}^{\theta_v I_g}$ consists in checking that the domain of $\swarrow\!\!\!\!\nearrow \bar{\mathbf{x}}$ is contained in $\theta_v I_g$. Indeed, the indices y are chosen in the correct range. Also, $l < \underline{\ell}^g(B) < \underline{\ell}^g(D)$. Finally, $\widetilde{w}, w + v \in B + v \subseteq D + v$. The same applies to the second part of the definition.

To prove the bijectivity of $\swarrow\!\!\!\!\nearrow = \swarrow\!\!\!\!\nearrow(v,g)$, we consider the inverse function, which is $\swarrow\!\!\!\!\nearrow(-v, \theta_v g)$. To check that this is true, remark that the two parts of the definition of $\swarrow\!\!\!\!\nearrow$ can be inverted separately; the responsible points are just reversed, and the responsibilities switched back. The points which are not responsible being identical, the values there get shifted back as well.

For the preservation of probability, note that $\swarrow\!\!\!\!\nearrow$ leaves the levels intact, and replicates the same number of zeros and ones, just at different places. Then, the stationarity of the Bernoulli–processes takes effect.

The last statement is a consequence of the concept of switching responsibilities described above. The operator moves the values of $\bar{\mathbf{x}}$ at any point responsible in g for the emplacement of streets of level l in B to the point which is in $\theta_v g$ responsible for the emplacement of streets of level l on $B + v$. If one translates the concept of responsibility into the construcion of the streetgrid, one sees that $\mathrm{SG}(\swarrow\!\!\!\!\nearrow, 0|_{J_{\theta_v g}})$ reconstitutes indeed the shifted g on the shifted domain.

The opposite inclusion follows from the above considerations on bijectivity. □

Theorem 3.3.9. $\mathrm{SG}(\mathbf{X})$ *is stationary.*

Proof. We need to show the invariance of $\mathrm{SG}(\mathbf{X})$'s finite-dimensional marginal distributions under the arbitrary shifts in $\mathbb{Z}^2$. Fortunately, we can restrict ourselves to distributions on boxes and shift-vectors inside these boxes: if we need a farther shift, we just take a bigger box.

Let $B \ni 0$ be a box, $v \in \mathbb{Z}^2$ such that $-v \in B$, and $g \in {\mathbb{N}_0}^B$.

In the following calculations, the first equality is due to Lemma 3.3.2, the second one is true because the smallest (w.r.t. the semi-order established by the subset-relation) block around B is unique. The fourth equality holds because the block property depends only on $\overline{D}$, and for the sixth one we apply Lemma 3.3.4 for one inclusion, the other one following directly from Notation 3.3.1. Lemma 3.3.4 also implies the disjointness of $I_{\widehat{g}}$ and $J_{\widehat{g}}$ leading to the independence used for the seventh equality. For the last equality, we apply Lemma 3.3.8.

$$
\begin{aligned}
&P\big(\mathrm{SG}(\mathbf{X})|_B = g\big) \\
&\quad = P\Big(\bigcup_{\substack{D \supseteq B \\ \text{box}}} \big\{\mathrm{SG}(\mathbf{X})|_B = g,\ D \text{ block w.r.t. } \mathrm{SG}(\mathbf{X})\big\}\Big) \\
&\quad = \sum_{\substack{D \supseteq B \\ \text{box}}} P\big(\mathrm{SG}(\mathbf{X})|_B = g,\ D \text{ is the smallest block w.r.t. } \mathrm{SG}(\mathbf{X}) \text{ containing } B\big) \\
&\quad = \sum_{\substack{D \supseteq B \\ \text{box}}} \sum_{\substack{\widehat{g} \in {\mathbb{N}_0}^{\overline{D}}: \\ \widehat{g}|_B = g}} P\big(\mathrm{SG}(\mathbf{X})|_{\overline{D}} = \widehat{g},\ D \text{ is the smallest block w.r.t. } \mathrm{SG}(\mathbf{X}) \text{ containing } B\big) \\
&\quad = \sum_{\substack{D \supseteq B \\ \text{box}}} \sum_{\substack{\widehat{g} \in {\mathbb{N}_0}^{\overline{D}}: \\ \widehat{g}|_B = g}} P\big(\mathrm{SG}(\mathbf{X})|_{\overline{D}} = \widehat{g},\ D \text{ is the smallest block w.r.t. } \widehat{g} \text{ containing } B\big) \\
&\quad = \sum_{\substack{D \supseteq B \\ \text{box}}} \sum_{\substack{\widehat{g} \in {\mathbb{N}_0}^{\overline{D}}: \\ \widehat{g}|_B = g}} \mathbb{1}_{D \text{ is the smallest block w.r.t. } \widehat{g} \text{ containing } B} P\big(\mathrm{SG}(\mathbf{X})|_{\overline{D}} = \widehat{g}\big) \\
&\quad = \sum_{\substack{D \supseteq B \\ \text{box}}} \sum_{\substack{\widehat{g} \in {\mathbb{N}_0}^{\overline{D}}: \\ \widehat{g}|_B = g}} \mathbb{1}_{D \text{ smallest block}} \sum_{\bar{\mathbf{x}} \in \{0,1\}^{I_{\widehat{g}}}} P\big(\mathrm{SG}(\bar{\mathbf{x}}, 0|_{J_{\widehat{g}}})|_{\overline{D}} = \widehat{g},\ \mathbf{X}|_{I_{\widehat{g}}} = \bar{\mathbf{x}},\ \mathbf{X}|_{J_{\widehat{g}}} \equiv 0\big) \\
&\quad = \sum_{\substack{D \supseteq B \\ \text{box}}} \sum_{\substack{\widehat{g} \in {\mathbb{N}_0}^{\overline{D}}: \\ \widehat{g}|_B = g}} \mathbb{1}_{D \text{ smallest block}} \sum_{\bar{\mathbf{x}} \in \{0,1\}^{I_{\widehat{g}}}} P(\mathbf{X}|_{I_{\widehat{g}}} = \bar{\mathbf{x}}) P\big(\mathbf{X}|_{J_{\widehat{g}}} \equiv 0\big) \mathbb{1}_{\mathrm{SG}(\bar{\mathbf{x}}, 0|_{J_{\widehat{g}}})|_{\overline{D}} = \widehat{g}} \\
&\quad = \sum_{\substack{D \supseteq B \\ \text{box}}} \sum_{\substack{\widehat{g} \in {\mathbb{N}_0}^{\overline{D}}: \\ \widehat{g}|_B = g}} \mathbb{1}_{D \text{ smallest block}} \sum_{\bar{\mathbf{x}} \in \{0,1\}^{I_{\widehat{g}}}} P\big(\mathbf{X}|_{\theta_v I_{\widehat{g}}} = \swarrow\!\!\!\!\nearrow \bar{\mathbf{x}}\big) P\big(\mathbf{X}|_{\vartheta_v J_{\widehat{g}}} \equiv 0\big) \mathbb{1}_{\mathrm{SG}(\swarrow\!\!\!\!\nearrow \bar{\mathbf{x}}, 0|_{\vartheta_v J_{\widehat{g}}})|_{\overline{D}+v} = \theta_v \widehat{g}}
\end{aligned}
$$

We continue by applying the bijectivity of $\swarrow\!\!\!\!\nearrow$, and reverting the steps which lead here, but with respect to the shifted sets.

$$
\begin{aligned}
&\quad = \sum_{\substack{D \supseteq B \\ \text{box}}} \sum_{\substack{\widehat{g} \in {\mathbb{N}_0}^{\overline{D}}: \\ \widehat{g}|_B = g}} \mathbb{1}_{D \text{ smallest block}} \sum_{\bar{\mathbf{y}} \in \{0,1\}^{\theta_v I_{\widehat{g}}}} P\big(\mathbf{X}|_{\theta_v I_{\widehat{g}}} = \bar{\mathbf{y}}\big) P\big(\mathbf{X}|_{\vartheta_v J_{\widehat{g}}} \equiv 0\big) \mathbb{1}_{\mathrm{SG}(\bar{\mathbf{y}}, 0|_{\vartheta_v J_{\widehat{g}}})|_{\overline{D}+v} = \theta_v \widehat{g}} \\
&\quad = \sum_{\substack{D \supseteq B \\ \text{box}}} \sum_{\substack{\widehat{g} \in {\mathbb{N}_0}^{\overline{D}}: \\ \widehat{g}|_B = g}} \mathbb{1}_{D \text{ smallest block}} \sum_{\bar{\mathbf{y}} \in \{0,1\}^{\theta_v I_{\widehat{g}}}} P\big(\mathrm{SG}(\bar{\mathbf{y}}, 0|_{\vartheta_v J_{\widehat{g}}})|_{\overline{D}+v} = \theta_v \widehat{g}, \mathbf{X}|_{\theta_v I_{\widehat{g}}} = \bar{\mathbf{y}}, \mathbf{X}|_{\vartheta_v J_{\widehat{g}}} \equiv 0\big)
\end{aligned}
$$

$$= \sum_{\substack{D \supseteq B \\ \text{box}}} \sum_{\substack{\widehat{g} \in \mathbb{N}_0{}^{\overline{D}}: \\ \widehat{g}|_B = g}} \mathbb{1}_{D \text{ smallest block w.r.t. } \widehat{g} \text{ containing } B} P\big(\operatorname{SG}(\mathbf{X})|_{\overline{D+v}} = \theta_v \widehat{g}\big)$$

At this point, we need to adjust the summation. We perform some trivial shift operations and change the indices of the sums:

$$\begin{aligned}
&= \sum_{\substack{D \supseteq B \\ \text{box}}} \sum_{\substack{\widehat{g} \in \mathbb{N}_0{}^{\overline{D}}: \\ \widehat{g}|_B = g}} \mathbb{1}_{D+v \text{ smallest block w.r.t. } \theta_v \widehat{g} \text{ containing } B+v} P\big(\operatorname{SG}(\mathbf{X})|_{\overline{D+v}} = \theta_v \widehat{g}\big) \\
&= \sum_{\substack{D \supseteq B \\ \text{box}}} \sum_{\substack{\widetilde{g} \in \mathbb{N}_0{}^{\overline{D+v}}: \\ \widetilde{g}|_{B+v} = \theta_v g}} \mathbb{1}_{D+v \text{ smallest block w.r.t. } \widetilde{g} \text{ containing } B+v} P\big(\operatorname{SG}(\mathbf{X})|_{\overline{D+v}} = \widetilde{g}\big) \\
&= \sum_{\substack{D' \supseteq B+v \\ \text{box}}} \sum_{\substack{\widetilde{g} \in \mathbb{N}_0{}^{\overline{D'}}: \\ \widetilde{g}|_{B+v} = \theta_v g}} \mathbb{1}_{D' \text{ smallest block w.r.t. } \widetilde{g} \text{ containing } B+v} P\big(\operatorname{SG}(\mathbf{X})|_{\overline{D'}} = \widetilde{g}\big). \qquad (3.8)
\end{aligned}$$

On the other hand, because $-v \in B$ implies $0 \in B + v$, we can apply Lemma 3.3.2 to the following (with subsequent steps similar to the ones just performed):

$$\begin{aligned}
&P\big(\operatorname{SG}(\mathbf{X})|_{B+v} = \theta_v g\big) \\
&\quad = P\Big(\bigcup_{\substack{D' \supseteq B+v \\ \text{box}}} \big\{\operatorname{SG}(\mathbf{X})|_{B+v} = \theta_v g,\ D' \text{ block w.r.t. } \operatorname{SG}(\mathbf{X})\big\}\Big) \\
&\quad = \sum_{\substack{D' \supseteq B+v \\ \text{box}}} P\big(\operatorname{SG}(\mathbf{X})|_{B+v} = \theta_v g,\ D' \text{ is the smallest block w.r.t. } \operatorname{SG}(\mathbf{X}) \text{ containing } B+v\big) \\
&\quad = \sum_{\substack{D' \supseteq B+v \\ \text{box}}} \sum_{\substack{\widetilde{g} \in \mathbb{N}_0{}^{\overline{D'}}: \\ \widetilde{g}|_{B+v} = \theta_v g}} P\big(\operatorname{SG}(\mathbf{X})|_{\overline{D'}} = \widetilde{g},\ D' \text{ is the smallest block w.r.t. } \operatorname{SG}(\mathbf{X}) \text{ containing } B+v\big) \\
&\quad = \sum_{\substack{D' \supseteq B+v \\ \text{box}}} \sum_{\substack{\widetilde{g} \in \mathbb{N}_0{}^{\overline{D'}}: \\ \widetilde{g}|_{B+v} = \theta_v g}} P\big(\operatorname{SG}(\mathbf{X})|_{\overline{D'}} = \widetilde{g},\ D' \text{ is the smallest block w.r.t. } \widetilde{g} \text{ containing } B+v\big),
\end{aligned}$$

which is equal to (3.8). □

3.4 Mixing and ergodic properties

Definition 3.4.1. *We say a family $(F_u)_{u \in \mathbb{Z}^2}$ of discrete random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is* mixing w.r.t. $\mathbb{P}$ *if for any $v \in \mathbb{Z}^2 \setminus 0$, any finite box $B \subseteq \mathbb{Z}^2$, and any realizations $f_1, f_2 : B \to \mathbb{R}$, it holds that*

$$\big|\mathbb{P}(F|_B = f_1, F|_{B+nv} = \theta_{nv} f_2) - \mathbb{P}(F|_B = f_1)\mathbb{P}(F|_{B+nv} = \theta_{nv} f_2)\big| \xrightarrow[n \to \infty]{} 0.$$

Theorem 3.4.2. *The streetgrid* $\operatorname{SG}(\mathbf{X})$ *is mixing w.r.t. P.*

Proof. Take B a box. Because of the stationarity of the process $\mathrm{SG}(\mathbf{X})$, we can suppose $0 \in B$ without loss of generality. Let $g \in {\mathbb{N}_0}^B$. As for the shift, take $v \in \mathbb{Z}^2$ such that $v_0 > 0$. The case $v_1 > 0$ can be proven analogously.

Define the *cutting–event*

$$C_n := \big\{\exists m > \lfloor \frac{\ell^{\mathrm{SG}}(B)}{2} \rfloor \vee \lfloor \frac{\ell^{\mathrm{SG}}(B+nv)}{2} \rfloor,\ \exists\, b_0'(B)+\beta_m \leq x \leq b_0(B+nv)-1 : \mathbf{X}(x,2m,0) = 1\big\}.$$

The meaning of this event is that between B and $B+nv$, there is a vertical street of higher level than any of the streets in g and h.

The event C_n satisfies, for $n \in \mathbb{N}$ large enough,

$$\begin{aligned}
P(C_n) &= P\Big(\bigcup_{m > \lfloor \frac{\ell^{\mathrm{SG}}(B)}{2} \rfloor \vee \lfloor \frac{\ell^{\mathrm{SG}}(B+nv)}{2} \rfloor} \ \bigcup_{b_0'(B)+\beta_m \leq x \leq b_0(B+nv)-1} \{\mathbf{X}(x,2m,0) = 1\}\Big) \\
&= \sum_{\widehat{m} \in \mathbb{N}_0} P\Big(\bigcup_{m > \widehat{m}} \ \bigcup_{b_0'(B)+\beta_m \leq x \leq b_0(B+nv)-1} \{\mathbf{X}(x,2m,0) = 1\}\Big) P\Big(\lfloor \frac{\ell^{\mathrm{SG}}(B)}{2} \rfloor \vee \lfloor \frac{\ell^{\mathrm{SG}}(B+nv)}{2} \rfloor = \widehat{m}\Big) \\
&\geq \sum_{\widehat{m} \in \mathbb{N}} P\Big(\bigcup_{b_0'(B)+\beta_{\widehat{m}} \leq x \leq b_0(B+nv)-1} \{\mathbf{X}(x,2\widehat{m},0) = 1\}\Big) P\Big(\lfloor \frac{\ell^{\mathrm{SG}}(B)}{2} \rfloor \vee \lfloor \frac{\ell^{\mathrm{SG}}(B+nv)}{2} \rfloor = \widehat{m}-1\Big) \\
&= \sum_{\widehat{m} \in \mathbb{N}} \Big(1 - (1-\lambda_{\widehat{m}})^{nv_0 + b_0(B) - b_0'(B) - \beta_{\widehat{m}}}\Big) P\Big(\lfloor \frac{\ell^{\mathrm{SG}}(B)}{2} \rfloor \vee \lfloor \frac{\ell^{\mathrm{SG}}(B+nv)}{2} \rfloor = \widehat{m}-1\Big) \xrightarrow[n\to\infty]{} 1.
\end{aligned}$$

C_n also has the property to render independent events happening on B and $B+nv$: it implies that the smallest block around B w.r.t. $\mathrm{SG}(\mathbf{X})$ and the smallest block around $B+nv$ w.r.t. $\mathrm{SG}(\mathbf{X})$ are disjoint, which means that different points are responsible for the two. We hence have, with the events

$$G := \{\mathrm{SG}(\mathbf{X})|_B = g\} \text{ and } H_v := \{\mathrm{SG}(\mathbf{X})|_{B+v} = \theta_v h\},$$

and if we denote by C_n^c the complement of C_n,

$$\begin{aligned}
&P(\mathrm{SG}(\mathbf{X})|_B = g, \mathrm{SG}(\mathbf{X})|_{B+nv} = \theta_{nv} h) \\
&\quad = P(G \cap H_{nv}) \\
&\quad = P(G \cap H_{nv} \cap C_n^c) + P(G \cap H_{nv} \cap C_n) \\
&\quad = P(G \cap H_{nv} \cap C_n^c) + P(G \cap H_{nv} | C_n) P(C_n) \\
&\quad = P(G \cap H_{nv} \cap C_n^c) + P(G | C_n) P(H_{nv} | C_n) P(C_n) \\
&\quad = P(G \cap H_{nv} \cap C_n^c) + P(G \cap C_n) P(H_{nv} \cap C_n) / P(C_n) \\
&\quad \xrightarrow[n\to\infty]{} P(G)P(H_0).
\end{aligned}$$

□

Corollary 3.4.3. *The streetgrid* $\mathrm{SG}(\mathbf{X})$ *is totally ergodic.*

3.5 Consequences

Corollary 3.5.1. *The environment* ω *is stationary.*

Proof. This is true because in order to determine every point $\omega(u)$, $u \in \mathbb{Z}^2$, the same function is applied to the SG–values around u in a local and stationary manner, and because $\mathrm{SG}(\mathbf{X})$ is stationary. □

Corollary 3.5.2. *The environment ω is mixing.*

Proof. To prove this, we would like to carry over the arguments from the proof of Theorem 3.4.2. However there is an issue about ω (as a function of SG) not being completely localized in the sense that in order to determine ω on a box B, one needs to know the width of the streets present in B. Recall that only if a street has its full planned width the biased transition probabilities are placed on it; else, the transition probabilities of a simple random walk are used.

Fortunately, it is possible to determine what ω looks like on B by knowing SG on a box

$$\overline{B}^{\beta_{\ell^{\mathrm{SG}}(B)}} := \{b_0(B) - \beta_{\ell^{\mathrm{SG}}(B)}, \dots, b_1(B) - \beta_{\ell^{\mathrm{SG}}(B)}\} \times \{b_0'(B) - \beta_{\ell^{\mathrm{SG}}(B)}, \dots, b_1'(B) + \beta_{\ell^{\mathrm{SG}}(B)}\};$$

one migth want to think of this box as a thicker closure, with thickness $\beta_{\ell^{\mathrm{SG}}(B)}$. In other words, $\omega|_B$ is $\mathrm{SG}(B)|_{\overline{B}^{\beta_{\ell^{\mathrm{SG}}(B)}}}$–measurable.

We will use this fact in the following calculations. Take $\widetilde{g}, \widetilde{h} \in (S^2)^B$.

$$\begin{aligned}
&P\big(\omega|_B = \widetilde{g},\ \omega|_{B+nv} = \theta_{nv}\widetilde{h}\big)\\
&\quad= \sum_{k,l\in\mathbb{N}} P\big(\omega|_B = \widetilde{g},\ \omega|_{B+nv} = \theta_{nv}\widetilde{h},\ \ell^{\mathrm{SG}}(B) = k,\ \ell^{\mathrm{SG}}(B+nv) = l\big)\\
&\quad= \sum_{k,l\in\mathbb{N}} \sum_{g\in\mathbb{N}^{\overline{B}^{\beta_k}},\ h\in\mathbb{N}^{\overline{B}^{\beta_l}}} P\big(\omega|_B = \widetilde{g},\ \omega|_{B+nv} = \theta_{nv}\widetilde{h},\ \ell^{\mathrm{SG}}(B) = k,\ \ell^{\mathrm{SG}}(B+nv) = l,\\
&\qquad\qquad\qquad\qquad \mathrm{SG}|_{\overline{B}^{\beta_k}} = g,\ \mathrm{SG}|_{\overline{B}^{\beta_l}+nv} = \theta_{nv}h\big)\\
&\quad= \sum_{k,l\in\mathbb{N}} \sum_{\substack{g\in\mathbb{N}^{\overline{B}^{\beta_k}}\\ h\in\mathbb{N}^{\overline{B}^{\beta_l}}}} P\left(\begin{matrix}\omega|_B = \widetilde{g}, & \ell^{\mathrm{SG}}(B) = k,\\ \omega|_{B+nv} = \theta_{nv}\widetilde{h}, & \ell^{\mathrm{SG}}(B+nv) = l\end{matrix}\ \middle|\ \begin{matrix}\mathrm{SG}|_{\overline{B}^{\beta_k}} = g,\\ \mathrm{SG}|_{\overline{B}^{\beta_l}+nv} = \theta_{nv}h\end{matrix}\right)\\
&\qquad\qquad\qquad\qquad P\big(\mathrm{SG}|_{\overline{B}^{\beta_k}} = g,\ \mathrm{SG}|_{\overline{B}^{\beta_l}+nv} = \theta_{nv}h\big)\\
&\quad= \sum_{k,l\in\mathbb{N}} \sum_{g\in\mathbb{N}^{\overline{B}^{\beta_k}},\ h\in\mathbb{N}^{\overline{B}^{\beta_l}}} \mathbb{1}_{\omega(g)|_B = \widetilde{g},\ \omega(\theta_{nv}h)|_{B+nv} = \theta_{nv}\widetilde{h},\ \ell^g(B) = k,\ \ell^{\theta_{nv}h}(B+nv) = l}\\
&\qquad\qquad\qquad\qquad P\big(\mathrm{SG}|_{\overline{B}^{\beta_k}} = g,\ \mathrm{SG}|_{\overline{B}^{\beta_l}+nv} = \theta_{nv}h\big)\\
&\quad\xrightarrow[n\to\infty]{} \sum_{k,l\in\mathbb{N}} \sum_{\substack{g\in\mathbb{N}^{\overline{B}^{\beta_k}}\\ h\in\mathbb{N}^{\overline{B}^{\beta_l}}}} \mathbb{1}_{\omega(g)|_B = \widetilde{g},\ \ell^g(B) = k} \mathbb{1}_{\omega(h)|_B = \widetilde{h},\ \ell^h(B) = l} P\big(\mathrm{SG}|_{\overline{B}^{\beta_k}} = g\big) P\big(\mathrm{SG}|_{\overline{B}^{\beta_l}} = h\big)\\
&\quad= P\big(\omega|_B = \widetilde{g}\big) P\big(\omega|_B = \widetilde{h}\big)
\end{aligned}$$

□

4 Properties of the random walk

4.1 The main theorem and the idea of its proof

Theorem 4.1.1.

$$PP_0^{\omega}(X_t \cdot \vec{1} \xrightarrow[t\to\infty]{} \infty) > 0,$$

where ω is the environment from Definition 2.3.2 *with its corresponding probability measure P, and (X_t, P_0^ω) the random walk from* (1.2).

A similar assertion holds with $\vec{1}$ replaced by $-\vec{1}$. The two together imply Theorem 1.0.1.

Recall the heuristical description at the beginning of Subsection 3.2. The idea of the proof of the Theorem is that the random walk X_t has positive probability to follow the streets in the initial grid InitGrid, at least from some starting point onwards. The starting point has positive probability to be reached directly from the origin. From there the random walk proceeds exactly like described, except for the "going straight" part: as it is a *random* walk, we have to take care of some fluctuations; but this is possible thanks to the streets growing nicely, see Corollary 3.2.7.

A complete proof of Theorem 4.1.1 will be given later. We start with a few technical

4.2 Definitions and Lemmata

Definition 4.2.1. *We define the hitting time of the random walk $(X_t)_t$ of the set $B \subseteq \mathbb{Z}^2$ as*

$$\tau_B := \inf\{t \geq 0 | X_t \in B\},$$

and the hitting time of the set $B' \subseteq \mathbb{Z}^2$ after hitting B as

$$\tau_{B,B'} = \tau(B, B') := \inf\{t \geq \tau_B | X_t \in B'\}.$$

τ_B and $\tau_{B,B'}$ are of course stopping times w.r.t. $\mathcal{G}_t := \sigma(X_s, s \leq t)$ the natural filtration.

Definition 4.2.2. *We define sequences of sets, some of which depend on the parameter $n \in \mathbb{N}$:*

$$\begin{aligned}
\mathcal{B}^{\mathsf{a}}_m(n) &:= \{-\frac{\beta_m}{16} + 1, \ldots, \frac{\beta_m}{16}\} \times \{-\frac{\beta_{m-1}}{16}, \ldots, n\}, \\
\mathcal{S}^{\mathsf{a}}_m &:= \text{B'twn}\big(0, \frac{\beta_m}{2} e_1\big), \\
\mathcal{E}^{\mathsf{a}}_m(n) &:= \{u \in \partial\mathcal{B}^{\mathsf{a}}_m(n) | u_1 \leq n\},\ m \geq 5.
\end{aligned}$$

"$\mathcal{S}$" and "$\mathcal{E}$" stand for "Start" and "Escape". Furthermore, define the "Target"-set

$$\mathcal{T}^{\mathsf{a}}_m(n) := \partial\mathcal{B}^{\mathsf{a}}_m(n) \setminus \mathcal{E}^{\mathsf{a}}_m(n) = \text{B'twn}\big((-\frac{\beta_m}{16} + 1, n+1), (\frac{\beta_m}{16}, n+1)\big),\ m \geq 5.$$

The reason for the restriction to $m \geq 5$ is the same as in (3.2).

Lemma 4.2.3. *Take some sequence $(n_m)_{m\geq 5}$ such that $\frac{\beta_m}{2} \leq n_m \leq \beta^\alpha_{m+2}$, $m \geq 5$. Also take a sequence of starting points $v_m \in \mathcal{S}^{\mathsf{a}}_m$, $m \geq 5$. We consider the (non-random) environment defined by setting*

$$\varpi^{\mathsf{a}}(u) := \begin{cases} \omega^0_{\nearrow} & \text{if } u_0 \leq 0, \\ \qquad \omega^0_{\nwarrow} & \text{if } u_0 > 0 \end{cases}$$

for all $u \in \mathbb{Z}^2$. It engenders the random walk $(X_t)_{t\geq 0}$ in the environment ϖ^{a}, starting in v_m, given by the measure $P^{\varpi^{\mathsf{a}}}_{v_m}$. It now holds that $P^{\varpi^{\mathsf{a}}}_{v_m}(\tau_{\mathcal{E}^{\mathsf{a}}_m(n_m)} < \tau_{\mathcal{T}^{\mathsf{a}}_m(n_m)})$ is summable in m, where $\tau_\cdot$ is from Definition 4.2.1.

A picture of the sets from Definition 4.2.2 and the environment of Lemma 4.2.3 can be found in Figure 9.

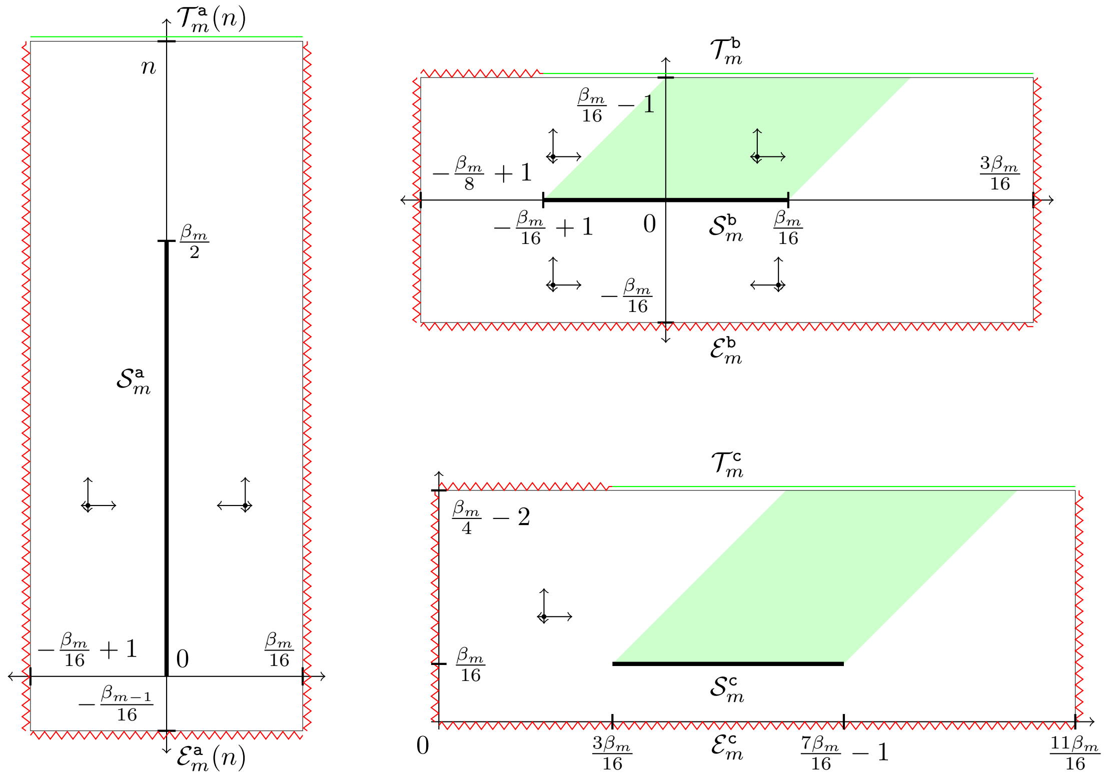

Figure 9: Escape and target sets used in Lemmas 4.2.3, 4.2.5, and 4.2.6, together with their corresponding environment. Nothing is to scale.

Proof. We split the movement of X_t into its two coordinates $X_t = (X_{t,0}, X_{t,1})$. $X_{t,1}$ is stochastically minorated by a random walk on $\mathbb{Z}$ with uniform drift to the right (and possibility to sometimes stand still). The probability of this random walk to hit some negative $-a$ before wandering off towards infinity decays exponentially in a.

Also, the time to reach some positive b grows linearly in b, in the sense that there is a positive, non-random constant c_1 such that the probability of not reaching b up to time $c_1 b$ decays exponentially fast in b.

As the probabilities set in ϖ^{a} to go left or right are uniformly bounded away from 1, the random walk $X_.$ will spend a nontrivial fraction of its time going left and right. This means that there is some positive, non-random constant $c_2 < 1$ such that the probability that the number of times $X_.$ goes left or right up to time t is greater than $c_2 t$ decays exponentially in t.

$|X_{t,0}|$ is stochastically dominated by a random walk reflected at 0 with negative drift. Each excursion from 0 of such a reflected random walk is stochastically dominated by a geometric random variable, and the excursions are independent; recall that the probability of a geometric random variable to be larger than a decays exponentially in a.

The number of excursions of $|X_{t,0}|$ up to some time can be estimated very crudely by the number of steps to the left or right up to that time.

If we put the pieces together, we find that the probability of escape to the left or right is for large m bounded by the probability of at least one out of $c_2 c_1 (\beta_{m+2})^\alpha \geq c_2 c_1 n_m$ independent geometric random variables being larger than $\frac{\beta_m}{16}$, which can be verified to be still exponentially small in m.

As we did not care to keep track of exact rates, we settle for a much weaker statement of summability. $\square$

Definition 4.2.4. *We need many more similar objects as the ones in Definition* 4.2.2*:*

$$\begin{aligned}
\mathcal{B}^{\mathtt{b}}_m &:= \{-\frac{\beta_m}{8}+1,\dots,\frac{3\beta_m}{16}\}\times\{-\frac{\beta_m}{16},\dots,\frac{\beta_m}{16}-1\},\\
\mathcal{S}^{\mathtt{b}}_m &:= \text{B'twn}\big((-\frac{\beta_m}{16}+1)e_0,\frac{\beta_m}{16}e_0\big),\\
\mathcal{E}^{\mathtt{b}}_m &:= \{u\in\partial\mathcal{B}^{\mathtt{b}}_m|u_0\le-\frac{\beta_m}{16}\ \textit{or}\ u_1\le\frac{\beta_m}{16}-1\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}^{\mathtt{c}}_m &:= \{0,\dots,\frac{11\beta_m}{16}\}\times\{0,\dots,\frac{\beta_m}{4}-2\},\\
\mathcal{S}^{\mathtt{c}}_m &:= \text{B'twn}\big((\frac{3\beta_m}{16},\frac{\beta_m}{16}),(\frac{7\beta_m}{16}-1,\frac{\beta_m}{16})\big),\\
\mathcal{E}^{\mathtt{c}}_m &:= \{u\in\partial\mathcal{B}^{\mathtt{c}}_m|u_0\le\frac{3\beta_m}{16}-1\ \textit{or}\ u_1\le\frac{\beta_m}{4}-2\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}^{\mathtt{A}}_m(n) &:= \{-\frac{\beta_m}{16},\dots,n\}\times\{-\frac{\beta_m}{16}+1,\dots,\frac{\beta_m}{16}\},\\
\mathcal{S}^{\mathtt{A}}_m &:= \text{B'twn}\big(0,\frac{\beta_m}{2}e_0\big),\\
\mathcal{E}^{\mathtt{A}}_m(n) &:= \{u\in\partial\mathcal{B}^{\mathtt{A}}_m(n)|u_0\le n\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}^{\mathtt{B}}_m &:= \{-\frac{\beta_m}{16},\dots,\frac{\beta_m}{16}-1\}\times\{-\frac{\beta_m}{8}+1,\dots,\frac{3\beta_m}{16}\},\\
\mathcal{S}^{\mathtt{B}}_m &:= \text{B'twn}\big((-\frac{\beta_m}{16}+1)e_1,\frac{\beta_m}{16}e_1\big),\\
\mathcal{E}^{\mathtt{B}}_m &:= \{u\in\partial\mathcal{B}^{\mathtt{B}}_m|u_0\le\frac{\beta_m}{16}-1\ \textit{or}\ u_1\le-\frac{\beta_m}{16}\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}^{\mathtt{C}}_m &:= \{0,\dots,\frac{\beta_{m+1}}{4}-2\}\times\{0,\dots,\frac{\beta_{m+1}}{2}+\frac{3\beta_m}{16}\},\\
\mathcal{S}^{\mathtt{C}}_m &:= \text{B'twn}\big((\frac{\beta_m}{16},\frac{3\beta_m}{16}),(\frac{\beta_m}{16},\frac{7\beta_m}{16}-1)\big),\\
\mathcal{E}^{\mathtt{C}}_m &:= \{u\in\partial\mathcal{B}^{\mathtt{C}}_m|u_0\le\frac{\beta_{m+1}}{4}-2\ \textit{or}\ u_1\le\frac{3\beta_m}{16}-1\},\ m\ge5.
\end{aligned}$$

The target sets are

$$\begin{aligned}
\mathcal{T}^{\dagger}_m &:= \partial\mathcal{B}^{\dagger}_m\setminus\mathcal{E}^{\dagger}_m,\ \dagger\in\{\text{“}\mathtt{b}\text{”},\text{“}\mathtt{c}\text{”},\text{“}\mathtt{B}\text{”},\text{“}\mathtt{C}\text{”}\},\\
\mathcal{T}^{\mathtt{A}}_m(n) &:= \partial\mathcal{B}^{\mathtt{A}}_m(n)\setminus\mathcal{E}^{\mathtt{A}}_m(n),\ m\ge5,\ n\in\mathbb{N},
\end{aligned}$$

and they compute as

$$\mathcal{T}^{\mathtt{b}}_m = \text{B'twn}\big((-\frac{\beta_m}{16}+1,\frac{\beta_m}{16}),(\frac{3\beta_m}{16},\frac{\beta_m}{16})\big),$$

$$\begin{aligned}
\mathcal{T}_m^{\mathsf{c}} &= \text{B'twn}\big((\frac{3\beta_m}{16}, \frac{\beta_m}{4}-1), (\frac{11\beta_m}{16}, \frac{\beta_m}{4}-1)\big),\\
\mathcal{T}_m^{\mathsf{A}}(n) &= \text{B'twn}\big((n+1, -\frac{\beta_m}{16}+1), (n+1, \frac{\beta_m}{16})\big),\\
\mathcal{T}_m^{\mathsf{B}} &= \text{B'twn}\big((\frac{\beta_m}{16}, -\frac{\beta_m}{16}+1), (\frac{\beta_m}{16}, \frac{3\beta_m}{16})\big),\\
\mathcal{T}_m^{\mathsf{C}} &= \text{B'twn}\big((\frac{\beta_{m+1}}{4}-1, \frac{3\beta_m}{16}), (\frac{\beta_{m+1}}{4}-1, \frac{\beta_{m+1}}{2}+\frac{3\beta_m}{16})\big),\ m \geq 5,\ n \in \mathbb{N}.
\end{aligned}$$

Visualizations of these events can be found in Figures 9 and 10. There, also the events of interest and the environments in the following Lemmata are shown.

Lemma 4.2.5. *Define an environment by setting, for $u \in \mathbb{Z}^2$,*

$$\varpi^{\mathsf{b}}(u) := \begin{cases} \omega_{\nwarrow} & \text{if } u_1 < 0,\ u_0 > 0, \\ \omega_{\nearrow} & \text{else} \end{cases}$$

which engenders the random walk (X_t) starting in v under $P_v^{\varpi^{\mathsf{b}}}$, $v \in \mathbb{Z}^2$. Let $v_m \in \mathcal{S}_m^{\mathsf{b}}$, $m \geq 5$, be an arbitrary sequence. It then holds that $P_{v_m}^{\varpi^{\mathsf{b}}}(\tau_{\mathcal{E}_m^{\mathsf{b}}} < \tau_{\mathcal{T}_m^{\mathsf{b}}})$ is summable in m.

Proof. The arguments will be quite the same as in the proof of the last Lemma.

There are four possibilities of escape to $\mathcal{E}_m^{\mathsf{b}}$, namely

- to the south, which is exponentially becoming unlikely as the box grows with m, because of the uniform drift to the north.
- to the west, which is exponentially becoming unlikely because of the uniform drift pushing in the opposite direction on the western half–plane.
- to the east, which is exponentially becoming unlikely because the drift to the north is in the eastern half–plane at least as strong as the drift to the east, which means that the linear speed of $X_{\cdot,1}$ is at least the same as the one of $X_{\cdot,0}$. With the box growing large, even if $X_\cdot$ starts at the easternmost possible point $\frac{\beta_m}{16}e_0$, by the time $X_{\cdot,1}$ reaches $\frac{\beta_m}{16}$, $X_{\cdot,0}$ will not have reached $\frac{3\beta_m}{16}+1$.
- to the horizontal piece of ∂B_a in the northern west, which is exponentially becoming unlikely because the drift to the north provides that the probability of $X_{\cdot,1}$ being smaller than 0 at the time $X_{\cdot,0}$ hits $\frac{\beta_m}{16}$ is decaying fastly.

□

Lemma 4.2.6. *$P_{v_m}^{\omega_{\nearrow}}(\tau_{\mathcal{E}_m^{\mathsf{c}}} < \tau_{\mathcal{T}_m^{\mathsf{c}}})$ is summable in m for any arbitrary sequence $v_m \in \mathcal{S}_m^{\mathsf{c}}$, $m \geq 5$.*

Lemma 4.2.7. *Take some sequence $(n_m)_{m\geq 5}$ such that $\frac{\beta_m}{2} \leq n_m \leq \beta_{m+2}^{\alpha}$, $m \geq 5$. Also take a sequence of starting points $v_m \in \mathcal{S}_m^{\mathsf{A}}$, $m \geq 5$. Define the environment by setting*

$$\varpi^{\mathsf{A}}(u) := \begin{cases} \omega_{\searrow} & \text{if } u_1 > 0, \\ \omega_{\nearrow} & \text{if } u_1 \leq 0,\ u \in \mathbb{Z}^2. \end{cases}$$

It now holds that $P_{v_m}^{\varpi^{\mathsf{A}}}(\tau_{\mathcal{E}_m^{\mathsf{A}}(n_m)} < \tau_{\mathcal{T}_m^{\mathsf{A}}(n_m)})$ is summable in m.

Lemma 4.2.8. *Define an environment by setting, for $u \in \mathbb{Z}^2$,*

$$\varpi^{\mathtt{B}}(u) := \begin{cases} \omega_{\searrow} & \text{if } u_0 < 0,\ u_1 > 0, \\ \omega_{\nearrow} & \text{else,} \end{cases}$$

which engenders the random walk (X_t) starting in v under $P_v^{\varpi^{\mathtt{B}}}$, $v \in \mathbb{Z}^2$. Let $v_m \in \mathcal{S}_m^{\mathtt{B}}$ be an arbitrary sequence. It then holds that $P_{v_m}^{\varpi^{\mathtt{B}}}(\tau_{\mathcal{E}_m^{\mathtt{B}}} < \tau_{\mathcal{T}_m^{\mathtt{B}}})$ is summable in m.

Lemma 4.2.9. *$P_{v_m}^{\omega_{\nearrow}}(\tau_{\mathcal{E}_m^{\mathtt{C}}} < \tau_{\mathcal{T}_m^{\mathtt{C}}})$ is summable in m for any arbitrary sequence $v_m \in \mathcal{S}_m^{\mathtt{C}}$.*

The arguments needed for the proofs of these last four Lemmata are the same as in the two preceeding proofs, which is why we omit them here.

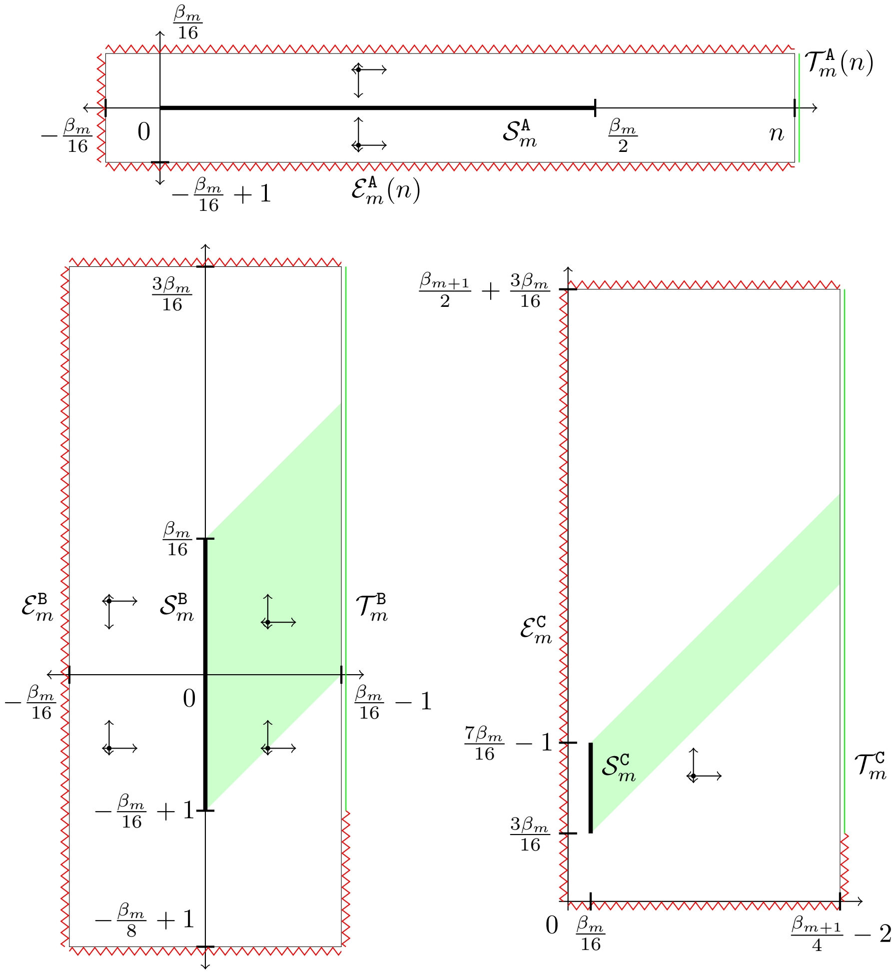

Figure 10: Escape and target sets used in Lemmas 4.2.7, 4.2.8, and 4.2.9.

4.3 Proof of the Theorem

We prove Theorem 4.1.1 by showing that the random walk has positive probability to hit a certain sequence of target sets leading to infinity in a prescribed order, while not hitting the succession of escape–sets we define at the same time. The sets will be based on the ones who have just been treated in the Lemmas 4.2.3, 4.2.5, 4.2.6, 4.2.7, 4.2.8, and 4.2.9.

Definition 4.3.1. *We will shift the sets defined in Definition* 4.2.2 *by the vectors*

$$
\begin{aligned}
\mathcal{O}^{\mathsf{a}}_m &:= \Rsh\mathrm{Lane}_{+,+}(B^1_{m-1}) + (\frac{\beta_m}{4}, -\frac{\beta_{m-1}}{16} + 1),\\
\mathcal{O}^{\mathsf{b}}_m &:= \Rsh\mathrm{Lane}_{+,+}(B^0_m) + e_1,\\
\mathcal{O}^{\mathsf{c}}_m &:= \Rsh\mathrm{Lane}_{+,+}(B^0_m) + (-\frac{\beta_m}{4} + 1, 1),\\
\mathcal{O}^{\mathsf{A}}_m &:= \Rsh\mathrm{Lane}_{+,+}(B^0_m) + (-\frac{\beta_m}{16} + 1, \frac{\beta_m}{4}),\\
\mathcal{O}^{\mathsf{B}}_m &:= \Rsh\mathrm{Lane}_{+,+}(B^1_m) + e_0,\\
\mathcal{O}^{\mathsf{C}}_m &:= \Rsh\mathrm{Lane}_{+,+}(B^1_m) + (1, -\frac{\beta_m}{4} + 1), m \geq 5;
\end{aligned}
$$

"$\mathcal{O}$" stands for the shifted "Origin".

Also define

$$
\begin{aligned}
n^{\mathsf{a}}_m &:= (\Rsh\mathrm{Lane}_{+,+}(B^0_m))_1 - (\Rsh\mathrm{Lane}_{+,+}(B^1_{m-1}))_1 + \frac{\beta_{m-1}}{16} - 1,\\
n^{\mathsf{A}}_m &:= (\Rsh\mathrm{Lane}_{+,+}(B^1_m))_0 - (\Rsh\mathrm{Lane}_{+,+}(B^0_m))_0 + \frac{\beta_m}{16} - 1, \;\; m \geq 5.
\end{aligned}
$$

The next lemma shows how each shifted target set coincides with the next shifted starting set.

Lemma 4.3.2.

$$
\begin{aligned}
\mathcal{T}^{\mathsf{a}}_m(n^{\mathsf{a}}_m) + \mathcal{O}^{\mathsf{a}}_m &= \mathcal{S}^{\mathsf{b}}_m + \mathcal{O}^{\mathsf{b}}_m,\\
\mathcal{T}^{\mathsf{b}}_m + \mathcal{O}^{\mathsf{b}}_m &= \mathcal{S}^{\mathsf{c}}_m + \mathcal{O}^{\mathsf{c}}_m,\\
\mathcal{T}^{\mathsf{c}}_m + \mathcal{O}^{\mathsf{c}}_m &= \mathcal{S}^{\mathsf{A}}_m + \mathcal{O}^{\mathsf{A}}_m,\\
\mathcal{T}^{\mathsf{A}}_m(n^{\mathsf{A}}_m) + \mathcal{O}^{\mathsf{A}}_m &= \mathcal{S}^{\mathsf{B}}_m + \mathcal{O}^{\mathsf{B}}_m,\\
\mathcal{T}^{\mathsf{B}}_m + \mathcal{O}^{\mathsf{B}}_m &= \mathcal{S}^{\mathsf{C}}_m + \mathcal{O}^{\mathsf{C}}_m,\\
\mathcal{T}^{\mathsf{C}}_m + \mathcal{O}^{\mathsf{C}}_m &= \mathcal{S}^{\mathsf{a}}_{m+1} + \mathcal{O}^{\mathsf{a}}_{m+1}.
\end{aligned}
$$

Proof. We prove the first line, the others being similar.

$$
\begin{aligned}
&\mathcal{T}^{\mathsf{a}}_m(n^{\mathsf{a}}_m) + \mathcal{O}^{\mathsf{a}}_m\\
&\quad = \mathrm{B'twn}\big((-\frac{\beta_m}{16} + 1, n^{\mathsf{a}}_m + 1), (\frac{\beta_m}{16}, n^{\mathsf{a}}_m + 1)\big) + \Rsh\mathrm{Lane}_{+,+}(B^1_{m-1}) + (\frac{\beta_m}{4}, -\frac{\beta_{m-1}}{16} + 1)\\
&\quad = \mathrm{B'twn}\big((-\frac{\beta_m}{16} + 1, (\Rsh\mathrm{Lane}_{+,+}(B^0_m))_1 - (\Rsh\mathrm{Lane}_{+,+}(B^1_{m-1}))_1 + \frac{\beta_{m-1}}{16}),\\
&\qquad\qquad (\frac{\beta_m}{16}, (\Rsh\mathrm{Lane}_{+,+}(B^0_m))_1 - (\Rsh\mathrm{Lane}_{+,+}(B^1_{m-1}))_1 + \frac{\beta_{m-1}}{16})\big)\\
&\qquad\qquad\qquad + \Rsh\mathrm{Lane}_{+,+}(B^1_{m-1}) + (\frac{\beta_m}{4}, -\frac{\beta_{m-1}16}{8} + 1)
\end{aligned}
$$

$$= \text{B'twn}\left(((\Rsh\text{Lane}_{+,+}(B^1_{m-1}))_0 + \frac{\beta_m}{4} - \frac{\beta_m}{16} + 1, (\Rsh\text{Lane}_{+,+}(B^0_m))_1 + 1), ((\Rsh\text{Lane}_{+,+}(B^1_{m-1}))_0 + \frac{\beta_m}{4} + \frac{\beta_m}{16}, (\Rsh\text{Lane}_{+,+}(B^0_m))_1 + 1)\right)$$

$$= \text{B'twn}\Big(((\Rsh\text{Lane}_{+,+}(B^0_m))_0 - \frac{\beta_m}{16} + 1, (\Rsh\text{Lane}_{+,+}(B^0_m))_1 + 1), ((\Rsh\text{Lane}_{+,+}(B^0_m))_0 + \frac{\beta_m}{16}, (\Rsh\text{Lane}_{+,+}(B^0_m))_1 + 1)\Big)$$

$$= \text{B'twn}\left((-\frac{\beta_m}{16} + 1)e_0, \frac{\beta_m}{16}e_0\right) + \Rsh\text{Lane}_{+,+}(B^0_m) + e_1 = \mathcal{S}^{\mathtt{b}}_m + \mathcal{O}^{\mathtt{b}}_m.$$

□

Proof of Theorem 4.1.1. Out of convenience, we set

$$\mathcal{T}^\dagger_m := \mathcal{T}^\dagger_m(n^\dagger_m),\ \mathcal{E}^\dagger_m := \mathcal{E}^\dagger_m(n^\dagger_m),\ \mathcal{B}^\dagger_m := \mathcal{B}^\dagger_m(n^\dagger_m),\ \dagger \in \{\text{“}\mathtt{a}\text{”}, \text{“}\mathtt{A}\text{”}\},\ m \geq 5,$$

and $\text{ABC} := \{\text{“}\mathtt{a}\text{”}, \text{“}\mathtt{b}\text{”}, \text{“}\mathtt{c}\text{”}, \text{“}\mathtt{A}\text{”}, \text{“}\mathtt{B}\text{”}, \text{“}\mathtt{C}\text{”}\}$. Also define the initial target– and escape–sets

$$\mathcal{T}^0 := \mathcal{S}^{\mathtt{a}}_{M+1}\mathcal{O}^{\mathtt{a}}_{M+1} \text{ and } \mathcal{E}^0 := \left(\partial\,\text{B'twn}(0, \Rsh\mathcal{T}^0 - e_0)\right) \setminus \mathcal{T}^0.$$

The event

$$\{\tau_{\mathcal{T}^0} < \tau_{\mathcal{E}^0}\} \cap \bigcap_{m \geq M+1} \bigcap_{\dagger \in \text{ABC}} \{\tau(\mathcal{S}^\dagger_m + \mathcal{O}^\dagger_m, \mathcal{T}^\dagger_m + \mathcal{O}^\dagger_m) < \tau(\mathcal{S}^\dagger_m + \mathcal{O}^\dagger_m, \mathcal{E}^\dagger_m + \mathcal{O}^\dagger_m)\}$$

implies $X_t \cdot \vec{1} \to \infty$, $t \to \infty$: it describes the path of a random walk that hits a target set, from this target set moves to the next target set, and so on. As, roughly speaking, these target sets "lead to infinity in the direction of the vector $\vec{1} = (1,1)$", they help describing a path of a random walk the scalar product with $\vec{1}$ of which is diverging to $+\infty$. A picture of a piece of such a path with the corresponding target sets is available in Figure 11.

With the help of Lemma 4.3.2, we can successively apply the strong Markov property for $X.$, and see that P–a.s.,

$$P^\omega_0(X_t \cdot \vec{1} \xrightarrow[t\to\infty]{} \infty)$$
$$\geq P^\omega_0\Big(\{\tau_{\mathcal{T}^0} < \tau_{\mathcal{E}^0}\} \cap \bigcap_{m \geq M+1} \bigcap_{\dagger \in \text{ABC}} \{\tau(\mathcal{S}^\dagger_m + \mathcal{O}^\dagger_m, \mathcal{T}^\dagger_m + \mathcal{O}^\dagger_m) < \tau(\mathcal{S}^\dagger_m + \mathcal{O}^\dagger_m, \mathcal{E}^\dagger_m + \mathcal{O}^\dagger_m)\}\Big)$$

$$= P^\omega_0\left(\tau_{\mathcal{T}^0} < \tau_{\mathcal{E}^0}\right) \prod_{m \geq M+1} \prod_{\dagger \in \text{ABC}} P^\omega_0\Big(\tau\big(\mathcal{S}^\dagger_m + \mathcal{O}^\dagger_m, \mathcal{T}^\dagger_m + \mathcal{O}^\dagger_m\big) < \tau\big(\mathcal{S}^\dagger_m + \mathcal{O}^\dagger_m, \mathcal{E}^\dagger_m + \mathcal{O}^\dagger_m\big)\Big).$$

Because of the ellipticity of the random environment, and because M from (3.2) is P–a.s. finite, the first probability on the right hand side is strictly larger than 0.

The product being larger than 0 is thus equivalent to

$$\sum_{\dagger \in \text{ABC}} \sum_{m \geq M+1} P^\omega_0\Big(\tau\big(\mathcal{S}^\dagger_m + \mathcal{O}^\dagger_m, \mathcal{T}^\dagger_m + \mathcal{O}^\dagger_m\big) > \tau\big(\mathcal{S}^\dagger_m + \mathcal{O}^\dagger_m, \mathcal{E}^\dagger_m + \mathcal{O}^\dagger_m\big)\Big) < \infty.$$

The case "=" cannot occur because the target– and escape–sets are disjoint. Hence, what we need to show is the P–almost sure summability in m of

$$P^\omega_0\Big(\tau\big(\mathcal{S}^\dagger_m + \mathcal{O}^\dagger_m, \mathcal{E}^\dagger_m + \mathcal{O}^\dagger_m\big) < \tau\big(\mathcal{S}^\dagger_m + \mathcal{O}^\dagger_m, \mathcal{T}^\dagger_m + \mathcal{O}^\dagger_m\big)\Big),\ \dagger \in \text{ABC}.$$

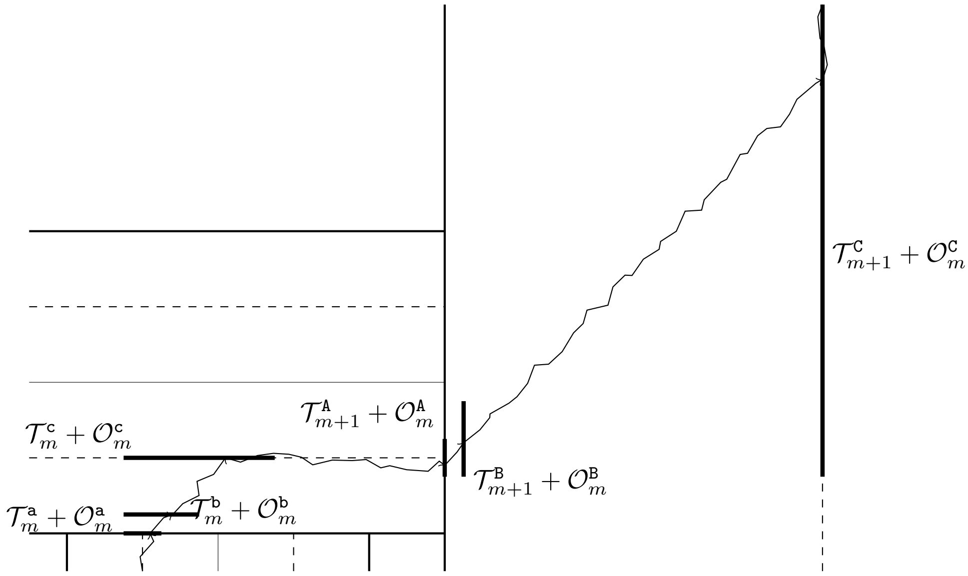

Figure 11: Target areas. The path has positive probability to hit them in that order.

Let us look at the case $\dagger =$ "$\mathtt{b}$". Note that

$$X_t \in \mathcal{B}^{\mathtt{b}}_m + \mathcal{O}^{\mathtt{b}}_m \text{ for all } t \in \Big\{\tau\big(\mathcal{S}^{\mathtt{b}}_m + \mathcal{O}^{\mathtt{b}}_m\big), \ldots, \big[\tau\big(\mathcal{S}^{\mathtt{b}}_m + \mathcal{O}^{\mathtt{b}}_m, \mathcal{T}^{\mathtt{b}}_m + \mathcal{O}^{\mathtt{b}}_m\big) \wedge \tau\big(\mathcal{S}^{\mathtt{b}}_m + \mathcal{O}^{\mathtt{b}}_m, \mathcal{E}^{\mathtt{b}}_m + \mathcal{O}^{\mathtt{b}}_m\big)\big] - 1\Big\}.$$

Also, $\omega(u) = (\theta_{\mathcal{O}^{\mathtt{b}}_m} \varpi^{\mathtt{b}})(u)$ for all $u \in \mathcal{B}^{\mathtt{b}}_m + \mathcal{O}^{\mathtt{b}}_m$, where $\varpi^{\mathtt{b}}$ is the one defined in Lemma 4.2.5. This is true because of the placements of $\mathcal{O}^{\mathtt{b}}_m$, and Corollary 3.2.7.

So, the probability is the same as the one in Lemma 4.2.5, which yields summability.

The other cases in ABC can be treated the same way using Lemmas 4.2.3, 4.2.6, 4.2.7, 4.2.8, and 4.2.9; for "$\mathtt{a}$" and "$\mathtt{A}$", we need to remark that $(n^{\mathtt{a}}_m)_m$ and $(n^{\mathtt{A}}_m)_m$ satisfy the necessary conditions. □

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