Spherical Quasihomogeneous $SL(2)$-Varieties

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Contents

Introduction

The n-dimensional projective space $\mathbb{P}^n$ over $\mathbb{C}$ can be defined as the quotient $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$. A similar quotient construction was proposed by David Cox for any toric variety X [Cox95]. Let X be an n-dimensional toric variety defined by a rational polyhedral fan $\triangle$. We denote by $\triangle(1)$ the set of all 1-dimensional cones in $\triangle$ and by $\mathrm{Cl}(X)$ the divisor class group of X. Then the algebraic group (quasitorus)

$$\mathbb{T} := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X), \mathbb{C}^*)$$

has a natural linear action on the affine space $\mathbb{C}^{\triangle(1)}$ such that the categorical $(\mathbb{C}^{\triangle(1)} \setminus Z)/\mathbb{T}$ exists and it is isomorphic to X where Z is Zariski closed subset defined by some homogeneous ideal in the coordinate ring $\mathbb{C}[x_0, \cdots, x_n]$.

A quasihomogeneous $SL(2)$-variety is a normal 3-dimensional algebraic variety X over an algebraic closed field k together with a regular action of $SL(2)$ which has an open dense orbit in X. For the simplicity we consider only the case $k = \mathbb{C}$. In this work we give a geometric method to construct a special class of $SL(2)$-varieties X as a categorical quotient.

As a first step of our investigation we consider the case of affine $SL(2)$-varieties [BH08]. These varieties were classified by Popov [P73]. Every affine $SL(2)$-variety E is uniquely determined by a pair of numbers: a rational number $h = p/q$ ($\gcd(p, q) = 1$, $0 < h \leq 1$) called the *height* of E and a natural number m called the *degree* of E. The corresponding variety $SL(2)$-variety will be denoted by $E_{h,m}$. We prove that $E_{h,m}$ is isomorphic to the categorical quotient of the affine hypersurface $H_{q-p} \subset \mathbb{C}^5$ defined by the equation

$$X_0^{q-p} = X_1X_4 - X_2X_3$$

modulo the action of the diagonalizable group $G_0 \times G_m \subset D(5, \mathbb{C})$, where

$$G_0 \cong \mathbb{C}^* \cong \{\mathrm{diag}\,(t, t^{-p}, t^{-p}, t^q, t^q);\; t \in \mathbb{C}^*\},\; G_m \cong \mu_m = \langle \zeta_m \rangle,$$

where $G_m \subset D(5, \mathbb{C})$ is generated by diag $(1, \zeta_m^{-1}, \zeta_m^{-1}, \zeta_m, \zeta_m)$.

In order to make our next step we remark that the affine $SL(2)$-varieties belong to a larger class of $SL(2)$-varieties having an additional $\mathbb{C}^*$-action which commutes with the $SL(2)$-action. We will call these varieties by *$SL(2)$-varieties with $\mathbb{C}^*$-action*. It is very important that $SL(2)$-varieties with $\mathbb{C}^*$-action can be considered as spherical G-varieties with respect to a regular action of the reductive 4-dimensional algebraic group

$$G := SL(2) \times \mathbb{C}^*,$$

i.e., the stabilizer $H \subset G$ of a point in the open $SL(2)$-orbit is a 1-dimensional spherical subgroup of G. For this reason we call $SL(2)$-varieties with $\mathbb{C}^*$-action also *spherical quasihomogeneous $SL(2)$-varieties*.

Spherical varieties are generalizations of toric varieties and they are classified by colored fans of strictly convex colored cones. Using the combinatorics of these fans one can determine certain geometric properties of these varieties for example: smoothness, completeness, or projectivity (see [K91]). We remark that the open dense $SL(2)$-orbit in a spherical $SL(2)$-variety X is isomorphic to $SL(2)/\mathcal{C}_m$, where $\mathcal{C}_m$ is a cyclic group of order m. This number is a generalization of the degree of an affine $SL(2)$-variety $E_{h,m}$ to an arbitrary spherical quasihomogeneous $SL(2)$-variety X.

Using the theory of spherical varieties, we can describe an arbitrary spherical quasihomogeneous $SL(2)$-variety $X = X(\Sigma)$ by a 2-dimensional colored fan Σ in $\mathbb{R}^2$. Let $v_1, \ldots, v_r \in \mathbb{Z}^2$ be the set of all lattice generators of 1-dimensional cones in Σ where $v_i = (-p_i, -q_i)$ such that $\gcd(p_i, q_i) = 1$; $1 \leq i \leq r$. Then we show that $X(\Sigma)$ can be obtained as a GIT-quotient of the affine hypersurface in $\mathbb{C}^{r+4}$:

$$Y_1^{p_1+q_1} \cdots Y_r^{p_r+q_r} = X_1X_4 - X_2X_4$$

modulo the action of the diagonalizable group $G_0 \times G_m \subset D(r+4, \mathbb{C})$, where $G_0 \cong (\mathbb{C}^*)^r$ and G_m is a cyclic group of order m.

From this construction it was not difficult to show that Cox ring of such varieties is defined by a unique equation. Similar examples of algebraic varieties whose Cox ring is defined by a unique equation were considered in [BH07]. In the affine case this description of Cox ring can be used as a good illustration of more general recent results of Brion on Cox ring of spherical varieties [B07].

D. Luna and Th. Vust in [LV83] have discovered combinatorial diagrams for describing arbitrary normal $SL(2)$-embedding X (we call them *Luna-Vust diagrams*). These diagrams give information about the local rings of $SL(2)$-orbits in X. In this work, for any spherical $SL(2)$-variety $X = X(\Sigma)$ we give a method to construct the corresponding Luna-Vust diagram from the 2-dimensional colored fan Σ.

2-dimensional colored fans defining $SL(2)$-varieties with $\mathbb{C}^*$-actions are very convenient for the investigation of their birational morphisms. Using colored fans, we give a classification of all smooth $SL(2)$-varieties with $\mathbb{C}^*$-action with the Picard number ≤ 3. From these varieties we have classified all minimal smooth varieties, i.e., varieties which are not the blow-up of another varieties. This generalizes the results of L. M. Jauslin [Ja87] in the special case minimal smooth $SL(2)$ and $PGL(2)$-embeddings. Furthermore we have found minimal smooth $SL(2)$-varieties with $\mathbb{C}^*$-action which are toric.

In chapter 1 we give an overview of quotients by a torus action and explain the notion Cox ring of a quasiprojective varieties. For this purpose we introduce the language of geometric invariant theory (GIT-quotients).

In chapter 2 we consider the normal affine $SL(2)$-variety $E_{h,m}$. Then define

$$k := \gcd(q-p, m),\ b := (q-p)/k.$$

We prove that the variety $E_{h,m}$ is also isomorphic to the categorical quotient of the affine hypersurface $H_b \subset \mathbb{C}^5$ defined by the equation

$$Y_0^b = X_1X_4 - X_2X_3$$

modulo the action of the diagonalizable group $G_0^{'} \times G_a \subset D(5, \mathbb{C})$, where

$$G_0^{'} \cong \mathbb{C}^* \cong \{\text{diag}\ (t^k, t^{-p}, t^{-p}, t^q, t^q);\ t \in \mathbb{C}^*\},\ G_a \cong \mu_a = \langle \zeta_a \rangle,$$

where G_a is generated by diag $(1, \zeta_a^{-1}, \zeta_a^{-1}, \zeta_a, \zeta_a)$. From this alternative construction we could prove that Cox ring of the variety $E_{h,m}$ is defined by the following equation

$$\mathbb{C}[Y_0, X_1, X_2, X_3, X_4]/(Y_0^b - X_1X_4 + X_2X_3).$$

This quotient construction together with the first one are new and were overlooked in the classical theory of affine $SL(2)$-varieties [Kr84].

Furthermore, we give a geometric description of $SL(2)$-equivariant flips in the case that $E_{h,m}$ is an arbitrary singular normal affine quasihomogeneous $SL(2)$-variety.

The chapter 3 contains a basic survey of Luna-Vust theory for G-equivariant embeddings X of spherical homogeneous spaces G/H where G is a connected reductive algebraic group and $H \subset G$ is spherical algebraic subgroup of G [LV83] (see also [K91]). These embeddings are corresponded to colored fans consisting of colored cones. We apply this theory to the classification of affine $SL(2)$-varieties, to $SL(2)$-flips and to the Panyushev minimal resolution. In particular we describe $SL(2)$-flips from the point of view of the theory of spherical varieties developed by Luna and Vust and give a spherical description of the resolution of singularities of affine normal quasihomogeneous $SL(2)$-varieties proposed by Panyushev [Pa91].

In chapter 4 we prove that the open $SL(2)$-orbit in $SL(2)$-varieties with $\mathbb{C}^*$-action is isomorphic to $SL(2)/\mathcal{C}_m$, where $\mathcal{C}_m$ is a cyclic group of order m and these varieties are spherical where we can describe them combinatorial in 2-dimensional colored fans. These fans are used in two quotient constructions which generalize the two quotient constructions in the affine case. We give simple examples of this construction for some classical smooth projective $SL(2)$-varieties. Furthermore as in the case affine $SL(2)$-varieties we show that Cox ring of these varieties is defined by the following single equation:

$$\mathbb{C}[x_1, x_2, x_3, x_4, y_1, \dots, y_r]/(x_1x_4 - x_2x_3 - \prod_{i=1}^{r} y_i^{p_i+q_i}),$$

where r is the number of the 1-dimensional colored cones in the colored fan corresponding to the considered variety.

In chapter 5 we introduce the generalized Luna-Vust diagrams of $SL(2)/\Gamma$-embeddings in the case when $\Gamma = \mathcal{C}_m$ is a cyclic group of order m. These generalized diagrams were introduced by Jauslin-Moser in [Ja87]. After this we give a method to get generalized Luna-Vust diagrams from colored fans of $SL(2)$-varieties with $\mathbb{C}^*$-action. Our next purpose is to classify all smooth projective spherical quasihomogeneous $SL(2)$-varieties which a minimal. Such a classification problem was considered by Mukai and Umemura [MU83], Jauslin-Moser [Ja87, Ja90], Nakano [N87], and Kebekus [K00]. Our classification of minimal models uses the language of 2-dimensional colored fans. We show that the Picard number of a minimal spherical $SL(2)$-variety is at most 3. Therefore we first classify all 2-dimensional colored fans of smooth $SL(2)$-varieties with $\mathbb{C}^*$-action with Picard number ≤ 3 and afterwards determine those varieties which are minimal. Finally, we use Cox rings in order to find all minimal smooth spherical quasihomogeneous $SL(2)$-varieties which are toric.

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Chapter 1

Quotients by a torus action

1.1 Categorical quotients

Let G be a connected reductive affine algebraic group and X be a G-variety over the algebraically closed field $\mathbb{C}$, where G acts on X via the following regular morphism

$$\mu : G \times X \to X;\ (g, x) \mapsto \mu(g, x).$$

A morphism $\varphi : X \to Y$ is called **G-invariant** if it is constant along the orbits, i.e., if

$$\varphi(gx) = \varphi(x)\ \ \forall x \in X \text{ and } \forall g \in G.$$

The morphism φ is called **affine** if for any open affine subset $V \subseteq Y$ the preimage $\varphi^{-1}(V)$ is again affine subset in X. Let $\mathcal{O}(X)$ be the ring of regular functions of X. Then define the following subset of $\mathcal{O}(X)$

$$\mathcal{O}(X)^G := \{f \in \mathcal{O}(X) \mid f(gx) = f(x)\ \forall\ g \in G\ and\ \forall x \in X\}.$$

This set is called the **G-invariant ring** of $\mathcal{O}(X)$.

Definition 1.1.1. Let G be a reductive affine algebraic group and X be a G-variety. An algebraic variety Y together with a morphism $\varphi : X \to Y$ is called a **good quotient** of X over $\mathbb{C}$ by G if it satisfies the following properties

(i) φ is affine and G-invariant,

(ii) For every open subset $V \subseteq Y$, the pullback

$$\varphi_V^* : \mathcal{O}(V) \to \mathcal{O}(\varphi^{-1}(V))^G$$

is an isomorphism.

The variety Y together with a morphism $\varphi : X \to Y$ is called a **geometric quotient** if it is a good quotient and its fibers are precisely the G-orbits. We denote the good quotient of X by G by $X/\!\!/G$ and the geometric quotient by X/G.

Remark 1.1.2. Let X be an affine algebraic G-variety and G a reductive algebraic group. Then the $\mathcal{O}(X)^G \subseteq \mathcal{O}(X)$ is a finitely generated $\mathbb{C}$-algebra [Kr84, II.3.2]. This guarantess existence of a good quotient $\varphi : X \to Y$, where $Y := \mathrm{Spec}(\mathcal{O}(X)^G)$.

Remark 1.1.3. [ADHL09] If G is not reductive but $\mathcal{O}(X)^G$ is finitely generated, then the good quotient still exists.

Example 1.1.4. Consider the following $\mathbb{C}^*$-action on $X := \mathbb{C}^2$

$$\mathbb{C}^* \times X \longrightarrow X;$$

$$(t, (x, y)) \longmapsto (t^a x, t^b y).$$

(i) If $a = b = 1$. Then $\mathbb{C}^*$-invariant functions are the constants and the map $\varphi : X \longrightarrow \{pt\}$ defines a good quotient.

(ii) If $a = 0$ and $b = 1$. Then the ring of $\mathbb{C}^*$-invariant functions is generated by x and the following map

$$\varphi : X \to \mathbb{C};\ (x, y) \mapsto x$$

defines a good quotient.

(iii) If $a = 1$ and $b = -1$. Then the ring of $\mathbb{C}^*$-invariant functions is generated by xy and the map

$$\varphi : X \to \mathbb{C};\ (x, y) \mapsto xy$$

is a good quotient.

All these cases are not geometric quotients.

Example 1.1.5. Consider the $\mathbb{C}^*$-action on $X := \mathbb{C}^2\backslash\{0\}$ as follows:

$$\mathbb{C}^* \times X \longrightarrow X;\ (t, (u, v)) \mapsto (tu, tv).$$

The map

$$\varphi : X \to \mathbb{P}^1;\ (u, v) \mapsto [u : v]$$

defines a geometric quotient, because the smallest closed and G-invariant subsets are exactly the G-orbits and two different G-orbits are mapped to two different points.

The following theorem lists the properties of good quotients:

Theorem 1.1.6. *Let G be a reductive algebraic group acts on the variety X. Then every good quotient $\varphi : X \to Y$ has the following properties:*

(i) G-closeness: If $Z \subseteq X$ is a closed G-invariant set, then $\varphi(Z)$ is closed in Y.

(ii) G-separation: If $Z_1, Z_2 \subseteq X$ are closed, G-invariant sets, where $Z_1 \cap Z_2 = \emptyset$, then $\varphi(Z_1) \cap \varphi(Z_2) = \emptyset$.

Proof. It is suffices to prove this theorem in case X is affine, because φ is affine and the statments are local with respect to Y. This is done in [Kr84, II.3.2]. $\square$

Corollary 1.1.7. Let G be a reductive algebraic group and X be a G-variety. If $\varphi : X \longrightarrow Y$ is a good quotient, then φ is surjective and for every $y \in Y$ we have:

(a) The fiber $\varphi^{-1}(y)$ contains exactly a unique closed G-orbit Gx.

(b) Every orbit in $\varphi^{-1}(y)$ contains Gx in its closure.

Proof. The proof of this corollary depends on the above theorem and on the truth that for every action $G : X$ there is a closed G-orbit. $\square$

Corollary 1.1.8. Let G be a reductive algebraic group acts on the variety X, and $\varphi : X \to Y$ be a good quotient. Then:

(a) The quotient space Y carries the quotient topology with respect to the map φ.

(b) For every G-invariant morphism $\phi : X \to Z$, there is a unique morphism $\psi : Y \to Z$ such that $\phi = \psi \circ \varphi$.

Proof. The proof of (a) follows from the theorem 1.1.6 and the proof of (b) follows from the corollary 1.1.7 and from property (ii) in 1.1.1 and from the first statement. $\square$

Definition 1.1.9. A good quotient $\varphi : X \to Y$ satisfies the property (b) in 1.1.8 is called **categorical quotient**. This implies that the categorical quotient is unique up to isomorphism.

Example 1.1.10. Let the additive group $\mathbb{C}$ acts on $\mathbb{C}^2$ by

$$(t,(c_1,c_2)) \mapsto (c_1, c_2 + tc_1);\ t \in \mathbb{C},\ (c_1,c_2) \in \mathbb{C}^2.$$

This induces $\mathbb{C}$-action on $\mathbb{C}[x_1,x_2]$ as follows:

$$(t, p(x_1,x_2)) \mapsto p(x_1, x_2 + tx_1).$$

The invariant functions are $\mathbb{C}[x_1,x_2]^{\mathbb{C}} = \mathbb{C}[x_1]$. Thus the projection onto the first coordinate is a categorical quotient, but it is not a geometric quotient, because the two closed $\mathbb{C}$-invariant points $(0,0)$ and $(0,1)$ are projected onto the same point.

We consider the following two examples, which were studied by A. A'Campo-Neuen and J. Hausen in [AH06].

Example 1.1.11. Let X be the smooth four-dimensional toric variety obtained by glueing the two affine charts

$$U_1 := \mathbb{C}^4,\ \ U_2 := \mathbb{C}^3 \times \mathbb{C}^*$$

along the common subset $(\mathbb{C} \times \mathbb{C}^*)^2$ by the map

$$(t_1,t_2,t_3,t_4) \longmapsto (t_1t_2^2, t_2^{-1}, t_3, t_4).$$

Let $\mathbb{T} := (\mathbb{C}^*)^4$ be the torus acts on X and let

$$H := \{(t^{-2}, 1, t, t);\ t \in \mathbb{C}^*\} \subset \mathbb{T}$$

be the one-dimensional subtorus. If one consider the H-action on X, then there is no categorical quotient by this action on X.

Example 1.1.12. Consider the subvariety $X := \mathbb{C}^2 \times (\mathbb{C}^*)^2 \cup (\mathbb{C}^*)^2 \times \mathbb{C}^2$ of $\mathbb{C}^4$ with the regular $\mathbb{C}^*$-action given by

$$t \cdot (x_1,x_2,x_3,x_4) := (tx_1, tx_2, x_3, t^{-1}x_4).$$

It was shown that there is no categorical quotient by this action on X in the category of algebraic varieties.

Proposition 1.1.13. *([ADHL09]) Let G be a reductive algebraic group acts on the variety X, and let $\varphi : X \to Y$ be a good quotient. Then*

(i) If $V \subseteq Y$ is an open subset, then the restriction

$$\varphi : \varphi^{-1}(V) \to V$$

is a good quotient by the restricted G-action.

(ii) If $Z \subseteq X$ is a closed G-invariant subset, then the restriction

$$\varphi : Z \to \varphi(Z)$$

is a good quotient by the restricted G-action.

1.2 Linearized line bundles

We keep all notations in the last section; G is a connected reductive affine algebraic group acts on the normal algebraic variety X by the regular morphism μ. We denote by p_X the natural projection on X;

$$p_X : G \times X \to X.$$

Definition 1.2.1. A **linear fibration** over X is an algebraic variety E together with a surjective morphism $\pi : E \to X$ of algebraic varieties such that: for every $x \in X$, there is a vector space structure on the fiber $E_x = \pi^{-1}(x)$. The variety E is called the **total space** of the fibration and the fibre over x is denoted by E_x. The bundle

$$X \times \mathbb{C}^r \to X$$

defined by projection on the first factor is called the **trivial fibration** of rank r. Let $\pi : E \to X$, $\pi' : E' \to X$ be two fibrations. Then a morphism $f : E \to E'$ of varieties is called a **morphism of linear fibrations**, if it satisfies the properties:

(i) $\pi' \circ f = \pi$

(ii) For every $x \in X$, the induced map $f_x : E_x \to E'_x$ is linear.

For every $U \subset X$, we denote by $E\mid_U$ the fibration

$$\pi^{-1}(U) \to U$$

defined by ristriction on U. An **algebraic vector bundle** of rank r on X is a linear fibration $E \to X$ such that: For every $x \in X$, there exists an open neighbourhood U of x and an isomorphism of fibrations

$$\sigma : E\mid_U \to U \times \mathbb{C}^r.$$

A vector bundle L of rank one on X, $\pi : L \to X$, will be called a **line bundle**.

Definition 1.2.2. A G-**linearization** of the line bundle L is a G-action $\phi : G \times L \to L$ on L with the following properties:

(i) $\pi : L \to X$ is G-equivariant, i.e., $\pi(gy) = g\pi(y)$ holds $\forall\, g \in G$, $\forall\, y \in L$;

(ii) ϕ is linear on the fibers, i.e., for every $g \in G$ and every $x \in X$, the map $\phi_x : L_x \to L_{gx}$ is linear.

Example 1.2.3. Let the algebraic group G acts on X by μ. Then consider the trivial line bundle $L := X \times \mathbb{C}$. A linearisation of L can be given by choosing a character χ of G, i.e., $\chi \in \mathrm{Hom}(G, \mathbb{C}^*)$, with a G-action ϕ on L defined by

$$\phi(g, (x, t)) := (\mu(g, x), \chi(g)t)$$

where $g \in G$ and $(x, t) \in X \times \mathbb{C}$.

Corollary 1.2.4. From the last example follows that a G-linearisation is not unique.

Remark 1.2.5. For any G-linearization of L, the following diagram

$$\begin{tikzcd} G \times L \arrow[r, "\phi"] \arrow[d, "id\times\pi"'] & L \arrow[d, "\pi"] \\ G \times X \arrow[r, "\mu"] & X \end{tikzcd}$$

is commutative. This diagramm is a pull-back, i.e., it induces an isomorphism

$$G \times L = p_X^*(L) \xrightarrow{\sim} \mu^*(L)$$

of line bundles on $G \times X$.

The following lemma shows that the invers is also true.

Lemma 1.2.6. *[KKLV] Let $\phi : G \times L \to L$ be a morphism. Assume that the diagram*

$$\begin{tikzcd} G \times L \arrow[r, "\phi"] \arrow[d, "id\times\pi"'] & L \arrow[d, "\pi"] \\ G \times X \arrow[r, "\mu"] & X \end{tikzcd}$$

is a pull-back diagram, such that $\phi(e, z) = z$ for all $z \in L$ and that $\phi(g, *)$ maps the zero section of L into itself for all $g \in G$. Then ϕ is a G-linearization of L.

Lemma 1.2.7 ([KKLV])**.** *The line bundle L is G-linearizable if and only if the two bundles $\mu^*(L)$ and $p_X^*(L)$ on $G \times X$ are isomorphic.*

Proposition 1.2.8 ([KKLV])**.** *Let L be a line bundle on a normal G-variety X. Then there is a number $d > 0$ such that $L^{\otimes d}$ is G-linearizable.*

1.3 Semistable points and GIT-quotients

Let G be a connected reductive linear algebraic group acts on a normal algebraic variety X and $\pi : L \to X$ be a G-linearised line bundle over X. For any G-invariant section f of $L^{\otimes d}$ $(d > 0)$, we define the following set X_f
$$X_f = \{x \in X \mid f(x) \neq 0\}.$$
Then the set X_f is an open G-invariant subset of X.

Definition 1.3.1. A point $x \in X$ is called **semistable** point if for some positive integer d, there exists a G-invariant section f of $L^{\otimes d}$ such that $f(x) \neq 0$ and X_f is affine. The point x is called **stable** if it is semistable and has a finite stabilizer and the G-action is closed on X_f. We denoted by X^{ss} the set of all semistable points of X and by X^s the set of all stable points of X.

Example 1.3.2. Consider the general linear group $G := GL(n)$ over $\mathbb{C}$ and the set $X := M(n)$ of all $(n \times n)$-matrices over $\mathbb{C}$. Let G acts on X by conjugation. Then
$$X^{ss} = \{\text{non-nilpotent matrices}\},$$
$$X^s = \{\text{non-nilpotent diagonalizable matrices}\}.$$

Example 1.3.3. Let $G := SL(2)$ the special linear group of 2×2-matrices over $\mathbb{C}$ and $X := M(2 \times n)$ the set of all $(2 \times n)$-matrices over $\mathbb{C}$. Let G acts on X by the left multiplication. Then
$$X^{ss} = X^s = \{(2 \times n) - \text{matrices of rank } 2\}.$$

Example 1.3.4. Let $X \subseteq \mathbb{P}^n$ be a projective variety and L the restriction to X of the hyperplane bundle $\mathcal{O}(1)$. Any linear action induces an L-action and
$$X^{ss}(L) = X^{ss},\ X^s(L) = X^s.$$

Theorem 1.3.5 ([KKMS74])**.** *Let X be a variety and L a line bundle over X. Then for any G-linearization of L of a reductive group G on X holds:*

(i) There exists a good quotient $\varphi : X^{ss}(L) \to Y$ and $Y = X^{ss}(L)/\!\!/G$ is quasi-projective.

(ii) There exists a Zariski-open subset Y^s of Y such that $\varphi^{-1}(Y^s) = X^s(L)$ and $Y^s = X^s(L)/G$ is a geometric quotient of $X^s(L)$.

(iii) *Let $x_1, x_2 \in X^{ss}(L)$. Then*

$$\varphi(x_1) = \varphi(x_2) \text{ if and only if } \overline{Gx_1} \cap \overline{Gx_2} \cap X^{ss}(L) \neq \emptyset.$$

(iv) *Let $x \in X^{ss}(L)$. Then*

$$x \text{ is stable} \Leftrightarrow x \text{ has finite stabilizer and } Gx \text{ is closed in } X^{ss}(L).$$

Example 1.3.6. Consider the following $\mathbb{C}^*$-action on the projective space $\mathbb{P}^n$ $(n \geq 2)$:

$$(t, [x_0 : \cdots : x_n]) \mapsto [tx_0 : \cdots : tx_{n-1} : t^{-n}x_n].$$

Given $x := [x_0 : \cdots : x_n] \in \mathbb{P}^n$, then:

(1) x is stable if $x_n \neq 0$ and $x_0, \cdots, x_{n-1}$ are not all zero.

(2) x is not semistable if $x = [x_0 : \cdots : x_{n-1} : 0]$ or $x = [0 : \cdots : 0 : 1]$.

(3) There are no strictly semistable points.

Thus $(\mathbb{P}^n)^s = (\mathbb{P}^n)^{ss}$ can be identified with $\mathbb{C}^n \setminus \{0\}$ and the action of $\mathbb{C}^*$ becomes

$$(t, [x_0 : \cdots : x_n]) \mapsto [t^{n+1}x_0, \cdots, t^{n+1}x_{n-1}].$$

Hence

$$(\mathbb{P}^n)^s/\mathbb{C}^* = (\mathbb{P}^n)^{ss}/\!\!/\mathbb{C}^* = \mathbb{P}^{n-1}.$$

1.4 Cox ring of quasiprojective varieties

Let X be a normal algebraic variety over $\mathbb{C}$ with a free finitely generated divisor class group $\mathrm{Cl}(X)$. Denote by WDiv(X) the free abelian group generated by all irreducible divisors.

Definition 1.4.1. Let K be the subgroup of WDiv(X) such that the canonical map

$$K \to \mathrm{Cl}(X);\ D \mapsto [D]$$

is an isomorphism. Then the **Cox ring** of X is the algebra of global sections;

$$\mathcal{R}(X) := \bigoplus_{[D] \in \mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)).$$

Example 1.4.2. Let X be the projective space $\mathbb{P}^n$ and $D \subseteq \mathbb{P}^n$ be a hyperplane. Then $[D]$ generates $\mathrm{Cl}(\mathbb{P}^n)$ freely. Define K to be the subgroup of $\mathrm{WDiv}(\mathbb{P}^n)$ generated by D. Then Cox ring is the ring:

$$\mathcal{R}(\mathbb{P}^n) \cong \mathbb{C}[t_0, t_1, \cdots, t_n].$$

Remark 1.4.3. Let $X \subseteq \mathbb{P}^n$ is a normal subvariety whose divisor class group is generated by a hyperplane section and let $\overline{X} \subseteq \mathbb{C}^{n+1}$ be the cone over X. Then $\mathcal{R}(X)$ coincides with $\Gamma(\overline{X}, \mathcal{O})$ if and only if X is projectively normal.

Remark 1.4.4. Let s denote the rank of Cl(X). Then the Cox ring $\mathcal{R}(X)$ is realized as a subring of the Laurent polynomial ring

$$\mathcal{R}(X) \subseteq \mathbb{C}(X)[T_1^{\pm 1}, \cdots, T_s^{\pm 1}].$$

This inclusion gives rise to an isomorphism of the quotient fields

$$\mathrm{Quot}(\mathcal{R}(\mathrm{X})) \cong \mathbb{C}(\mathrm{X})(\mathrm{T}_1, \cdots, \mathrm{T_s}).$$

Proposition 1.4.5 ([ADHL09])**.** *Let X be a normal variety with free finitely generated divisor class group. Then*

(i) The Cox ring $\mathcal{R}(X)$ is a unique factorization domain.

(ii) The units of the Cox ring are given by $\mathcal{R}(X)^* = \Gamma(X, \mathcal{O}^*)$.

Now we extend the definition of Cox to normal varieties X with a finitely generated divisor class group Cl(X), i.e., Cl(X) may have a nontrivial torsion.

Definition 1.4.6. [ADHL09] Let X be a normal variety with $\Gamma(X, \mathcal{O}^*)$ $= \mathbb{C}^*$. Fix a finitely generated subgroup $K \subseteq \mathrm{WDiv(X)}$ projecting onto Cl(X), let $K^0 \subseteq K$ be the kernel of the map $c : K \to \mathrm{Cl(X)}$ sending $D \in K$ to its class $[D] \in \mathrm{Cl(X)}$ and fix a character of K^0 with values in the field of rational functions on X, i.e., a group homomorphism, $\chi : K^0 \to \mathbb{C}(X)^*$ such that

$$\mathrm{div} \circ \chi(\mathrm{E}) = \mathrm{id}_{\mathrm{K}^0}.$$

Let

$$\mathcal{S} = \bigoplus_{D \in K} \mathcal{S}_D, \ \ \mathcal{S}_D := \mathcal{O}_X(D)$$

be the divisorial sheaf associated to K and denote by $\mathcal{I}$ the sheaf of ideals of $\mathcal{S}$ locally generated by the sections $1 - \chi(E)$, where $E \in K^0$ and $\chi(E)$ is homogeneous of degree $-E$. Consider the quotient sheaf

$\mathcal{R} := \mathcal{S}/\mathcal{I}$ and the projection $\pi : \mathcal{S} \to \mathcal{R}$. We define the **Cox sheaf** of X to be $\mathcal{R}$ together with the Cl(X)-grading

$$\mathcal{R} := \bigoplus_{[D]\in \mathrm{Cl}(X)} \pi\left(\bigoplus_{D'\in c^{-1}([D])} \mathcal{S}_{D'}\right).$$

The **Cox ring** of X is the ring of global sections

$$\mathcal{R}(X) := \bigoplus_{[D]\in \mathrm{Cl}(X)} \Gamma(X, \mathcal{R}_{[D]}).$$

Remark 1.4.7. The above definitions of Cox sheaf and Cox ring are independent of choices of K and the character $\chi : K^0 \to \mathbb{C}(X)^*$. For any open set $U \subseteq X$, the canonical homomorphism

$$\Gamma(U, \mathcal{S})/\Gamma(U, \mathcal{I}) \to \Gamma(U, \mathcal{R})$$

is an isomorphism. In particular, we have

$$\mathcal{R}(X) \cong \Gamma(X, \mathcal{S})/\Gamma(X, \mathcal{I}).$$

The assumption $\Gamma(X, \mathcal{O}^*) = \mathbb{C}^*$ is crucial for the uniqueness of Cox sheaves and rings.

1.5 Toric varieties and their Cox ring

An n-dimensional toric variety is a normal variety with an open orbit T isomorphic to the algebraic torus $(\mathbb{C}^*)^n$ such that the action of the torus on itself can be extended to a regular action on X. We describe the Cox ring $\mathcal{R}(X)$ of a toric variety X and show how to reconstruct X from $\mathcal{R}(X)$.

Definition 1.5.1. Let $N \cong \mathbb{Z}^r$ be a lattice and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ its real extension. A **strongly convex rational polyhedral** cone σ in $N_{\mathbb{R}}$ is a cone with apex at the origin, generated by a finite number of vectors in N and contains no line through the origin. Let $M = \mathrm{Hom}(N, \mathbb{Z})$ be the dual lattice of N. We denote the dual cone of σ by $\sigma^\vee$ which is also a strongly convex rational polyhedral cone in $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. A **fan** $\triangle$ in N is a collection of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ satisfying the conditions: every face of a cone in $\triangle$ is also a cone in $\triangle$, and the intersection of two cones in $\triangle$ is a face of each cone.

Remark 1.5.2. A rational polyhedral fan $\triangle$ in N specifies a toric variety $X(\triangle)$ as follows:

(1) Every cone $\sigma \in \triangle$ defines an affine set

$$U_\sigma := \mathrm{Spec}\,(\mathbb{C}[\sigma^\vee \cap M]).$$

(2) Every cone $\sigma \in \triangle$ and a face $\tau \preceq \sigma$ define

$$(U_\sigma)_\tau := \mathrm{Spec}\,(\mathbb{C}[\sigma^\vee \cap M])_w,$$

where $w \in \sigma^\vee \cap M$ determines a supporting hyperplane for τ.

(3) For σ_1, $\sigma_2 \in \triangle$ define $(U_{\sigma_1})_{\sigma_2} := U_{\sigma_1 \cap \sigma_2}$. Then $X(\triangle)$ is the variety obtained by glueing the U_σ along the isomorphisms obtained by composing

$$(U_{\sigma_1})_{\sigma_2} \longrightarrow U_{\sigma_1 \cap \sigma_2} \longrightarrow (U_{\sigma_2})_{\sigma_1}.$$

Remark 1.5.3. The set $U_0 = \mathrm{Spec}\,(\mathbb{C}[M]) = (\mathbb{C}^*)^n$ is an open subvariety of $X(\triangle)$ and denoted by T_N.

Conversely: Every toric variety X can be specified by a fan $X(\triangle)$ as above.

Remark 1.5.4. The lattices M, N and the fan $\triangle \in N$ have natural geometric interpretations

(i) The points of the torus T_N are $\mathbb{C}$-algebra homomorphisms $\phi : \mathbb{C}[M] \to \mathbb{C}$ and every element $m \in M$ corresponds to coordinates on the torus via a group homomorphisms $\chi_m(\phi) = \phi(m)$ to $\mathbb{C}^*$ called **characters**. Thus the lattice M is lattice the characters of the torus.

(ii) The elements $n \in N$ correspond to one-parameter subgroups λ_n. The coordinates of the point $\lambda_n(t)$ are given by $\chi_m(\lambda_n(t)) = t^{m(n)}$.

(iii) Let $\triangle(1)$ denote the set of rays of the fan $\triangle$. A ray $v \in \triangle(1)$ determines an irreducible, torus-invariant, one-codimension subvariety $Y(v)$ of $X(\triangle)$. Moreover, the subvarieties $Y(v)$ generate the group of Weil divisors modulo principal divisors, i.e., generate $\mathrm{Cl}(X)$ and we have the following exact sequence:

$$0 \to M \to \mathbb{Z}^{\triangle(1)} \to \mathrm{Cl}(X) \to 0,$$

where $\mathbb{Z}^{\triangle(1)}$ is the free group generated by $Y(v)$ for $v \in \triangle(1)$.

Example 1.5.5. Consider the variety $X := \mathbb{P}^1 \times \mathbb{P}^1$ with the action of $T := (\mathbb{C}^*)^2$ given by

$$(t_1, t_2) \cdot ([u_0 : u_1], [v_0 : v_1]) := ([t_1 u_0 : u_1], [t_2 v_0 : v_1]).$$

The fan $\triangle(X)$ has four maximal cones with the rays:

$$e_1 := (1, 0),\ e_2 := (0, 1),\ -e_1,\ -e_2$$

of $N \cong \mathbb{Z}^2$. Moreover $\mathrm{Cl}(X) \cong \mathbb{Z}^2$.

Theorem 1.5.6. *([Cox95]) The Cox ring $\mathcal{R}(X)$ of the toric variety X is the polynomial ring*

$$\mathcal{R}(X) := \mathbb{C}[\{v \mid v \in \triangle(1)\}]$$

multigraded by $\deg(v_i) = [v_i]$, where $[v_i]$ denotes the class of $Y(v_i) \in \mathrm{Cl}(X)$.

Definition 1.5.7. Let X be a toric variety and $\triangle$ is the corresponding fan of X. For every $\sigma \in \triangle$ let $v^{\overline{\sigma}}$ be the product of the variables v corresponding to rays not contained in σ. The **irrelevant ideal** of $\mathcal{R}(X)$ is the ideal:

$$J_X := \langle \{v^{\overline{\sigma}} \mid \sigma \in \triangle\} \rangle.$$

$(\mathbb{C}^*)^{\triangle(1)} = \mathrm{Hom}(\mathbb{Z}^{\triangle(1)}, \mathbb{C}^*)$ acts diagonally on $\mathcal{R}(X)$. Thus induces an action of the group $G := \mathrm{Hom}(\mathrm{Cl}(X), \mathbb{C}^*)$ via the inclusion

$$0 \to \mathrm{Hom}(\mathrm{Cl}(X), \mathbb{C}^*) \to \mathrm{Hom}(\mathbb{Z}^{\triangle(1)}, \mathbb{C}^*) \to \mathrm{Hom}(M, \mathbb{C}^*) \to 0,$$

which we obtain from applying $\mathrm{Hom}(;\mathbb{C}^*)$ on the sequence

$$0 \to M \to \mathbb{Z}^{\triangle(1)} \to \mathrm{Cl}(X) \to 0.$$

Theorem 1.5.8. *([LV]) The toric variety X is the categorical quotient of the open set $\mathrm{Spec}(\mathcal{R}(X)) \setminus V(J_X)$ by the action of G.*

Example 1.5.9. Consider the fan $\triangle$ with a single cone $\sigma \subset \mathbb{Z}^2$;

$$\sigma := \mathbb{R}_{>0} v_1 + \mathbb{R}_{>0} v_2,$$

such that $v_1 := 2e_1 - e_2$ and $v_2 := e_2$. Cox ring $\mathcal{R}(X)$ of the variety X of $\triangle$ is $\mathbb{C}[v_1, v_2]$. The irrelevant ideal is $J_X = \langle 1 \rangle$ and $\mathcal{R}(X)^G$ is isomorphic to $\mathbb{C}[x^2, xy, y^2]$, i.e., $X = \mathrm{Spec}(\mathrm{R(X)^G})$ is the cone over a smooth quadric.

Example 1.5.10. Consider the fan $\triangle_n$ of the toric variety of the Hirzebruch surface $\mathbb{F}_n$. The Cox ring $\mathcal{R}(\mathbb{F}_n)$ is the $\mathbb{Z}_2$-graded polynomial ring $k[v_1, \cdots, v_4]$, where $\deg(v_1) = \deg(v_3) = (1, 0)$, $\deg(v_2) = (-n, 1)$ and $\deg(v_4) = (0, 1)$. The irrelevant ideal is

$$J_X = \langle v_1 v_2, v_2 v_3, v_3 v_4, v_4 v_1 \rangle.$$

Chapter 2

Affine $SL(2)$-varieties

2.1 The Popov classification of affine $SL(2)$-varieties

The complete classification of normal affine quasihomogeneous $SL(2)$-varieties has been obtained by Popov [P73] but shorter modern presentation of this classification is contained in the book of Kraft [Kr84, III.4.1].

Let $\mu_n = \langle \zeta_n \rangle$ be the cyclic group of n-th roots of unity. Then define the following closed subgroups of $SL(2)$

$$T := \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \;:\; t \in \mathbb{C}^* \right\}, \quad B := \left\{ \begin{pmatrix} t & \alpha \\ 0 & t^{-1} \end{pmatrix} \;:\; t \in \mathbb{C}^*, \alpha \in \mathbb{C} \right\},$$

$$U_n := \left\{ \begin{pmatrix} \xi & \alpha \\ 0 & \xi^{-1} \end{pmatrix} \;:\; \alpha \in \mathbb{C},\ \xi^n = 1 \right\}, \quad U := \left\{ \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \;:\; \beta \in \mathbb{C} \right\}.$$

Lemma 2.1.1. *[Kr84, III.4.1] Any one-dimensional subgroup of the algebraic group $SL(2)$ is conjugated to one of the above mentioned subgroups.*

Theorem 2.1.2. *[P73] Every 3-dimensional normal affine quasihomogeneous $SL(2)$-variety containing more than one orbit is uniquely determined by a pair of numbers $(h, m) \in \{\mathbb{Q} \cap (0, 1]\} \times \mathbb{N}$. The corresponding variety denoted by $E_{h,m}$.*

Definition 2.1.3. The rational number h is called the **height** of $E_{h,m}$ and we write it as $h = \frac{p}{q}$, where p and q are relative prime numbers. The number m is called the **degree** of $E_{h,m}$ and it equals the order of the stabilizer of a point in the open dense $SL(2)$-orbit $\mathcal{U} \subset E_{h,m}$. This stabilizer is always a cyclic group.

Remark 2.1.4. If $h = 1$, then $E_{1,m}$ is smooth and it contains two $SL(2)$-orbits:

$$\mathcal{U} \cong SL(2)/C_m \text{ and } \mathcal{D} \cong SL(2)/T.$$

The geometric description of $E_{1,m}$ is easy and well-known

$$E_{1,m} \cong SL(2) \times_T \mathbb{C},$$

where the torus T acts on $\mathbb{C}$ by character $\chi_m : t \to t^m$. Thus $E_{1,m}$ can be considered as a line bundle over $SL(2)/T$.

Remark 2.1.5. If $0 < h < 1$, then $E_{h,m}$ contains a unique $SL(2)$-invariant singular point O. If $h = \frac{p}{q}$, where $\gcd(p, q) = 1$, then we define

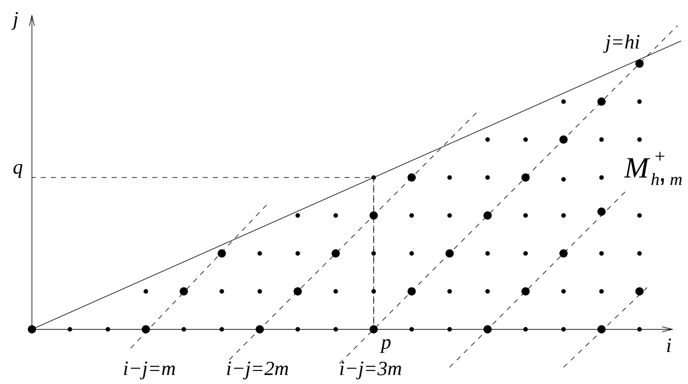
$$k := \gcd(m, q - p),\ a := \frac{m}{k}.$$

Then $E_{h,m}$ contains three $SL(2)$-orbits

$$\mathcal{U} \cong SL(2)/C_m,\ \mathcal{D} \cong SL(2)/U_{a(p+q)},\ \text{and } \{O\}.$$

The explicit construction of $E_{h,m}$ given in [P73] and [Kr84] involves finding a system of generators of the following semigroup

$$M_{h,m} := \{(i, j) \in \mathbb{Z}^2_{\geq 0}\ :\ j \leq hi,\ m|(i - j)\}.$$



Remark 2.1.6. Let V_n be the standard $(n+1)$-dimensional irreducible representation of $SL(2)$ in the space of binary forms of degree n. Denote by $(i_1, j_1), \dots, (i_r, j_r)$ a system of generators of the semigroup $M_{h,m}$. Then $E_{h,m}$ is isomorphic to the closure $\overline{SL(2)v}$ of the $SL(2)$-orbit of the vector

$$v := (X^{i_1}Y^{j_1}, \dots, X^{i_r}Y^{j_r}) \in V_{i_1+j_1} \oplus \dots \oplus V_{i_r+j_r}.$$

Example 2.1.7. Let $m = (q-p)a$, where $a \in \mathbb{N}$, then the semigroup $M_{h,m}$ is minimally generated by $ap+1$ elements

$$\{(m,0),(m+1,1),(m+2,2),\dots,(aq,ap)\}$$

and

$$v := (X^m, X^{m+1}Y, \dots, X^{aq}Y^{ap}) \in V_m \oplus V_{m+2} \oplus \cdots \oplus V_{aq+ap} \cong V_{aq} \otimes V_{ap}.$$

Remark 2.1.8. It is easy to see that the numbers h and m are uniquely determined by the embedding of the monoid $M_{h,m}$ into $\mathbb{Z}^2_{\geq 0}$.

Conversely: If $v \in V_{n_1} \oplus \cdots \oplus V_{n_r}$, for some $n_i \in \mathbb{N}$, is a vector such that $O \in \overline{SL(2)v}$ and v has a stabilizer which is isomorphic to the cyclic group μ_n, where $\overline{SL(2)v}$ is an $SL(2)$-affine variety $E_{h,m}$, then in order to appoint the height of $E_{h,m}$ write the n_i component $v_i \in V_{n_i}$ of v as follows

$$v_i = a_i x^{t_i} y^{s_i} + \sum_{j>0} a_{ij} x^{t_i+j} y^{s_i-j}$$

with $n_i = t_i + s_i$, $a_i \neq 0$ and $t_i > s_i$. Then define

$$h(v) = max_{i=1}^{r} \frac{s_i}{t_i}.$$

Then the variety $E_{h,m}$ has the height $h(v)$.

There exists another relation between the submonoid $M_{h,m} \subset \mathbb{Z}^2_{\geq 0}$ and $E_{h,m}$ which is described through the following theorem

Theorem 2.1.9. *[Kr84, III,4.3] Let E be a normal affine 3-dimensional quasihomogeneous $SL(2)$-variety with the affine coordinate ring $\mathbb{C}[E]$. Denote by $\mathbb{C}[E]^U$ the U-invariant subring. We can consider $\mathbb{C}[E]^U$ as a subring of $\mathbb{C}[SL(2)]^U \cong \mathbb{C}[X,Y]$, where $\mathbb{C}[X,Y]$ is the algebra of regular functions on $SL(2)/U \cong \mathbb{C}^2 \setminus \{(0,0)\}$. Then the monomials $\{X^iY^j \mid (i,j) \in M_{h,m}\}$ form a $\mathbb{C}$-basis of $\mathbb{C}[E]^U$, i.e.*

$$A_{h,m} = \langle X^iY^j \ : \ j \leq hi, \ m|(i-j)\rangle \subset \mathbb{C}[X,Y] = \mathbb{C}[SL_2]^U,$$

where we have denoted $\mathbb{C}[E_{h,m}]^U$ by $A_{h,m}$.

2.2 Affine $SL(2)$-varieties as a categorical quotient

In this section we show that the quasihomogeneous affine $SL(2)$-variety $E_{h,m}$ ($h := \frac{p}{q}$, where $\gcd(p,q) = 1$) is isomorphic to a categorical quotient of a 4-dimensional affine hypersurface H_{q-p} in $\mathbb{C}^5$. This description of this variety is new and simple [BH08].

Denote by V_n the standard $(n+1)$-dimensional irreducible representation of $SL(2)$ in the space of binary forms of degree n. We consider $\mathbb{C}^5$ with the coordinates X_0, X_1, X_2, X_3, X_4 as $V_0 \oplus V_1 \oplus V_1$. Identify the coordinates X_1, X_2, X_3, X_4 with the coefficients of the 2×2-matrix

$$\begin{pmatrix} X_1 & X_3 \\ X_2 & X_4 \end{pmatrix}.$$

Let the algebraic group $SL(2)$ acts on $\mathbb{C}^5$ by left multiplication on X_1, X_2, X_3, X_4 and trivial on X_0. We denote by $D(5,\mathbb{C})$ the group of diagonal matrices of order 5 acting on $\mathbb{C}^5$.

The next theorem includes our discription of $E_{h,m}$ as a categorical quotient:

Theorem 2.2.1. *Let $E_{h,m}$ be a normal affine $SL(2)$-variety of height $h = p/q \leq 1$ $(\gcd(p,q) = 1)$ and of degree m. Then $E_{h,m}$ is isomorphic to the categorical quotient of the affine hypersurface $H_{q-p} \subset \mathbb{C}^5$ defined by the equation*

$$X_0^{q-p} = X_1X_4 - X_2X_3$$

modulo the action of the diagonalizable group $G_0 \times G_m \subset D(5,\mathbb{C})$, where $G_0 \cong \mathbb{C}^*$ consists of diagonal matrices $\{diag\,(t, t^{-p}, t^{-p}, t^q, t^q)\ ;\ t \in \mathbb{C}^*\}$ and $G_m \cong \mu_m = \langle \zeta_m \rangle$ is generated by $diag\,(1, \zeta_m^{-1}, \zeta_m^{-1}, \zeta_m, \zeta_m)$.

Proof. Case 1: $h = 1$. Then $p = q = 1$,

$$G_0 := \{\mathrm{diag}\,(t, t^{-1}, t^{-1}, t, t)\ ;\ t \in \mathbb{C}^*\},$$

$$G_m := \{\mathrm{diag}\,(1, \zeta^{-1}, \zeta^{-1}, \zeta, \zeta)\ ;\ \zeta \in \mu_m\},$$

and the hypersurface H_0 is defined by the following equation

$$1 = X_1X_4 - X_2X_3.$$

The algebraic group $G_0 \times G_m$ can be written as a direct product in another way:

$$G_0 \times G_m = G_0 \times G'_m,$$

where $G'_m := \{\mathrm{diag}\ (\zeta, 1, 1, 1, 1)\ ;\ \zeta \in \mu_m\}$. We remark that the hypersurface

$$H_0 = \{(x_0, x_1, x_2, x_3, x_4) \in \mathbb{C}^5; x_1x_4 - x_2x_3 = 1\}$$

is isomorphic to the product $SL(2)\times\mathbb{C}$. Moreover, the G_0-action on the first factor $SL(2)$ is the same as the action of the maximal torus T by right multiplication. On the other hand, $H_0/\!\!/G'_m$ is again isomorphic to $SL(2)\times\mathbb{C}$, because G'_m acts trivially on $SL(2)$ and $\mathbb{C}/\!\!/G'_m \cong \mathbb{C}$ (one replaces the coordinate X_0 on $\mathbb{C}$ by a new G'_m-invariant coordinate $Y_0 = X_0^m$). So the G_0-action on the second factor $\mathbb{C}$ in $SL(2) \times \mathbb{C} \cong H_0/\!\!/G'_m$ is defined by the character $\chi_m : \ t \to t^m$. Thus, we come to the already known description of $E_{1,m}$ as a T-quotient (see Remark 2.1.4):

$$E_{1,m} \cong SL(2) \times_T \mathbb{C}.$$

CASE 2: $m = 1$, $h = p/q < 1$. The $SL(2)$-action on $\mathbb{C}^5$ commutes with the G_0-action and the hypersuface H_{q-p} defined by the equation

$$X_0^{q-p} = X_1X_4 - X_2X_3.$$

is invariant under this $G_0 \times SL(2)$-action. Moreover, the $G_0 \times SL(2)$-stabilizer in the point $x := (1,1,0,0,1) \in H_{q-p}$ is trivial. Therefore H_{q-p} is the closure of the $G_0 \times SL(2)$-orbit of x in $\mathbb{C}^5$ and

$$X_{p,q} := \operatorname{Spec} \mathbb{C}[H_{q-p}]^{G_0}$$

is an affine $SL(2)$-embedding. One can identify the open dense $SL(2)$-orbit $\mathcal{U}$ in $X_{p,q}$ with the G_0-quotient of the open subset in H_{q-p} defined by the condition $X_0 \neq 0$. Moreover, the affine coordinate ring $\mathbb{C}[\mathcal{U}]$ is generated by the G_0-invariant monomials

$$X := X_0^p X_1, \ \ Y := X_0^{-q} X_3, \ \ Z := X_0^p X_2, \ \ W := X_0^{-q} X_4$$

satisfying the equation

$$\det \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = X_0^{p-q}X_1X_4 - X_0^{p-q}X_2X_3 = 1.$$

By a theorem of Luna-Vust [Kr84, III,3.3], the normality of

$$X_{p,q} := \operatorname{Spec} \mathbb{C}[H_{q-p}]^{G_0}$$

follows from the normality of $\mathbb{C}[X_{p,q}]^U = \mathbb{C}[H_{q-p}]^{G_0 \times U}$. It is easy to see that

$$\mathbb{C}[H_{q-p}]^U \cong \mathbb{C}[X_0, X_1, X_3].$$

Since U-action and G_0-action commute, it remains to compute the G_0-invariant subring $\mathbb{C}[X_0, X_1, X_3]^{G_0}$ under the $\mathbb{C}^*$-action of G_0 on $\mathbb{C}^3$ defined by diag (t, t^{-p}, t^q). Straightforward calculations show that the ring $\mathbb{C}[X_0, X_1, X_3]^{G_0}$ has a $\mathbb{C}$-basis consisting of all monomials $X^iY^j = X_0^{pi-qj}X_1^iX_3^j \in \mathbb{C}[\mathcal{U}]^U = \mathbb{C}[X, Y]$ such that $pi - qj \geq 0$, $i \geq 0$, $j \geq 0$, i.e. , $(i,j) \in M_{h,1}^+$. By 2.1.8 and 2.1.9, we obtain simultaniously that $X_{p,q}$ is normal and that $X_{p,q} \cong E_{h,1}$.

Case 3: $m > 1$, $h = p/q < 1$. Let $X^m_{p,q}$ be the categorical quotient of H_{q-p} by $G_0 \times G_m$ where $G_m \cong \mu_m = \langle \zeta_m \rangle$ acts by

$$\text{diag}\,(1, \zeta_m^{-1}, \zeta_m^{-1}, \zeta_m, \zeta_m).$$

By the same arguments as above, one obtains that

$$\mathbb{C}[X^m_{p,q}]^U \cong \mathbb{C}[X_0, X_1, X_3]^{G_0 \times G_m},$$

where $G_m \cong \mu_m$ acts on $\mathbb{C}^3$ by diag $(1, \zeta_m^{-1}, \zeta_m)$ and G_0 on $\mathbb{C}^3$ by diag (t, t^{-p}, t^q). Therefore the ring $\mathbb{C}[X^m_{p,q}]^U \subset \mathbb{C}[\mathcal{U}]^U = \mathbb{C}[X, Y]$ has a $\mathbb{C}$-basis consisting of all monomials of the form:

$$X^i Y^j = X_0^{pi-qj} X_1^i X_3^j;\ (i, j) \in M^+_{h,m}.$$

The condition $m|(j - i)$ follows from the G_m-invariance of monomials $X_0^{pi-qj} X_1^i X_3^j$. By 2.1.8 and 2.1.9 we get that $X^m_{p,q} \cong E_{h,m}$. $\square$

It will be important to have the following another similar description of an arbitrary affine normal quasihomogeneous $SL(2)$-variety $E_{h,m}$ as a categorical quotient of an affine hypersurface:

Theorem 2.2.2. *Let $E_{h,m}$ be a normal affine $SL(2)$-variety of height $h = p/q \leq 1$ $(\gcd(p, q) = 1)$ and of degree m. We define*

$$k := \gcd(q - p, m) \textit{ and } b := (q - p)/k.$$

Then $E_{h,m}$ is isomorphic to the categorical quotient of the affine hypersurface $H_b \subset \mathbb{C}^5$ defined by the equation

$$Y_0^b = X_1 X_4 - X_2 X_3$$

modulo the action of the diagonalizable group $G := G'_0 \times G_a \subset D(5, \mathbb{C})$, where $G'_0 \cong \mathbb{C}^*$ consists of diagonal matrices

$$\{diag\,(t^k, t^{-p}, t^{-p}, t^q, t^q);\ t \in \mathbb{C}^*\}$$

and $G_a \cong \mu_a = \langle \zeta_a \rangle$ is generated by

$$\{diag\,(1, \zeta_a^{-1}, \zeta_a^{-1}, \zeta_a, \zeta_a)\}.$$

Proof. By Theorem 2.2.1, we have $E_{h,m} = H_{q-p}/\!\!/(G_0 \times G_m)$. We note that the conditions $k = \gcd(q - p, m)$ and $\gcd(q, p) = 1$ imply that $\gcd(k, p) = \gcd(k, q) = 1$. Since ζ_m^a is a generator of μ_k and since the maps $z \to z^p$ and $z \to z^q$ are bijective on μ_k we can find another generator $\xi \in \mu_k$ such that $\xi^p \zeta_m^a = \xi^q \zeta_m^a = 1$. Thus, $G_0 \times G_m$ contains the following element

$$g = \mathrm{diag}\,(\xi, 1, 1, 1, 1) = (\xi, \xi^{-p}, \xi^{-p}, \xi^q, \xi^q) \cdot (1, \zeta_m^{-a}, \zeta_m^{-a}, \zeta_m^a, \zeta_m^a).$$

Consider the homomorphism

$$\psi_k \ : \ D(5, \mathbb{C}) \to D(5, \mathbb{C}), \quad (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \mapsto (\lambda_0^k, \lambda_1, \lambda_2, \lambda_3, \lambda_4).$$

Then $\psi_k(G_0) = G_0'$ and

$$G_k' := \mathrm{Ker}\,\psi_k \cap (G_0 \times G_m) = \langle g \rangle = \{\mathrm{diag}\,(\zeta, 1, 1, 1, 1)\ ;\ \zeta \in \mu_k\}.$$

So we obtain a short exact sequence

$$1 \to G_k' \to G_0 \times G_m \to G_0' \times G_a \to 1,$$

where

$$G_a = \{\mathrm{diag}\,(1, \zeta^{-1}, \zeta^{-1}, \zeta, \zeta)\ :\ \zeta \in \mu_a\}.$$

Therefore the categorical G-quotient of H_{q-p} can be divided in two steps. First we divide H_{q-p} by the subgroup $G_k' \subset G_0 \times G_m$ and after that divide by the group $G_0' \times G_a$. Using a new G_k'-invariant coordinate $Y_0 = X_0^k$, we see that $H_{q-p} /\!\!/ G_k'$ is isomorphic to the hypersurface H_b defined by the equation

$$Y_0^b = X_1 X_4 - X_2 X_3.$$

Since G_0 acts on Y_0 by character $t \to t^k$,

$$E_{h,m} \cong X_{p,q}^m = H_{q-p} /\!\!/ (G_0 \times G_m)$$

is isomorphic to the categorical quotient of H_b modulo the above $G_0' \times G_a$-action. $\square$

2.3 Cox ring of an affine $SL(2)$-variety

Let A be a finitely generated abelian group. We need the following criterion for a finitely generated factorial A-graded $\mathbb{C}$-algebra R with $R^\times = \mathbb{C}^*$ to be a Cox ring of a normal quasiprojective algebraic variety X with $A \cong \mathrm{Cl}(X)$.

Theorem 2.3.1. *Let Y be an irreducible affine algebraic variety over $\mathbb{C}$ with a factorial coordinate ring $R = \mathbb{C}[Y]$. We assume that $\Gamma(Y, \mathcal{O}_Y^*) = \mathbb{C}^*$ and that Y admits a regular action $G \times Y \to Y$ of a diagonalizable group G, or, equivalently, R admits an A-grading by the group $A = \mathrm{Hom}_{\mathrm{alg}}(G, \mathbb{C}^*)$ of algebraic characters of G. Then R is a Cox ring of some normal quasiprojective algebraic variety X such that $\mathrm{Cl}(X) \cong A$ and $\Gamma(X, \mathcal{O}_X^*) = \mathbb{C}^*$ if and only if the following conditions are satisfied:*

(i) there exists an open nonsingular G-invariant subset $U \subset Y$ such that $\operatorname{codim}_Y(Y \setminus U) \geq 2$ *and G acts freely on U;*

(ii) there exists a character $\chi \in \operatorname{Hom}_{\mathrm{alg}}(G, \mathbb{C}^*)$ such that U is contained in $Y^{ss}(L)$ as a saturated open subset with respect to the quotient $Y^{ss}(L) \to Y^{ss}(L)/\!\!/G$, where L is the G-linearization of the trivial line bundle over Y corresponding to χ.

Proof. Assume that Y admits a regular G-action such that conditions (i), (ii) are satisfied. We define X to be $Y^{ss}(L)/\!\!/G$. Then X is a normal irreducible quasiprojective variety and $\Gamma(X, \mathcal{O}_X^*) = \Gamma(Y, \mathcal{O}_Y^*)^G = \mathbb{C}^*$. Moreover, $\overline{U} := U/G$ is a smooth open subset of X, because U is a saturated open subset of $Y^{ss}(L)$. Let us show that $\operatorname{Cl}(X) \cong A$, where $A = \operatorname{Hom}_{\mathrm{alg}}(G, \mathbb{C}^*)$. Since R is factorial and U is a smooth open subset of Y, we have $\operatorname{Pic}(U) = \operatorname{Cl}(U) = 0$. By a general result in [KKV89, 5.1], the Picard group of $\overline{U}$ is isomorphic to the group of G-linearizations of the trivial line bundle over U. On the other hand, since $\operatorname{codim}_Y (Y \setminus U) \geq 2$ and Y is normal, all invertible regular functions on U extend to invertible regular functions on Y, i.e., they are constant. By [KKV89], the latter implies that the group of G-linearizations of the trivial line bundle over U is isomorphic to the group of characters of G, i.e.,

$$\operatorname{Pic}(\overline{U}) \cong \operatorname{Hom}_{\mathrm{alg}}(G, \mathbb{C}^*) = A.$$

Since $\operatorname{Pic}(\overline{U}) = \operatorname{Cl}(\overline{U})$, it remains to show that $\operatorname{codim}_X(X \setminus \overline{U}) \geq 2$. Assume that there exists an irreducible nonempty divisor $Z \subset X$ such that $\overline{U} \cap Z = \emptyset$. Since X is normal, the local ring $\mathcal{O}_{X,Z}$ is a discrete valuation ring, i.e., there exists an affine open subset $U' \subset X$ such that $Z' := U' \cap Z \neq \emptyset$, $\overline{U} \cap Z' = \emptyset$, and Z' is a principle divisor in U' defined by a regular function $g \in \mathbb{C}[U']$. Consider the morphism

$$\pi \ : \ Y^{ss}(L) \to X.$$

We can assume $\widetilde{U'} := \pi^{-1}(U')$ is an affine open subset in $Y^{ss}(L)$ and $\mathbb{C}[U'] = \mathbb{C}[\widetilde{U'}]^G$. Then the element $\tilde{g} := \pi^*(g) \in \mathbb{C}[\widetilde{U'}]$ defines a principle divisor $\widetilde{Z'} := (\tilde{g}) \subset \widetilde{U'}$ such that $\widetilde{Z'} \cap U = \emptyset$ and $\widetilde{Z'} \neq \emptyset$. The latter contradicts to $\operatorname{codim}_{\widetilde{U'}}(\widetilde{U'} \setminus (\widetilde{U'} \cap U)) \geq \operatorname{codim}_Y(Y \setminus U) \geq 2$, i.e., we must have $Z' = \emptyset$.

In order to identify $R = \bigoplus_{a \in A} R_a$ with the Cox ring of X we consider a finite subset $\{a_1, \ldots, a_r\} \subset A$ such that the homogeneous components $R_{a_1}, \ldots, R_{a_r}$ generate the algebra R and $R_{a_i} \neq 0$ for all $i \in \{1, \ldots, r\}$. Since the class of any effective divisor in X is a nonnegative integral linear combination of $a_1, \ldots, a_r$, we obtain that $a_1, \ldots, a_r$ are generators of A. We choose r nonzero elements $g_j \in R_{a_j}$, $j \in \{1, \dots, r\}$ which define r effective principal divisors $\widetilde{D_j} = (g_j)$ in Y. Then we obtain r effective divisors in X:

$$D_j := (\widetilde{D_j} \cap Y^{ss}(L))/\!\!/G, \ \ j \in \{1, \dots, r\}.$$

Consider the epimorphism $\varphi : \mathbb{Z}^r \to A$. For any $k = (k_1, \dots, k_r) \in \mathbb{Z}^r$ we define a rational function

$$g(k) := g_1^{k_1} \cdots g_r^{k_r} \in \mathbb{C}(Y)$$

and a divisor

$$D(k) := k_1 D_1 + \cdots + k_r D_r \in \mathrm{Div}(X).$$

If $a'_1, \dots, a'_s$ is a $\mathbb{Z}$-basis of $\mathrm{Ker}\,\varphi$, then s rational functions $f_i := g(a'_i)$ $(i = 1, \dots, s)$ are G-invariant, i.e., elements of $\mathbb{C}(X)$. So we obtain s principle divisors $D'_i := D(a'_i) = (f_i)$ in X. On the other hand, for any $k \in \mathbb{Z}^r$, one has

$$\mathcal{L}(D(k)) = \left\{ \frac{h}{g(k)} \in \mathbb{C}(X) \ : \ h \in R_{\varphi(k)} \right\}.$$

Consider the $\mathbb{Z}^r$-graded ring

$$\mathcal{R} := \bigoplus_{k \in \mathbb{Z}^r} \mathcal{L}(D(k))$$

together with the surjective homogeneous homomorphism

$$\beta \ : \ \mathcal{R} \to R = \bigoplus_{a \in A} R_a$$

whose restriction to k-th homogeneous component is an isomorphism

$$\beta_k \ : \ \mathcal{L}(D(k)) \xrightarrow{\cong} R_{\varphi(k)}$$

defined by multiplication with $g(k)$. Then the elements

$$\left(\frac{h}{g(k)} - \frac{h}{g(k + a'_i)} \right) = \left(\frac{h}{g(k)} - \frac{h}{g(k) f_i} \right) \in \mathcal{R}_k \oplus \mathcal{R}_{k + a'_i},$$

$$\forall k \in \mathbb{Z}^r, \ \forall h \in R_{\varphi(k)} = R_{\varphi(k + a'_i)}, \ \forall i \in \{1, \dots, s\}$$

are contained in $\mathrm{Ker}\,\beta$. Therefore, β induces a surjective homogeneous homomorphism of the Cox ring $\mathcal{R}/\mathcal{I}$ to R. By comparing the homogeneous components of $\mathcal{R}/\mathcal{I}$ and R, we obtain that $\mathcal{R}/\mathcal{I} \cong R$.

Now assume that a factorial A-graded $\mathbb{C}$-algebra R is the Cox ring of some normal irreducible quasiprojective variety X with $\mathrm{Cl}(X) \cong A$. Using the same idea as in Cox ring definition, we can define a sheaf-theoretical version of the Cox ring of X (see [H08, Section 2]):

$$\widetilde{R} = \bigoplus_{a \in A} \mathcal{O}_X(a)$$

which is a A-graded $\mathcal{O}_X$-algebra such that $\Gamma(X, \widetilde{R}) = R$. Define $Y' := \mathrm{Spec}_X(\widetilde{R})$ as a relative spectrum over X. By [H08, Prop.2.2], $Y' \subset Y := \mathrm{Spec}_{\mathbb{C}}(R)$ is an open embedding and the morphism $\pi : Y' \to X$ is a categorical quotient by the action of $G := \mathrm{Spec}\,\mathbb{C}[A]$. Moreover, G acts freely on the open subset $U := \pi^{-1}(\overline{U})$, where $\overline{U} := X \setminus \mathrm{Sing}(X) \subset X$ the set of all smooth points of X and $\mathrm{codim}_Y(Y \setminus U) \geq 2$. We consider a locally closed embedding $\jmath : X \to \mathbb{P}^n$ and define $\mathcal{L} := \jmath^*\mathcal{O}(1)$. Since $\mathrm{Cl}(Y') = 0$, the pullback $L := \pi^*\mathcal{L}$ is a trivial line bundle over Y' having a G-linearization. Since all invertible global regular functions on Y' are constants, this G-linearization is determined by a character $\chi \in \mathrm{Hom}_{\mathrm{alg}}(G, \mathbb{C}^*) \cong A$. Since $\pi : Y' \to X$ is a good G-quotient and X is quasiprojective, by [H04, Theorem 3.3(ii)], Y' is contained in $Y^{ss}(L)$ is an open saturated subset. So U ($U \subseteq Y' \subseteq Y^{ss}(L)$) is open and saturated in $Y^{ss}(L)$. Theorem is proved. $\square$

Remark 2.3.2. Methods in [H08] allow to formulate and prove a more general version of Theorem 2.3.1 for algebraic varieties X which are not necessary quasiprojective. Moreover, in Theorem 2.3.1 it is enough to assume only A-graded factoriality of R, i.e., that every A-homogeneous divisorial ideal is principal.

Now we begin with the following observation:

Proposition 2.3.3. *The affine coordinate ring $\mathbb{C}[H_b]$ of the hypersurface $H_b \subset \mathbb{C}^5$ is factorial. Invertible elements in $\mathbb{C}[H_b]$ are exactly nonzero constants.*

Proof. Consider the open subset $U_2^+ \subset H_b$ defined by $X_2 \neq 0$. Since U_2^+ is isomorphic to a Zariski open subset in $\mathbb{C}^4$, we obtain $\mathrm{Cl}(U_2^+) = 0$. The complement $\widetilde{S^+} := H_b \setminus U_2^+$ is a principle divisor (X_2). We note that $\widetilde{S^+}$ defined by the binomial equation $Y_0^b = X_1X_4$ which shows that $\widetilde{S^+}$ is isomorphic to the product of $\mathbb{C}$ (with the coordinate X_3) and a 2-dimensional affine toric variety with a A_{b-1}-singularity defined by the equation $Y_0^b = X_1X_4$. Therefore, $\widetilde{S^+}$ is irreducible and the short exact localization sequence

$$\begin{array}{ccccccc} \mathbb{Z} & \to & \mathrm{Cl}(H_b) & \to & \mathrm{Cl}(U_2^+) & \to & 0 \\ 1 & \mapsto & [\widetilde{S^+}] & & & & \end{array}$$

shows that $[\widetilde{S^+}] = 0 \in \mathrm{Cl}(H_b)$, i.e., the image of $\mathbb{Z}$ in $\mathrm{Cl}(H_b)$ is zero. Thus, we obtain $\mathrm{Cl}(H_b) = 0$. In order to prove the second statement we consider the following two cases.

Case 1: $b = 0$. Then $H_b \cong SL(2) \times \mathbb{C}$. Since all invertible elements in the coordinate ring of $SL(2)$ are constants, we obtain the same property for the coordinate ring of $SL(2) \times \mathbb{C}$.

Case 2: $b > 0$. Then we can define a $\mathbb{Z}_{\geq 0}$-grading of $\mathbb{C}[H_b]$ by setting $\deg X_1 = \deg X_2 = \deg X_3 = \deg X_4 = b$ and $\deg Y_0 = 2$. Since the 0-degree component of $\mathbb{C}[H_b]$ is $\mathbb{C}$, we obtain that all invertible elements in $\mathbb{C}[H_b]$ are nonzero constants. □

Proposition 2.3.4. *Consider the following Zariski open subset in H_{q-p}:*

$$U := H_{q-p} \setminus (\{X_1 = X_2 = 0\} \cup \{X_3 = X_4 = 0\}).$$

Denote $U_{h,m} := E_{h,m} \setminus \mathrm{Sing}(E_{h,m}) \subseteq E_{h,m}$, where $\mathrm{Sing}(E_{h,m}) = \emptyset$ if $h = 1$ and $\mathrm{Sing}(E_{h,m}) = \{O\}$ if $h < 1$. Then the group $(G_0 \times G_m)/G'_k \cong G'_0 \times G_a$ acts freely on $U/G'_k \subset H_b$ and $U_{h,m} \cong U/(G_0 \times G_m)$.

Proof. Let $x = (y_0, x_1, x_2, x_3, x_4)$ be a point in U and $g \in G = G_0 \times G_m$ an element such that $gx = x$. We write g as a product of two elements

$$g = \mathrm{diag}\,(t, t^{-p}, t^{-p}, t^q, t^q) \cdot \mathrm{diag}\,(1, \zeta_m^{-s}, \zeta_m^{-s}, \zeta_m^s, \zeta_m^s), \quad t \in \mathbb{C}^*, \zeta_m \in \mu_m.$$

Then $t^{-p}\zeta_m^{-s} = 1$ (because at least one of x_1 and x_2 is nonzero), and $t^q\zeta_m^s = 1$ (because at least one of x_3 and x_4 is nonzero). Therefore, $t^{-p}\zeta_m^{-s}t^q\zeta_m^s = t^{q-p} = 1$. We consider two cases:

Case 1: $y_0 \neq 0$. Then we have $t = 1$. Then $\zeta_m^s = 1$ and $g = 1$.

Case 2: $y_0 = 0$. Then we have

$$t^{-p}\zeta_m^{-s} = t^q\zeta_m^s = 1 = t^{q-p}.$$

Therefore $t \in \mu_{q-p}$. Since $\gcd(p, q-p) = 1$ the element t^p has the same order as t. On the other hand, $t^p = \zeta_m^{-s} \in \mu_m$. So the order of t is a common divisor of $q - p$ and m. In particular, we have $t^k = 1$. This implies that $g \in G'_k$.

Now we remark that the open subset $U/G'_k \subset H_b$ is $SL(2)$-invariant and has nonempty intersection with the $SL(2)$-invariant divisor $\widetilde{D} := \{Y_0 = 0\} \subset H_b$. Therefore, the smooth $SL(2)$-variety $U/(G_0 \times G_m)$ contains more than one $SL(2)$-orbit. So $U/(G_0 \times G_m)$ coincides with $E_{h,m} \setminus \mathrm{Sing}(E_{h,m}) = U_{h,m}$ (see Remarks). □

Corollary 2.3.5. For any affine $SL(2)$-variety $E_{h,m}$, one has

$$\mathrm{Cox}(E_{h,m}) \cong \mathbb{C}[H_b] = \mathbb{C}[Y_0, X_1, X_2, X_3, X_4]/(Y_0^b - X_1X_4 + X_2X_3).$$

Proof. Let L_0 be trivial G-linearized line bundle over H_b, i.e., $\mathcal{O}_{H_b} \cong \mathcal{O}_{H_b}(L_0)$ as G-bundles. Then $H_b^{ss}(L_0) = H_b$ and $H_b^{ss}(L_0)/\!\!/G \cong E_{h,m}$. By 2.3.4, the group $(G_0 \times G_m)/G'_k \cong G'_0 \times G_a$ acts freely on $U/G'_k \subset H_b$ and $\mathrm{codim}_{H_b}(H_b \setminus U/G'_k \subset H_b) = 2$. By 2.3.1, the affine coordinate ring of H_b is isomorphic to the Cox ring of $E_{h,m}$. $\square$

Corollary 2.3.6. [Ga08] An affine $SL(2)$-variety $E_{h,m}$ is toric if and only if $b = 1$, i.e., $q - p$ divides m.

Proof. If $b = 0$ (i.e. $h = 1$), then $E_{1,m}$ is smooth and $\mathrm{Cl}(E_{1,m}) \cong \mathbb{Z}$. However, the divisor class group of any smooth affine toric variety is trivial. Hence, $E_{1,m}$ is not toric.

In general, if X is a normal affine toric variety such that all invertible elements in $\mathbb{C}[X]$ are constant, then $\mathrm{Cox}(X)$ is a polynomial ring [Cox95]. In particular, the spectrum of $\mathrm{Cox}(X)$ is nonsingular. On the other hand, if $b > 1$, then the hypersurface $H_b \subset \mathbb{C}^5$ defined by the equation $Y_0^b - X_1X_4 + X_2X_3 = 0$ is singular. Therefore, $E_{h,m}$ is not toric if $b > 1$. If $b = 1$, then $H_b \cong \mathbb{C}^4$, so $E_{h,m} \cong \mathbb{C}^4/\!\!/G$ is toric. $\square$

Using 2.3.5, we obtain a simple interpretation of the following computation of $\mathrm{Cl}(E_{h,m})$ due to Panyushev:

Proposition 2.3.7. *[Pa92, Th.2] For any normal affine $SL(2)$-variety $E_{h,m}$, one has*

$$\mathrm{Cl}(E_{h,m}) \cong \mathbb{Z} \oplus C_a.$$

Let $D \subset E_{h,m}$ be the closure of the unique 2-dimensional $SL(2)$-orbit $\mathcal{D}$. Denote by $S^+ \subset E_{h,m}$(resp. by $S^- \subset E_{h,m}$) the closure in $E_{h,m}$ of the B-orbit in $\mathcal{U} \cong SL(2)/C_m$ defined by the equation $Z^m = 0$ (resp. by $W^m = 0$). Then $\mathrm{Cl}(E_{h,m})$ is generated by two elements $[D]$ and $[S^+]$ (respectively, by $[D]$ and $[S^-]$) satisfying the unique relation:

$$ap[D] + m[S^+] = 0,$$

or, respectively,

$$-aq[D] + m[S^-] = 0.$$

Proof. The isomorphisms

$$\mathrm{Cl}(E_{h,m}) \cong \mathrm{Hom}_{\mathrm{alg}}(G, \mathbb{C}^*) \cong \mathrm{Hom}_{\mathrm{alg}}(G_0', \mathbb{C}^*) \oplus \mathrm{Hom}_{\mathrm{alg}}(G_a, \mathbb{C}^*) \cong \mathbb{Z} \oplus C_a.$$

follow immediately from 2.3.5. Let $D' \subset E_{h,m}$ be an arbitrary nonzero effective irreducible divisor. Consider the surjective morphism $\pi : U/G_k' \to U/(G_0 \times G_m) = U_{h,m}$. Then the support of D' has a nonempty intersection with $U_{h,m}$, because $\mathrm{codim}_{E_{h,m}} \mathrm{Sing}(E_{h,m}) \geq 2$. Then the closure $\widetilde{D'}$ of $\pi^{-1}(D' \cap U_{h,m}) \subset H_b$ is a G-invariant principal irreducible divisor (see 2.3.3). Therefore, $\widetilde{D'}$ is defined by zeros of a polynomial $\tilde{f}(Y_0, X_1, X_2, X_3, X_4)$ such that $\tilde{f}(gx) = \tilde{\chi}(g)\tilde{f}(x)$ and $\tilde{\chi} = \chi_{D'} \in \mathrm{Hom}_{\mathrm{alg}}(G, \mathbb{C}^*)$ is the character representing the class $[D'] \in \mathrm{Cl}(E_{h,m})$. It is easy to see that the irreducible divisors $\widetilde{D}, \widetilde{S^+}, \widetilde{S^-} \subset H_b$ are defined respectively by polynomials Y_0, X_2, X_4. The corresponding characters $\tilde{\chi}$ of $G \cong \mathbb{C}^* \times \mu_a$ are :

$$\chi_D(t, \zeta) = t^k, \quad \chi_{S^+}(t, \zeta) = t^{-p}\zeta^{-1}, \quad \chi_{S^-}(t, \zeta) = t^q \zeta.$$

Since $\gcd(p, k) = \gcd(q, k) = 1$ each pair $\{\chi_D, \chi_{S^+}\}$ and $\{\chi_D, \chi_{S^-}\}$ generate the character group of $\mathbb{C}^* \times \mu_a$. Moreover, we have

$$\chi_D^{ap}(t, \zeta)\chi_{S^+}^m(t, \zeta) = \chi_D^{-aq}(t, \zeta)\chi_{S^-}^m(t, \zeta) = 1 \ \ \forall t \in \mathbb{C}^*, \ \forall \zeta \in \mu_a.$$

This implies the following two relations in $\mathrm{Cl}(E_{h,m})$:

$$ap[D] + m[S^+] = -aq[D] + m[S^-] = 0.$$

Consider two natural surjective homomorphisms

$$\psi^+ \ : \ \mathbb{Z}^2 \to \mathrm{Cl}(E_{h,m}), \ (k_1, k_2) \mapsto k_1[D] + k_2[S^+],$$

$$\psi^- \ : \ \mathbb{Z}^2 \to \mathrm{Cl}(E_{h,m}), \ (k_1, k_2) \mapsto k_1[D] + k_2[S^-].$$

Then

$$\mathrm{Ker}\, \psi^+ = \langle (ap, m) \rangle, \quad \mathrm{Ker}\, \psi^- = \langle (-aq, m) \rangle,$$

because each of two elements $(p, k), (-q, k) \in \mathbb{Z}^2$ generates a direct summand of $\mathbb{Z}^2$, and, by $ka = m$, we have

$$\mathbb{Z}^2 / \langle (pa, m) \rangle \cong \mathbb{Z} \oplus C_a \cong \mathbb{Z}^2 / \langle (-qa, m) \rangle.$$

□

2.4 $SL(2)$-equivariant flips

Let us start with toric $SL(2)$-equivariant flips. It is known that if $m = a(q-p)$ then the toric variety $E_{h,m}$ is isomorphic to the closure of the orbit of the highest vector in the irreducible $SL(2) \times SL(2)$-module $V_{ap} \otimes V_{aq}$ [Pa92, Prop.2]. In this case, $E_{h,m}$ is isomorphic to the affine cone in $V_{ap} \otimes V_{aq} \cong \mathbb{C}^{(ap+1)\times(aq+1)}$ with vertex 0 over the projective embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into a projective space by the global sections of the ample sheaf $\mathcal{O}(ap, aq)$. The closure D of the 2-dimensional $SL(2)$-orbit $\mathcal{D}$ in $E_{h,m}$ is isomorphic to the affine cone over $a(p+q)$-th Veronese embedding of $\mathbb{P}^1$ considered as diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$. If e_1, e_2, e_3 is a standard basis of $\mathbb{R}^3$ then the toric variety $E_{h,m}$ is defined by the cone $\sigma = \sum_{i=1}^{4} \mathbb{R}_{\geq 0} v_i$ where

$$v_1 = e_1,\ v_2 = -e_1 + aqe_3,\ v_3 = e_2, v_4 = -e_2 + ape_3,$$

i.e., v_1, v_2, v_3, v_4 satisfy the equation $pv_1 + pv_2 = qv_3 + qv_4$. Let $E'_{h,m}$ be the blow up of $0 \in E_{h,m} \subset \mathbb{C}^{(ap+1)\times(aq+1)}$. It corresponds to the subdivion of σ into 4 simplicial cones having a new common ray $\mathbb{R}_{\geq 0} v_5$ ($v_5 = e_3$) and generated by the following 4 sets of lattice vectors

$$\{v_1, v_3, v_5\},\ \{v_2, v_3, v_5\},\ \{v_2, v_4, v_5\},\ \{v_4, v_1, v_5\}.$$

The exceptional divisor D' over 0 corresponding to the new lattice vector v_5 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Moreover, the whole variety $E'_{h,m}$ is smooth and can be considered as a line bundle of bidegree $(-aq, -ap)$ over $\mathbb{P}^1 \times \mathbb{P}^1$. Consider two 2-dimensional simplical cones

$$\sigma^+ = \mathbb{R}_{\geq 0} v_3 + \mathbb{R}_{\geq 0} v_4, \text{ and } \sigma^- = \mathbb{R}_{\geq 0} v_1 + \mathbb{R}_{\geq 0} v_2.$$

There exist two different subdivisons of σ into pairs of simplicial cones

$$\sigma = (\mathbb{R}_{\geq 0} v_1 + \sigma^+) \cup (\mathbb{R}_{\geq 0} v_2 + \sigma^+) \text{ and } \sigma = (\mathbb{R}_{\geq 0} v_3 + \sigma^-) \cup (\mathbb{R}_{\geq 0} v_4 + \sigma^-).$$

We denote toric varieties corresponding to these subdivisions by $E^-_{h,m}$ and $E^+_{h,m}$ respectively. Then one obtains the following diagram of toric morphisms:

$$\begin{tikzcd}
 & E'_{h,m} \arrow[dl, "\gamma^-"'] \arrow[dr, "\gamma^+"] & \\
E^-_{h,m} \arrow[dr, "\varphi^-"'] & & E^+_{h,m} \arrow[dl, "\varphi^+"] \\
 & E_{h,m} &
\end{tikzcd}$$

The morphisms γ^- and γ^+ restriced to D' are projections of $\mathbb{P}^1 \times \mathbb{P}^1$ onto first and second factors. We denote by C^- (reps. C^+) the γ^--image (resp. γ^+-image) of D' in $E^-_{h,m}$ (resp. $E^+_{h,m}$). Then singularities along C^- (reps. along C^+) are determined by the 2-dimensional cone σ^- (resp. σ^+). The relations

$$v_3 + v_4 = apv_5, \;\; v_1 + v_2 = aqv_5$$

show that the 2-dimensional affine toric variety X_{σ^-} (resp. X_{σ^+}) is an affine cone over $\mathbb{P}^1$ embedded by $\mathcal{O}(ap)$ (resp. by $\mathcal{O}(aq)$) to $\mathbb{P}^{ap}$ (resp. $\mathbb{P}^{aq}$). By $1 \le p < q$, we obtain that $E^-_{h,m}$ is always singular and $E^+_{h,m}$ is nonsingular if and only if $ap = 1$. Simple calculations in Chow rings of toric varieties $E^-_{h,m}$ and $E^+_{h,m}$ show that

$$C^- \cdot K_{E^-_{h,m}} = \frac{2(p-q)}{aq^2} < 0, \quad C^+ \cdot K_{E^+_{h,m}} = \frac{2(q-p)}{ap^2} > 0.$$

So the birational map

$$E^-_{h,m} \dashrightarrow E^+_{h,m}$$

is a toric flip.

Now we consider a general case for an affine $SL(2)$-variety $E_{h,m}$. Let us begin with the calculation of the canonical class of an arbitrary $SL(2)$-variety $E_{h,m}$ which has been done by Panyushev in [Pa92, Prop.4 and 5]:

Proposition 2.4.1. *For any normal affine $SL(2)$-variety $E_{h,m}$, one has*

$$K_{E_{h,m}} = -(1+b)[D].$$

Proof. Using the description of $E_{h,m}$ as a categorical quotient $H_b /\!\!/ G$ of the hypersurface $H_b \subset \mathbb{C}^5$, we can consider $E_{h,m}$ as a hypersurface in the 4-dimensional affine toric variety $\mathcal{T}_{h,m} := \mathbb{C}^5 /\!\!/ G$. It is well-known that the canonical divisor of any toric variety consists of irreducible divisors in the complement to the open torus orbit taken with the multiplicity -1. If we consider Y_0, X_1, X_2, X_3, X_4 as homogeneous coordinates of the toric variety $\mathcal{T}_{h,m}$, then the canonical class of $\mathcal{T}_{h,m}$ corresponds to the character $\chi : G \to \mathbb{C}^*$

$$\chi(t,\zeta) = t^{-k}(t^p\zeta)^2(t^{-q}\zeta^{-1})^2 = t^{-k+2p-2q}.$$

On the other hand, G acts on the polynomial $Y_0^b - X_1X_4 + X_2X_3$ by the character

$$\chi'(t,\zeta) = t^{q-p}.$$

Therefore, by adjunction formula, the canonical class of $E_{h,m}$ corresponds to the character $\chi^+ = \chi + \chi'$:

$$\chi^+(t, \zeta) = t^{-k+p-q}.$$

Since the class $[D] \in \mathrm{Cl}(E_{h,m})$ is defined by the character $\chi_D(t, \zeta) = t^k$, we obtain that

$$K_{E_{h,m}} = \frac{-k + p - q}{k}[D] = -(1 + b)[D].$$

$\square$

Proposition 2.4.2. *Let L^+ be the trivial line bundle over H_b together with the linearization corresponding to the character χ^+, then*

$$H_b^{ss}(L^+) = U^+ := H_b \setminus \{X_1 = X_2 = 0\}.$$

Proof. The space $\Gamma(H_b, (L^+)^{\otimes n})^G$ consists of all regular functions f on H_b such that $f(gx) = (\chi^+(g))^n f(x)$. It is easy to see that $\Gamma(H_b, (L^+)^{\otimes n})^G$ is generated as a $\mathbb{C}$-space by restrictions of monomials $Y_0^{k_0} X_1^{k_1} X_2^{k_2} X_3^{k_3} X_4^{k_4}$ satisfying the above homogeneity condition, i.e.,

$$t^{k_0 k - k_1 p - k_2 p + k_3 q + k_4 q} \zeta^{-k_1 - k_2 + k_3 + k_4} = t^{n(-k+p-q)} \quad \forall t \in \mathbb{C}^*, \quad \forall \zeta \in \mu_a.$$

The last condition implies $a|(k_3 + k_4 - k_1 - k_2)$ and

$$k_0 k - k_1 p - k_2 p + k_3 q + k_4 q = n(-k + p - q).$$

Since $n(-k + p - q) < 0$ and $k_i \geq 0$ $(0 \leq i \leq 4)$, we obtain that at least one of the integers k_1 and k_2 must be positive, i.e., all monomials $Y_0^{k_0} X_1^{k_1} X_2^{k_2} X_3^{k_3} X_4^{k_4} \in \Gamma(H_b, (L^+)^{\otimes n})^G$ vanish on the subset $\{X_1 = X_2 = 0\} \cap H_b$. On the other hand, if at least one of two coordinates X_1 and X_2 of a point $x \in H_b$ is not zero, then one of the monomials

$$X_1^{a(q-p+k)}, \quad X_2^{a(q-p+k)} \in \Gamma(H_b, (L^+)^{\otimes p})^G$$

does not vanish in x. Hence, $H_b^{ss}(L^+) = U^+$. $\square$

Proposition 2.4.3. *Let L^- be the trivial line bundle over H_b together with the linearization corresponding to the character $\chi^- = -\chi^+$, then*

$$H_b^{ss}(L^-) = U^- := H_b \setminus \{X_3 = X_4 = 0\}.$$

Proof. The condition $f(gx) = (\chi^-(g))^n f(x)$ for a monomial

$$f = Y_0^{k_0} X_1^{k_1} X_2^{k_2} X_3^{k_3} X_4^{k_4} \in \Gamma(H_b, (L^-)^{\otimes n})^G$$

implies that

$$t^{k_0k-k_1p-k_2p+k_3q+k_4q}\zeta^{-k_1-k_2+k_3+k_4} = t^{n(k+q-p)} \ \ \forall t \in \mathbb{C}^*, \ \ \forall \zeta \in \mu_a.$$

Since $n(k+q-p) > 0$, we obtain that at least one of three integers k_0, k_3, k_4 must be positive. Therefore, all monomials

$$Y_0^{k_0}X_1^{k_1}X_2^{k_2}X_3^{k_3}X_4^{k_4} \in \Gamma(H_b, (L^-)^{\otimes n})^G$$

vanish on the subset $\{Y_0 = X_3 = X_4 = 0\} \cap H_b = \{X_3 = X_4 = 0\} \cap H_b$. On the other hand, if at least one of two coordinates X_3 and X_4 of a point $x \in H_b$ is not zero, then one of the monomials

$$X_3^{a(q-p+k)}, \ \ X_4^{a(q-p+k)} \in \Gamma(H_b, (L^-)^{\otimes q})^G$$

does not vanish in x. Hence, $H_b^{ss}(L^-) = U^-$. □

Theorem 2.4.4. *Define*

$$E^-_{h,m} := H_b^{ss}(L^-)/\!\!/G, \ \ E^+_{h,m} := H_b^{ss}(L^+)/\!\!/G.$$

Then the open embeddings

$$H_b^{ss}(L^-) = U^- \subset H_b, \ \ H_b^{ss}(L^+) = U^+ \subset H_b,$$

define two natural birational morphisms

$$\varphi^- \ : \ E^-_{h,m} \to E_{h,m}, \ \ \varphi^+ \ : \ E^+_{h,m} \to E_{h,m},$$

and the $SL(2)$-equivariant flip

$$\begin{tikzcd}
E^-_{h,m} \arrow[rr, dashed] \arrow[dr, "\varphi^-"'] & & E^+_{h,m} \arrow[dl, "\varphi^+"] \\
& E_{h,m} &
\end{tikzcd}$$

Proof. The statement follows immediately from the isomorphisms

$$E^-_{h,m} \cong \text{Proj} \bigoplus_{n\geq 0} \Gamma(H_b, (L^-)^{\otimes n})^G \cong \text{Proj} \bigoplus_{n\geq 0} \Gamma(E_{h,m}, \mathcal{O}(-nK_{E_{h,m}}))$$

and

$$E^+_{h,m} \cong \text{Proj} \bigoplus_{n\geq 0} \Gamma(H_b, (L^+)^{\otimes n})^G \cong \text{Proj} \bigoplus_{n\geq 0} \Gamma(E_{h,m}, \mathcal{O}(nK_{E_{h,m}})).$$

□

Corollary 2.4.5. One has the following isomorphisms:

$$E^-_{h,m} \cong \operatorname{Proj} \bigoplus_{n\geq 0} \Gamma(E_{h,m}, \mathcal{O}(-nD)), \quad E^+_{h,m} \cong \operatorname{Proj} \bigoplus_{n\geq 0} \Gamma(E_{h,m}, \mathcal{O}(nD)).$$

Proof. These isomorphisms follow from the equation $K_{E_{h,m}} = -(1+b)[D]$ (2.4.1) and from the isomorphism

$$\operatorname{Proj} \bigoplus_{n\geq 0} R_n \cong \operatorname{Proj} \bigoplus_{n\geq 0} R_{nl}$$

for any noetherian graded ring $R = \bigoplus_{n\geq 0} R_n$ and for any positive integer l. $\square$

In order to describe the geometry $E^-_{h,m}$ and $E^+_{h,m}$ in more detail we need two 2-dimensional affine varieties S^+ and S^- having regular B-actions (see also 2.3.7).

Proposition 2.4.6. *Let $S^+ \subset E_{h,m}$ be the closure of an B-orbit obtained as categorical quotient of $W^+ := H_{q-p} \cap \{X_2 = 0\}$ by $G_0 \times G_m$. Then S^+ is isomorphic to the normal affine toric surface* $\operatorname{Spec} \mathbb{C}[M^+_{h,m}]$.

Proof. We note that $W^+ = H_{q-p} \cap \{X_2 = 0\} \subset \mathbb{C}^5$ is a 3-dimesional affine toric variety which is a product of $\mathbb{C}$ and a 2-dimensional affine toric variety defined by the binomial equation $X_0^{q-p} = X_1X_4$. Let us compute the categorical quotient $W^+/\!\!/G_0$. Since the group G_0 acts on X_0, X_1, X_3, X_4 by diag (t, t^{-p}, t^q, t^q) for every nonconstant G_0-invariant monomial $X_0^{k_0}X_1^{k_1}X_3^{k_3}X_4^{k_4}$ ($k_i \in \mathbb{Z}_{\geq 0}$) the condition $k_0 - pk_1 + qk_3 + qk_4 = 0$ implies $k_1 > 0$. If at the same time $k_4 > 0$, then

$$X_0^{k_0}X_1^{k_1}X_3^{k_3}X_4^{k_4} - X_0^{k_0+q-p}X_1^{k_1-1}X_3^{k_3}X_4^{k_4-1} \in I(W^+).$$

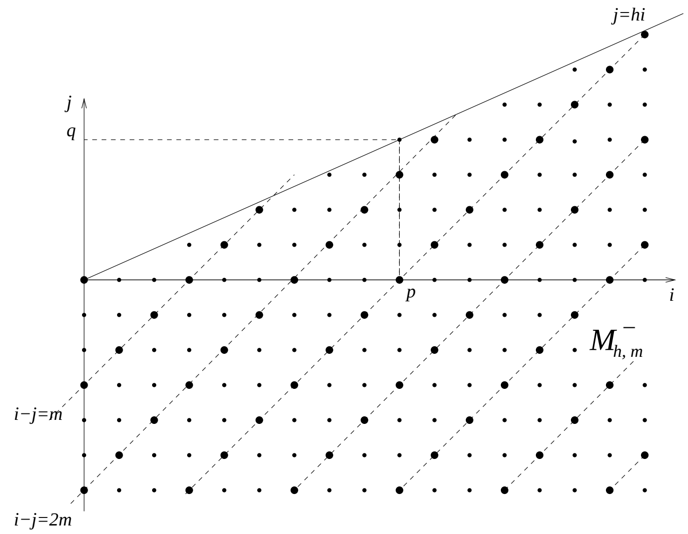
Using the equation $X_0^{q-p} = X_1X_4$ several times, we can get another monomial $X_0^{k_0'}X_1^{k_1'}X_3^{k_3'}$ such that $X_0^{k_0}X_1^{k_1}X_3^{k_3}X_4^{k_4} - X_0^{k_0'}X_1^{k_1'}X_3^{k_3'} \in I(W^+)$, i.e., vanishes on W^+. Therefore, the coordinate ring of $W^+/\!\!/G_0$ contains a $\mathbb{C}$-basis consisting of all G_0-invariant monomials in X_0, X_1, X_3. These monomials have form $X_0^{pk_1-qk_3}X_1^{k_1}X_3^{k_3} = X^{k_1}Y^{k_3}$ where $pk_1 - qk_3 \geq 0$, (i.e., $(k_1, k_3) \in M^+_{h,1})$). So the coordinate ring of $S^+ = W^+/\!\!/(G_0 \times G_m)$ has a $\mathbb{C}$-basis consisting of G_m-invariants monomials $X^{k_1}Y^{k_3} = X_0^{pk_1-qk_2}X_1^{k_1}X_3^{k_3}$ which correspond to lattice points $(k_1, k_3) \in M^+_{h,m} = M^+_{h,1} \cap \{(k_1, k_3) \in \mathbb{Z}^2_{\geq 0} : m|(k_1 - k_3)\}$, i.e. $S^+ \cong \operatorname{Spec} \mathbb{C}[M^+_{h,m}]$. $\square$

Proposition 2.4.7. *Let $S^- \subset E_{h,m}$ be the closure of an B-orbit obtained as categorical quotient of $W^- := H_{q-p} \cap \{X_4 = 0\}$ by $G_0 \times G_m$.*

Then S^- is isomorphic to the normal affine toric surface $\operatorname{Spec}\mathbb{C}[M^-_{h,m}]$, *where the monoid $M^-_{h,m} \subset \mathbb{Z}^2$ is defined as follows:*

$$M^-_{h,m} := \{(i,j) \in \mathbb{Z}^2 \ : \ j \le hi, \ i \ge 0, \ m|(i-j)\}.$$

See the following Figure:



Proof. We note that $W^- = H_{q-p} \cap \{X_4 = 0\} \subset \mathbb{C}^5$ is a 3-dimesional toric variety which is a product of $\mathbb{C}$ and a 2-dimensional toric variety defined by the equation $X_0^{q-p} = -X_2X_3$. Again the computation of the categorical quotient $W^- /\!\!/ G_0$ reduces to finding all G_0-invariant monomials $X_0^{k_0}X_1^{k_1}X_2^{k_2}X_3^{k_3}$. Under the condition $X_0^{q-p} = -X_2X_3$ we can assume that at least one of two variables X_2, or X_3 does not appear in $X_0^{k_0}X_1^{k_1}X_2^{k_2}X_3^{k_3}$ (i.e., $k_2 = 0$, or $k_3 = 0$). If $k_2 = 0$, then we come to the same situation as in 2.4.6 and obtain G_0-invariant monomials $X^{k_1}Y^{k_3} = X_0^{pk_1-qk_3}X_1^{k_1}X_3^{k_3}$ $(k_1,k_3) \in M^+_{h,1}$. If $k_3 = 0$, then we obtain G_0-invariant monomials $X_0^{pk_1+pk_2}X_1^{k_1}X_2^{k_2} = X^{k_1}Z^{k_2}$, $(k_1,k_2 \in \mathbb{Z}_{\ge 0})$. The equation $X_0^{q-p} = -X_2X_3$ implies that on $W^- /\!\!/ G_0$ we have $YZ = X_0^{-q}X_2X_0^pX_3 = -1$. So in case $k_2 = 0$ we obtain the monomials in $X^{k_1}(Y^{-1})^{k_2}$, $(k_1,k_2 \in \mathbb{Z}_{\ge 0})$. Unifying both cases, we get all G_0-invariant monomials X^iY^j, $(i,j) \in \widetilde{M}^1_h$. The action of the finite group G_m on X and Y gives rise to an additional restriction: $m|(i-j)$. Therefore, $G_0 \times G_m$-invariant monomials can be identified with the set of all lattice points $(i,j) \in M^-_{h,m}$. $\square$

Remark 2.4.8. If $m = a(q-p)$ (i.e. $E_{h,m}$ is toric), then $S^- \cong X_{\sigma^-}$ and $S^+ \cong X_{\sigma^+}$, where σ^- and σ^+ are 2-dimensional cones as above.

Definition 2.4.9. Let S be an algebraic surface with a regular action $B \times S \to S$ of a Borel subgroup $B \subset SL(2)$. We denote by $SL(2) \times_B S$ the $SL(2)$-variety $(SL(2) \times S)/B$, where B is considered to act on $SL(2)$ by right multiplication:

$$\left(\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}, \begin{pmatrix} t & \alpha \\ 0 & t^{-1} \end{pmatrix}\right) \mapsto \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \cdot \begin{pmatrix} t & \alpha \\ 0 & t^{-1} \end{pmatrix}^{-1},$$

$$\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \in SL(2).$$

Theorem 2.4.10. *One has the following isomorphisms*

$$E^-_{h,m} \cong SL(2) \times_B S^-, \ \ E^+_{h,m} \cong SL(2) \times_B S^+.$$

Proof. Since $U^+ = H_b \setminus \{X_1 = X_2 = 0\}$ and G acts on (X_1, X_2) by scalar matrices, we obtain a natural $SL(2)$-equivariant morphism

$$\alpha^+ \ : \ E^+_{h,m} \cong U^+ /\!\!/ G \to \mathbb{P}^1, \ \ (Y_0, X_1, X_2, X_3, X_4) \mapsto [X_1 : X_2]$$

Analogously, we obtain a natural $SL(2)$-equivariant morphism

$$\alpha^- \ : \ E^-_{h,m} \cong U^- /\!\!/ G \to \mathbb{P}^1, \ \ (Y_0, X_1, X_2, X_3, X_4) \mapsto [X_3 : X_4].$$

By 2.4.6 and 2.4.7, we have

$$S^+ = (\alpha^+)^{-1}[1:0], \ \ S^- = (\alpha^+)^{-1}[1:0].$$

Since the morphisms α^+ and α^- are $SL(2)$-equivariant and $SL(2)$ acts transitively on $\mathbb{P}^1$, we have

$$S^+ \cong (\alpha^+)^{-1}(z), \ S^- \cong (\alpha^-)^{-1}(z) \ \ \forall z \in \mathbb{P}^1,$$

i.e., $E^+_{h,m}$ (resp. $E^-_{h,m}$) is a fibration over $\mathbb{P}^1$ with fiber S^+ (resp. S^-). On the other hand, the projection $SL(2) \times S^{\pm} \to SL(2)$ defines two natural morphisms $SL(2)$-equivariant morphisms

$$\pi^+ \ : \ SL(2) \times_B S^+ \to SL(2)/B \cong \mathbb{P}^1,$$

$$\pi^- \ : \ SL(2) \times_B S^- \to SL(2)/B \cong \mathbb{P}^1,$$

such that $SL(2) \times_B S^+$ (resp. $SL(2) \times_B S^-$) is a fibration over $\mathbb{P}^1$ with fiber S^+ (resp. S^-). Consider the morphisms

$$\tilde{\beta}^+ \ : \ SL(2) \times S^+ \to U^+ /\!\!/ G, \ \ \tilde{\beta}^- \ : \ SL(2) \times S^- \to U^- /\!\!/ G$$

defined by

$$\tilde{\beta}^{\pm}(g,x) = gx, \ \ \forall g \in SL(2), \ \forall x \in S^{\pm} = (\alpha^{\pm})^{-1}[1:0].$$

Since $\tilde{\beta}^{\pm}(gb^{-1}, bx) = \tilde{\beta}^{\pm}(g,x) = gx \ \forall b \in B$, the morphism $\tilde{\beta}^{\pm}$ descends to a morphism

$$\beta^{\pm} \ : \ SL(2) \times_B S^{\pm} \to U^{\pm} /\!\!/ G \cong E^{\pm}_{h,m}.$$

The latter is an isomorphism, because $\beta^{\pm}$ is $SL(2)$-equivariant and it maps isomorphically the fiber of $\pi^{\pm}$ over $[1:0]$ to the fiber of $\alpha^{\pm}$ over the B-fixed point $[B] \in SL(2)/B$. $\square$

Remark 2.4.11. Since $M^{+}_{h,m}$ is a submonoid of $M^{-}_{h,m}$ we obtain a birational morphism $\psi : \ S^{-} \to S^{+}$ of 2-dimensional normal affine toric varieties S^{-} and S^{+}. However, ψ is not B-equivariant, because an element

$$\begin{pmatrix} t & \alpha \\ 0 & t^{-1} \end{pmatrix} \in B$$

sends $X^iY^j \in \mathbb{C}[M^{+}_{h,m}]$ to

$$(tX)^i(tY + \alpha X^{-1})^j \in \mathbb{C}[M^{+}_{h,m}]$$

and sends $X^rY^s \in \mathbb{C}[M^{-}_{h,m}]$ to

$$(tX - \alpha Y^{-1})^r(tY)^s \in \mathbb{C}[M^{-}_{h,m}].$$

This is the reason why there is no any birational $SL(2)$-equivariant morphism from $SL(2) \times_B S^{-}$ to $SL(2) \times_B S^{+}$, but only a flip.

Remark 2.4.12. Let $E_{h,m} \hookrightarrow V$ be a closed embedding, where V is an affine space isomorphic to $V_{i_1+j_1} \oplus \cdots \oplus V_{i_r+j_r}$ (see 2.1.5). We define a $\mathbb{C}^*$-action on V such that $t \in \mathbb{C}^*$ acts by multiplication with t^{j-i} on V_{i+j}. The affine subvariety $E_{h,m} \subset V$ is invariant under this $\mathbb{C}^*$-action, because $\mathbb{C}^*$ acts on the vector v (see 1.3) in the same way as the maximal torus of $SL(2)$. Consider the weighted blow up $\delta : \widetilde{V} \to V$ of $0 \in V$ with respect to weights of this $\mathbb{C}^*$-action. The birational pullback of $E_{h,m}$ under $\delta : \ \widetilde{V} \to V$ is a $SL(2)$-variety $E'_{h,m}$ together with surjective morphisms $\gamma^{-} : E'_{h,m} \to E^{-}_{h,m}$ and $\gamma^{+} : E'_{h,m} \to E^{+}_{h,m}$ such that the following diagram commutes

$$\begin{tikzcd} & E'_{h,m} \arrow[dl, "\gamma^{-}"'] \arrow[dr, "\gamma^{+}"] & \\ E^{-}_{h,m} \arrow[rr, dashed] & & E^{+}_{h,m} \end{tikzcd}$$

The variety $E'_{h,m}$ contains two $SL(2)$-invariant divisors $D' \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\widetilde{D} := \delta^*(D)$ whose intersection $C = D' \cap \widetilde{D} \cong \mathbb{P}^1$ is the unique 1-dimensional closed $SL(2)$-orbit in $E'_{h,m}$. The morphism $\gamma^{\pm}$ contracts D' to $C^{\pm} \subset E^{\pm}_{h,m}$. The divisor D' corresponds to the $SL(2)$-invariant discrete valuation of the function field $\mathbb{C}(SL(2))$ defined by above $\mathbb{C}^*$-action on $E_{h,m}$ such that $\mathbb{C}(D')$ is the $\mathbb{C}^*$-invariant subfield $\mathbb{C}(SL(2))^{\mathbb{C}^*} \cong \mathbb{C}(SL(2)/\mathbb{C}^*)$. We note that the $SL(2)$-variety $E'_{h,m}$ has also a toroidal structure, i.e., along the closed 1-dimensional $SL(2)$-orbit C, it is locally isomorphic to a product of an affine line $\mathbb{A}^1$ and a 2-dimensional affine toric surface S' which is isomorphic to $\operatorname{Spec} \mathbb{C}[M'_{h,m}]$ where

$$M'_{h,m} := \{(i,j) \in \mathbb{Z}^2 \ : \ pj - qi \geq 0, j - i \in m\mathbb{Z}_{\geq 0}\}.$$

In particular, $S' \cong \mathbb{A}^2/\mu_b$ and $E'_{h,m}$ is nonsingular along C if and only if $b = 1$, i.e., iff $E_{h,m}$ is toric.

Proposition 2.4.13. *The canonical divisor of $E^{\pm}_{h,m}$ has the following intersection numbers with the 1-dimensional $SL(2)$-orbits $C^{\pm} \subset E^{\pm}_{h,m}$:*

$$K_{E^-_{h,m}} \cdot C^- = -\frac{(1+b)k}{aq^2}, \quad K_{E^+_{h,m}} \cdot C^+ = \frac{(1+b)k}{ap^2}.$$

Proof. Since $E_{h,m}, E^-_{h,m}$ and $E^+_{h,m}$ have the same divisor class group, we can use 2.3.7 and obtain that

$$ap[D] + m[S^+] = 0 \in \operatorname{Cl}(E^+_{h,m}).$$

The divisor $S^+ \subset E^+_{h,m}$ intersects the curve C^+ transversally, but this intersection point is an isolated cyclic quotient singularity of type A_{ap-1} in S^+. Therefore, we have $S^+ \cdot C^+ = \frac{1}{ap}$ and

$$D \cdot C^+ = -\left(\frac{m}{ap} S^+\right) \cdot C^+ = -\frac{k}{ap^2}.$$

By 2.4.1, we get

$$K_{E^+_{h,m}} \cdot C^+ = \frac{(1+b)k}{ap^2}.$$

Similarly, the intersection point of C^- and $S^- \subset E^-_{h,m}$ is an isolated cyclic quotient singularity of type A_{aq-1} in S^-. Therefore, we have $S^- \cdot C^- = \frac{1}{aq}$ and, by

$$-aq[D] + m[S^-] = 0 \in \operatorname{Cl}(E^-_{h,m}),$$

we obtain

$$D \cdot C^- = \left(\frac{m}{aq} S^- \right) \cdot C^- = \frac{k}{aq^2}.$$

By 2.4.1, this implies

$$K_{E^-_{h,m}} \cdot C^- = -\frac{(1+b)k}{aq^2}.$$

$\square$

Chapter 3

Spherical varieties

3.1 Spherical subgroups $H \subset G$

Let G be an algebraic group over $\mathbb{C}$ and denote by $R_u(G)$ the largest normal unipotent subgroup of G. Recall that the group G is called **reductive** if $R_u(G) = \{e\}$. Let H be an algebraic subgroup of G. Consider the homogeneous space G/H.

Definition 3.1.1. An **embedding** of the homogeneous space G/H is a normal G-variety X together with a G-equivariant open embedding $G/H \hookrightarrow X$, i.e., the stabilizer of a point in the open orbit is isomorphic to H. A normal embedding X of G/H is called **simple**, if X has only one closed G-orbit. The open embedding $G/H \hookrightarrow X$ induces an isomorphism between fields of the rational functions on G/H and X, i.e., $k(X) \cong k(G/H)$.

Definition 3.1.2. A **Borel subgroup** $B \subset G$ is a maximal connected solvable subgroup of G. The homogeneous space G/H is called **spherical** if it contains an open dense B-orbit. In this case the group H is called **spherical subgroup** of G. Similarly, a **spherical embedding** X of the homogeneous space G/H is an embedding containing an open dense B-orbit.

Example 3.1.3. A simple example of a spherical homogeneous variety is an algebraic torus. In this case $G = B \cong (\mathbb{C}^*)^n$ is a torus and $H = \{e\}$ is the trivial subgroup of G.

Example 3.1.4. Let $SL(2)$ be the linear group of 2×2-matrices over $\mathbb{C}$ with determinant one. Consider the reductive group $G := SL(2) \times \mathbb{C}^*$. Let $(n, m) \in \mathbb{Z}^2$ such that $\gcd(n, m) = 1$ and define the following closed algebraic subgroup of G

$$H_{(n,m)} := \left\{ \left(\begin{pmatrix} \lambda^n & 0 \\ 0 & \lambda^{-n} \end{pmatrix}, \lambda^m \right) \mid \lambda \in \mathbb{C}^* \right\} \subset SL(2) \times \mathbb{C}^*.$$

Let

$$\tilde{B} := \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}^* \right\}$$

be a Borel subgroup of $SL(2)$. Then the group $B := \tilde{B} \times \mathbb{C}^*$ is a 3-dimensionl Borel subgroup of G. Let $\overline{e} := eH_{(n,m)} \in G/H_{(n,m)}$ where

$$e := \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, 1 \right).$$

We determine the B-stabilizer in $\overline{e}$

$$\mathrm{st}_B(\overline{e}) := \{b \in B;\ b\overline{e} = \overline{e}\}.$$

Let

$$b := \left(\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}, \gamma \right)$$

be an arbitrary elemnet of B. Then

$$\begin{aligned} b \cdot \overline{e} &= \left(\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}, \gamma \right) \cdot \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, 1 \right) \cdot \left(\begin{pmatrix} \lambda^n & 0 \\ 0 & \lambda^{-n} \end{pmatrix}, \lambda^m \right) \\ &= \left(\begin{pmatrix} (\alpha+\beta)\lambda^n & \beta\lambda^{-n} \\ \alpha^{-1}\lambda^n & \alpha^{-1}\lambda^{-n} \end{pmatrix}, \gamma\lambda^m \right) \end{aligned}$$

for some $\left(\begin{pmatrix} \lambda^n & 0 \\ 0 & \lambda^{-n} \end{pmatrix}, \lambda^m \right) \in H_{(n,m)}$.

$$\overline{e} = \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, 1 \right) \cdot \left(\begin{pmatrix} \mu^n & 0 \\ 0 & \mu^{-n} \end{pmatrix}, \mu^m \right) = \left(\begin{pmatrix} \mu^n & 0 \\ \mu^n & \mu^{-n} \end{pmatrix}, \mu^m \right)$$

for some $\left(\begin{pmatrix} \mu^n & 0 \\ 0 & \mu^{-n} \end{pmatrix}, \mu^m \right) \in H_{(n,m)}$. The equation $\overline{e} = b{\cdot}\overline{e}$ is satisfied, when

$$(\alpha+\beta)\lambda^n = \mu^n,\ \alpha^{-1}\lambda^n = \mu^n,\ \beta\lambda^{-n} = 0,\ \alpha^{-1}\lambda^{-n} = \mu^{-n},\ \gamma\lambda^m = \mu^m.$$

This gives: $\beta = 0,\ \alpha = \pm 1$ and then $\mu^n = \pm\lambda^n$. Thus $(\mu^{2n})^m = (\lambda^{2n})^m$. On the other side: $\gamma^{2n}\lambda^{2nm} = \mu^{2nm}$. Therefore $\gamma^{2n} = 1$, i.e., $st_B(\overline{e})$ is a cyclic subgroup of G. We have then

$$\dim B\overline{e} = \dim B - \dim \mathrm{st}_B(\overline{e}) = 3 - 0 = \dim B.$$

Then $B\overline{e}$ is an open dense B-orbit. This means that the homogeneous space $G/H_{(n,m)}$ is spherical.

Example 3.1.5. Let $X := \mathbb{P}^2 \times \mathbb{P}^1$. Define the following $SL(2)$-action on X

$$SL(2) \times X \longrightarrow X,$$

$$\left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}, ([u_0 : u_1 : u_2], [v_1 : v_2]) \right) \mapsto ([u_0 : xu_1 + yu_2 : zu_1 + wu_2],$$
$$[xv_1 + yv_2 : zv_1 + wv_2]).$$

Define also the following $\mathbb{C}^*$-action on X

$$\mathbb{C}^* \times X \longrightarrow X,$$

$$(t, ([u_0 : u_1 : u_2], [v_1 : v_2])) \mapsto ([u_0 : tu_1 : tu_2], [t^{-1}v_1 : t^{-1}v_2]).$$

Then consider the group $G := SL(2) \times \mathbb{C}^*$. Since the $\mathbb{C}^*$-action commutes with $SL(2)$-action, one can regard X as a G-variety. Define the 1-dimensional algebraic subgroup H of G

$$H := \left\{ \left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \right) : \lambda \in \mathbb{C}^* \right\}.$$

Let $\left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}, t \right) \in G$ and $x_0 := ([0 : 0 : 1], [1 : 0]) \in X$. Then the image of the element

$$\left(\left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}, t \right), ([0 : 0 : 1], [1 : 0]) \right)$$

under the G-action is

$$([0 : ty : tw], [t^{-1}x : t^{-1}z])$$

and $\left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}, t \right) \in \mathrm{st}_G(x_0)$, when

$$([0 : ty : tw], [t^{-1}x : t^{-1}z]) = ([0 : 0 : 1], [1 : 0]).$$

This gives $y = z = 0$ and $x = w^{-1} = t$. Therefore $\mathrm{st}_G(x_0) = H$. Therefore X defines an G/H-embedding. Consider the Borel subgroup $\tilde{B}$ of the upper triangular matrices of $SL(2)$. Then the group $B = \tilde{B} \times \mathbb{C}^*$ is a Borel subgroup of G. It is easy to prove that the B-stabilizer in the point $p := ([0 : 0 : 1], [0 : 1])$ is trivial, i.e., the B-orbit of p is an open dense., i.e., X is a spherical G/H-embedding.

3.2 G-invariant valuations and divisors

Let X be an embedding of the homogeneous space G/H and let $\mathbb{C}(X)$ be the rational function field of X.

Definition 3.2.1. A **discrete valuation** of $\mathbb{C}(X)$ is a map

$$\upsilon : \mathbb{C}(X)^* \to \mathbb{Q}$$

which satisfies the properties:

(i) $\upsilon(f_1 + f_2) \geq min\{\upsilon(f_1), \upsilon(f_2)\}$; $f_1, f_2, f_1 + f_2 \in \mathbb{C}(X)^*$.

(ii) $\upsilon(f_1 f_2) = \upsilon(f_1) + \upsilon(f_2)$.

(iii) $\upsilon(\mathbb{C}^*) = 0$.

If $\upsilon(f) = 0$ for all $f \in \mathbb{C}(X)^*$, then υ is called **trivial valuation**. We associate to every valuation υ a normal local subring $\mathcal{O}_\upsilon$ of $\mathbb{C}(X)$ as

$$\mathcal{O}_\upsilon := \{\{f \in \mathbb{C}(X)^* | \upsilon(f) \geq 0\} \cup \{0\}\} \subset \mathbb{C}(X).$$

The ring $\mathcal{O}_\upsilon$ is called the **valuation ring** of υ. The **maximal ideal** m_υ of $\mathcal{O}_\upsilon$ is

$$m_\upsilon := \{f \in \mathbb{C}(X)^* | \upsilon(f) > 0\} \cup \{0\}.$$

Definition 3.2.2. The valuation υ is called G-**invariant** if it satisfies the relation

$$\upsilon(g \cdot f) = \upsilon(f) \text{ for all } f \in \mathbb{C}(X)^* \text{ and all } g \in G.$$

We denote by $\mathcal{V}$ the set of all nontrivial G-invariant discrete valuations of $\mathbb{C}(X)$.

Let B be a Borel subgroup of G. Then we define

$$Div(X)^B := \{D \subset X;\ D \text{ is a } B\text{-invariant irreducible Weil divisor}\}.$$

Every $D \in Div(X)^B$ is associated to a B-invariant valuation:

$$\upsilon_D : \mathbb{C}(X)^* \longrightarrow \mathbb{Q},\ f \mapsto \upsilon_D(f)$$

where $\upsilon_D(f)$ denotes the **order** of pole or zero of f on D.

Remark 3.2.3. $|Div(X)^B| < \infty$, because every B-invariant divisor D in $Div(X)^B$ is an irreducible component of the complement to X of the open dense B-orbit Bx_0.

Remark 3.2.4. Let D be a divisor of $Div(X)^B$, then

(i) Either: $D \cap (G/H) \neq \phi$. Then there is some $D_0 \in Div(G/H)^B$, where $D = \overline{D}_0$. In this case D_0 is not G-stable. These divisors will be denoted by $\mathcal{F}(X)$ and elements of $\mathcal{F}(X)$ will be called **colores** of X.

(ii) Or: $D \cap (G/H) = \phi$. Then D is an irreducible component of the complement to X of (G/H). Since G is connected, then the divisor D is even G-stable. The set of these divisors will be denoted by $Div(X)^G$.

Therefore
$$Div(X)^B = \mathcal{F}(X) \sqcup Div(X)^G.$$

Example 3.2.5. In the case $X := G = \mathbb{T} \cong (\mathbb{C}^*)^n$ and H is trivial, we have:
$$\mathcal{F}(\mathbb{T}) = Div(\mathbb{T})^{\mathbb{T}} = \emptyset, \;\; Div(X)^G = \emptyset,$$
because $G = B = \mathbb{T}$ and the complement of (G/H) to X is the empty set.

Example 3.2.6. In example 3.1.4 consider the following points
$$\overline{p}_1 := p_1 H_{(n,m)},\; \overline{p}_2 := p_2 H_{(n,m)} \;\in G/H_{(n,m)},$$
where
$$p_1 := \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right),\; p_2 := \left(\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, 1\right) \in G.$$
We detremine the B-stabilizer in $\overline{p}_1$ and the in $\overline{p}_2$. Let
$$b := \left(\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}, \gamma\right)$$
be an arbitrary elemnet of the B-stabilizer in $\overline{p}_1$. Then every element
$$h := \left(\begin{pmatrix} \lambda^n & 0 \\ 0 & \lambda^{-n} \end{pmatrix}, \lambda^m\right) \in H_{(n,m)}$$
satisfies the equation $bp_1 = p_1 h$ where
$$bp_1 = \left(\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}, \gamma\right) \cdot \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right) = \left(\begin{pmatrix} \alpha & \alpha+\beta \\ 0 & \alpha^{-1} \end{pmatrix}, \gamma\right),$$
$$p_1 h = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right) \cdot \left(\begin{pmatrix} \lambda^n & 0 \\ 0 & \lambda^{-n} \end{pmatrix}, \lambda^m\right) = \left(\begin{pmatrix} \lambda^n & \lambda^{-n} \\ 0 & \lambda^{-n} \end{pmatrix}, \lambda^m\right).$$
From the above equation b is of the form
$$\left(\begin{pmatrix} \lambda^n & \lambda^{-n} - \lambda^n \\ 0 & \lambda^{-n} \end{pmatrix}, \lambda^m\right).$$

This gives that the B-orbit in $\overline{p}_1$ is of codimension one. Similary one gets that the B-orbit in $\overline{p}_2$ is of codimension one. Therefore, the set of colores of $G/H_{(n,m)}$ is

$$\mathcal{F}(G/H_{(n,m)}) = \{D_1 := B\overline{p}_1,\ D_2 := B\overline{p}_2\}.$$

Since the complement of $G/H_{(n,m)}$ to X is the empty set , then

$$Div(G/H_{(n,m)})^G = \{\emptyset\}.$$

Example 3.2.7. In example 3.1.5 ($X := \mathbb{P}^2 \times \mathbb{P}^1$) we consider the points

$$\overline{p}_1 := ([0:1:0],[1:1]),\ \overline{p}_2 := ([0:1:1],[1:0]) \ \in X.$$

Let

$$b := \left(\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}, t\right) \in B.$$

Then $b\overline{p}_1 = ([0:\alpha t:0],[(\alpha+\beta)t^{-1}:\alpha^{-1}t^{-1}])$. The equation $b\overline{p}_1 = \overline{p}_1$ is satisfied, when

$$\alpha t = 1,\ (\alpha+\beta)t^{-1} = 1,\ \alpha^{-1}t^{-1} = 1.$$

This means that b is of the form

$$\left(\begin{pmatrix} t^{-1} & t - t^{-1} \\ 0 & t \end{pmatrix}, t\right).$$

Therefore the B-orbit in $\overline{p}_1$ is of codimension one. Similary one gets the B-orbit in $\overline{p}_2$ is of codimension one. We get that

$$\mathcal{F}(X) = \{D_1 := B\overline{p}_1,\ D_2 := B\overline{p}_2\}.$$

Let

$$\overline{q}_1 := ([0:1:1],[1:1]),\ \overline{q}_2 := ([1:1:1],[1:1]) \ \in X.$$

It is not difficult to prove that the G-orbits in these points are of codimension one and then

$$Div(X)^G = \{D_1^{'} := G\overline{q}_1,\ D_2^{'} := G\overline{q}_2\}.$$

3.3 Colored cones and fans

We keep all previous notations in this section. Denote by $\chi(B)$ the character group of the Borel subgroup B of the algebraic group G, i.e., the set of all algebraic homomorphisms from B to $\mathbb{C}^*$. Define $\mathbb{C}(G/H)^B$ the set of B-eigenfunctions as follows

$$\mathbb{C}(G/H)^B := \{f \in \mathbb{C}(G/H)^* \mid \exists \ \textit{character} \ \chi_f \in \chi(B);$$

$$f(bx) = \chi_f(b) \cdot f(x) \ \ \forall\, b \in B, \ \ \forall\, x \in G/H\}.$$

Then $\mathbb{C}(G/H)^B$ is an abelian subgroup of the multiplicative group of the field $\mathbb{C}(G/H)$. This induced the following map

$$\psi : \mathbb{C}(G/H)^B \to \chi(B), \ f \mapsto \chi_f.$$

The homomorphism ψ is not surjective in general. The image of ψ is

$$\chi(G/H) := \{\chi_f \ \mid \ f \in \mathbb{C}(G/H)^B\} \subseteq \chi(B).$$

We denote $\chi(G/H)$ by Λ. Kernel of ψ is exactly the B-invariant functions, i.e., the constants. Thus we get the following short exact sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \mathbb{C}(G/H)^B \longrightarrow \Lambda \longrightarrow 0.$$

The image Λ of ψ is finitely generated free abelian subgroup of $\chi(B)$ and its rank is called the **rank** of G/H.

Example 3.3.1. If $X = G = \mathbb{T} \cong (\mathbb{C}^*)^n$, then define

$$\chi_i : \mathbb{T} \longrightarrow \mathbb{C}^*, \ (t_1, \ldots, t_n) \longmapsto t_i$$

characters of $\mathbb{T}$. Since $\mathbb{C}(\mathbb{T})^{\mathbb{T}}$ is generated from monoms of the form $x_1^{a_1} \ldots x_n^{a_n}$, then $f \in \mathbb{C}(\mathbb{T})^{\mathbb{T}}$ is associated to $\chi_f := (a_1, \ldots, a_n)$. The map

$$\mathbb{Z}^n \to \chi(\mathbb{T}), \ (a_1, \ldots, a_n) \mapsto \chi_1^{a_1} \ldots \chi_n^{a_n}$$

is an isomorphism of algebraic groups. Thus $\chi(\mathbb{T}) \cong \mathbb{Z}^n$.

Example 3.3.2. Let us determine the lattice Λ in 3.1.5. Since $\mathrm{st}_G(([0 : 0 : 1], [1 : 0])) \cong H$, then we get the isomorphism

$$G/H \longrightarrow SL(2)$$

$$\left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}, t\right) H \longmapsto \begin{pmatrix} xt & yt^{-1} \\ zt & wt^{-1} \end{pmatrix}.$$

Then the affine coordinate ring of $SL(2)$ can be identified with the subring H-invariants in $\mathbb{C}[G]$ which is generated by 4 regular functions:

$$\tilde{x} := xt, \ \ \tilde{y} := yt^{-1}, \ \ \tilde{z} := zt, \ \ \tilde{w} := wt^{-1}$$

and the B-eigenfunctions are all monomials of the form

$$\tilde{z}^r \tilde{w}^s, \quad r, s \in \mathbb{Z}.$$

Let

$$A := \left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}, t \right) H \in G/H, \; C := \left(\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}, \mu \right) \in B,$$

then:

$$CA = \left(\begin{pmatrix} \alpha x + \beta z & \alpha y + \beta w \\ \alpha^{-1} z & \alpha^{-1} w \end{pmatrix}, \mu t \right) H.$$

Images of $\tilde{z}, \; \tilde{w}$ under the B-action are:

$$\tilde{z} = zt \mapsto \alpha^{-1} \tilde{z} \mu, \;\; \tilde{w} = wt^{-1} \mapsto \alpha^{-1} \tilde{w} \mu^{-1}.$$

Therefore

$$\Lambda = \{(-1, 1)\mathbb{Z} + (-1, -1)\mathbb{Z}\} \subseteq \chi(B) \cong \mathbb{Z}^2.$$

Remark 3.3.3. Every valuation υ of $\mathbb{C}(G/H)$ induces a homomorphism as follows

$$\mathbb{C}(G/H)^B \to \mathbb{Q}, \; f \mapsto \upsilon(f).$$

So it induces a lattice element

$$\rho_\upsilon \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Q}).$$

Corollary 3.3.4. The map

$$\mathcal{V} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Q}), \; \upsilon \longmapsto \rho_\upsilon.$$

is injective on the set of G-invariant valuations $\mathcal{V}$ (in general it is not injective). Therefore we identify $\mathcal{V}$ with its image in $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Q})$. We denote by $\Lambda^\vee$ the $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Q})$.

Example 3.3.5. Since $Div(\mathbb{T})^{\mathbb{T}} = \emptyset$ in 3.1.3, then the imag of $\mathcal{V}$ in $\mathrm{Hom}_{\mathbb{Z}}(\chi(\mathbb{T}), \mathbb{Q})$ is the empty set.

Example 3.3.6. We determine the vectors $\rho_{\upsilon_1}, \; \rho_{\upsilon_2}$ in 3.1.5. The B-invariant divisors are defined by the equations

$$D_1 := V(u_2), \; D_2 := V(v_2).$$

The G-invariant divisors are defined by the equations

$$D_1' := V(u_0), \; D_2' := V(u_1 v_2 - v_1 u_2).$$

We define the isomorphism

$$\mathcal{U} := X \setminus (D_1^{'} \cup D_2^{'}) \to SL(2),$$

$$([u_0 : u_1 : u_2], [v_1 : v_2]) \longmapsto \begin{pmatrix} \frac{u_0 v_1}{u_1 v_2 - v_1 u_2} & \frac{u_1}{u_0} \\ \frac{u_0 v_2}{u_1 v_2 - v_1 u_2} & \frac{u_2}{u_0} \end{pmatrix}.$$

The B-eigenfunctions are

$$\tilde{z} := \frac{u_0 v_1}{u_1 v_2 - v_1 u_2}, \ \tilde{w} := \frac{u_1}{u_0}.$$

The function $\tilde{z}$ has a zero of order one along D_1' and a pole of order one along D_2'. The function $\tilde{w}$ has a pole of order one along $D_1^{'}$. Thus

$$\rho_{v_1} = (1, -1), \quad \rho_{v_2} = (-1, 0)$$

and then

$$\Lambda^\vee := \{(1, -1)\mathbb{Z} + (-1, 0)\mathbb{Z}\}.$$

Proposition 3.3.7. *([K91]) Let X be a spherical embedding of the homogeneous space G/H. Then the number of G-orbits in X is finite and each orbit is spherical.*

Corollary 3.3.8. Any embedding X of G/H is covered by finitely many simple open subembeddings.

Remark 3.3.9. Let X be an embedding of G/H and let $Y \subseteq X$ be an orbit. Then define the set

$$Div_Y(X)^B := \{D \in Div(X)^B \ ; \ Y \subseteq D\}.$$

The orbit Y determines the following two sets

$$\mathcal{B}_Y(X) := \{v_D \in \mathcal{V} \ ; \ D \in Div_Y(X)^B \text{ is } G\text{-invariant}\},$$

$$\mathcal{F}_Y(X) := \{D \cap (G/H) \in Div(G/H)^B \ ; \\ D \in Div_Y(X)^B \text{ is not } G\text{-invariant}\}.$$

Theorem 3.3.10. *([K91]) A simple embedding X of a homogeneous space G/H with the closed orbit Y is uniquely determined by the pair $(\mathcal{B}_Y(X), \mathcal{F}_Y(X))$.*

Definition 3.3.11. A subset $\mathcal{C}$ of $\Lambda^\vee$ is called a **cone** if it is closed under addition and multiplication by $\mathbb{Q}^+ := \{q \in \mathbb{Q} \mid q \geq 0\}$. The **dual cone** of $\mathcal{C}$ is defined as follows

$$\mathcal{C}^\vee := \{\alpha \in \Lambda \mid \alpha(v) \geq 0 \ \ \forall \, v \in \mathcal{C}\}.$$

The cone $\mathcal{C}$ is called **strictly convex** if $\mathcal{C} \cap (-\mathcal{C}) = 0$. If there are finitely many elements $v_1, \cdots, v_s$ in $\Lambda^\vee$ such that $\mathcal{C} = \mathbb{Q}^+ v_1 + \cdots + \mathbb{Q}^+ v_s$, then we call $\mathcal{C}$ **finitely generated** cone. A **face** of the cone $\mathcal{C}$ is defined as follows

$$\{v \in \mathcal{C} \mid \alpha(v) = 0 \ ; \ \alpha \in \mathcal{C}^\vee\}.$$

The **dimension** of the cone $\mathcal{C}$ is the dimension of its linear span. A face of dimension one is called an **extremal ray**. The **relative interior** of $\mathcal{C}$ is $\mathcal{C}$ with all proper faces removed and it will be denoted by $\mathcal{C}^\circ$.

Definition 3.3.12. There is a natural map

$$\rho : Div(G/H)^B \to \Lambda^\vee ; \ D \mapsto \rho_{v_D}.$$

which is not injective in general. Define $\mathcal{C}_Y(X) \subseteq \Lambda^\vee$ the cone which is generated by $\rho(\mathcal{F}_Y(X))$ and $\mathcal{B}_Y(X)$.

Definition 3.3.13. A **colored cone** is a pair $(\mathcal{C}, \mathcal{F})$ with $\mathcal{C} \subseteq \Lambda^\vee$ and $\mathcal{F} \subseteq Div(G/H)^B$ where the following properties are satisfied:

(i) The cone $\mathcal{C}$ is generated by all ρ_{v_D} such that $D \in \mathcal{F}$ and by finitely many elements ρ_v such that $v \in \mathcal{V}$.

(ii) There is v of $\mathcal{V}$ such that $\rho_v \in \mathcal{C}^0$.

The elements of the set $\mathcal{F}$ is called **colors** of the cone $\mathcal{C}$ and it may be empty. A colored cone $(\mathcal{C}, \mathcal{F})$ is called **strictly convex** if $\mathcal{C}$ is a strictly convex cone.

Theorem 3.3.14. *([K91]) There is a bijection between isomorphism classes of simple embeddings $G/H \hookrightarrow X$ with the closed orbit Y and strictly convex colored cones $(\mathcal{C}_Y(X), \mathcal{F}_Y(X))$.*

Definition 3.3.15. A **colored fan** is a nonempty finite set Σ of colored cones satisfies the properties:

(i) For every $(\mathcal{C}, \mathcal{F}) \in \Sigma$ and every face $\mathcal{C}'$ in the maximal face $\mathcal{C}^c$ of $\mathcal{C}$, the colored cone $(\mathcal{C}', \emptyset)$ belongs again to Σ.

(ii) For every $v \in \mathcal{V}$ there is at most one colored cone $(\mathcal{C}, \mathcal{F}) \in \Sigma$ such that $v \in \mathcal{C}^\circ$.

A colored fan Σ is called **strictly convex** if all elements $\mathcal{C}$ of Σ are strictly convex.

Theorem 3.3.16. *([K91]) There is a bijection between isomorphism classes of embeddings $G/H \hookrightarrow X$ and strictly convex colored fans $\Sigma(X)$.*

Example 3.3.17. Let us determine the colored fan of the spherical variety $X = \mathbb{P}^2 \times \mathbb{P}^1$ (see 3.1.5). We write X as follows

$$X = SL(2) \sqcup (D_1^{'} \cup D_2^{'}),$$

where $D_1^{'} := V(u_0)$ and $D_2^{'} := V(u_1 v_2 - v_1 u_2)$. It easy to show that $D_1^{'} \cong \mathbb{P}^1 \times \mathbb{P}^1$. Define

$$\Delta := \{(u, v) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid \exists\, g \in G;\ gu = gv\} \cong \mathbb{P}^1 \subset D_1^{'}.$$

The complement of $\mathbb{P}^1$ to $D_1^{'}$ is an 2-dimensional orbit with a stabilizer isomorphic to a maximal torus $\mathbb{T}$ in $SL(2)$. So

$$D_1^{'} \cong \mathbb{P}^1 \sqcup SL(2)/\mathbb{T}.$$

The divisor $D_2^{'}$ intersects $D_1^{'}$ in $\mathbb{P}^1$ and the complement of $\mathbb{P}^1$ to $D_2^{'}$ is isomorphic to $\mathbb{C}^2 \times \mathbb{P}^1$ which is a blow-up in $(0, 0) \in \mathbb{C}^2$. So

$$D_2^{'} \cong \mathbb{P}^1 \sqcup \mathbb{C}^2 \setminus \{0\} \sqcup \mathbb{P}^1.$$

Therefore we write X as

$$X = SL(2) \sqcup SL(2)/\mathbb{T} \sqcup \mathbb{P}^1 \sqcup \mathbb{C}^2 \setminus \{0\} \sqcup \mathbb{P}^1.$$

Since X has two 1-dimensional G-orbits, then the colored fan Σ of X includes two colored cones. Since one of these orbits is $D_1^{'} \cap D_2^{'}$ and the divisors $D_1^{'}$, $D_2^{'}$ are associated to the vectors $v_1 := \rho_{v_1} = (1, -1)$, $v_2 := \rho_{v_2} = (-1, 0)$ respectively, then one of the colored cones σ_1 is generated of these vectors, i.e.,

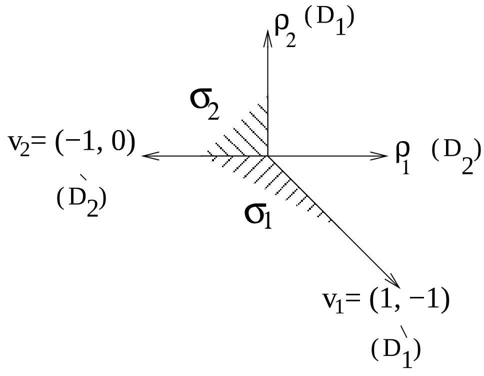
$$\sigma_1 := \mathbb{R}_{\geq 0} v_1 + \mathbb{R}_{\geq 0} v_2.$$

Since the other 1-dimensional G-orbit lies in the B-divisor $D_1 := V(u_2)$ (which is associated to the vector ρ_2), then the other colored cone σ_2 is defined as follows:

$$\sigma_2 := \mathbb{R}_{\geq 0} v_2 + \mathbb{R}_{\geq 0} \rho_2.$$

Therefore the colored fan Σ of X is

$$\Sigma := \{(\sigma_1, \emptyset),\ (\sigma_2, \{\rho_2\})\}.$$



3.4 Birational morphisms between spherical embeddings

Let $G/H,\ G/H^{'}$ be spherical homogeneous spaces and let $\varphi : G/H \longrightarrow G/H^{'}$ be a dominant G-equivariant morphism. This induces the injection

$$\varphi^* : \Lambda \hookrightarrow \Lambda^{'}$$

where $\Lambda := \chi(G/H),\ \Lambda^{'} := \chi(G/H^{'})$. So

$$\varphi_* : \Lambda^{\vee} \twoheadrightarrow \Lambda^{'\vee}.$$

It clear that $\varphi_*(\mathcal{V}(G/H)) = \mathcal{V}(G/H^{'})$. Let $\mathcal{F}_{\varphi}$ be the subset of-$Div(G/H)^B$ which maps dominantly to $G/H^{'}$. Then we define:

Definition 3.4.1. Let $(\mathcal{C}, \mathcal{F}),\ (\mathcal{C}^{'}, \mathcal{F}^{'})$ be the colored cones of G/H, $G/H^{'}$ respectively. Then $(\mathcal{C}, \mathcal{F})$ **maps to** $(\mathcal{C}^{'}, \mathcal{F}^{'})$ if it holds

(i) $\varphi_*(\mathcal{C}) \subseteq \mathcal{C}^{'}$.

(ii) $\varphi_*(\mathcal{F} \setminus \mathcal{F}_{\varphi}) \subseteq \mathcal{F}^{'}$.

If $\Sigma,\ \Sigma^{'}$ be the colored fans of $G/H,\ G/H^{'}$ respectively. Then Σ **maps to** $\Sigma^{'}$ if every element of Σ maps to some element of $\Sigma^{'}$.

Theorem 3.4.2. *([K91]) Let $X,\ X^{'}$ be embeddings of $G/H,\ G/H^{'}$ respectively. Then the morphism $\varphi : G/H \longrightarrow G/H^{'}$ extends to a morphism $X \longrightarrow X^{'}$ if and only if $\Sigma(X)$ maps to $\Sigma(X^{'})$.*

Now we apply the language of spherical varieties to the case of affine $SL(2)$-varieties and corresponding $SL(2)$-filps. Let us consider the $\mathbb{C}^*$-action on hypersurface

$$H_b\ :\ \{X_0^b = X_1X_4 - X_2X_3\} \subset \mathbb{C}^5$$

defined by the diagonal matrices

$$\operatorname{diag}(1, s^{-1}, s^{-1}, s, s), \quad s \in \mathbb{C}^*.$$

We note that this $\mathbb{C}^*$-action commutes with the $SL(2)$-action and with the action of $G = G_0' \times G_a$. So we obtain a natural $\mathbb{C}^*$-action on the categorical quotient $H_b /\!\!/ G \cong E_{h,m}$ which commutes with the $SL(2)$-action. We note that this $\mathbb{C}^*$-action has been already constructed in 2.4.12 using a closed embedding $E_{h,m} \hookrightarrow V$. This allows to consider $E_{h,m}$ as an affine $SL(2) \times \mathbb{C}^*$-variety.

Proposition 3.4.3. *The affine variety $E_{h,m}$ is spherical with respect to the above $SL(2) \times \mathbb{C}^*$-action.*

Proof. The open subset $\mathcal{U} = (H_b \cap \{Y_0 \neq 0\})/G \subset E_{h,m}$ is obviously $SL(2) \times \mathbb{C}^*$-invariant. Since $SL(2) \times \mathbb{C}^*$ acts transitively on $\mathcal{U}$, we have $\mathcal{U} \cong (SL(2) \times \mathbb{C}^*)/H$ for some closed subgroup $H \subset SL(2) \times \mathbb{C}^*$. It is easy to see that

$$(H_b \cap \{Y_0 X_2 X_4 \neq 0\})/G \subset \mathcal{U}$$

is an open dense orbit of the 3-dimensional Borel subgroup $\widetilde{B} := B \times \mathbb{C}^*$ in $SL(2) \times \mathbb{C}^*$. Hence, $E_{h,m}$ is a spherical embedding corresponding to the spherical homogeneous space $(SL(2) \times \mathbb{C}^*)/H$. □

Remark 3.4.4. There exists one more way to define the same $\mathbb{C}^*$-action on $E_{h,m}$. We identify $\mathbb{C}^*$ with the maximal torus $T \subset SL(2)$ which acts on $SL(2)$ by right multiplication. Then this action extends to a regular action on $E_{h,m}$ and commutes with the $SL(2)$-action by left multiplication so that we obtain a regular action of $SL(2) \times T$ on $E_{h,m}$. By [Kr84, III, 4.8], even a more general statement is true: $E_{h,m}$ admits a regular action of $SL(2) \times B$.

If we identify $\mathcal{U}$ with $SL(2)/C_m$ and consider the subgroup $H \subset SL(2) \times \mathbb{C}^*$ as a stabilizer of the class of unit matrix in $SL(2)/C_m$, then

$$H = \{(\operatorname{diag}(t, t^{-1}), t^m) \; : \; t \in \mathbb{C}^*\} \subset SL(2) \times \mathbb{C}^*.$$

The lattice Λ of rational $\widetilde{B}$-eigenfunctions on $\mathcal{U}$ (up to multiplication with a nonzero constant) consists of all Laurent monomials $Z^i W^j \in \mathbb{C}[SL(2)]^{\mu_m}$ such that $m | (i-j)$. Therefore, $E_{h,m}$ is a spherical embedding of rank 2. This rank equals also the minimal codimension of U-orbits in $E_{h,m}$ (we identify U with the maximal unipotent subgroup in $SL(2) \times \mathbb{C}^*$). Let us consider only the case $h < 1$. In order to describe spherical varieties $E_{h,m}$, $E_{h,m}^+$, and $E_{h,m}^-$ by combinatorial data, we remark that they contain exactly three $\widetilde{B}$-invariant divisors:

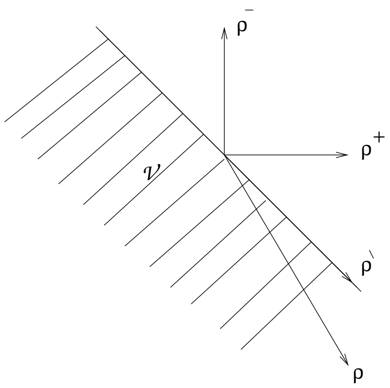
$$D = H_b \cap \{Y_0 = 0\} /\!\!/ G,$$

$$S^+ = H_b \cap \{X_2 = 0\}/\!\!/G, \;\; S^- = H_b \cap \{X_4 = 0\}/\!\!/G.$$

The restrictions of the corresponding descrete valuations $\mathbb{C}(\mathcal{U})^* \to \mathbb{Z}$ to the lattice Λ define lattice vectors $\rho, \rho^+, \rho^- \in \Lambda^*$ in the dual space $\mathcal{Q} := \mathrm{Hom}(\Lambda, \mathbb{Q})$. We can consider ρ^+, ρ^- as a $\mathbb{Q}$-basis of $\mathcal{Q}$. Then the set of all $SL(2) \times \mathbb{C}^*$-invariant valuations generate so called **valuation cone** $\mathcal{V} \subset \mathcal{Q}$, $\mathcal{V} = \{x\rho^+ + y\rho^- \in \mathcal{Q} \;:\; x + y \leq 0\}$. The equations $Z = X_0^p X_2$, $W = X_0^{-q} X_4$ imply

$$\rho = p\rho^+ - q\rho^- \in \mathcal{V}.$$

It is easy to see that $E_{h,m}$, $E^-_{h,m}$, and $E^+_{h,m}$ are simple spherical embeddings (i.e., they contain exactly one closed $SL(2) \times \mathbb{C}^*$-orbit of dimension 1, or 0). Therefore, they can be described by **colored cones** $(\mathcal{C}, \mathcal{F})$, where $\mathcal{F}$ is a subset of $\{\rho^+, \rho^-\}$ and $\mathcal{C} \subset \mathcal{Q}$ is a strictly convex cone generated by $\mathcal{F}$ and ρ.



More precisely we have:

$$\mathcal{C}(E_{h,m}) = \mathbb{Q}_{\geq 0}\rho + \mathbb{Q}_{\geq 0}\rho^-, \;\; \mathcal{F}(E_{h,m}) = \{\rho^+, \rho^-\},$$

$$\mathcal{C}(E^-_{h,m}) = \mathbb{Q}_{\geq 0}\rho + \mathbb{Q}_{\geq 0}\rho^+, \;\; \mathcal{F}(E^-_{h,m}) = \{\rho^+\},$$

$$\mathcal{C}(E^+_{h,m}) = \mathbb{Q}_{\geq 0}\rho + \mathbb{Q}_{\geq 0}\rho^-, \;\; \mathcal{F}(E^+_{h,m}) = \{\rho^-\}.$$

Moreover, the spherical variety $E'_{h,m}$ is also simple. However, $E'_{h,m}$ contains one more $SL(2) \times \mathbb{C}^*$-invariant divisor D' such that the restrictions of the corresponding discrete valuations to Λ defines a lattice vector $\rho' = \rho^+ - \rho^- \in \mathcal{V}$. In this case, we have

$$\mathcal{C}(E'_{h,m}) = \mathbb{Q}_{\geq 0}\rho + \mathbb{Q}_{\geq 0}\rho', \;\; \mathcal{F}(E'_{h,m}) = \emptyset.$$

Remark 3.4.5. We note that birational morphisms $f : W' \to W$ of simple spherical varieties W', W where $f \in \{\varphi^-, \varphi^+, \gamma^-, \gamma^+\}$ has an interpretation in terms of colors. In our situation, we see that the set of colors $\mathcal{F}(W)$ is strictly larger than $\mathcal{F}(W')$. In particular, the birational morphism $\varphi^- : E^-_{h,m} \to E_{h,m}$ combinatorially means that the cone $\mathcal{C}(E^-_{h,m}) = \mathcal{C}(E_{h,m})$ remains unchanged, but it gets an additional color ρ^+: $\mathcal{F}(E_{h,m}) = \mathcal{F}(E^-_{h,m}) \cup \{\rho^+\}$. On the other hand, the birational morphism $\varphi^+ : E^+_{h,m} \to E_{h,m}$ also adds an additional color ρ^-: $\mathcal{F}(E_{h,m}) = \mathcal{F}(E^+_{h,m}) \cup \{\rho^-\}$ such that the color ρ^+ becomes an interior point of $\mathcal{C}(E_{h,m})$. This agree with a general description of Mori contractions in [B93, 3.4, 4.4].

Remark 3.4.6. According to Alexeev and Brion [AB04], every spherical G-variety $\mathcal{X}$ admits a flat degeneration to a toric variety $\mathcal{X}_0$. In general case, there exist several degenerations depending on different reduced decompositions of the longest element w_0 in the Weyl group of the reductive group G. However, in the case $G = SL(2) \times \mathbb{C}^*$ the choice of such a decomposition is unique. A simplest example of such a toric degeneration appears in the case $\mathcal{X} := SL(2)$ considered as a spherical homogeneous space of $SL(2) \times \mathbb{C}^*$. Then $\mathcal{X}_0 = \{X_1X_4 - X_2X_3 = 0\}$ is a singular affine 3-dimensional toric quadric. The corresponding deformation is $\mathcal{X}_0 = \lim_{t\to 0} \mathcal{X}_t$ where $\mathcal{X}_t := \{X_1X_4 - X_2X_3 = t\}$.

Let $T_{h,m}$ be the toric degeneration of $E_{h,m}$. Then

$$T_{h,m} := \operatorname{Spec} \mathbb{C}[\widetilde{M}_{h,m}]$$

where the semigroup

$$\widetilde{M}_{h,m} := \{(i,j,k) \in \mathbb{Z}^3_{\geq 0} \ : \ m|(j-i), \ jp - qi \geq 0, \ i+j \geq k\}$$

has surjective homomorphism $\pi : (i,j,k) \mapsto (i,j)$ onto $M^+_{h,m}$ where elements (i,j) can be identified with the highest vector $X^iY^j \in V_{i+j}$ and the lattice points $\pi^{-1}(i,j) \subset \widetilde{M}_{h,m}$ correspond to the standard basis of V_{i+j}. So the toric degeneration $T_{h,m}$ of $E_{h,m}$ is defined by a 3-dimensional cone

$$\sigma_0 = \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2 + \mathbb{R}_{\geq 0}v_3 + \mathbb{R}_{\geq 0}v_4$$

where $v_1 = (0,0,1)$, $v_2 = (1,1,-1)$, $v_3 = (0,1,0)$, $v_4 = (p,-q,0)$ satisfying the relation

$$pv_1 + pv_2 = (p+q)v_3 + v_4.$$

In the notations of [AB04], the dual 3-dimensional cone $\check{\sigma}_0$ has a surjective projection onto 2-dimensional **momentum cone** $\check{\sigma}$ where $\sigma = \mathcal{C}(E_{h,m}) = \mathbb{R}_{\geq 0}v_3 + \mathbb{R}_{\geq 0}v_4$. The fibers of this projection are 1-dimensional **string polytopes**. Since $p+q \neq 1$, the affine toric variety $T_{h,m}$ does not admit a quasihomogeneous $SL(2)$-action (see also a remark in [Ga08, Section 8]).

Remark 3.4.7. It is not easy to describe the behavior of toric degenerations under equivariant morphisms of spherical varieties. The simplest example in 3.4.6 shows that toric degenerations do not preserve equivariant open embeddings: toric geneneration $\mathcal{U}_0$ of the open orbit $\mathcal{U} \subset E_{h,m}$ is not an open subset in $T_{h,m}$, the corresponding birational morphism $\mathcal{U}_0 \to T_{h,m}$ contracts a divisor in $\mathcal{U}_0$. We remark that if $m = 1$ then $T^+_{h,m}$ locally isomorphic to product $\mathbb{A}^2/\mu_p \times \mathbb{A}^1$. Therefore, toric degeneration $T^+_{h,m}$ of $E^+_{h,m}$ has the same type of toroidal singularity along the curve $C^+_T \subset T^+_{h,m}$ as $C^+ \subset E^+_{h,m}$. However, the same is not true for the toric degeneration $T^-_{h,m}$ of $E^-_{h,m}$. For instance, if $m = 1$ then $T^-_{h,1}$ has only a single isolated singularity, but singular locus of $E^-_{h,1}$ is the whole curve $C^- \subset E^-_{h,1}$.

3.5 Panyushev minimal resolution

Let $E_{h,m}$ be an $SL(2)$-affine variety contains more than two orbits. Then $h < 1$ and $E_{h,m}$ contains a single singular point O. Panyushev has described a minimal $SL(2)$-equivariant resolution of this singular point ([Pa88]).

Definition 3.5.1. A **resoultion of singularities** is a proper birational morphism $\alpha : \widetilde{E}_{h,m} \to E_{h,m}$ such that:

(i) $\widetilde{E}_{h,m}$ is a smooth variety,

(ii) α is an isomorphism over the set of regular points of $E_{h,m}$, i.e., it means α is isomorphism over $E_{h,m}\backslash O$.

The fiber $\alpha^{-1}(O)$ will be called the **singular fiber**.

Theorem 3.5.2. *[Pa91] Let*

$$\lambda(t) := \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in T$$

and let f be a point from the open orbit in $E_{h,m}$ such that $\lim_{t\to 0} \lambda(t)f$ exists (this limit is the unique orbit O in $E_{h,m}$). If we define $Y_{h,m} :=$ $\operatorname{Spec} A_{h,m}$ ($A_{h,m}$ was defined in 2.1.9), then $Y_{h,m} \cong \overline{Bf} \subset E_{h,m}$ for B a Borel subgroup of $SL(2)$. More precisely, the isomorphism is determined by the restriction on $\overline{Bf}$ of the natural morphism $E_{h,m} \to Y_{h,m}$.

We remark that $Y_{h,m}$ is an affine toric variety corresponding to the semigroup algebra $A_{h,m}$ associated with the semigroup

$$M_{h,m} = M' \cap M_{h,1},$$

where

$$M' := \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \ : \ m|(i-j)\} \cong \mathbb{Z} \times \mathbb{Z}.$$

Let $\{e_1, e_2\}$ is the canonical basis of $M := \mathbb{Z}^2$. Then

$$\{w_1 = me_1 = (m,0), \ \ w_2 = e_1 + e_2 = (1,1)\}$$

is a basis of M'. Then

$$qe_1 + pe_2 = (q-p)e_1 + p(e_1 + e_2) = \frac{q-p}{m} w_1 + pw_2$$

is a generator of the 2-dimensional cone:

$$\check{\sigma} = \mathbb{R}_{\geq 0} w_1 + \mathbb{R}_{\geq 0} \left(\frac{q-p}{m} w_1 + pw_2 \right)$$

where $\check{\sigma} \cap M' = M_{h,m}$. The dual cone $\sigma \subset N_{\mathbb{R}}$ of $\check{\sigma}$ is generated by the vectors

$$\tilde{w_2}, \ \ p\tilde{w_1} - \frac{q-p}{m} \tilde{w_2},$$

where $\{\tilde{w_1} = (1/m, -1/m), \ \tilde{w_2} = (0,1)\} \in N_{\mathbb{R}}$ is the dual basis of $\{w_1, w_2\} \in M_{\mathbb{R}}$. This determines the corresponding cone σ of the affine toric variety $X_\sigma = Y_{h,m}$.

Remark 3.5.3. For this choice of the basis of M' the cones $\check{\sigma}$ and σ are determined only by the rational number

$$h' = \frac{pm}{q-p} = \frac{hm}{1-h}.$$

One can easy see that $h' \neq 1$.

Remark 3.5.4. The variety $Y_{h,m}$ is smooth if and only if $\check{\sigma}$ is generated by a $\mathbb{Z}$-basis of M', i.e., if $p = 1$ and $m \mid (q-p)$ and $h' = m/(q-1) = 1/n$ for some $n \in \mathbb{N}$. Then the natural morphism

$$\beta : \ SL(2) *_B Y_{h,m} \to E_{h,m}; \ \ (g * y) \mapsto gy$$

is a resolution of singularities with singular fiber $SL(2)/B = \mathbb{P}^1$. We note that $SL(2) *_B Y_{h,m}$ is isomorphic to the spherical variety $E^+_{h,m}$.

If the variety $Y_{h,m}$ is not smooth then one constructs a B-equivariant resolution of singularities $\gamma : \tilde{Y}_{h,m} \to Y_{h,m}$ by a subdivision of the cone $\sigma \subset N_{\mathbb{R}}$ using n vectors $v_i = (p_i, -q_i)$ $(i = 1, \ldots, n)$. We denote $h_i = p_i/q_i$ $(i = 1, \ldots, n)$. The resolution of singularities of $E_{h,m}$ is a composition of the morphisms:

$$\alpha : \widetilde{E}_{h,m} = SL(2) *_B \tilde{Y}_{h,m} \xrightarrow{\gamma} SL(2) *_B Y_{h,m} \cong E^+_{h,m} \xrightarrow{\beta} E_{h,m}.$$

Remark 3.5.5. The singular fiber $\alpha^{-1}(O)$ is irreducible if and only if $h' = p'_0/q'_0 < 1$ and $q'_0 = 1 \pmod{p'_0}$, or h' is an integer.

The following theorem of Panyushev includes a description of the resolution of the singular fiber $\beta^{-1}(O)$.

Theorem 3.5.6. *[Pa88] For any affine normal quasihomogeneous $SL(2)$-variety $E_{h,m}$, there exists a minimal equivariant resolution of singularities $\beta : Z \to E_{h,m}$, and its singular fiber has the following structure. Let $h' = hm/(1-h) = p'_0/q'_0$, where $p'_0, q'_0 \in \mathbb{N}$. Then:*

(i) If $p'_0 = 1$, then $\beta^{-1}(O) = \mathbb{P}^1$.

(ii) If $p'_0 > 1$, then there exists a unique sequence of rational numbers

$$h' = h'_0 < h'_1 < \cdots < h'_n \leq \infty,$$

where $h'_i = p'_i/q'_i = \frac{h_i m}{1-h_i}$ such that $p'_i > p'_{i+1}$, $p'_n = 1$ and $p'_{i+1}q'_i - p'_i q'_{i+1} = 1$ (it is possible that $q'_n = 0$, i.e., $h'_n = 1/0 = \infty$). Then the singular fiber $\beta^{-1}(O)$ has n irreducible components such that each component is a ruled surface $\mathbb{F}_{e_i}$ (by fibration over $\mathbb{P}^1$ with fiber $\mathbb{P}^1$), moreover

$$e_i = \begin{cases} 2p'_i + mq'_i, & \text{if } h'_i \neq \infty; \\ 0, & \text{if } h'_i = \infty \end{cases}$$

and

$$F_{e_i} \cap F_{e_j} = \begin{cases} \emptyset, & |i-j| > 1; \\ \mathbb{P}^1, & |i-j| = 1. \end{cases}$$

In the last case the intersection $F_{e_i} \cap F_{e_{i+1}}$ is a fiber in each of these fibrations.

Remark 3.5.7. In order to find the numbers $h'_i = \frac{p'_i}{q'_i}$ one uses the following algorithm:

(i) If $h' < 1$, then decompose h' into a continued fraction as follows

$$h' = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} = [a_1, a_2 \ldots, a_s].$$

The number s is called the length of the fraction. Then if the length is even discard a_s and consider $[a_1, \ldots, a_{s-1}]$ but if the length is odd decrease a_s by one and consider $[a_0, a_1, \ldots, a_s - 1]$. Continuing thus until to get the term $[a_1] = \frac{1}{a_1}$. This produces the required ascending sequence of rational numbers h'_i.

(ii) If $h' > 1$, then decompose h' into a continued fraction as

$$h' = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} = [a_0; a_1, a_2, \ldots, a_s].$$

In this case one uses the same procedure but get at the end $h'_n = \frac{1}{0} = \infty$.

Example 3.5.8. Let $h = \frac{30}{217}$ and $m = 1$. Then

$$h' = h_0 = 30/187 = \cfrac{1}{6 + \cfrac{1}{4 + \cfrac{1}{3 + \cfrac{1}{2}}}} = [0; 6, 4, 3, 2].$$

Then

$$h'_1 = [6, 4, 3] = \frac{13}{81},\ h'_2 = [6, 4, 2] = \frac{9}{56},\ h'_3 = [6, 4, 1] = [6, 5] = 5/31$$

and $h'_4 = [6] = \frac{1}{6}$.

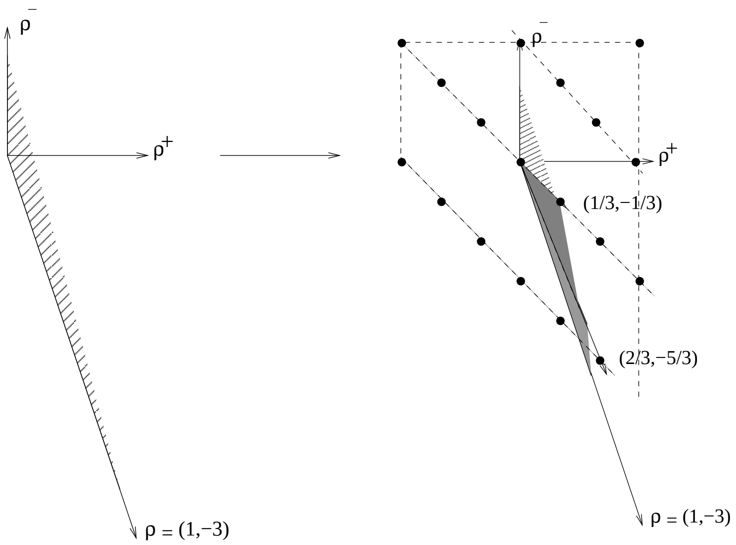
Example 3.5.9. Let $h = \frac{1}{3}$ and $m = 3$. Then:

$$h' = h'_0 = \frac{3}{2} = [1; 2],\ h'_1 = [1; 1] = \frac{2}{1} = 2 \textit{ and } h'_2 = \infty.$$

Therefore we gets the following rational numbers sequence

$$h'_0 = \frac{3}{2} < h'_1 = 2 < h'_2 = \infty.$$

Panyushev resolution is a subdivision of the cone of the toric variety $Y_{1/3,3}$. From the theory of spherical varieties we can consider this subdivision also as a subdivision of the colored cone of the spherical variety $E^+_{1/3,3}$.



The subdivision uses the following vectors

$$v_0 = \rho = (p, -q) := (1, -3), v_1 = (p_1, -q_1) := (2/3, -5/3),$$

$$v_2 = (p_2, -q_2) =: (1/3, -1/3).$$

Then we obtain

$$h_0 = \frac{1}{3},\ h_1 = \frac{2}{5},\ h_2 = 1.$$

Using $h_i' = \frac{h_i m}{1-h_i}$, we get again the numbers

$$h_0' = \frac{3}{2},\ h_1' = 2,\ h_2' = \infty.$$

Chapter 4

$SL(2)$-varieties with $\mathbb{C}^*$-action

4.1 Preliminary statements

A quasihomogeneous algebraic $SL(2)$-variety X over the field $\mathbb{C}$ is a variety with a regular action of the group $SL(2)$ includes an open dense orbit. In this chapter we consider a special class of these varieties, namely $SL(2)$-varieties with $\mathbb{C}^*$-action. It will be shown that these varieties are spherical. This allows to describe $SL(2)$-variety with $\mathbb{C}^*$-action as a quotient of a hypersurface in the affine space modulo an algebraic group.

Definition 4.1.1. A $SL(2)$-variety X with $\mathbb{C}^*$-action is a quasihomogeneous algebraic $SL(2)$-variety with an effective additional $\mathbb{C}^*$-action on X (i.e., $\forall\ t_1 \neq t_2 \in \mathbb{C}^*,\ \exists\ x \in X,\quad t_1x \neq t_2x$). This $\mathbb{C}^*$-action commutes with the $SL(2)$-action. The variety X can be considered as a G-variety, where G is the reductive group $SL(2) \times \mathbb{C}^*$.

First we need some statements about general G-equivariant actions on G/H.

Proposition 4.1.2. *Let G be an arbitrary group and $H \subseteq G$ a subgroup. If $\varphi : G/H \to G/H$ is a map which commutes with the left G-action on G/H, then there exists an element $x \in N_G(H)$ such that*

$$\varphi(gH) = gxH,\ \forall g \in G.$$

The element $x \in N_G(H)$ is uniquely determined modulo H.

Proof. Consider $H = \{e\} \in G/H$ and let $x \in G$ be the element such that $\varphi(H) = xH$. Since φ commutes with the left multiplication by H we obtain

$$xH = \varphi(H) = \varphi(HeH) = H\varphi(eH) = H\varphi(H) = HxH.$$

Thus $xH = Hx$, i.e., $x \in N_G(H)$. □

Corollary 4.1.3. Let Q be a group acts on G/H,

$$Q \times G/H \to G/H$$

where this action commutes with the left G-action on G/H. Then the map $Q \longrightarrow N_G(H)/H$ is a group homomorphism.

Corollary 4.1.4. If $H = \{e\}$ is the trivial subgroup. Then an Q-action on $G/H = G$ which commutes with the left G-action is determined by a homomorphism $Q \to G$.

Corollary 4.1.5. Every automorphism of the variety $SL(2)$ commutes with left $SL(2)$-action is uniquely determined by the image of the unit matrix E_2 of $SL(2)$.

Corollary 4.1.6. Since $Aut_{SL(2)}(SL(2)) \cong SL(2)$, then an effective $\mathbb{C}^*$-action on $SL(2)$ acts as a maximal torus $\mathbb{T}$ of $SL(2)$, i.e.,

$$\mathbb{C}^* \to Aut_{SL(2)}(SL(2)) \cong SL(2).$$

Proposition 4.1.7. *Let X be a $SL(2)$-variety with $\mathbb{C}^*$-action then the open dense $SL(2)$-orbit in X is isomorphic to $SL(2)/\mathcal{C}_m$, where $\mathcal{C}_m$ is a cyclic group of a finite order m.*

Proof. Since X is a $SL(2)$-variety, then the open dense $SL(2)$-orbit in X is isomorphic to $SL(2)/\Gamma$, where Γ is a finite subgroup of $SL(2)$. Since $\mathbb{C}^*$ acts on X, then $\mathbb{C}^*$ acts also on $SL(2)/\Gamma$. By 4.1.3 one has the homomorphism

$$\mathbb{C}^* \longrightarrow N_{SL(2)}(\Gamma)/\Gamma.$$

Since $\mathbb{C}^*$ is connected, we have the homomorphism

$$\mathbb{C}^* \longrightarrow N_{SL(2)}(\Gamma)^\circ$$

where the connected group $N_{SL(2)}(\Gamma)^\circ$ is either U or B or T.

(i) If $N_{SL(2)}(\Gamma)^\circ = U$, then we obtain an embedding $\mathbb{C}^* \hookrightarrow U$ which is impossible, because U is unipotent.

(ii) If $N_{SL(2)}(\Gamma)^\circ = B$, then B is connected component of $N_{SL(2)}(\Gamma)$. It is well-known that $N_{SL(2)}(B) = B$ and $N_{SL(2)}(\Gamma)^\circ$ is normal in $N_{SL(2)}(\Gamma)$. So we get $N_{SL(2)}(\Gamma) = B$. Since $\Gamma \cap U = \{e\}$, then we get the following commutative diagram

$$\begin{tikzcd} N_{SL(2)}(\Gamma) = B \arrow[r, two heads] & \mathbb{T} \cong \mathbb{C}^* \\ \Gamma \arrow[u, hook] \arrow[ur, hook] & \end{tikzcd}$$

It follows that Γ is a finite subgroup of $\mathbb{C}^*$, i.e., Γ is cyclic.

(iii) If $N_{SL(2)}(\Gamma)^\circ = T$, then

$$\Gamma \subset N_{SL(2)}(T) = T \cup \left\{ \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix} \right\}.$$

If $\Gamma \subset T \cong \mathbb{C}^*$, then Γ is cyclic. Otherwise there is an element $\begin{pmatrix} 0 & s \\ -s^{-1} & 0 \end{pmatrix} \in \Gamma$ such that:

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 & s \\ -s^{-1} & 0 \end{pmatrix} \cdot \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 0 & t^2 s \\ (t^{-1})^2 s^{-1} & 0 \end{pmatrix} \in \Gamma$$

for all $t \in \mathbb{C}^*$. This contradicts that Γ is finite. Therefore Γ is isomorphic to some cyclic group $\mathcal{C}_m$ of order m.

□

Lemma 4.1.8. *Let X be a $SL(2)$-embedding with $\mathbb{C}^*$-action. Let $G := SL(2) \times \mathbb{C}^*$, then X is a spherical G-variety.*

Proof. From the Corollary 4.1.6 follows that the G-action on X is isomorphic to a $SL(2) \times \mathbb{T}$-action. Since the open orbit of X is isomorphic to $SL(2)$, we examine the G-action on $SL(2)$. Let $(g,t) \in G$ and $h \in SL(2)$. Since the $SL(2)$-action commutes with $\mathbb{T}$-action, then

$$((g,t),h) \mapsto gth = ght.$$

Since all Borel subgroups of $SL(2)$ are conjugate and their intersection equals to $\{\pm E_2\}$, then there is a Borel subgroup B of $SL(2)$, which does not include maximal torus $\mathbb{T}$ of $SL(2)$. Thus there is an element $h \in SL(2)$, where $Bh\mathbb{T}$ is of 3-dimensional. Therefore X has an open dense B-orbit and thus X is spherical. □

Proposition 4.1.9. *Let X be a $SL(2)$-variety with $\mathbb{C}^*$-action. Then there exists a $SL(2)$-embedding Y with $\mathbb{C}^*$-action together with a canonical $SL(2) \times \mathbb{C}^*$-equivariant finite surjective morphism $Y \to X$ of spherical $SL(2) \times \mathbb{C}^*$-varieties.*

Proof. Since X is a $SL(2)$-variety with $\mathbb{C}^*$ action, then the open dense $SL(2)$-orbit in X is isomorphic to $SL(2)/\mathcal{C}_m$, where $\mathcal{C}_m$ is a cyclic subgroup of $SL(2)$ of the order m (see 4.1.7). The cyclic group $\mathcal{C}_m$ is contained in some maximal torus $\mathbb{T} \cong \mathbb{C}^* \subset SL(2)$. Thus we obtain a canonical $SL(2) \times \mathbb{C}^*$-equivariant finite surjective morphism

$$SL(2) \longrightarrow SL(2)/\mathcal{C}_m.$$

The action of $\mathbb{T} \cong \mathbb{C}^*$ on $SL(2)/\mathcal{C}_m$ is well defined as

$$\{g\mathcal{C}_m\} \mapsto \{gt\mathcal{C}_m\}, \;\; t \in \mathbb{T}, \;\; g \in SL(2),$$

because the commutative group $\mathbb{T}$ contains $\mathcal{C}_m$. In particular, by 4.1.8, $SL(2)/\mathcal{C}_m$ is a spherical $SL(2) \times \mathbb{C}^*$-variety. Then we define Y to be the integral closure of $SL(2)$ over X. More precisely, if $X = \bigcup_{i=1}^k U_i$ is a union of affine varieties U_i. Then $Y = \bigcup_{i=1}^k U_i'$, where the coordinate ring $K[U_i']$ consists of all rational functions on $SL(2)$ which are integral over the coordinate ring $K[U_i]$. So that we have the following commutative diagram

$$\begin{tikzcd} Y \arrow[r, two heads] & X \\ SL(2) \arrow[u, hook] \arrow[r, two heads] & SL(2)/\mathcal{C}_m \arrow[u, hook] \end{tikzcd}$$

with $SL(2) \times \mathbb{C}^*$-equivariant finite surjective morphism $Y \twoheadrightarrow X$. $\square$

4.2 2-dimensional colored fans

Let us describe the combinatorial data defining the spherical variety X. We start with the spherical homogeneous variety G/H ($G = SL(2) \times \mathbb{C}^*$, $H \subset SL(2) \times \mathbb{C}^*$) and identify it with $SL(2)/\mathcal{C}_m$, where the cyclic group

$$\mathcal{C}_m = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} ; \;\; \lambda^m = 1 \right\}$$

is contained in the maximal torus

$$\mathbb{T} = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \;\; t \in \mathbb{C}^* \right\}.$$

Now we have a regular action of $SL(2) \times \mathbb{T}$ on $SL(2)/\mathcal{C}_m$:

$$(A, t) \times \{g\mathcal{C}_m\} \mapsto \{Agt\mathcal{C}_m\}.$$

However the action of the torus $\mathbb{T}$ is not effective, because elements of the cyclic subgroup $\mathcal{C}_m \subset \mathbb{T}$ act trivially on $SL(2)/\mathcal{C}_m$. So we consider another 1-dimensional torus $\mathbb{T}' := \mathbb{T}/\mathcal{C}_m$ and the corresponding action of $SL(2) \times \mathbb{T}'$ on $SL(2)/\mathcal{C}_m$. The surjective homomorphism $\mathbb{T} \to \mathbb{T}'$ is defined by the homomorphism $t \mapsto t^m$, ($t \in \mathbb{C}^*$). Fix now a Borel subroup of $SL(2)$ as

$$\tilde{B} := \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}^* \right\},$$

then the group $B := \tilde{B} \times \mathbb{C}^*$ is a Borel subgroup of $G \cong SL(2) \times \mathbb{C}^*$ of dimension 3.

Let us first consider the case $m = 1$. Then $SL(2)$ acts on its self by left multiplication and $\mathbb{T}$ atcs on $SL(2)$ by right multiplication. The orbit of $E_2 \in SL(2)$ with respect to $G = SL(2) \times \mathbb{T}$-action defines the surjective morphism

$$G = SL(2) \times \mathbb{T} \longrightarrow SL(2),$$

$$\left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}, \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) \longmapsto \begin{pmatrix} xt & yt^{-1} \\ zt & wt^{-1} \end{pmatrix}.$$

The stabilizer $\mathrm{st}_G(E_2)$ is the subgroup

$$H = \left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}, \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right), \quad t \in \mathbb{C}^*.$$

So we get the isomorphism

$$G/H \longrightarrow SL(2).$$

Then the affine coordinate ring of $SL(2)$ can be identified with the subring H-invariants in $\mathbb{C}[G]$ which is generated by 4 regular functions:

$$\tilde{x} := xt, \tilde{y} := yt^{-1}, \tilde{z} := zt, \tilde{w} := wt^{-1}.$$

Consider the B-action on $SL(2)$:

$$\begin{pmatrix} \tilde{x} & \tilde{y} \\ \tilde{z} & \tilde{w} \end{pmatrix} \longmapsto \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} \tilde{x} & \tilde{y} \\ \tilde{z} & \tilde{w} \end{pmatrix} \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha\mu\tilde{x} + \beta\mu\tilde{z} & \alpha\mu^{-1}\tilde{y} + \beta\mu^{-1}\tilde{w} \\ \alpha^{-1}\mu\tilde{z} & \alpha^{-1}\mu^{-1}\tilde{w} \end{pmatrix}.$$

So the B-eigenfunctions are all monomials of the form

$$\tilde{z}^r \tilde{w}^s, \quad r, s \in \mathbb{Z}$$

and we obtain the lattice

$$\Lambda = \{(-1, 1)\mathbb{Z} + (-1, -1)\mathbb{Z}\} \subseteq \chi(B)$$

with the canonical basis:

$$e_1 = \alpha^{-1}\mu^{-1}, \; e_2 = \alpha^{-1}\mu$$

which defines two colors in the dual lattice.

In the case of arbitrary m we have the surjective morphism

$$SL(2) \twoheadrightarrow SL(2)/\mathcal{C}_m$$

of degree m. So we get the sublattice $\Lambda_m \subset \Lambda$ corresponding to characters $\alpha^k\mu^l$ of B such that m divides l, i.e., lattice vectors $ie_1 + je_2 = (-i-j, i-j)$ such that $m|(i-j)$. Thus we have

$$\Lambda_m = \{ie_1 + je_2 \in \Lambda \ : \ m|(i-j)\} \subseteq \Lambda.$$

Definition 4.2.1. Let $N = \mathbb{Z}^2$ be the dual lattice of Λ. We denote by ρ_1, ρ_2 the dual basis of N corresponding to the basis $e_1, e_2 \in \Lambda$. For any positive integer m, define the **dual lattice** of Λ_m:

$$N_m := \{\mathbb{Z}^2 + \mathbb{Z}(\frac{1}{m}, -\frac{1}{m})\}$$

with a $\mathbb{Z}$-basis:

$$\{(1,0), (\frac{1}{m}, -\frac{1}{m})\}.$$

Then we have $(N_m)_{\mathbb{R}} = \mathbb{R}\rho_1 \oplus \mathbb{R}\rho_2$, where $(N_m)_{\mathbb{R}} := N_m \otimes_{\mathbb{Z}} \mathbb{R}$ the real vector space corresponding to N_m.

Remark 4.2.2. Let X be $SL(2)$-variety with a $\mathbb{C}^*$-action. Then the set of colors consists exactly of two B-stable divisors

$$\mathcal{F}(X) := \{\rho_1, \rho_2\}.$$

Lemma 4.2.3. *Let $W \subset \mathbb{C}[G/H]$ be an irreducible G-invariant subspace and let $v : \mathbb{C}(G/H) \to \mathbb{Z}$ be a G-invariant valuation of the field of rational functions on G/H. Then v is constant on $W \setminus \{0\}$.*

Proof. Let $f, f' \in W \setminus \{0\}$ be two arbitrary nonzero elements of W. Then for any elements $g_i \in G_i$ and $\lambda_i \in \mathbb{C}$ one has

$$v(\sum_i \lambda_i g_i(f)) \geq \min_i v(\lambda_i g_i(f)) = v(f).$$

Since W is irreducible, the set of linear combinations $\sum_i \lambda_i g_i(f)$ coincides with W. In particular, we obtain $v(f') \geq v(f)$. By applying the same arguments to f', we obtain $v(f) \geq v(f')$. $\square$

Lemma 4.2.4. *Let $v : \mathbb{C}(SL(2)) \to \mathbb{Z}$ be a $SL(2) \times \mathbb{C}^*$-invariant valuation. We put $n_1 := v(z)$, $n_2 := v(w)$. Then*

$$n_1 + n_2 \leq 0.$$

Proof. Let W_1 (resp. W_2) the subspace in $\mathbb{C}[SL(2)]$ generated by x and z (resp. by y and w). Then W_i is G-invariant and irreducible. By 4.2.3, we have

$$n_1 = v(x) = v(z), \quad n_2 = v(y) = v(w).$$

This implies

$$0 = v(1) = v(xw - yz) \geq \min\{v(xw), v(yz)\} = n_1 + n_2.$$

□

Proposition 4.2.5. *The set of G-invarianten valuations of $\mathbb{C}(SL(2))$ coming from some G-invariant compactification of $SL(2)$ one-to-one corresponds to the set of primitive lattice vectors $(n_1, n_2) \in \mathbb{Z}^2$ such that $n_1 + n_2 \leq 0$.*

Proof. By 4.2.4, it remains to show that for any primitive lattice vector $n = (n_1, n_2) \in \mathbb{Z}^2$ with the condition $n_1 + n_2 \leq 0$ there exists a compactification X of $SL(2)$ and a divisor $D \subset X$ which defines a G-invariant valuation v_D such that $n_1 = v_D(z)$ and $n_2 = v_D(w)$. We define X as a good quotient of the hypersurface

$$u^{-n_1-n_2} = xw - yz$$

in $\mathbb{C}^5$ with respect to the $\mathbb{C}^*$-action

$$t(x, z, y, w, u) = (t^{-n_1}x, t^{-n_1}z, t^{-n_2}y, t^{-n_2}z, tu), \quad t \in \mathbb{C}^*.$$

Without loss of generality we can assume that $n_1 < 0$. Then we distinguish 3 cases:

CASE 1: $n_2 < 0$. Then X is a hypersurface in the weighted projective space

$$\mathbb{P}(-n_1, -n_1, -n_2, -n_2, 1) = (\mathbb{C}^5 \setminus \{0\})/\mathbb{C}^*.$$

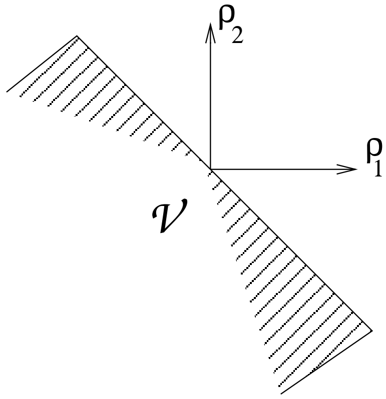
CASE 2: $n_2 = 0$, $n_1 = -1$. Then $X \cong \mathbb{P}^1 \times \mathbb{C}^2$ together with the divisor $\mathbb{F}_1 \subset X$.

CASE 3: $n_2 > 0$. Then X can be choosen as the affine variety. □

Definition 4.2.6. Let X be a spherical $SL(2)$-variety. The subset of $(N_m)_{\mathbb{R}}$ defined as follows:

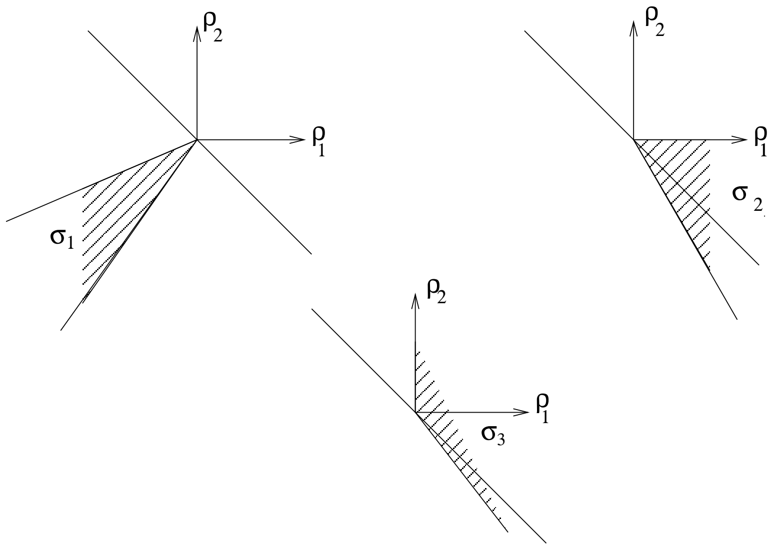
$$\mathcal{V} := \{(n_1, n_2) \in (N_m)_{\mathbb{R}}; \;\; n_1 + n_2 \leq 0\} \subset (N_m)_{\mathbb{R}}$$

is called the **valuation cone** of X.



Corollary 4.2.7. A colored cone σ in a colored fan $\Sigma(X)$ of a spherical $SL(2)$-variety X is a cone of one of these forms

$$(\sigma_1, \mathcal{F}_1 = \emptyset),\ (\sigma_2, \mathcal{F}_2 = \{\rho_1\}),\ (\sigma_3, \mathcal{F}_3 = \{\rho_1\}),\ (\sigma_3, \mathcal{F}_3 = \{\rho_1,\ \rho_2\}).$$



4.3 Quotient constructions

Definition 4.3.1. Let $\Sigma(X)$ be a strictly convex colored fan of a spherical $SL(2)$-variety X. We consider the lattice $N \subseteq N_m$ as above. Let $\{v_1, \dots, v_r\} \subset \mathcal{V}$ be the set of all N-**primitive generators** of 1-dimensional colored cones $(\sigma, \mathcal{F})$ in $\Sigma(X)$ such that $\mathcal{F} = \emptyset$. By

$\{\tilde{v}_1, \ldots, \tilde{v}_r\}$ we denote the set of all N_m-**primitive generators** of 1-dimensional colored cones $(\sigma, \mathcal{F})$ in $\Sigma(X)$. We have

$$v_i = k_i \tilde{v}_i, \ \ k_i \in \mathbb{Z}_{>0}, \ k_i | m \ (i = 1, \ldots, r).$$

We write the N-primitive lattice generators as

$$v_i := (-p_i, -q_i) \in N, \ \ i = 1, \ldots, r$$

and consider $r + 4$ variables $x_1, x_2, x_3, \ x_4, y_1, \ldots, y_r$ as coordinates on the affine space $\mathbb{C}^{r+4}$ together with a bijection between the lattice vectors $v_1, \ldots, v_r$ and the variables $y_1, \ldots, y_r$. Let us denote by $Y(\Sigma)$ the hypersurface in $\mathbb{C}^{r+4}$ defined by the equation

$$x_1 x_4 - x_2 x_3 = y_1^{p_1+q_1} y_2^{p_2+q_2} \ldots . y_r^{p_r+q_r}.$$

The N_m-primitive lattice generators $\tilde{v}_i = (1/k_i) v_i$ $(i = 1, \ldots, k)$ have rational coordinates

$$\tilde{v}_i = (-\tilde{p}_i, -\tilde{q}_i) = (\frac{-p_i}{k_i}, \frac{-q_i}{k_i}) \in N_m.$$

We define another hypersurface $\tilde{Y}(\Sigma) \subset \mathbb{C}^{r+4}$ by the equation

$$x_1 x_4 - x_2 x_3 = y_1^{\tilde{p}_1+\tilde{q}_1} y_2^{\tilde{p}_2+\tilde{q}_2} \ldots . y_r^{\tilde{p}_r+\tilde{q}_r}.$$

Remark 4.3.2. The sum $\tilde{p}_i + \tilde{q}_i \in \mathbb{Z}_{\geq 0}$, because if we write $\tilde{v}_i$ as a linear combination of the base vectors of the lattice N_m as

$$\tilde{v}_i = (-\tilde{p}_i, -\tilde{q}_i) = c(1,0) + b(\frac{1}{m}, -\frac{1}{m}) = (c + \frac{b}{m}, -\frac{b}{m}); \ c, b \in \mathbb{Z}.$$

Then

$$\tilde{p}_i + \tilde{q}_i = -c - \frac{b}{m} + \frac{b}{m} = -c \in \mathbb{Z}$$

In particular, $k_i | (p_i + q_i)$. We show that

$$k_i = \gcd(p_i + q_i, m).$$

We have seen that k_i is a common divisor of $p_i + q_i$ and m. Let l be another common divisor of $p_i + q_i$ and m. Then we write

$$\frac{1}{l} v_i = \frac{1}{l}(-p_i, -q_i) = \frac{-p_i - q_i}{l}(1,0) + \frac{q_i}{l}(1,-1) \in N_m.$$

Therefore $l | k_i$. The equality $k_i = \gcd(p_i + q_i, m)$ together with $\gcd(p_i, q_i) = 1$ implies that

$$\gcd(p_i, k_i) = \gcd(q_i, k_i) = 1.$$

We explain two constructions of the 3-dimensional $SL(2)$-variety $X(\Sigma)$ as quotients of two hypersurfaces $Y(\Sigma)$ and $\tilde{Y}(\Sigma)$:

$$X(\Sigma) := U(\Sigma) /\!\!/ G(\Sigma)$$

and

$$X(\Sigma) := \tilde{U}(\Sigma) /\!\!/ \tilde{G}(\Sigma)$$

where $U(\Sigma)$ (resp.$\tilde{U}(\Sigma)$) is an open subset of the hypersurface $Y(\Sigma) \subset \mathbb{C}^{r+4}$ (resp. $\tilde{Y}(\Sigma) \subset \mathbb{C}^{r+4}$) and $G(\Sigma) \cong (\mathbb{C}^*)^r \times \mu_m$ (resp. $\tilde{G}(\Sigma) \cong (\mathbb{C}^*)^r \times \mu_a$)). The order a of the cyclic group μ_a is defined as follows:

$$a := \gcd(a_1, \dots, a_r), \ \ a_i := \frac{m}{k_i}, \ (i = 1, \dots, r).$$

Definition 4.3.3. We define the following algebraic subsets of the affine space $\mathbb{C}^{r+4}$:

$$W_1 := \{x_1 = x_2 = 0\}, \ \ W_2 := \{x_3 = x_4 = 0\}.$$

and define for every $i \in \{1, \dots, r\}$ the set:

$$V_i := \{y_i = 0\} \subset \mathbb{C}^{r+4}.$$

If $(\sigma, \mathcal{F}) \in \Sigma(X)$ is a colored cone then we set

$$U(\sigma, \mathcal{F}) := \mathbb{C}^{r+4} \setminus \left(\left(\bigcup_{v_i \notin \sigma} V_i \right) \cup \left(\bigcup_{\rho_j \notin \mathcal{F}} W_j \right) \right)$$

and define

$$U(\Sigma) := Y(\Sigma) \cap \left(\bigcup_{(\sigma,\mathcal{F}) \in \Sigma} U(\sigma, \mathcal{F}) \right),$$

$$\tilde{U}(\Sigma) := \tilde{Y}(\Sigma) \cap \left(\bigcup_{(\sigma,\mathcal{F}) \in \Sigma} U(\sigma, \mathcal{F}) \right).$$

First quotient construction:

Definition 4.3.4. Let

$$G_0 := (\mathbb{C}^*)^r = \{\mathbf{t} = (t_1, \dots, t_r), \ t_i \in \mathbb{C}^*\}.$$

We define a linear action of the group

$$G(\Sigma) := G_0 \times \mu_m$$

on $(\mathbb{C}^*)^{r+4}$ as follows: The group G_0 acts on the coordinates $x_1, \dots, x_4$, $y_1, \dots y_r$ of the affine space $\mathbb{C}^{r+4}$ by the formulas:

$$\mathbf{t}(x_1) = \left(\prod_{i=1}^{r} t_i^{p_i}\right) x_1, \quad \mathbf{t}(x_2) = \left(\prod_{i=1}^{r} t_i^{p_i}\right) x_2,$$

$$\mathbf{t}(x_3) = \left(\prod_{i=1}^{r} t_i^{q_i}\right) x_3, \quad \mathbf{t}(x_4) = \left(\prod_{i=1}^{r} t_i^{q_i}\right) x_4,$$

and

$$\mathbf{t}(y_i) = t_i y_i, \quad i = 1, \dots, r.$$

The cyclic group $\mu_m = \langle \zeta_m \rangle$ acts on the coordinates of $\mathbb{C}^{r+4}$ by the formula

$$(x_1, \dots, x_4, y_1, \dots, y_r) \mapsto (\zeta_m^{-1} x_1, \zeta_m^{-1} x_2, \zeta_m x_3, \zeta_m x_4, y_1, \dots, y_r).$$

Remark 4.3.5. We identify the affine coordinates $x_1, \cdots, x_4$ with the coefficients of the 2×2-matrix

$$\begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}$$

and act with $SL(2)$ by the left multiplication on these coordinates.

Theorem 4.3.6. *Given a colored fan $\Sigma \subset N_m$. Then the categorical quotient $U(\Sigma)/\!\!/G(\Sigma)$ is isomorphic to a quasihomogeneous spherical $SL(2)$-variety $X(\Sigma)$, i.e., $X(\Sigma)$ is a $SL(2)$-variety with additional $\mathbb{C}^*$-action.*

Proof. CASE 1: $m = 1$. In this case μ_1 acts trivialy on the affine coordinates $x_1, \cdots, x_4, y_1, \cdots, y_r$. Since the hypersurface $Y(\Sigma)$ and the set $\left(\bigcup_{(\sigma,\mathcal{F})\in\Sigma} U(\sigma, \mathcal{F})\right)$ are invariant under the $G_0 \times SL(2)$-action, then the intersection $U(\Sigma)$ is also $G_0 \times SL(2)$-invariant. Morover, the $G_0 \times SL(2)$-stabilizer in the point $x = (1, 0, 0, 1, \cdots, 1) \in U(\Sigma)$ is trivial. So the $G_0 \times SL(2)$-orbit of x in $\mathbb{C}^{r+4}$ is Zariski dense in $U(\Sigma)$ and its quotient $\mathcal{O}$ modulo G_0 is isomorphic to $SL(2)$, i.e., $U(\Sigma)/\!/G_0$ is a $SL(2)$-embedding. One can identify the open dense $SL(2)$-orbit $\mathcal{O}$ in $U(\Sigma)$ by the condition $y_1 \cdots y_r \neq 0$. Moreover, the coordinate ring $\mathbb{C}[\mathcal{O}]$ is generated by the G_0-invariant monomials

$$X := y_1^{-p_1} \cdots y_r^{-p_r} x_1, \quad Y := y_1^{-q_1} \cdots y_r^{-q_r} x_3,$$

$$Z := y_1^{-p_1} \cdots y_r^{-p_r} x_2, \quad W := y_1^{-q_1} \cdots y_r^{-q_r} x_4$$

satisfying the equation

$$\det \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = y_1^{-p_1-q_1} \cdots y_r^{-p_r-q_r} x_1 x_4 - y_1^{-p_1-q_1} \cdots y_r^{-p_r-q_r} x_2 x_3$$

$$= y_1^{-p_1-q_1} \cdots y_r^{-p_r-q_r} (x_1 x_4 - x_2 x_3) = 1.$$

We remark that the group $\mathbb{C}^*$ acts on $\mathbb{C}^{r+4}$ by

$$(x_1, \ldots, x_4, y_1, \ldots, y_r) \mapsto (\lambda_m^{-1} x_1, \lambda_m^{-1} x_2, \lambda_m x_3, \lambda_m x_4, y_1, \ldots, y_r)$$

commutes with the $G_0 \times SL(2)$-action and $U(\Sigma)$ is $\mathbb{C}^*$-invariant. Thus we obtain a $SL(2) \times \mathbb{C}^*$-action on $U(\Sigma)//G_0$. In order to identify $U(\Sigma)//G_0$ with $X(\Sigma)$ we remark that $U(\Sigma)//G_0$ is covered by open subsets

$$(Y(\Sigma) \cap U(\sigma, \mathcal{F})) //G_0$$

and every quotient $U(\sigma, \mathcal{F})//G_0$ is a simple spherical variety corresponding to the colored cone $(\sigma, \mathcal{F})$. This can be checked case by case for every type of colored cone as follows. For example consider a cone $(\sigma, \mathcal{F} = \emptyset)$ without colors generated by v_k and v_{k+1}. Then we have an open subset

$$U(\sigma, \emptyset) = \mathbb{C}^{r+4} \setminus \left(\left(\bigcup_{i \neq k, k+1} V_i \right) \cup (W_1 \cup W_2) \right).$$

We write G_0 as the product of two tori $G_\sigma \times G'_\sigma$:

$$G_\sigma := \{\mathbf{t} \in G_0 \ : \ t_i = 1 \ \ \forall i \neq k, k+1\}$$

$$G'_\sigma := \{\mathbf{t} \in G_0 \ : \ t_k = t_{k+1} = 1\}.$$

Then the quotient $U(\sigma, \emptyset)//G_0$ is isomorphic to the quotient of

$$\mathbb{C}^6 \setminus (W_1 \cup W_2)$$

modulo G_σ, because the quotient by G'_σ eliminates $r-2$ coordinates y_i $(i \neq k, k+1)$. The last quotient is easy to compute explicitly as a 4-dimensional toric variety. The 3-dimensional variety

$$(Y(\Sigma) \cap U(\sigma, \mathcal{F})) //G_0$$

is a hypersurface defined by the equation

$$x_1 x_4 - x_2 x_3 = y_k^{p_k + q_k} y_{k+1}^{p_{k+1} + q_{k+1}}$$

in this 4-dimensional toric variety. If σ is a colored cone generated by a lattice vector v_k and a color ρ_1, then

$$U(\sigma, \mathcal{F}) = \mathbb{C}^{r+4} \setminus \left(W_2 \cup \bigcup_{i \neq k} V_i \right)$$

and we can split G_0 as the product of two tori $G_\sigma \times G'_\sigma$:

$$G_\sigma := \{ \mathbf{t} \in G_0 \ : \ t_i = 1 \ \ \forall i \neq k \}$$

$$G'_\sigma := \{ \mathbf{t} \in G_0 \ : \ t_k = 1 \}.$$

The quotient $U(\sigma, \mathcal{F})//G_0$ is isomorphic to a 4-dimensional toric variety obtained as quotient of

$$\mathbb{C}^5 \setminus W_2$$

modulo G_σ, because the quotient by G'_σ eliminates $r-1$ coordinates y_i $(i \neq k)$. The 3-dimensional variety $(Y(\Sigma) \cap U(\sigma, \mathcal{F}))\, //G_0$ is a hypersurface defined by the equation

$$x_1 x_4 - x_2 x_3 = y_k^{p_k + q_k}$$

in this 4-dimensional toric variety.

Case 2: $m > 1$. In this case μ_m acts on $x_1, x_2, x_3, x_4, y_1, \ldots, y_r$ by

$$\mathrm{diag}(\zeta_m^{-1}, \zeta_m^{-1}, \zeta_m, \zeta_m, 1, \ldots, 1).$$

Since $U(\Sigma)$ is invariant under the μ_m-action and the point

$$x = (1, 0, 0, 1, \cdots, 1) \in U(\Sigma)$$

has a cyclic group $\mathcal{C}_m$ of order m as a G-stabilizer, then $U(\Sigma)//G$ is a $SL(2)/\mathcal{C}_m$-embedding. By the same argument as above, $U(\Sigma)//G(\Sigma)$ has a $SL(2) \times \mathbb{C}^*$-action. The quotient $U(\Sigma)//G(\Sigma)$, in this case, is covered by open subsets

$$(Y(\Sigma) \cap U(\sigma, \mathcal{F}))\, //G_0 \times \mu_m$$

and every quotient $U(\sigma, \mathcal{F})//G_0 \times \mu_m$ is a simple spherical variety corresponding to the colored cone $(\sigma, \mathcal{F})$ as above. As an example the case of a cone $(\sigma, \mathcal{F} = \emptyset)$ without colors generated by v_k and v_{k+1} one spilt $G(\Sigma)$, as above, as the product of the two tori $G_\sigma \times G'_\sigma$, because μ_m acts trivialy on the coordinates $y_1, \ldots, y_r$. By the same argument the quotient $U(\sigma, \emptyset)//G(\Sigma)$ is isomorphic to the quotient of

$$\mathbb{C}^6 \setminus (W_1 \cup W_2)$$

modulo G_σ, which is 4-dimensional toric variety and the 3-dimensional variety

$$(Y(\Sigma) \cap U(\sigma, \mathcal{F})) \,//G_0$$

is a hypersurface defined by the equation

$$x_1x_4 - x_2x_3 = y_k^{p_k+q_k} y_{k+1}^{p_{k+1}+q_{k+1}}$$

in this 4-dimensional toric variety. Other cases can be dealed the same. □

Second quotient construction: We preserve all the previous notations.

Definition 4.3.7. We define a linear action of the group

$$\tilde{G}(\Sigma) := G_0 \times \mu_a$$

on $(\mathbb{C}^*)^{r+4}$ such that the group G_0 acts on the coordinates $x_1, \ldots, x_4$, $y_1, \ldots y_r$ of the affine space $\mathbb{C}^{r+4}$ by:

$$\mathbf{t}(x_1) = \left(\prod_{i=1}^{r} t_i^{p_i}\right) x_1, \quad \mathbf{t}(x_2) = \left(\prod_{i=1}^{r} t_i^{p_i}\right) x_2,$$

$$\mathbf{t}(x_3) = \left(\prod_{i=1}^{r} t_i^{q_i}\right) x_3, \quad \mathbf{t}(x_4) = \left(\prod_{i=1}^{r} t_i^{q_i}\right) x_4,$$

$$\mathbf{t}(y_i) = t_i^{k_i} y_i, \quad i = 1, \ldots, r$$

and the cyclic group μ_a acts on $(x_1, \ldots, x_4, y_1, \ldots, y_r)$ by

$$\mathrm{diag}(\zeta_a^{-1}, \zeta_a^{-1}, \zeta_a, \zeta_a, 1, \ldots, 1).$$

Theorem 4.3.8. *The categorical quotient $\tilde{U}(\Sigma)/\!\!/\tilde{G}(\Sigma)$ is isomorphic to a quasihomogeneous spherical $SL(2)$-variety $X(\Sigma)$, which also defined through the categorical quotient $U(\Sigma)/\!\!/G(\Sigma)$.*

Proof. By Theorem 4.3.6, $U(\Sigma)/\!\!/G(\Sigma)$ is isomorphic to a quasihomogeneous spherical $SL(2)$-variety $X(\Sigma)$. As in the affine case, we split the quotient in two steps. Let $G_m \subset D(r+4, \mathbb{C})$ be the cyclic group of order m generated by

$$g := (\zeta_m, \zeta_m, \zeta_m^{-1}, \zeta_m^{-1}, 1, \ldots, 1),$$

where ζ_m is a primitive m-th root of unity. Then $\zeta_m^{a_i}$ generates a cyclic group μ_{k_i} of order k_i (we use the equality $a_ik_i = m$). Since $\gcd(p_i, k_i) =$

$\gcd(q_i, k_i) = 1$ and $k_i | p_i + q_i$ there exists another generator ξ_i of μ_{k_i} such that

$$\zeta_m^{a_i} \xi_i^{p_i} = \zeta_m^{-a_i} \xi_i^{q_i} = 1.$$

Consider the element

$$g_i := (1, 1, 1, 1, \ldots, \underbrace{\xi_i}_{i+4}, \ldots) =$$

$$= (\xi_i^{p_i}, \xi_i^{p_i}, \xi_i^{q_i}, \xi_i^{q_i}, \ldots, \underbrace{\xi_i}_{i+4}, \ldots) \cdot (\zeta_m^{a_i}, \zeta_m^{a_i}, \zeta_m^{-a_i}, \zeta_m^{-a_i}, 1 \ldots, 1) \in G_0 \times G_m.$$

Let us define the r-dimensional torus

$$G_0' := \{\mathrm{diag}(\prod_{i=1}^{r} t_i^{p_i}, \prod_{i=1}^{r} t_i^{p_i}, \prod_{i=1}^{r} t_i^{q_i}, \prod_{i=1}^{r} t_i^{q_i}, t_1^{k_1}, \ldots, t_r^{k_r});\ t_i \in \mathbb{C}^*\}.$$

Then consider the homomorphism

$$\psi\ :\ D(r+4, \mathbb{C}) \to D(r+4, \mathbb{C}),$$

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \alpha_1, \ldots, \alpha_r) \mapsto (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \alpha_1^{k_1}, \ldots, \alpha_r^{k_r}).$$

Then $\psi(G_0) = G_0'$. However, the elements $g_1, \ldots, g_r \in G_0 \times G_m$ belong to $\mathrm{Ker}\, \psi$ and they generate a finite subgroup in $D(r+4, \mathbb{C})$ of order $k_1 k_2 \cdots k_r$. On the other hand, $\mathrm{Ker}\, \psi \subset D(r+4, \mathbb{C})$ consists of all elements

$$\{\mathbf{t} := (1, 1, 1, 1, t_1, \ldots, t_r)\ :\ t_i^{k_i} = 1, \ \ i = 1, \ldots, r\}.$$

Therefore $\mathrm{Ker}\, \psi = \mathrm{Ker}\, \psi|_{G_0 \times G_m}$ and we denote this group by G'. So we have the short exact sequence

$$1 \to G' \to G_0 \times G_m \to \psi(G_0 \times G_m) \to 1,$$

Define

$$k := \mathrm{lcm}\,(k_1, \ldots, k_r).$$

Then $ak = m$. We show that $\psi(G_0 \times G_m) \cong G_0' \times \mu_a$. This follows from the computation of the surjective homomorphism

$$\overline{\psi}\ :\ G_m \cong (G_0 \times G_m)/G_0 \to \psi(G_0 \times G_m)/G_0'.$$

Since $g_i g^{-a_i} \in G_0$ and $\psi(g_i) = 1$, then we obtain that $\overline{\psi}(g^{a_i}) = 1$ in $\psi(G_0 \times G_m)/G_0'$. The elements $g^{a_1}, \ldots, g^{a_r}$ generate a cyclic subgroup of order k which is generated by g^a. Moreover, we have $G_m/\langle g^a \rangle \cong \mu_a$.

It remains to show $\operatorname{Ker}\overline{\psi} = \langle g^a \rangle$, i.e., the inclusion $\operatorname{Ker}\overline{\psi} \subseteq \langle g^a \rangle$. Let $g^l \in \operatorname{Ker}\overline{\psi}$. Then $\psi(g^l) \in G_0'$. So we have

$$g^l := (\zeta_m^l, \zeta_m^l, \zeta_m^{-l}, \zeta_m^{-l}, 1, \dots, 1) = (\prod_{i=1}^r t_i^{p_i}, \prod_{i=1}^r t_i^{p_i}, \prod_{i=1}^r t_i^{q_i}, \prod_{i=1}^r t_i^{q_i}, t_1^{k_1}, \dots, t_r^{k_r})$$

for some $t_i \in \mathbb{C}^*$. This implies that $t_i^{k_i} = 1$ for $1 \leq i \leq r$. In particular, $g^l \in \operatorname{Ker}\psi$. On the other hand, $\operatorname{Ker}\psi$ is generated by $g_1, \dots, g_r$. So we can write

$$g^l = \prod_{i=1}^r g_i^{l_i}.$$

We write $g_i = g^{a_i} h_i$ for some elements $h_i \in G_0$. So we get

$$g^l = \prod_{i=1}^r g^{a_i l_i} h_i^{l_i}.$$

We see that the image of g^l in $(G_0 \times G_m)/G_0 \cong G_m$ is contained in the subgroup $\langle g^a \rangle$ generated by $g^{a_1}, \dots, g^{a_r}$. Therefore

$$\psi(G_0 \times G_m)/G_0' = (\psi(G_0) \times \psi(G_m))/G_0' = (G_0' \times \psi(G_m))/G_0' = \psi(G_m),$$

$$\psi(G_0 \times G_m)/G_0' = \overline{\psi}(G_m) \cong G_m/\langle g^a \rangle \cong \mu_a.$$

i.e., $\psi(G_m) \cong \mu_a$. Thus we get the short exact sequence

$$1 \to G' \to G_0 \times G_m \to G_0' \times \mu_a \to 1,$$

Therefore the categorical $G_0 \times G_m$-quotient of $U(\Sigma)$ can be divided in two steps. First we divide $U(\Sigma)$ by the subgroup $G' \subset G_0 \times G_m$ and after that we divide by the group $G_0' \times \mu_a$. The first quotient is simple, because one uses new G'-invariant coordinates $y_i' = y_i^{k_i}$ $(1 \leq i \leq r)$. The hypersurface $Y(\Sigma)/\!\!/G'$ is isomorphic to the hypersurface $\tilde{Y}(\Sigma)$. Since G_0 acts on y_i by character $t \to t^{k_i}$,

$$X(\Sigma) = U(\Sigma)/\!\!/(G_0 \times G_m)$$

is isomorphic to the categorical quotient of $\tilde{U}(\Sigma)$ modulo the above $G_0' \times \mu_a$-action. $\square$

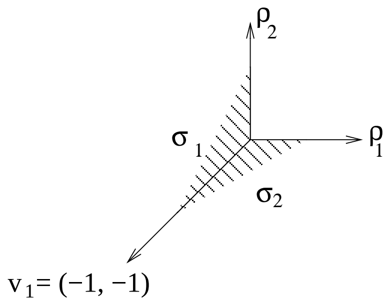
4.4 Examples

Example 4.4.1. (*Quadric in* $\mathbb{P}^4$). Consider the vector $v_1 := (-1, -1) \in N_1 = \mathbb{Z}^2$ and the colored cones:

$$\sigma_1 := \mathbb{R}_{\geq 0}\rho_2 + \mathbb{R}_{\geq 0}v_1, \ \sigma_2 := \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}\rho_1.$$

Then define the following strictly convex colored fan:

$$\Sigma := \{(\sigma_1, \rho_2), (\sigma_2, \rho_1)\}.$$



The hypersurface $Y(\Sigma)$ in $\mathbb{C}^5$ is defined by the equation

$$x_1x_4 - x_2x_3 - y_1^2 = 0.$$

The algebraic torus $G(\Sigma) \cong \mathbb{C}^*$ acts on $\mathbb{C}^5$ as follows:

$$\mathbb{C}^* \times \mathbb{C}^5 \longrightarrow \mathbb{C}^5,$$

$$(t, (x_1, x_2, x_3, x_4, y_1)) \longmapsto (x_1t, x_2t, x_3t, x_4t, y_1t),$$

and $SL(2)$ acts on $\mathbb{C}^5$ as this:

$$SL(2) \times \mathbb{C}^5 \longrightarrow \mathbb{C}^5,$$

$$\left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}, (x_1, x_2, x_3, x_4, y_1)\right) \longmapsto (xx_1 + yx_3, xx_2 + yx_4,$$

$$zx_1 + wx_3, zx_2 + wx_4, y_1).$$

It is clear that $SL(2)$-action commutes with the $\mathbb{C}^*$-action. Then

$$U(\sigma_1, \rho_2) := \mathbb{C}^5 \setminus \{x_1 = x_2 = 0\},$$

$$U(\sigma_2, \rho_1) := \mathbb{C}^5 \setminus \{x_3 = x_4 = 0\}.$$

Thus:

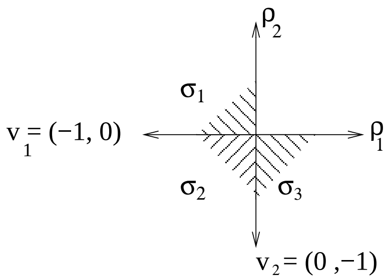
$$U(\Sigma) := Y(\Sigma) \cap (\{x_1 = x_2 = 0\} \cup \{x_3 = x_4 = 0\}).$$

The quotient $U(\Sigma)/\!\!/\mathbb{C}^*$ is the projective quadric in $\mathbb{P}^4$.

Example 4.4.2. (*The flag variety $SL(3)/B \subset \mathbb{P}^2 \times \mathbb{P}^2$*). Consider the vectors $v_1 := (-1, 0), v_1 := (0, -1) \in \mathbb{Z}^2$ and the colored cones:

$$\sigma_1 := \mathbb{R}_{\geq 0}\rho_2 + \mathbb{R}_{\geq 0}v_1,\ \sigma_2 := \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2,\ \sigma_3 := \mathbb{R}_{\geq 0}v_2 + \mathbb{R}_{\geq 0}\rho_1.$$

Let Σ be the colored fan generated by the cones $(\sigma_1, \rho_2), (\sigma_2, \emptyset), (\sigma_3, \rho_1)$.



The hypersurface $Y(\Sigma) \subset \mathbb{C}^6$ is defined by the equation:

$$x_1x_4 - x_2x_3 - y_1y_2 = 0.$$

Define the $(\mathbb{C}^*)^2$-action μ on $\mathbb{C}^6$ by

$$\mu\left((t_1, t_2), (x_1, \cdots, y_2)\right) = (t_1x_1, t_1x_2, t_2x_3, t_2x_4, t_1y_1, t_2y_2).$$

We consider the left multiplication $SL(2)$-action on x_1, x_2, x_3, x_4 and the trivial action on y_1, y_2. We have:

$$U(\sigma_1, \rho_2) := \mathbb{C}^5 \setminus \left(\{x_1 = x_2 = 0\} \cup \{y_2 = 0\}\right),$$

$$U(\sigma_2, \emptyset) := \mathbb{C}^5 \setminus \left(\{x_1 = x_2 = 0\} \cup \{x_3 = x_4 = 0\}\right),$$

$$U(\sigma_3, \rho_1) := \mathbb{C}^5 \setminus \left(\{x_3 = x_4 = 0\} \cup \{y_1 = 0\}\right).$$

Then $U(\Sigma)$ is the complement in $Y(\Sigma)$ of the closed set:

$$\{x_1 = x_2 = y_1 = 0\} \cup \{x_3 = x_4 = y_2 = 0\}.$$

Consider two copies of $\mathbb{P}^2$ with the homogeneous coordinates x_1, x_2, y_2 and x_3, x_4, y_1. Then the quotient:

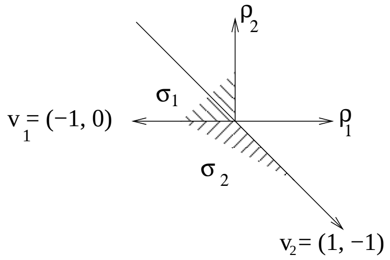
$$X(\Sigma) = U(\Sigma)/(\mathbb{C}^*)^2$$

is the quadric hypersurface in the product $\mathbb{P}^2 \times \mathbb{P}^2$, i.e., the flag variety $SL(3)/B$.

Example 4.4.3. (*The Toric variety* $X(\Sigma) := \mathbb{P}^1 \times \mathbb{P}^2$). Take $v_1 = (-1,0), v_2 = (1,-1) \in \mathbb{Z}^2$ and let

$$\sigma_1 := \mathbb{R}_{\geq 0}\rho_2 + \mathbb{R}_{\geq 0}v_1, \quad \sigma_2 := \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2$$

be two colored cones. Then define the fan $\Sigma := \{(\sigma_1, \rho_2),\ (\sigma_2, \emptyset)\}$.



The hypersurface $Y(\Sigma) \subset \mathbb{C}^6$ is defined by the equation

$$x_1x_4 - x_2x_3 = y_1.$$

The group $G(\Sigma) = (\mathbb{C}^*)^2$ acts on $(x_1, x_2, x_3, x_4, y_1, y_2)$ by

$$\mathrm{diag}\,(t_1t_2^{-1}, t_1t_2^{-1}, t_2, t_2, t_1, t_2).$$

We have:

$$U(\sigma_1, \rho_2) := \mathbb{C}^6 \setminus (\{x_1 = x_2 = 0\} \cup \{y_2 = 0\}),$$
$$U(\sigma_2, \emptyset) := \mathbb{C}^6 \setminus (\{x_1 = x_2 = 0\} \cup \{x_3 = x_4 = 0\}).$$

Since the hypersurface $Y(\Sigma)$ is isomorphic to the affine space $\mathbb{C}^5$ with the coordinates x_1, x_2, x_3, x_4, y_2, then the open subset $U(\Sigma) \subset Y(\Sigma)$ is the complement to

$$\{x_3 = x_4 = y_2 = 0\} \cup \{x_1 = x_2 = 0\}.$$

Thus the quotient $X(\Sigma) = U(\Sigma)/\!\!/(\mathbb{C}^*)^2$ is isomorphic to the projective toric variety $\mathbb{P}^1 \times \mathbb{P}^2$.

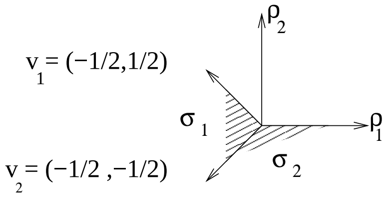
Example 4.4.4. Take the lattice $N_2 := \{(1,0)\mathbb{Z} + (-\frac{1}{2}, -\frac{1}{2})\mathbb{Z}\}$. Then consider in N_2 the vectors

$$\tilde{v}_1 = (-\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}(-1,1) = \frac{1}{2}v_1,$$
$$\tilde{v}_2 = (-\frac{1}{2}, -\frac{1}{2}) = \frac{1}{2}(-1,-1) = \frac{1}{2}v_2$$

and the colored cones

$$\sigma_1 := \mathbb{R}_{\geq 0}\tilde{v}_1 + \mathbb{R}_{\geq 0}\tilde{v}_2, \quad \sigma_2 := \mathbb{R}_{\geq 0}\tilde{v}_2 + \mathbb{R}_{\geq 0}\rho_1.$$

Define the colored fan Σ generated by $(\sigma_1, \emptyset),\ (\sigma_2, \rho_1)$. For the simplicity we denote $\tilde{v}_1, \tilde{v}_2$ in the next Figure by v_1, v_2 respectively.



We apply the second quotient construction. Then $m = k_1 = k_2 = 2$ and $a = 1$. The equation $x_1x_4 - x_2x_3 = y_2$ is defined the hypersurface $\tilde{Y}(\Sigma) \subset \mathbb{C}^6$, which is isomorphic to $\mathbb{C}^5$ with the coordinates x_1, x_2, x_3, x_4, y_1. The algebraic torus $G'_0 \cong (\mathbb{C}^*)^2$ acts on the variables x_1, x_2, x_3, x_4, y_1 by

$$\operatorname{diag}\,(t_1t_2, t_1t_2, t_1^{-1}t_2, t_1^{-1}t_2, t_1^2).$$

We have two open subsets in $\mathbb{C}^6$:

$$U(\sigma_1, \emptyset) := \mathbb{C}^6 \setminus (\{x_1 = x_2 = 0\} \cup \{x_3 = x_4 = 0\}),$$

$$U(\sigma_2, \rho_1) := \mathbb{C}^6 \setminus (\{x_3 = x_4 = 0\} \cup \{y_1 = 0\}).$$

Then

$$U(\sigma_1, \emptyset) \cup U(\sigma_2, \rho_1) = \mathbb{C}^6 \setminus (\{x_1 = x_2 = y_1 = 0\} \cup \{x_3 = x_4 = 0\}).$$

The open subset $\tilde{U}(\Sigma) \subset \tilde{Y}(\Sigma)$ is the complement to $\mathbb{C}^5$ of the set

$$\{x_1 = x_2 = y_1 = 0\} \cup \{x_3 = x_4 = 0\}$$

and the variety $X(\Sigma) = \tilde{U}(\Sigma)/\!\!/\tilde{G}(\Sigma)$ is a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ which is a toric variety obtained from $\mathbb{P}^3$ by blow up of a $SL(2) \times \mathbb{C}^*$-invariant line in $\mathbb{P}^3$.

4.5 The Cox ring

Definition 4.5.1. Define the lattice

$$N' := \mathbb{Z}^2 + \sum_{i=1}^{r} \mathbb{Z}\tilde{v}_i.$$

Then it is clear that $\mathbb{Z}^2 \subset N' \subset N_m$ and $\mid N_m/\mathbb{Z}^2 \mid = m$.

Proposition 4.5.2. *$\mid N_m/N' \mid = a$.*

Proof. We have the short exact sequence:

$$0 \to N' \to N_m \to N'/N_m \to 0.$$

By the isomorphism theorem, we get

$$N_m/N' \cong (N_m/\mathbb{Z}^2)/(N'/\mathbb{Z}^2).$$

Therefore it remains to compute the cyclic subgroup $N'/\mathbb{Z}^2$ in the cyclic group $N_m/\mathbb{Z}^2 \cong \mathbb{Z}/m\mathbb{Z}$. The subgroup $N'/\mathbb{Z}^2 \subseteq N_m/\mathbb{Z}^2$ is generated by the elements $x_i := \tilde{v}_i + \mathbb{Z}^2 \in N_m/\mathbb{Z}^2$ $(i = 1, \dots r)$. By the definition, x_i has order k_i in $N_m/\mathbb{Z}^2$. Therefore, $x_1, \dots, x_k$ generate the cyclic subgroup of order $k := \mathrm{lcm}\,(k_1, \dots, k_r)$. So we get $|N'/\mathbb{Z}^2| = k$ and $|N_m/N'| = m/k = a$. □

Proposition 4.5.3. *The coordinate ring $\mathbb{C}[\tilde{Y}(\Sigma)]$ of the hypersurface $\tilde{Y}(\Sigma) \subset \mathbb{C}^{4+r}$ is factorial.*

Proof. Consider the open subset $U_2 \subset \tilde{Y}(\Sigma)$ defined by $x_2 \neq 0$. Since U_2 is isomorphic to a Zariski open subset in $\mathbb{C}^{3+r}$, we obtain $\mathrm{Cl}(U_2) = 0$. The complement $\widetilde{S} := \tilde{Y}(\Sigma) \backslash U_2$ is a principle divisor (x_2). We note that the divisor $\widetilde{S}$ defined by the binomial equation $x_1 x_4 = y_1^{\tilde{p}_1 + \tilde{q}_1} \cdots y_r^{\tilde{p}_r + \tilde{q}_r}$ which shows that $\widetilde{S}$ is isomorphic to the product of $\mathbb{C}$ (with the coordinate x_3) and a $r+1$-dimensional affine toric variety with the equation $x_1 x_4 = y_1^{\tilde{p}_1 + \tilde{q}_1} \cdots y_r^{\tilde{p}_r + \tilde{q}_r}$. Therefore, $\widetilde{S}$ is irreducible and the short exact localization sequence

$$\begin{array}{ccccccc} \mathbb{Z} & \to & \mathrm{Cl}(\tilde{Y}(\Sigma)) & \to & \mathrm{Cl}(U_2) & \to & 0 \\ 1 & \mapsto & [\widetilde{S}] & & & & \end{array}$$

shows that $[\widetilde{S}] = 0 \in \mathrm{Cl}(\tilde{Y}(\Sigma))$, i.e., the image of $\mathbb{Z}$ in $\mathrm{Cl}(\tilde{Y}(\Sigma))$ is zero. Thus, we obtain $\mathrm{Cl}(\tilde{Y}(\Sigma)) = 0$ and $\mathbb{C}[\tilde{Y}(\Sigma)]$ is factorial. □

Proposition 4.5.4. *The divisor class group of $X(\Sigma)$ is isomorphic to*

$$\mathbb{Z}^r \oplus \mathbb{Z}/a\mathbb{Z}.$$

Proof. We identify the lattice Λ_m with the lattice of B-eigenfunctions. Every element $e \in \Lambda_m$ defines a principal B-invariant divisor on $X(\Sigma)$. We prove the following short exact sequence:

$$0 \to \Lambda_m \to \mathbb{Z}^{r+2} \to \mathrm{Cl}(X(\Sigma)) \to 0$$

where the map $\Lambda_m \to \mathbb{Z}^{r+2}$ sends an element $e \in \Lambda_m$ to

$$(\langle e, \rho_1 \rangle, \langle e, \rho_2 \rangle, \langle e, v_1 \rangle, \dots, \langle e, v_r \rangle).$$

We apply $\mathrm{Hom}(*, \mathbb{Z})$ to the above exact sequence and get

$$0 \to \mathrm{Hom}(\mathrm{Cl}(X(\Sigma)), \mathbb{Z}) \to \mathbb{Z}^{r+2} \to N_m \to \mathrm{Ext}^1(\mathrm{Cl}(X(\Sigma)), \mathbb{Z})) \to 0,$$

where $\mathrm{Ext}^1(\mathrm{Cl}(X(\Sigma)), \mathbb{Z})) \cong N_m/N' \cong \mathbb{Z}/a\mathbb{Z}$ is isomorphic to the torsion in $\mathrm{Cl}(X(\Sigma))$. This implies that

$$\mathrm{Cl}(X(\Sigma)) \cong \mathbb{Z}^r \oplus \mathbb{Z}/a\mathbb{Z}.$$

□

Proposition 4.5.5. *Consider the following Zariski open subset in $Y(\Sigma)$:*

$$U := Y(\Sigma) \setminus (W_1 \cup W_2 \cup \bigcup_{i<j} (V_i \cap V_j)).$$

Then the group $(G_0 \times G_m)/G' \cong G_0' \times \mu_a$ acts freely on $U/G' \subset \tilde{Y}(\Sigma)$.

Proof. Let $x = (x_1, x_2, x_3, x_4, y_1, \ldots, y_r)$ be a point in U and $g \in G = G_0 \times G_m$ an element such that $gx = x$. We write g as a product of two elements

$$\mathrm{diag}\,(\prod_{i=1}^r t_i^{p_i}, \prod_{i=1}^r t_i^{p_i}, \prod_{i=1}^r t_i^{q_i}, \prod_{i=1}^r t_i^{q_i}, t_1, \ldots, t_r),\ t_i \in \mathbb{C}^*,$$

and

$$\mathrm{diag}\,(\zeta^s, \zeta^s, \zeta^{-s}, \zeta^{-s}, 1, \ldots, 1),\ \zeta \in \mu_m.$$

Then $\prod_{i=1}^r t_i^{p_i} \zeta^s = 1$ (because at least one of x_1 and x_2 is nonzero), and $\prod_{i=1}^r t_i^{q_i} \zeta^{-s} = 1$ (because at least one of x_3 and x_4 is nonzero). Therefore, $\prod_{i=1}^r t_i^{p_i} \zeta^s \prod_{i=1}^r t_i^{q_i} \zeta^{-s} = \prod_{i=1}^r t_i^{p_i+q_i} = 1$. Moreover, if $y_i \neq 0$ for some i then $t_i = 1$. It follows from the definition of U that at most one of the coordinates $y_1, \ldots, y_r$ equals 0. We consider two cases.

Case 1: All coordinates $y_1, \ldots, y_r$ are nonzero. Thus $t_1 = \cdots = t_r = 1$. Then $\zeta^s = 1$ and $g = 1$.

Case 2: Only one coordinate $y_j = 0$ and $y_i \neq 0$ for $i \neq j$. Then we have $t_i = 1$ for all $i \neq j$. Then we have

$$t_j^{p_j} \zeta^s = t_j^{q_j} \zeta^{-s} = 1 = t_j^{p_j+q_j}.$$

Therefore $t_j \in \mu_{p_j+q_j}$. Since $\gcd(p_j,\ p_j + q_j) = 1$ the element $t_j^{p_j}$ has the same order as t_j. On the other hand, $t_j^{p_j} = \zeta^{-s} \in \mu_m$. So the order of t_j is a common divisor of $p_j + q_j$ and m. In particular, we have $t_j^{k_j} = 1$. This implies that $g \in G'$. □

Corollary 4.5.6. For arbitrary spherical quasihomogeneous $SL(2)$-variety $X(\Sigma)$, the Cox ring is isomorphic to the affine coordinate ring of $\tilde{Y}(\Sigma)$

$$\mathbb{C}[x_1, x_2, x_3, x_4, y_1, \ldots, y_k]/(x_1x_4 - x_2x_3 - \prod_{i=1}^{k} y_i^{\tilde{p}_i+\tilde{q}_i}),$$

i.e., it is defined by a single equation.

Proof. We apply 2.3.1 and use the statements 4.5.3, 4.5.4 and 4.5.5. $\square$

Chapter 5

Minimal models of spherical $SL(2)$-varieties

5.1 Luna-Vust diagrams of $SL(2)$-varieties

In this section we represent the combinatorial description of the normal embeddings of a homogeneous space G/H in the case; $G = SL(2)$ over $\mathbb{C}$ and $H = \mathcal{C}_m$ is a cyclic group of order m ([Ja87]).

Definition 5.1.1. An **embedding** of the homogeneous space $SL(2)/\mathcal{C}_m$ is a reduced irreducible normal algebraic $SL(2)$-variety X with an equivariant open injective morphism $i : SL(2)/\mathcal{C}_m \hookrightarrow X$.

A normal embedding is characterized by the local rings of its orbits. Therefore to find all normal embeddings of $SL(2)/\mathcal{C}_m$ one has to find the set of possible local rings of orbits and then to find which of such local rings can be combined to form a variety.

Let $\mathbb{C}[SL(2)]$ be the ring of regular functions on $SL(2)$ and $\mathbb{C}(SL(2))$ be its quotient field. There is an action of $SL(2)$ (resp. $\mathcal{C}_m$) on $\mathbb{C}(SL(2))$ induced by left (resp. right) translation. Let $\mathbb{C}(SL(2))^{\mathcal{C}_m}$ be the subfield of $\mathbb{C}(SL(2))$ of invariants by right translation by $\mathcal{C}_m$. We define:

$$\mathcal{V}(SL(2)/\mathcal{C}_m) = \{\text{discrete normalized geometric valuations of } \mathbb{C}(SL(2))^{\mathcal{C}_m} \text{ over } \mathbb{C} \text{ stable by } SL(2)\},$$

where a **geometric** valuation here is a valuation such that its valuation ring is a localization of an algebra of finite type. Define also:

$$\mathcal{V}_1(SL(2)/\mathcal{C}_m) = \{v \in \mathcal{V}(SL(2)/\mathcal{C}_m) \mid {}^{SL(2)}\mathbb{C}_v \cong \mathbb{C}\},$$

where $\mathbb{C}_v$ is the residue field of v and ${}^{SL(2)}\mathbb{C}_v$ is the subfield of $SL(2)$-invariants.

Fix a Borel subgroup B of $SL(2)$. Then we denote:

$$^B\mathcal{D}(SL(2)/\mathcal{C}_m) = \{\text{irreducible divisors of } SL(2)/\mathcal{C}_m \text{ stable by } B\}.$$

Remark 5.1.2. Since B is of codimension one in $SL(2)$, $^B\mathcal{D}(SL(2)/\mathcal{C}_m)$ is the set of B-orbits in $SL(2)/\mathcal{C}_m$. If $\mathcal{C}_m = \{e\}$, then $^B\mathcal{D}(SL(2)) \cong B \setminus SL(2) \cong \mathbb{P}^1$ (For the simplicity we write $\mathbb{P}^1$ instead of $\mathbb{P}^1(\mathbb{C})$). Therefore in general we identify $^B\mathcal{D}(SL(2)/\mathcal{C}_m)$ with $\mathbb{P}^1/\mathcal{C}_m$.

Let $D \in^B \mathcal{D}(SL(2)/\mathcal{C}_m) \cong \mathbb{P}^1/\mathcal{C}_m$ and denote by $\widetilde{D}$ the inverse image of D by the canonical morphism:

$$SL(2) \to SL(2)/\mathcal{C}_m.$$

For each $D \in^B \mathcal{D}(SL(2)/\mathcal{C}_m)$, there is some $g_D \in \mathbb{C}[SL(2)]$ such that g_D generates the ideal of functions of $\mathbb{C}[SL(2)]$, which vanish on $\widetilde{D}$. (Since the divisor class group of $SL(2)$ is trivial, then there is such g_D)

For such D, we denote

$$a(D) := \text{the number of irreducible components of } \widetilde{D},$$

$$m(D) := \frac{m}{a(D)};\ m := \text{card}\, \mathcal{C}_m,$$

$$f_D := g_D^{m(D)} \in \mathbb{C}[SL(2)]^{\mathcal{C}_m},$$

$$r(D) := \frac{2}{a(D)} - 1.$$

Any valuation $v \in \mathcal{V}(SL(2)/\mathcal{C}_m)$ is determined by its values $\{v(f_D)\}$, where $D \in \mathbb{P}^1/\mathcal{C}_m$. We normalize the elements of $\mathcal{V}(SL(2)/\mathcal{C}_m)$ such that their minimal value is (-1). The following proposition classifies the set $\mathcal{V}(SL(2)/\mathcal{C}_m)$ as follows:

Proposition 5.1.3. *([Ja87])*

(i) *Given $D \in \mathbb{P}^1/\mathcal{C}_m$ and an $r \in (-1, r(D)] \cap \mathbb{Q}$, then there exists a unique valuation $v(D,r) \in \mathcal{V}(SL(2)/\mathcal{C}_m)$ defined by*

$$v(D,r)(f_{D_0}) = \begin{cases} r, & \text{if } D = D_0; \\ -1, & \text{if } D \neq D_0. \end{cases}$$

(ii) $\mathcal{V}_1(SL(2)/\mathcal{C}_m) = \{v(D,r) \mid D \in \mathbb{P}^1/\mathcal{C}_m,\ r \in (-1, r(D)] \cap \mathbb{Q})\}$,

(iii) *$\mathcal{V}(SL(2)/\mathcal{C}_m) - \mathcal{V}_1(SL(2)/\mathcal{C}_m)$ consists of one element $v(\ ,-1)$ such that $v(\ ,-1)(f_D) = -1$ for all $D \in \mathbb{P}^1/\mathcal{C}_m$.*

Remark 5.1.4. There are two fixed elements $D_1,\ D_2$ of $\mathbb{P}^1$ and if m is odd (resp. even), then each other orbit is of order m (resp. $\frac{m}{2}$). Therefore for each $D \subset \mathbb{P}^1/\mathcal{C}_m$ either $\widetilde{D} = D_1$ or D_2, in this case $a(D) = r(D) = 1$ or $\widetilde{D}$ consists of m elements (resp. of $\frac{m}{2}$), in this case $a(D) = m$ and $r(D) = \frac{2}{m} - 1$ (resp. $a(D) = \frac{m}{2}$ and $r(D) = \frac{4}{m} - 1$).

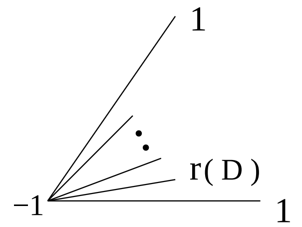
One can draw a diagram called a **Luna-Vust diagram** of the set of $\mathcal{V}(SL(2)/\mathcal{C}_m)$ which is based on the set

$$\mathcal{H} := \{[-1, r(D)] \cap \mathbb{Q}\} \times \mathbb{P}^1/\mathcal{C}_m \sim$$

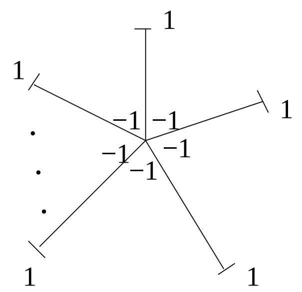
of rational intervals $[-1, r(D)] \cap \mathbb{Q}$ parametrized by complex points of $\mathbb{P}^1/\mathcal{C}_m$ and glued together at the point $\{-1\} \in \{[-1, r(D)] \cap \mathbb{Q}\}$,

$$\{-1\} \times x \sim \{-1\} \times y \quad \forall x, y \in \mathbb{P}^1/\mathcal{C}_m,$$

one get such a figure:



Remark 5.1.5. In case $\mathcal{C}_m = \{e\}$, i.e., the case of $SL(2)$-embeddings, there are two fixed elements and each other orbit is of order one. So either $\widetilde{D}$ one of the fixed elements or consists of one element. Thus we have alwayes $a(D) = r(D) = 1$. The Luna-Vust diagram of $\mathcal{V}(SL(2))$ in this case looks like the following one:



Definition 5.1.6. Define:

$\mathcal{L}_1^n(SL(2)/\mathcal{C}_m) = \{$local rings of non-open orbits of normal

$SL(2)/\mathcal{C}_m$-embeddings}.

An element $l \in \mathcal{L}_1^n(SL(2)/\mathcal{C}_m)$ is called a **locality**. l corresponds to the local ring $\mathcal{O}_l$ with the maximal ideal m_l. This ring is determined by its essential valuations stable by B. The set of B-stable valuations is (see [LV83])

$$\mathcal{V}(SL(2)/\mathcal{C}_m) \cup \{v_D\}_{D \in \mathbb{P}^1/\mathcal{C}_m},$$

where v_D is the valuation associated to D on $SL(2)/\mathcal{C}_m$. For l in $\mathcal{L}_1^n(SL(2)/\mathcal{C}_m)$, we denote by $\mathcal{V}_l$ the set of essentail valuations of $\mathcal{O}_l$ in $\mathcal{V}(SL(2)/\mathcal{C}_m)$ and by ${}^B\mathcal{D}_l$ the subset of $\mathbb{P}^1/\mathcal{C}_m$, whose valuations are essential for $\mathcal{O}_l$.

Proposition 5.1.7. *([Ja90]) Suppose X is an $SL(2)/\mathcal{C}_m$-embedding. Let Y be an orbit in X. Then there exists an affine B-stable open subset of X which intersects Y.*

Let $\mathcal{D}$ be a cofinite subset of $\mathbb{P}^1/\mathcal{C}_m$. We define

$$A(\mathcal{D}) := \{f \in \mathbb{C}(SL(2))^{\mathcal{C}_m} \mid f = gh \,;\, g \in \mathbb{C}(SL(2)),$$

$$h \text{ is an eigenvector of } \mathcal{C}_m \text{ and } v_D(h) = 0 \; \forall D \in \mathcal{D}\},$$

where v_D is the valuation of $\mathbb{C}(SL(2))^{\mathcal{C}_m}$ associated to the divisor D of $SL(2)/\mathcal{C}_m$. For $\mathcal{W} = \{w_1, \ldots, w_n\} \subset \mathcal{V}(SL(2)/\mathcal{C}_m)$ a finite set, define

$$A(\mathcal{D}, \mathcal{W}) = A(\mathcal{D}) \cap \mathcal{O}_{w_1} \cap \cdots \cap \mathcal{O}_{w_n},$$

where $\mathcal{O}_{w_i}$ is the valuation ring of $w_i \in \mathcal{V}(SL(2)/\mathcal{C}_m)$ $(i = 1, \ldots, n)$.

Remark 5.1.8. [Ja87]

(i) $A(\mathcal{D})$ is the ring of regular functions on the set

$$SL(2)/\mathcal{C}_m - \bigsqcup_{D \in \mathcal{D}} D.$$

(ii) $A(\mathcal{D}, \mathcal{W})$ is an integrally closed subalgebra of a finite type of $\mathbb{C}(SL(2)/\mathcal{C}_m)$.

Corollary 5.1.9. The proposition 5.1.7 gives that there exists a cofinite subset of ${}^B\mathcal{D}$ which contains ${}^B\mathcal{D}_l$, denoted $\mathcal{D}$, such that $\mathcal{O}_l$ is the localization of $A(\mathcal{D}, \mathcal{V}_l)$ in a prime ideal $m := A(\mathcal{D}, \mathcal{V}_l) \cap m_l$.

Remark 5.1.10. Let $\mathcal{D} \subset^B \mathcal{D}(SL(2)/\mathcal{C}_m)$ and $\mathcal{W} \subset \mathcal{V}(SL(2)/\mathcal{C}_m)$, then there is at most one $l \in \mathcal{L}_1^n(SL(2)/\mathcal{C}_m)$ such that ${}^B\mathcal{D}_l = \mathcal{D}$ and $\mathcal{V}_l = \mathcal{W}$. This means that the locality $l \in \mathcal{L}_1^n$ is characterized by the triple $(\mathcal{D}, \mathcal{W}, v)$, where $\mathcal{D}$ is cofinite and $v \in \mathcal{V}_1(SL(2)/\mathcal{C}_m)$.

The next proposition describes which triples give rise to localities of $\mathcal{L}_1^n(SL(2)/\mathcal{C}_m)$:

Proposition 5.1.11. *([LV83] or [Ja87]) Let $\mathcal{D}$ be a cofinite subset of $\mathbb{P}^1/\mathcal{C}_m$, $\mathcal{W} = \{w_1, \ldots, w_n\} \subset \mathcal{V}(SL(2)/\mathcal{C}_m)\}$ with $w_j = v(D_j, r_j)$, $(j = 1, \ldots, n)$, and $v \in \mathcal{V}_1(SL(2)/\mathcal{C}_m)$. Then $(\mathcal{D}, \mathcal{W}, v)$ represents an element of $\mathcal{L}_1^n(SL(2)/\mathcal{C}_m)$ if and only if it is one of the following types:*

(1) Type A_n $(n \geq 1)$

$$\mathcal{D} = \mathbb{P}^1/\mathcal{C}_m - \{D_1, \ldots, D_n\} \text{ and } D_i \neq D_j \text{ if } i \neq j;\ -1 < r_j \leq r(D_j)$$

$$\text{and} \sum_{j=1}^{n} \frac{1}{1+r_j} > 1;\ v \in \bigcup_{D \in \mathcal{D}} v(D,]-1, r(D)]) \cup \bigcup_{j=1}^{n} v(D_j,]-1, r_j[).$$

(2) Type AB $(n = 2)$

$$D_1 \notin \mathcal{D} \text{ and } \mathcal{D} \neq \mathbb{P}^1/\mathcal{C}_m - \{D_1\};\ D_1 = D_2 \text{ and}$$

$$-1 \leq r_1 < r_2 \leq r(D_1);\ v \in v(D_1,]r_1, r_2[).$$

(3) Type B_+ $(n = 1)$

$$D_1 \in \mathcal{D} \neq \mathbb{P}^1/\mathcal{C}_m;\ -1 \leq r_1 < r(D_1);\ v \in v(D_1,]r_1, r(D_1)]).$$

(4) Type B_- $(n = 1)$

$$\mathcal{D} = \mathbb{P}^1/\mathcal{C}_m - \{D_1\};\ 0 < r_1 < r(D_1);\ v \in v(D_1,]r_1, r(D_1)]).$$

(5) Type B_0 $(n = 1)$

$$\mathcal{D} = \mathbb{P}^1/\mathcal{C}_m;\ 0 < r_1 < r(D_1);\ v \in v(D_1,]r_1, r(D_1)]).$$

(6) Type C

$$\mathcal{W} = \{v\}$$

Definition 5.1.12. Let X be an $SL(2)/\mathcal{C}_m$-embedding. Then define

$$L(X) := \{l \in \mathcal{L}_1^n(SL(2)/\mathcal{C}_m) \mid l \text{ is a locality in } X\}.$$

The set $L(X)$ is open in the topological space $\mathcal{L}_1^n(SL(2)/\mathcal{C}_m)$ (topology of Zariski) and noetherian. If $l \in \mathcal{L}_1^n(SL(2)/\mathcal{C}_m)$, then the set:

$$\mathcal{F}_l := \{v \in \mathcal{V}_1(SL(2)/\mathcal{C}_m);\ v \text{ dominates } \mathcal{O}_l\}$$

is called the **facette** of l.

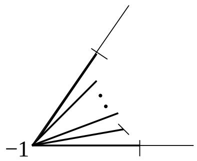
Remark 5.1.13. For each $l \in \mathcal{L}_1^n(SL(2)/\mathcal{C}_m)$, one draw a Luna-Vust diagram represents l as follows:

(i) Dark the facette of l in the diagram for $\mathcal{V}_l(SL(2)/\mathcal{C}_m)$.

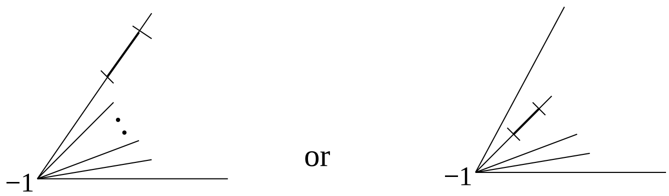
(ii) Elements of types B_+, B_- and B_0 will be distinguished by labelling the facette with a sign $+, -$ and 0 respectively.

The diagrams of the elements of $\mathcal{L}_l(SL(2)/\mathcal{C}_m)$ are the followings:

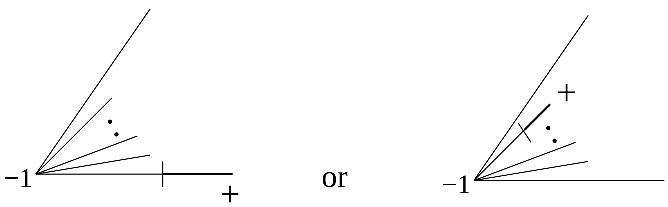
Type A n



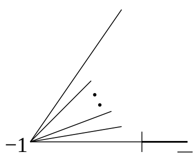
Type AB



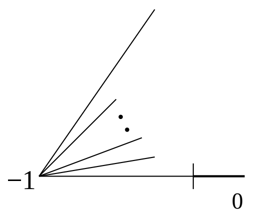
Type B₊



Type B₋

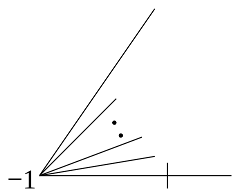


Type B_0

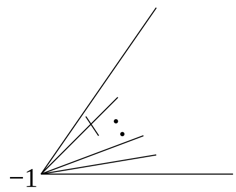


or

Type C



or



Remark 5.1.14. One notes that orbits of type C are exactly the 2-dimensional orbits. Orbits of type B_0 are fixed points and the remaining 4 types are 1-dimensional orbits which are isomorphic to $\mathbb{P}^1/\mathcal{C}_m$.

Definition 5.1.15. Define the following set

$$\mathcal{L}'(SL(2)/\mathcal{C}_m) = \{l \in \mathcal{L}_l^n(SL(2)/\mathcal{C}_m) \mid l \text{ is of the type } B_+ \text{ and}$$

$$\mathcal{V}_l = \{v(\ ,-1)\}\}.$$

The following proposition determines for which L of $\mathcal{L}_1^n(SL(2)/\mathcal{C}_m)$ there exists an $SL(2)/\mathcal{C}_m$-embedding X such that $L = L(X)$:

Proposition 5.1.16. *[Ja87] Let L be a subset of $\mathcal{L}_1^n(SL(2)/\mathcal{C}_m)$. Then L is a normal embedding if and only if it is satisfies the following properties:*

(1) If $l \in L$, then $\mathcal{V}_l \cap \mathcal{V}_1(SL(2)/\mathcal{C}_m) \subset L$.

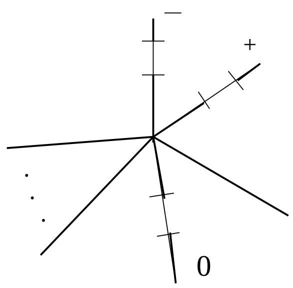
(2) If $l \in L$ and $v(,-1) \in \mathcal{V}_l$, then L contains a subset cofinite in $\mathcal{L}'(SL(2)/\mathcal{C}_m)$.

(3) $L - \mathcal{L}'(SL(2)/\mathcal{C}_m)$ is finite.

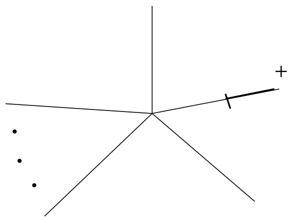
(4) the facettes of the elements in L are disjoint.

Furthermore an embedding L is **complete** *if and only if the union of the facettes of the localities in L is $\mathcal{V}_l(SL(2)/\mathcal{C}_m)$.*

Example 5.1.17. (a) Here is an example of a Luna-Vust diagram, which represents an $SL(2)$-embedding with 11 orbits: a fixed point of type B_0, three 1-dimensional orbits of types A_3, B_+, and B_-, six 2-dimensional orbits of type C and the open orbit:



(b) The following diagram does not represent an embedding, because the first condition in the last proposition is not verified.



5.2 From a colored fan to the Luna-Vust diagram

We explain in this section our method to move from a colored fan Σ of a spherical $SL(2)$-variety X to its corresponding Luna-Vust diagram. To do this it will be enough to explain how to go from an arbitrary colored cone to its corresponding facette in Luna-Vust diagram.

Remark 5.2.1. In Luna-Vust diagram corresponding $SL(2)/\mathcal{C}_m$-embedding with $\mathbb{C}^*$-action, we always have $a(D) = r(D) = 1$, because in this case every divisor D defines a $\mathbb{C}^*$-fixed point in $\mathbb{P}^1/\mathcal{C}_m$ and the finite morphism $\mathbb{P}^1 \to \mathbb{P}^1/\mathcal{C}_m$ is totaly ramified over the $\mathbb{C}^*$-fixed points in $\mathbb{P}^1/\mathcal{C}_m$, i.e., the divisor $\tilde{D}$ consists of a unique irreducible component.

Definition 5.2.2. Define the **square** S in the vector space $\mathbb{R}^2$ as follows:

$$S := [-1, 1] \times [-1, 1].$$

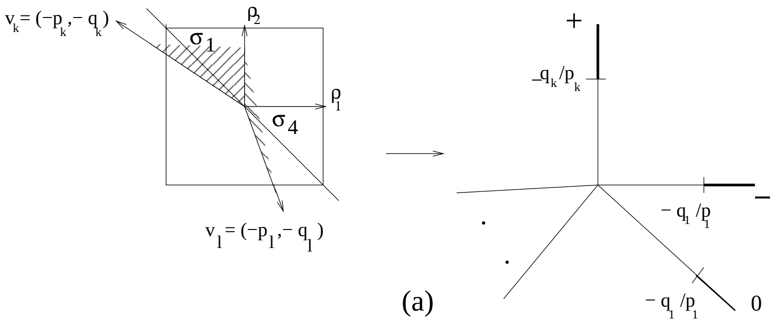
Denote by a the side $\{[-1,1]\times\{-1\}\}\subset S$ and by b the side $\{\{-1\}\times [-1,1]\}\subset S$.

The Method: We distinguish the cases:

(i) Let $v_k := (-p_k, -q_k) \neq (-1,1)$ be a vector of N_m and consider the colored cone $(\sigma_1, \mathcal{F}_1)$, where

$$\sigma_1 := \mathbb{R}_{\geq 0}\rho_2 + \mathbb{R}_{\geq 0}v_k, \text{ and } \mathcal{F}_1 := \{\rho_2\}.$$

A cone such σ_1 is corresponding to a segment $[-\frac{q_k}{p_k}, 1]$ of Luna-Vust diagram with $(+)$ marking end, i.e., σ_1 corresponds an orbit of the type B_+(see Figure (a)).

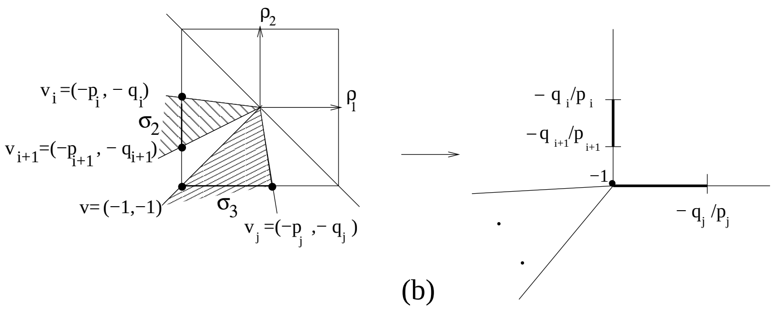


(a)

(ii) Let $v_i := (-p_i, -q_i)$, $v_{i+1} := (-p_{i+1}, -q_{i+1}) \in N_m$ two vectors which are diffrent from the vector $(-1,-1)$, then consider the colored cone $(\sigma_2, \mathcal{F}_2)$, where

$$\sigma_2 := \mathbb{R}_{\geq 0}v_i + \mathbb{R}_{\geq 0}v_{i+1}, \text{ and } \mathcal{F}_2 := \{\emptyset\}.$$

A cone such σ_2 is corresponding in Luna-Vust diagram to a segment $[-\frac{q_{i+1}}{p_{i+1}}, -\frac{q_i}{p_i}]$, i.e., σ_2 correspods to an orbit of the type AB of this diagram (see Figure (b)).



(b)

(iii) In Figure (b) consider the colored cone $(\sigma_3, \mathcal{F}_3)$, where

$$\sigma_3 := \mathbb{R}_{\geq 0}(-1,-1) + \mathbb{R}_{\geq 0} v_j, \text{ and } \mathcal{F}_3 := \{\emptyset\}.$$

The cone σ_3 corresponds to the segment $[-1, -\frac{q_j}{p_j}]$ of Luna-Vust diagram. That means it corresponds to an orbit of the type A_1.

(iv) In Figure (a) take the colored cone $(\sigma_4, \mathcal{F}_4')$, where

$$\sigma_4 := \mathbb{R}_{\geq 0} v_l + \mathbb{R}_{\geq 0}\rho_2 \text{ and } \mathcal{F}_4' := \{\rho_2\}.$$

This cone corresponds to the segment $[-\frac{q_l}{p_l}, 1]$ of Luna-Vust diagram with $(-)$ marking end, i.e., it corresponds to an orbit of the type B_-.

(v) In Figure (a) consider the colored cone $(\sigma_4, \mathcal{F}_4)$, where

$$\sigma_4 := \mathbb{R}_{\geq 0} v_l + \mathbb{R}_{\geq 0}\rho_2 \text{ and } \mathcal{F}_4 := \{\rho_1, \rho_2\}.$$

This cone corresponds to the segment $[-\frac{q_l}{p_l}, 1]$ with (0) marking end. This means it corresponds to an orbit of the type B_0.

Remark 5.2.3. If the colored cone of the form $(\sigma, \mathcal{F})$ where $\mathcal{F} \neq \emptyset$, then if $\mathcal{F}$ includes just one color (resp. two colors) then we mark the end of the segment of the Luna-Vust diagram with $(+)$ (resp. (-)).

Remark 5.2.4. The corresponding Luna-Vust diagram of a colored fan is a digaram consists of all segments which are corresponding all 2-dimensional colored cones in this fan.

5.3 Smooth $SL(2)$-embeddings and $PGL(2)$-embeddings

Let X be an $SL(2)/H$-embedding. We consider the case $H := \{e\}$ or the case $H := \{\pm E_2\}$, i.e., the case $SL(2)$ or $PGL(2)$-embedding. Since the set of singular points $\mathrm{Sing}(X)$ of X is stable by $SL(2)/H$, then every orbit is either contained in $\mathrm{Sing}(X)$ or in its complement $\mathrm{Reg}(X)$. We denote by

$$\mathcal{R}(G) := \{l \in \mathcal{L}_l^n(G);\ \mathcal{O}_l \text{ is a regular local ring}\}.$$

An embedding is smooth if an only if the localities of all its orbits are in $\mathcal{R}(G)$. The next theorems determine exactly $\mathcal{R}(G)$ in case $G = SL(2)$ and $G = PGL(2)$.

Lemma 5.3.1. *([Ja90]) Let l be a locality of $\mathcal{L}_l^n(G)$, then*

(i) If l is of the type B_0, then $l \notin \mathcal{R}(G)$.

(ii) If l is of the type C, then $l \in \mathcal{R}(G)$.

Proof. Since X is normal, then any orbit of codimension one is in Reg$X$ which prove the first statment (i). For (ii) Popov has showed in [P73] that fixed points are always singular. □

Theorem 5.3.2. *([Ja90]) Let l be a locality of $\mathcal{L}_l^n(SL(2))$, then $l \in \mathcal{R}(SL(2))$ if and only if it is one of the following types:*

(1) Type A_1 with $r_1 = -\frac{1}{q}$ and $q \in \mathbb{N}^*$.

(2) Type A_2 with $r_1 = r_2 = 0$ or with $r_i = 1$ and $r_j = \frac{q-1}{q}$; $q \in \mathbb{N}^*$.

(3) Type AB with $r_i = \frac{p_i}{q_i}$ $(i = 1,\ 2)$, where p_i and q_i are relatively prime from $\mathbb{Z}$, and $p_1q_2 - p_2q_1 = 1$.

(4) Type B_+ with $r_1 = 0$ or $r_1 = -1$.

(5) Type B_- with $r_1 = \frac{1}{q}$ and $q \in \mathbb{N}^*$.

(6) Type C.

Theorem 5.3.3. *([Ja90]) Let l be a locality of $\mathcal{L}_l^n(PGL(2))$, then $l \in \mathcal{R}(PGL(2))$ if and only if it is one of the following types:*

(1) Type A_1 with $r_1 = -\frac{1}{2n+1}$ and $n \in \mathbb{N}^*$.

(2) Type A_2 with $r_i = 1$ and $r_j = \frac{q-2}{q}$; $q \geq 2$.

(3) Type A_3 with $r_1 = r_2 = r_3 = 1$.

(4) Type AB with $r_i = \frac{p_i}{q_i}$ $(i = 1,\ 2)$, (p_i and q_i in lowest term such that $q_i - p_i$ is even and $q_i > 0$), and $p_1q_2 - p_2q_1 = 2$.

(5) Type B_+ with $r_1 = -1$.

(6) Type B_- with $r_1 = \frac{1}{2n+1}$ and $n \in \mathbb{N}^*$.

(7) Type C.

5.4 Morphisms and blow-ups of embeddings

The following proposition shows how to read off morphisms between G-embeddings (G is either $SL(2)$ or $PGL(2)$) directly from their diagrams.

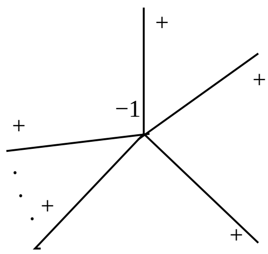
Proposition 5.4.1. *[Ja90] Let X, $X^{'}$ be two G-embeddings. Then the identity map on G extends unique to a G-morphism $\varphi : X \to X^{'}$ if and only if it for each $l \in L(X)$, $L(X) \subset \mathcal{L}_l^n(G)$ is the set of localities of orbits of X, there exists $l^{'} \in L(X^{'})$ such that $\mathcal{F}_l \subset \mathcal{F}_{l'}$ and if l is of the type B_0 then so is $l^{'}$, and if l is of the type B_+ (resp. B_-) then $l^{'}$ is either of the same type or of the type B_0. If φ exists, then it is proper if and only if:*

$$\cup_{l \in L(X)} \mathcal{F}_l = \cup_{l' \in L(X')} \mathcal{F}_{l'}$$

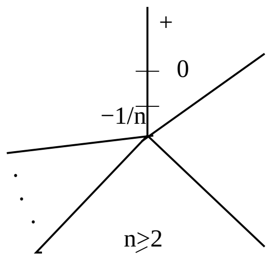
Proposition 5.4.2. *[Ja90] Let X and X' are smooth G-embeddings with a G-morphism $\varphi : X \to X'$. Then φ is a composition of blow-downs.*

Definition 5.4.3. A smooth complete embedding is called **minimal embedding** if it is not the blow-up of another smooth embedding.

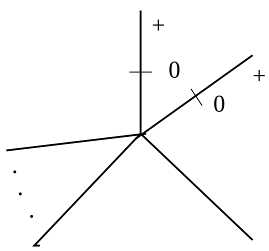
Minimal smooth $SL(2)$-embeddings:



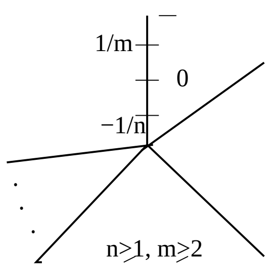
(a)



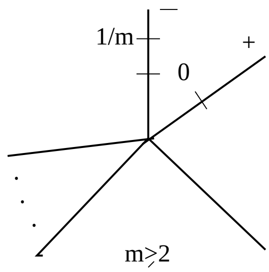
(d)



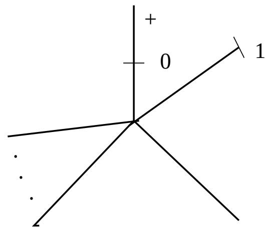
(e)



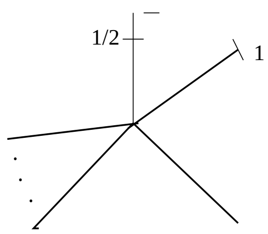
(f)



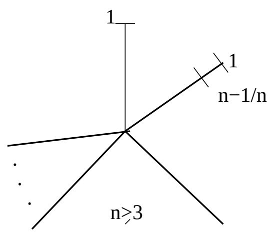
(g)



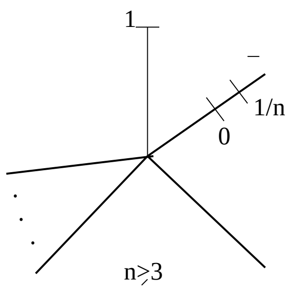
(h)



(i)

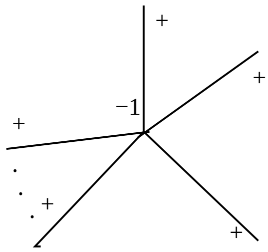


(j)

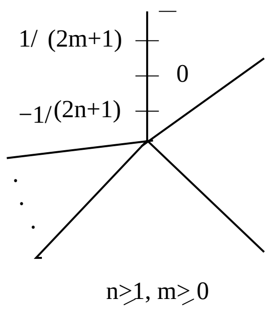


(k)

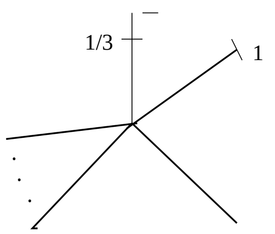
Minimal smooth $PGL(2)$-embeddings:



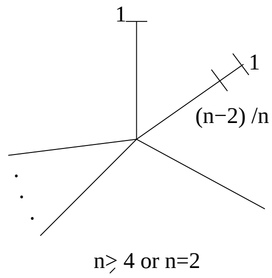
(a1)



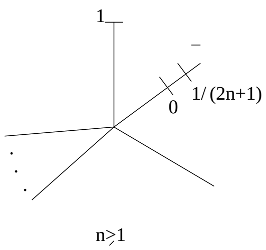
(c1)



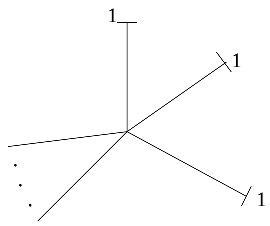
(d1)



(e1)



(f1)



(g1)

5.5 A classification of the smooth varieties with the Picard number ≤ 3

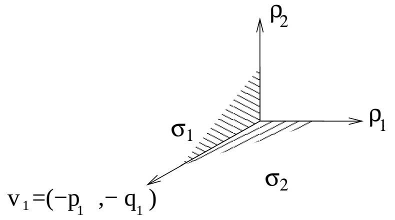
In this section we classify the smooth $SL(2)/\mathcal{C}_m$-embeddings from their colored fans. So we distinguish the following cases:

X has the Picard number 1: Let $v_1 := (-p_1, -q_1)$ be a vector of the lattice N_m. Then consider the colored complete fan

$$\Sigma := \{(\sigma_1, \rho_2),\ (\sigma_2, \rho_1)\},$$

where

$$\sigma_1 := \mathbb{R}_{\geq 0}\rho_2 + \mathbb{R}_{\geq 0}v_1,\ \sigma_2 := \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}\rho_1.$$



We determine the vector v_1 in case X is a smooth variety. Therefore we apply the smoothness condition on every colored cone in Σ, i.e., on σ_1, σ_2:

$$\begin{vmatrix} 0 & -p_1 \\ 1 & -q_1 \end{vmatrix} = \frac{1}{m},\ \begin{vmatrix} -p_1 & 1 \\ -q_1 & 0 \end{vmatrix} = \frac{1}{m}.$$

It follows that: $p_1 = q_1 = \frac{1}{m}$. Thus

$$v_1 = (-\frac{1}{m}, -\frac{1}{m}).$$

Since $v_1 \in N_m$, we write it as a linear combination in the base vectors

$$v_1 = (-\frac{1}{m}, -\frac{1}{m}) = a(1,0) + b(\frac{1}{m}, -\frac{1}{m}).$$

Then

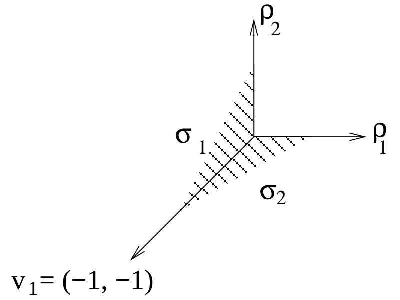
$$a + \frac{b}{m} = -\frac{1}{m},\ -\frac{b}{m} = -\frac{1}{m}.$$

It follows that $b = 1$ and $ma = -2$, which means $m \mid 2$. Therefore either $m = 1$ or $m = 2$.

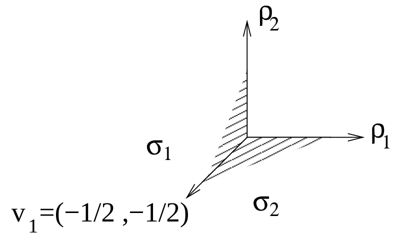
$m = 1$: We get a fan corresponds to a $SL(2)$-embedding. The equation

$$x_1x_4 - x_2x_3 = y_1^2$$

is defined the corresponding hypersurface to this fan. It clear that this embedding is isomorphic to a quadric in $\mathbb{P}^4$



$m = 2$: We get a fan of a $PGL(2)$-embedding:



The corresponding hypersurface is defined by the equation:

$$x_1x_4 - x_2x_3 = y_1.$$

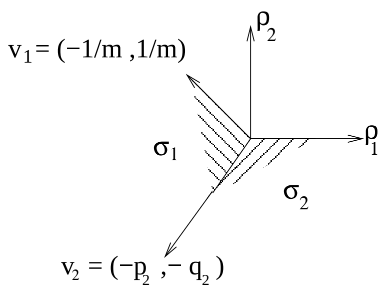
The embedding is isomorphic to the projective space $\mathbb{P}^3$.

X **has the Picard number** 2**:** One distinguishs the cases

CASE 1. Let $v_1 := (-\frac{1}{m}, \frac{1}{m}), v_2 := (-p_2, -q_2)$ be two vectors of N_m. We define the fan

$$\Sigma := \{(\sigma_1, \emptyset), (\sigma_2, \rho_1)\},$$

where $\sigma_1 := \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2,\ \ \sigma_2 := \mathbb{R}_{\geq 0}v_2 + \mathbb{R}_{\geq 0}\rho_1$.



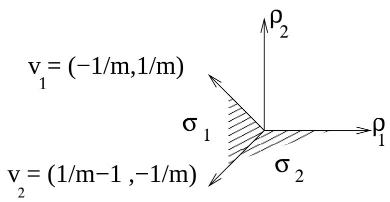
We determine the vector v_2, when X is smooth. We apply the smoothness condition on σ_1 and σ_2

$$\begin{vmatrix} -\frac{1}{m} & -p_2 \\ \frac{1}{m} & -q_2 \end{vmatrix} = \frac{1}{m}, \quad \begin{vmatrix} -p_2 & 1 \\ -q_2 & 0 \end{vmatrix} = \frac{1}{m}.$$

It follows that $q_2 + p_2 = 1$, $q_2 = \frac{1}{m}$. Then $p_2 = 1 - \frac{1}{m}$ and

$$v_2 = (\frac{1}{m} - 1, -\frac{1}{m}) \in N_m.$$

So we get the fan:



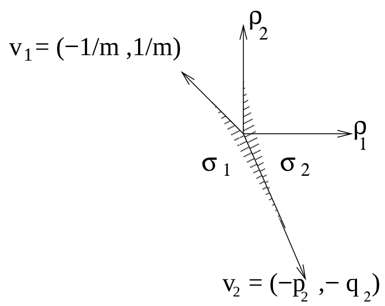
The corresponding hypersurface to this fan is defined by the following equation:

$$x_1x_4 - x_2x_3 = y_2.$$

CASE 2. Let $v_1 = (-\frac{1}{m}, \frac{1}{m}), v_2 = (-p_2, -q_2) \in N_m$. Consider the fan

$$\Sigma := \{(\sigma_1, \emptyset), \quad (\sigma_2, \{\rho_1, \rho_2\})\},$$

such that: $\sigma_1 := \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2$, $\sigma_2 := \mathbb{R}_{\geq 0}v_2 + \mathbb{R}_{\geq 0}\rho_2$.



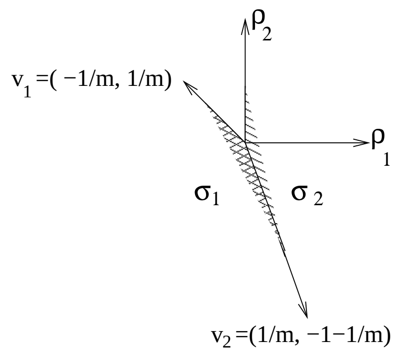
Apply the smoothness condition on σ_1, σ_2:

$$\begin{vmatrix} -\frac{1}{m} & -p_2 \\ \frac{1}{m} & -q_2 \end{vmatrix} = \frac{1}{m}, \quad \begin{vmatrix} -p_2 & 0 \\ -q_2 & 1 \end{vmatrix} = \frac{1}{m}.$$

This gives: $q_2 + p_2 = 1$, $p_2 = -\frac{1}{m}$. Then $q_2 = 1 + \frac{1}{m}$. Therefore,

$$v_2 = (\frac{1}{m}, -1 - \frac{1}{m}) \in N_m$$

and we get the fan:



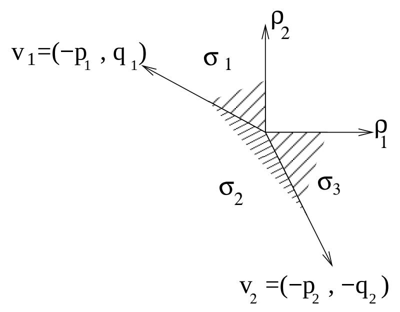
The corresponding hypersurface is defined by the equation:

$$x_1x_4 - x_2x_3 = y_2.$$

Case 3. Let $v_1 = (-p_1, -q_1), v_2 = (-p_2, -q_2) \in N_m$, where $v_1 \neq (-\frac{1}{m}, \frac{1}{m})$ and $v_2 \neq (\frac{1}{m}, -\frac{1}{m})$. Then consider the fan

$$\Sigma := \{(\sigma_1, \rho_2),\ (\sigma_2, \emptyset),\ (\sigma_3, \rho_1)\},$$

where $\sigma_1 := \mathbb{R}_{\geq 0}\rho_2 + \mathbb{R}_{\geq 0}v_1$, $\sigma_2 := \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2$ and $\sigma_3 := \mathbb{R}_{\geq 0}v_2 + \mathbb{R}_{\geq 0}\rho_1$.



We determine v_1, v_2 when X is smooth. The smoothness condition on every cone in Σ gives:

$$\begin{vmatrix} 0 & -p_1 \\ 1 & -q_1 \end{vmatrix} = \frac{1}{m}, \quad \begin{vmatrix} -p_1 & -p_2 \\ -q_1 & -q_2 \end{vmatrix} = \frac{1}{m}, \quad \begin{vmatrix} -p_2 & 1 \\ -q_2 & 0 \end{vmatrix} = \frac{1}{m}.$$

This implies: $p_1 = q_2 = \frac{1}{m}$. Since $v_1,\ v_2 \in N_m$, then they must be of the form:

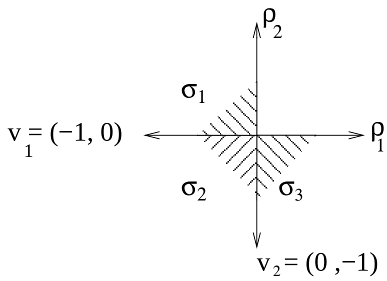
$$v_1 = (-\frac{1}{m}, \frac{1}{m} - a),\ v_2 = (\frac{1}{m} - b, -\frac{1}{m}),$$

where $a,\ b$ are strictly positive arbitrary integers. If we replace the coordinates of v_1, v_2 in the equation $p_1q_2 - p_2q_1 = \frac{1}{m}$, we get

$$m = \frac{1}{a} + \frac{1}{b} - \frac{1}{ab}$$

($ab \neq 0$ because a, b are strictly positive integers). This implies that $m = 1$, which occurs in the following cases:

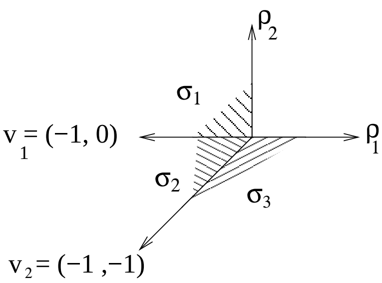
(I) $a = b = 1$. Then: $v_1 = (-1, 0),\ v_2 = (0, -1)$



The corresponding hypersurface is defined by the following equation:

$$x_1x_4 - x_2x_3 = y_1y_2.$$

(II) $a = 1$ and $b = 2$. Then: $v_1 = (-1, 0),\ v_2 = (-1, -1)$.



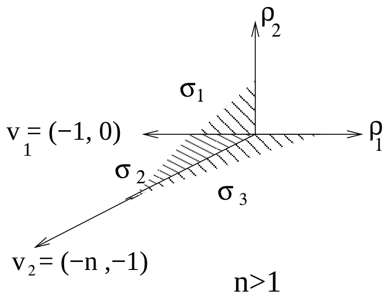
The corresponding hypersurface is:

$$x_1x_4 - x_2x_3 = y_1y_2^2.$$

(III) $a = 1$ and $b \in \mathbb{Z}_{>2}$. Denote by $n := -(1 - b)$. Then we get the vectors:

$$v_1 = (-1, 0),\ v_2 = (-n, -1),$$

where $n \in \mathbb{Z}_{>1}$, (because $b \in \mathbb{Z}_{>2}$).



The corresponding hypersurface is defined by the equation:

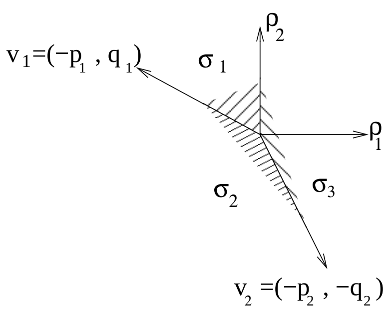
$$x_1x_4 - x_2x_3 = y_1y_2^{n+1}.$$

Case 4. Let $v_1 = (-p_1, -q_1), v_2 = (-p_2, -q_2) \in N_m$, where $v_1 \neq (-\frac{1}{m}, \frac{1}{m})$ and $v_2 \neq (\frac{1}{m}, -\frac{1}{m})$. Then consider the fan

$$\Sigma := \{(\sigma_1, \rho_2),\ (\sigma_2, \emptyset),\ (\sigma_3, \{\rho_1, \rho_2\})\,\},$$

where

$$\sigma_1 := \mathbb{R}_{\geq 0}\rho_2 + \mathbb{R}_{\geq 0}v_1,\ \sigma_2 := \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2$$
$$\sigma_3 := \mathbb{R}_{\geq 0}v_2 + \mathbb{R}_{\geq 0}\rho_2.$$



Determine v_1, v_2 in case when X is smooth:

$$\begin{vmatrix} 0 & -p_1 \\ 1 & -q_1 \end{vmatrix} = \frac{1}{m}, \quad \begin{vmatrix} -p_1 & -p_2 \\ -q_1 & -q_2 \end{vmatrix} = \frac{1}{m}, \quad \begin{vmatrix} -p_2 & 0 \\ -q_2 & 1 \end{vmatrix} = \frac{1}{m}.$$

It implies that $p_1 = -p_2 = \frac{1}{m}$. Replace this in the second determinant equation:

$$q_2 + q_1 = 1.$$

Since $v_1, v_2 \in N_m$, then q_1, q_2 must be of the following form

$$q_1 = -\frac{1}{m} + a,\ q_2 = \frac{1}{m} + b,$$

where a, b are strictly positive integers. From the equation $q_2 + q_1 = 1$ we get $b = 1 - a$. Thus

$$v_1 = (-\frac{1}{m}, \frac{1}{m} - a),\ v_2 = (\frac{1}{m}, -\frac{1}{m} + a - 1).$$

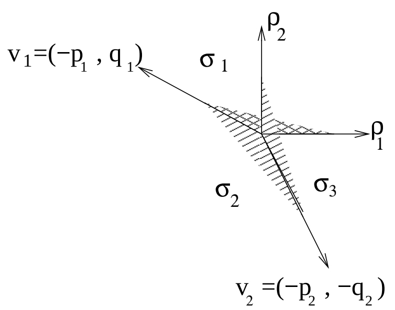
Since $v_1, v_2 \in \mathcal{V}$, then a is an integer such that: $0 \leq a \leq 1$. Either $a = 0$ or $a = 1$. This implies that either $b = 1$ or $b = 0$, which contradicts that a, b are strictly positive integers. Therefore there is no variety corresponds to this fan.

CASE 5. Let $v_1 = (-p_1, -q_1),\ v_2 = (-p_2, -q_2) \in N_m$, where $v_1 \neq (-\frac{1}{m}, \frac{1}{m})$ and $v_2 \neq (\frac{1}{m}, -\frac{1}{m})$. Consider the following fan

$$\Sigma := \{(\sigma_1, \{\rho_1, \rho_2\}),\ (\sigma_2, \emptyset),\ (\sigma_3, \{\rho_1, \rho_2\})\},$$

where

$$\sigma_1 := \mathbb{R}_{\geq 0}\rho_1 + \mathbb{R}_{\geq 0}v_1,\ \sigma_2 := \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2$$
$$\sigma_3 := \mathbb{R}_{\geq 0}v_2 + \mathbb{R}_{\geq 0}\rho_2.$$



Smoothness conditions give:

$$\begin{vmatrix} 1 & -p_1 \\ 0 & -q_1 \end{vmatrix} = \frac{1}{m},\quad \begin{vmatrix} -p_1 & -p_2 \\ -q_1 & -q_2 \end{vmatrix} = \frac{1}{m},\quad \begin{vmatrix} -p_2 & 0 \\ -q_2 & 1 \end{vmatrix} = \frac{1}{m}.$$

It follows that $q_1 = p_2 = -\frac{1}{m}$. Then

$$p_1 = \frac{1}{m} + a,\quad q_2 = \frac{1}{m} + b,$$

where a, b are strictly positive integers. Replacing p_1, q_2 in the second determinant equation gives:

$$(\frac{1}{m} + a)(\frac{1}{m} + b) - (\frac{-1}{m} \cdot \frac{-1}{m}) = \frac{1}{m}.$$

It follows:

$$m = \frac{1-a-b}{ab}, \quad \text{(here } ab \neq 0\text{)}.$$

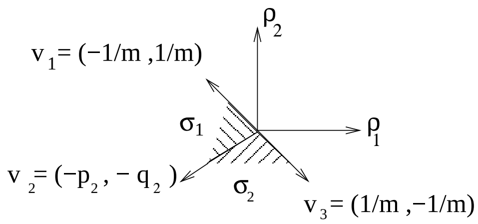
Since $a, b \in \mathbb{Z}_{>0}$, then $ab > 0$ and $1-a-b < 0$. This implies that $m < 0$, which contradicts that $m \in \mathbb{Z}_{>0}$. So there is no variety corresponds to the this fan in this case.

X has the Picard number 3: We distinguish the cases:

Case 1. Let $v_1 = (-\frac{1}{m}, \frac{1}{m})$, $v_2 = (-p_2, -q_2)$, $v_3 = (\frac{1}{m}, -\frac{1}{m})$ are three vectors of the lattice N_m. We consider the colored fan:

$$\Sigma := \{(\sigma_1, \emptyset),\ (\sigma_2, \emptyset)\}$$

such that: $\sigma_1 := \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2$, $\sigma_2 := \mathbb{R}_{\geq 0}v_2 + \mathbb{R}_{\geq 0}v_3$.



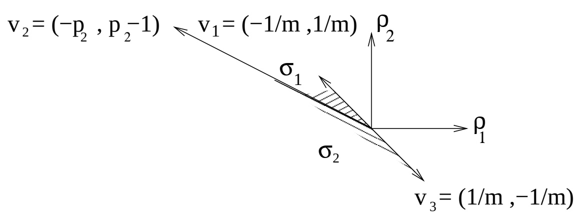
We determine v_2 in case when X is smooth. Therefore we apply the smoothness condition on all cones in Σ:

$$\begin{vmatrix} -\frac{1}{m} & -p_2 \\ \frac{1}{m} & -q_2 \end{vmatrix} = \frac{1}{m}, \quad \begin{vmatrix} -p_2 & \frac{1}{m} \\ -q_2 & -\frac{1}{m} \end{vmatrix} = \frac{1}{m}.$$

This implies: $p_2 + q_2 = 1$. Then $q_2 = 1 - p_2$ and we get

$$v_2 = (-p_2, -(1-p_2)),$$

where p_2 is an arbitrary integer.



The hypersurface corresponds to this fan is defined by the equation:

$$x_1x_4 - x_2x_3 = y_2.$$

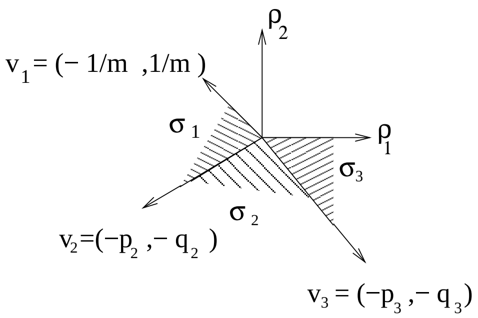
Case 2. Let $v_1 = (-\frac{1}{m}, \frac{1}{m}),\ v_2 = (-p_2, -q_2),\ v_3 = (-p_3, -q_3) \in N_m$, such that $v_3 \neq (\frac{1}{m}, -\frac{1}{m})$. Then define the cones:

$$\sigma_1 := \mathbb{R}_{\geq 0} v_1 + \mathbb{R}_{\geq 0} v_2,\ \sigma_2 := \mathbb{R}_{\geq 0} v_2 + \mathbb{R}_{\geq 0} v_3,$$

$$\sigma_3 := \mathbb{R}_{\geq 0} v_3 + \mathbb{R}_{\geq 0} \rho_1.$$

and consider the fan:

$$\Sigma := \{(\sigma_1, \emptyset),\ (\sigma_2, \emptyset),\ (\sigma_3, \rho_1)\}.$$



We compute v_2, v_3, when X is a smooth variety:

$$\begin{vmatrix} -\frac{1}{m} & -p_2 \\ \frac{1}{m} & -q_2 \end{vmatrix} = \frac{1}{m},\quad \begin{vmatrix} -p_2 & -p_3 \\ -q_2 & -q_3 \end{vmatrix} = \frac{1}{m},\quad \begin{vmatrix} -p_3 & 1 \\ -q_3 & 0 \end{vmatrix} = \frac{1}{m}.$$

This gives:

$$q_2 + p_2 = 1,\ \ p_2 q_3 - p_3 q_2 = \frac{1}{m},\ \ q_3 = \frac{1}{m}.$$

Then $q_2 = 1 - p_2$. Since $v_3 \in N_m$ and $q_3 = \frac{1}{m}$, then p_3 is of the form:

$$p_3 = -\frac{1}{m} + a,$$

where a is an strictly positive arbitrary integer. By substitution in the second determinant equation we get:

$$-ma(1 - p_2) = 0\ \Rightarrow 1 - p_2 = 0\ \Rightarrow p_2 = 1.$$

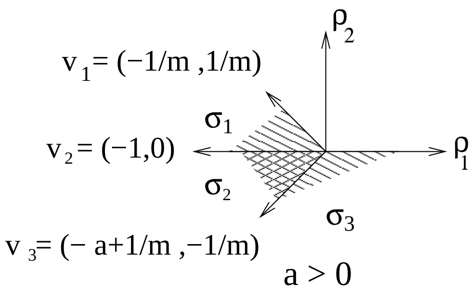
This implies: $q_2 = 1 - 1 = 0$. Therefore

$$v_1 = (-\frac{1}{m}, \frac{1}{m}),\ \ v_2 = (-1, 0),\ \ v_3 = (\frac{1}{m} - a, -\frac{1}{m}),$$

where a is a strictly positive integer. The hypersurface is:

$$x_1 x_4 - x_2 x_3 = y_2 y_3^a;\ a \in \mathbb{Z}_{>0}.$$

The corresponding fan is:



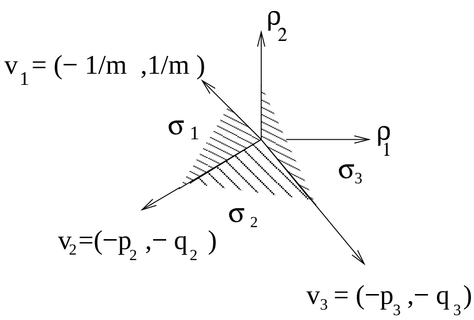
Case 3. Let $v_1 = (-\frac{1}{m}, \frac{1}{m})$, $v_2 = (-p_2, -q_2)$, $v_3 = (-p_3, -q_3) \in N_m$, where $v_3 \neq (\frac{1}{m}, -\frac{1}{m})$. Then define in N_m the cones:

$$\sigma_1 := \mathbb{R}_{\geq 0} v_1 + \mathbb{R}_{\geq 0} v_2, \quad \sigma_2 := \mathbb{R}_{\geq 0} v_2 + \mathbb{R}_{\geq 0} v_3,$$

$$\sigma_3 := \mathbb{R}_{\geq 0} v_3 + \mathbb{R}_{\geq 0} \rho_2.$$

We consider the fan:

$$\Sigma := \{(\sigma_1, \emptyset),\ (\sigma_2, \emptyset),\ (\sigma_3, \{\rho_1, \rho_2\})\}.$$



The smoothness condition on each cone in this fan gives:

$$\begin{vmatrix} -\frac{1}{m} & -p_2 \\ \frac{1}{m} & -q_2 \end{vmatrix} = \frac{1}{m}, \quad \begin{vmatrix} -p_2 & -p_3 \\ -q_2 & -q_3 \end{vmatrix} = \frac{1}{m}, \quad \begin{vmatrix} -p_3 & 0 \\ -q_3 & 1 \end{vmatrix} = \frac{1}{m}.$$

It implies:

$$q_2 + p_2 = 1, \quad p_2 q_3 - p_3 q_2 = \frac{1}{m}, \quad p_3 = -\frac{1}{m}.$$

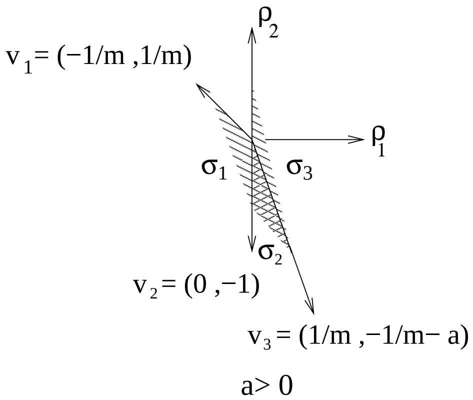
Since $v_3 \in N_m$ and $p_3 = -\frac{1}{m}$, then $q_3 = \frac{1}{m} + a$, where a is some strictly positive integer. By substitution in the second equation we get:

$$p_2(\frac{1}{m} + a) - (\frac{-1}{m})(1 - p_2) = \frac{1}{m}.$$

So $ap_2 = 0$ which implies: $p_2 = 0$ and then $q_2 = 1$. Therefore we get the vectors:

$$v_1 = (-\frac{1}{m}, \frac{1}{m}),\ v_2 = (0, -1),\ v_3 = (\frac{1}{m}, -\frac{1}{m} - a),$$

where $a \in \mathbb{Z}_{>0}$.



The hypersurface is defined as follows:

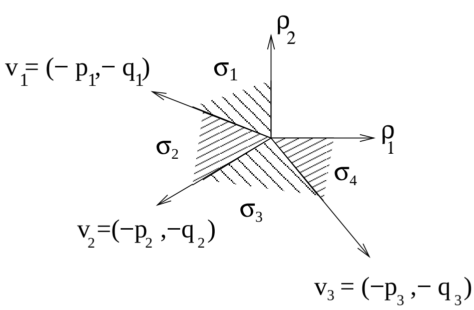
$$x_1x_4 - x_2x_3 = y_2y_3^a;\ a \in \mathbb{Z}_{>0}.$$

CASE 4. Let $v_1 = (-p_1, -q_1),\ v_2 = (-p_2, -q_2),\ v_3 = (-p_3, -q_3) \in N_m$, where $v_1 \neq (-\frac{1}{m}, \frac{1}{m}), v_3 \neq (\frac{1}{m}, -\frac{1}{m})$. Then consider the fan

$$\Sigma := \{(\sigma_1, \rho_2),\ (\sigma_2, \emptyset),\ (\sigma_3, \emptyset),\ (\sigma_4, \rho_1)\},$$

where $\quad \sigma_1 := \mathbb{R}_{\geq 0}\rho_2 + \mathbb{R}_{\geq 0}v_1,\ \ \sigma_2 := \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2,$

$$\sigma_3 := \mathbb{R}_{\geq 0}v_2 + \mathbb{R}_{\geq 0}v_3,\ \ \sigma_4 := \mathbb{R}_{\geq 0}v_3 + \mathbb{R}_{\geq 0}\rho_1.$$



From the smoothness condition on every cone we get:

$$\begin{vmatrix} 0 & -p_1 \\ 1 & -q_1 \end{vmatrix} = \frac{1}{m}, \quad \begin{vmatrix} -p_1 & -p_2 \\ -q_1 & -q_2 \end{vmatrix} = \frac{1}{m},$$

$$\begin{vmatrix} -p_2 & -p_3 \\ -q_2 & -q_3 \end{vmatrix} = \frac{1}{m}, \quad \begin{vmatrix} -p_3 & 1 \\ -q_3 & 0 \end{vmatrix} = \frac{1}{m}.$$

It implies:

$$p_1 = q_3 = \frac{1}{m}, \ \ p_1q_2 - p_2q_1 = \frac{1}{m}, \ \ p_2q_3 - p_3q_2 = \frac{1}{m}.$$

Since $v_1, v_3 \in N_m$, then

$$q_1 = -\frac{1}{m} + a, \ p_3 = -\frac{1}{m} + b,$$

where a, b are some strictly positive integers. By replacing q_1, p_3 in the second and the third determinant equations we get:

$$p_2 + q_2 - p_2am = 1, \ \ p_2 + q_2 - q_2bm = 1.$$

This implies that:

$$m(q_2b - p_2a) = 0.$$

Since $m \neq 0$, then $q_2b - p_2a = 0$. So $q_2 = \frac{a}{b}p_2$. Replacing in the smoothness condition equation of σ_2 (or σ_3) gives

$$m = \frac{1}{a} + \frac{1}{b} - \frac{1}{ap_2}.$$

The fact $v_2 \in \mathcal{V}$ gives $-p_2 - q_2 = -p_2(1 + \frac{a}{b}) \leqslant 0$. It means: $p_2 > 0$ (if $p_2 = 0$, then $v_2 = (0,0)$). Therefore:

$$m = \frac{1}{a} + \frac{1}{b} - \frac{1}{ap_2} < \frac{1}{a} + \frac{1}{b} \leq 2.$$

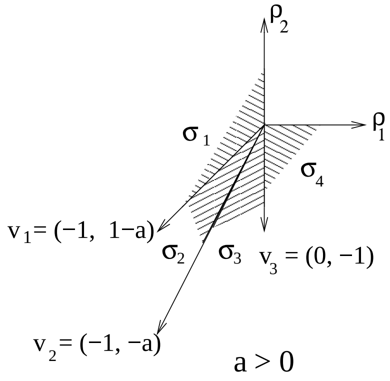
So $m = 1$, which occurs when $b = p_2 = 1$ and a is an arbitrary positive integer. So we get the vectors:

$$v_1 = (-1, 1-a), \ v_2 = (-1, -a), \ v_3 = (0, -1); \ a \in \mathbb{Z}_{>0}.$$

Then the hypersurface is:

$$x_1x_4 - x_2x_3 = y_1^a y_2^{1+a} y_3.$$

The corresponding fan is:



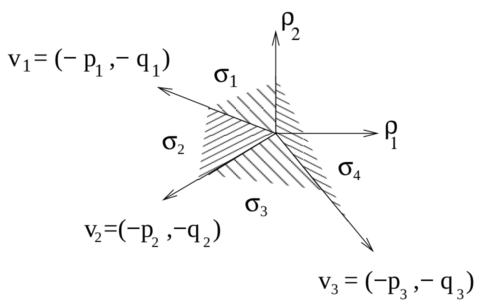
CASE 5. Let $v_1 = (-p_1, -q_1), v_2 = (-p_2, -q_2), v_3 = (-p_3, -q_3) \in N_m$, where $v_1 \neq (-\frac{1}{m}, \frac{1}{m}), v_3 \neq (\frac{1}{m}, -\frac{1}{m})$. We define the fan

$$\Sigma := \{(\sigma_1, \rho_2),\ (\sigma_2, \emptyset),\ (\sigma_3, \emptyset),\ (\sigma_4, \{\rho_1, \rho_2\})\},$$

such that:

$$\sigma_1 := \mathbb{R}_{\geq 0}\rho_2 + \mathbb{R}_{\geq 0}v_1, \quad \sigma_2 := \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2,$$

$$\sigma_3 := \mathbb{R}_{\geq 0}v_2 + \mathbb{R}_{\geq 0}v_3, \quad \sigma_4 := \mathbb{R}_{\geq 0}v_3 + \mathbb{R}_{\geq 0}\rho_2.$$



Aplly the smoothness condition on all cones in Σ:

$$\begin{vmatrix} 0 & -p_1 \\ 1 & -q_1 \end{vmatrix} = \frac{1}{m}, \quad \begin{vmatrix} -p_1 & -p_2 \\ -q_1 & -q_2 \end{vmatrix} = \frac{1}{m},$$

$$\begin{vmatrix} -p_2 & -p_3 \\ -q_2 & -q_3 \end{vmatrix} = \frac{1}{m}, \quad \begin{vmatrix} -p_3 & 0 \\ -q_3 & 1 \end{vmatrix} = \frac{1}{m}.$$

It gives:

$$p_1 = -p_3 = \frac{1}{m}, \quad p_1 q_2 - p_2 q_1 = \frac{1}{m}, \quad p_2 q_3 - p_3 q_2 = \frac{1}{m}.$$

Since $v_1, v_3 \in N_m$, then q_1, q_3 are of the from

$$q_1 = -\frac{1}{m} + a, \quad q_3 = \frac{1}{m} + b,$$

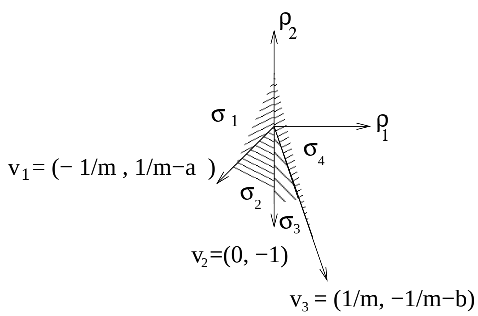
where a, b are some strictly positive integers. Replace p_1, q_1 in the second determinant equation and p_3, q_3 in the third one:

$$p_2 + q_2 - p_2 am = 1, \quad p_2 + q_2 + p_2 bm = 1.$$

Subtract one of them from the other one: $mp_2(a+b) = 0$. Either $a+b=0$ which contradicts that a, b are strictly positive integers. Or $p_2 = 0$. So $q_2 = 1$ and then

$$v_1 = (-\frac{1}{m}, \frac{1}{m} - a), \quad v_2 = (0, -1),$$

$$v_3 = (\frac{1}{m}, -\frac{1}{m} - b); \quad a, b \in \mathbb{Z}_{>0}.$$



The corresponding hypersurface is defined by:

$$x_1x_4 - x_2x_3 = y_1^a y_2 y_3^b; \quad a, b \in \mathbb{Z}_{>0}.$$

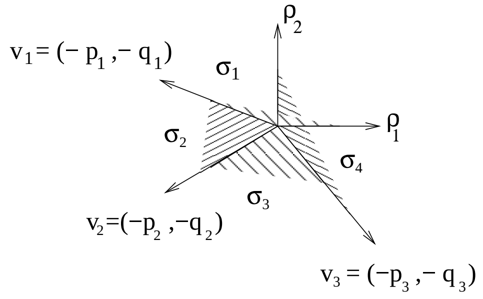
Case 6. Let $v_1 = (-p_1, -q_1), v_2 = (-p_2, -q_2), v_3 = (-p_3, -q_3) \in N_m$. Take the fan

$$\Sigma := \{(\sigma_1, \{\rho_1, \rho_2\}),\ (\sigma_2, \emptyset),\ (\sigma_3, \emptyset),\ (\sigma_4, \{\rho_1, \rho_2\})\},$$

where

$$\sigma_1 := \mathbb{R}_{\geq 0}\rho_1 + \mathbb{R}_{\geq 0}v_1, \quad \sigma_2 := \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2,$$

$$\sigma_3 := \mathbb{R}_{\geq 0}v_2 + \mathbb{R}_{\geq 0}v_3, \quad \sigma_4 := \mathbb{R}_{\geq 0}v_3 + \mathbb{R}_{\geq 0}\rho_2.$$



The smoothness condition on every cone in Σ gives:

$$\begin{vmatrix} 1 & -p_1 \\ 0 & -q_1 \end{vmatrix} = \frac{1}{m}, \quad \begin{vmatrix} -p_1 & -p_2 \\ -q_1 & -q_2 \end{vmatrix} = \frac{1}{m},$$

$$\begin{vmatrix} -p_2 & -p_3 \\ -q_2 & -q_3 \end{vmatrix} = \frac{1}{m}, \quad \begin{vmatrix} -p_3 & 0 \\ -q_3 & 1 \end{vmatrix} = \frac{1}{m}.$$

This implies:

$$q_1 = p_3 = -\frac{1}{m}, \quad p_1 q_2 - p_2 q_1 = \frac{1}{m}, \quad p_2 q_3 - p_3 q_2 = \frac{1}{m}.$$

Since $v_1, v_3 \in N_m$, then

$$p_1 = \frac{1}{m} + a, \quad q_3 = \frac{1}{m} + b,$$

where a, b are strictly positive integers. On the other hand one can write v_2 as follows:

$$v_2 = -c(1,0) + d(\frac{1}{m}, -\frac{1}{m}) = (+\frac{d}{m} - c, -\frac{d}{m}),$$

where $d \in \mathbb{Z}$ and $c \in \mathbb{Z}_{>0}$. So

$$p_2 = -\frac{d}{m} + c, \quad q_2 = \frac{d}{m}.$$

If we replace p_1, q_1, p_2, q_2 in the second and third determinant equations, we get: $ad + c = 1$, $m = \frac{1}{cb} + \frac{d}{c} - \frac{1}{b}$ respectively. So $d = \frac{1-c}{a}$ and then

$$m = \frac{1}{cb} + \frac{1}{ac} - \frac{1}{a} - \frac{1}{b} = (\frac{1}{c} - 1)(\frac{1}{b} + \frac{1}{a}).$$

Since $m, b, a \in \mathbb{Z}_{>0}$, then

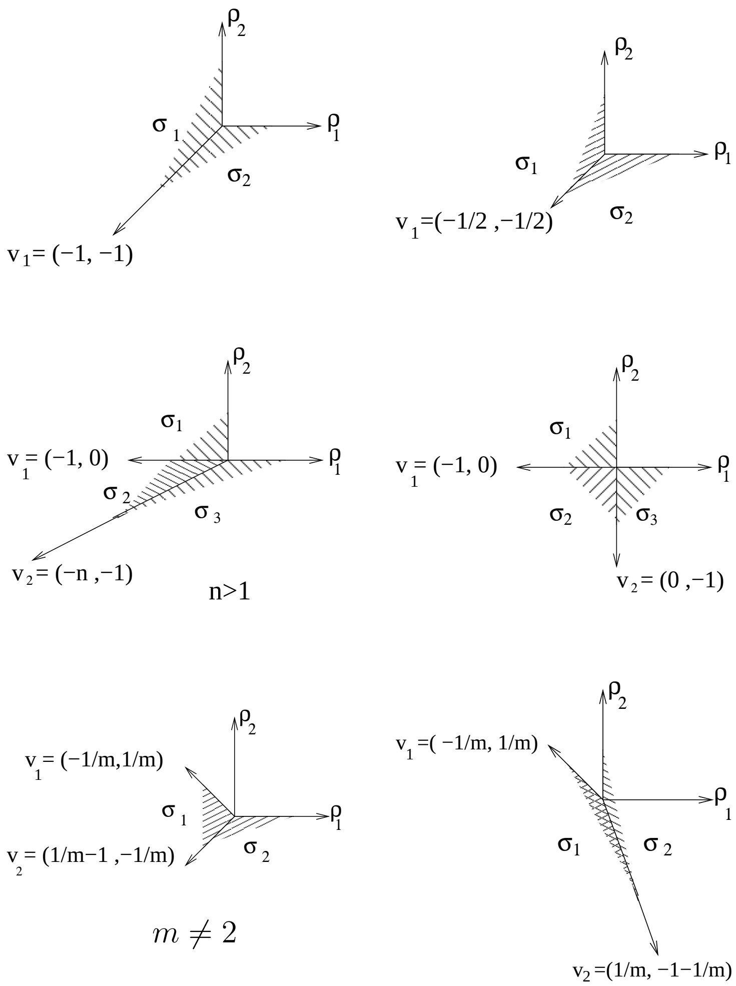
$$\frac{1}{c} - 1 > 0 \Rightarrow \frac{1}{c} > 1.$$

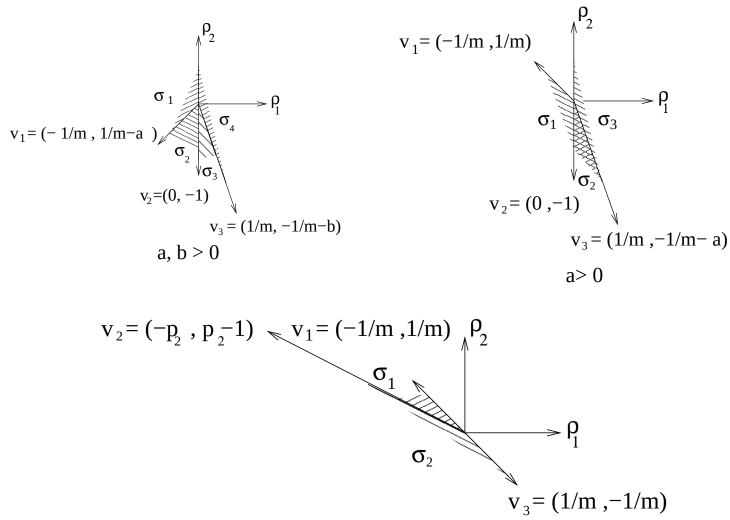
It follows that $c < 1$. This contradicts $c \in \mathbb{Z}_{>0}$. Therefore there is no variety corresponds to such a fan.

5.6 A classification of minimal smooth varieties

In this section we classify smooth minimal varieties of $SL(2)$-varieties with $\mathbb{C}^*$-action. Recall a smooth complete embedding is called minimal if it is not the blow-up of another smooth embedding.

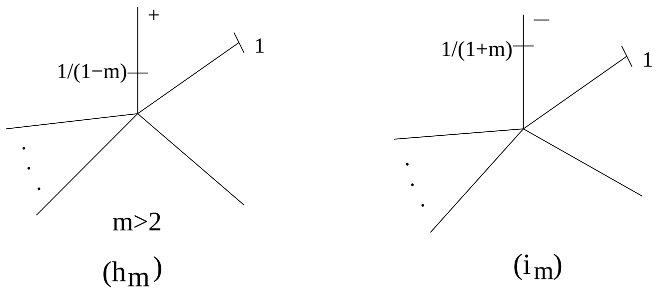
(I). The following fans are fans of minimal models:

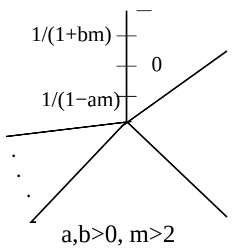




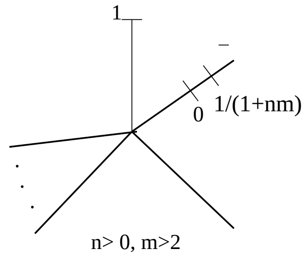
$$p_2 \in \mathbb{Z}_{>0}$$

Here are Luna-Vust diagrams correspond to fans of minimal models which have not been done by Moser Jauslin:

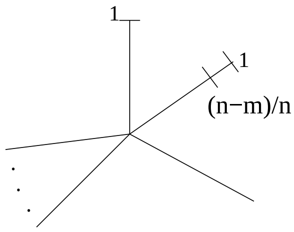




(f_m)

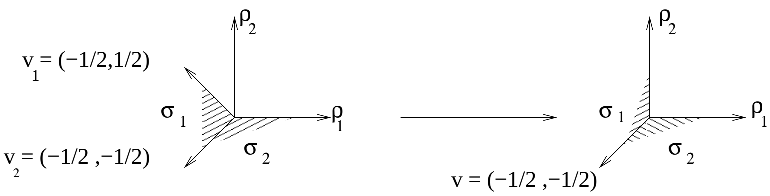


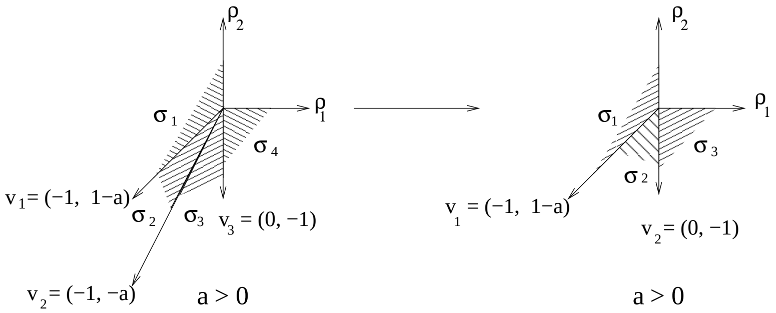
(k_m)

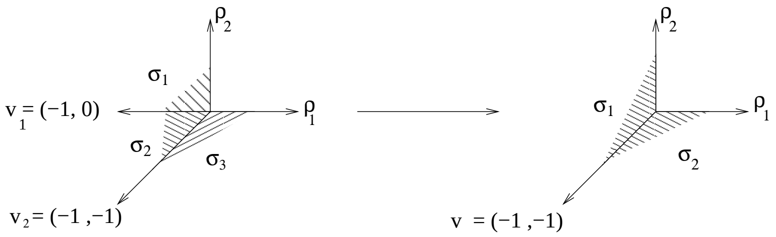


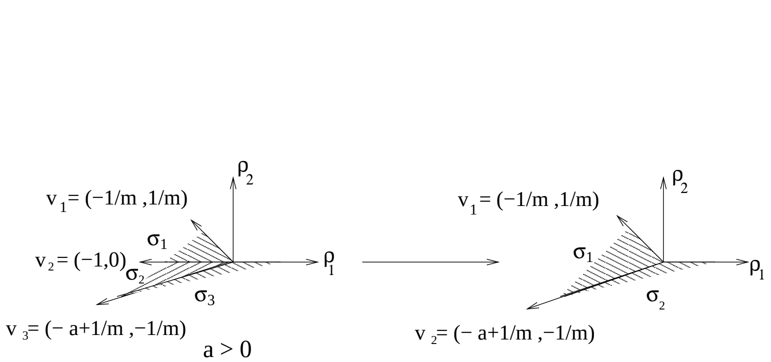
(e_m)

(II). The following blow-ups of the following colored fans show that these fans do not represent minmal models of smooth $SL(2)/\mathcal{C}_m$-embeddings:




ρ2
ρ1
σ1
σ4
σ2
σ3
v1= (−1, 1−a)
v3 = (0, −1)
v2= (−1, −a)
a > 0
ρ2
ρ1
σ1
σ3
σ2
v1 = (−1, 1−a)
v2= (0, −1)
a > 0


ρ2
σ1
v1= (−1, 0)
ρ1
σ2
σ3
v2= (−1 ,−1)
ρ2
σ1
ρ1
σ2
v = (−1 ,−1)


ρ2
v1= (−1/m ,1/m)
σ1
v2= (−1,0)
σ2
ρ1
σ3
v3= (− a+1/m ,−1/m)
a > 0
v1= (−1/m ,1/m)
ρ2
σ1
ρ1
σ2
v2= (− a+1/m ,−1/m)

5.7 Smooth spherical $SL(2)$-varieties which are toric

In order to classify all $SL(2)$-varieties with $\mathbb{C}^*$-action which are toric we use the characterization of toric variety via its Cox rings which must be a polynomial ring. On the other hand, the Cox ring is defined by the unique equation

$$X_1X_4 - X_2X_3 = \prod_{i=1}^{r} Y_i^{\tilde{p}_i+\tilde{q}_i}$$

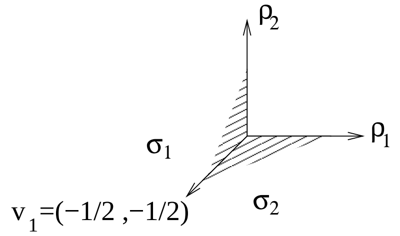
in $\mathbb{C}[X_1, \ldots, X_4, Y_1, \ldots, Y_r]$. Therefore, $X(\Sigma)$ is toric if and only if there exists a single index $i \in \{1, \ldots, r\}$ such that $\tilde{p}_i + \tilde{q}_i > 0$. In this case $\tilde{p}_i + \tilde{q}_i = 1$ and

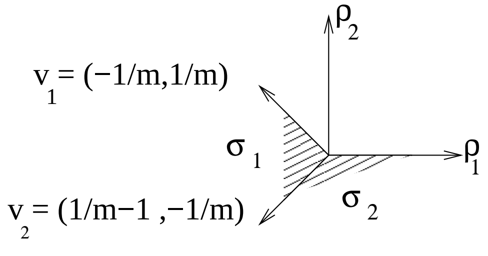
$$\tilde{p}_j + \tilde{q}_j = 0, \ \ \forall j \neq i.$$

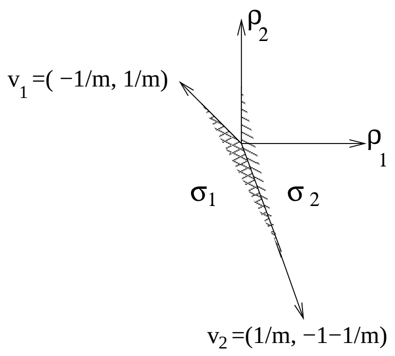
So we have the equation

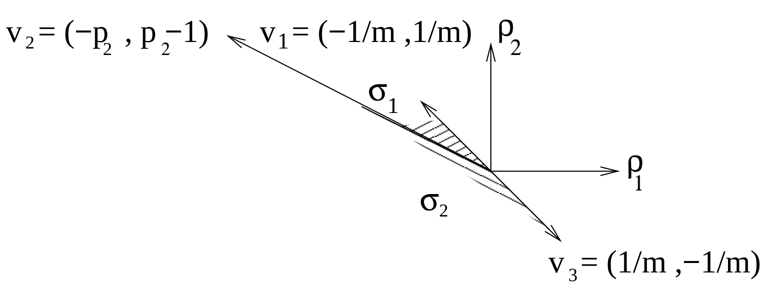
$$X_1X_4 - X_2X_3 = Y_i.$$

Hence $r \leq 3$ and we can apply our classification of smooth spherical $SL(2)$-varieties with the Picard number ≤ 3. By our classification of spherical $SL(2)$-varieties, we obtain the following 4 possibilities for the colored fan $X(\Sigma)$:






ρ2
v1 =(−1/m, 1/m)
ρ1
σ1
σ 2
v2 =(1/m, −1−1/m)


v 2 = (−p2 , p 2−1)
v1 = (−1/m ,1/m)
ρ2
σ1
ρ1
σ2
v 3 = (1/m ,−1/m)

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Zusammenfassung in deutscher Sprache

Man kann den n-dimensionalen projektiven Raum $\mathbb{P}^n$ über $\mathbb{C}$ als den folgenden Quotienten:

$$(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$$

definieren. Eine ähnliche Quotientenkonstruktion für eine beliebige torische Varietät X wurde von David Cox vorgeschlagen [Cox95]. Sei X eine n-dimensionale torische Varietät mit einem rationalen polyhedralen Fächer $\triangle$. Sei $\triangle(1)$ die Menge aller 1-dimensionaler Kegel von $\triangle$ und sei $\mathrm{Cl}(X)$ die Divisorenklassengruppe von X. Dann wirkt die algebraische Gruppe (Quasitorus):

$$\mathbb{T} := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X), \mathbb{C}^*)$$

auf natürliche Weise auf dem affinen Raum $\mathbb{C}^{\triangle(1)}$, so dass der kategorische Quotient $(\mathbb{C}^{\triangle(1)} \setminus Z)/\mathbb{T}$ existiert und isomorph ist zu X. Hierbei ist Z eine Zariski abgeschlossene Menge, die durch ein homogenes Ideal im Koordinatenring $\mathbb{C}[x_0, \ldots, x_n]$ definiert ist.

Eine quasihomogene $SL(2)$-Varietät ist eine normale 3-dimensionale algebraische Varietät X über einem algebraisch abgeschlossenen Körper k zusammen mit einer regulären $SL(2)$-Wirkung, so dass X eine offene dichte Bahn hat. Zur Vereinfachung betrachten wir den Fall $k = \mathbb{C}$. In dieser Arbeit geben wir eine geometrische Methode zur Konstruktion einer speziellen Klasse von $SL(2)$-Varietäten X als kategorische Quotienten an.

Als ersten Schritt haben wir den affinen Fall von $SL(2)$-Varietäten betrachtet [BH08]. Diese Varietäten wurden von Popov [P73] klassifiziert. Jede affine $SL(2)$-Varietät E ist durch zwei Zahlen eindeutig bestimmt; einer rationalen Zahl $h = p/q$ ($\gcd(p, q) = 1$, $0 < h \leq 1$), welche die *Höhe* von E genannt wird, sowie einer natürlichen Zahl m, dem sogenannten *Grad* von E. Die entsprechende affine $SL(2)$-Varietät wird mit $E_{h,m}$ bezeichnet. Wir haben gezeigt, dass die Varietät $E_{h,m}$ zum kategorischen Quotienten der affinen Hyperfläche:

$$H_{q-p} : \{X_0^{q-p} = X_1X_4 - X_2X_3\} \subset \mathbb{C}^5$$

nach der Wirkung der diagonalisierbaren Gruppe $G_0 \times G_m \subset D(5, \mathbb{C})$ isomorph ist, wobei

$$G_0 \cong \mathbb{C}^* \cong \{\text{diag}\,(t, t^{-p}, t^{-p}, t^q, t^q);\ t \in \mathbb{C}^*\},\ G_m \cong \mu_m = \langle \zeta_m \rangle,$$

so dass $G_m \subset D(5, \mathbb{C})$ von diag $(1, \zeta_m^{-1}, \zeta_m^{-1}, \zeta_m, \zeta_m)$ erzeugt ist.

Wir haben dann bemerkt, dass affine $SL(2)$-Varietäten zu einer Klasse von $SL(2)$-Varietäten gehören, die eine zusätzliche mit der $SL(2)$-Wirkung kommutierende $\mathbb{C}^*$-Wirkung besitzen. Diese Varietäten werden wir *$SL(2)$-Varietäten mit $\mathbb{C}^*$-Wirkung* nennen. Es ist sehr wichtig, dass $SL(2)$-Varietäten mit $\mathbb{C}^*$-Wirkung als *sphärische G-Varietäten* bezüglich der regulären Wirkung der reduktiven 4-dimensionalen algebraischen Gruppe

$$G := SL(2) \times \mathbb{C}^*,$$

betrachtet werden können, d.h., der Stabilizator $H \subset G$ eines Punkts der offenen $SL(2)$-Bahn 1-dimensionale sphärische Untergruppe von G ist. Deshalb werden wir $SL(2)$-Varietäten mit $\mathbb{C}^*$-Wirkung auch *sphärische quasihomogene $SL(2)$-Varietäten* nennen.

Sphärische Varietäten sind eine Verallgemeinerung von torischen Varietäten und wurden durch gefärbte Fächer von strikt konvexen Kegeln klassifiziert. Durch diese kombinatorische Beschreibung kann man einige geometrische Eigenschaften von diesen Varietäten wie beispielsweise Glattheit, Vollständigkeit oder Projektivität bestimmen ([K91]). Wir bemerken, dass die offene dichte $SL(2)$-Bahn einer sphärischen $SL(2)$-Varietät X isomorph ist zu $SL(2)/\mathcal{C}_m$, wobei $\mathcal{C}_m$ eine zyklische Gruppe der Ordnung m ist. Die Zahl m ist eine Verallgemeinarung des Grads im affinen Fall.

Mit Hilfe der Theorie der sphärischen Varietäten können wir eine beliebige sphärische quasihomogene $SL(2)$-Varietät $X = X(\Sigma)$ durch einen 2-dimensionalen gefärbten Fächer Σ in $\mathbb{R}^2$ beschreiben. Seien $v_1, \ldots, v_r \in \mathbb{Z}^2$ die Menge aller Erzeuger der Gitter von 1-dimensionalen Kegeln in Σ, und $v_i = (-p_i, -q_i)$, $\gcd(p_i,\ q_i) = 1$ $(1 \le i \le r)$. Wir zeigen, dass $X(\Sigma)$ als GIT-Quotient der folgenden affinen Hyperfläche in $\mathbb{C}^{r+4}$:

$$Y_1^{p_1+q_1} \cdots Y_r^{p_r+q_r} = X_1X_4 - X_2X_4$$

nach der Wirkung der diagonalisierbaren Gruppe $G_0 \times G_m \subset D(r+4, \mathbb{C})$ realisiert werden kann. Hierbei ist $G_0 \cong (\mathbb{C}^*)^r$ und G_m eine zyklische Gruppe der Ordnung m.

Mit dieser Konstruktion war es nicht schwierig zu zeigen, dass der Cox Ring dieser Varietäten durch eine einzige Gleichung definiert ist. Ähnliche Beispiele von algebraischen Varietäten, deren Cox Ring durch eine einzige Gleichung definiert wird, wurden in [BH07] betrachtet. Im affinen Fall kann diese Beschreibung von Cox Ringen als eine Illustration von allgemeineren neuen Resultaten von Brion über Cox Ringe von sphärischen Varietäten verwendet werden [B07].

D. Luna und Th. Vust haben in [LV83] kombinatorische Diagramme entdeckt um eine beliebige normale $SL(2)$-Einbettung X zu beschreiben (wir nennen diese Diagramme *Luna-Vust Diagramme*). Diese Diagramme geben Informationen über die lokalen Ringe von $SL(2)$-Bahnen in X. In dieser Arbeit finden wir für beliebige sphärische $SL(2)$-Varietäten $X = X(\Sigma)$ eine Methode zur Konstruktion des korrespondierenden Luna-Vust Diagramms aus dem 2-dimensionalen gefärbten Fächer Σ.

2-dimensionale gefärbte Fächer, welche $SL(2)$-Varietäten mit $\mathbb{C}^*$-Wirkungen definieren, sind sehr gut geeignet für die Untersuchung ihrer birationalen Morphismen. Durch die gefärbten Fächer werden wir alle glatten $SL(2)$-Varietäten mit $\mathbb{C}^*$-Wirkungen und Picard Zahl ≤ 3 klassifizieren. Von diesen Varietäten haben wir alle minimalen glatten Varietäten bestimmt, das sind Varietäten, die keine Aufblasung anderer Varietäten sind. Dies verallgemeinert die Resultate der L. M. Jauslin [Ja87] in dem speziellen Fall minimaler glatter $SL(2)$- und $PGL(2)$-Einbettungen. Ferner haben wir minimale glatte $SL(2)$-Varietäten mit $\mathbb{C}^*$-Wirkung, welche zusätzlich torisch sind, bestimmt.

Appendix-Lebenslauf

Lebenslauf

19.12.1972	geboren in Aleppo, Syrien
1978–1988	Schulbesuch in Althaorastadt: 1978–1983 Grundschule 1983–1988 Gymnasium
1989	Abitur
1989–1994	Studium der Mathematik an der Universität Aleppo
1994–1996	Diplomarbeit in der abstrakten Algebra und Analysis an der Universität Aleppo
1996–1998	Magisterarbeit "Studium über graduierte Ringe", betreut von Prof. Dr. S. Saad, Universität Aleppo
1998–2001	Lehrerin am Gymnasium Der-Algemal
seit 2001	Wissenschaftliche Angestellte an der Mathematischen Fakultät der Universität Damaskus
2004–2008	Promotionsstipendium der Universität Damaskus an der Universität Tübingen
seit 2006	Doktorarbeit "Spherical quasihomogeneous $SL(2)$-varieties", betreut von Prof. Dr. V. Batyrev, Universität Tübingen
seit 2008	Wissenschaftliche Angestellte am Mathematischen Institut der Universität Tübingen