Gorenstein toric Fano varieties

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Dissertation

der Fakultät für Mathematik und Physik
der Eberhard-Karls-Universität Tübingen
zur Erlangung des Grades eines
Doktors der Naturwissenschaften vorgelegt

2005

Tag der mündlichen Qualifikation:	22.07.2005
Dekan:	Prof. Dr. P. Schmid
1. Berichterstatter:	Prof. Dr. V. Batyrev
2. Berichterstatter:	Prof. Dr. J. Hausen

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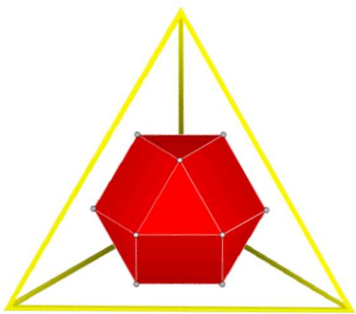

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Für Jule

Contents

Introduction

In this thesis we concern ourselves with Gorenstein toric Fano varieties, that is, with complete normal toric varieties whose anticanonical divisor is an ample Cartier divisor. These algebraic-geometric objects correspond to reflexive polytopes introduced by Batyrev in [Bat94]. Reflexive polytopes are lattice polytopes containing the origin in their interior such that the dual polytope also is a lattice polytope. It was shown by Batyrev that the associated varieties are ambient spaces of Calabi-Yau hypersurfaces and together with their duals naturally yield candidates for mirror symmetry pairs. This has raised a lot of interest in this special class of lattice polytopes among physicists and mathematicians. It is known that in fixed dimension d there only are a finite number of isomorphism classes of d-dimensional reflexive polytopes. Using their computer program PALP [KS04a] Kreuzer and Skarke succeeded in classifying d-dimensional reflexive polytopes for $d \leq 4$ [KS98, KS00, KS04b]. They found 16 isomorphism classes for $d = 2$, 4319 for $d = 3$, and 473800776 for $d = 4$.

While there are many papers devoted to the study and classification of nonsingular toric Fano varieties [WW82, Bat82a, Bat82b, Bat99, Sat00, Deb03, Cas03a, Cas03b], in the singular case there has not yet been done so much, especially in higher dimensions. This can be explained by several difficulties: First many algebraic-geometric methods like birational factorization, Riemann-Roch or intersection theory cannot simply be applied, especially since there need not exist a crepant toric resolution. Second most convex-geometric proofs relied on the vertices of a facet forming a lattice basis, a fact which is no longer true for reflexive polytopes, where facets can even contain lattice points in their interior. Third the huge number of reflexive polytopes causes any classification approach to depend heavily on computer calculations, hence often we do not get mathematically satisfying proofs even when restricting to low dimensions.

The aim of this thesis is to give a first systematic mathematical investigation of Gorenstein toric Fano varieties by thorougly examining the combinatorial and geometric properties of their convex-geometric counterparts, that is, reflexive polytopes. We would like to generalize useful tools and theorems previously only known to hold for nonsingular toric Fano varieties to the case of mild singularities and to prove classification theorems in important cases and in arbitrary dimension. Moreover we are interested in finding constraints on the combinatorics of reflexive polytopes and conjectures and sharp bounds on invariants that can explain interesting observations made in the large computer data.

As we will see most of these aims are actually not out of reach, for instance it will unexpectedly turn out that $\mathbb{Q}$-factorial Gorenstein toric Fano varieties are in many aspects nearly as benign as nonsingular toric Fano varieties. Moreover by these generalizations with a strong focus on combinatorics even results previously already proven in the nonsingular case become more transparent.

Above all this work provides useful tools, several conjectures to prove in the future and many results that show reflexive polytopes to be truly interesting objects - not only from a physicist's but also from a pure mathematician's point of view.

This thesis is organized in six chapters. Any major chapter (3-6) starts with an introductory section, in which also an explicit list of the most important new results is contained. Furthermore the reader will find right after this introduction a summary of notation and at the end of this thesis an index as well as a comprehensive bibliography.

The first two chapters cover the notions that are basic for this work.

Chapter 1 fixes the main notation and gives a survey of important results from toric geometry.

Chapter 2 gives an exposition of toric Fano varieties and classes of singularities that appear naturally when trying to desingularize toric Fano varieties. Here we set up the dictionary of convex-geometric and algebraic-geometric notions: Fano polytopes correspond to toric Fano varieties, smooth Fano polytopes to nonsingular toric Fano varieties and canonical (respectively terminal) Fano polytopes to toric Fano varieties with canonical (respectively terminal) singularities. Moreover simplicial Fano polytopes are associated to $\mathbb{Q}$-factorial toric Fano varieties.

Chapter 3 is the heart of this thesis. Here the main objects of study are introduced: Reflexive polytopes corresponding to Gorenstein toric Fano varieties.

At the beginning two elementary technical tools are investigated and generalized that have already been used to successfully investigate and classify nonsingular toric Fano varieties [Bat99, Sat00, Deb03, Cas03b]. The first one, that is especially useful in lower dimensions, is the projection map. We prove some general facts about projections of reflexive polytopes (Prop. 3.2.2), thereby we can relate the properties of Gorenstein toric Fano varieties to that of lower-dimensional toric Fano varieties. As an application we present generalizations to mild singularities of an algebraic-geometric result due to Batyrev [Bat99] stating that the anticanonical class of a torus-invariant prime divisor of a nonsingular toric Fano variety is always numerically effective (Cor. 3.2.7, Prop. 3.2.9). Moreover we get as a trivial corollary (Lemma 3.5.6) that a lattice point on the boundary that has lattice distance one from a facet F has to be contained in a facet F' intersecting F in a codimension two face. Previously this observation could be proven by Debarre in [Deb03] only in the case of a smooth Fano polytope by using a lattice basis among the vertices of F.

The second important tool is the notion of primitive collections and relations. It was introduced by Batyrev in [Bat91] to completely describe smooth Fano polytopes and has been essential for his classification of nonsingular toric Fano 4-folds [Bat99]. In general this tool is not applicable for reflexive polytopes, since it uses the existence of lattice bases among the vertices. The special case of a primitive collection of length two corresponds to a pair of lattice points on the boundary that do not lie in a common facet. This situation is extremely important and was investigated by Casagrande in [Cas03a] to prove some strong restrictions on smooth Fano polytopes. Now the author has been able to suitably generalize this notion to reflexive polytopes (Prop. 3.3.1) and apply it successfully to show that the same restrictions also hold for simplicial reflexive polytopes. In algebraic-geometric language the corresponding statement reads as follows: the Picard number of a $\mathbb{Q}$-factorial Gorenstein toric Fano variety exceeds the Picard number of a torus-invariant prime divisor at most by three (Cor. 3.5.17). As another application we prove that any pair of vertices of a simplicial reflexive polytope can be connected by at most three edges, where the case that less than three edges do not suffice can only occur for a centrally symmetric pair of vertices (Cor. 3.3.2). In graph-theoretic language this yields that the diameter of the edge-graph of a simplicial reflexive polytope is at most three. From this we easily get that certain combinatorial types cannot be realized as reflexive polytopes (Cor. 3.3.4).

The main part of the third chapter deals with upper bounds on the volume and the number of lattice points and vertices of a reflexive polytope.

We state in Conjecture 3.5.2 that a d-dimensional reflexive polytope has at most $6^{d/2}$ vertices, where equality holds only for one special even-dimensional reflexive polytope. We also present some preliminary coarse bounds depending on the combinatorics of the facets (Prop. 3.5.5). In the last chapter the conjecture will be proven to hold for centrally symmetric simple reflexive polytopes. Here in the third chapter the main focus is on the simplicial case: We extend the long-standing conjecture of Batyrev on the maximal number of vertices of smooth Fano polytopes to simplicial reflexive polytopes (Conj. 3.5.7): It states that a d-dimensional simplicial reflexive polytope has at most $3d$ vertices, if d is even, and $3d-1$ vertices, otherwise. We have also included in this conjecture the extra statement that the number of $3d$ vertices should only be obtained in one unique case. Now using the previously described tools we give a proof in the case of a centrally symmetric pair of vertices of the dual polytope (Thm. 3.5.11). Based on these results that were published in the preprint [Nil04a] Casagrande has been able to successfully prove Conjecture 3.5.7 in [Cas04].

In low dimensions the maximal number of lattice points of a reflexive polytope is achieved only by some reflexive simplices. Hence it makes sense to restrict to this situation in order to find a good general upper bound. A possible way to do this is to prove a sharp upper bound on the volume of a reflexive simplex. This has been achieved by the author in Theorem 3.7.13. To this end we describe in section 3.6 the common approach of Batyrev [Bat94] and Conrads [Con02] to determine lattice simplices by so called weight systems. In the case of reflexive simplices they correspond to unit fractions summing up to one, e.g., $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$. Now the proof follows from upper bounds on the denominators of these unit fractions (Prop. 3.6.29). Moreover using this approach we can prove in Thm. 3.7.19 an observation of Haase and Melnikov [HM04] stating that there is a unique d-dimensional reflexive simplex with the maximal number of lattice points on an edge, namely $2y_{d-1}-1$, for the sequence (y_n) defined as $y_0 = 2$ and $y_n = 1 + y_0 \cdots y_{n-1}$.

So we see that in this situation convex-geometric questions naturally correspond to non-trivial problems in elementary number theory. However these results may also be translated into algebraic geometry: For instance the bound on the volume yields a sharp upper bound on the anticanonical degree of a d-dimensional Gorenstein toric Fano variety X with class number one, in particular on weighted projective spaces with Gorenstein singularities (Thm. 3.7.7): For $d = 2$, respectively $d = 3$, the degree of X is at most 9, respectively 72; for $d \geq 4$ the degree of X is at most $2(y_{d-1}-1)^2$, and equality holds only for one special variety. This bound for $d = 3$ (called the Fano-Iskovskikh conjecture) has recently been proven by Prokhorov [Pro04] for three-dimensional Gorenstein Fano varieties with canonical singularities. Now the previous result yields enough evidence to motivate a generalization of the Fano-Iskovskikh conjecture to higher dimensions. This shows again that toric varieties are fertile testing grounds for conjectures on more general varieties.

We also describe as another useful tool "counting modulo a natural number" as introduced by Batyrev in [Bat82a]. This simple method can be successfully applied to prove a sharp bound on the number of lattice points in terminal reflexive polytopes (Cor. 3.7.23) and will also be used in the last chapter.

Chapter 4 gives a first application of the techniques developed in the previous chapter: a complete, explicit, computer-independent classification of three-dimensional Gorenstein toric Fano varieties with terminal singularities. There are precisely 100 isomorphism classes, corresponding to so called quasi-smooth Fano polytopes (Thm. 4.3.2). The main idea of the proof is to show that in most cases there exists a vertex that is centrally symmetric to some other vertex and at the same time the sum of two other vertices. Then we use the result (Prop. 4.2.17) that the corresponding quasi-smooth Fano polytope is uniquely determined by a small set of special relations among its vertices, these are called quasi-primitive relations. To this end we include a general discussion of various notions of primitive collections and relations at the beginning of this chapter.

Chapter 5 is concerned with a special set of lattice points that can be associated to a complete fan, called the set of roots. For the fan of normals of a reflexive polytope the set of roots is precisely the set of lattice points in the interior of facets. Roots are important in order to describe the automorphism group of the associated toric variety, in particular its dimension. Centrally symmetric roots are called semisimple roots. The automorphism group is reductive if and only if any root is semisimple. There are two different approaches to describe the set of roots: The more algebraic one due to Cox [Cox95] uses the notion of the homogeneous coordinate ring and can be applied to complete toric varieties. The more geometric one is due to Bruns and Gubeladze [BG99] and directly concerned with lattice polytopes. The author first applied to the set of roots the results in the third chapter about pairs of lattice points in reflexive polytopes, here it turned out Bruns and Gubeladze had already derived similar observations. Later the author gave a generalization to complete toric varieties using the approach of Cox. The main idea is to introduce so called facet bases and root bases that parametrize the set of facets containing roots and the set of semisimple roots in a geometrically convenient way (Def. 5.1.6, Prop. 5.1.22).

As an application of this notion we derive that a complete toric variety is already a product of projective spaces, if the set of semisimple roots span the vectorspace (Prop. 5.1.19). In the case of a reflexive polytope we even get that the intersection of the reflexive polytope with the space spanned by all semisimple roots is again a reflexive polytope associated to a product of projective spaces (Thm. 5.2.12). The figure on the title of this work illustrates for $\mathbb{P}^3$ the general phenomenon 5.1.18 that the convex hull of all semisimple roots is also always a reflexive polytope. Further we obtain that a d-dimensional reflexive polytope has at most $2d$ facets containing roots, where equality implies the toric variety to be a product of projective lines (Cor. 5.2.4). As one of the main results we prove that the reductive automorphism group of a d-dimensional complete toric variety that is not a product of projective spaces has at most dimension 2 for $d = 2$, respectively $d^2 - 2d + 4$ for $d \geq 3$ (Thm. 5.1.25). This sharp bound yields an explanation for observations on the number of roots of reflexive polytopes due to Kreuzer and the author in the database [KS04b].

One of the main motivation to study the set of roots was given by the goal to find combinatorial criteria for the automorphism group to be reductive. There is a well-known result due to Matsushima that the automorphism group of a nonsingular Fano variety is reductive, if the variety admits an Einstein-Kähler metric. In [BS99] Batyrev and Selivanova showed that such a metric exists, if the reflexive polytope, whose fan of normals is associated to the toric variety, is symmetric, i.e., the group of linear automorphisms of the polytope has no nonzero fixpoint. Hence they got that symmetry implies any root to be semisimple, and they asked for a purely convex-geometric proof of this corollary. Moreover the existence of an Einstein-Kähler metric implies the barycenter of this reflexive polytope to be zero, the reverse implication could only be proven very recently in [WZ04] by Wang and Zhu. However at the beginning of this research it was not clear that even in the case of a nonsingular toric Fano variety the vanishing of the barycenter implies all roots to be semisimple, yet this was observed by Batyrev and Kreuzer even for any reflexive polytope up to dimension four in the computer database and they conjectured that this should hold in any dimension.

Now this chapter contains purely convex-geometric proofs of these conjectures and implications in greatest possible generality, as well as some other combinatorial criteria that are sufficient for the automorphism group to be reductive (Thm. 5.3.1). We illustrate these results by several examples and classify all three-dimensional symmetric reflexive polytopes (Thm. 5.4.5). Moreover we show, how these criteria relate to the approach of Kreuzer [Kre03a, Kre03b] to investigate sums of lattice points in multiples of a reflexive polytope by a variant of the Erhart polynomial.

Chapter 6 gives as a final application of the results achieved some insight in centrally symmetric reflexive polytopes. A main result states that the unit lattice cube $[-1,1]^d$ is the one and only d-dimensional centrally symmetric reflexive polytope with the maximal number of lattice points (Thm. 6.5.1). For the proof we use the fact that the unit lattice cube is also the only such polytope with the maximal number of $2d$ roots (Thm. 6.1.1).

In the case of a simple centrally symmetric reflexive polytope we can prove the general Conjecture 3.5.2 on the maximal number of vertices (Thm. 6.2.2). This is actually an application of the other main result of this chapter: a complete classification of arbitrary-dimensional simplicial reflexive polytopes having a centrally symmetric pair of facets (Thm. 6.3.1, Cor. 6.3.3). This is a unifying generalization of results of Ewald [Ewa88] and his students [Wag95, Wir97]; the proofs here are considerably simpler since we use dual bases rather than calculations with determinants. Applying to $d \leq 5$ yields $4, 5, 15, 20$ isomorphism classes of these d-dimensional polytopes (Thm. 6.3.12). As a corollary we get that any d-dimensional simplicial reflexive polytope with a centrally symmetric pair of facets can be embedded in the unit lattice cube $[-1,1]^d$, while the dual polytope can be embedded in $\lfloor d/2 \rfloor \; [-1,1]^d$. Moreover we prove a general result on embedding a reflexive polytope into a multiple of the unit lattice cube under some mild assumptions (Thm. 6.4.4). Finally we determine the maximal number of lattice points, namely $2d^2 + 1$, a simplicial reflexive polytope with a centrally symmetric pair of facets can have. Again there is only one such polytope with this number of vertices (Thm. 6.5.3).

Acknowledgment

First of all I want to express my gratitude to my thesis advisor Professor Victor V. Batyrev for posing problems, his advice and encouragement.

I am thankful to Professor Klaus Altmann for kind hospitality at the FU Berlin and the possibility of giving a talk. I would also like to thank the DFG Forschungsschwerpunkt "Globale Methoden in der komplexen Geometrie" for financial support.

I am grateful to Professor Christian Haase for his genuine interest, fruitful discussions resolving in Example 5.1.26, his invitation to the Algebraic Geometry seminar at Duke University and for the warm hospitality I experienced there. I am thankful to the Duke University for financial support. It is also a pleasure to thank Milena Hering for her invitation to the student Algebraic Geometry seminar at the University of Michigan and for the nice time I had there.

Further I would like to express my gratitude to Professor Maximillian Kreuzer for his kind support with the classification data and his efforts with the computer package PALP, and for many motivating examples and observations. The database [KS04b] has been of enormous use to me, since it gives mathematicians the possibility to test and find many interesting conjectures. Furthermore Professor Kreuzer initiated and participated in studying the properties of successive sums of lattice points in reflexive polytopes as documented in the manuscripts [Kre03a, Kre03b] that were kindly given to my disposal.

Thanks also go to Professor Annette A'Campo-Neuen for giving reference to [Büh96], Professor Günter Ewald for giving reference to [Wir97], Professor Xiaohua Zhu for giving reference to [WZ04], and Professor Olivier Debarre for an exposition of the proof of [Deb03, Theorem 8]. Moreover I would like to thank Professor Günter Ziegler for giving smart answers to stupid questions.

Last but not least I wish to thank Mark Blume, Tobias Döhler, Andreas Hermann and Juliane Rost for proof-reading and the Research Group Algebra for the entirely stimulating environment. The axiom of choice was not harmed during the making of this thesis.

I am forever indebted to my family for unbreakable faith, unbelievable support and uncountable signs of love.

Notation

General

$\mathbb{N}$	$=\{0,1,2,\ldots\}$; 0 is a natural number but not positive
α_k	calculating modulo k (p. 88)
$\lfloor x \rfloor$ ($\lceil x \rceil$)	the greatest (lowest) integer lower (greater) or equal to $x \in \mathbb{R}$
$\ldots, \hat{a}_i, \ldots$	the "hat"-symbol means that the ith entry a_i is left out
N, M, $\langle \cdot, \cdot \rangle$	dual lattices N and M with pairing $\langle \cdot, \cdot \rangle$ (p. 19)
$N_{\mathbb{R}}, M_{\mathbb{R}}$	associated real vectorspaces
d	usually the dimension of N, M, $N_{\mathbb{R}}$, $M_{\mathbb{R}}$
lin, aff	the linear span, respectively the affine span (p. 19)
pos, conv	the positive hull, respectively the convex hull (p. 19)
$< A >_{\mathbb{Z}}$	the set of integer linear combinations of elements in $A \subseteq M$

Fans

$\triangle$	usually a (complete) fan (often in N) (p. 19)
$\triangle(k)$	the set of k-dimensional cones of $\triangle$
$\mathrm{supp}(\triangle)$	the support of $\triangle$ (p. 21)
$X(N, \triangle)$	the toric variety associated to lattice N and fan $\triangle$ (p. 20)
σ	usually a (full-dimensional) cone
$\sigma^{\vee}$	the dual cone of σ (p. 19)
τ	usually a ray, i.e., a one-dimensional cone in $\triangle$
$\mathcal{V}_\tau$	the torus-invariant primedivisor associated to ray τ (p. 21)
v_τ	the unique primitive lattice point on ray τ (p. 22)
$\mathrm{SF}(N, \triangle)$	the set of piecewise linear functions (p. 22)
h	often an element in $\mathrm{SF}(N, \triangle)$
D_h	the Cartier divisor associated to h (p. 22)
P_h	the polytope of global sections of D_h (p. 26)
G_σ, $G_\triangle$	set of primitive lattice points on rays of σ, resp. $\triangle$ (p. 35, 37)
$Q_\triangle$, $P_\triangle$	$Q_\triangle = \mathrm{conv}(G_\triangle)$, $P_\triangle = {Q_\triangle}^*$ (p. 37, 37)
$\triangle_v$	$= \Sigma_{P_v}$ (p. 49)
$\overline{\triangle}$	fan associated to some ray τ (p. 49)

Important polytopes and varieties

E_d	the d-dimensional reflexive simplex with $X(M, \Sigma_{E_d}) \cong \mathbb{P}^d$ (p. 126)
$\mathcal{Z}_d$	the d-dimensional standard lattice zonotope (p. 58)
S_3	the two-dimensional del Pezzo surface $X(M, \Sigma_{\mathcal{Z}_2})$ (p. 58)
$\mathcal{F}_d$	the d-dimensional del Pezzo polytope (p. 153)
W_d	the d-dimensional del Pezzo variety $X(M, \Sigma_{\mathcal{F}_d})$ (p. 153)
$\tilde{\mathcal{F}}_d$	the d-dimensional pseudo del Pezzo polytope (p. 153)
$\tilde{W}_d$	the d-dimensional pseudo del Pezzo variety $X(M, \Sigma_{\tilde{\mathcal{F}}_d})$ (p. 153)
D_d	d-dimensional simplicial reflexive polytope with $2d$ vertices (p. 155)
$\mathcal{D}_d$	$= X(M, \Sigma_{D_d})$ (p. 155)

Polytopes

P	usually a d-dimensional polytope (often in $M_{\mathbb{R}}$)
P^*	the dual polytope of P (p. 24)
$\mathcal{N}_P$	the fan of normals of P (p. 24)
Σ_P	the fan spanned by the faces of P
X_P	the toric variety associated to the normal fan of P
b_P	the barycenter of P (p. 25)
wb_P	the weighted barycenter of P (p. 140)
e_P	the Ehrhart polynomial of P (p. 25)
s_P	the lattice point sum polynomial of P (p. 140)
r_P	the root sum polynomial of P (p. 143)
$\mathrm{vol}(P)$	the volume of P, if P is full-dimensional
$\mathcal{W}(P)$	the graph of lattice points on the boundary of P (p. 53)
$\mathrm{Aut}_M(P)$	automorphisms of M leaving $P \subseteq M_{\mathbb{R}}$ invariant
∂P	the boundary of P
$\mathrm{int}P$	the interior of P, if P is full-dimensional
$G \leq P$	G is a face of P
$\mathcal{V}(P)$	the set of vertices of P
$\mathcal{F}(P)$	the set of facets of P
F	usually a facet of P
η_F	the unique inner normal of F, i.e., $\langle \zeta_F, F \rangle = -1$ (p. 24)
ζ_F	the unique primitive inner normal of F (p. 25)
ν_F	$= -\eta_F$
$\mathrm{relint}F$	the relative interior of F
$\mathrm{rvol}(F)$	the relative volume of F (p. 25)
$\det \mathrm{aff}(F)$	determinant of affine sublattice generated by F
v, w, x, y	usually lattice points
$[x, y]$	$= \mathrm{conv}(x, y)$, similarly $]x, y]$, $]x, y[$
$v \sim w$	v, w are contained in a common facet
$v \vdash w$	v is away from w for v, w lattice points on ∂P (p. 47)
$\mathrm{st}(v)$	the star set of v for $v \in \partial P$ (p. 47)
∂v	the link of v for $v \in \partial P$ (p. 47)
π_v	the projection map along a lattice point v (p. 48)
ι_v	the inverse map of π_v from P_v onto $\mathrm{st}(v)$ (p. 48)
M_v, P_v	$M_v = M/\mathbb{Z}v$, $P_v = \pi_v(P)$
X_v	$= X(M_v, \triangle_v)$ (p. 49)
$z(v, w)$	lattice point in ∂P, if $v \not\sim w$ and $v + w \neq 0$ (p. 52)

Quasi-smooth Fano polytopes

$\mathcal{P}$	a primitive collection of P (p. 91)
$\sigma(\mathcal{P})$	the sum of the elements in $\mathcal{P}$ (p. 91)
$\deg(\mathcal{P})$	the degree of $\mathcal{P}$ (p. 92)
$LR(P)$	group of linear relations of lattice points in P (p. 93)
π_i,P_i,M_i,ι_i	$= \pi_{v_i}$, $\pi_{v_i}(P)$, M_{v_i}, ι_{v_i} for vertex v_i (p. 97)
$\partial_M(v_i)$	$= \partial(v_i) \cap M$ (p. 97)
$\deg(v_i)$	$= \lvert \partial_M(v_i) \rvert$, the degree of the vertex v_i (p. 97)
$m_i(v)$, $b_i(v)$	numbers associated to vertices v_i, v (p. 104)
n	$= \lvert \mathcal{V}(P) \rvert$ for P quasi-smooth Fano polytope (p. 113)
f_i, p	number of i-dim. faces, parallelogram facets (p. 113)
$\lambda(P)$, $M(P)$	$\lambda(P) = n - 3 - \rho_X$, rank of the matrix $M(P)$ (p. 113)

Toric varieties

X	usually a (toric) variety of dimension d (p. 20)
$\mathrm{Aut}(X)$	automorphism group, $\mathrm{Aut}^\circ(X)$ is connected component
K_X, $-K_X$	canonical, respectively anticanonical, divisor of X
$\deg(X)$	$= (-K_X)^d$, the (anticanonical) degree of X
$\mathrm{Cl}(X)$	The class group of X (p. 21)
$\equiv$	linear equivalence of divisors (p. 21)
$\mathrm{Pic}(X)$	the Picard group of X (p. 22)
ρ_X	the Picard number of X (p. 22)
$\mathrm{NE}(X)$	the Mori cone of X (p. 23)
$\mathrm{discr}(X)$	discrepancy of X (p. 34)
j_X	the Gorenstein index of X (p. 33)
$E := \mathrm{Exc}(f)$	the exceptional locus of a birational morphism f
$E^{(1)}$	the union of the exceptional divisors
X^*	the "dual" toric variety (p. 84)

Weight systems

$\mathrm{Herm}(d, \lambda)$	Hermite normal form matrices of size d, det λ (p. 69)
Q	here a weight system, i.e., an element of $\mathbb{Q}^{d+1}_{>0}$ (p. 68)
Q_{red}	the reduction of the weight system Q (p. 68)
$\|Q\|$	the total weight of the weight system Q (p. 68)
λ_Q	the factor of the weight system Q (p. 68)
m_Q	an invariant of the weight system Q (p. 70)
$\mathbb{P}(Q)$	weighted projective space with weight system Q
Q_P	weight system associated to lattice polytope P (p. 69)
λ_P	the factor of the weight system Q_P (p. 69)
M_P	the lattice generated by the vertices of P
P_Q	simplex associated to reduced weight system Q (p. 69)
S_Q	polytope associated to reduced weight system Q (p. 72)
Q_d	Sylvester weight system of length d (p. 73)
Q'_d	enlarged Sylvester weight system of length d (p. 73)
y_n	Sylvester sequence $2, 3, 7, 43, \ldots$ (p. 73)
t_n	$= y_n - 1 = y_0 \cdots y_{n-1}$, sequence $1, 2, 6, 42, \ldots$ (p. 73)

Roots

$\mathcal{R}$	set of roots of fan or reflexive polytope (p. 122, 130)
$\mathcal{S}$	set of semisimple roots (p. 122, 130)
$\mathcal{U}$	$= \mathcal{R} \backslash \mathcal{S}$, set of unipotent roots
τ_m, x_m	associated ray, resp. monomial, to a root m (p. 124)
$\mathcal{S}_1$	$= \{m \in \mathcal{S} \,:\, \tau_m$ not associated to some unipotent root$\}$
$\mathcal{S}_2$	$= \mathcal{S} \backslash \mathcal{S}_1$
$v \perp w$	v and w are orthogonal (p. 123, 132)
$p(v, w)$	root associated to roots v, w (p. 124)
$v \equiv w$	v and w are equivalent semisimple roots (p. 124)
$\mathcal{F}_v$	facet containing root v of a reflexive polytope P
η_v	$= \eta_{\mathcal{F}_v}$ for a reflexive polytope, in general see p. 122
S	often the homogeneous coordinate ring (p. 123)
Y,$\mathcal{M}$	indeterminates, respectively monomials, in S (p. 124)
Y_i	equivalence class of indeterminates in S (p. 125)
p, q, r, s	number of special equivalence classes Y_i (p. 125)

Chapter 1

Fans, polytopes and toric varieties

Introduction

The main purpose of this chapter is to fix the notation and to give an overview of the most important results about toric varieties and lattice polytopes. All toric varieties are normal, but may be singular. Proofs are usually left out, since they can be easily found in the literature, i.e., [Ewa96], [Ful93] and [Oda88].

1.1 Cones and fans

Let $N \cong \mathbb{Z}^d$ be a d-dimensional *lattice* and $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \cong \mathbb{Z}^d$ the *dual lattice* with $\langle \cdot, \cdot \rangle$ the nondegenerate symmetric pairing. As usual, $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^d$ and $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^d$ (respectively $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$) will denote the rational (respectively real) scalar extensions.

Throughout this work the roles of N and M are interchangeable.

For a subset S of $N_{\mathbb{R}}$ let $\mathrm{lin}(S)$, resp. $\mathrm{aff}(S)$, be the *linear span*, resp. *affine span*, of S. We denote by $\mathrm{pos}(S)$ the *positive hull* of S, i.e., the set of positive linear combinations, and by $\mathrm{conv}(S) = \mathrm{pos}(S) \cap \mathrm{aff}(S)$ the *convex hull* of S.

A *polyhedral cone* $\sigma \subseteq N_{\mathbb{R}}$ is the positive hull $\mathrm{pos}(G)$ of finitely many points G in $N_{\mathbb{R}}$. If additionally the generators G are lattice points, i.e., $G \subseteq N$, then σ is called a *polyhedral lattice cone*. Finally a cone σ is called *strongly convex*, if it does not contain a linear subspace, i.e., $0 \in N$ is an apex of σ.

We define further the *dual cone* $\sigma^\vee = \{u \in M_{\mathbb{R}} \,:\, \langle u, v \rangle \geq 0 \;\forall\, v \in \sigma\}$. It is fully ($d$-)dimensional if and only if σ is strongly convex. If σ is a lattice cone, then also $\sigma^\vee$ is a lattice cone (Farkas Lemma).

A *face* of a cone σ is the intersection of σ with an affine hyperplane such that σ is contained in one of the two halfspaces. A defining vector for such an hyperplane is called a defining *outer*, respectively *inner*, *normal* for that face. The normal of a codimension one face can be uniquely defined by some normalizing condition.

A *fan* $\triangle$ is here defined to be a collection of finitely many strongly convex polyhedral lattice cones such that any face of a cone in $\triangle$ is also an element of

$\triangle$ and the intersection of two cones in $\triangle$ is a face in each. For $k \in \mathbb{N}$ we let $\triangle(k)$ denote the set of k-dimensional cones in $\triangle$.

1.2 The classical construction of a toric variety from a fan

Let σ be a cone in a fan $\triangle$. Then the set $S_\sigma = M \cap \sigma^\vee$ of lattice points in its dual cone is a finitely generated saturated additive submonoid of M generating M, i.e., $< S_\sigma >_{\mathbb{Z}} = M$. Here saturation for a submonoid W of M means that any lattice point that can be written as a rational positive combination of elements in W, i.e., a lattice point in $\mathrm{pos}(W) \cap M_{\mathbb{Q}}$, must already be contained in W.

Now we define the finitely generated $\mathbb{C}$-Algebra $A_\sigma := \mathbb{C}[S_\sigma]$, and the corresponding affine scheme $U_\sigma := \mathrm{Spec}\, \mathbb{C}[S_\sigma]$. U_σ is called the *affine toric variety* associated to σ. When looking at its closed points, i.e., the maximal spectrum of A_σ, the Hilbertscher Nullstellensatz implies $\mathrm{Spm}\, \mathbb{C}[S_\sigma] = \mathrm{Hom}_{\mathbb{C}-alg.}(\mathbb{C}[S_\sigma], \mathbb{C}) = \mathrm{Hom}_{\mathrm{mon}}(S_\sigma, \mathbb{C}) = \{\phi : S_\sigma \to \mathbb{C},\ \phi(0) = 1,\ \phi(m + m') = \phi(m) \cdot \phi(m')\}$.

$A_{\{0\}} = \mathbb{C}[M]$ is isomorphic to the ring of laurent polynomials and $T_N := U_{\{0\}}$ is an *algebraic torus*, i.e., the closed points of T_N are $\mathrm{Spm}\, \mathbb{C}[M] = \mathrm{Hom}_{\mathrm{mon}}(M, \mathbb{C})$ $= \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) = N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^d$. The *characters* $\mathrm{e}(m) \in A_{\{0\}}$ of the torus are just the monomials for $m \in M$.

If $\tau \leq \sigma$ is a face of σ, then one has $A_\tau = (A_\sigma)_{\mathrm{e}(u)}$ for some $u \in S_\sigma$ with $\tau = \sigma \cap u^\perp$. Especially the natural map $U_\tau \to U_\sigma$, on the closed points given by $\phi \mapsto \phi|_{S_\sigma}$, is an open immersion as the elementary open set $\mathcal{D}_{U_\sigma}(\mathrm{e}(u))$. The embedding of T_N is called *the big torus*.

There is a natural torus action of T_N given on the closed points of U_σ by $T_N \times U_\sigma \to U_\sigma$, $(t, \phi) \mapsto t \cdot \phi$ pointwise, induced by the map $A_\sigma \to T_N \otimes A_\sigma$, $\mathrm{e}(u) \to \mathrm{e}(u) \otimes \mathrm{e}(u)$. This extends the action of the big torus T_N on itself.

Finally we define the *toric variety* $X(\triangle) = X(N, \triangle)$ associated to the fan $\triangle$ as the disjoint union of the affine toric varieties U_σ ($\sigma \in \triangle$), where for $\sigma_1, \sigma_2 \in \triangle$ one glues (as schemes) along the in U_{σ_1} und U_{σ_2} mutually open subvariety $U_{\sigma_1 \cap \sigma_2}$.

An alternative construction of a toric variety X as a *categorical quotient* was given by Cox in [Cox95], here X can even be described as a geometric quotient if and only if any cone in the fan defining X is a simplex.

1.3 The category of toric varieties

Let $\triangle$ be a fan. Then the previously constructed $X(\triangle)$ is a variety in the sense of [Har77, II.4.10], i.e., an integral separated scheme of finite type over $\mathbb{C}$, $X(\triangle)$ is normal and equipped with a natural torus action by T_N extending its inherent diagonal action. Of course $X(\triangle)$ is a disjoint union of the orbits under the torus action, this can be read off explicitly from the fan, see [Ful93, 3.1]. Every cone $\sigma \in \triangle$ corresponds exactly to a $\dim(\sigma)$-codimensional torus orbit closure $\mathcal{V}_\sigma$ in $X(\triangle)$.

We generally define a *toric variety* X as a normal irreducible algebraic variety over $\mathbb{C}$ with an open embedded algebraic torus $T = (\mathbb{C}^*)^d$ acting on X in extension of its own action.

To get a category of toric varieties one needs a suitable notion of a morphism: So let X and Y be two toric varieties with its embedded toruses T_X and T_Y. Then a morphism of varieties $f : Y \to X$ is called a *morphism of toric varieties*, if additionally a morphism of algebraic groups $f' : T_Y \to T_X$ exists, so that f is *equivariant* with respect to f', i.e., $f(t \cdot y) = f'(t) \cdot f(y)$ for all $t \in T_Y$ and $y \in Y$.

Now again let $\triangle$ be a fan with the lattice N, and also $\triangle'$ a fan with the lattice N'. A $\mathbb{Z}$-linear map $\phi : N' \to N$ is called a *map of fans*, if the image of any cone in $\triangle'$ under $\phi \otimes_{\mathbb{Z}} \mathbb{R}$ is mapped in a cone in $\triangle$. Then one can construct a morphism of toric varieties $\phi_* : X(N', \triangle') \to X(N, \triangle)$ that is equivariant with respect to $\phi_*|T_{N'} = \phi \otimes 1 : T_{N'} = N' \otimes_{\mathbb{Z}} \mathbb{C}^* \to T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^*$.

This gives a natural map from the category of fans to the category of toric varieties. One has the theorem that this is a *covariant isomorphism of categories*. Under this correspondence an affine toric variety corresponds to the fan of the faces of a strongly convex polyhedral lattice cone. Also proper toric varieties correspond to *complete* fans. A complete fan $\triangle$ is a fan such that its support is the whole space $N_{\mathbb{R}}$, where its *support* $\mathrm{supp}(\triangle)$ is defined as the union of all cones contained in $\triangle$.

In [Oda88, Prop. 1.33] it is explained, when an equivariant morphism between toric varieties corresponds to a *toric fibre-bundle*; in particular products of toric varieties correspond to products of the associated fans. In the same way a *toric blow-up*, i.e., a blow-up of a toric variety along a torus-invariant subvariety, is associated to a fan that is a star subdivision of the corresponding cone, this can be found in [Oda88, Prop. 1.26].

1.4 The class group, the Picard group and the Mori cone

As typical in the case of toric varieties their invariants can be extracted from the fan. Let $X = X(N, \triangle)$.

A *k-cycle* is defined to be an element of the free abelian group on the k-dimensional irreducible closed subvarieties of X. Define the *Chow group* $\mathrm{A}_k(X)$ of X as the quotient of the group of k-cycles modulo rational equivalence (see [Har77, A.1]). Then $\mathrm{A}_k(X)$ is generated by the classes of orbit closures $\mathcal{V}_\sigma$ for $\sigma \in \triangle(d-k)$.

Let's look at the *class group* $\mathrm{Cl}(X) = \mathrm{A}_{d-1}$ of X, i.e., the group of Weil divisors modulo linear equivalence $\equiv$. Any torus-invariant primedivisor $\mathcal{V}_\tau$ corresponds exactly to one *ray* $\tau \in \triangle(1)$, i.e., a one-dimensional cone in $\triangle$. These divisors generate the group of torus-invariant Weil divisors $T_N\mathrm{Div}(X)$. Further a torus-invariant principal Weil divisor is exactly given by $\mathrm{div}(\mathrm{e}(u))$ for $u \in M$. We have the following right exact sequence, which is exact, if every element of $N_{\mathbb{R}}$ is a linear combination of elements in $\mathrm{supp}(\triangle)$, i.e., $\mathrm{lin}(\mathrm{supp}(\triangle)) = N_{\mathbb{R}}$:

$$0 \to M \to T_N\mathrm{Div}(X) = \bigoplus_{\tau \in \triangle(1)} \mathbb{Z}\mathcal{V}_\tau \to \mathrm{Cl}(X) \to 0. \tag{1.1}$$

In particular $\mathrm{Cl}(X)$ is a finitely generated abelian group, its rank is called the *class number* of X. When the rays of $\triangle$ span $N_{\mathbb{R}}$, we have $\mathrm{rank}(\mathrm{Cl}(X)) = r - d$. $\mathrm{Cl}(X)$ is in general not torsionfree, even if X is proper.

The calculation of the *Picard group* $\mathrm{Pic}(X)$, i.e., the group of Cartier divisors modulo linear equivalence, is based on the following notion: Let $\mathrm{SF}(N, \triangle)$ denote the set of all functions $h : \mathrm{supp}(\triangle) \to \mathbb{R}$ being $\mathbb{Z}$-valued on $N \cap \mathrm{supp}(\triangle)$ and linear on $\mathrm{supp}(\triangle)$, i.e., for every $\sigma \in \triangle$ there exists $l_\sigma \in M$ with $h\,|_\sigma = l_\sigma\,|_\sigma$. Let $T_N\mathrm{CDiv}(X)$ denote the set of torus-invariant Cartier divisors. For a ray $\tau \in \triangle(1)$ we define v_τ to be the (unique) *primitive* lattice point on τ, i.e., the first lattice point on τ. For $h \in \mathrm{SF}(N, \triangle)$ we set

$$D_h := -\sum_{\tau \in \triangle(1)} h(v_\tau)\mathcal{V}_\tau \in T_N\mathrm{CDiv}(X).$$

For $\sigma \in \triangle$ we get $D_h\,|_{U_\sigma} = \mathrm{div}(\mathrm{e}(-l_\sigma))$. The map $h \mapsto D_h$ defines an isomorphism of free abelian groups:

$$\mathrm{SF}(N, \triangle) \cong T_N\mathrm{CDiv}(X).$$

Note: A Weil divisor D on X is Cartier iff $D\,|_{U_\sigma}$ is principal for all $\sigma \in \triangle$. Again we have a right exact sequence which is exact if $\mathrm{lin}(\mathrm{supp}(\triangle)) = N_\mathbb{R}$:

$$0 \to M \to \mathrm{SF}(N, \triangle) \to \mathrm{Pic}(X) \to 0. \tag{1.2}$$

$\mathrm{Pic}(X)$ is also a finitely generated abelian group. Define the *Picard number* ρ_X of X to be the rank of $\mathrm{Pic}(X)$. We have $\rho_X \leq \mathrm{rank}(\mathrm{Cl}(X)) = |\triangle(1)| - d$. If $\triangle$ contains at least one d-dimensional cone, then $\mathrm{Pic}(X)$ is (torsion-)free which is equivalent to the splitting of this sequence. If X is proper, i.e., $\triangle$ is complete, then any maximal cone of $\triangle$ is d-dimensional. In the proper case one can explicitly compute the Picard number from the fan (see [Ewa96, V.5.9] and [Eik93]).

Between the above two sequences we have a natural commutative diagram enduced by the embedding $T_N\mathrm{CDiv}(X) \subseteq T_N\mathrm{Div}(X)$. Exactly in the case of a nonsingular variety these groups are equal and the two diagrams are naturally isomorphic.

If any maximal cone in a fan is d-dimensional, e.g., $\triangle$ complete, we have:

$$\mathrm{Pic}(X) \cong H^2(X; \mathbb{Z}).$$

This implies

$$\mathrm{Pic}(X) \cong \mathrm{NS}(X), \tag{1.3}$$

where $\mathrm{NS}(X)$ is the *Néron-Severi-Group*, i.e., the group of Cartier divisors modulo algebraic equivalence.

From now on let X be proper. Define $N^1(X)$ as the group of Cartier divisors modulo numerical equivalence. Two Cartier divisors are called *numerically equivalent*, if they have the same intersection number (see [Deb01]) with every curve, i.e., both associated invertible sheafs have the same degree on of their restrictions to any curve. Now we simply get:

$$\mathrm{Pic}(X) \cong N^1(X).$$

Especially the above defined Picard number is in the proper case the same as the usual Picard number $\rho_X = \mathrm{rank}(N^1(X))$ (see [Deb01, 1.3]).

In the same way let's look at curves for X proper. $\mathrm{A}_1(X)$ is generated by the classes of the one-dimensional orbit closures $\mathcal{V}_\rho$ for $\rho \in \triangle(d-1)$, such a ρ

is called a *wall*. By [Rei83] or [Mat02] we have for every irreducible curve C in X that C is rationally equivalent to $\sum_{\rho\in\triangle(d-1)} a_\rho[\mathcal{V}_\rho]$ for $a_\rho \in \mathbb{N}$.

Now one defines $N_1(X)$ as the quotient of the group of 1-cycles modulo numerical equivalence, where two 1-cycles are *numerically equivalent* if they have the same intersection number with every Cartier divisor. This gives a nondegenerate *intersection pairing*

$$\mathrm{Pic}(X) \times N_1(X) \to \mathbb{Z}.$$

Hence if X is nonsingular, then by (1.2) we get that $N_1(X)$ is isomorphic to the group of integer relations among the primitive generators of the rays of $\triangle$.

Let $\mathrm{NE}(X)$ denote the *Mori cone of curves*, i.e. the set of classes of effective 1-cycles in $N_1(X) \otimes_{\mathbb{Z}} \mathbb{R}$. Because rational equivalence implies numerical equivalence, the above result on rational equivalence of curves yields that

$$\mathrm{NE}(X) = \sum_{\rho\in\triangle(d-1)} \mathbb{R}_{\geq 0}[\mathcal{V}_\rho] \tag{1.4}$$

is a closed polyhedral cone generated by the classes of the walls of the fan.

1.5 Polytopes and lattice points

First two important results from convex geometry (e.g., see [Ewa96, Zie95]).

Let $S \subseteq M_{\mathbb{R}}$ be any subset.

Theorem 1.5.1 (Helly's theorem). *Any point in* $\mathrm{conv}(S)$ *is in the convex hull of at most* $d+1$ *points in* S*.*

Theorem 1.5.2 (Steinitz's theorem). *Any point in the interior of* $\mathrm{conv}(S)$ *is in the interior of the convex hull of at most* $2d$ *points in* S*.*

Now let's define the main objects:

A *polyhedron* is a finite intersection of halfspaces in $M_{\mathbb{R}}$. A bounded polyhedron is called a *polytope*. A polytope P can also be characterized as the convex hull of finitely many points.

The boundary of P is denoted by ∂P, the relative interior of P by $\mathrm{relint}P$. When P is full-dimensional, its relative interior is also denoted by $\mathrm{int}P$. Analogously to cones one can define *faces* and *normals* of a polytope. A k-dimensional face of P is called a *vertex* for $k=0$, an *edge* for $k=1$, or a *facet*, if it is has codimension one. A face F of P is denoted by $F \leq P$, the vertices of P form the set $\mathcal{V}(P)$, the facets of P the set $\mathcal{F}(P)$. A polytope is always the convex hull of its vertices. The boundary is covered by facets. There is the so called *diamond property* stating that any $(d-2)$-dimensional face of a d-dimensional polytope is contained in exactly two different facets.

P is called a *lattice polytope*, respectively *rational polytope*, if $\mathcal{V}(P) \subseteq M$, respectively $\mathcal{V}(P) \subseteq M_{\mathbb{Q}}$. A homomorphism (resp. iso-) of lattice polytopes is a homomorphism (resp. iso-) of the associated lattices such that the induced real linear homomorphism maps the one polytope to (resp. onto) the other.

There is the so called *support function* h_P of a (rational) polytope $P \subseteq M_{\mathbb{R}}$ defined by

$$h_P \,:\, N_{\mathbb{R}} \to \mathbb{R},\ y \mapsto \inf\{\langle y, x\rangle \,:\, x \in P\}.$$

h_P is a positive homogeneous (see [Ewa96, Def. 5.5]) upper convex function. There exists a coarsest complete fan $\mathcal{N}_P$ in $N_\mathbb{R}$ called the *normal fan* of P such that h_P is linear on each cone. This terminology comes from the following observation: Define for a face F of P the *normal cone* $\mathcal{N}_P(F)$ as the union of $\{0\}$ and the set of inner normals of F. In terms of h_P this means

$$\overline{\mathcal{N}_P(F)} = \{y \in N_\mathbb{R} \,:\, h_P(y) = \langle y, x\rangle \;\forall\, x \in F\},$$

where we take the closure of $\mathcal{N}_P(F)$. Then

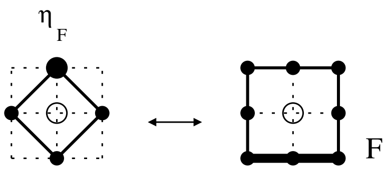
$$\mathcal{N}_P = \{\overline{\mathcal{N}_P(F)} \,:\, F \leq P\}.$$

Now let P be a d-dimensional lattice polytope with $0 \in \mathrm{int}P$. Apart from the normal fan there is a more obvious possibility to define a complete fan from P: Let $\Sigma_P := \{\mathrm{pos}(F) \,:\, F \leq P\}$ be the *fan spanned by P*.

These two constructions can be related by the notion of the *dual polytope*

$$P^* \;:=\; \{y \in N_\mathbb{R} \,:\, \langle y, x\rangle \geq -1 \;\forall\, y \in P\}.$$

P^* is a d-dimensional rational polytope with $0 \in \mathrm{int}P$ and vertices in $N_\mathbb{Q}$. P^* does not have to be a lattice polytope again.



We have the *duality*

$$(P^*)^* = P.$$

There is an inclusion-reversing *combinatorial* correspondence between k-dimensional faces of P and $d-1-k$-dimensional faces of P^*. For instance the dual of a *simplicial* polytope, where any facet is a *simplex*, i.e., the convex hull of d vertices, is a *simple* polytope, where any vertex is only contained in d facets.

Geometrically, if F is a face of P, then its corresponding face of P^* is given by all inner normals of F which lie in the affine hyperplane $\{y \in N_\mathbb{R} \,:\, \langle y, x\rangle = -1 \,\forall\, y \in F\}$. For a facet $F \leq P$ we let $\eta_F \in N_\mathbb{Q}$ denote the *unique inner normal* satisfying $\langle \eta_F, F\rangle = -1$. Hence we have

$$\mathcal{V}(P^*) = \{\eta_F \,:\, F \in \mathcal{F}(P)\}.$$

We have now the following important relation

$$\mathcal{N}_P = \Sigma_{P^*} \;\text{ and vice versa }\; \mathcal{N}_{P^*} = \Sigma_P.$$

The dual of the product of d_i-dimensional polytopes $P_i \subseteq \mathbb{R}^{d_i}$ with $0 \in \mathrm{int}P_i$ for $i = 1, 2$ is given by

$$(P_1 \times P_2)^* = \mathrm{conv}(P_1^* \times \{0\}, \{0\} \times P_2^*) \subseteq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}. \tag{1.5}$$

It is in general an unsolved problem to find equations or inequalities between non-combinatorial invariants of P and P^* such as the number of lattice points or the volume. For some partial relations in low dimensions see 3.7.1.

Now let's do a short excursion into some more or less well-known convex and enumerative geometry (see for instance [Sta86]).

So let from now on $P \subseteq M_{\mathbb{R}}$ be a lattice polytope of dimension $n \leq d$.

Choose a $\mathbb{Z}$-basis of $M_{\mathbb{R}}$, so one can measure lengths and volumes, and also a regular parametrization $\phi : \mathbb{R}^n \to P$ where the image of the canonical basis $e_1, \ldots, e_n$ is a lattice basis of the affine sublattice $\operatorname{aff}(P) \cap M$. Then the Jacobian of ϕ is denoted by $\det \operatorname{aff}(P)$, this is just the volume of the fundamental paralloped of the affine sublattice $\operatorname{aff}(P) \cap M$. If F is a facet of P and $n = d$, then one can also prove that $\det \operatorname{aff}(F)$ is the length of ζ_F, where $\zeta_F \in N$ is defined as the *unique primitive inner normal* of F, i.e., ζ_F is the first non-zero lattice point on the one-dimensional cone of inner normals of F.

Now the *relative volume* or *lattice volume* of P is well-defined as

$$\operatorname{rvol}(P) := \operatorname{vol}_{\mathbb{R}^n}(\phi^{-1}(P)) = \frac{\operatorname{vol}_n(P)}{\det \operatorname{aff}(P)},$$

where $\operatorname{vol}_n(P)$ is just the volume of P in the sense of differential geometry. $\operatorname{rvol}(P)$ is thereby invariant under unimodular transformations of $M_{\mathbb{R}}$, so independent of the chosen $\mathbb{Z}$-basis. The relative volume is just the ordinary one, if P is full-dimensional, i.e., $n = d$.

Now recall the definition of the analytical *barycenter* b_P of P

$$b_P := \left(\frac{\int_P x_1 \, dx}{\operatorname{vol}_n(P)}, \ldots, \frac{\int_P x_n \, dx}{\operatorname{vol}_n(P)}) \right),$$

where the integral must be understood in the sense of differential geometry. Obviously $S(P) \in M_{\mathbb{Q}}$ is invariant of the chosen $\mathbb{Z}$-basis.

In the case of $b_P = 0$ there is the following (coarse) inequality called *Blaschke-Santaló inequality* (see [Lut93, p.165]):

$$\operatorname{vol}(P)\operatorname{vol}(P^*) < \omega_d^2, \tag{1.6}$$

where ω_d is the volume of the d-dimensional unit ball in $\mathbb{R}^d$.

An important definition is the *lattice point enumerator* of P:

$$e_P(k) := |\, kP \cap M \,| \quad \text{for } k \in \mathbb{N}.$$

There is the following classical result:

Theorem 1.5.3 (Theorem of Ehrhart). *There exists a unique polynomial $e(X) \in \mathbb{Q}[X]$ called* Ehrhart polynomial *of P such that $e(k) = e_P(k)$ for all $k \in \mathbb{N}$. It has degree $\deg(e) = n = \dim(P)$. Denote $\operatorname{coeff}_i(e) \in \mathbb{Q}$ for the coefficient of $e(x)$ of degree $i \in \mathbb{N}$. Then one has*

$$\operatorname{coeff}_n(e) = \operatorname{rvol}(P),$$

$$\operatorname{coeff}_{n-1}(e) = \frac{1}{2} \sum_{F \in \mathcal{F}(P)} \operatorname{rvol}(F),$$

$$\operatorname{coeff}_0(e) = 1.$$

And the following reciprocity law *holds:*

$$|\operatorname{relint}(kP) \cap M\,| = (-1)^d e(-k) \; \forall k \in \mathbb{N}_{>0}.$$

Especially the relative volume of a lattice polytope is a rational number. By augmentation this is also true for polytopes with vertices in $M_{\mathbb{Q}}$.

As an application let's prove the *theorem of Pick*: Let P be a two-dimensional lattice polytope in the plane. Then $\mathrm{vol}_2(P) = \mathrm{coeff}_2(e_P) = 1/2((e_P(1)-1) + (e_P(-1)-1)) = 1/2(|P \cap M| + |\mathrm{int}P \cap M| - 2) = 1/2|\partial P \cap M| + |\mathrm{int}P \cap M| - 1$.

Recently the theorem of Ehrhart was substantially generalized by the following theorem of Brion and Vergne [BV97, Prop.4.1]:

Theorem 1.5.4. *Let $P \subseteq M_{\mathbb{R}}$ be a d-dimensional lattice polytope. Let ϕ be a homogeneous polynomial function of degree g. Define for $k \in \mathbb{N}$*

$$i(\phi, P)(k) := \sum_{m \in kP \cap M} \phi(m) \quad \text{and} \quad i(\phi, \mathrm{int}P)(k) := \sum_{m \in (k\mathrm{int}P) \cap M} \phi(m).$$

Then one has

$$k \mapsto i(\phi, P)(k) \quad \text{and } k \mapsto i(\phi, \mathrm{int}P)(k)$$

are polynomials of degree $\leq d + g$. There is the reciprocity law:

$$i(\phi, \mathrm{int}P)(k) = (-1)^{d+g} i(\phi, P)(-k) \; \forall \, k \in \mathbb{N}_{>0}.$$

To recover the main parts of the theorem of Ehrhart just choose $\phi = 1$.

1.6 Big and nef Cartier divisors

Let $X = X(N, \triangle)$ and $h \in \mathrm{SF}(N, \triangle)$. We define the polyhedron

$$P_h := \{x \in M_{\mathbb{R}} \; : \; \langle x, y \rangle \geq h(y) \; \forall \, y \in \mathrm{supp}(\triangle)\}.$$

If $\mathrm{supp}(\triangle)$ generates $N_{\mathbb{R}}$ as a cone, i.e., $\mathrm{pos}(\mathrm{supp}(\triangle)) = N_{\mathbb{R}}$, e.g., $\triangle$ complete, then P_h is a (possibly empty) polytope of dimension $n \leq d$. The *global sections* of D_h are

$$H^0(X, O_X(D_h)) = \bigoplus_{u \in P_h \cap M} \mathbb{C}\mathrm{e}(u). \tag{1.7}$$

Especially the dimension over $\mathbb{C}$ of the global sections of $O_X(D_h)$ is just the number of lattice points in P_h.

Let X from now on be proper and h be (uniquely) given by $l_\sigma \in M$ on every cone $\sigma \in \triangle(d)$.

It follows from 1.7 that, if $P_h \neq \emptyset$, then

$$e_{P_h} \; : \; k \mapsto \dim_{\mathbb{C}} H^0(X, O_X(kD_h))$$

is just the Ehrhardt polynomial of the n-dimensional polytope P_h of degree n with leading coefficient $\mathrm{rvol}(P_h)$.

Generally a Cartier divisor D on X is called *big*, if

$$\liminf_{k \to +\infty} \frac{\dim_{\mathbb{C}} H^0(X, kD)}{k^d} > 0.$$

It follows therefore that

$$D_h \text{ big } \iff P_h \text{ has dimension } n = d.$$

One calls a Cartier divisor D *nef*, if $D \in \mathrm{NE}(X)^\vee$, i.e., $D.C \geq 0$ for all curves C on X.

Proposition 1.6.1. *There are the following equivalences:*

1. D_h *is base-point-free, i.e.,* $O_X(D_h)$ *is generated by its global sections*
2. h *is* upper convex, *i.e.,* $h(x) + h(y) \leq h(x + y)$ *for all* $x, y \in N_{\mathbb{R}}$
3. $l_\sigma \in P_h$ *for all* $\sigma \in \triangle(d)$
4. $P_h = \mathrm{conv}(l_\sigma \ : \ \sigma \in \triangle(d))$
5. $D_h.\mathcal{V}_\rho \geq 0$ *for all* $\rho \in \triangle(d-1)$
6. D_h *is nef*

If this holds, then $h(v) = \inf\{\langle u, v\rangle \ : \ u \in P_h \cap M\}$ *for* $v \in N$ *and* h *is the support function of the non-empty* n*-dimensional lattice polytope* P_h*.*

These equivalences can be regarded as a strong version of the *base-point-freeness theorem* (see [Deb01, 7.32]) for proper toric varieties.

In this context we should mention, how to compute the intersection number in the fifth equivalence:

Let $\rho \in \triangle(d-1)$. There exist $\sigma_1, \sigma_2 \in \triangle(d)$ with $\rho = \sigma_1 \cap \sigma_2$. Now let $\overline{N} := N / < \rho \cap N >_{\mathbb{Z}}$. One has the canonically projected fan $\overline{\triangle} = \{\{0\}, \overline{\sigma_1}, \overline{\sigma_2}\}$ where $\overline{\sigma_1} = \mathrm{pos}(v_{\sigma_1})$ and $\overline{\sigma_2} = \mathrm{pos}(v_{\sigma_2})$ for primitive $v_{\sigma_1}, v_{\sigma_2} \in \overline{N}$. We have $\mathcal{V}_\rho = X(\overline{N}, \overline{\triangle}) \cong \mathbb{P}^1$. For $\overline{h} = h - l_{\sigma_1} \in \mathrm{SF}(\overline{N}, \overline{\triangle})$ we have

$$D_h.\mathcal{V}_\rho = \deg(O_{\mathcal{V}_\rho}(D_{\overline{h}})) = -\overline{h}(v_{\sigma_1}) - \overline{h}(v_{\sigma_2}).$$

Let's look at the *self-intersection number* of a Cartier divisor D on X: D^d is generally defined as $d!$-times the leading coefficient of the polynomial

$$q_D \ : \ k \mapsto \chi(X, O_X(kD))$$

of degree $\leq d$.

For D_h nef, there is the following *vanishing theorem* (that is special for the toric case):

$$H^i(X, O_X(D_h)) = 0 \text{ for all } i > 0. \tag{1.8}$$

If D_h is nef, then this implies

$$q_{D_h} = e_{P_h}$$

and therefore we get (no vanishing theorem is necessary for this, see [Deb01, 1.31])

$${D_h}^d = d! \,\mathrm{vol}_d(P_h),$$

and D_h is also big iff this number is not zero.

Going further into intersection theory and mixed volumes one can prove the *theorem of Bernstein* (see [Ful93, 5.5]).

1.7 Ample Cartier divisors and projective toric varieties

Let $X = X(N, \triangle)$ be a proper toric variety and $h \in \mathrm{SF}(N, \triangle)$ given by elements $l_\sigma \in M$ for $\sigma \in \triangle(d)$.

Let $E = \{m_0, \ldots, m_s\} \subseteq P_h \cap M$ be a non-empty set of global sections of D_h (under the usual identification). Then we have:

$$E \text{ generates } O_X(D_h) \text{ iff } l_\sigma \in E \text{ for all } \sigma \in \triangle(d).$$

Such an E exists iff D_h is base-point-free, i.e., nef. In this case there is the following morphism of toric varieties:

$$\Psi_E \ : \ X \to \mathbb{P}^s(\mathbb{C}),\ x \mapsto [\mathrm{e}(m_0)(x) : \cdots : \mathrm{e}(m_s)(x)] \tag{1.9}$$

Proposition 1.7.1. *There are the following equivalences:*

1. *E generates $O_X(D_h)$ and Ψ_E is a closed immersion*
2. *For all $\sigma \in \triangle(d)$ is $\Psi_E|_{U_\sigma}$ a closed immersion and $\Psi_E^{-1}(\{x_j \neq 0\}) = U_\sigma$ for $m_j = l_\sigma$*
3. *S_σ is generated by $E - l_\sigma$ for all $\sigma \in \triangle(d)$ and h is* strictly upper convex, *i.e., $h(x) + h(y) \leq h(x + y)$ for all $x, y \in N_\mathbb{R}$ with equality iff x and y are contained in a common cone $\sigma \in \triangle(d)$*
4. *D_h is ample and S_σ is generated by $E - l_\sigma$ for all $\sigma \in \triangle(d)$*

For any such E the degree *of X under the closed immersion Ψ_E is $d!\mathrm{vol}_d(P_h)$.*

To get a criterion for very ampleness, just set $E = P_h \cap M$:

D_h is *very ample*	iff	D_h is nef and $\Psi_{P_h \cap M}$ a closed immersion
	iff	S_σ is generated by $P_h \cap M - l_\sigma$ for all $\sigma \in \triangle(d)$ and h is strictly upper convex.

However ampleness is a more combinatorial condition:

Proposition 1.7.2. *There are the following equivalences:*

1. *h is strictly upper convex*
2. *$l_\sigma \in P_h$ for all $\sigma \in \triangle(d)$ and $l_{\sigma_1} \neq l_{\sigma_2}$ for $\sigma_1, \sigma_2 \in \triangle(d)$ with $\sigma_1 \neq \sigma_2$*
3. *P_h has as vertices exactly the pairwise different elements l_σ for $\sigma \in \triangle(d)$*
4. *$D_h.\mathcal{V}_\rho > 0$ for all $\rho \in \triangle(d-1)$*
5. *D_h is* ample, *i.e., there exists a $k > 0$ such that D_{kh} is very ample*

If D_h is ample, then D_h is also nef and big, especially D_h is base-point-free and P_h is a d-dimensional lattice polytope with $\mathcal{N}_{P_h} = \triangle$.

So in the toric case *Kleiman's criterion* holds also for proper varieties. One gets as a corollary:

X is *projective*	iff	$\triangle$ is spanned by a d-dimensional lattice polytope $Q \subseteq N_\mathbb{R}$ with $0 \in \mathrm{int}Q$
	iff	$\triangle$ is the normal fan of a d-dimensional lattice polytope $P \subseteq M_\mathbb{R}$.

Definition 1.7.3. Let $P \subseteq M_{\mathbb{R}}$ be a rational polytope. We define the associated projective toric variety as
$$X_P := X(N, \mathcal{N}_P).$$

A concrete way to describe X_P is to take E as the set of lattice points in a sufficiently large multiple of P and define X_P as the closure of $\Psi_E(\mathbb{C}^*)^d$ in (1.9). An alternative construction is also possible by describing X_P as the projective spectrum of a semigroup algebra (see [Bat94]).

For d-dimensional rational polytopes P_1, P_2 equation (1.5) implies
$$X_{P_1} \times X_{P_2} \cong X_{P_1 \times P_2}.$$

Remark 1.7.4. As was observed in [Con02, Prop. 2.1] *projective* toric varieties are isomorphic as *abstract* varieties if and only if they are as *toric* varieties.

Chapter 2

Singularities and toric Fano varieties

Introduction

In this section desingularization or resolution of singularities is described and how different classes of singularities can be derived from this. Since this subject is not explicitly contained in the standard literature of algebraic geometry such as Hartshorne [Har77], it is a treated here in a more detailed manner than above. We refer to [Mat02] and [Deb01]. Furthermore we set up the correspondence of toric Fano varieties and Fano polytopes, this section is essential for the next chapters. Here we refer to [Dai02], [Deb03] and [Sat02].

2.1 Resolution of singularities and discrepancy

Let X be a normal complex variety. X is called *globally factorial* iff any Weil divisor is principal ($\mathrm{Cl}(X) = 0$). In the affine case this is just the fact that the coordinate ring is factorial. Recall that a Cartier divisor is a locally principal Weil divisor. There is the analogous definition of a *factorial* variety iff any Weil divisor is Cartier. This is equivalent to the fact that any localization is a factorial ring. This holds for X nonsingular. Analogous one can define a Weil divisor D to be $\mathbb{Q}$-*principal*, respectively $\mathbb{Q}$-*Cartier*, if there exists a positive natural number k such that kD is principal, respectively Cartier. If any Weil divisor is $\mathbb{Q}$-principal, respectively $\mathbb{Q}$-Cartier, X is called *globally* $\mathbb{Q}$-*factorial*, respectively $\mathbb{Q}$-*factorial*. Finally an element in $\mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ is called $\mathbb{Q}$-*divisor*.

Remark 2.1.1. It would be more systematic to call globally ($\mathbb{Q}$-)factorial varieties ($\mathbb{Q}$-)factorial varieties, and ($\mathbb{Q}$-)factorial varieties locally ($\mathbb{Q}$-)factorial varieties (this is done in [Deb01]). However in *toric* literature the notion of $\mathbb{Q}$-factorial varieties is already established (see [Mat02] or [Cas04]).

Let X,Y be complex varieties of dimension d.

Generally a *desingularization* or *resolution of singularities* of X is defined as a proper birational morphism $f : Y \to X$ with Y nonsingular. The existence of such a resolution is part of the next theorem which is somewhat stronger and needs some more notation.

First let $f : Y \to X$ be a proper birational morphism with X normal. Define the *exceptional locus* $E := \mathrm{Exc}(f)$ as the closed subset of Y where f is not a local isomorphism. Then f is surjective, $E = f^{-1}(f(E))$ and $\mathrm{codim}_X f(E) \geq 2$. $f(E)$ is the set of elements of X with positive dimensional fibre and $X - f(E)$ is the *domain* of f^{-1}, i.e., the largest open set over which $f^{-1} : X \dashrightarrow Y$ is defined. Here it is important to note that, if X is $\mathbb{Q}$-factorial every irreducible component of $\mathrm{Exc}(f)$ has codimension 1, i.e., the exceptional locus is the union of primedivisors. In general this might not be true. (For these results see [Deb01, 1.40]).

Second let Y be nonsingular and D an effective divisor on Y. D has *simple normal crossings* iff each irreducible component of D_{red} is nonsingular and whenever some irreducible components meet at a point y, their local equations form a part of a regular system of parameters in $O_{Y,y}$. This just means that any subset of irreducible components of D_{red} *intersect transversally*, i.e., $\mathrm{edim}\, O_{Y,y}/(f_1, \ldots, f_r) = d - r$ for $f_1, \ldots, f_r$ locally defining equations of irreducible components $D_1, \ldots, D_r$ of D_{red} for $\cap_{i=1,\ldots,r} D_i \neq \emptyset$. Especially any non-empty intersection of irreducible components of D_{red} is nonsingular.

Now there is the following fundamental result (see [Deb01, 7.22]):

Theorem 2.1.2 (Hironaka's theorem on embedded resolution). *Let X be a complex variety and Z a subscheme of X. There exists a nonsingular complex variety Y and a projective birational morphism $f : Y \dashrightarrow X$ such that $\mathrm{Exc}(f) \cup f^{-1}(Z)$ is an effective divisor with simple normal crossings.*

Next let's define a hierarchy of singularities on a normal complex variety X of dimension d:

For this we first construct the canonical divisor on a possibly singular X by reduction to the nonsingular case. Let $X_{\mathrm{reg}} := X - \mathrm{Sing}(X)$ be the *nonsingular locus*. Then there exists $\Omega^d_{X_{\mathrm{reg}}}$ the sheaf of differentials on X_{reg}, a locally free sheaf of rang d. Therefore $\omega_{X_{\mathrm{reg}}} := \wedge^d \Omega^d_{X_{\mathrm{reg}}}$ the canonical sheaf on X_{reg} is an invertible sheaf. Choose a Weil divisor $K_{X_{\mathrm{reg}}}$ with $O_{X_{\mathrm{reg}}}(K_{X_{\mathrm{reg}}}) = \omega_{X_{\mathrm{reg}}}$. Because of $\mathrm{codim}_X \mathrm{Sing}(X) \geq 2$ there exists a unique Weil divisor K_X called *canonical divisor* on X with $K_X\,|_{X_{\mathrm{reg}}} = K_{X_{\mathrm{reg}}}$. Note that K_X only is unique up to linear equivalence. $-K_X$ is called *anticanonical divisor*.

For the construction of the so called ramification formula the next lemma (see [Deb01, 7.11] and proof on page 177) and its corollaries are essential:

Lemma 2.1.3. *Let $f : Y \to X$ be a proper birational morphism with X normal. D a Cartier divisor on X and F an effective Cartier divisor on Y whose support is contained in $E := \mathrm{Exc}(f)$. Then*

$$H^0(X, D) \cong H^0(Y, f^*D + F) \cong H^0(Y - E, f^*D + F) \cong H^0(X - f(E), D).$$

Corollary 2.1.4. *Let f, X, Y, E as in the lemma.*

1. *The (global) sections of the structure sheafs on $Y, Y - E, X, X - f(E)$ are the same.*

2. *There are no non-zero principal divisors on Y whose support is contained in E.*

3. *Let F be a Cartier divisor on Y whose support is contained in E. If F is effective, then*
$$f_*O_Y(F) \cong O_X.$$
The reverse implication holds, if Y is nonsingular.

4. *Let Y be nonsingular. If D, D' are Cartier divisors on X and F,F' are divisors on Y whose support is contained in E with*
$$f^*D + F \equiv f^*D' + F',$$
then $D \equiv D'$ and $F = F'$.

Let $f : Y \to X$ be a resolution of singularities, and let K' be an arbitrary canonical divisor on Y. We have
$$Y - E \cong X - f(E), \quad O_{Y-E}(K'|_{Y-E}) \cong O_{X-f(E)}(K_X|_{X-f(E)}).$$
Therefore there exists a rational function r on Y such that for the canonical divisor $K_Y := K' + \mathrm{div}(r)$ the Weil divisor $K_Y|_{Y-E}$ maps to $K_X|_{X-f(E)}$. Because of the second point in the corollary K_Y is uniquely determined by f and K_X.

Finally there is the following definition:

Definition 2.1.5. Let K_X be $\mathbb{Q}$-Cartier, i.e., there exists a positive integer j such that jK_X is a Cartier divisor. The smallest such j is called the *(Gorenstein) index* j_X of X. This definition is of course independent of the chosen canonical divisor K_X.

In this case there exist unique numbers $a_i \in \mathbb{Z}$ such that
$$j_X K_Y = f^*(j_X K_X) + \sum_i a_i j_X E_i,$$
where E_i are the *exceptional divisors*, i.e., the irreducible components of E of dimension $d-1$.

As an equality of $\mathbb{Q}$-divisors we get the so called *ramification formula*
$$K_Y = f^*K_X + \sum_i a_i E_i.$$
where $\sum_i a_i E_i$ is called the *ramification divisor.*

If C_X, C_Y were any canonical divisors on X, respectively Y, with
$$C_Y \equiv f^*C_X + \sum_i a'_i E_i,$$
then $a' = a_i$ by the last point of the corollary. Especially the a_i and the ramification divisor are independent of the chosen canonical divisor on X.

So in the ramification formula we can replace equality = by linear equivalence $\equiv$. If X and Y are both projective, then by using [Deb01, 7.19] we can show that a Cartier divisor on Y whose support is contained in E is equal to 0 iff it is numerically equivalent to 0. So it would be well-defined even to replace linear by numerical equivalence in the ramification formula.

If for a resolution f the ramification divisor vanishes, i.e., $a_i = 0$ for all exceptional divisors E_i, e.g., no exceptional divisor exists, then f is called a *crepant resolution.* This does not imply other resolutions also to be crepant.

Now we are ready to define terminal and canonical singularities.

Definition 2.1.6. A complex variety X has *terminal* (respectively *canonical*) singularities, if X is normal, the canonical divisor is $\mathbb{Q}$-Cartier, and $a_i > 0$ (respectively $a_i \geq 0$) for all i in a given resolution of singularities f.

The last condition is equivalent to the intrinsic condition $f_* O_Y(j_X K_Y - E^{(1)}) \cong O_X(j_X K_X)$ for $E^{(1)}$ the union of the exceptional divisors (respectively $f_* O_Y(j_X K_Y) \cong O_X(j_X K_X)$) as can be seen from the third point of the corollary by using the projection formula.

Because two desingularizations can always be dominated by a third, one can show as in [Deb01, 7.14] again by using above corollary that this definition does not depend on the chosen resolution of singularities.

What we have seen so far implies immediately that if X is $\mathbb{Q}$-factorial and has terminal singularities and there exists a crepant resolution, then X must already be nonsingular.

Now by considering so called boundary-divisors one can do very much the same in the so-called *log*-situation. But instead of repeating here this more general construction we use the unifying notion of discrepancy:

Definition 2.1.7. Let X be normal with $\mathbb{Q}$-Cartier canonical divisor. The *discrepancy* $\mathrm{discr}(X)$ of X is the minimum of all a_i and 1, if $a_i \geq -1$ for all i, and $-\infty$ otherwise, where the a_i are defined in the ramification formula for some resolution of singularities f such that $E^{(1)} = \sum_i E_i$ is an effective divisor with simple normal crossings.

Such a resolution exists by Hironaka's theorem and this notion is independent of the chosen desingularization.

X has terminal (respectively canonical) singularities iff $\mathrm{discr}(X) > 0$ (respectively $\mathrm{discr} \geq 0$). Eventually we also define:

Definition 2.1.8. X has *log terminal* (respectively *log canonical*) singularities, if $\mathrm{discr}(X) > -1$ (respectively $\mathrm{discr} \geq -1$).

Some remarks concerning the notion 'log terminal' should be made. There are many technical definitions around. They all involve a log-pair (X, D) where D is a so called boundary divisor, i.e., a $\mathbb{Q}$-divisor where all coefficients are in $[0, 1]$. To get the case above just set $D = 0$. In general one defines the discrepancy slightly different in an intrinsic manner (see [Mat02, 4.4.1]) to get the strongest notion called 'purely log terminal' as in [Mat02, 4.4.2]. The above notion is actually the weaker notion 'log terminal' which is in general not independent of the given resolution, for this see [Mat02, 4.3.2,4.3.3]. On the other hand if all coefficients of the boundary divisor are < 1 (as in the case here) these two definitions coincide with one called 'kawamata log terminal' (see [Mat02, 4.4.3] and [Deb01, 7.24]), a notion which is again independent of the given resolution, as described in [Deb01, 7.25].

Finally one should remark that these definitions given above are indeed *local*.

In dimension two a normal surface has terminal singularities iff it is nonsingular ([Mat02, 4.6.5]). In dimension three or higher a normal variety with terminal singularities satisfies $\mathrm{codim}\,\mathrm{Sing}(X) \geq 3$ ([Mat02, 4.6.6]).

2.2 Singularities on toric varieties

In this section it will be shortly described, how to characterize above defined singularities on toric varieties using the results of the previous chapter (see [Bor00] and [Dai02]).

So let $X = X(N, \triangle)$. We define $G_\sigma := \{v_\tau \,:\, \tau \in \sigma(1)\}$ for $\sigma \in \triangle$.

The strongest condition is actually X being *globally factorial*. This is equivalent to the fact that $\{v_\tau \,:\, \tau \in \triangle(1)\}$ is part of a $\mathbb{Z}$-basis of N. This holds iff $\mathcal{V}_\tau$ is a principal divisor for all $\tau \in \triangle(1)$.

In the toric case there are the following special equivalences:

Proposition 2.2.1. *The following conditions are equivalent:*

1. X *is* nonsingular
2. G_σ *is part of a* $\mathbb{Z}$*-basis of* N *for all* $\sigma \in \triangle$
3. $U_\sigma \cong \mathbb{C}^{\dim\sigma} \times (\mathbb{C}^*)^{d-\dim\sigma}$ *for all* $\sigma \in \triangle$
4. U_σ *is globally factorial for all* $\sigma \in \triangle$
5. $\mathcal{V}_\tau$ *is Cartier for all* $\tau \in \triangle(1)$
6. X *is* factorial

In low dimensions one can try to find classification results of isomorphism classes of proper nonsingular toric varieties up to some Picard number $\rho_X = |\triangle(1)| - d$. In the two-dimensional case the set of isomorphism classes are in bijection with a special set of so-called weighted circular graphs, see [Oda88, 1.29]. Using these results three-dimensional proper nonsingular toric varieties with Picard number five or less which are minimal in the sense of equivariant blow-ups were completely described in [Oda88, 1.34].

Analogously let's examine the '$\mathbb{Q}$-case':

X is *globally* $\mathbb{Q}$*-factorial* iff $\{v_\tau \,:\, \tau \in \triangle(1)\}$ is linearly independent iff $\mathcal{V}_\tau$ is $\mathbb{Q}$-principal for all $\tau \in \triangle(1)$.

Proposition 2.2.2. *The following conditions are equivalent:*

1. $\triangle$ *is* simplicial, *i.e.,* $|\sigma(1)| = \dim(\sigma)$ *for all* $\sigma \in \triangle$
2. G_σ *is linearly independent for all* $\sigma \in \triangle$
3. U_σ *is globally* $\mathbb{Q}$*-factorial for all* $\sigma \in \triangle$
4. $\mathcal{V}_\tau$ *is* $\mathbb{Q}$*-Cartier for all* $\tau \in \triangle(1)$
5. X *is* $\mathbb{Q}$-factorial

Our next goal is to describe the resolution of singularities in the toric case. For this one needs a toric description of the canonical divisor:

$$K_X := -\sum_{\tau \in \triangle(1)} \mathcal{V}_\tau$$

is a *canonical divisor* (see [Mat02, 14.3.1]). Here one should note that any toric variety is Cohen-Macaulay, and this canonical divisor defines the *dualizing sheaf* of X. Furthermore X always has *rational* singularities. For these notions see [Ful93, pp. 30,76,89] and [Dai02].

Remark 2.2.3. K_X is $\mathbb{Q}$-Cartier iff for all $\sigma \in \triangle$ the set G_σ is contained in an affine hyperplane, or equivalently, iff for all $\sigma \in \triangle$ there exists an $u_\sigma \in M_\mathbb{Q}$ such that $\langle u_\sigma, v_\tau \rangle = 1$ for all $\tau \in \sigma(1)$.

As described in [Ewa96, VI.8.5] one can refine by finitely many so called stellar subdivisions the fan $\triangle$ into a nonsingular fan $\triangle'$ with $\triangle(1) \subseteq \triangle'(1)$ so that $f : X' := X(N, \triangle') \to X$ is a toric morphism and moreover a *resolution of singularities*. Then we get that

$$E^{(1)} = \sum_{\tau' \in \triangle'(1) - \triangle(1)} \mathcal{V}_{\tau'}$$

is an effective divisor on X' with simple normal crossings.

It should be noted that there is a *toric Chow's lemma*, i.e., if X is proper, one can choose f such that Y is projective.

Let K_X be $\mathbb{Q}$-Cartier. Then the *ramification formula* looks like this:

$$-j_X \sum_{\tau' \in \triangle'(1)} \mathcal{V}_{\tau'} = j_X f^* K_X + \sum_{\tau' \in \triangle'(1) - \triangle(1),\, \tau' \subset \sigma \in \triangle} j_X(-1 + \langle u_\sigma, v_{\tau'} \rangle) \mathcal{V}_{\tau'},$$

because $j_X K_X |_{U_\sigma} = \mathrm{div}(j_X \mathrm{e}(u_\sigma))$ for all $\sigma \in \triangle$ and u_σ as in 2.2.3. Therefore

$$\mathrm{discr}(X) = -1 + \min_{\sigma \in \triangle \text{maximal}, 0 \neq v \in \sigma \cap N - G_\sigma} \langle u_\sigma, v \rangle$$

is a rational number in $]-1, 1]$ (see also [Deb03, Prop.12]).

Proposition 2.2.4. *Let K_X be $\mathbb{Q}$-Cartier (and u_σ defined as in 2.2.3). Then*

X has log terminal singularities.

X has terminal singularities *iff* $\{x \in \sigma \,:\, \langle u_\sigma, x \rangle \leq 1\} \cap N = \{0\} \cup G_\sigma$ *for all $\sigma \in \triangle$ maximal.*

X has canonical singularities *iff* $\{x \in \sigma \,:\, \langle u_\sigma, x \rangle < 1\} \cap N = \{0\}$ *for all $\sigma \in \triangle$ maximal.*

The resolution f above is *crepant* iff for all exceptional divisors $\mathcal{V}_{\tau'}$ we have $v_{\tau'} \in \mathrm{aff}(G_\sigma)$ for $\tau' \subset \mathrm{relint}(\sigma)$ and some $\sigma \in \triangle$.

If all maximal cones of $\triangle$ are d-dimensional, then the *Gorenstein index* j_X is the least common multiple of all (positive) denominators of the coefficients of the vectors $\{u_\sigma\}_{\sigma \in \triangle(d)}$.

The existence of a crepant resolution implies $j_X = 1$.

2.3 Toric Fano varieties

Here we lay out the basic notions of toric Fano varieties. For a survey see [Deb03] and [Sat02].

Definition 2.3.1. A complex variety X is called *Fano variety* (respectively *weak Fano variety*), if X is projective, normal and the anticanonical divisor $-K_X$ is an ample (respectively nef and big) $\mathbb{Q}$-Cartier divisor.

Because Fano varieties are in some sense opposed to varieties of general type, i.e., where the canonical divisor K_X is ample, they are quite rare and objects of major studies (see [Deb01, Deb03, PSG99]).

In some sources (e.g., [Deb03]) a Fano variety is assumed to be log terminal. Since we are only concerned with the toric case, this assumption is redundant by Prop. 2.2.4.

Now to the toric case: Let $\triangle$ be a complete fan in $N_\mathbb{R}$, and $X := X(N, \triangle)$ the associated complete normal toric variety.

We set $G_\triangle := \{v_\tau \ : \ \tau \in \triangle(1)\}$ and $G_\sigma := G_\triangle \cap \sigma$ for $\sigma \in \triangle$ as in the previous section.

Now we define the lattice polytope $Q_\triangle := \mathrm{conv}(G_\triangle) \subseteq N_\mathbb{R}$ with $0 \in \mathrm{int} Q_\triangle$, and the rational dual polytope $P_\triangle := Q_\triangle^* \subseteq M_\mathbb{R}$.

From the results of the last chapter we can immediately derive the following propositions:

Proposition 2.3.2. *The following conditions are equivalent:*

1. *X is a* toric weak Fano variety
2. *$\sum_{\tau \in \triangle(1)} \mathcal{V}_\tau$ is a nef $\mathbb{Q}$-Cartier divisor*
3. *There exists $k \in \mathbb{N}_{>0}$ and an upper convex $h \in \mathrm{SF}(N, \triangle)$ such that $h(v_\tau) = -k$ for all $\tau \in \triangle(1)$*
4. *$\mathrm{conv}(G_\sigma)$ generates an affine hyperplane for all $\sigma \in \triangle(d)$ and any facet of $Q_\triangle$ consists of an union of $\mathrm{conv}(G_\sigma)$ for some $\sigma \in \triangle(d)$*

If this holds, then the minimal k is just the Gorenstein index j_X, and when h is given by $\{l_\sigma\}$, then

$$j_X P_\triangle = \mathrm{conv}(l_\sigma \ : \ \sigma \in \triangle(d)) = P_h,$$

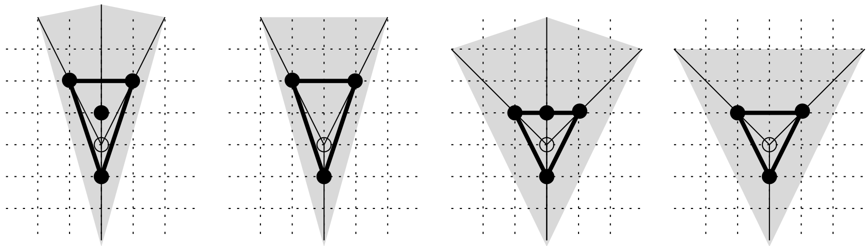
with $V(P_\triangle) = \{l_\sigma / j_X \ : \ \sigma \in \triangle(d)\}$. Especially j_X is also the minimal k such that $kP_\triangle$ is a lattice polytope.

Proposition 2.3.3. *The following conditions are equivalent:*

1. *X is a* toric Fano variety
2. *$\sum_{\tau \in \triangle(1)} \mathcal{V}_\tau$ is an ample $\mathbb{Q}$-Cartier divisor*
3. *There exists $k \in \mathbb{N}_{>0}$ and a strictly upper convex $h \in \mathrm{SF}(N, \triangle)$ such that $h(v_\tau) = -k$ for all $\tau \in \triangle(1)$*
4. *For all $\sigma \in \triangle(d)$ the polytope $F_\sigma := \mathrm{conv}(G_\sigma)$ is a facet of $Q_\triangle$*
5. *$\triangle = \Sigma_{Q_\triangle}$*
6. *$\triangle = \mathcal{N}_{P_\triangle}$*

In this case $\mathcal{V}(Q_\triangle) = G_\triangle$ and $V(P_\triangle) = \{l_\sigma / j_X \ : \ \sigma \in \triangle(d)\}$ with $|\mathcal{V}(P_\triangle)| = |\triangle(d)|$.

Example 2.3.4. In the following figure we have fans that define in this order: not a weak Fano variety; a Fano variety; just a weak Fano variety; a Fano variety. Hereby the black polygon is $Q_\triangle$ and the black dots form the set $G_\triangle$:



The following (hierarchically descending) definitions are now convenient:

Definition 2.3.5. Let $Q \subseteq N_{\mathbb{R}}$ be a d-dimensional lattice polytope containing the origin in its interior.

- Q is called a *Fano polytope*, if the vertices are primitive lattice points.
- Q is called a *canonical Fano polytope*, if $\mathrm{int} Q \cap N = \{0\}$.
- Q is called a *terminal Fano polytope*, if $Q \cap N = \{0\} \cup \mathcal{V}(Q)$.
- Q is called a *smooth Fano polytope*, if the vertices of any facet of Q form a $\mathbb{Z}$-basis of the lattice N.

Remark 2.3.6. *Beware:* In most papers (see [Bat99, Sat00]) a Fano polytope is assumed to be already a smooth Fano polytope. This more systematic notation that will be used throughout this thesis was partly introduced in [Deb03]. Moreover in polytope literature sometimes a polytope is called 'smooth', if it has a special symmetry property or even if the dual is a smooth Fano polytope. So one always has to check the definitions carefully!

We have the following correspondence theorem (see 1.7.4 and 2.2.4):

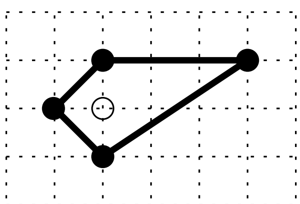
Proposition 2.3.7. *There is a correspondence between isomorphism classes of Fano polytopes and isomorphism classes of toric Fano varieties.*

Thereby canonical Fano polytopes correspond to toric Fano varieties with canonical singularities, i.e., $\mathrm{discr} X \geq 0$*;*

terminal Fano polytopes correspond to toric Fano varieties with terminal singularities, i.e., $\mathrm{discr} X > 0$*;*

smooth Fano polytopes correspond to nonsingular toric Fano varieties.

Example 2.3.8. There are infinitely many non-isomorphic d-dimensional Fano polytopes for $d \geq 2$ as the following examples for $d = 2$ show. Hereby the polygon is terminal and smooth for $k = 1$, canonical but not terminal for $k = 2$, and not canonical for $k \geq 3$:



In any dimension there is up to isomorphism always only a finite number of canonical Fano polytopes, as follows from the following important finiteness theorem (see [Bor00] for a survey, this result can be deduced from 3.7.14):

Theorem 2.3.9. *For $\epsilon > 0$ there exist only finitely many isomorphism classes of d-dimensional toric Fano varieties with discrepancy greater than $-1+\epsilon$.*

For the weak case we define:

Definition 2.3.10. Let $Q \subseteq M_{\mathbb{R}}$ be a Fano polytope spanning $\triangle$. Then a fan $\triangle'$ is called a *crepant refinement* of $\triangle$, if $\triangle'$ is a refinement of $\triangle$ in the usual sense and additionally for any $\tau' \in \triangle'(1)$ there exists a $\sigma \in \triangle$ such that $v_{\tau'} \subseteq \operatorname{conv}(G_\sigma)$. When the toric variety associated to the fan $\triangle'$ is again projective, the fan $\triangle'$ is called a *coherent crepant refinement.*

Using the ramification formula (as on page 36) we see that such crepant refinements correspond to equivariant proper birational morphisms
$f : X' = X(M, \triangle') \to X = X(M, \triangle)$ with $K_{X'} = f^* K_X$.

Proposition 2.3.11. *Toric weak Fano varieties correspond uniquely up to isomorphism to coherent crepant refinements of fans spanned by Fano polytopes.*

For the next result (see Prop. 2.2.2 for the first and [Bat94, Thm. 2.2.24] for the second part) recall that a polytope is simplicial, if any facet is a simplex:

Proposition 2.3.12. *$\mathbb{Q}$-factorial toric Fano varieties correspond uniquely up to isomorphism to simplicial Fano polytopes.*

There exists a coherent crepant refinement by stellar subdivisions that resolves a weak toric Fano variety X with canonical singularities to a $\mathbb{Q}$-factorial weak toric Fano variety X' with terminal singularities.

Such a morphism $X' \to X$ is called a *MPCP-desingularization* in [Bat94]. X is said to *admit a coherent crepant resolution*, if such an X' can be chosen to be nonsingular.

Finally we come to the essential notion:

Definition 2.3.13. A complex variety X is called *Gorenstein*, iff $j_X = 1$, i.e., K_X is a Cartier divisor.

There are other characterizations of the Gorenstein property in terms of the local rings [Eis94, Ch. 21] and the affine semigroup rings [Oda88, p.126, Remark (i)]. For our purposes this definition is the most suitable one, we refer to [Bat94] and [CK99, App. A].

Now let's look a little bit more closely at the local situation:

Proposition 2.3.14. *The following conditions are equivalent for $\sigma \in \triangle(d)$:*

1. *U_σ is* Gorenstein
2. *$\sum_{\tau\in\sigma(1)} \mathcal{V}_\tau$ is a principal divisor*
3. *There exists $l_\sigma \in M$ such that $\langle l_\sigma, v_\tau \rangle = -1$ for all $\tau \in \sigma(1)$*
4. *The unique inner normal η_F of the facet $F := \operatorname{conv}(G_\sigma)$ of $\operatorname{conv}(0, G_\sigma)$ is a lattice point*

5. *There exists a $\mathbb{Z}$-basis $e_1, \ldots, e_d$ of N such that $G_\sigma \subseteq \{x \in N_{\mathbb{R}} \,:\, x_d = -1\}$*

6. *Any $v \in G_\sigma$ can be extended to a $\mathbb{Z}$-basis $e_1, \ldots, e_{d-1}, v$ such that $G_\sigma \subseteq \{x \in N_{\mathbb{R}} \,:\, x_d = 1\}$*

Especially $N \cap \{x \in \sigma \,:\, \langle l_\sigma, x\rangle > -1\} = \{0\}$.

Proof. 3. $\Rightarrow$ 4., 3. $\Leftrightarrow$ 6. stems from the splitting of the short exact sequence $0 \to \{x \in N \,:\, \langle l_\sigma, x\rangle = 0\} \to N \stackrel{\langle l_\sigma, \cdot\rangle}{\to} \mathbb{Z} \to 0$.

The other implications are immediate.

□

In the global case we have the following equivalences:

Proposition 2.3.15. *The following conditions are equivalent:*

1. *X is a* Gorenstein toric Fano variety

2. *$\sum_{\tau \in \triangle(1)} \mathcal{V}_\tau$ is an ample Cartier divisor*

3. *There exists a strictly upper convex $h \in \mathrm{SF}(N, \triangle)$ such that $h(v_\tau) = -1$ for all $\tau \in \triangle(1)$*

4. *$Q_\triangle$ spans the fan $\triangle$ and $P_\triangle$ is a lattice polytope*

Then $V(P_\triangle) = \{l_\sigma \,:\, \sigma \in \triangle(d)\}$ with $|\mathcal{V}(P_\triangle)| = |\triangle(d)|$.

We have by (1.6) for the anticanonical degree *of X:*

$$\deg(X) := (-K_X)^d = d!\mathrm{vol}(P_\triangle).$$

As an important corollary from 2.3.14 and 2.3.9 we get:

Corollary 2.3.16. *Gorenstein toric (weak) Fano varieties have canonical singularities. In fixed dimension d there is only a finite number of isomorphism classes of d-dimensional Gorenstein toric Fano varieties.*

Chapter 3

Reflexive polytopes

Introduction

In this chapter the main objects of study are introduced: Gorenstein toric Fano varieties. We investigate this class of varieties by combinatorial methods using the notion of a reflexive polytope which appeared in connection to mirror symmetry (see [Bat94] and [CK99]). A reflexive polytope is just a lattice polytope, that contains as the only lattice point the origin in its interior, such that the dual polytope is also a lattice polytope. There is up to isomorphism only a finite number of reflexive polytopes in fixed dimension.

This chapter contains generalizations of tools and results previously known only for nonsingular toric Fano varieties (due to Casagrande and Debarre), as well as a detailed treatment of an approach by Batyrev and Conrads. As applications we obtain new classification results and bounds of invariants, and we formulate conjectures concerning combinatorial and geometrical properties of arbitrary-dimensional reflexive polytopes. In the remaining parts of this thesis the results achieved here will be applied.

In section 1 we give the definition and several characterizations of a reflexive polytope (see Prop. 3.1.4) and describe its basic properties.

In sections 2 and 3 two elementary technical tools are investigated and generalized that were already previously used to successfully investigate and classify nonsingular toric Fano varieties [Bat99, Sat00, Cas03b].

In section 2 we investigate the projection of a reflexive polytope along a lattice point on the boundary (see Prop. 3.2.2). Thereby we can relate properties of a Gorenstein toric Fano variety to that of a lower-dimensional toric Fano variety (e.g., Cor. 3.2.5). As an application we give a new, purely combinatorial proof (Cor. 3.2.8) of a result due to Batyrev [Bat99, Prop. 2.4.4] stating that the anticanonical class of a torus-invariant prime divisor of a nonsingular toric Fano variety is always numerically effective, and we generalize this result to the case of locally complete intersections (Prop. 3.2.9).

In section 3 we consider pairs of lattice points on the boundary of a reflexive polytope and show that in this case there exists a generalization of the notion of a primitive relation as introduced by Batyrev in [Bat91] (Prop. 3.3.1). As an application we prove that there are constraints on the combinatorics of a reflexive polytope, in particular on the diameter of the edge-graph of a simplicial reflexive polytope (Cor. 3.3.2). Thereby we get that certain combinatorial types of polytopes cannot be realized as reflexive polytopes (Cor. 3.3.3, Cor. 3.3.4).

In section 4 we give a short review of classification results of reflexive polytopes in low dimensions. In particular we give a concise proof of the classification of reflexive polygons (Prop. 3.4.1). Furthermore we describe the algorithm that was used by Kreuzer and Skarke for the computer classification of reflexive polytopes in dimension three and four (see [KS97, KS98, KS00]).

In section 5 we are concerned with the maximal number of vertices of a reflexive polytope, respectively a simplicial reflexive polytope. In the general case we formulate a conjecture (Conj. 3.5.2), that will be proven in the case of a centrally symmetric simple reflexive polytope in the last chapter (Thm. 6.2.2). Here we give coarse bounds that depend on the maximal number of vertices of a facet by generalizing results of Voskresenskij, Klyachko and Debarre (see Prop. 3.5.5). We also extend the conjecture on the maximal number of vertices of a smooth Fano polytope to the case of a simplicial reflexive polytope (Conj. 3.5.7). One of the main results of this section, that is even new in the smooth case, is the verification of this conjecture under the assumption of an additional symmetry of the polytope (Thm. 3.5.11). For this we generalize a result due to Casagrande [Cas03a, Thm. 2.4] to the case of a $\mathbb{Q}$-factorial Gorenstein toric Fano variety (Cor. 3.5.17) stating that the Picard number of a nonsingular toric Fano variety exceeds the Picard number of a torus-invariant prime divisor at most by three. These results were published in [Nil04a] as a preprint in May 2004. In November 2004 Casagrande published in [Cas04] the proof of the conjecture in the case of a smooth Fano polytope. The author pointed out to her a possible simplification in her argument and in December 2004 Casagrande published a complete proof of Conj. 3.5.7 partly relying on results of this section, especially on Thm. 3.5.11. From her proof we can also derive a bound on the number of vertices of a general reflexive polytope (Cor. 3.5.13), thereby improving on Prop. 3.5.5.

In section 6 we deal with reflexive simplices. We describe the approach of Batyrev and Conrads how to determine these simplices by so called weight systems. Then we prove sharp upper bounds on the total weight of these weight systems (Thm. 3.6.22). The proof relies on some number-theoretic inequalities concerning sums of unit fractions.

In section 7 we are concerned with the number of lattice points in a reflexive polytope. Reflexive polytopes can be characterized by its Ehrhart polynomial that counts lattice points in multiples of the polytope (Prop. 3.7.2). The main theorem of this section is a sharp upper bound on the volume of a reflexive simplex (Thm. 3.7.13), the corresponding algebraic-geometric result yields a general conjecture on the maximal anticanonical degree of a Gorenstein Fano variety with canonical singularities. We also obtain a bound on the number of lattice points of a reflexive simplex, and furthermore we prove that in any dimension there is a unique reflexive simplex with the maximal number of lattice points on an edge (Thm. 3.7.19), this was observed in low dimensions in [HM04]. At the end of this section we count lattice points modulo a natural number, and show how this is useful in the case of terminal reflexive polytopes (Cor. 3.7.23).

Summary of most important new results of this chapter:

· Properties of projecting reflexive polytopes (Prop. 3.2.2, p. 48)

- The anticanonical class of a torus-invariant prime divisor of a $\mathbb{Q}$-factorial Gorenstein toric Fano variety with terminal singularities is a nef $\mathbb{Q}$-Cartier divisor (Cor. 3.2.7, p. 50).
- The anticanonical class of a torus-invariant prime divisor of a toric Fano variety with terminal singularities and locally complete intersections is a nef Cartier divisor (Prop. 3.2.9, p. 51).
- Properties of pairs of lattice points on the boundary (Prop. 3.3.1, p. 52)
- The diameter of the edge-graph of a simplicial reflexive polytopes is at most three (Cor. 3.3.2, p. 53)
- Proof of sharp upper bound on the number of vertices of a simplicial reflexive polytope in the case of a centrally symmetric pair of vertices of the dual polytope (Thm. 3.5.11, p. 61)
- The Picard number of a $\mathbb{Q}$-factorial Gorenstein toric Fano variety exceeds the Picard number of a torus-invariant prime divisor at most by three (Cor. 3.5.17, p. 66)
- Let V be the anticanonical degree of a Gorenstein toric Fano variety X of class number one, and V^* the anticanonical degree of the "dual" variety X^*. Then we prove sharp bounds on V (Thm. 3.7.7, p. 83), on $V \cdot V^*$ (Thm. 3.7.11, p. 84), and on the anticanonical degree of a torus-invariant curve on X (Thm. 3.7.9, p. 84).

3.1 Basic properties

Reflexive polytopes naturally enter the picture as the combinatorical counterparts of Gorenstein toric Fano varieties.

Definition 3.1.1. A Fano polytope whose dual is a lattice polytope is called *reflexive polytope.*

By Proposition 2.3.15 the next result is straightforward:

Theorem 3.1.2. *Gorenstein toric Fano varieties correspond uniquely up to isomorphism to reflexive polytopes.*

Gorenstein toric weak Fano varieties correspond uniquely up to isomorphism to coherent crepant refinements of fans spanned by reflexive polytopes.

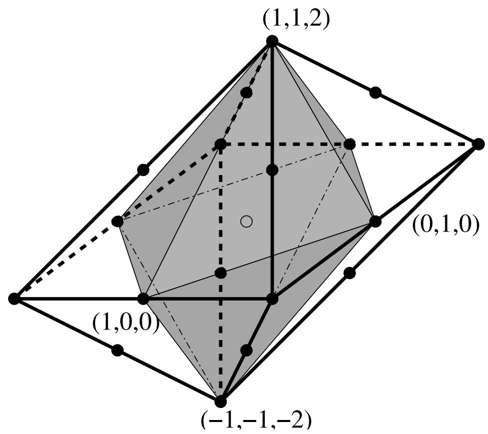
From 2.3.16 we get:

Theorem 3.1.3. *A reflexive polytope is canonical. There are only finitely many equivalence classes of reflexive polytopes in any given dimension.*

This finiteness result is one of the main reasons why the study of reflexive polytopes is on the one hand very motivating, because classification results should be at least in very low dimensions achievable, and on the other hand so difficult because of the 'sporadic' nature of reflexive polytopes.

It would be interesting to find a direct proof of this result that only uses the definition of a reflexive polytope.

Here is an example to illustrate the difference of canonical Fano polytopes and reflexive polytopes, it is taken from [KS96] (see 3.2.6 for another example):



The outer polytope is a centrally symmetric reflexive polytope, however the interior one is a centrally symmetric terminal polytope that is not reflexive, in fact all facets have lattice distance two from the origin, i.e., the unique primitive outer normal has value 2 on the defining facet.

It is obvious that lattice points on the boundary of a canonical Fano polytope are primitive, for a reflexive polytope however even more is true. The next result summarizes the most important equivalences of reflexivity (8. and 12. seem to be not yet written down somewhere else):

Proposition 3.1.4. *Let $P \subseteq M_{\mathbb{R}}$ be a d-dimensional lattice polytope with $0 \in \mathrm{int} P$.*

1. *P is reflexive*
2. *P^* is a lattice polytope*
3. *P^* is reflexive*
4. *$X(M, \Sigma_P)$ is a (Gorenstein toric) Fano variety*
5. *$X(N, \mathcal{N}_P)$ is a (Gorenstein toric) Fano variety*
6. *All facets have integral lattice distance one from the origin (i.e., for any $F \in \mathcal{F}(P)$ there exists $\nu_F \in N$ such that $\langle \nu_F, m \rangle = 1$ for all $m \in F$)*
7. *There are no lattice points lying between the affine hyperplane spanned by a facet and its parallel through the origin*
8. *Any lattice point on an affine hyperplane spanned by a facet is primitive*
9. *$\eta_F \in N$ for any $F \in \mathcal{F}(P)$*
10. *$\zeta_F = \eta_F$ for any $F \in \mathcal{F}(P)$*
11. *For any $F \in \mathcal{F}(P)$ and $m \in F \cap M$ there is a $\mathbb{Z}$-basis $e_1, \ldots, e_d$ of M such that $e_d = m$ and $F \subseteq \{x \in M_{\mathbb{R}} : x_d = 1\}$ (in particular $M \cap \eta_F^{\perp}$ has as a $\mathbb{Z}$-basis $e_1, \ldots, e_{d-1}$, and $\eta_F = -e_d^*$ in the dual $\mathbb{Z}$-basis $e_1^*, \ldots, e_d^*$ of N)*
12. *For any $F \in \mathcal{F}(P)$ there is a $\mathbb{Z}$-basis $e_1, \ldots, e_d$ of M such that $\|\eta_F\| = 1$ and $\|m\| = 1$ for some $m \in F \cap M$ (the norm taken with respect to this basis, respectively its dual)*

Proof. 8. $\Rightarrow$ 6.: Let $\nu_F := -\zeta_F \in N$ be the unique primitive outer normal, so there exists a primitive lattice point $m \in M$ such that $\langle \nu_F, m\rangle = 1$. Let $c := \langle \nu_F, F\rangle \in \mathbb{N}_{>0}$. Then cm is a lattice point in $\mathrm{aff}(F)$, so primitive, hence $c = 1$.

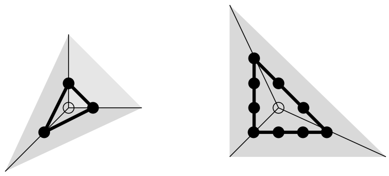
The other implications and equivalences are obvious or follow from the results of the last section in the previous chapter, in particular from Propositions 2.3.14 and 2.3.15.

□

For important algebraic-geometric characterizations of reflexive polytopes (generic anticanonical hypersurfaces are Calabi-Yau) see [Bat94, Thm. 4.1.9] or [HM04, 2.1], for its consequences in mirror symmetry see [Bat94] or [CK99], for other combinatorial equivalences see subsection 3.7.1.

Corollary 3.1.5. $P \mapsto X_P := X(N, \mathcal{N}_P)$ *induces a correspondence of isomorphism classes of reflexive polytopes and Gorenstein toric Fano varieties.*

In particular there is a complete *duality* of reflexive polytopes. This implies a natural (mirror) symmetry of isomorphism classes of Gorenstein toric Fano varieties. In the following example the left fan defines $\mathbb{P}^2$, the right a quotient $\mathbb{P}^2/\mathbb{Z}_2$:



If P is a reflexive polytope, then X_P is a nonsingular toric Fano variety if and only if P^* is a smooth Fano polytope.

It is also important to note (e.g., see (1.5)) that products of reflexive polytopes are again reflexive.

In low dimensions reflexive polytopes have a very important and peculiar property (see [Bat94]):

Proposition 3.1.6. *For* $d \leq 3$ *any* d*-dimensional Gorenstein toric Fano variety admits a coherent crepant resolution (see 2.3.12).*

For a four-dimensional counterexample see 3.2.6.

We recall a convenient definition for the proof:

Definition 3.1.7. A lattice polytope $P \subseteq M_{\mathbb{R}}$ is called *empty*, if $P \cap M = \mathcal{V}(P)$.

Now 3.1.6 follows from 2.3.12 and the next well-known lemma:

Lemma 3.1.8. *Let* P *be a* d*-dimensional lattice polytope with* $0 \in \mathrm{int} P$.

1. *Let* $d = 2$. *Lattice points* x, y *form a* $\mathbb{Z}$*-basis if and only if* $\mathrm{conv}(0, x, y)$ *is an empty two-dimensional polytope. If* P *is a canonical Fano polytope, this holds, if* x, y *are lattice points on the boundary that are not contained in a common facet of* P *and* $x + y \neq 0$.

   In particular any two-dimensional terminal Fano polytope is a smooth Fano polytope, and any two-dimensional canonical Fano polytope is a reflexive polytope.

2. *Let $d = 3$ and P be reflexive. Three lattice points x, y, z in a common facet of P form a $\mathbb{Z}$-basis if and only if* $\operatorname{conv}(x, y, z)$ *is empty.*

   In particular any three-dimensional simplicial terminal reflexive polytope is a smooth Fano polytope.

It is well-known that ample Cartier divisors on a complete nonsingular toric variety are already very ample (see [Oda88, Cor.2.15]). For a general bound see [EW91] or [Pay04], for a related conjecture see [Kan98]. Here we prove a result that seems to be folklore:

Proposition 3.1.9. *Let X be a d-dimensional Gorenstein toric Fano variety with $d \leq 3$.*

Then the ample anticanonical divisor $-K_X$ is very ample.

Proof. Let $P \subseteq M_{\mathbb{R}}$ the corresponding reflexive polytope, $u \in \mathcal{V}(P^*)$ arbitrary.

Let $n \in \operatorname{pos}(P^* - u) \cap N$. By the results in section 1.7 we have to show that n is a linear combination with positive integers of elements in $(P^* \cap N) - u$.

It is easy to see that there exist $F \in \mathcal{F}(P^*)$ and $x_1, x_2 \in F \cap N$ such that $\operatorname{conv}(u, x_1, x_2)$ is an empty lattice polytope and $n \in \operatorname{pos}(0 - u, x_1 - u, x_2 - u)$. By 3.1.8(2) we get that $\{u, x_1, x_2\}$ is a $\mathbb{Z}$-basis of N. In particular n is a linear combination of $\{-u, x_1 - u, x_2 - u\}$ with positive integers. □

3.2 Projecting along lattice points on the boundary

Throughout the section let P be a d-dimensional reflexive polytope in $M_{\mathbb{R}}$.

The projection map along a vertex of P is an essential tool in investigating toric Fano varieties, since one hopes to get some information from the corresponding lower-dimensional variety (see [Bat99, Cas03a]). In the case of a reflexive polytope it is also useful to consider projecting along lattice points on the boundary of P that are not necessarily vertices.

The following definitions will be used throughout this work:

Definition 3.2.1. Let $x, y \in \partial P$ with $x \neq y$.

- $[x, y] := \operatorname{conv}(x, y)$, $\;]x, y] := [x, y] \backslash \{x\}$, $\;]x, y[:= [x, y] \backslash \{x, y\}$.
- $x \sim y$, if $[x, y]$ is contained in a face of P, i.e., x and y are contained in a common facet of P.
- The *star set* of x is the set

$$\operatorname{st}(x) := \{y \in \partial P \,:\, x \sim y\} = \bigcup \{F \in \mathcal{F}(P) \,:\, x \in F\}.$$

- The *link* of x is the set

$$\partial x := \partial \operatorname{st}(x) = \bigcup \{G \leq P \,:\, G \subseteq \operatorname{st}(x),\, x \notin G\}.$$

- $y \vdash x$ (y is said to be *away from* x), if y is not in the relative interior of $\mathrm{st}(x)$. Hence
$$\partial x = \{y \in \mathrm{st}(x) \,:\, y \vdash x\}.$$
y is away from x iff there exists a facet that contains y but not x, e.g., if y is a vertex or $x \not\sim y$. There is also a local criterion:
$$y \vdash x \iff x + \lambda(y - x) \notin P \;\; \forall\, \lambda > 1.$$

The next proposition shows the important properties of the projection of a reflexive polytope along some lattice point on the boundary.

Proposition 3.2.2. *Let $P \subseteq M_{\mathbb{R}}$ be a reflexive polytope. Let $v \in \partial P \cap M$.*

We define the quotient lattice $M_v := M/\mathbb{Z}v$ and the canonical projection map along v
$$\pi_v \,:\, M_{\mathbb{R}} \to (M_v)_{\mathbb{R}} = M_{\mathbb{R}}/\mathbb{R}v.$$
Then $P_v := \pi_v(P)$ is a lattice polytope in $(M_v)_{\mathbb{R}}$ with $\mathcal{V}(P_v) \subseteq \pi_v(\mathcal{V}(P))$ that contains the origin in its interior.

1. *Let U be the set of elements $x \in P$ such that $x + \lambda v \notin P$ for all $\lambda > 0$.*

   The restriction of π_v to U induces a bijection
   $$U \to P_v.$$
   We denote the inverse map by ι_v.

   For $S := \mathrm{st}(v)$ we have
   $$U = S$$
   and thus
   $$P_v = \mathrm{conv}(\pi_v(\mathcal{V}(P) \cap \partial v)).$$
2. *The projection map induces a bijection $S \cap M \to P_v \cap M_v$.*
3. *The projection map induces a bijection $\partial v \to \partial P_v$. ∂P_v is covered by the projection of all $(d-2)$-dimensional faces C such that $C \in \mathcal{F}(F)$ for some facet F of P with $v \in F$ and $v \notin C$; $\pi_v(C)$ is contained in a facet of P_v.*
4. $$\iota_v(\mathcal{V}(P_v)) \subseteq \mathcal{V}(P) \cap \partial v.$$
   Let $z \in \mathcal{V}(P) \cap \partial v$. Then $\pi_v(z) \in \partial P_v$. If $]v, z[$ is contained in the relative interior of a facet of P, then $\pi_v(z) \in \mathcal{V}(P_v)$.
5. *Let $F \in \mathcal{F}(P)$ with $v \in F$. Then*
   $$\mathrm{pos}(\pi_v(F)) \cap P_v = \pi_v(F).$$
6. *The image $\pi_v(F)$ of a facet F parallel to v, i.e., $\langle \eta_F, v\rangle = 0$, is a facet of P_v. It is $\pi_v^{-1}(\pi_v(F)) \cap P = F$. There are at least $|\,\mathcal{V}(\pi_v(F))\,|$-vertices of F in S. Any point in $F \cap S$ is contained in a facet that contains v and intersects F in a $(d-2)$-dimensional face.*

   The preimage $\Gamma := \pi_v^{-1}(F') \cap P$ of a facet F' of P_v is either a facet of P parallel to v or a $(d-2)$-dimensional face of P. In the last case $\Gamma \to F'$ is an isomorphism, and there exists exactly one facet of P that contains Γ and v.

7. *Suppose $-v \in P$. Then any facet of P either contains v, or $-v$, or is parallel to v, i.e., a facet of the form $\pi_v^{-1}(F') \cap P$ for $F' \in \mathcal{F}(P_v)$.*

8. $$(P_v)^* \cong P^* \cap v^\perp \text{ as lattice polytopes.}$$

   P_v is reflexive if and only if $P^* \cap v^\perp$ is a lattice polytope.

Proof. 1. Let F be a facet of P containing v and x. If $\lambda > 0$, then $\langle \eta_F, x+\lambda v\rangle = -1 - \lambda < -1$, so $x + \lambda v \notin P$. Hence $S \subseteq U$.

On the other hand let $x \in U$. Considering the polytope $P \cap \mathrm{lin}(v, x)$ we see there is a facet F of P not parallel to v that contains x with $\langle \eta_F, v\rangle < 0$. Since P is reflexive, we have $\langle \eta_F, v\rangle = -1$, hence $v \in F$ and $x \sim v$. This implies $S = U$.

2. Let $m' \in P_v \cap M_v$. We have $u := \iota_v(m') \in U = S$. So there exists a facet $F \in \mathcal{F}(P)$ with $v, u \in F$. By 3.1.4(11) there is a $\mathbb{Z}$-basis $e_1, \ldots, e_{d-1}, e_d = v$ of M such that $\mathcal{V}(F) \subseteq \{x \in M_{\mathbb{R}} : x_d = 1\}$, i.e., η_F is the dual vector $-e_d^*$. Let $u = \lambda_1 e_1 + \cdots + \lambda_d e_d$ for $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$. Now $e_1, \ldots, e_{d-1}$ is a $\mathbb{Z}$-basis of M_v. Therefore $\lambda_1, \ldots, \lambda_{d-1} \in \mathbb{Z}$. Since $u \in F$, we get $\lambda_d = u_d = \langle -\eta_F, u\rangle = 1$, hence $u \in M$.

3. The projection induces an homeomorphism $S \to P_v$, and hence a bijection of their boundaries.

4. The first statements follow from the first and the third point. Let $z \in \mathcal{V}(P) \cap \partial v$ be such that $]v, z[$ is contained in the relative interior of a facet of P. Then there is only one facet $F \in \mathcal{F}(P)$ that contains v and z. Since $z \in \mathcal{V}(F)$ we can choose an affine hyperplane H that intersects F only in z and is parallel to v. For $P_v := \pi_v(P)$ and $H' := \pi_v(H)$ let $x' \in H' \cap P_v$. It remains to show that $x' = z' := \pi_v(z)$. So assume not. H' intersects $\pi_v(F)$ only in z'. Therefore $\iota_v(y') \notin F$ for all $y' \in]z', x']$. Finiteness of $\mathcal{F}(P)$ implies that z is contained in another facet $\neq F$ containing v, a contradiction.

5. This is proven in the same manner as the third point.

6. The first statements follow from the third and the fourth point.

For the second statement let $\dim(\Gamma) = d - 2$. Now observe that if $\Gamma \to F'$ were not injective, a facet containing Γ necessarily would be parallel to v, so its image a facet containing F', a contradiction. Therefore $\Gamma \to F'$ is an isomorphism of polytopes with respect to their affine spans. We choose $x \in \mathrm{relint} F'$. Let $y := \iota_v(x) \in S \cap \Gamma$. By assumption also $y \in \mathrm{relint}\Gamma$. Let $G \in \mathcal{F}(P)$ with $v, y \in G$. Then $\Gamma \subseteq G$, hence G is one of the two facets containing Γ.

7. Let $-v \in P$. Any facet $F \in \mathcal{F}(P)$ satiesfies $-1 \leq \langle \eta_F, v\rangle = -\langle \eta_F, -v\rangle \leq 1$. From this the statements follow.

8. Choose again a facet F of P with $v \in F$, and a $\mathbb{Z}$-basis $e_1, \ldots, e_d$ of M such that $e_d = v$ and $e_1, \ldots, e_{d-1}$ is a $\mathbb{Z}$-basis of $M \cap \eta_F^\perp \cong M_v$. In these coordinates we get $(P_v)^* = \{y' \in \mathbb{R}^{d-1} : \langle y', x'\rangle \geq -1 \ \forall x' \in P_v\} = \{y' \in \mathbb{R}^{d-1} : \langle (y'_1, \ldots, y'_{d-1}, 0), (x_1, \ldots, x_d)\rangle \geq -1 \ \forall x \in P\} = P^* \cap v^\perp$.

The last point and its proof are taken from [AKMS97, theorem in section 3]. □

Let's consider the *algebraic-geometric interpretation* of the projection map:

Throughout let $X := X(M, \triangle)$ for $\triangle := \Sigma_P$ and $P \subseteq M_{\mathbb{R}}$ reflexive.

Let $v \in \mathcal{V}(P)$, $\tau := \mathrm{pos}(v) \in \triangle(1)$, and π_v as in the previous proposition. As in [Oda88, 1.7] the fan $\overline{\triangle} := \{\pi_v(\sigma) \,:\, \sigma \in \triangle, \tau \leq \sigma\} = \{\mathrm{pos}(\pi_v(F)) \,:\, F \in \mathcal{F}(P), v \in F\}$ defines the projective toric variety $\mathcal{V}_\tau$ that is the torus invariant prime divisor corresponding to the ray τ.

On the other hand there is the projected polytope $P_v := \pi_v(P)$ that spans a fan $\triangle_v$ in $(M_v)_{\mathbb{R}}$, we denote the corresponding projective toric variety by X_v. In the following we discuss, how and when $\mathcal{V}_\tau$ and X_v are related.

We choose a triangulation $\mathcal{T} := \{T_k\}$ of ∂v into simplicial lattice polytopes. Then $\triangle_{\mathcal{T}} := \{\mathrm{pos}(\pi_v(T_k))\}$ is a simplicial fan in $(M_v)_{\mathbb{R}}$ with corresponding $\mathbb{Q}$-factorial complete toric variety $X_{\mathcal{T}}$. From Proposition 3.2.2(5) it follows that $\triangle_{\mathcal{T}}$ is a *common refinement* of $\overline{\triangle}$ and $\triangle_v$. Especially there are induced proper birational morphisms $X_{\mathcal{T}} \to \mathcal{V}_\tau$ and $X_{\mathcal{T}} \to X_v$.

In general $\overline{\triangle}$ is not a refinement of $\triangle_v$ (and vice versa). However in the case that P is simplicial, we can choose $\mathcal{T}$ obviously in such a way that $\triangle_{\mathcal{T}} = \overline{\triangle}$, in particular this implies that $\overline{\triangle}$ is a refinement of $\triangle_v$.

In order to draw conclusions about the canonical divisor and singularities of these lower-dimensional toric varieties there is the following sufficient assumption:

$$\exists\, f \in \mathbb{N}_{>0} \;\;:\;\; \forall\, w \in \mathcal{V}(P) \cap \partial v \;:\; |\,[v, w] \cap M\,| - 1 = f. \tag{3.1}$$

Suppose this condition holds. For any $w \in \mathcal{V}(P) \cap \partial v$ the second point in the proposition implies $|\,[0, \pi_v(w)] \cap M_v\,| - 1 = f$. Furthermore let $w' \in \mathcal{V}(P_v)$, $\tau' := \mathrm{pos}(w')$. Since proposition 3.2.2(4) implies $\iota_v(w') \in \mathcal{V}(P) \cap \partial v$, the previous consideration yields $v_{\tau'} = (1/f)w'$, hence $Q_{\triangle_v} = (1/f)P_v$ (for this notation see p. 37). Therefore $-K_{X_v}$ is an ample $\mathbb{Q}$-Cartier divisor, i.e., X_v is a *toric Fano variety.*

Definition 3.2.3. X, respectively P, is called *semi-terminal*, if for all $v \in \mathcal{V}(P)$ condition (3.1) holds for $f = 1$, i.e., $[v, w] \cap M = \{v, w\}$ for all $w \in \mathcal{V}(P) \cap \partial v$.

Proposition 3.2.4. *Let $P \subseteq M_{\mathbb{R}}$ be a reflexive polytope.*

1. *P is semi-terminal iff P_v is a Fano polytope for all $v \in \mathcal{V}(P)$.*
2. *P is terminal iff P_v is a canonical Fano polytope for all $v \in \mathcal{V}(P)$.*

Proof. 1. From left to right: This holds, since $Q_{\triangle_v} = P_v$ is a Fano polytope.

From right to left: Let $v \neq w \in \mathcal{V}(P)$ with $v \sim w$. Choose C and F as in Proposition 3.2.2(3) such that $w \in \mathcal{V}(C)$. Therefore we see that $F' := \mathrm{aff}(\pi_v(C)) \cap P_v$ is a facet of P_v. Hence by 3.2.2(6) for $\Gamma := \pi_v^{-1}(F') \cap P$ we have either $\Gamma = G$ for a facet $G \in \mathcal{F}(P)$ that is parallel to v or $\Gamma = C$. In the first case $\langle \eta_G, v\rangle = 0$ and $\langle \eta_G, w\rangle = -1$, so $|\,[v, w] \cap M\,| = 2$. In the second case obviously $F' = \pi_v(C)$, so $\pi_v(w) \in \mathcal{V}(F')$, hence by assumption a primitive lattice point. From 3.2.2(2) we get $|\,[v, w] \cap M\,| = 2$.

2. From left to right: Let $v \in \mathcal{V}(P)$, $0 \neq m' \in M_v \cap P_v$. In the notation of Proposition 3.2.2(1,2) we get $\iota_v(m') \in M \cap \partial P \subseteq \mathcal{V}(P)$ by assumption. Hence $m' = \pi_v(\iota_v(m')) \in \partial P_v$ by Proposition 3.2.2(4).

From right to left: Assume there is a $w \in \partial P \cap M$, $w \notin \mathcal{V}(P)$. Then w is a proper convex combination of vertices of P contained in a common facet. Let v be one of them. Then $\pi_v(w)$ is in the interior of P_v, a contradiction.

□

Corollary 3.2.5. *Let X be a Gorenstein toric Fano variety. Then the following two statements are equivalent:*

1. *X has terminal singularities*

2. *X is semi-terminal and X_v has canonical singularities for any $v \in \mathcal{V}(P)$*

If this holds, then X_v is a toric Fano variety for any $v \in \mathcal{V}(P)$.

In particular we see that terminality is a necessary condition for obtaining a reflexive polytope under projection, however not sufficient for $d \geq 4$.

Example 3.2.6. Let $d = 4$, and $e_1, \ldots, e_4$ a $\mathbb{Z}$-basis of M. We define the simplicial centrally symmetric reflexive polytope $P := \text{conv}(\pm(2e_1 + e_2 + e_3 + e_4), \pm e_2, \pm e_3, \pm e_4)$. Then P is combinatorially a crosspolytope, has 8 vertices and 16 facets. It is a terminal Fano polytope but not a smooth Fano polytope, so especially it admits no crepant resolution. The projection P_{e_4} along the vertex e_4 has 6 vertices, P_{e_4} is even a terminal Fano polytope but not reflexive. This polytope is taken from [Wir97] where it is used in a different context.

Now we consider $\mathcal{V}_\tau$: To ensure that the canonical divisor of $\mathcal{V}_\tau$ is $\mathbb{Q}$-Cartier, we need in general the $\mathbb{Q}$-factoriality of X. So let P be simplicial and assume again that condition (3.1) holds. Then $\overline{\triangle}$ is a coherent crepant refinement of $\triangle_v$. Hence $-K_{\mathcal{V}_\tau}$ is a nef $\mathbb{Q}$-Cartier divisor, i.e., $\mathcal{V}_\tau$ is a *toric weak Fano variety.* From Cor. 3.2.5 we get:

Corollary 3.2.7. *Let X be a $\mathbb{Q}$-factorial Gorenstein toric Fano variety. Then the following two statements are equivalent:*

1. *X has terminal singularities*

2. *X is semi-terminal and $\mathcal{V}_\tau$ has terminal singularities for any $\tau \in \triangle(1)$*

If this holds, then $\mathcal{V}_\tau$ is a $\mathbb{Q}$-factorial toric weak Fano variety for any $\tau \in \triangle(1)$.

Finally to additionally derive the Gorenstein property, i.e., that the canonical divisor is $\mathbb{Z}$-Cartier, we need a stronger assumption, that is trivial in the case of a smooth Fano polytope:

$$\begin{gathered}\text{For any } F \in \mathcal{F}(P) \text{ with } v \in F \text{ and } C \in \mathcal{F}(F) \text{ with } v \notin C \\ \text{there exist } w_1, \ldots, w_{d-1} \in C \cap M \text{ such that} \\ w_1, \ldots, w_{d-1}, v \text{ is a } \mathbb{Z}\text{-basis of } M.\end{gathered} \tag{3.2}$$

If this condition is fulfilled, then (3.1) holds for $f = 1$, P_v is reflexive by Prop. 3.2.2(3), and X_v is a *Gorenstein toric Fano variety.* If P is also simplicial, then $\mathcal{V}_\tau$ is a *Gorenstein toric weak Fano variety.*

Suppose now X is semi-terminal, simplicial and $\mathcal{V}_\tau$ is nonsingular for any $\tau \in \triangle(1)$. It follows from 3.1.4(11) that (3.2) holds for any $v \in \mathcal{V}(P)$. Then P_v is reflexive, in particular canonical for any $v \in \mathcal{V}(P)$, hence Cor. 3.2.5 implies that P is terminal. Since P is also simplicial, the assumption implies that P is already a smooth Fano polytope. We have proven the following corollary:

Corollary 3.2.8. *Let X be a $\mathbb{Q}$-factorial Gorenstein toric Fano variety. Then the following two statements are equivalent:*

1. *X is nonsingular*

2. *X is semi-terminal and $\mathcal{V}_\tau$ is nonsingular for any $\tau \in \triangle(1)$*

If this holds, then $\mathcal{V}_\tau$ is a $\mathbb{Q}$-factorial toric weak Fano variety for any $\tau \in \triangle(1)$, and X_v is a Gorenstein toric Fano variety admitting the coherent crepant resolution $\mathcal{V}_\tau \to X_v$ for any $v \in \mathcal{V}(P)$, $\tau = \mathrm{pos}(v)$.

The important fact that the projection of a smooth Fano polytope is reflexive was already proven by Batyrev in [Bat99, Prop. 2.4.4], however he used the notion of a primitive relation [Bat91] and results of Reid about the Mori cone [Rei83]. Another proof that is essentially equivalent to the one of Batyrev was done by Evertz [Eve88] by explicitly calculating equations of facets.

There is now a generalization of this result to the class of toric Fano varieties with locally complete intersections. These varieties were thoroughly investigated in [DHZ01], where it was proven that they admit coherent crepant resolutions.

Proposition 3.2.9. *Let X be a Gorenstein toric Fano variety that has singularities that are locally complete intersections.*
Then the following three statements are equivalent:

1. *X is semi-terminal*

2. *X has terminal singularities*

3. *Any facet of P can be embedded as a lattice polytope into $[0,1]^{d-1}$*

If this holds, then X_v is a Gorenstein toric Fano variety for any $v \in \mathcal{V}(P)$. If additionally X is $\mathbb{Q}$-factorial, then X is nonsingular.

Proof. The facets of the corresponding reflexive polytope P are so called *Nakajima* polytopes, a comprehensive description can be found in [DHZ01]. Using their results it is straightforward to prove the following statement by induction on n:

Let v be a vertex of an n-dimensional Nakajima polytope F in a lattice M such that $|\,[w, w'] \cap M\,| = 2$ for all $w, w' \in \mathcal{V}(P)$, $w \neq w'$. Then F is empty, can be embedded as a lattice polytope in $[0,1]^n$, and for any facet $C \in \mathcal{F}(F)$ with $v \notin C$ there exist n vertices $w_1, \ldots, w_n$ of C such that $w_1 - v, \ldots, w_n - v$ is a $\mathbb{Z}$-basis.

From this result the proposition is obvious using condition (3.2).

□

In Example 3.2.6 we have the situation that any facet satisfies the third condition of the previous proposition. The corresponding variety X is $\mathbb{Q}$-factorial, has terminal singularities, but X_v is never a Gorenstein toric Fano variety, so X does not have locally complete intersections.

3.3 Pairs of lattice points on the boundary

Throughout the section let P be a d-dimensional reflexive polytope in $M_{\mathbb{R}}$.

In [Bat91] the important notions of *primitive collections* and *primitive relations* were defined for nonsingular projective toric varieties and used in [Bat99] for the classification of four-dimensional smooth Fano polytopes (for a more general definition and treatment of these notions see section 4.1). Here primitive relations are in some sense minimal integer relations among the vertices of a smooth Fano polytope, hence they give 1-cycles on the variety (see p. 23), that even generate the Mori cone (see [Rei83] and [Bat91]). Unfortunately these useful tools cannot simply be generalized to the class of reflexive polytopes, since they rely on the existence of lattice bases among the vertices of a facet. However the next proposition shows that in the simplest yet most important case of a primitive collection of order two, i.e., a pair of lattice points on the boundary that are not contained in a common face (see Def. 4.1.4), we still have a kind of *generalized* primitive relation ($v + w = 0$ or $av + bw - z = 0$) that (unlike the trivial choice in Def. 4.1.6) mimics the good properties of the nonsingular case:

Proposition 3.3.1. *Let $P \subseteq M_{\mathbb{R}}$ be a reflexive polytope, $v, w \in \partial P \cap M$, $v \neq w$. Exactly one of the following three statements holds:*

1. $v \sim w$
2. $v + w = 0$
3. $v + w \in \partial P$

Let the third condition be satisfied. Then the following statements hold:

v, w is a $\mathbb{Z}$-basis of $\operatorname{lin}(v, w) \cap M$. *There exists exactly one pair $(a, b) \in \mathbb{N}^2_{>0}$ such that, if we set $z := z(v, w) := av + bw$, then $z \in \partial P$, $v \sim z$ and $w \sim z$. Moreover:*

i. *$a = 1$ or $b = 1$. $a = |\,[w, z] \cap M\,| - 1$ and $b = |\,[v, z] \cap M\,| - 1$. If $F \in \mathcal{F}(P)$ with $v, z \in F$, then $\langle \eta_F, w \rangle = \frac{a-1}{b}$.*

ii. *Any facet containing z (or $v + w$) contains exactly one of the points v or w.*

iii. *For any $F \in \mathcal{F}(P)$ containing v and z there exists a facet $G \in \mathcal{F}(P)$ containing w and z such that $F \cap G$ is a $(d-2)$-dimensional face of P.*

iv. *If $z \in \mathcal{V}(P)$, $b = 1$, and $]v, z[$ is contained in the relative interior of a facet of P, then $[w, z]$ is contained in an edge.*

Proof. Let $v \not\sim w$ and $v + w \neq 0$. The first condition implies that for any facet $F \in \mathcal{F}(P)$ we have $\langle \eta_F, v + w \rangle = \langle \eta_F, v \rangle + \langle \eta_F, w \rangle > -2$. However reflexivity of P implies that this must be a natural number greater or equal to -1, so $v + w \in P$ by duality. We get $0 \neq v + w \in \partial P$, because P is canonical.

Since $\operatorname{conv}(0, v, w)$ is an empty two-dimensional polytope, by 3.1.8(1) we see that v, w is a $\mathbb{Z}$-basis.

Let F be a facet of P containing $v + w$. We may assume $\langle \eta_F, v \rangle = -1$ and $\langle \eta_F, w \rangle = 0$. This implies $v \sim v + w$.

We can use this consideration again for the pair $v+w, w$. Since $F \cap M$ is finite, this eventually yields a natural number $b \in \mathbb{N}_{>0}$ such that $z = v + bw \in F$ and $w \sim z$. In particular $a = 1$. This proves the existence of z and i.

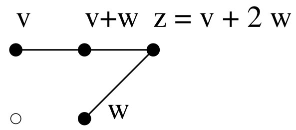
ii. Let $F' \in \mathcal{F}(P)$ with $z \in F'$. Assume $v, w \notin F'$, hence $-1 = \langle \eta_{F'}, z\rangle = a\langle \eta_{F'}, v\rangle + b\langle \eta_{F'}, w\rangle \geq 0$, a contradiction.

iii. Let $F \in \mathcal{F}(P)$ with $v, z \in F$. Since $z \in \partial F$ and $z \vdash v$, there exists a face $H \in \mathcal{F}(F)$ with $z \in H$ and $v \notin H$. By the diamond property (p. 23) there exists a facet $G \in \mathcal{F}(P)$ such that $G \cap F = H$. In particular $z \in G$ and $v \notin G$, so by the second point $w \in G$.

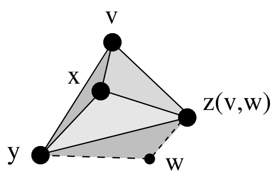
iv. Follows from Prop. 3.2.2(4) applied to π_v.

□

The statement and the proof shall be illustrated by the following figure:



For another example note that the following situation cannot occur for a three-dimensional reflexive polytope, since $\mathrm{conv}(x, y, z(v, w))$ does not contain v or w, a contradiction to 3.3.1(ii); the reader should not try to prove this by explicitly calculating facets:



The symmetric relation $\sim$ defines a *graph* $\mathcal{W}(P)$ on $\partial P \cap M$, where $\partial P \cap M$ are the vertices of $\mathcal{W}(P)$, and $\{v, w\}$ is an edge of $\mathcal{W}(P)$ if and only if $v \sim w$.

From the previous proposition we can now easily derive the following corollary about combinatorial properties of reflexive polytopes:

Corollary 3.3.2. *Let $P \subseteq M_{\mathbb{R}}$ be a reflexive polytope.*

1. *Any pair of points in $\partial P \cap M$ can be connected by at most three edges of the graph $\mathcal{W}(P)$, with equality only possibly occuring for a centrally symmetric pair of points.*

2. *The previous statement also holds for the subgraph of $\mathcal{W}(P)$ whose vertices consist of $\mathcal{V}(P)$; this is a purely combinatorial object. In the case of a simplicial polytope this is just the usual edge-graph on the vertices of P.*

3. *By dualizing we get that a pair of facets of a reflexive polytope is either parallel, contains a common vertex, or does have mutually non-trivial intersection with another facet.*

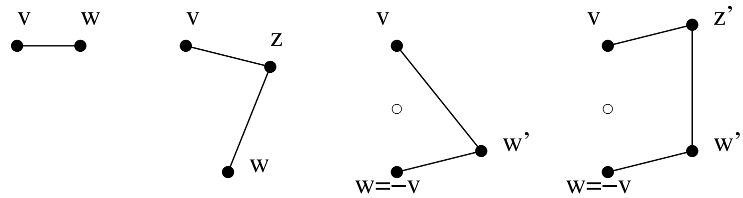
Proof. 1. Let $v, w \in \partial P \cap M$, $v \neq w$, $v \not\sim w$. If $v + w \neq 0$, then 3.3.1 yields $z := z(v, w) \in \partial P \cap M$ such that $v \sim z \sim w$. If $v + w = 0$, then we can choose $w' \in \mathcal{V}(P)$, $w' \neq w$, with $w' \sim w$. Now apply 3.3.1 for the lattice points v, w'.

2. Let $v, w \in \mathcal{V}(P)$, $v \neq w$, $v \not\sim w$. We can again assume $v + w \neq 0$, and $z := z(v, w) \in \partial P \cap M$. Now 3.3.1(iii) implies that there exists $z' \in \mathcal{V}(P)$ such that $v \sim z' \sim w$.

3. This is obvious from the second point.

□

The statement and the proof of the first point is illustrated by the following figures:



Without using the existing classification of two-dimensional reflexive polytopes (see Prop. 3.4.1) the proposition and the corollary yield an immediate application in the case of $d = 2$ (for the proof of the second point use statement i of the proposition).

Corollary 3.3.3. *Let P be a two-dimensional reflexive polytope.*

1. *P has at most six vertices; equality occurs iff P is of type* 6a *in Prop. 3.4.1.*

2. *Any facet of P contains at most five lattice points; there exists a facet with five lattice points iff P is of type* 8c *in Prop. 3.4.1.*

This first point is also a direct consequence of [PR00, Thm. 1] stating that $|\,\partial P \cap M\,| + |\,\partial P^* \cap N\,| = 12$ for a two-dimensional reflexive polytope, where however no direct combinatorial proof is known that does not use some kind of induction (see also Prop. 3.4.1 below). In higher dimensions there is no such direct relation between the lattice points in the dual pair of reflexive polytopes.

Another application is to show that certain combinatorial types of polytopes cannot be realized as reflexive polytopes. As an example we have a look at the *regular polyhedra* (see for instance [Sti01]).

These contain the d-simplex, the d-cube, and its dual, the d-crosspolytope; for $d = 3$ these are the tetrahedron, the cube, and the octahedron. In any dimension these combinatorial types can be realized as reflexive polytopes. However apart from these three infinite families there are five sporadic cases. For $d = 3$ we also have the icosahedron (simplicial, with 12 vertices and 20 facets), and its dual, the dodecahedron. Moreover for $d = 4$ there are the 24-cell (self-dual, with 24 vertices, facets are octahedra), the 600-cell (simplicial, with 120 vertices and 600 facets), and its dual, the 120-cell. Now there is the following result:

Corollary 3.3.4. *There is no reflexive polytope that is combinatorially isomorphic to the dodecahedron or the icosahedron. There is no reflexive polytope that is combinatorially isomorphic to the 120-cell or the 600-cell.*

Proof. Let P be a reflexive polytope and $d = 3$. By duality we can assume that P is combinatorially isomorphic to an icosahedron. Corollary 3.3.2(2) immediately yields by looking at the edge-graph that P is centrally symmetric. However any three-dimensional centrally symmetric simplicial reflexive polytope has at most 8 vertices as will be proven in Theorem 3.5.11.

Finally by Corollary 3.3.2(2) and duality it is enough to note that there are pairs of vertices of the simplicial 600-cell that cannot be connected by at most three edges (see [Sti01, Fig. 5]).

□

It is now an astonishing observation (see [KS02]) that the self-dual 24-cell *can* be uniquely realized as a reflexive polytope with vertices $\{\pm e_i \,:\, i = 1, \ldots, 4\} \cup \{\pm(e_i - e_j) \,:\, i = 1,2,\, j = 2,3,4,\, j > i\} \cup \{\pm(e_i - e_3 - e_4) \,:\, i = 1,2\} \cup \{\pm(e_1 + e_2 - e_3 - e_4)\}$ for $e_1, \ldots, e_4$ a $\mathbb{Z}$-basis of M. It is even centrally symmetric and terminal. Here it is interesting to note the necessity of these conditions:

Corollary 3.3.5. *Let P be a four-dimensional reflexive polytope P that is combinatorially a 24-cell. Then P has to be centrally symmetric and terminal.*

Proof. Let v be a vertex of P. Now choose the vertex $w \in \mathcal{V}(P)$ corresponding to the usual antipodal point. Assume $v + w \neq 0$. It is easy to see (see [Sti01, Fig. 4]) that the intersection of a facet containing v and a facet containing w is empty or consists of a unique vertex z where $]v, z[$ and $]w, z[$ are contained in the relative interiors of these facets. This implies $v \not\sim w$ and $z(v, w) = z \in \mathcal{V}(P)$, a contradiction to the last point of Prop. 3.3.1.

The terminality of P can be proven in an analogous way.

□

Due to the list [KS04b] the previously described polytope is even the only four-dimensional reflexive polytope with 24 vertices and 24 facets such that any vertex is contained in 6 facets.

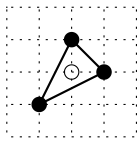
3.4 Classification results in low dimensions

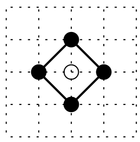
In this section we will give a survey of special classes of reflexive polytopes and previously achieved classification results in low dimensions.

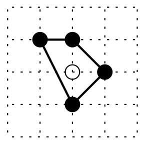
Smooth Fano polytopes, as they form the most important class of reflexive polytopes, were intensively studied over the last decade by Batyrev [Bat91, Bat99], Casagrande [Cas03a, Cas03b], Debarre [Deb03], Sato [Sat00], et al. It could be proven that there are 18 smooth Fano polytopes for $d = 3$ (see [Bat82a, WW82]) and 124 for $d = 4$ (see [Bat99, Sat00]) up to isomorphism. Here we will have a look at recent classification results of reflexive polytopes in low dimensions.

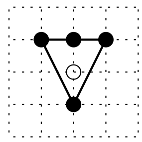
For $d = 1$ the polytope $[-1, 1]$ corresponding to $\mathbb{P}^1$ is the only Fano polytope. For $d = 2$ any canonical Fano polytope is reflexive by 3.1.8(1), and these isomorphism classes can be easily classified (e.g., see [KS97] or [Sat00, Thm. 6.22]). For the convenience of the reader and later reference we will give the list of the 16 isomorphism classes of reflexive polygons as well as a simple proof.

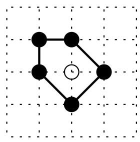
Proposition 3.4.1. *There are exactly* 16 *isomorphism classes of two-dimensional reflexive polytopes (the number in the labels denotes the number of lattice points on the boundary):*

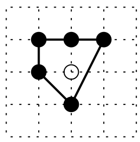
 3

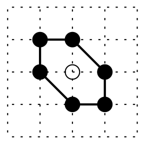
 4a

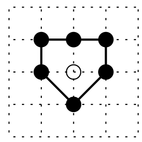
 4b

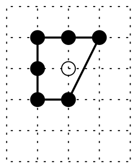
 4c

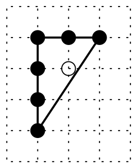
 5a

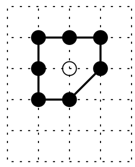
 5b

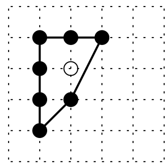
 6a

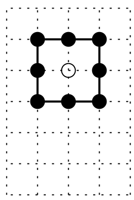
 6b

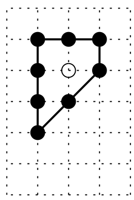
 6c

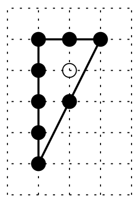
 6d

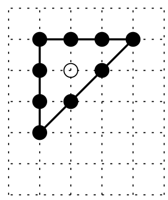
 7a

 7b

 8a

 8b

 8c

 9

There exist the following dual pairs of reflexive polygons:
(3,9), (4a,8a), (4b,8b), (4c,8c), (5a,7a), (5b,7b).
Each of the reflexive polygons 6a, 6b, 6c, 6d *is isomorphic to its dual.*

Proof. Let P be a two-dimensional reflexive polytope. We distinguish three different cases:

1. Any facet of P contains only two lattice points, i.e., P is a terminal Fano polytope. There are three different cases (see Prop. 3.3.1):

   (a) P is combinatorially a triangle.
   By 3.1.8(1) we may assume that $(1,0)$, $(0,1)$ are vertices of P. Let x be the third vertex. From Prop. 3.2.4(2) it follows that the projection of P along $(1,0)$ is a canonical Fano polytope, i.e., isomorphic to $[-1,1]$, hence $x_2 = -1$. By projecting along $(0,1)$ we get $x_1 = -1$, so P is of type 3.

   (b) There exist three vertices $u, v, w \in \mathcal{V}(P)$ with $u + w = v$.
   Since P is a terminal Fano polytope, Prop. 3.3.1 implies $u \sim v$ and $w \sim v$, and we may assume $u = (-1,1)$, $v = (0,1)$, $w = (1,0)$. Again projecting along v yields $P \cap \{(-1,x) \,:\, x \in \mathbb{Z}\} \subseteq \{(-1,0),(-1,1)\}$, $P \cap \{(0,x) \,:\, x \in \mathbb{Z}\} \subseteq \{(0,-1),(0,0),(0,1)\}$, $P \cap \{(1,x) \,:\, x \in \mathbb{Z}\} \subseteq \{(1,-1),(1,0)\}$. We get as possible types 4b, 5a, 6a.

(c) Any two vertices that are no neighbours are centrally symmetric. This immediately implies that P is of type $4a$.

2. There exists a facet F containing exactly one lattice point in relint$F$.

   We may assume $\mathcal{V}(F) = \{(-1,1),(1,1)\}$. Then by Prop. 3.2.2(1) we have $P \subseteq \{x \in \mathbb{R}^2 : -1 \leq x_1 \leq 1,\, x_2 \leq 1\}$. Since $(0,-1)$ is not contained in int$P$, we get $P \subseteq \{x \in \mathbb{R}^2 : x_2 \geq -3\}$. From this we readily derive the next ten isomorphism types $4c$,$5b$,$6b$,$6c$,$6d$,$7a$,$7b$,$8a$,$8b$,$8c$.

3. The remaining case.

   We may assume $\mathcal{V}(F) = \{(-1,1),(a,1)\}$ for $a \in \mathbb{N}$, $a \geq 2$. Let $v \in \mathcal{V}(P)$ with $v_2 \leq -1$ minimal. As $(1,0)$ is not in the interior of P, we have $v_1 \leq 0$. Then by assumption necessarily $\text{conv}((-1,1),(-1,-2),(2,1)) \subseteq P$. This must be an equality, hence P is of type 9.

□

The proof includes the statement that there are exactly five *toric Del Pezzo surfaces* , i.e., two-dimensional nonsingular toric Fano varieties: Type 3 corresponds to $\mathbb{P}^2$, $4a$ to $\mathbb{P}^1 \times \mathbb{P}^1$, $4b$ to the Hierzebruch surface, $5a$ to $\mathbb{P}^1 \times \mathbb{P}^1$ blown up in one torus invariant point, $6a$ to $\mathbb{P}^1 \times \mathbb{P}^1$ blown up in two torus invariant points. This result can also be proven by birational factorisation [Oda88, Prop. 2.21], primitive relations [Bat91] or determinants [Ewa96, Thm. V.8.2].

In general even for $d = 3$ there are too many reflexive polytopes to give a classification by pencil and paper. However by restricting to smaller classes of reflexive polytopes it is still interesting to find rigorous mathematical proofs of observations and classification results by directly using their *intrinsic* properties:

Recently Kasprzyk classified in [Kas03] all 634 *three-dimensional terminal polytopes* by first describing the minimal cases purely mathematically and then using a computer program for the remaining ones. In [Kas04] a complete list is available, where for each polytope the Gorenstein index is specified, so that it is possible to recover all 100 terminal reflexive three-dimensional polytopes (note that in [Kas03, Kas04] terminal Fano polytopes are called Fano polytopes, and the Gorenstein index is referred to as the Fano index). Independently (already in 2002) the author has obtained this list (see Thm. 4.3.1), this classification is explained in the next chapter. Wagner classified three-dimensional centrally symmetric reflexive polytopes (see [Wag95]). Moreover the author gave a classification of three-dimensional reflexive polytopes where the linear automorphism group does not have non-trivial fixpoints (see Thm. 5.4.5).

Kreuzer and Skarke described in [KS97, KS98, KS00] a *general* algorithm to classify reflexive polytopes in fixed dimension d. Using their computer program PALP (see [KS04a]) they applied their method for $d \leq 4$, and found 4319 reflexive polytopes for $d = 3$ and 473800776 for $d = 4$. They also described how to find a normal form of lattice polytopes, toric fibrations and symmetries. The complete list of three-dimensional reflexive polytopes and a searchable database of the four-dimensional reflexive polytopes can be found on their webpage [KS04b].

Here the ideas of their algorithm shall be shortly sketched (in the case of a reflexive simplex an explicit approach due to Conrads is described in section 3.6): First they showed that any lattice polytope $Q \subseteq N_{\mathbb{R}}$ that is minimal with respect to the so called IP-property, i.e., it contains the origin in its interior, can be combinatorially decomposed into simplices, a so called *IP simplex structure*. There are 2, 3, 9 such structures for $d = 2, 3, 4$. For any simplex we have a unique *weight system* (see section 3.6), hence an IP-simplex structure yields a so called *combined weight system* (CWS). Let N_{coarsest} be the lattice generated by the vertices of Q, and M_{finest} its dual lattice. Then the CWS q of Q determines Q^* and hence Q up to an isomorphism of M_{finest} respectively N_{coarsest}. Now we define the lattice polytope $Q(q) := \text{conv}(Q^* \cap M_{\text{finest}})$. They proved that if $Q(q)$ has the IP-property, then this is true for any weight system in the CWS. For $d \leq 4$ this even implies $Q(q)$ to be reflexive. Now they determined any such possible CWS, and hence classified any lattice polytope Q that is minimal with respect to the IP-property. When we have a reflexive polytope $P \subseteq M_{\mathbb{R}}$, we can always find such a $Q \subseteq N_{\mathbb{R}}$ with $\mathcal{V}(Q) \subseteq \mathcal{V}(P^*)$. Hence $P \subseteq Q^*$, where the vertices of Q^* generate M_{finest}. Now in order to recover our reflexive polytope we just have to choose some lattice subpolytope P of Q^* (with respect to M_{finest}). Then we check, if the *vertex-pairing-matrix* is integral, and in this case we again find M as a a lattice $M \subseteq M_{\text{finest}}$ containing the vertices of P. This can be done in a systematic way using Hermite normal forms (see Thm. 3.6.6)

Using this list we find 194, respectively 5450, classes of three-dimensional, respectively four-dimensional, simplicial reflexive polytopes. Moreover this yields 151 four-dimensional terminal simplicial reflexive polytopes (for $d = 3$ any such polytope is smooth, see 3.1.8(2)).

3.5 Sharp bounds on the number of vertices

Throughout the section let P be a d-dimensional reflexive polytope in $M_{\mathbb{R}}$.

Since in higher dimensions only in very special cases classification results exist, one tries to find at least sharp bounds on invariants and to characterize the case of equality. Here we examine the number of vertices of a reflexive polytope. While for general reflexive polytopes only a conjecture due to the author exists, in the simplicial case we have an almost complete answer. This was achieved by Casagrande after the publication of the preprint [Nil04a]. Moreover the author could generalize previous non-trivial results of Debarre and Casagrande about smooth Fano polytopes to simplicial reflexive polytopes.

The number of vertices corresponds to the rank of the class group of the toric variety $X := X(M, \Sigma_P)$ associated to the fan spanned, we have by (1.1)

$$\text{rank}\, \text{Cl}(X) = |\,\mathcal{V}(P)\,| - d. \tag{3.3}$$

The computer classification of Kreuzer and Skarke (see [KS04b]) yields that the maximal number of vertices of a d-dimensional reflexive polytope is 6 for $d = 2$, 14 for $d = 3$ and 36 for $d = 4$ (note that there is a misprint on page 1220 of [KS00] stating 33 as the maximal number of vertices).

Definition 3.5.1. $\mathcal{Z}_d := \text{conv}([0,1]^d, -[0,1]^d)$ is called the n-dimensional *standard lattice zonotope*. It is a centrally symmetric terminal reflexive polytope, for $d \geq 3$ it is not simplicial (see [DHZ01, Proof of Thm. 3.21] where $\mathcal{Z}^{(d)} = \mathcal{Z}_{d-1}$). A picture of $\mathcal{Z}_3$ can be found on p. 118.

However $\mathcal{Z}_2 = \text{conv}(\pm[0,1]^2)$ is the (up to isomorphism) unique centrally symmetric self-dual smooth Fano polytope with 6 vertices (of type $6a$ in Prop. 3.4.1). We denote by $S_3 := X(M, \Sigma_{\mathcal{Z}_2})$ the associated nonsingular toric Del Pezzo surface that is the blow up of $\mathbb{P}^2$ in three torus-invariant points.

Conjecture 3.5.2. Let P be a d-dimensional reflexive polytope. Then

$$|\mathcal{V}(P)| \leq 6^{\frac{d}{2}},$$

where equality occurs if and only if d is even and $P \cong (\mathcal{Z}_2)^{\frac{d}{2}}$.

Remark 3.5.3. *It would be enough to prove this conjecture for d even*, because products of reflexive polytopes are again reflexive.

For $d = 2$ a rigorous proof is known (see Cor. 3.3.3(1) and Prop. 3.5.5(1) below). For $d \leq 4$ the author checked this conjecture using the classification data [KS04b] and the computer program PALP (see [KS04a]). In higher dimensions this conjecture will be proven for centrally symmetric simple reflexive polytopes in the last chapter, see Theorem 6.2.2 on page 149.

Remark 3.5.4. In the paper [HM04] Haase introduces the notion of the *reflexive dimension* refldim(Q) of a lattice polytope Q as the least dimension d a reflexive polytope P can have that contains Q as a face. He proves that refldim(Q) is finite. Assuming the previous conjecture we get

$$\operatorname{refldim}(Q) > 2 \log_6 \mathcal{V}(Q).$$

If d is odd, there is no such obvious candidate as $\mathcal{Z}^{d/2}$ is for the even case. For $d = 3$ the maximal number of vertices is 14, and $\mathcal{Z}_3$ is the only reflexive polytope with this number of vertices, *it is not simple.* For odd $d \geq 5$ the reflexive polytope $S_3^{(d-3)/2} \times \mathcal{Z}_3$ has $6^{(d-3)/2}14$ vertices. Of course one could conjecture that this were the maximal number of vertices. Let P have the maximal number of vertices for d odd, then we get assuming the previous conjecture

$$6^{(d-1)/2}\frac{14}{6} \leq |\mathcal{V}(P)| < 6^{(d-1)/2}\sqrt{6} = 6^{d/2}.$$

For $d = 5$, this would imply $84 \leq |\mathcal{V}(P)| \leq 88$. So even assuming the correctness of above conjecture there could exist a reflexive polytope with more than 84 vertices without implying an obvious contradiction.

The next result yields two coarse upper bounds on the number of vertices of a reflexive polytope in terms of some combinatorial invariants of the facets. The first bound is a straightforward generalization of a bound due to Voskresenskij and Klyachko [VK85, Thm.1] originally proven in the setting of a smooth Fano polytope. The second upper bound is a generalization of [Deb03, Thm. 8], where Debarre improved from a bound of order $O(d^2)$ on the number of vertices of a smooth Fano polytope to a bound of order $O(d^{3/2})$. We recover the original results for simplicial reflexive polytopes. Of course by dualizing one can derive upper bounds on the number of facets. These results will be further improved in the remainder of this section (see Theorem 3.5.12 and Corollary 3.5.13).

Proposition 3.5.5. *Let P be a reflexive polytope.*
Define $\alpha := \max(\mathcal{V}(F) \,:\, F \in \mathcal{F}(P))$ and $\beta := \max(\mathcal{F}(F) \,:\, F \in \mathcal{F}(P))$.

1. $|\mathcal{V}(P)| \leq 2d\alpha$.

   More precisely we distinguish two cases:

   If $\alpha \geq 2d - 3$, then $|\mathcal{V}(P)| \leq 2d(\alpha - d + 2) - 2$.

If $\alpha \leq 2d-3$*, then* $|\mathcal{V}(P)| \leq d\alpha + \alpha - d + 1$*.*
If P *is simplicial, i.e.,* $\alpha = d$*, and* $d \geq 3$*, this yields*

$$|\mathcal{V}(P)| \leq d^2 + 1.$$

2. $|\mathcal{V}(P)| \leq (\alpha - d + 1)\beta + 2 + 2\sqrt{(\alpha-1)(d+1)((\alpha-1)+(\alpha-d+1)\beta))}$*.*
If P *is simplicial, i.e.,* $\alpha = d = \beta$*, this yields*

$$|\mathcal{V}(P)| \leq d + 2 + 2\sqrt{(d^2-1)(2d-1)}.$$

Proof. Analyzing the proofs of Thm. 1 in [VK85] and Thm. 8 in [Deb03] in the more general setting of a reflexive polytope, we see that by taking the general invariants α and β into account we just have to reprove remark 5(2) in section 2.3 of [Deb03], because only there explicitly a lattice basis was used. That result is essentially the first part of the next lemma. □

Lemma 3.5.6. *Let* $P \subseteq M_{\mathbb{R}}$ *be a reflexive polytope.*

Let $F \in \mathcal{F}(P)$*,* $u := \eta_F \in \mathcal{V}(P^*)$ *and* $\{F_i\}_{i \in I}$ *the facets that intersect* F *in a* $(d-2)$*-dimensional face. Let* $m \in \partial P \cap M$ *with* $\langle u, m\rangle = 0$*.*

Then $m \in \cup_{i\in I} F_i$*.*

Let additionally F *be a simplex with* $\mathcal{V}(F) = \{e_1, \ldots, e_d\}$*. Let* $e_1^*, \ldots, e_d^*$ *be the dual* $\mathbb{R}$*-basis of* $N_{\mathbb{R}}$*. For* $i = 1, \ldots, d$ *denote by* F_i *the facet of* P *such that* $F_i \cap F = \operatorname{conv}(e_j \,:\, j \neq i)$*.*

1. *For* $i \in \{1, \ldots, d\}$ *we have*

$$m \notin F_i \iff \langle e_i^*, m\rangle \geq 0.$$

2. *If there exists* $i \in \{1, \ldots, d\}$ *such that* $m \in F_i$ *and* $m \notin F_j$ *for all* $j \in \{1, \ldots, d\}$*,* $j \neq i$*, then* $m \not\sim e_i$*.*

3. *Assume that for every* $i = 1, \ldots, d-1$ *there exists a lattice point* m^i *on* F_i *such that* $\langle u, m^i\rangle = 0$ *and* $\langle e_i^*, m^i\rangle = -1$*. Then* $e_1, \ldots, e_d$ *is a* $\mathbb{Z}$*-basis of* M*.*

Proof. The first part follows from Prop. 3.2.2(6) for π_m.

Now let F be a simplex. Then $u = \sum_{j=1}^d (-e_j^*) \in N$. Let $i \in \{1, \ldots, d\}$. Since $m^i \notin F$ and 0 is in the interior of P, the number $\alpha_i := \frac{-1-\langle u, m^i\rangle}{\langle e_i^*, m^i\rangle} > 0$ is well-defined. We get $\eta_{F_i} = u + \alpha_i e_i^*$. From this 1. is readily derived. 2. is just a corollary. In 3. we get $\alpha_i = 1$ and $e_i^* = \eta_{F_i} - u \in N$ for $i = 1, \ldots, d-1$ and $e_d^* = -u - e_1^* - \ldots - e_{d-1}^* \in N$.

This proof is inspired by remark 5(2) in section 2.3 of [Deb03]. □

In the following we will focus on the class of *simplicial* reflexive polytopes, i.e., where the corresponding varieties are $\mathbb{Q}$-factorial, or equivalently, the class number equals the Picard number. The previous proposition already gave a hint that simplicial reflexive polytopes are actually quite close to smooth Fano polytopes at least when considering only the number of vertices. This motivated the author to state the following explicit conjecture:

Conjecture 3.5.7 (Nill 5/2004). Let P be a d-dimensional simplicial reflexive polytope. Then

$$|\mathcal{V}(P)| \leq \begin{cases} 3d & , \quad d \text{ even}, \\ 3d-1 & , \quad d \text{ odd}. \end{cases}$$

For d even equality holds if and only if $P^* \cong (\mathcal{Z}_2)^{\frac{d}{2}}$, i.e., $X \cong (S_3)^{\frac{d}{2}}$.

Remark 3.5.8. *It would be enough to prove this conjecture for d even:* Assume there were a simplicial reflexive polytope P with d odd and $|\mathcal{V}(P)| \geq 3d$. Then necessarily $P^* \times P^* \cong (\mathcal{Z}_2)^d$, this would imply P to be centrally symmetric with $|\mathcal{V}(P)| = 3d$, a contradiction to d odd.

From [Kas03] we get that reflexivity is essential, because the maximal number of vertices a three-dimensional simplicial terminal Fano polytope can have is 10.

The previous conjecture was originally proposed in the case of smooth Fano polytopes by Batyrev [Ewa96, p. 337], and was up to 2003 only rigorously proven to hold for (up to) five-dimensional smooth Fano polytopes by Casagrande in [Cas03a, Thm. 3.2].

The bound is also sharp in the odd-dimensional case, take $X = \mathbb{P}^1 \times (S_3)^{\frac{d-1}{2}}$. However even for $d = 3$ there is exactly one another simplicial reflexive polytope with 8 vertices, it is a smooth Fano polytope, not centrally symmetric, and the corresponding toric variety X is an equivariant S_3-fibre bundle over $\mathbb{P}^1$. So it is also tempting to formulate an explicit conjecture:

Conjecture 3.5.9. In Conjecture 3.5.7 equality holds for d odd if and only if the reflexive polytope defines a toric variety that is a (nonsingular) equivariant $(S_3)^{\frac{d-1}{2}}$-fibre bundle over $\mathbb{P}^1$.

This was proven in the nonsingular case by Casagrande for $d \leq 5$, in this general form however it could only be confirmed in the case of a centrally symmetric pair of facets, this will be proven in Theorem 6.2.4.

Another observation for $d \leq 4$ is that the maximal number of facets a simplicial reflexive polytope can have ($d = 2$: 6, $d = 3$: 12, $d = 4$: 36) is achieved by the ones with the maximal number of vertices. In even dimension this is of course just a corollary of conjectures 3.5.2 and 3.5.7.

Remark 3.5.10. For the proof of these conjectures in low dimensions the Dehn-Sommerville equations were used together with the following theorem by Batyrev [Bat99, Prop. 2.3.7] for nonsingular toric Fano varieties, where f_i is the number of i-dimensional faces of the corresponding smooth Fano polytope:

$$12 f_{d-3} \geq (3d-4) f_{d-2}.$$

It is astonishing to observe that for $d \leq 4$ by the classification of Kreuzer and Skarke this relation is also valid for simplicial reflexive polytopes. However there is not yet an algebraic-geometric explanation for this phenomenon!

The main goal of this section is to give a proof of Conjecture 3.5.7 in the case of additional symmetries of the polytope:

Theorem 3.5.11. *Conjecture 3.5.7 holds in the case of a simplicial reflexive polytope P where P^* contains a vertex $u \in \mathcal{V}(P^*)$ such that $-u \in P^*$.*

The results in this section were published in the preprint [Nil04a] in May 2004. However in October 2004 Casagrande proved Conjecture 3.5.7 for the class of smooth Fano polytopes. For this she needed the fact that the sum of all vertices in the dual polytope is zero, she proved this using birational factorization. Then she applied a simple and neat enumerating argument to get the upper bound, for the proof she only used the previous lemma in the case of a smooth Fano polytope as given in [Deb03]. For the equality case she refered to Theorem 3.5.11 of the author. In November 2004 the author pointed out to her that the first fact is actually a folklore result in convex geometry (see Lemma 5.3.8) and can for instance be found in another preprint [Nil04b] of the author published in July 2004. In December 2004 Casagrande published a new version of her paper [Cas04] where she simplified her approach to get the following final result [Cas04, Thm.1(i)]:

Theorem 3.5.12 (Casagrande 12/2004). *Conjecture 3.5.7 holds.*

Not only does this result render the coarse bounds in Prop. 3.5.5 obsolete in the simplicial case, but analyzing the proof in [Cas04] it is possible to get also an improvement of the general bound in Prop. 3.5.5(2):

Corollary 3.5.13. *Let P be a reflexive polytope.*
Define $\alpha := \max(\mathcal{V}(F) \,:\, F \in \mathcal{F}(P))$ and $\beta := \max(\mathcal{F}(F) \,:\, F \in \mathcal{F}(P))$.

$$|\,\mathcal{V}(P)\,| \leq 2\alpha + (\alpha - d + 1)\beta.$$

Using the ideas of Casagrande's proof the author realized that Theorem 3.5.11 could actually be reduced to the centrally symmetric case, where an easier proof is possible, even more, a complete classification is now available, see section 6.3. However in the remainder of this section we give the original proof as published in [Nil04a], since it uses a technique (Lemma 3.5.15) that is interesting in itself.

The main result for analyzing smooth Fano polytopes is a theorem of Reid about extremal rays of the Mori cone and primitive relations (see [Rei83] and [Cas03a, Thm. 1.3]). Although for simplicial reflexive polytopes there is no general notion of a primitive relation, for the simplest case as defined in Prop. 3.3.1 we still have an analogous result (recall Definition 3.2.1):

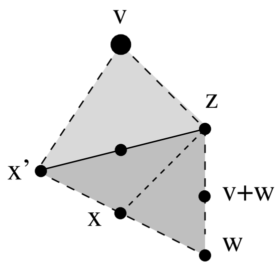
Lemma 3.5.14. *Let P be a simplicial reflexive polytope.*
Let $v \in \mathcal{V}(P)$, $w \in \partial P \cap M$ with $v + w \in \partial P$ and $z := z(v, w)$.
Let $x \in \partial P$, $x \notin \{v, w, z\}$, with $x \sim z$ and x away from v.
Then $\mathrm{conv}(x, z, w)$ is contained in a face.
Moreover exactly one of the following two conditions holds:

1. *Any facet containing x and z contains also w.*
2. *There exists a facet F with $x, v, z \in F$.*

The second case must occur, if $w \in \mathcal{V}(P)$ and x is away from w.
If the second case occurs, we have:
For any such F there exists a unique facet G with $x, w, z \in G$ such that $F \cap G$ is a $(d-2)$-dimensional face of P. $F \cap G$ consists of those elements of F that are away from v, respectively those elements of G that are away from the (unique) vertex not in F. Obviously $w \notin F$ and $v \notin G$.

Proof. Assume the first case does not hold. Prop. 3.3.1(ii) implies that there exists a facet $F \in \mathcal{F}(P)$ with $x, z, v \in F$. By 3.3.1(iii) there is a facet $G \in \mathcal{F}(P)$ containing w and z such that $F \cap G$ is a $(d-2)$-dimensional face. Since F, G are simplices and $v \in \mathcal{V}(F)$, the remaining statements are now straightforward. □

The two cases are illustrated in the following figure for a three-dimensional reflexive polytope (where x as in the first, x' as in the second case):



The next result is a generalization of a lemma proven by Casagrande [Cas03a, Lemma 2.3] for smooth Fano polytopes, here we recover the original statement in the more general setting of a terminal simplicial reflexive polytope.

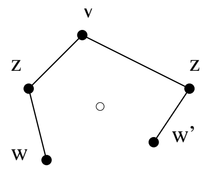
Lemma 3.5.15. *Let P be a simplicial reflexive polytope.*

Let $v, w \in \mathcal{V}(P)$, $w' \in \partial P \cap M$ away from w. Furthermore let $v + w \in \partial P$ and $v + w' \in \partial P$, $z := z(v, w)$ and $z' := z(v, w')$.

We define $K := P \cap \operatorname{lin}(v, w, w')$. Then K is a two-dimensional *reflexive polytope (of possible types* 5*a*, 6*a*, 6*b*, 7*a* *in Prop. 3.4.1).*

If K is terminal, then $z = v + w$, $w' = -v - w = -z$, $z' = v + w' = -w$; and either $\partial K \cap M = \{v, w, z, w', z'\}$ or $\partial K \cap M = \{v, w, z, w', z', -v = w + w' = z(w, w')\}$.

Essentially the lemma states that the following figure is *two*-dimensional:



Proof. Let $z = av + bw$ and $z' = a'v + b'w'$ as in 3.3.1. We note that w' and z are away from v and w; z' is away from v and w'.

Assume $w' \sim z$. Since w' is away from $w \in \mathcal{V}(P)$, it follows from 3.5.14 that there exists a facet that contains w', z, v, hence $w' \sim v$, a contradiction.

Thus $w' \not\sim z$; in particular, $z \neq z'$. There are now two different cases, and it must be shown that the second one cannot occur.

1. v, w, w' are linearly dependent.

   By 3.3.1 there are three possibilities:

   If $w \sim w'$, then $K = \operatorname{conv}(v, z, w, z', w')$. If $w + w' = 0$, then $v \in \operatorname{conv}(v + w, v + w')$, a contradiction. If $w + w' \in \partial P$, then $K = \operatorname{conv}(v, z, w, z', w', z(w, w'))$.

Thus in any case K is a lattice polytope with 5 or 6 vertices, canonical, hence by 3.1.8(1) reflexive. By analyzing the cases in 3.4.1 we get the remaining statements.

2. v, w, w' are linearly independent.

   Hence also z, z', v are linearly independent.

   Assume $z' \sim z$. 3.5.14 implies that $\operatorname{conv}(z', z, w)$ is contained in a facet $F \in \mathcal{F}(P)$. Since $v \notin F$ and $z' \in F$, 3.3.1(ii) implies $w' \in F$, a contradiction to $w' \not\sim z$.

   Thus $z \not\sim z'$.

   By assumption $z + z' \neq 0$, hence $z + z' \in \partial P$. Let $y := z(z', z) = kz' + lz \in \partial P \cap M$. We have $y \notin \{z, z', v, w, w'\}$, because v, w, w' are linearly independent.

   Choose $y' \in \partial P$ with $y' = v + \lambda(y - v)$ for $\lambda \geq 1$ maximal, so that y' is away from v. Furthermore $y' \sim z'$ and $y' \sim z$. So by 3.5.14 there exist facets $F_1, F_2 \in \mathcal{F}(P)$ such that $\operatorname{conv}(y', z, w) \subseteq F_1$ and $\operatorname{conv}(y', z', w') \subseteq F_2$; $v \notin F_1, F_2$.

   Now choose $y'' = w + \mu(y' - w) \in P$ for $\mu \geq 1$ maximal; so y'' is away from w. Furthermore $\operatorname{conv}(y'', y', z, w) \subseteq F_1$ and y'' away from v, so by 3.5.14 there exists a facet $G \in \mathcal{F}(P)$ that contains y'', v, z and intersects F_1 in a $(d-2)$-dimensional face. Hence necessarily $\langle \eta_{F_1}, v \rangle = \frac{b-1}{a}$ and $\langle \eta_G, w \rangle = \frac{a-1}{b}$.

   K is a three-dimensional polytope. Any face of K is contained in a face of P. Since y', z, w (resp. y', z', w') are linearly independent, $F_1 \cap K$ (resp. $F_2 \cap K$) is a facet of K. Moreover $F_1 \cap K \neq F_2 \cap K$, because $w' \not\sim z$. So $C := F_1 \cap F_2 \cap K$ is a vertex or edge of K containing y'. Since also $w' + z \neq 0$ and $w' \not\sim z$, we get $w' + z \in \partial P$. We set $x := z(w', z) \in \partial P \cap M$. Since z, w', w (resp. z, w', y) are linearly independent, we have $x \neq w$ (resp. $x \neq y$).

   We distinguish several cases:

   (a) $y' = y$.
   In the $\mathbb{R}$-basis w, w', z of $\mathbb{R}^3$ we see that y has negative first and nonnegative second and third coordinate, so $\operatorname{pos}(w', z) \cap [w, y]$ consists of one point x'. We have $x' \in\,]w, y[\, \subseteq F_1$. Moreover since $x' \sim z$, we get $x' \in\,]z, x]$.

   i. $w \in F_2$.
   The vertices of C consist of y'' and w. Since $x' \in C$, we have $x' \sim w'$, hence $x = x' \in\,]w, y[\, \subseteq C$, thus also $x \neq y''$. If $a = 1$, it were $0 = \langle \eta_G, w \rangle > \langle \eta_G, x \rangle > \langle \eta_G, y'' \rangle = -1$, a contradiction.
   Therefore $a \geq 2$, $b = 1$, $\langle \eta_G, w \rangle = a - 1 > 0$.
   Then $\langle \eta_{F_2}, v \rangle = \frac{b'-1}{a'}$ and $-1 = \langle \eta_{F_2}, y \rangle = -k + la\frac{b'-1}{a'} - lb$, thus, since $b = 1$,
   $$\frac{k-1}{l} = a\frac{b'-1}{a'} - 1 \in \mathbb{N}. \tag{3.4}$$
   Since $\frac{b'-1}{a'} \neq 0$, $b' \geq 2$ and $a' = 1$.
   If $k = 1$, then (3.4) yields $1 = a(b'-1) \geq a \geq 2$, a contradiction.

If $l=1$, then $\langle \eta_G, y\rangle = -k-1+kb'\langle \eta_G, w'\rangle$. So $-1 = \langle \eta_G, y''\rangle = (1-\mu)(a-1)+\mu\langle \eta_G, y\rangle = (1-\mu)(a-1)-\mu k-\mu+\mu kb'\langle \eta_G, w'\rangle$. This implies $\langle \eta_G, w'\rangle = \frac{\frac{-a}{\mu}+k+a}{kb'}$. Since (3.4) yields $k = a(b'-1)$, this implies

$$\langle \eta_G, w'\rangle = \frac{\frac{-1}{\mu b'}+1}{b'-1} \in \mathbb{N}.$$

On the other hand $\mu \geq 1$ and $b' \geq 2$ yields $0 < \frac{-1}{\mu b'}+1 < 1$, this contradicts the previous equation.

ii. $w \notin F_2$.
This immediately implies $y'' = y$. Assume $x' \neq x$. This yields $x \in F_1$. Let $x'' \in F_1$ away from w such that $x \in [w, x'']$. By assumption w, z, x', y, x, x'' are contained in F_1 and $\operatorname{aff}(z,y) \cap F_1 = F_1 \cap G \cap K$. Now 3.5.14 yields $x'' \in F_1 \cap G$, so $y \in\,]z, x''[$. However $w' \sim y$, so $w' \sim z$, a contradiction. Hence $x = x' \in\,]w, y[$.
Since $w' \sim x$ there exists a facet $H \in \mathcal{F}(P)$ containing w', w, y; furthermore $H \neq F_2$, since $w \notin F_2$. Hence there are edges $F_2 \cap H \supseteq [w', y]$ and $F_1 \cap H = [w, y]$ of K.
Since $w \not\sim z'$ we can define in a double recursion $x_r^0 := x$, $x_l^0 := x(z', w)$, $x_r^i := x(x_l^{i-1}, z)$, $x_l^i := x(z', x_r^{i-1})$ for $i \in \mathbb{N}$, $i \geq 1$. As w', y, z and w, y, z' are linearly independent, we easily see that this procedure is well-defined, and $x_l^0, x_l^1, x_l^2, \ldots$ are pairwise different lattice points in $]w', y[$ and $x_r^0, x_r^1, x_r^2, \ldots$ are pairwise different lattice points in $]w, y[$. Hence we have constructed infinitely many lattice points in P, a contradiction.

(b) $y' \neq y$.
If $y'' \neq y'$, then y is a lattice point in the interior of K, a contradiction.

Thus $y'' = y'$. This implies $y \in\,]v, y'[$, so $y \in G$, and $\operatorname{conv}(z', y', y, v)$ is contained in a facet $F' \in \mathcal{F}(P)$. 3.5.14 implies that there exists a unique facet $G' \in \mathcal{F}(P)$ that contains w', z', y' such that $F' \cap G'$ is a $(d-2)$-dimensional face.

Furthermore $\frac{b-1}{a} = \langle \eta_{F_1}, v\rangle > \langle \eta_{F_1}, y\rangle > \langle \eta_{F_1}, y'\rangle = -1$, hence $b \geq 2$ and $a = 1$. Especially we get $\langle \eta_G, w\rangle = 0$.

i. $w \in G'$.
Since $w \in \mathcal{V}(P)$, $w \notin F'$ and w' is away from $w \in \mathcal{V}(G')$, 3.5.14 implies that $w' \in G' \cap F'$, a contradiction.

ii. $w \notin G'$.
Let $C' := G' \cap F_1 \cap K$. Then y' is a vertex of C'.
Assume C' were an edge. Let $v' \in \mathcal{V}(C')$ with $v' \neq y'$. This implies $v' \neq w$. Then v' is away from w, hence by 3.5.14 $v' \in G$, therefore $v' = y'$, a contradiction.
So $C' = \{y'\}$, and the same way we see that $[w, y']$ is an edge of K.
Obviously $x \in [w, y'] \cup [y', z']$, however because $x \sim z$, this yields $x \in\,]w, y']$. Since $0 = \langle \eta_G, w\rangle > \langle \eta_G, x\rangle \geq \langle \eta_G, y'\rangle = -1$, this implies $x = y'$.

Furthermore we have $y' = (1-\lambda)v + \lambda y = ((1-\lambda) + \lambda(ka' + la))v + \lambda kb'w' + \lambda lbw$, where $\lambda > 1$. On the other hand $x = rw' + sz = rw' + sav + sbw$ for $r, s \in \mathbb{N}$, $r, s \geq 1$.
Comparing the coefficients for w' and w this yields

$$\lambda l = s, \quad \lambda kb' = r.$$

From the first equation we get $s = \lambda l > l \geq 1$, so $s \geq 2$. This implies $1 = r = \lambda kb' > kb' \geq 1$, a contradiction.

□

Using Prop. 3.2.2(1-4) and analyzing the possible cases in Prop. 3.4.1 it is straightforward to prove a corollary of the previous lemma:

Corollary 3.5.16. *Let P be a simplicial reflexive polytope and $v \in \mathcal{V}(P)$.*
There are at most three vertices of P not in the star set of v; equality implies that $-v \in \mathcal{V}(P)$. For $P_v := \pi_v(P)$ and $M_v := M/\mathbb{Z}v$ we have

$$|\,\mathcal{V}(P)\,| \leq |\,\partial P_v \cap M_v\,| + 4,$$

where equality implies $-v \in \mathcal{V}(P)$. There are now two cases:

1. *Let $w \in \mathcal{V}(P)$ with $w \neq -v$ and $w \not\sim v$.*
   Then any lattice point on the boundary of P is in the star set of v or *in the star set of w but not away from w* or *in* $\mathrm{lin}(v, w)$*. This implies*
   $$|\,P_v \cap M_v\,| + |\,\mathrm{int} P_w \cap M_w\,| \leq |\,\partial P \cap M\,| \leq |\,P_v \cap M_v\,| + |\,\mathrm{int} P_w \cap M_w\,| + 2;$$
   if the second equality holds, then $-v \in P$.
2. *No such w as in 1. exists. Then:*
   $$|\,\mathcal{V}(P)\,| \leq |\,\partial P_v \cap M_v\,| + 2.$$

Going back to algebraic geometry we derive a generalization of a theorem proven by Casagrande in the nonsingular case [Cas03a, Thm. 2.4]:

Corollary 3.5.17. *If X is a $\mathbb{Q}$-factorial Gorenstein toric Fano variety with torus-invariant prime divisor $\mathcal{V}_\tau$, then the Picard numbers satisfy the inequality*

$$\rho_X - \rho_{\mathcal{V}_\tau} \leq 3.$$

Finally using Lemmas 3.5.6 and 3.5.15 we are now ready to prove the main theorem.

Proof of theorem 3.5.11. Let P be a simplicial reflexive polytope such that there exists a vertex $u \in \mathcal{V}(P^*)$ with $-u \in P^*$. Let F be the facet corresponding to u and F' the face defined by $-u$. Now define the set $\{v^1, \ldots, v^d\}$ of vertices not in F but in facets intersecting F in a codimension two face.

Lemma 3.5.6 immediately implies that $\mathcal{V}(P)\backslash(\mathcal{V}(F) \cup \mathcal{V}(F')) = \{v \in \mathcal{V}(P) : \langle u, v\rangle = 0\} \subseteq \{v^1, \ldots, v^d\}$. This yields the bound $|\mathcal{V}(P)| \leq 3d$.

In order to prove Conjecture 3.5.7 and thereby finish the proof of Theorem 3.5.11 we may assume that $|\mathcal{V}(P)| = 3d$ and d is even by Remark 3.5.8.

Let $e_1, \ldots, e_d$ be the vertices of the facet F such that $F_i := \mathrm{conv}(v^i, e_j : j \neq i)$ is a facet for $i = 1, \ldots, d$. Since by 3.5.6 any lattice point x in ∂P with $\langle u, x\rangle = 0$ is contained in some F_i, however v^i is the only vertex of F_i with $\langle u, v^i\rangle \geq 0$ (resp. $\langle u, v^i\rangle = 0$), this necessarily implies $x = v^i$. Hence we get

$$\{x \in \partial P \cap M \,:\, \langle u, x\rangle = 0\} = \{v^1, \ldots, v^d\}.$$

Analogously let $b_1, \ldots, b_d$ be the vertices of F' such that $\mathrm{conv}(v^i, b_j : j \neq i)$ is a facet for $i = 1, \ldots, d$. Then we get the following three facts for $i, j, k \in \{1, \ldots, d\}$:

· *Fact 1:* For $i \neq j$: $v^i \sim v^j$ or $v^i + v^j = 0$.

(*Proof:* Assume not. Then there exists a k such that $v^i + v^j = v^k \in F_k$. By 3.3.1(ii) this implies $v^i \in F_k$ or $v^j \in F_k$, giving $v^i = v^k$ or $v^j = v^k$, a contradiction.)

· *Fact 2:* For i: $e_i + v^i \in \partial P$ and $b_i + v^i \in \partial P$.

(*Proof:* Since $v^i \notin F_j$ for all $j \neq i$ and $v^i \in F_i$, 3.5.6(2) yields $e_i \not\sim v^i$. By symmetry the same holds for b_i.)

· *Fact 3:* Let i, j such that $e_i + b_j \in \partial P$. Then $z(e_i, b_j) = e_i + b_j = v^k$ for some $i \neq k \neq j$.

(*Proof:* Since $\langle u, e_i + b_j\rangle = 0$, let $e_i + b_j = v^k$ for some k. Assume $e_i \not\sim v^k$. By 3.3.1 then also $2e_i + b_j \in \partial P$. This implies $v^k = 1/2(b_j + (2e_i + b_j))$, a contradiction to $v^k \in \mathcal{V}(P)$. By symmetry we get $e_i \sim v^k \sim b_j$. By fact 2 necessarily $i \neq k \neq j$.)

Let $i \in \{1, \ldots, d\}$. By fact 2 we can apply Lemma 3.5.15 to the vertices v^i, e_i, b_i. From fact 3 and analyzing the possible types in 3.4.1 we get that $P \cap \mathrm{lin}(v^i, e_i, b_i)$ must be a terminal two-dimensional reflexive polytope, so $-e_i = v^i + b_i \in F'$, $-b_i = v^i + e_i \in F$, $v^i = z(-e_i, -b_i) = -e_i + (-b_i)$. As this is true for all $i = 1, \ldots, d$, we get $F' = -F$ and $-e_i, -b_i \in \mathcal{V}(P)$.

This gives a map

$$s \,:\, \{1, \ldots, d\} \to \{1, \ldots, d\}, \;\; i \mapsto s(i), \;\; \text{such that } b_{s(i)} := -e_i.$$

1. s is injective, hence a permutation.

2. There are no fixpoints under s, i.e., $s(i) \neq i$ for all $\{1, \ldots, d\}$.

3. $-v^i = e_i + b_i \in \partial P$ for all $i \in \{1, \ldots, d\}$.

   (*Proof:* By 3.5.15 it is enough to show that $-v^i \in P$. Assume not. Fact 1 implies then $v^i \sim v^{s(i)} = z(-e_{s(i)}, -b_{s(i)}) = z(-e_{s(i)}, e_i)$, so by 3.5.14 $v^i \sim e_i$, a contradiction to fact 2.)

4. $s \circ s = \mathrm{id}$.

   (*Proof:* Assume there exists an $i \in \{1, \ldots, d\}$ such that for $j := s(i)$ we have $b_{s(j)} \neq b_i$. This implies $b_i \sim v^{s(i)} = z(-e_{s(i)}, -b_{s(i)}) = z(b_{s(j)}, e_i)$, so by assumption and 3.5.14 $b_i \sim e_i$. This is a contradiction to (c).)

Property (d) implies that P is centrally symmetric. Furthermore s is a product of $\frac{d}{2}$ disjoint transpositions in the symmetric group of $\{1, \ldots, d\}$. This permutation s and the set $\{e_1, \ldots, e_d\}$ of vertices of F uniquely determine P, because $F' = -F$ and $v^i = -e_i + e_{s(i)}$ for all $i \in \{1, \ldots, d\}$.

For any $i \in \{1, \ldots, d\}$ we get $\langle u, v^i \rangle = 0$ and $\langle e_i^*, v^i \rangle = \langle e_i^*, -e_i + e_{s(i)} \rangle = -1$. Hence 3.5.6(3) implies that $e_1, \ldots, e_d$ is a $\mathbb{Z}$-basis of M. This immediately yields the uniqueness of P up to isomorphism of the lattice.

□

3.6 Reflexive simplices

A lattice polytope is determined by the relations among the vertices and the coordinates of some linearly independent family of vertices. Often the first information can be derived from the combinatorial data and the second information is encoded in some matrix normal forms. In the case of centrally symmetric simplicial reflexive polytopes this is described and illustrated in the last chapter.

The most general approach to describe the combinatorics is to use the notion of the *weight system* of a polytope, especially prominent in the general approach of Kreuzer and Skarke [KS97] (see p. 57). Here we examine the simplest case, that is, the case of a reflexive simplex in detail. It is well-known that there is a direct relation to elementary number theory in this setting.

In the first subsection we summarize in a new and unifying way the results of Conrads in [Con02] (based upon observations of Batyrev in [Bat94]) about lattice simplices and their weight systems. Here weighted projective spaces with Gorenstein singularities correspond uniquely to so called *reflexive weight systems*.

In the second subsection we give the correspondence of reflexive weight systems and *unit partitions*, that is, unit fractions that sum up to one. Using new number-theoretic results we get as the main result upper bounds on the total weight of weight systems of reflexive simplices.

3.6.1 Weight systems of simplices

This subsection summarizes results of Batyrev in [Bat94, 5.4,5.5] and Conrads in [Con02].

The following notion is essential:

Definition 3.6.1. A family of positive rational numbers $Q := (q_0, \ldots, q_d)$ is called a *weight system* of *length* d and *total weight* $|\,Q\,| := \sum_{i=0}^d q_i$. The *reduction* $Q_{\rm red}$ of Q is defined as the unique primitive lattice point in $\mathrm{pos}(Q)$, and the *factor* $\lambda_Q \in \mathbb{Q}_{>0}$ of Q is defined by $Q = \lambda_Q Q_{\rm red}$. Two weight systems are regarded to be *isomorphic*, if their entries are just permutated.

Let Q consist only of natural numbers. Then we have $\lambda_Q = \gcd(q_0, \ldots, q_d)$. Such a Q is called *reduced*, if $\lambda_Q = 1$, and *normalized*, if after removing an arbitrary weight we still have a reduced weight system.

There is now the following connection to lattice simplices:

Definition 3.6.2. Let $P = \mathrm{conv}(v_0, \ldots, v_d) \subseteq M_{\mathbb{R}}$ be a d-dimensional rational simplex with $0 \in \mathrm{int} P$.

Then we define $q_i := |\det(v_0, \ldots, \hat{v_i}, \ldots, v_n)| \in \mathbb{Q}_{>0}$ for $i = 0, \ldots, n$ and the family $Q_P := (q_0, \ldots, q_d)$. Q_P is called *the associated weight system* of P of factor $\lambda_P := \lambda_{Q_P}$.

Furthermore we need the following definition:

Definition 3.6.3. For a lattice polytope $P \subseteq M_{\mathbb{R}}$ we generally define the sublattice M_P of M as the lattice generated by the vertices of P, i.e., the *coarsest* lattice such that P is a lattice polytope.

A d-dimensional lattice simplex $P \subseteq M_{\mathbb{R}}$ is called *spanning*, if $0 \in \mathrm{int}P$ and $M = M_P$.

We have the following characterization:

Lemma 3.6.4. *Let P be a d-dimensional simplex with $0 \in \mathrm{int}P$. Then for $Q_P = (q_0, \ldots, q_n)$ we have*

$$\sum_{i=0}^{d} q_i v_i = 0.$$

Furthermore let P be a lattice simplex. Then $(Q_P)_{\mathrm{red}} = Q_P/\lambda_P = (q'_0, \ldots, q'_n)$ is the unique reduced weight system satisfying

$$\sum_{i=0}^{d} q'_i v_i = 0.$$

Moreover $\lambda_P = \det M_P = |M/M_P|$. Especially P is a spanning lattice simplex if and only if the associated weight system is reduced.

Proof. This is the content of [Con02, Lemma 2.4]. Also observe that the kernel of the surjective map $\mathbb{Z}^{d+1} \to M_P \cong \mathbb{Z}^d$, $x \mapsto \sum_{i=0}^{d} x_i v_i$ is free of rank one. □

When considering sublattices the following definition is very convenient:

Definition 3.6.5 (Hermite normal form). For $d, \lambda \in \mathbb{N}_{\geq 1}$ we denote by $\mathrm{Herm}(d, \lambda)$ the finite set of lower triangular matrices $H \in \mathrm{Mat}_d(\mathbb{N})$ with determinant λ satisfying $h_{i,j} < h_{i,j}$ for all $j = 1, \ldots, d-1$ and $i > j$.

Theorem 3.6.6. *For any $U \in \mathrm{Mat}_d(\mathbb{Z})$ with determinant $\lambda \neq 0$ there exists a matrix $L \in \mathrm{GL}_d(\mathbb{Z})$ and a Hermite normal form matrix $H \in \mathrm{Herm}(d, \lambda)$ such that $LU = H$.*

This is [Con02, Thm. 4.2].

There is now the following result due to Conrads [Con02, 3.6-3.8, 4.4-4.7], see also [Bat94, Thm. 5.4.5]. As usually we denote by $\mathbb{P}(Q)$ the *weighted projective space* associated to a weight system $Q \in \mathbb{Q}_{>0}^{d+1}$, e.g., $\mathbb{P}(1, \ldots, 1) = \mathbb{P}^d$.

Theorem 3.6.7 (Batyrev, Conrads).

1. *$P \mapsto Q_P$ yields a well-defined correspondence of isomorphism classes of spanning lattice simplices and reduced weight systems. We denote the reverse map by $Q \mapsto P_Q$.*

2. *Hereby P is a a Fano polytope if and only if Q_P is a normalized weight system.*

   Furthermore $P \mapsto X(\Sigma_P, M) \cong \mathbb{P}(Q_P)$ gives a well-defined correspondence of isomorphism classes of spanning Fano simplices and isomorphism classes of weighted projective spaces.

3. *Any lattice simplex containing the origin in its interior with weight system Q is the image of $P_{Q_{\text{red}}}$ under a Hermite normal form matrix of determinant λ_P. In particular there are only finitely many lattice simplices containing the origin in its interior and having the same associated weight system.*

   Such a lattice simplex $P \subseteq M_{\mathbb{R}}$ defines a toric variety that is the quotient of the weighted projective space $\mathbb{P}(Q)$ by the action of the finite group M/M_P of order λ_P.

How to explicitly construct the unique spanning simplex associated to a reduced weight system is described in [Con02]. Note however that for general lattice simplices the associated weight system does not have to determine the lattice simplex uniquely!

There is now an important invariant of a weight system, this generalizes [Con02, Def. 5.4] as will be seen in 3.6.26(1):

Definition 3.6.8. Let Q a weight system of length d. Then we define

$$m_Q := \frac{|Q|^{d-1}}{q_0 \cdots q_d} \in \mathbb{Q}_{>0}.$$

For the proof of next result we need a lemma that is proven by an explicit calculation:

Lemma 3.6.9.

$$\det \begin{pmatrix} n_1 - 1 & -1 & \cdots & & & \cdots & -1 \\ & \ddots & \ddots & \ddots & & & \\ -1 & \cdots & -1 & n_i - 1 & -1 & \cdots & -1 \\ & & & \ddots & \ddots & \ddots & \\ -1 & \cdots & & & \cdots & -1 & n_d - 1 \end{pmatrix}$$

$$= n_1 \cdots n_d - \sum_{j=1}^{d} n_1 \cdots \hat{n_j} \cdots n_d$$

Now we can add some interesting additional information that is missing in the paper [Con02]:

Proposition 3.6.10. *Let $P \subseteq M_{\mathbb{R}}$ a d-dimensional simplex with $0 \in \text{int}P$. Then*

$$Q_{P^*} = m_{Q_P} Q_P.$$

Proof. Let $Q := Q_P$, $t := |Q|$, $\mathcal{V}(P) = \{v_0, \ldots, v_d\}$ and for $i = 0, \ldots, d$ we denote by $F_i = \text{conv}(v_0, \ldots, \hat{v_i}, \ldots, v_d)$. Fix a basis of M and its dual basis of N. Now fix $i \in \{0, \ldots, d\}$, and let A_i be the matrix consisting of the coordinates of $v_1, \ldots, \hat{v_i}, \ldots, v_d$ as rows, and let B_i be the matrix consisting of the coordinates of $\eta_{F_1}, \ldots, \hat{\eta_{F_i}}, \ldots, \eta_{F_d}$ as columns. Since $v_i = -\sum_{j \neq i} \frac{q_j}{q_i} v_j$, we get $\langle \eta_{F_i}, v_i \rangle = \sum_{j \neq i} \frac{q_j}{q_i} = \frac{t - q_i}{q_i} = \frac{t}{q_i} - 1$.

Without restriction we set $i = 0$. Applying the previous lemma to $A_0 B_0$, we get $\det(A_0 B_0) = \frac{t}{q_1} \cdots \frac{t}{q_d} - \sum_{j=1}^d \frac{t}{q_1} \cdots \hat{\frac{t}{q_j}} \cdots \frac{t}{q_d} = \frac{t^{d-1}}{q_1 \cdots q_d}(t - \sum_{j=1}^d q_j) = \frac{t^{d-1}}{q_1 \cdots q_d} q_0$. Therefore $\det(B_0) = \frac{\det(A_0 B_0)}{\det(A_0)} = \frac{t^{d-1}}{q_1 \cdots q_d}$.

□

In particular we observe that there is an *involution*:

$$(\mathbb{R}\backslash\{0\})^{d+1} \to (\mathbb{R}\backslash\{0\})^{d+1} \; : \; x \mapsto \frac{(x_0 + \cdots + x_d)^{d-1}}{x_0 \cdots x_d} x.$$

So this motivates to define a duality also on the level of weight systems:

Definition 3.6.11. For a weight system Q we define the *dual* weight system $Q^* := m_Q Q$ for m_Q as above. This is a duality in the sense $(Q^*)^* = Q$.

If $Q = \lambda_Q Q_{\text{red}}$ is a weight system, then

$$Q^* = \frac{m_{Q_{\text{red}}}}{\lambda_Q} Q_{\text{red}}. \tag{3.5}$$

The proposition can now be reformulated as $Q_{P^*} = (Q_P)^*$.

This yields a condition for self-duality (use above theorem):

Corollary 3.6.12. *Let P be a lattice simplex with $0 \in \text{int}P$, $Q := Q_P$.*
If P is self-dual, i.e., $P \cong P^*$, then $Q^* = Q$, i.e., $\lambda_P = \sqrt{m_{Q_{\text{red}}}}$.
If P is a spanning lattice simplex, the reverse holds (with $\lambda_P = 1 = m_{Q_{\text{red}}}$).

In the situation of a reflexive simplex the following definition turns out to be convenient:

Definition 3.6.13. A weight system is called *reflexive*, if it is reduced and any weight is a divisor of the total weight. Especially it has to be normalized.

The notion of a reflexive weight system is motivated by the following result [Con02, Prop. 5.1] (partially [Bat94, Thm. 5.4.3]):

Theorem 3.6.14 (Batyrev, Conrads). *Under the correspondence of Theorem 3.6.7 we get correspondences of isomorphism classes of*

- *reflexive simplices whose vertices span the lattice*
- *reflexive weight systems*
- *weighted projective spaces with Gorenstein singularities*

Summing this discussion up we get a generalization of [Con02, 5.3, 5.5]:

Proposition 3.6.15. *Let $P \subseteq M_{\mathbb{R}}$ be a reflexive simplex with associated weight system $Q := Q_P = \lambda_P Q_{\mathrm{red}}$. Then Q_{red} is a reflexive weight system.*

Let $P_{\mathrm{red}} \subseteq M_{\mathbb{R}}$ be the reflexive simplex corresponding to Q_{red}. Then:

1. $(Q_{P^*})_{\mathrm{red}} = Q_{\mathrm{red}}$, $\lambda_{P^*} = m_{Q_P}\lambda_P = \frac{m_{Q_{\mathrm{red}}}}{\lambda_P}$, $m_{Q_{\mathrm{red}}} = \lambda_{(P_{\mathrm{red}})^*} \in \mathbb{N}_{>0}$.

2. $$P_{\mathrm{red}} \overset{\lambda_P}{\to} P \overset{\frac{m_{Q_{\mathrm{red}}}}{\lambda_P}}{\to} (P_{\mathrm{red}})^*.$$
   for injective lattice homomorphisms of the given integer determinant.

3. $\lambda_P \mid m_{Q_{\mathrm{red}}}$. *Furthermore*
   $$\lambda_P = 1 \iff P \cong P_{\mathrm{red}}, \quad \lambda_P = m_{Q_{\mathrm{red}}} \iff P \cong (P_{\mathrm{red}})^*.$$

Proof. From [Con02, 5.1] we get that Q_{red} is a reflexive weight system (an easy calculation).

1. Follows from 3.6.10 and 3.6.4. 2. First apply 3.6.7 to P. Then apply 3.6.7 to the lattice simplex P^*, use 1. and dualize. 3. from 2. □

From (3.3) and 3.6.7(3) we get the following corollary:

Corollary 3.6.16. *Gorenstein toric Fano varieties with class number one are uniquely associated to fans spanned by the faces of a reflexive simplex.*

Any such variety is the quotient of a weighted projective space $\mathbb{P}(Q)$ with Gorenstein singularities by the action of a finite abelian group of order less or equal to m_Q. Equality holds iff the variety is associated to the fan spanned by the faces of $(P_Q)^*$.

Note however, that if a spanning lattice simplex P has a (reduced) weight system Q such that Q^* is also a weight system with only natural numbers, i.e., $m_Q \in \mathbb{N}$, then P does not necessarily has to be reflexive, i.e., Q does not have to be reflexive, e.g., $Q = (1,1,1,9)$ with $m_Q = 16$ and $Q^* = (16,16,16,144)$.

Next we are concerned with the question how to construct lattice simplices from weight systems. This is actually a non-trivial task, however at first glance there seems to be a naive way of doing it:

Definition 3.6.17. Let $Q = (q_0, \ldots, q_d)$ be a reduced weight system. Define $(k_0, \ldots, k_d) := (|Q|/q_0, \ldots, |Q|/q_d) \in \mathbb{Q}_{>0}^{d+1}$. We assume $k_d = \max(k_0, \ldots, k_d)$.

We define $C_Q := \mathrm{conv}(\mathrm{conv}(0, k_0 e_0, \ldots, k_{d-1}e_{d-1}) \cap M) \subseteq M_{\mathbb{R}}$, where e_0, $\ldots$, e_{d-1} is an arbitrary but fixed $\mathbb{Z}$-basis of M. For $e := e_0 + \cdots + e_{d-1}$ we denote by $S_Q := C_Q - e$ an (up to lattice isomorphism well-defined) lattice polytope associated to the weight system Q.

Note that S_Q may not be a simplex anymore (e.g., $Q = (6,2,1)$)! However:

Proposition 3.6.18. *Let Q be a reflexive weight system with minimal weight q_d. Then we have in the notation of the previous definition that*

$$S_Q \cong \mathrm{conv}(k_0 e_0 - e, \ldots, k_{d-1}e_{d-1} - e, -e)$$

is a lattice simplex with associated weight system $q_d m_Q Q$.

Furthermore

$$S_Q \text{ is reflexive } \iff q_d = 1 \iff S_Q \cong (P_Q)^*.$$

In this case

$$P_Q \cong \mathrm{conv}(e_0, \ldots, e_{d-1}, -q_0 e_0 - \cdots - q_{d-1} e_{d-1}).$$

Proof. Obviously $S := S_Q = \mathrm{conv}(k_0 e_0 - e, \ldots, k_{d-1} e_{d-1} - e, -e)$ is a lattice simplex satisfying $1/k_0(k_0 e_0 - e) + \cdots + 1/k_{d-1}(k_{d-1} e_{d-1} - e) + 1/k_d(-e) = 0$. Using Lemma 3.6.4 we get $Q_S = \lambda_S Q$. Using Lemma 3.6.9 we get $\lambda_S q_0 = |\det(k_1 e_1 - e, \ldots, k_{d-1} e_{d-1} - e, -e)| = k_1 \cdots k_{d-1}$, hence $\lambda_S = \frac{k_0 \cdots k_{d-1}}{|Q|} = \frac{|Q|^{d-1}}{q_0 \cdots q_{d-1}} = q_d m_Q$.

For the second part we can assume by the previous proposition that $q_d = 1$, and we have to show that S is reflexive. Now we simply observe that $-q_0 e_0 - \cdots - q_{d-1} e_{d-1}, e_0, \ldots, e_{d-1}$ are the inner normals of S.

□

So we see that under this naive construction reflexive weight systems not necessarily yield reflexive simplices, e.g., look at $Q = (4, 3, 3, 2)$; only in the (more or less trivial) case where the vertices of a facet of the corresponding spanning reflexive simplex form a lattice basis. So we get:

Corollary 3.6.19. $Q \mapsto S_Q \cong (P_Q)^*$ *yields a correspondence of reflexive weight systems containing* 1 *as an entry and the duals of reflexive simplices containing a facet whose vertices form a lattice basis.*

3.6.2 The main result

We need the following well-known sequence (e.g., see [AS70]):

Definition 3.6.20. The recursive sequence [Slo04, A000058] of pairwise coprime natural numbers $y_0 := 2$, $y_n := 1 + y_0 \cdots y_{n-1}$ is called *Sylvester sequence*. It satisfies $y_n = y_{n-1}^2 - y_{n-1} + 1$ and starts as $y_0 = 2$, $y_1 = 3$, $y_2 = 7$, $y_3 = 43$, $y_4 = 1807$. We also define $t_n := y_n - 1 = y_0 \cdots y_{n-1}$.

Using these numbers we define two special sets of reflexive weight systems:

Definition 3.6.21.

- The $d+1$-tuple of natural numbers

  $$Q_d := (\frac{t_d}{y_0}, \ldots, \frac{t_d}{y_{d-1}}, 1)$$

  is called *Sylvester weight system* of length d.

- The $d+1$-tuple of natural numbers

  $$Q'_d := (\frac{2t_{d-1}}{y_0}, \ldots, \frac{2t_{d-1}}{y_{d-2}}, 1, 1)$$

  is called *enlarged Sylvester weight system* of length d.

The goal of this section is to prove the following theorem:

Theorem 3.6.22.

1. *If Q is a reflexive weight system of length d, then*
$$|\,Q\,| \leq t_d, \text{ with equality iff } Q \text{ is isomorphic to } Q_d.$$
2. *If P is a reflexive simplex for $d \geq 3$, then*
$$|\,Q_P\,| \leq 2t_{d-1}^2,$$
with equality iff $P \cong S_{Q_{\mathrm{red}}}$ for Q_{red} isomorphic to Q'_d or $(3,1,1,1)$.
3. *If P is a reflexive simplex, then*
$$|\,Q_P\,||\,Q_{P^*}\,| \leq t_d^2,$$
with equality iff $P \cong S_{Q_d}(\cong S^*_{Q_d})$.

Since by Theorem 3.6.7(3) (see also 3.6.15 and 3.6.16) there are only finitely many reflexive polytopes having as the reduction of their associated weight system the same reflexive weight system we get from the first point a direct proof of a fact that is known to hold in general for reflexive polytopes:

Corollary 3.6.23. *There is only a finite number of isomorphism classes of d-dimensional reflexive simplices.*

In particular there is a *classification algorithm* as described in [Con02]: First determine all reflexive weight systems Q of length d. Then construct the associated spanning reflexive simplex P_Q. Eventually we look for reflexive simplices in the images of P_Q under any Hermite normal form matrix of determinant λ for λ a divisor of m_Q, where $\lambda \leq m_Q/2$ suffices by duality.

Now the proof of the theorem is essentially pure number theory.

For this we need the notion of a unit partition that is closely related to that of an Egyptian fraction, e.g., see [Epp04]:

Definition 3.6.24. A family of positive natural numbers $(k_0, \ldots, k_d)$ is called a *unit partition of total weight* $\mathrm{lcm}(k_0, \ldots, k_d)$, if $\sum_{i=0}^d 1/k_i = 1$.

The crucial observation is the following result (essentially due to Batyrev in [Bat94, 5.4]):

Proposition 3.6.25. *There is a a bijection between reflexive weight systems and unit partitions, given by mapping $Q \in \mathbb{Q}_{>0}^{d+1}$ to $(\frac{|Q|}{q_0}, \ldots, \frac{|Q|}{q_d})$, respectively mapping $(k_0, \ldots, k_d)$ to $(\frac{\mathrm{lcm}(k_0,\ldots,k_d)}{k_0}, \ldots, \frac{\mathrm{lcm}(k_0,\ldots,k_d)}{k_d})$.*

Hereby the length and the total weight of the reflexive weight system and the corresponding unit partition are the same.

From Propositions 3.6.15 and 3.6.12 we get the following corollary:

Corollary 3.6.26. *Let $P \subseteq M_{\mathbb{R}}$ be a reflexive simplex with $Q := Q_P$. Let $P_{\mathrm{red}} \subseteq M_{\mathbb{R}}$ be the reflexive simplex corresponding to Q_{red} and $(k_0, \ldots, k_d)$ the associated unit partition. Then*

1.
$$m_{Q_{\mathrm{red}}} = \frac{k_0 \cdots k_d}{\operatorname{lcm}(k_0, \ldots, k_d)^2} = \lambda_{(P_{\mathrm{red}})^*} \in \mathbb{N}_{>0}.$$

2. If P is self-dual (i.e., $P^* \cong P$), then $k_0 \cdots k_d$ has to be a square. P_{red} is self-dual if and only if $k_0 \cdots k_d = \operatorname{lcm}(k_0, \ldots, k_d)^2$.

In particular the first point yields the purely number-theoretic corollary that the square of the least common divisor of a unit partition divides the product. The first part of this corollary was already proven by Batyrev in [Bat94, Cor. 5.5.4] when investigating fundamental groups.

Now we define the unit partitions corresponding to above weights systems:

Definition 3.6.27.

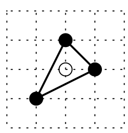
- $(y_0, \ldots, y_{d-1}, t_d)$ is called *Sylvester partition* of length d (corresponding to Q_d). It is a unit partition of total weight $t_d = t_{d-1} y_{d-1}$. Since $m_{Q_d} = 1$, it corresponds to a *self-dual* reflexive simplex $S_Q \cong P_Q$.
- $(y_0, \ldots, y_{d-2}, 2t_{d-1}, 2t_{d-1})$ is called *enlarged Sylvester partition* of length d (corresponding to Q'_d). Note that $m_{Q'_d} = t_{d-1}$.

These two unit partitions were defined upon observations by Haase and Melnikov in [HM04] (in turn basing on [LZ91], [Hen83], [PWZ82]).

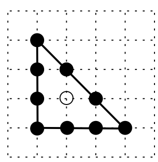
As an illustration of the previous notions we classify the five two-dimensional reflexive simplices:

Example 3.6.28. Consider the case $d = 2$. We have three unit partitions:

1. $(3, 3, 3)$ corresponding to $Q := (1, 1, 1)$. This yields $m_Q = 3$. So we have $P_Q := \operatorname{conv}((1, 0), (0, 1), (-1, -1))$ (corresponding to $\mathbb{P}^2$) and $S_Q = (P_Q)^* \cong \operatorname{conv}((2, -1), (-1, 2), (-1, -1))$ as the only reflexive simplices P with $(Q_P)_{\mathrm{red}} = Q$ (due to 3.6.15(3)):



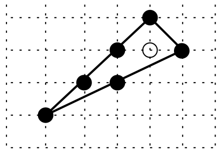
3



(index 3)

9=3*

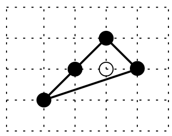
2. The Sylvester partition $(2, 3, 6)$ corresponding to $Q := Q_2 = (3, 2, 1)$. This yields $m_Q = 1$. So we get the self-dual reflexive simplex $P_Q = S_Q \cong \operatorname{conv}((1, 0), (0, 1), (-3, -2))$ (corresponding to $\mathbb{P}(Q)$):



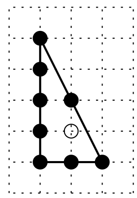
(index 1)

6d=(6d)*

3. The enlarged Sylvester partition $(2, 4, 4)$ corresponding to $Q := Q'_2 = (2, 1, 1)$. This yields $m_Q = 2$. So we have $P_Q := \operatorname{conv}((1, 0), (0, 1), (-2, -1))$ (corresponding to $\mathbb{P}(Q)$) and $S_Q = (P_Q)^* \cong \operatorname{conv}((1, -1), (-1, 3), (-1, -1))$ as the only reflexive simplices P with $(Q_P)_{\text{red}} = Q$:



4c



(index 2)

8c=(4c)*

Using Proposition 3.6.15, Corollary 3.6.26 and Equation (3.5) on p. 71 it is straightforward to deduce Theorem 3.6.22 from the second and third statements of the following number-theoretic proposition:

Proposition 3.6.29. *Let $(k_0, \ldots, k_d)$ be a unit partition.*

1.

$$d + 1 \leq \max(k_0, \ldots, k_d) \leq t_d,$$

with equality in the second case only for the Sylvester partition.

Furthermore: If $k_0 \leq \ldots \leq k_d$, then

$$k_j \leq (d - j + 1)t_j$$

for $j \in \{0, \ldots, d\}$.

2.

$$(d + 1)^2 \leq \operatorname{lcm}(k_0, \ldots, k_d)^2 \leq k_0 \cdots k_d \leq t_d^2,$$

with equality in the last case only for the Sylvester partition.

3. Let $d \geq 3$ and $k_0 \leq \ldots \leq k_d$. Then

$$\frac{k_0 \cdots k_d}{\operatorname{lcm}(k_0, \ldots, k_d)} \leq k_0 \cdots k_{d-1} \leq 2t_{d-1}^2,$$

where the first equality holds iff $k_d = \operatorname{lcm}(k_0, \ldots, k_d)$, and the second equality iff $(k_0, \ldots, k_d)$ is the enlarged Sylvester partition or $(2, 6, 6, 6)$.

For the proof we need several results about sums of unit fractions.

The most important one was given by Curtiss [Cur22, Thm. I], however his proof is rather complicated.

Theorem 3.6.30 (Curtiss). *Let $x_1, \ldots, x_n$ be positive integers such that $s := \sum_{i=1}^n \frac{1}{x_i} < 1$. Then*

$$s \leq \sum_{i=0}^{n-1} \frac{1}{y_i} = 1 - \frac{1}{t_n},$$

with equality if and only if $\{x_1, \ldots, x_n\} = \{y_0, \ldots, y_{n-1}\}$.

In the case of a unit partition there is also a very insightful and easy proof that is straightforward to deduce from the following nice result [IK95, Lemma 1]. Here we have included statements that are implicit in their proof.

Lemma 3.6.31 (Izhboldin, Kurliandchik). *Let $x_1, \ldots, x_n$ be real numbers satisfying $x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$, $x_1 + \cdots + x_n = 1$ and $x_1 \cdots x_k \leq x_{k+1} + \cdots + x_n$ for $k = 1, \ldots, n-1$. Then*

$$x_n \geq \frac{1}{t_{n-1}}, \qquad x_1 \cdots x_n \geq \frac{1}{t_{n-1}^2},$$

where equality in the first case holds iff equality in the second case holds iff $x_i = \frac{1}{y_{i-1}}$ for $i = 1, \ldots, n-1$.

We can now make an addition to this result. For this we need the following inequality:

Lemma 3.6.32. *Let $n \geq 4$, $1 \leq k \leq n-1$. Then*

$$(k+1)^k t_{n-k-1}^{k+1} \leq 2t_{n-2}^2,$$

with equality iff $k = 1$ or $(n,k) = (4,2)$.

Proof. Proof by induction on n. By explicitly checking $n = 4, 5$, we can assume $n \geq 6$.

For $k = 1$ the statement is trivial, so let $k \geq 2$. By induction hypothesis for $(n-1, k-1)$ we have $k^{k-1} t_{n-k-1}^k \leq 2t_{n-3}^2$, this yields $(k+1)^k t_{n-k-1}^{k+1} \leq 2t_{n-3}^2 t_{n-k-1} (\frac{k+1}{k})^k k$. Since $(\frac{k+1}{k})^k < e$, it is enough to show $t_{n-3}^2 t_{n-k-1} ek \leq t_{n-2}^2$, or equivalently, $t_{n-k-1} ek \leq y_{n-3}^2$.

For $n \geq 6$ it is easy to see that $e(n-1) \leq y_{n-3}$ (e.g., by 3.6.33 below). Hence $t_{n-k-1} ek \leq t_{n-3} e(n-1) \leq t_{n-3} y_{n-3} < y_{n-3}^2$.

□

The following lemma gives the asymptotical behavior of the Sylvester sequence (e.g., [GKP89, (4.17)]):

Lemma 3.6.33. *There is a constant $c \approx 1.2640847353\cdots$ (called* Vardi *constant, see [Slo04, A076393]) such that for any $n \in \mathbb{N}$*

$$y_n = \left\lfloor c^{2^{n+1}} + \frac{1}{2} \right\rfloor.$$

Using 3.6.32 and the ideas of the proof of 3.6.31 we can now show:

Lemma 3.6.34. *Let $n \geq 4$, $x := (x_1, \ldots, x_n)$ as in Lemma 3.6.31. Then*

$$x_1 \cdots x_{n-1} \geq \frac{1}{2t_{n-2}^2},$$

with equality iff $(1/x_1, \ldots, 1/x_n)$ equals $(2,6,6,6)$ or the enlarged Sylvester partition of length $n-1$.

Proof.
STEP I:
Let A denote the set of n-tupels x satisfying the conditions of the lemma. It is easy to see that we have for $x \in A$ necessarily $1 > x_1$ and $x_n > 0$.

Since A is compact, there exists some $x \in A$ with $x_1 \cdots x_{n-1}$ minimal. Because of $(\frac{1}{y_0}, \ldots, \frac{1}{y_{n-3}}, \frac{1}{2t_{n-2}}, \frac{1}{2t_{n-2}}) \in A$ we have $x_1 \cdots x_{n-1} \leq \frac{1}{2t_{n-2}^2}$.

Let us assume that $x_{n-1} > x_n$.
Claim: In this case we have

$$x_1 > x_2 > \cdots > x_{n-1} > x_n,$$
$$x_1 \cdots x_k = x_{k+1} + \cdots + x_n \text{ for } k = 1, \ldots, n-2.$$

Proof of claim:
By convention we let $x_0 := 1$ and $x_{n+1} := 0$. We proceed by induction on $l = 1, \ldots, n-1$, and assume by induction hypothesis that $x_1 > x_2 > \cdots > x_l \geq x_{l+1}$ and $x_1 \cdots x_k = x_{k+1} + \cdots + x_n$ for $k = 1, \ldots, l-2$.
We distinguish three cases:

1. $x_l > x_{l+1}$ and $x_1 \cdots x_{l-1} = x_l + \cdots + x_n$.

   In this case we can proceed.

2. $x_l > x_{l+1}$ and $x_1 \cdots x_{l-1} < x_l + \cdots + x_n$.

   This implies $l \geq 2$. Then we can find some $\delta > 0$ s.t. $x' \in A$ with $x'_{l-1} := x_{l-1} + \delta$, $x'_l := x_l - \delta$ and $x'_j := x_j$ for $j \in \{1, \ldots, n\} \backslash \{l-1, l\}$. Hence $x'_1 \cdots x'_{n-1} = \frac{x_1 \cdots x_{n-1}}{x_{l-1} x_l}(x_{l-1} x_l + \delta(x_l - x_{l-1}) - \delta^2) < x_1 \cdots x_{n-1}$, a contradiction.

3. $x_l = x_{l+1} = \cdots = x_i > x_{i+1}$ for $l + 1 \leq i \leq n-1$.

   This implies $l \leq n-2$. Again we find some $\delta > 0$ s.t. $x' \in A$ with $x'_l := x_l + \delta$, $x'_i := x_i - \delta$ and $x'_j := x_j$ for $j \in \{1, \ldots, n\} \backslash \{l, i\}$.

   This can be done, since otherwise there has to exist $l \leq j < i$ such that $x_1 \cdots x_j = x_{j+1} + x_{j+2} + \cdots + x_n$. Since $x_j = x_{j+1}$ we have $0 = (1 - x_1 \cdots x_{j-1}) x_j + x_{j+2} + \cdots + x_n$, a contradiction.

   Since again $x'_1 \cdots x'_{n-1} < x_1 \cdots x_{n-1}$, we get a contradiction.

End of proof of claim.

So we have $x_1 = x_2 + \cdots + x_n = 1 - x_1$, hence $x_1 = \frac{1}{2} = \frac{1}{y_0}$. By induction on $k = 2, \ldots, n-2$ we get $x_1 \cdots x_k = x_{k+1} + \cdots + x_n = 1 - x_1 - \cdots - x_k$, hence $\frac{1}{t_{k-1}} x_k = 1 - \frac{1}{y_0} - \cdots - \frac{1}{y_{k-2}} - x_k = \frac{1}{t_{k-1}} - x_k$, so $x_k = 1 - t_{k-1} x_k$. This implies $x_k = \frac{1}{1+t_{k-1}} = \frac{1}{y_{k-1}}$. So this yields $x_1 = \frac{1}{y_0}$, $\ldots$, $x_{n-2} = \frac{1}{y_{n-3}}$.
Furthermore $x_{n-1} + x_n = 1 - x_1 - \cdots - x_{n-2} = \frac{1}{t_{n-2}}$. Since $x_{n-1} > x_n$, we get $x_{n-1} > \frac{1}{2t_{n-2}}$. Therefore we have proven $x_1 \cdots x_{n-1} > \frac{1}{2t_{n-2}^2}$, a contradiction.

So this step yields $x_{n-1} = x_n$.

STEP II:

Let A' denote the set of $(n-1)$-tupels $w \in \mathbb{R}^{n-1}$ satisfying the following conditions: $w_1 \geq w_2 \geq \cdots \geq w_{n-2} \geq \frac{w_{n-1}}{2} \geq 0$, $w_1 + \cdots + w_{n-1} = 1$ and $w_1 \cdots w_k \leq w_{k+1} + \cdots + w_{n-1}$ for $k = 1, \ldots, n-2$.

Now let $w \in A'$ be fixed with $w_1 \cdots w_{n-1}$ minimal.

Since $(\frac{1}{y_0}, \ldots, \frac{1}{y_{n-3}}, \frac{1}{t_{n-2}}) \in A'$, we have $w_1 \cdots w_{n-1} \leq \frac{1}{t_{n-2}^2}$.

Let $z := w_s = \cdots = w_{n-2} = \frac{w_{n-1}}{2}$ for $1 \leq s \leq n-1$ minimal.

We define $k := n - s$. There are three cases to consider:

1. $s = 1$, i.e., $k = n-1$.

   Then $nz = (n-2)z + 2z = w_1 + \cdots + w_{n-2} + w_{n-1} = 1$, so $z = 1/n$. This implies $w_1 \cdots w_{n-1} = \frac{2}{n^{n-1}}$. However $\frac{1}{t_{n-2}^2} < \frac{2}{n^{n-1}}$ for $n \geq 4$ by 3.6.32, a contradiction.

2. $s = 2$, i.e., $k = n-2$.

   Then $1 = w_1 + w_2 + \cdots + w_{n-2} + w_{n-1} = w_1 + (n-3)z + 2z = w_1 + (n-1)z$, hence $w_1 = 1 - (n-1)z$. Since $w_1 > z$, we get $z < \frac{1}{n}$. On the other hand $w_1 \leq w_2 + \cdots + w_{n-1} = (n-1)z$, hence $z \geq \frac{1}{2(n-1)}$.

   We have $w_1 \cdots w_{n-1} = (1-(n-1)z)2z^{n-2}$. This function attains its minimum on the interval $[\frac{1}{2(n-1)}, \frac{1}{n}[$ only for $z = \frac{1}{2(n-1)}$. (The proof of this statement is left to the reader.)

   Therefore $\frac{1}{t_{n-2}^2} \geq w_1 \cdots w_{n-1} \geq \frac{1}{(2(n-1))^{n-2}}$. However by 3.6.32 we have $\frac{1}{(2(n-1))^{n-2}} \geq \frac{1}{t_{n-2}^2}$, with equality only for $n = 4$. Hence we get $n = 4$, $z = \frac{1}{2(n-1)}$, and $w = (\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$.

3. $s \geq 3$, i.e., $k \leq n-3$.

   Now a similar reasoning as in the proof of above claim yields

   $$w_1 = \frac{1}{y_0}, \ \ldots, \ w_{s-2} = \frac{1}{y_{s-3}}.$$

   Then $1 = w_1 + \cdots + w_{s-2} + w_{s-1} + w_s + \cdots + w_{n-1} = 1 - \frac{1}{t_{s-2}} + w_{s-1} + (n-s+1)z$, hence $w_{s-1} = \frac{1}{t_{s-2}} - (k+1)z$.

   Since $w_{s-1} > z$, we get $z < \frac{1}{(k+2)t_{s-2}}$.

   Since $w_1 \cdots w_{s-2} w_{s-1} \leq w_s + \cdots + w_{n-1}$, we get $\frac{1}{t_{s-2}}(\frac{1}{t_{s-2}} - (k+1)z) \leq (k+1)z$, hence $z \geq \frac{1}{(k+1)t_{s-2}^2(1+\frac{1}{t_{s-2}})} = \frac{1}{(k+1)t_{s-1}}$.

   Now we have $w_1 \cdots w_{n-1} = \frac{1}{t_{s-2}}(\frac{1}{t_{s-2}} - (k+1)z)2z^{n-s} =: f(z)$.

   Since the function $f(z)$ is for $z > 0$ strictly monotone increasing up to some value and then strictly monotone decreasing, we see that
   $\frac{1}{t_{n-2}^2} \geq w_1 \cdots w_{n-1} \geq \min(f(\frac{1}{(k+1)t_{s-1}}), f(\frac{1}{(k+2)t_{s-2}})) =$
   $\min(\frac{2}{(k+1)^k t_{s-1}^{k+1}}, \frac{2}{(k+2)^{k+1} t_{s-2}^{k+2}})$.

There are two cases:

(a) $s = n-1$, i.e., $k = 1$.
Here $w_1 \cdots w_{n-1} \geq \min(\frac{1}{t_{n-2}^2}, \frac{2}{9t_{n-3}^3})$. Since $\frac{1}{t_{n-2}^2} \leq \frac{2}{9t_{n-3}^3}$ (by the recursive definition), we get $w_1 \cdots w_{n-1} = \frac{1}{t_{n-2}^2}$, and $z = \frac{1}{2t_{n-2}}$. Hence $w = (\frac{1}{y_0}, \ldots, \frac{1}{y_{n-3}}, \frac{1}{t_{n-2}})$.

(b) $s \leq n-2$, i.e., $k \geq 2$.
Here $\frac{2}{(k+1)^k t_{s-1}^{k+1}} \leq \frac{2}{(k+2)^{k+1} t_{s-2}^{k+2}}$ if and only if $\frac{y_{s-2}^{k+1}}{t_{s-2}} \geq \frac{(k+2)^{k+1}}{(k+1)^k}$. This is true for $k = 2$. For $k \geq 3$ we have $(k+2)(\frac{k+2}{k+1})^k < (k+2)e \leq 3^k \leq y_{s-2}^k < \frac{y_{s-2}^{k+1}}{t_{s-2}}$. Hence $\frac{1}{t_{n-2}^2} \geq w_1 \cdots w_{n-1} \geq \frac{2}{(k+1)^k t_{s-1}^{k+1}}$, a contradiction to 3.6.32.

STEP III:
Now we can finish the proof: Since $w' := (x_1, \ldots, x_{n-2}, 2x_{n-1}) \in A'$, we get

$$x_1 \cdots x_{n-1} = w'_1 \cdots w'_{n-2} \frac{w'_{n-1}}{2} \geq \frac{1}{2t_{n-2}^2},$$

where equality implies that $(x_1, \ldots, x_n) = (\frac{1}{y_0}, \ldots, \frac{1}{y_{n-3}}, \frac{1}{2t_{n-2}}, \frac{1}{2t_{n-2}})$ or $n = 4$ and $(x_1, x_2, x_3, x_4) = (1/2, 1/6, 1/6, 1/6)$.

□

It is now straightforward to prove the main result of this section:

Proof of Proposition 3.6.29. We can assume $k_0 \leq \ldots \leq k_d$. First we show that for $x_i := 1/k_i$ the conditions in 3.6.31 are satisfied: So let $i \in \{0, \ldots, d-1\}$. Then $0 < x_{i+1} + \cdots + x_d = 1 - x_0 - \cdots - x_i = \frac{k_0 \cdots k_i - \sum_{j=0}^{i} k_0 \cdots \hat{k_j} \cdots k_d}{k_0 \cdots k_i} \geq \frac{1}{k_0 \cdots k_i} = x_0 \cdots x_i$.

1. The first lower bound can be immediately derived from the partition property. The upper bounds are proven by showing

$$\frac{1}{k_j} \geq \frac{1}{d-j+1} \frac{1}{t_j}$$

for $j \in \{0, \ldots, d\}$.

For $j = 0$ the result is trivial since $t_0 = 1$, $1/k_0 + \cdots 1/k_d = 1$ and $k_0 = \min(k_0, \ldots, k_d)$. For $j \in \{1, \ldots, d\}$ Lemma 3.6.30 implies $1 - 1/k_j - \cdots - 1/k_d = \sum_{i=0}^{j-1} 1/k_i \leq 1 - 1/t_j$, so $1/k_j + \cdots + 1/k_d \geq 1/t_j$, hence the ordering assumption yields the desired statement.

2. The lower bound follows from 1., the middle bound from 3.6.26(1). The upper bound follows from 3.6.31.

3. This follows immediately from 3.6.34.

□

3.7 Lattice points in reflexive polytopes

It is interesting to try to find sharp upper bounds on the number of lattice points in a d-dimensional reflexive polytope. On the one hand this is motivated by convex geometry (see for instance [LZ91]) and geometry of numbers (see the results in the previous section), on the other hand there is also a direct algebraic-geometric interpretation: Let $P \subseteq M_{\mathbb{R}}$ be reflexive, then by (1.7)

$$|\, P \cap M \,| = h^0(X_P, -K_{X_P}). \tag{3.6}$$

Another important invariant of a lattice polytope is its volume; moreover bounding the volume of a lattice polytope also bounds the number of lattice points (e.g., [LZ91]):

Lemma 3.7.1 (Blichfeldt). *Let $P \subseteq M_{\mathbb{R}}$ be a d-dimensional lattice polytope. Then*

$$|\, P \cap M \,| \leq d + d! \operatorname{vol}(P).$$

If $P \subseteq M_{\mathbb{R}}$ is reflexive, then the normalized volume of P is just the (anti-canonical) degree of X_P (see Prop. 2.3.15):

$$\deg(X_P) = (-K_{X_P})^d = d! \operatorname{vol}(P). \tag{3.7}$$

In algebraic geometry one is interested in finding a sharp upper bound on the degree of a Gorenstein Fano variety with canonical singularities (see [Pro04]). The results achieved here can be used as a conjecture for this more general situation.

In the first subsection we show how the duality property of reflexive polytopes effects the Ehrhart polynomial. In the second we prove upper bounds on the volume and hence on the number of lattice points of a reflexive simplex and determine the maximal number of lattice points on an edge. In the last subsection we describe how to count lattice points modulo a natural number. This is useful in the case of centrally symmetric or terminal reflexive polytopes.

3.7.1 The Ehrhart polynomial

In this subsection the topic of lattice points in polytopes is continued from section 1.6. The well-known fact is presented, how the symmetry property of reflexive polytopes can be found again in the Ehrhart polynomial (see Thm. 1.5.3). In three dimensions a simple and also well-known Pick type formula for the volume of a reflexive polytope is derived. This result is certainly folklore, however the author could not find an explicit reference.

There is the following well-known result (e.g., see [Has00], [HM04]) with most parts originally due to Hibi [Hib92]:

Proposition 3.7.2. *Let $P \subseteq M_{\mathbb{R}}$ be d-dimensional lattice polytope with $0 \in \operatorname{int} P$. The following conditions are equivalent:*

1. *P is reflexive*
2. *$e_P(k) = |\operatorname{relint}((k+1)P)|$ for all $k \in \mathbb{N}$*

3. $e_P(k) = (-1)^d e_P(-k-1)$ *for all* $k \in \mathbb{N}$

4. $\operatorname{rvol}(P) = \frac{1}{d} \sum_{F \in \mathcal{F}(P)} \operatorname{rvol}(F)$

5. $\operatorname{coeff}_{d-1}(e_P) = \frac{d}{2} \operatorname{coeff}_d(e_P)$

For the proof first observe that any full-dimensional cone in $M_{\mathbb{R}}$ contains a lattice basis. From this we easily deduce:

Lemma 3.7.3. *P as in the proposition, $F \in \mathcal{F}(P)$, $\nu_F \in N_{\mathbb{Q}}$ with $\langle \nu_F, F \rangle = 1$. The following conditions are equivalent:*

1. $\nu_F \in N$

2. $\langle \nu_F, m \rangle \in \mathbb{Z}$ *for all* $m \in \operatorname{pos}(F) \cap M$

3. $\operatorname{pos}(F) \cap M \subseteq \cup_{k \in \mathbb{N}} kF$

Using the lemma and the reciprocity law in Theorem 1.5.3 the first three equivalences are straightforward to prove. The remaining equivalences are easy to see by 3.1.4 and the results in section 1.5.

In particular we get by the third equivalence in the proposition:

Corollary 3.7.4. *The Ehrhart polynomial of a d-dimensional reflexive polytope is determined by its values for $k = 1, \ldots, \lfloor \frac{d}{2} \rfloor$.*

For $d = 2, 3$ the corollary yields that e_P is determined by $e_P(1) = |\, P \cap M \,|$:

Corollary 3.7.5. *Let $P \subseteq M_{\mathbb{R}}$ be a d-dimensional reflexive polytope.*
If $d = 2$, then

$$e_P(x) = (\frac{|\, P \cap M \,|}{2} - \frac{1}{2})x^2 + (\frac{|\, P \cap M \,|}{2} - \frac{1}{2})x + 1,$$

In particular $\operatorname{vol}(P) = (|\, P \cap M \,| - 1)/2$.
If $d = 3$, then

$$e_P(x) = (\frac{|\, P \cap M \,|}{3} - 1)x^3 + (\frac{|\, P \cap M \,|}{2} - \frac{3}{2})x^2 + (\frac{|\, P \cap M \,|}{6} + \frac{3}{2})x + 1,$$

In particular $\operatorname{vol}(P) = |\, P \cap M \,|/3 - 1$.

Proof. Just check the equality of both sides using Proposition 3.7.2. □

In higher dimensions there is no such direct relation between the volume and the number of lattice points of a reflexive polytope (e.g., see page 88).

An alternative proof of this formula in the three-dimensional case could have been given by using the two-dimensional formula of Pick for lattice polygons and the fact that the facets have lattice distance one from the origin. A purely algebraic-geometric proof would be possible by proving the formula first for a smooth Fano polytope using Riemann-Roch and the double-weight-formula of Oda [Oda88, Cor. 1.32] and then using the existence of a crepant resolution in 3.1.6.

Applying the so called "Arithmetic Euler-Poincaré formula" as given in [Kan98, Theorem 6] by Kantor we get (recall Definition 3.1.7):

Corollary 3.7.6. *In any triangulation of the boundary of a three-dimensional reflexive polytope $P \subseteq M_{\mathbb{R}}$ of Volume V such that any two-dimensional simplex is an empty lattice polygon the numbers f_i $(i = 0, 1, 2)$ of i-dimensional simplices satisfy:*

$$f_0 = |\, P \cap M \,| - 1 = 3V + 2, \quad f_1 = 9V, \quad f_2 = 6V.$$

3.7.2 Bounds on the volume and lattice points

Throughout the section let $d \geq 2$.

In this subsection we will prove three results that will be first formulated in algebraic-geometric language and then in the convex-geometric setting. They give upper bounds on

1. the anticanonical degree of a Gorenstein toric Fano variety X of class number one (Thm. A), respectively the volume of a reflexive simplex P (Thm. A');
2. the anticanonical degree of a torus-invariant curve C on X (Thm. B), respectively the number of lattice points on an edge of a reflexive simplex P (Thm. B');
3. the product of the degrees of the dual pair X and X^* (Thm. C), respectively the product of the volumes of P and P^* (Thm. C').

Now we give the first algebraic-geometric result, here as in the whole subsection, we use the notation from the previous section:

Theorem 3.7.7 (A). *Let X be a d-dimensional Gorenstein toric Fano variety with class number one.*

1. *If $d = 2$, then*
$$(-K_X)^2 \leq 9,$$
with equality iff $X \cong \mathbb{P}^2$.
2. *If $d = 3$, then*
$$(-K_X)^3 \leq 72,$$
with equality iff $X \cong \mathbb{P}(3, 1, 1, 1)$ or $X \cong \mathbb{P}(Q'_3) = \mathbb{P}(6, 4, 1, 1)$.
3. *If $d \geq 4$, then*
$$(-K_X)^d \leq 2t_{d-1}^2,$$
with equality iff $X \cong \mathbb{P}(Q'_d)$.

This motivates the following conjecture:

Conjecture 3.7.8. The results of theorem A hold for Gorenstein Fano varieties with canonical singularities.

In the case of threefolds the bound in theorem A is the so-called Fano-Iskovskikh conjecture. It has very recently been proven by Prokhorov [Pro04].

Due to a sharp version of the Cone Theorem (p. 23) for projective toric varieties as given in [Fuj03] we know that there is always a torus-invariant integral curve on X such that its anticanonical degree is at most $d+1$. The next theorem shows that there is is a general upper bound in our setting:

Theorem 3.7.9 (B). *Let X be a d-dimensional Gorenstein toric Fano variety with class number one. Let C be a torus-invariant integral curve on X. Then*

$$-K_X.C \leq 2t_{d-1},$$

where equality implies $X \cong \mathbb{P}(Q'_d)$.

Let X be a d-dimensional Gorenstein toric Fano variety with class number one. Then $X = X_P$ for some reflexive simplex $P \subseteq M_{\mathbb{R}}$ (see Cor. 3.6.16).

Definition 3.7.10. We define X^* as the Gorenstein toric Fano variety with class number one that is associated to the fan spanned by the faces of P.

There is also the following result:

Theorem 3.7.11 (C). *Let X be a d-dimensional Gorenstein toric Fano variety with class number one. Then*

$$(-K_X)^d \, (-K_{X^*})^d \leq t_d^2,$$

with equality iff $X \cong \mathbb{P}(Q_d)$. In this case $X \cong X^*$.

Furthermore let X be a weighted projective space with Gorenstein singularities. Then

$$(-K_{X^*})^d \leq t_d,$$

with equality iff $X \cong \mathbb{P}(Q_d)$.

One could conjecture that theorem B and the first part of theorem C might also be true for Gorenstein toric Fano varieties with arbitrary class number.

Now the next observation shows how to apply the results of the previous section:

Lemma 3.7.12. *Let $P \subseteq M_{\mathbb{R}}$ be a lattice simplex with associated weight system $Q_P = (q_0, \ldots, q_d)$. Then*

$$\mathrm{vol}(P) = \sum_{i=0}^{d} \frac{q_d}{d!} = \frac{|Q_P|}{d!}.$$

From Prop. 3.6.15(2) we see that if Q is the associated reflexive weight system of P (i.e., Q is the reduction of Q_P), then $\mathrm{vol}(P) \leq \mathrm{vol}((P_Q)^*)$. Moreover if $\min(Q) = 1$, as for Q_d or Q'_d, then recall from Definition 3.6.17 and Prop. 3.6.18 that $S_Q \cong (P_Q)^*$.

By (3.7) on p. 81 theorem A can be derived from the following convex geometric result:

Theorem 3.7.13 (A').

1. $S_{(1,1,1)}$ *is the unique two-dimensional reflexive simplex with the largest volume* $\frac{9}{2}$*, respectively the largest number of lattice points* 10.
2. $S_{(3,1,1,1)}$ *and* $S_{Q'_3}$ *are the only three-dimensional reflexive simplices with the largest volume* 12*, respectively the largest number of lattice points* 39.
3. *Let* $d \geq 4$*. Then* $S_{Q'_d}$ *is the unique* d*-dimensional reflexive simplex with the largest volume of* $2t_{d-1}^2/d!$*. Any* d*-dimensional reflexive simplex contains at most* $d + 2t_{d-1}^2$ *lattice points.*

Proof of theorem A'. This is straightforward from Example 3.6.28, Corollary 3.7.5, Theorem 3.6.22(2), Lemma 3.7.1 and the previous lemma. □

These bounds vastly improve on more general bounds on lattice simplices containing only one lattice point in the interior as given (see [Pik00]). Here we cite the following theorem from [LZ91]:

Theorem 3.7.14 (Hensley, Lagarias, Ziegler). *Let* V *be the maximal volume of a canonical Fano polytope in* $M_{\mathbb{R}}$*. Then* V *is finite with*

$$V \leq 14^{d2^{d+1}}.$$

Any canonical Fano polytope can be embedded (i.e., is isomorphic as a lattice polytope to a canonical Fano polytope contained) in the lattice cube of side length at most $d \cdot d!V$ *(respectively* $d!V$*, if the polytope is a simplex).*

From this theorem we get the finiteness of isomorphism classes of canonical polytopes in a fixed dimension.

The bound in the theorem is extremely too large at least for reflexive polytopes in low dimensions, e.g., for $d = 2$ we get $14^{22^{d+1}} = 14^{16}$, however any canonical Fano polygon has at most a volume of 4.5, achieved by the polytope of type 9 in Prop. 3.4.1.

In general Theorem A' gives always a better upper bound on the volume of a reflexive simplex than the previous theorem!

Using a lower bound on the number of lattice points in $S_{Q'_d}$ due to [PWZ82] we see that there is still a small gap to bridge:

Corollary 3.7.15. *Let* J *denote the maximal number of lattice points some* d*-dimensional reflexive simplex can have. Then we get for* $d \geq 3$

$$\frac{1}{3(d-2)!} t_{d-1}^2 < J \leq d + 2t_{d-1}^2 \in O(c^{2^{d+1}}),$$

where $c \approx 1.26408$ *is the Vardi constant (see 3.6.33).*

The computer classification of Kreuzer and Skarke [KS04b] yields that the maximal number of lattice points in a d-dimensional reflexive polytope is 10 for $d = 2$, 39 for $d = 3$ and 680 for $d = 4$. The cases of equality are *precisely* the reflexive simplices given in theorem A'! This motivates the following conjecture:

Conjecture 3.7.16. Let $d \geq 4$: The reflexive simplex $S_{Q'_d}$ is the unique d-dimensional reflexive polytope with the largest number of lattice points.

As a corollary of the previous two theorems we can look at an embedding into a multiple of the *unit lattice cube* $[-1,1]^d$ (see also section 6.4):

Corollary 3.7.17. *Any d-dimensional reflexive simplex (for $d \geq 4$) can be embedded in $[-l,l]^d$ for some $l \in \mathbb{N}$ with $l \leq 2t_{d-1}^2 \leq 2c^{2^{d+1}}$ for $c \approx 1.26408$.*

Theorem C translates by (3.7) and Theorem 3.6.14 into:

Theorem 3.7.18 (C'). *Let P be a d-dimensional reflexive simplex. Then*

$$\mathrm{vol}(P)\,\mathrm{vol}(P^*) \leq \frac{t_d^2}{(d!)^2},$$

with equality iff $P \cong S_{Q_d}$. Here $S_{Q_d} \cong (S_{Q_d})^*$.

Furthermore let the vertices of P generate the lattice M. Then

$$\mathrm{vol}(P) \leq \frac{t_d}{d!},$$

with equality iff $P \cong S_{Q_d}$.

Proof of theorem C'. Follows immediately from Theorem 3.6.22(3) and Lemma 3.7.12.

□

Here the first inequality can be seen as a generalization of the Blaschke-Santaló inequality (1.6) in section 1.5.

Finally concerning theorem B we remark that any torus-invariant integral curve C is given by a *wall* $\rho \in \triangle(d-2)$, i.e., a $(d-2)$-dimensional cone of $\triangle$. Moreover ρ is obviously in correspondence with an edge e of P. There is the following observation (see [Lat96, Cor. 3.6]):

$$-K_X.C = |\, e \cap M \,| - 1. \tag{3.8}$$

Here the right side is called the *lattice length* of the edge e.

Using equation (3.8) we see that theorem B can be derived from the following combinatorial result:

Theorem 3.7.19 (B'). *The maximal number of lattice points on an edge of a d-dimensional reflexive simplex is $2t_{d-1}+1$, with equality attained only for $S_{Q'_d}$.*

This result has been observed by Haase and Melnikov in [HM04] for $d \leq 4$ and initialized this research.

Here one should state the following related result (see [Fuj03, 2.2]):

Proposition 3.7.20. *Let Q be a normalized weight system with associated simplex $P_Q = \mathrm{conv}(v_0, \ldots, v_d)$. Let $i, j \in \{0, \ldots, d\}$, $i \neq j$. Then we have for the torus-invariant integral curve C on $\mathbb{P}(Q)$ associated to the wall $\mathrm{pos}(v_k \,:\, i \neq k \neq j)$:*

$$(-K_{\mathbb{P}(Q)}).C = \frac{|\, Q \,|}{\mathrm{lcm}(q_i, q_j)}.$$

In particular, when Q is a reflexive weight system with corresponding unit partition $(k_0, \ldots, k_d)$, we get:

$$(-K_{\mathbb{P}(Q)}).C = \gcd(k_i, k_j).$$

Proof of theorem B'. The fact that $S_{Q'_d}$ satisfies the bound is trivial from the definition and 3.6.18.

Let $P = \mathrm{conv}(v_0, \ldots, v_d) \subseteq M_{\mathbb{R}}$ be a reflexive simplex with associated weight system Q_P, and let $Q := (Q_P)_{\mathrm{red}} = (q_0, \ldots, q_d)$ correspond to the unit partition $(k_0, \ldots, k_d)$. Let $[v_d, v_{d-1}]$ contain the maximal number of lattice points on an edge of P. Let $m \in M$ be the primitive lattice point such that $v_d - v_{d-1} = cm$ for $c \in \mathbb{N}_{>0}$. Obviously $|\,[v_d, v_{d-1}] \cap M\,| = c + 1$.

Assume $c \geq 2t_{d-1}^2$. Since $\sum_{i=0}^d q_i v_i = 0$, we have $c\langle \eta_{F_d}, m\rangle = \langle \eta_{F_d}, v_d\rangle + 1 = \frac{|Q|}{q_d} - 1 + 1 = k_d$. Furthermore $c\langle \eta_{F_{d-1}}, m\rangle = -k_{d-1}$. Hence

$$c \;\mid\; \gcd(k_d, k_{d-1}). \tag{3.9}$$

An alternative argument would have used Proposition 3.7.20.

As we have already seen in the proof of 3.6.29 (take $j = 1$) applying Lemma 3.6.30 yields

$$\frac{1}{k_{d-1}} + \frac{1}{k_d} \geq \frac{1}{t_{d-1}}. \tag{3.10}$$

From now on we assume without restriction that $k_d \geq k_{d-1}$. Hence the previous two equations yield $2t_{d-1} \leq c \leq k_{d-1} \leq 2t_{d-1}$, so we have $c = 2t_{d-1} = k_{d-1} = k_d$. In particular we get $\sum_{i=0}^{d-2} 1/k_i = 1 - 1/t_{d-1}$.

Now Lemma 3.6.30 implies $\{k_0, \ldots, k_{d-2}\} = \{y_0, \ldots, y_{d-2}\}$. Hence $Q = Q'_d$ corresponds to the enlarged Sylvester partition.

By 3.6.18 we can choose $\mathcal{V}(P_{\mathrm{red}}) = \{e_1, \ldots, e_d, e := -q_0 e_1 - \cdots - q_{d-1} e_d\}$ for some $\mathbb{Z}$-basis $e_1, \ldots, e_d$ of M. By Theorem 3.6.7 there is (up to unimodular equivalence) a matrix $H = \{h_{i,j}\}$ in Hermite normal form of determinant λ_P such that $H(e_1) = v_0, \ldots, H(e_d) = v_{d-1}, H(e) = v_d$. Recall that a quadratic matrix H is in Hermite normal form, if it is a lower triangular matrix with natural numbers as coefficients such that $h_{i,j} < h_{j,j}$ for $i > j$.

Since v_{d-1} is primitive, we get $h_{d,d} = 1$, so $v_{d-1} = e_d$. Because of

$$\frac{1}{2t_{d-1}}(H(e) - e_d) = m \in M \text{ and } \frac{q_i}{2t_{d-1}} = \frac{1}{y_i} \text{ for } i = 0, \ldots, d-2$$

we get the following $d - 1$ equations:

$$\begin{aligned} \frac{h_{1,1}}{y_0} &\in \mathbb{N} \\ y_0\left(\frac{h_{2,1}}{y_0} + \frac{h_{2,2}}{y_1}\right) &\in \mathbb{N} \\ &\cdots \\ y_0 \cdots y_{d-3}\left(\frac{h_{d-1,1}}{y_0} + \cdots + \frac{h_{d-1,d-1}}{y_{d-2}}\right) &\in \mathbb{N}. \end{aligned}$$

Using the fact that $\gcd(y_i, y_j) = 1$ for $i \neq j$, we deduce by induction that y_{i-1} divides $h_{i,i}$ for $i = 1, \ldots, d-1$. Hence we have

$$m_Q = t_{d-1} = y_0 \cdots y_{d-2} \leq h_{1,1} \cdots h_{d-1,d-1} h_{d,d} = \det H = \lambda_P \leq m_Q.$$

Now 3.6.15(3) implies $P \cong (P_{\mathrm{red}})^* = S_{Q'_d}$.

$\square$

3.7.3 Counting lattice points in residue classes

There is an easy method that is originally due to Batyrev [Bat82a, Lemma 1] how to get a sharp bound on the number of lattice points in some special cases: We simply count lattice points modulo a natural number k:

Definition 3.7.21. For $k \in \mathbb{N}$ we have the canonical homomorphism

$$\alpha_k \,:\, M \to M/kM \cong (\mathbb{Z}/k\mathbb{Z})^d.$$

For a convex set $C \subseteq M_\mathbb{R}$ with $C \cap M \neq \emptyset$ one easily sees that the minimal $k \in \mathbb{N}_{\geq 1}$ such that the restriction of α_k to $C \cap M$ is injective is just the maximal number of lattice points on an intersection of C with an affine line. This invariant minus one is called the *discrete diameter* of C in [Kan98].

Lemma 3.7.22. *Let $d \geq 2$, P a canonical Fano polytope and $B \subseteq \partial P \cap M$ with $|\,[x,y] \cap M\,| = 2$ for all $x, y \in B$, $x \neq y$, $x \sim y$. Let s denote the number of centrally symmetric pairs in B. Then*

$$|\,B\,| \leq 2^{d+1} - 2, \qquad s \geq |\,B\,| + 1 - 2^d.$$

Proof. We consider the restriction of α_2 to B. As P is canonical, the fibre of 0 is empty. Using the assumption it is also easy to see that the fibre of a non-zero element in $(\mathbb{Z}/2\mathbb{Z})^d$ has at most two elements, and in the case of equality it consists of one pair of centrally symmetric lattice points in B. From this the bounds can be derived.

□

For a semi-terminal canonical Fano polytope P (see Definition 3.2.3), the set $B = \mathcal{V}(P)$ satisfies the assumptions of lemma 3.7.22, so we immediately get a sharp bound on the number of vertices of P.

In particular we get a result that was proven in the case of a smooth Fano polytope in [Bat99, Prop. 2.1.11]:

Corollary 3.7.23. *Let P be a terminal Fano polytope. Then*

$$|\,\partial P \cap M\,| = |\,\mathcal{V}(P)\,| \leq 2^{d+1} - 2.$$

If equality holds, then P is centrally symmetric. This holds for the terminal reflexive d-dimensional standard lattice zonotope $\mathcal{Z}_d = \operatorname{conv}(\pm[0,1]^d)$.

The results in [Kas03] show that $\mathcal{Z}_d$ is even the only terminal Fano polytope with the maximal number of vertices for $d \leq 3$. However the computer classification of Kreuzer and Skarke yields two non-isomorphic four-dimensional terminal reflexive polytopes with $2^5 - 2 = 30$ vertices. They have different volumes, but of course the same number of lattice points!

The second case where counting modulo k works is the class of centrally symmetric reflexive polytopes as will described in the very last section of this thesis.

Chapter 4

Terminal Gorenstein toric Fano 3-folds

Introduction

In this chapter three-dimensional Gorenstein toric Fano varieties with terminal singularities are classified. They consist of 100 isomorphism classes. These varieties are particularly interesting, since due to the mild nature of their singularities they are still close to the well-known 18 smooth toric Fano 3-folds (see section 3.4). For instance in [Nam97] Namikawa proved that any Fano 3-fold with Gorenstein terminal singularities is smoothable by a flat deformation.

The classification uses the combinatorial description of these varieties as three-dimensional terminal reflexive polytopes. Very recently Kasprzyk classified in [Kas03] even all three-dimensional terminal Fano polytopes, however his proof relied partly on computer calculations, where in the reflexive case here no computer has been used and any calculation has been explicitly written down. For the classification we first observe that any three-dimensional terminal reflexive polytope has as facets either a triangle with vertices a, b, c or a parallelogram with vertices $a, b, c, a + b - c$ for a lattice basis a, b, c. This means geometrically that the singularities appearing are at most conifold singularities, i.e, they look locally like $z_1 z_2 - z_3 z_4 = 0$. Then we use properties of the projection map and some special relations among the vertices.

Many ideas for this approach are due to Batyrev. Furthermore Müller gave in his Diplomarbeit [Mül01] advised by Prof. Batyrev some preliminary results and presented a useful list containing vertices, facets and figures of these polytopes based on the computer database [KS04b]. However he did not yet describe an effective approach, the proofs were not rigorous and the list contained several errors, e.g., polytopes no. 8.12 and 8.21 (in his notation), as well as no. 8.7 and 8.14, were actually isomorphic.

In the first section of this chapter we discuss various notions of primitive collections and relations as introduced by Batyrev in [Bat91], since this tool was important for the classification of Fano 4-folds due to Batyrev [Bat99] and also enormously influenced the method of classification achieved in this chapter. Here we show how this data can in some cases be used to completely determine the polytope and its relations.

In the second section we define in the first subsection the notion of a quasi-smooth Fano polytope and show that this precisely gives the set of three-dimensional terminal reflexive polytopes. Moreover we introduce so called quasi-primitive collections and relations, these are especially suitable for describing quasi-smooth Fano polytopes. In the second subsection we define the notion of a symmetric vertex, where its antipodal point is also a vertex, and of an additive vertex, that is the sum of two other vertices. We examine their peculiar properties when projecting along such a vertex by using the list of two-dimensional reflexive polytopes in Prop. 3.4.1, as well as the results in Prop. 3.2.2(7) and Prop. 3.3.1.

In the third section we give the proof of the main classification theorem. It relies on the notion of an AS-point, i.e., a vertex that is both symmetric and additive. If no such AS-point exists, then we use Prop. 3.2.4(2) to show that the polytope has at most eight vertices, so by Prop. 3.1.6 we can use the classification of three-dimensional proper nonsingular toric varieties with Picard number five or less which are minimal in the sense of equivariant blow-ups as described in [Oda88, 1.34]. On the other hand if there exists an AS-point, then we can use Proposition 4.2.17 in the second section to completely classify the polytope by its quasi-primitive relations.

In the last section we give the list of 100 quasi-smooth Fano polytopes with some of their invariants. Moreover we show how to determine the Picard number of the corresponding toric varieties.

Summary of most important new results of this chapter:

- There are up to isomorphism 100 three-dimensional Gorenstein toric Fano varieties with terminal singularities (Prop. 4.3.2, p. 101)
- Their Picard numbers depend only on the combinatorics of the corresponding reflexive polytope (Prop. 4.4.2, p. 113)
- The corresponding reflexive polytopes can be uniquely determined by a small set of special relations among its vertices (Prop. 4.2.17, p. 101; Prop. 4.3.3, p. 102)

4.1 Primitive collections and relations

Here the notions of primitive collections and relations are described. For their important algebraic-geometric properties in particular concerning the associated 1-cycles in the Mori cone we refer to [Rei83], [Bat91], [BC94], [Bat99], [Sat00], and [Cas03c]. In the case of a primitive collection consisting of only two elements one should compare the definitions and results here with Proposition 3.3.1 and the remarks made there.

For any complete fan the following definition may be useful.

Definition 4.1.1. Let $\triangle$ be a complete fan. A subset $\mathcal{P} \subseteq \triangle(1)$ will be called a *primitive collection*, if it satisfies the conditions

1. For any $\sigma \in \triangle$ we have $\text{pos}(\mathcal{P}) \nsubseteq \sigma$,
2. For any $\tau \in \mathcal{P}$ there is a $\sigma \in \triangle$ such that $\text{pos}(\mathcal{P}\backslash\tau) \subseteq \sigma$.

This definition can also be found on p. 11 of the manuscript [Cox03]. When the fan consists only of simplicial cones, a set of rays is contained in a cone of the fan if and only if it generates some cone of the fan. Hence this definition coincides with the definition of a primitive collection as given in [Bat91] in the case of a nonsingular proper variety. By substituting $\subseteq$ with $=$ in any of the above two conditions, we get different generalizations of the notion of a primitive collection. However only the given definition yields the following essential observation:

Lemma 4.1.2. *Let C be a subset of rays of a complete fan $\triangle$. Then C is contained in a cone of $\triangle$ if and only if C does not contain a primitive collection.*

This holds, since *primitive collections are precisely the minimal elements among the subsets of rays where not all elements are contained in a common cone.* From this result we see that primitive collections suffice to determine the set of rays generating maximal cones. Since any cone in a complete fan is the intersection of maximal cones, we derive:

Proposition 4.1.3. *Two complete fans are combinatorially isomorphic, i.e., there is a bijection between the set of rays mapping generators of cones onto generators of cones, if and only if there is a bijection between the set of rays mapping primitive collections onto primitive collections.*

From now on let $P \subseteq M_{\mathbb{R}}$ be a reflexive polytope and $\triangle := \Sigma_P$.

Definition 4.1.4. A subset $\mathcal{P} \subseteq \partial P \cap M$ will be called a *primitive collection*, if it satisfies the conditions

1. For any $F \in \mathcal{F}(P)$ we have $\mathcal{P} \not\subseteq F$,
2. For any $m \in \mathcal{P}$ there is a $F \in \mathcal{F}(P)$ such that $\mathcal{P}\backslash\{m\} \subseteq F$.

When $\mathcal{P} \subseteq \mathcal{V}(P)$, $\mathcal{P}$ is called a $\mathcal{V}(P)$*-primitive collection*.

Hence $\mathcal{V}(P)$-primitive collections just correspond to primitive collections of $\triangle = \Sigma_P$. By 4.1.3 we get that *the set of $\mathcal{V}(P)$-primitive collections determines the combinatorial type of P*. For instance the d-dimensional crosspolytope can be uniquely combinatorially described as $2d$ vertices partitioned into d pairs of primitive collections. In the case of a terminal reflexive polytope the notions of $\mathcal{V}(P)$-primitive and primitive collections of course coincide.

As before we have:

Lemma 4.1.5. *Let C be a subset of $\partial P \cap M$. Then C is contained in a face of P if and only if C does not contain a primitive collection.*

To determine P as a lattice polytope we give a generalization of the notion of a primitive relation as defined in [Bat91].

Definition 4.1.6. Let $\mathcal{P} := \{w_1, \ldots, w_l\}$ be a chosen primitive collection of P. There is a unique face $G(\mathcal{P})$ of P such that

$$\sigma(\mathcal{P}) := \sum_{i=1}^{l} w_l \in \operatorname{relint}(\operatorname{pos}(G(\mathcal{P}))).$$

Then by Helly's theorem 1.5.1 we find an equation

$$\sigma(\mathcal{P}) = \sum_{j=1}^{t} \lambda_j v_j,$$

where $\lambda_j \in \mathbb{Q}_{>0}$ and $v_1, \ldots, v_t$ are lattice points on $G(\mathcal{P})$ and $1 \leq t \leq d$; this is called a *primitive relation associated to* $\mathcal{P}$. We have $\{v_1, \ldots, v_t\} = \emptyset$, iff $t = 0$, iff $\sigma(\mathcal{P}) = 0$.

Although as intrinsic as the notion of a primitive collection, the set of primitive relations associated to $\mathcal{P}$ contains in general more than one element. However this ambiguity can be resolved by explicitly choosing a triangulation of the polytope:

Let $\triangle'$ be a fixed crepant subdivision of $\triangle$ corresponding to a $\mathbb{Q}$-factorial weak toric Fano variety with terminal singularities as in Prop. 2.3.12. In this case by choosing $v_1, \ldots, v_t$ as the generators of a cone in $\triangle'$, there is a unique primitive relation called *the primitive relation of* $\mathcal{P}$ *with respect to* $\triangle'$.

Let F be a facet of P containing $v_1, \ldots, v_t$, set $\nu_F := -\eta_F$. Since not all $w_1, \ldots, w_l$ are contained in F, we have

$$0 < \sum_{j=1}^{t} \lambda_j = \langle \nu_F, \sigma(\mathcal{P}) \rangle = \sum_{i=1}^{l} \langle \nu_F, w_i \rangle \in \mathbb{Z}_{\leq l-1}.$$

Hence we can define the *degree* of a primitive relation as

$$\deg(\mathcal{P}) := l - \sum_{j=1}^{t} \lambda_j \in \mathbb{N}_{\geq 1}. \tag{4.1}$$

Under some assumptions there is a convenient property when determining primitive relations (generalizing [Bat91, Prop. 3.1]):

Lemma 4.1.7. *Let* $\sum_{i=1}^{l} w_i = \sum_{j=1}^{t} \lambda_j v_j$ *be a primitive relation such that* $\lambda_j \geq 1$ *for all* $j = 1, \ldots, t$.

Then $\{w_1, \ldots, w_l\} \cap \{v_1, \ldots, v_t\} = \emptyset$.

Proof. Assume $w_1 = v_1$. Let C be the smallest cone in $\triangle$ that contains $w_2, \ldots, w_l$. Define C' as the smallest cone in $\triangle$ containing $v_1, \ldots, v_t$, if $\lambda_1 > 1$, or containing $v_2, \ldots, v_t$ otherwise. Then

$$\sum_{i=2}^{l} w_i = (\lambda_1 - 1) v_1 + \sum_{j=2}^{t} \lambda_j v_j \in \operatorname{relint} C \cap \operatorname{relint} C'.$$

Hence $C = C'$. Let C'' be a cone of $\triangle$ containing $v_1, \ldots, v_t$. Then $w_2, \ldots, w_l \in C \subseteq C''$ and $w_1 = v_1 \in C''$, a contradiction.

□

We have the following result:

Proposition 4.1.8. *Assume for any primitive collection we can choose a primitive relation such that* $\lambda_j \geq 1$.

If the lattice points of a fixed facet of a reflexive polytope P *are given, then* P *is determined by the set of primitive collections and chosen primitive relations.*

Proof. Let $F \in \mathcal{F}(P)$ and $x_1, \ldots, x_d$ be a set of linearly independent vertices of F. Let $w \in \partial P \cap M$ with $a := \langle \nu_F, w \rangle \leq 0$ such that any $y \in \partial P \cap M$ with $\langle \nu_F, y \rangle > a$ is already determined.

Since $\{w, x_1, \ldots, x_d\}$ is not contained in a face of P, by 4.1.5 we can assume there is a primitive collection $\mathcal{P} = \{w, x_1, \ldots, x_s\}$ $(s \leq d)$. Now let $\sigma(\mathcal{P}) := w + \sum_{i=1}^{s} x_i = \sum_{j=1}^{t} \lambda_j v_j$ the associated primitive relation. If $\langle \nu_F, v_i \rangle > a$ for all $j = 1, \ldots, t$ we are finished.

So we may assume $\langle \nu_F, v_1 \rangle \leq a$. Hence $a + s = \langle \nu_F, \sigma(\mathcal{P}) \rangle = \sum_{j=1}^{t} \lambda_j \langle \nu_F, v_j \rangle$ $\leq \lambda_1 a + \sum_{j=2}^{t} \lambda_j \langle \nu_F, v_j \rangle \leq \lambda_1 a + \sum_{j=2}^{t} \lambda_j$. Since $s + 1 - \sum_{j=1}^{t} \lambda_j \geq 1$ by (4.1), this yields

$$0 < \lambda_1 \leq s - \sum_{j=2}^{t} \lambda_j \leq (\lambda_1 - 1)a.$$

Since $a \leq 0$, this implies $\lambda_1 < 1$, a contradiction.

□

An analogous result can be formulated for $\mathcal{V}(P)$-primitive collections.

If $\triangle'$ in Definition 4.1.6 can be chosen to be nonsingular, then the coefficients λ_j in a primitive relation with respect to $\triangle'$ are non-zero natural numbers, in particular they satisfy the condition of the previous proposition. Hence the intrinsic and finitely many conditions that for any primitive collection $\mathcal{P}$ there exist lattice points in the face of P that contains $\sigma(\mathcal{P})$ in the relative interior such that $\sigma(\mathcal{P})$ is a non-negative integer combination are important obstructions for the existence of a crepant resolution. However obviously they are not sufficient as can been seen in Example 3.2.6.

As a corollary we generalize a well-known result for nonsingular toric Fano varieties (see [Bat99]):

Corollary 4.1.9. *A reflexive polytope corresponding to a toric variety admitting a crepant resolution is uniquely determined by the lattice points of one arbitrary facet and the set of primitive collections and primitive relations (chosen with integer coefficients).*

We can also determine the group of linear relations:

Definition 4.1.10. Let $C \subseteq M_{\mathbb{R}}$ be a lattice polytope. We let $LR(C)$ denote the group of linear relations with integer coefficients among elements of $C \cap M$.

Proposition 4.1.11. *Assume for any primitive collection we can choose a primitive relation such that $\lambda_j \in \mathbb{N}$.*

Then the group $LR(\partial P)$ is generated by $\{LR(F) : F \in \mathcal{F}(P)\}$ and the chosen primitive relations.

Proof. Let

$$\sum_{i=1}^{s} \alpha_i w_i - \sum_{j=1}^{r} \beta_j u_j = 0$$

be a relation g such that $\{\alpha_i\}$, $\{\beta_j\}$ are non-zero natural numbers, $W := \{w_1, \ldots, w_s\} \subseteq \partial P \cap M$ and $U := \{u_1, \ldots, u_r\} \subseteq \partial P \cap M$ with $V \cap W = \emptyset$, and the 'absolute norm' $\mathrm{abs}(g) := \sum_{i=1}^{s} \alpha_i + \sum_{j=1}^{r} \beta_j \in \mathbb{N}_{>0}$ is minimal with the property that this integer relation g is not contained in the subgroup H of $LR(\partial P)$ generated by $\{LR(F) : F \in \mathcal{F}(P)\}$ and the set of primitive relations.

If W and U are each contained in a face of P, then their relative interiors intersect, hence W and U are contained in a common facet of P, however this is a contradiction to $g \notin H$. So we may assume W is not contained in a face of P, so by 4.1.5 we can assume that there is a primitive collection $\mathcal{P} = \{w_1, \ldots, w_l\}$ ($l \leq s$) and a corresponding primitive relation

$$\sum_{i=1}^{l} w_i - \sum_{j=1}^{t} \lambda_j v_j = 0$$

with $\lambda_j \in \mathbb{N}_{>0}$. The difference of the previous two equations yields a relation g'

$$\sum_{i=1}^{l} (\alpha_i - 1) w_i + \sum_{i=l+1}^{s} \alpha_i w_i + \sum_{j=1}^{t} \lambda_j v_j - \sum_{j=1}^{r} \beta_j u_j = 0.$$

Necessarily $g' \notin H$. Regrouping the elements yields

$$\mathrm{abs}(g') \leq \sum_{i=1}^{l} (\alpha_i - 1) + \sum_{i=l+1}^{s} \alpha_i + \sum_{j=1}^{t} \lambda_j + \sum_{j=1}^{r} \beta_j = \mathrm{abs}(g) - \deg(\mathcal{P}) < \mathrm{abs}(g),$$

by (4.1), a contradiction.

□

4.2 Combinatorics of quasi-smooth Fano polytopes

4.2.1 Definition and basic properties

Definition 4.2.1. Let $d = 3$ and F a subset of $M_{\mathbb{R}}$. We call F a *smooth triangle*, if $F = \mathrm{conv}(v_1, v_2, v_3)$ such that v_1, v_2, v_3 is a $\mathbb{Z}$-basis of M.

Definition 4.2.2. Let $d = 3$ and F a subset of $M_{\mathbb{R}}$. We call F a *smooth lattice parallelogram*, if there exist lattice elements $v_1, v_2, v_3, v_4 \in M$ having F as its convex hull such that v_1, v_2, v_3 form a $\mathbb{Z}$-basis of the lattice M and $v_1 + v_3 = v_2 + v_4$. In this case v_1, v_2, v_3, v_4 are the vertices of the lattice polytope F and any subset of its vertices having three elements form a $\mathbb{Z}$-basis of the lattice M. $[v_1, v_3]$ and $[v_2, v_4]$ are called the *diagonals* of the parallelogram.

Definition 4.2.3. Let $d = 3$ and P a polytope in $M_{\mathbb{R}}$ with $0 \in \mathrm{int}P$. We call P a *quasi-smooth Fano polytope*, if each facet F of P is either a smooth triangle or a smooth lattice parallelogram.

P has only triangles as facets iff it is a smooth Fano polytope, or equivalently $X(M, \Sigma_P)$ is nonsingular. Otherwise P contains at least one parallelogram as a facet, we call P *singular* in this case.

Here is the motivation for this definition:

Proposition 4.2.4. *The set of quasi-smooth Fano polytopes is precisely the set of three-dimensional terminal reflexive polytopes.*

Isomorphism classes of three-dimensional toric Fano varieties with at most conifold singularities correspond to isomorphism classes of Gorenstein toric Fano varieties with terminal singularities.

Proof. Obviously any quasi-smooth Fano polytope is a terminal reflexive polytope. On the other hand let P be a terminal reflexive polytope. Since P is terminal, any facet of P is an empty two-dimensional lattice polytope. It is elementary to see that any such two-dimensional polytope is lattice equivalent to $\text{conv}(0, e_1, e_2)$ or $\text{conv}(0, e_1, e_2, e_1 + e_2)$ for a two-dimensional lattice basis e_1, e_2. Since P is reflexive, 3.1.8(2) shows that P is quasi-smooth.

Let e_1, e_2, e_3 be a $\mathbb{Z}$-basis of M. If $\sigma = \text{pos}(e_1, e_2, e_3)$, then we get $\sigma^\vee = \text{pos}(e_1^*, e_2^*, e_3^*)$. If $\sigma = \text{pos}(e_1, e_2, e_3, e_1 + e_3 - e_2)$, then $\sigma^\vee = \text{pos}(e_1^*, e_3^*, e_1^* + e_2^*, e_2^* + e_3^*)$. In particular the only singularities of a toric variety corresponding to the fan spanned by a quasi-smooth Fano polytope are isolated singularities of the type $z_1 z_3 - z_2 z_4 = 0$.

□

Applying Lemma 3.7.22 yields:

Proposition 4.2.5. *The number of vertices $n = n(P)$ of any quasi-smooth Fano polytope P is not greater than 14. The number of symmetric pairs of vertices is at least $n - 7$.*

Let $P \subseteq M_{\mathbb{R}}$ be a reflexive polytope and $\triangle := \Sigma_P$.

Since by Prop. 3.1.6 any three-dimensional reflexive polytope has a crepant resolution, by 4.1.8 it is determined by the vertices of a facet, primitive collections and primitive relations with integer coefficients. For our purposes it will make even sense to define another type of primitive collection:

Definition 4.2.6. A subset $\mathcal{P} \subseteq \partial P \cap M$ will be called a *quasi-primitive collection*, if it satisfies the conditions

1. For any face G of P we have $\text{conv}(\mathcal{P}) \neq G$,
2. For any $m \in \mathcal{P}$ there is a face G of P such that $\text{conv}(\mathcal{P}\backslash\{m\}) = G$.

Quasi-primitive collections of order ≥ 2 are precisely the minimal elements among the subsets of $\mathcal{V}(P)$ that are not the vertex set of some face of P.

Hence we only get one direction of Lemma 4.1.5: *Any subset of $\mathcal{V}(P)$ that is not a face of P contains a quasi-primitive collection.* Since a quasi-primitive collection that is not contained in a facet is also a primitive collection, there are fewer collections of this kind. On the other hand, when a quasi-primitive collection is contained in a face, the degree of a corresponding primitive relation as in (4.1) is zero. So most results of the previous section cannot be generalized.

We can 'artificially' make sense of the notion of a quasi-primitive relation by urging Lemma 4.1.7 to hold:

Definition 4.2.7. Let P be a reflexive polytope and $\mathcal{P} = \{w_1, \ldots, w_l\}$ a quasi-primitive collection. If we can find an equation

$$\sigma(\mathcal{P}) := \sum_{i=1}^{l} w_i = \sum_{j=1}^{t} \lambda_j v_j,$$

where $\lambda_j \in \mathbb{Q}_{>0}$, and $v_1, \ldots, v_t$ are the generators of the unique face whose positive hull contains $\sigma(\mathcal{P})$ in its relative interior, $1 \leq t \leq d$ and $\{w_1, \ldots, w_l\} \cap \{v_1, \ldots, v_t\} = \emptyset$, then it is called a *quasi-primitive relation associated to $\mathcal{P}$.*

In general these conditions might be too strong, so no quasi-primitive relation can be associated to a given quasi-primitive relation. However in the case of a quasi-smooth Fano polytope we have existence and uniqueness:

Proposition 4.2.8. *Let P be a quasi-smooth Fano polytope with $|\mathcal{V}(P)| > 4$, and $\mathcal{P} = \{w_1, \dots, w_l\}$ a primitive (respectively quasi-primitive) collection.*

Then exactly one primitive (respectively quasi-primitive) relation can be associated to $\mathcal{P}$. There are the following possible cases:

1. *For $l = 2$:*
   (a) $w_1 + w_2 = 0$
   (b) $w_1 + w_2 = v_1$ *for* $v_1 \in \mathcal{V}(P)$*, in this case* $w_1 \sim v_1 \sim w_2$
   (c) $w_1 + w_2 = v_1 + v_2$ *where* w_1, w_2, v_1, v_2 *are the vertices of a parallelogram facet, this is the only case where* $\mathcal{P} = \{w_1, w_2\}$ *is contained in a common face, so* $\mathcal{P}$ *is not a primitive collection.*
2. *For $l = 3$:*
   (a) $w_1 + w_2 + w_3 = 0$
   (b) $w_1 + w_2 + w_3 = cv_1$ *for* $c \in \{1, 2\}$ *and* $v_1 \in \mathcal{V}(P)$.

Proof. Let $l = 2$. The four cases follow immediately from 3.3.1, terminality of P and the parallelogram relation. Let $l = 3$. We can assume that $\{w_1, w_2, w_3\}$ is not contained in a facet. By (4.1) it is enough to assume there were vertices v_1, v_2 in a facet F of P such that $w_1 + w_2 + w_3 = v_1 + v_2$. Then necessarily $\langle \eta_F, w_1 \rangle + \langle \eta_F, w_1 \rangle + \langle \eta_F, w_1 \rangle = -2$, so by reflexivity we can assume $w_1, w_2 \in F$ and $\langle \eta_F, w_3 \rangle = 0$. So by definition $\{w_1, w_2, v_1, v_2\}$ are the vertices of a parallelogram facet, hence they satisfy the parallelogram relation, from this an easy calculation yields a contradiction. $l \geq 4$ were only possible for the smooth Fano polytope that is the simplex spanning the fan corresponding to three-dimensional projective space.

□

In 4.2.16 the case (1b) will be further investigated.

Since any primitive relation is necessarily integral, we get as a corollary from 4.1.11:

Corollary 4.2.9. *The group of linear relations with integer coefficients among the vertices of a quasi-smooth Fano polytope is generated by the parallelogram relations of facets and the set of unique primitive relations associated to the set of primitive collections.*

When looking at a the simplest singular quasi-smooth Fano polytope, namely the pyramide with parallelogram basis $\operatorname{conv}(e_1, e_2, -e_1 + e_2 + e_3, e_3)$ and apex $-e_2 - e_3$, we see that there is only one quasi-primitive relation, i.e., the parallelogram relation, however the group of linear relations has rank 2, so the previous corollary does not hold for quasi-primitive relations.

From this example we see that it is not at all obvious that the set of quasi-primitive relations should suffice to determine the isomorphism type of a quasi-smooth Fano polytope as a lattice polytope. However by the classification 4.3.2 this result holds, in particular see Proposition 4.2.17 for a partial explanation.

4.2.2 Projections of quasi-smooth Fano polytopes

Let P be always a quasi-smooth Fano polytope and $v_i \in \mathcal{V}(P)$ a vertex of P.

Definition 4.2.10. Let $\pi_i := \pi_{v_i}$, $P_i := \pi_i(P)$, $M_i := M_{v_i}$, $\iota_i := \iota_{v_i}$ as in 3.2.2.
Let $\partial_M(v_i) := \partial(v_i) \cap M = (\mathrm{st}(v_i) \cap \mathcal{V}(P)) \backslash \{v_i\}$.

The number $\deg(v_i) := |\,\partial_M(v_i)\,|$ of vertices $\neq v_i$ in the star set of v_i is called the *degree* of v_i.

Two elements $v_j, v_k \in \partial_M(v_i)$ are called *contiguous* with regard to π_i, if $\pi_i(v_j), \pi_i(v_k)$ are contained in a facet of P_i such that $[\pi_i(v_j), \pi_i(v_k)] \cap M_i = \{\pi_i(v_j), \pi_i(v_k)\}$.

It is important to remark that it is possible to determine the star set and thereby the degree of a vertex by knowing only the quasi-primitive collections and quasi-primitive relations of P as follows immediately from the definition and 4.2.8(1).

Since by 3.1.8 canonical Fano polygons are reflexive, we get by applying 3.2.4(2) and 3.2.2 the following result:

Proposition 4.2.11. *A three-dimensional reflexive polytope is a quasi-smooth Fano polytope if and only if the projection along any vertex yields a canonical Fano polytope.*

We have the following properties:

1. *P_i is a two-dimensional reflexive polytope.*
2. *If $0 \neq w \in M_i \cap P_i$, then ${\pi_i}^{-1}(w) \cap \mathcal{V}(P) = \{v_j, v_k\}$, with $v_j = v_k \in \partial_M(v_i)$ or $v_j + v_i = v_k \in \partial_M(v_i)$ or $v_k + v_i = v_j \in \partial_M(v_i)$.*
3. *The elements of $\partial_M(v_i)$ can be (up to reversion and cyclic permutation uniquely) ordered as $s_{l+1} = s_1, s_2, \ldots, s_l$ with $l = \deg(v_i)$ such that s_j, s_{j+1} are contiguous for $j = 1, \ldots, l$; then $\mathrm{conv}(v_i, s_j, s_{j+1})$ is contained in a facet of P for $j = 1, \ldots, l$; if $[v_i, s_j]$ is not an edge, then necessarily $\mathrm{conv}(v_i, s_{j-1}, s_j, s_{j+1})$ is a smooth parallelogram facet.*

This motivates the following definition:

Definition 4.2.12. A nonzero lattice point $w \in P_i$ is called *double point* with regard to π_i, if there exist two different vertices $v_j, v_k \in \mathcal{V}(P)$ such that $\pi_i(v_j) = w = \pi_i(v_k)$. Otherwise w is called *simple point* with regard to π_i.

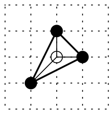
By 4.2.11 it is possible to determine the double points of π_i by knowing only the quasi-primitive collections and quasi-primitive relations of P.

As another application of 3.2.2 there is the following observation (due to Batyrev, see [Mül01]).

Proposition 4.2.13. *Let Γ be a facet of P_i. Then $G := {\pi_i}^{-1}(\Gamma) \cap P$ is a face of P, we have $|\,\Gamma \cap M_i\,| \leq 3$.*

1. *If $|\,\Gamma \cap M_i\,| = 2$, then G is an edge, a triangle or a parallelogram, respective Γ having no, one or two double points.*
2. *Let $|\,\Gamma \cap M_i\,| = 3$ and $\Gamma \cap M_i = \{x, y, z\}$ with $y = (x+z)/2$. Then G is a facet, $[v_i, \iota_i(y)]$ is an edge, and x, z are simple points. G is a parallelogram iff y is a double point.*

In particular there are up to isomorphism of the lattice M_i exactly 11 possible types of π_i-images of P as given in the following list ordered by the degree of v_i. Lines between 0 and a lattice point w denote that $[v_i, \iota_i(w)]$ necessarily has to be an edge of P.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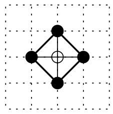
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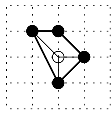
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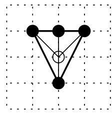
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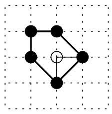
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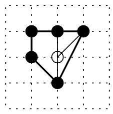
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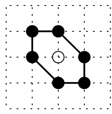
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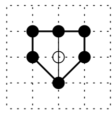
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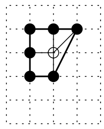
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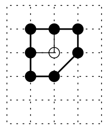
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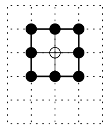
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Proof. Since by 3.2.2 and terminality of P the set of vertices of G maps surjective on the set of lattice points in Γ, the edge Γ has at most four lattice points. By 4.2.11(2) the case $|\,\Gamma \cap M_i\,| = 2$ is trivial.

Let $|\,\Gamma \cap M_i\,| \geq 3$. Then by 3.2.2(6) G has to be a triangle or a parallelogram facet that is parallel to v_i. By 3.2.2(2) the set of vertices of G in the star set of v_i is in bijection with the set of lattice points in Γ. However $|\,\partial_M(v_i) \cap \mathcal{V}(G)\,| = 3$. If G is a parallelogram, then by definition obviously the middle point y has to be a double point.

For the list use 3.4.1 and 4.2.11(3), and observe that, if $v_j \in \partial_M(v_i) \cap \mathcal{V}(P)$ with $[v_i, v_j]$ not an edge, then $\pi_i(v_j)$ is additive by the parallelogram relation. $\square$

There are three special classes of vertices of P:

Definition 4.2.14.

v_i is called *additive*, if v_i is the sum of two other vertices of P, i.e. there exists a quasi-primitive relation $v_j + v_k - v_i = 0$ for vertices $v_j, v_k \in \mathcal{V}(P)$.

v_i is called *symmetric*, if $-v_i \in \mathcal{V}(P)$ is also a vertex, i.e. there exists a quasi-primitive relation $v_i + v_j = 0$ for a vertex $v_j \in \mathcal{V}(P)$;

v_i is called *AS-point*, if v_i is additive and symmetric.

In the symmetric case there is more to say about the double points of π_i, the combinatorics of P and quasi-primitive collections among elements of $\partial_M(v_i)$.

Lemma 4.2.15. *Let $v_i \in \mathcal{V}(P)$ be a* symmetric *vertex of P.*

1. *$\mathcal{V}(P) = (\mathrm{st}(v_i) \cup \mathrm{st}(-v_i)) \cap \mathcal{V}(P)$; $\partial_M(v_i) \cap \partial_M(-v_i)$ contains the vertices v_k of P where $\pi_i(v_k)$ is a simple point; $\partial_M(v_i) \backslash \partial_M(-v_i)$ contains the vertices v_k of P where $\pi_i(v_k)$ is a double point, the preimage is $\{v_k, v_k - v_i\}$, where $v_k - v_i \in \partial_M(-v_i) \backslash \partial_M(v_i)$. A vertex in $\mathcal{V}(P)$ whose projection is a double point is additive.*

2. *Any facet G of P either contains v_i or $-v_i$ as a vertex or is the intersection of P and the π_i-preimage of a facet Γ of P_i. If in the last case $|\Gamma \cap M_i| = 3$, then G is a parallelogram (the middle point of $\Gamma \cap M_i$ is a double point, the other two lattice points are simple points). We have for $v_k, v_l \in \partial_M(v_i)$:*

   $[v_k, v_l]$ is an edge of P if and only if v_k, v_l are contiguous.

   Therefore:

   $\{v_k, v_l\}$ is a quasi-primitive collection if and only if v_k, v_l are not contiguous. If in this case $v_k \sim v_l$, then $[v_k, v_l]$ is a diagonal of a parallelogram facet (not necessarily containing v_i), so there is another vertex $v_t \in \partial_M(v_i)$ contagious to both v_k and v_l.

3. *Let $v_k, v_l \in \partial_M(v_i)$ be two vertices with $\pi_i(v_k) = -\pi_i(v_l)$. Then $\{v_k, v_l\}$ is a quasi-primitive collection with $v_k + v_l \in \{v_i, 0, -v_i\}$.*

   If $\pi_i(v_k)$ and $\pi_i(v_l)$ are double points, then $v_k + v_l = v_i$.

   If $v_k + v_l = -v_i$, then $\pi_i(v_k)$ and $\pi_i(v_l)$ are simple points.

   If $v_k + v_l \neq 0$, then v_i or $-v_i$ is an AS-point.

   If $\pi_i(v_k)$ is a double point, then v_k or $v_k - v_i$ is an AS-point.

These statements can be easily verified using 3.2.2(7), 4.2.11, 4.2.8.

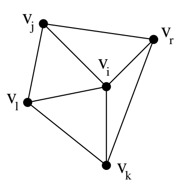
In the additive case there are strong restrictions on the structure of $\mathrm{st}(v_i)$:

Lemma 4.2.16. *Let $v_i \in \mathcal{V}(P)$ be an* additive *vertex of P with $v_j + v_k = v_i$ for vertices $v_j, v_k \in \mathcal{V}(P)$.*

There are exactly two vertices $v_l, v_r \in \partial_M(v_i) \backslash \{v_j, v_k\}$ such that $[v_i, v_l]$ and $[v_i, v_r]$ are edges of P.

Precisely the following 10 cases can occur (up to exchanging of v_j and v_k, respectively v_l and v_r), where the labels denote the degree of v_i:

1. *$[v_i, v_j]$ and $[v_i, v_k]$ are edges:*



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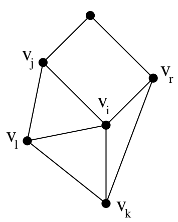

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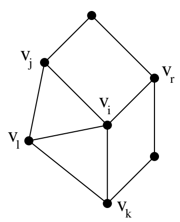

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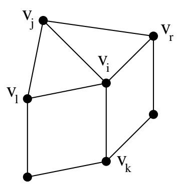

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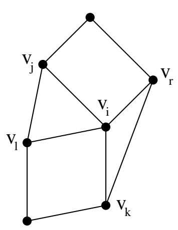

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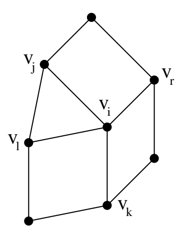

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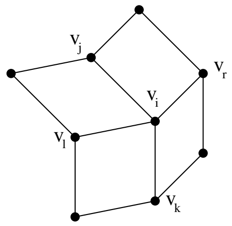

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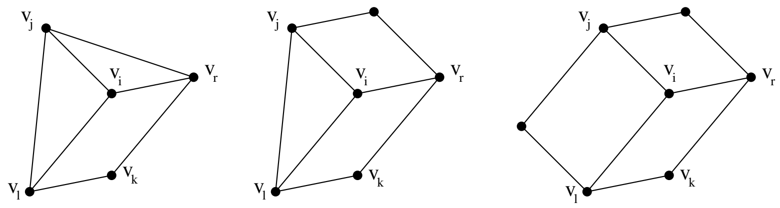
2. Either $[v_i, v_j]$ or $[v_i, v_k]$ is an diagonal (here without restriction $[v_i, v_k]$):



Proof. Since P is a three-dimensional polytope, there are at most two facets containing $[x, y]$ for vertices x, y of P, with equality iff $[x, y]$ is an edge.

So let F be a facet of P containing v_i. By 4.2.8 F contains $[v_i, v_j]$ or $[v_i, v_k]$. So there are at most four facets containing v_i, especially any vertex in the star set of v_i has to be a vertex of one of these facets.

Assume $[v_i, v_j]$ and $[v_i, v_k]$ were diagonals, i.e., there are exactly two facets containing v_i. This implies that v_j, v_r, v_k, v_l is an ordering of $\partial_M(v_i)$ as in 4.2.11(3), where $\text{conv}(v_i, v_j, v_l, v_r)$ and $\text{conv}(v_i, v_k, v_l, v_r)$ are parallelograms. Necessarily $v_i + v_j = v_l + v_r$ and $v_i + v_k = v_l + v_r$, an obvious contradiction. Alternatively use 3.3.1(iv).

□

Combining the previous two propositions we get a result that will be the key factor in classifying quasi-smooth Fano polytopes:

Proposition 4.2.17. *Quasi-smooth Fano polytopes having AS-points are uniquely determined by their quasi-primitive relations.*

More precisely the following data uniquely determines the quasi-smooth Fano polytope P:

1. *The existence of an AS-point $v_i \in \mathcal{V}(P)$ (with quasi-primitive relation $v_j + v_k = v_i$)*

2. *The associated quasi-primitive relation of the quasi-primitive collection $\{v_l, v_r\}$ for the unique vertices $v_l, v_r \in \mathcal{V}(P)\backslash\{v_j, v_k\}$ such that $[v_i, v_l]$ and $[v_i, v_r]$ are edges*

3. *What case in 4.2.16 occurs, i.e. which of the lines $[v_j, v_r]$, $[v_j, v_i]$, $[v_j, v_l]$, $[v_k, v_r]$, $[v_k, v_i]$, $[v_k, v_l]$ are no diagonals*

4. *Which of the boundary lattice points in P_i are double points*

The vertices occuring in the quasi-primitive relation associated to $\{v_l, v_r\}$ *are contained in*

$$\{v_l, v_j, v_j - v_i, v_i, 0, -v_i, v_r, v_k, v_k - v_i\}.$$

Proof. Let v_i as in 1. By 4.2.16 v_l, v_r as in 2. exist and are not contagious, so $\{v_l, v_r\}$ is a quasi-primitive collection by 4.2.15(2). By 4.2.16 $\text{conv}(v_i, v_j, v_r)$ is contained in a face of P, so it is possible to choose v_r, v_j, v_i as a $\mathbb{Z}$-basis of the lattice. Obviously $-v_i$ and v_k are determined by v_r, v_j, v_i. Now the additional statement about the vertices occuring in the quasi-primitive relation of $\{v_l, v_r\}$ follows from analyzing the possible types of P_i in 4.2.13 and using 4.2.8 and 4.2.11. So v_l can be determined by v_i, v_j, v_r and the relation in 2. By 3. and 4.2.16 *all* vertices in the star set of v_i are determined by v_i, v_j, v_k, v_r, v_l, and therefore by v_r, v_j, v_i. Now take a vertex $v \in \mathcal{V}(P)$, $v \notin \{v_i, -v_i\}$, then by 4.2.11(2) there is a unique vertex $v_h \in \partial_M(v_i)$, and by 4. it is known whether $v = v_h$ or $v = v_h - v_i$. Therefore all vertices of P are uniquely determined by 1. - 4. up to an isomorphism of the lattice, and obviously the conditions 1. to 4. are determined by quasi-primitive relations.

□

4.3 Classification of quasi-smooth Fano polytopes

4.3.1 The main theorem

The goal of this section is to prove the following theorem:

Theorem 4.3.1. *There exist exactly* 100 *isomorphism types of three-dimensional Gorenstein toric Fano varieties with terminal singularities. Of these are* 18 *non-singular and* 82 *singular.*

This is a corollary of the following convex-geometric formulation:

Theorem 4.3.2. *There exist exactly* 100 *isomorphism types of quasi-smooth Fano polytopes. Of these are* 18 *smooth and* 82 *singular. Quasi-smooth Fano polytopes are uniquely determined by the quasi-primitive relations associated to their quasi-primitive collections.*

number of vertices	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*
number of polytopes	*1*	*5*	*11*	*18*	*23*	*18*	*13*	*6*	*3*	*1*	*1*
of these are smooth	*1*	*4*	*7*	*4*	*2*	*0*	*0*	*0*	*0*	*0*	*0*
of these are singular	*0*	*1*	*4*	*14*	*21*	*18*	*13*	*6*	*3*	*1*	*1*

The strategy of the classification process is based upon the observations in the previous section. There are essentially two different parts:

If a quasi-smooth Fano polytope P has an AS-point, then applying the key result Proposition 4.2.17 gives us an explicit algorithm for uniquely determining P. This is done in subsection 4.3.3.

On the other hand if no AS-point of P exists, then we first deal with the case when there are no symmetric vertices, here we can apply an existing classification of Oda in [Oda88]. If there exists no AS-point but a symmetric vertex v, then we use the fact that there are only 11 types of polytopes that can occur when projecting along v to prove very strong restrictions on the polytope that will suffice to determine P.

4.3.2 Classification when no AS-points exist

In this subsection we are going to prove the following proposition:

Proposition 4.3.3. *There exist exactly* 17 *quasi-smooth Fano polytopes having no AS-points. Of these are* 8 *smooth (namely* $\mathbb{P}^3$, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{B}_4$, $\mathcal{C}_3$, $\mathcal{D}_1$, $\mathcal{D}_2$ *in the notation of [Bat99, 2.5]) and* 9 *singular (namely 5.1, 6.2, 6.4, 7.2, 7.3, 7.11, 7.13, 8.2, 8.21). They are uniquely determined by the quasi-primitive relations associated to their quasi-primitive collections. In the following list the vertices of the* 9 *singular polytopes are given relative to a* $\mathbb{Z}$*-basis* v, v', v'' *of the lattice:*

n^0	*vertices of P*
5.1	$v, v', v'', -v-v'', -v'-v''$
6.2	$v, v', v'', -v-v'', -v'-v'', v+v'+v''$
6.4	$v, v', v'', -v'', -v', -v+v'+v''$
7.2	$v, v', v'', -v'', v-v'', v'-v'', -v-v'$
7.3	$v, v', v'', -v'', v-v'', v'-v'', -v-v'+v''$
7.11	$v, v', v'', -v'', -v, -v', -v-v'+v''$
7.13	$v, v', v'', -v-v'', -v'-v'', -v-v', -v-v'-v''$
8.2	$v, v', v'', -v'', v-v'', v'-v'', -v-v'+v'', -v-v',$
8.21	$v, v', v'', v-v'+v'', -v, -v', -v'', -v+v'-v''$

The first case is handled using the existing classification of nonsingular proper toric varieties with a very small Picard number:

Lemma 4.3.4. *There exist exactly three quasi-smooth Fano polytopes that are singular and have no symmetric vertices. These are 5.1, 6.2, 7.13.*

Proof. By 4.2.5 the number of vertices of P is seven or less. As listed in [Oda88, Thm. 1.34] there are exactly three three-dimensional compact nonsingular toric varieties with Picard number four or less being minimal in the sense of equivariant blowing-ups where the associated fan has no symmetric pair of generators of one-dimensional cones. In the notation given there these are $3^4, (3^2 4^3)'', 3^1 4^3 5^3$. The last fan corresponds already to a quasi-smooth Fano polytope that is singular, namely 7.13. Now one has just to check that by equivariantly blowing-up the first two varities at most three, respectively two times one can only get 5.1, a pyramide with a parallelogram basis, or 6.2, a stacked simplex on a pyramide, as associated quasi-smooth Fano polytopes that are singular and have no symmetric vertices. This is an easy but tedious calculation that will be omitted. □

The next lemma will give restrictions on quasi-smooth Fano polytopes to have no AS-points:

Lemma 4.3.5. *Let P be a quasi-smooth Fano polytope having no AS-point. Let $v_i \in \mathcal{V}(P)$ be a symmetric vertex. If there exist w_1, w_2 nonzero lattice points in P_i with $w_1 + w_2 = 0$, then w_1 and w_2 are simple points. Let $v_1, v_2 \in \mathcal{V}(P)$ be the unique vertices with $\pi_i(v_1) = w_1$ and $\pi_i(v_2) = w_2$. Then $v_1 + v_2 = 0$.*

Proof. Because P has no AS-points it follows from the statements in 4.2.15(3) that $v_1 + v_2 = 0$ and w_1, w_2 are simple points.

□

Now the proof of the proposition could also be done by the classification of Oda, because in the proof it will be shown that $n(P) \leq 8$ is a necessary condition for a quasi-smooth Fano polytope to have no AS-points. Instead the polytopes will be classified using the language of quasi-primitive relations as this seems to be more instructive.

Proof of Proposition 4.3.3. By [Bat99, 2.5] it is easy to check that there exist exactly 8 smooth Fano polytopes having no AS-points. So let P be singular. By 4.3.4 we can assume that there exist vertices $v_i, v_j \in \mathcal{V}(P)$ with $v_i + v_j = 0$. By 4.2.15(2) a facet of P_i having three lattice points contains the middle point as the only double point. Also by 4.3.5 symmetric nonzero lattice points of P_i are simple points and their unique preimages in $\mathcal{V}(P)$ are symmetric. Looking up all 11 possible projection types in 4.2.13 one sees that there are only the following cases left:

3: In order for P not being simplicial there have to exist at least two double points of P_i.

If the remaining nonzero lattice point of P_i is no double point, then one can choose a $\mathbb{Z}$-basis $e_1, e_2, e_3 = v_i$ of M such that P consists of a Fano pyramide with basis $e_1, e_2, e_1 - e_3, e_2 - e_3$ and apex $v \in \mathcal{V}(P)$ with $\pi_i(v) = (-1, -1)$ and two stacked simplices with vertices v, e_3, e_1, e_2, respective $v, -e_3, e_1 - e_3, e_2 - e_3$. By 4.2.8(2) $v + e_1 + e_2 \in \{0, e_3, 2e_3\}$ and $v + (e_1 - e_3) + (e_2 - e_3) \in \{0, -e_3, -2e_3\}$. By symmetry this gives two non-isomorphic quasi-smooth Fano polytopes with seven vertices depending on whether one of these two quasi-primitive relations equals 0; if one does, P is isomorphic to 7.2, otherwise 7.3.

If the remaining nonzero lattice point of P_i is a double point, then one shows similarily that P is even uniquely determined as a quasi-smooth Fano polytope with eight vertices consisting of a prisma with two stacked simplices; P is isomorphic to 8.2.

4a: Because by 4.3.5 P then consists of six symmetric vertices, P would be an octahedron, which is simplicial. A contradiction.

4b: By 4.3.5 there exist $v_j, v_k \in \mathcal{V}(P)$ with $v_j + v_k = 0$, $\pi_i(v_j)$, $\pi_i(v_k)$ simple points and $v_l, v_r \in \partial_M(v_i)$ with $\pi_i(v_l + v_r) = \pi_i(v_k)$. By 4.2.15 $\{v_l, v_r\}$ is a quasi-primitive collection. Assume $v_l + v_r \in \mathcal{V}(P)$, then $v_l + v_r = v_k$, because $\pi_i(v_k)$ is a simple point. This means v_k is an AS-point, a contradiction. Therefore $[v_l, v_r]$ is a diagonal, by 4.2.15 $v_l + v_r = v_i + v_k$ without restriction (substitute $-v_i$ for v_i otherwise). Assume $\pi_i(v_l)$ were a double point. Then $v_l - v_i \in \mathcal{V}(P)$ and $(v_l - v_i) + v_r = v_k$, contradiction. The same is true for $\pi_i(v_r)$. So there are no double points in P_i and all six vertices in P are uniquely defined. P is isomorphic to 6.4.

5a: By 4.3.5 take an ordering v_j, v_r, v_t, v_k, v_l of $\partial_M(v_i)$ such that $v_j + v_k = 0$ and $v_l + v_r = 0$ and all points except possibly $\pi_i(v_t)$ are simple points of P_i.

Exactly as in the previous case one can assume that $v_l + v_t = v_i + v_k$ and $\pi_i(v_t)$ is a simple point. This yields 7.11.

6a: By 4.3.5 P is a completely symmetric polytope with eight vertices. Because P contains a smooth lattice polytope, by symmetry it follows immediately that P is the unique quasi-smooth Fano cube 8.21.

Finally it is easy to check that all these quasi-smooth Fano polytopes are uniquely determined by their quasi-primitive relations.

□

4.3.3 Classification when AS-points exist

In this subsection the following proposition will be proved which will yield together with Proposition 4.3.3 the proof of Theorem 4.3.2:

Proposition 4.3.6. *There exist exactly* 83 *quasi-smooth Fano polytopes having an AS-point. Of these are* 10 *smooth and* 73 *singular. They are uniquely determined by the quasi-primitive relations associated to their quasi-primitive collections.*

The second part and the main idea of the proof is contained in the key result Proposition 4.2.17.

Now the classification of singular quasi-smooth Fano polytopes that have an AS-point is split up into four Lemmas 4.3.9 to 4.3.14 depending on the minimal degree of an AS-point. From the proofs of these lemmas the coordinates of the vertices of the polytopes can be immediately read off by choosing v_r, v_j, v_i (as in 4.2.17) as a $\mathbb{Z}$-basis of the lattice.

Let P be always a quasi-smooth Fano polytope.

To simplify the proofs we need two definitions:

Definition 4.3.7. Let $v_i \in \mathcal{V}(P)$ and $v \in \mathcal{V}(P)$.

- We set
$$m_i(v) := |\,{\pi_i}^{-1}(\pi_i(v)) \cap \mathcal{V}(P)\,|.$$
For $v \in \partial_M(v_i)$ we get:
$m_i(v) = 1$, iff $\pi_i(v)$ is a simple point, and $m_i(v) = 2$, iff $\pi_i(v)$ is a double point. Hence 4.2.17(4) means exactly to determine $m_i(v)$ for all $v \in \partial_M(v_i)$.
- Assume v_i is symmetric. Then there is a unique vertex $b_i(v) \in \partial_M(-v_i)$ such that $\pi_i(b_i(v)) = \pi_i(v)$. For $v \in \partial_M(v_i)$ we get:
If $m_i(v) = 1$, then $b_i(v) = v$; otherwise $b_i(v) = v - v_i$ by 4.2.15.

The following technical lemma that will be subsequently used shows how to deduce in some cases the combinatorics of $\mathrm{st}(-v_i)$ from $\mathrm{st}(v_i)$.

Lemma 4.3.8. *Let $v_i \in \mathcal{V}(P)$ be a symmetric vertex of P. Also let $v_l, v_j, v_r \in \partial_M(v_i)$ such that v_l, v_j and v_j, v_r are contiguous. We set*
$$x := (m_i(v_l), m_i(v_j), m_i(v_r)) \in \{0, 1\}^3.$$

1. *Let $[v_i, v_j]$ be an edge of P and $v_l + v_r = v_j$ a quasi-primitive relation.*
Then $x \neq (2, 1, 2)$. $\mathrm{conv}(-v_i, b_i(v_l), b_i(v_j), b_i(v_r))$ is a parallelogram facet of P iff $x \in \{(1, 1, 2), (2, 1, 1), (2, 2, 2)\}$.

2. *Let $[v_i, v_j]$ be an edge of P and $v_l + v_r = v_j - v_i$ a quasi-primitive relation.*

   If $m_i(v_j) = 1$, then $x = (1, 1, 1)$ and $\operatorname{conv}(-v_i, v_l, v_j, v_r)$ is a parallelogram facet of P.

   If $m_i(v_j) = 2$, then $x \neq (2, 2, 2)$. $\operatorname{conv}(-v_i, b_i(v_l), v_j - v_i, b_i(v_r))$ is a parallelogram facet of P iff $x \in \{(2, 2, 1), (1, 2, 2)\}$.

3. *Let $\pi_i(v_l) + \pi_i(v_r) = \pi_i(v_j)$. Then $x = (2, 1, 2)$ iff $\operatorname{conv}(v_i, v_l, v_j, v_r)$ and $\operatorname{conv}(-v_i, b_i(v_l), b_i(v_j), b_i(v_r))$ are parallelogram facets of P.*

4. *Let $\operatorname{conv}(v_i, v_l, v_j, v_r)$ be a parallelogram facet of P. Then there exists no quasi-primitive relation $v_t + v_l = v_r - v_i$ for a vertex $v_t \in \mathcal{V}(P)$.*

Proof. 1.,2.,3. are easy consequences of 4.2.8, 4.2.11 and 4.2.15, so a proof will be omitted.

4. Assume $v_t + v_l = v_r - v_i$ were a quasi-primitive relation. If $m_i(v_r) = 1$, then $\operatorname{conv}(-v_i, v_t, v_r, v_l)$ were a parallelogram facet containing $-v_i$, so by 4.2.11(3) v_l would be contagious to v_r; a contradiction. So let $m_i(v_r) = 2$, then by 4.2.8 $v_l \in \operatorname{st}(v_r - v_i)$. By 4.2.15(2) it follows that $[v_l, v_r - v_i]$ is a diagonal of a parallelogram facet containing $-v_i$, also $v_l \in \partial_M(-v_i)$, so $m_i(v_l) = 1$; this is a contradiction to 3.

□

Finally the four lemmas will be proved.

Lemma 4.3.9. *There exist exactly* 34 *quasi-smooth Fano polytopes having an AS-point of degree* ≤ 4*. Of these are* 10 *smooth and* 24 *singular.*

Proof. By [Bat99, 2.5] and 4.3.3 there are exactly 10 smooth Fano polytopes left having AS-points, by 4.2.16 necessarily of degree 4. So let P be singular. Let $v_i \in \mathcal{V}(P)$ be an AS-point (as in 4.2.17(1)) with $\deg(v_i) = 4$ and $v_j + v_k = v_i$ a quasi-primitive relation for $v_j, v_k \in \mathcal{V}(P)$. Choose also v_l, v_r as in 4.2.17(2) with quasi-primitive relation $\mathcal{P}$. By 4.2.16 one can distinguish without restriction the following two cases (corresponding to 4.2.17(3)):

CASE I: $[v_i, v_j]$ and $[v_i, v_k]$ are edges of P.

By 4.2.8(1) and 4.2.13 there are without restriction the following possibilities for $\mathcal{P}$ left (corresponding to 4.2.17(2)):

Subcase (a): $v_l + v_r = 0$. (P_i is of type 4a.) In order for P not to be simplicial there have to be at least two contagious lattice points of P_i by 4.2.13 and 4.2.15, so let without restriction $m_i(v_j) = 2 = m_i(v_r)$. By 4.2.15(3) then $m_i(v_l) = 1$. The following two cases for the determination of double points of P_i (corresponding to 4.2.17(4)) are now only left:

$m_i(v_k)$	type of P
1	8.4
2	9.17

Subcase (b): $v_l + v_r = v_i$. (P_i is of type 4a.) For P not being simplicial let again without restriction $m_i(v_j) = 2$ and $m_i(v_r) = 2$. By symmetry also $m_i(v_k) \geq m_i(v_l)$. The following cases are then only left:

$m_i(v_k)$	$m_i(v_l)$	type of P
1	1	8.3
2	1	$-v_i$ AS-pt: case I(a)
2	2	10.1

Subcase (c): $v_l + v_r = -v_i$. (P_i is of type 4a.) By 4.2.15(3) $m_i(v_l) = 1 = m_i(v_r)$. Then P must be simplicial, contradiction.

Subcase (d): $v_l + v_r = v_j$. (P_i is of type 4b.) By symmetry let $m_i(v_r) \geq m_i(v_l)$. In order for P to have a parallelogram facet, the following cases are only left by 4.3.8(1):

$m_i(v_j)$	$m_i(v_k)$	$m_i(v_r)$	$m_i(v_l)$	type of P
1	1	2	1	7.8
1	2	2	1	8.8
2	1	2	1	8.14
2	1	2	2	9.8
2	2	2	1	9.18
2	2	2	2	10.4

Subcase (e): $v_l + v_r = v_j - v_i \in \mathcal{V}(P)$. ($P_i$ is of type 4b.) Especially $m_i(v_j) = 2$. By symmetry let $m_i(v_r) \geq m_i(v_l)$. In order for P to have a parallelogram facet, the following cases are only left by 4.3.8(2):

$m_i(v_k)$	$m_i(v_r)$	$m_i(v_l)$	type of P
1	2	1	8.11
2	2	1	9.14

Subcase (f): $v_l + v_r = v_j + (v_j - v_i)$ is the parallelogram relation of a facet. (P_i is of type 4c.) Especially $m_i(v_j) = 2$, and by 4.2.15(2) $m_i(v_l) = 1 = m_i(v_r)$. So the following two cases are left:

$m_i(v_k)$	type of P
1	7.1
2	8.1

Subcase (g): $v_l + v_r = v_j - v_i$ is the parallelogram relation of a facet. (P_i is of type 4b.) By 4.2.15(1) especially $m_i(v_j) = m_i(v_l) = m_i(v_r) = 1$. So the following two cases are left:

$m_i(v_k)$	type of P
1	6.3
2	7.14

Subcase (h): $v_l + v_r = (v_j - v_i) - v_i$ is the parallelogram relation of a facet. (P_i is of type 4b.) By 4.2.15(1) especially $m_i(v_j) = 2$, $m_i(v_l) = m_i(v_r) = 1$. So the following two cases are left:

$m_i(v_k)$	type of P
1	7.7
2	v_j AS-pt: case I(f)

CASE II: $[v_i, v_j]$ is an edge of P and $\mathrm{conv}(v_i, v_r, v_k, v_l)$ is a parallelogram facet with diagonals $[v_i, v_k]$ and $[v_l, v_r]$. (P_i is of type 4b.)

Then all vertices in $\partial_M(v_i)$ are determined. Let $m_i(v_r) \geq m_i(v_l)$. If $m_i(v_j) = 2 = m_i(v_k)$, then $-v_i$ is an AS-point of case I because of 4.3.8(3). So the following possibilities are left:

$m_i(v_j)$	$m_i(v_k)$	$m_i(v_r)$	$m_i(v_l)$	type of P
1	1	1	1	6.1
1	1	2	1	7.5
1	1	2	2	8.9
2	1	1	1	$v_j - v_i$ AS-pt: case I
2	1	2	1	8.7
2	1	2	2	9.3
1	2	1	1	$v_k - v_i$ AS-pt: case I (by 4.3.8(3))
1	2	2	1	8.5
1	2	2	2	9.6

□

Lemma 4.3.10. *There exist exactly* 37 *quasi-smooth Fano polytopes having an AS-point of minimal degree* 5. *They are all singular.*

Proof. Let $v_i = v_j + v_k$ be an AS-point as in 4.3.9 and $\{v_l, v_r\}$ the quasi-primitive collection with $[v_l, v_i]$, $[v_i, v_r]$ edges of P.

Distinguish two cases:

CASE I: $[v_i, v_j]$ and $[v_i, v_k]$ are edges of P.

Without restriction by 4.2.16 let $\mathrm{conv}(v_i, v_j, v_r)$ be a facet of P and $\mathrm{conv}(v_i, v_r, v_r - v_j, v_k)$ a parallelogram facet of P. By 4.2.15(2) $\{v_l, v_r - v_j\}$ is also a quasi-primitive collection.

Subcase (a): $v_l + v_r = 0$. (P_i is of type 5a.) Then $v_j + (v_r - v_j) = v_r$ and $v_l + (v_r - v_j) = v_k - v_i$. So by 4.2.15(3) and 4.3.8(1,2,3) there are the following cases left:

$m_i(v_j)$	$m_i(v_r)$	$m_i(v_r - v_j)$	$m_i(v_k)$	$m_i(v_l)$	type of P
1	1	1	1	1	7.12
2	1	1	1	1	8.12
1	2	1	1	1	8.13
2	2	1	1	1	9.11
·	1	1	2	1	$v_k - v_i$ AS-pt: deg. 4
1	2	1	2	1	9.15
2	2	1	2	1	10.12
1	1	2	2	1	9.4
1	2	2	2	1	10.5
2	2	2	2	1	11.1
1	1	1	2	2	9.13
2	1	1	2	2	10.10

Subcase (b): $v_l + v_r = v_i$. (P_i is of type 5a.) By symmetry let $m_i(v_r) \geq m_i(v_k)$, and $m_i(v_j) \geq m_i(v_l)$, if $m_i(v_r) = m_i(v_k)$. It is $v_j + (v_r - v_j) = v_r$ and $v_l + (v_r - v_j) = v_k$. By 4.3.8(1,3) the following cases are then only left:

$m_i(v_j)$	$m_i(v_r)$	$m_i(v_r - v_j)$	$m_i(v_k)$	$m_i(v_l)$	type of P
1	1	1	1	1	7.9
2	1	1	1	1	v_k AS-pt: case I(a)
2	1	1	1	2	9.5
1	1	2	1	1	8.18
1	2	1	1	$\cdot$	$v_r - v_i$ AS-pt: deg. 4
2	2	1	1	1	$v_j - v_i$ AS-pt: case I(a)
2	2	1	1	2	10.11
1	2	2	1	1	9.10
2	2	2	1	1	10.7
1	2	1	2	1	9.16
2	2	1	2	1	$v_j - v_i$ AS-pt: case I(a)
2	2	1	2	2	11.4
1	2	2	2	1	10.9
2	2	2	2	1	11.2
2	2	2	2	2	12.1

Subcase (c): $v_l + v_r = -v_i$. (P_i is of type 5a) By 4.2.15(3) $m_i(v_l) = 1 = m_i(v_r)$. Then $v_l + (v_r - v_j) = (v_k - v_i) - v_i$. Therefore $m_i(v_k) = 2$ and $m_i(v_r - v_j) = 1$. Then there are the following cases left:

$m_i(v_j)$	type of P
1	$v_k - v_i$ AS-pt: deg. 4
2	v_j AS-pt: deg. 4

Subcase (d): $v_l + v_r = v_j$. (P_i is still of type 5a.) Then $v_l + (v_r - v_j) = 0$. By 4.2.15(3), 4.3.8(1,3) the following cases are left:

$m_i(v_j)$	$m_i(v_r)$	$m_i(v_r - v_j)$	$m_i(v_k)$	$m_i(v_l)$	type of P
1	1	1	1	1	7.4
1	1	2	1	1	$v_r - v_j$ AS-pt: deg. 4
$\cdot$	1	1	2	1	v_j AS-pt: deg. 4
1	1	2	2	1	v_j AS-pt: case I(b)
2	1	1	1	1	8.15
1	2	1	1	1	8.17
1	2	2	1	1	9.9
1	2	1	2	1	$v_k - v_i$ AS-pt: case I(a)
1	2	2	2	1	10.6
2	2	1	1	1	$v_j - v_i$ AS-pt: case I(b)
2	2	2	1	1	$-v_i$ AS-pt: case I(a)
2	2	$\cdot$	2	1	v_j AS-pt: case I(b)
1	1	1	1	2	v_l AS-pt: deg. 4
1	1	1	2	2	9.12
2	1	1	$\cdot$	2	$-v_i$ AS-pt: case I(a/b)
2	2	1	1	2	again 10.6
2	2	1	2	2	11.3

Subcase (e): $v_l + v_r = v_j - v_i \in \mathcal{V}(P)$. ($P_i$ is of type 5a.) Especially $m_i(v_j) = 2$. Then $v_l + (v_r - v_j) = -v_i$, so by 4.2.15(3) $m_i(v_l) = 1 = m_i(v_r - v_j)$. So it follows by 4.3.8(2):

$m_i(v_r)$	$m_i(v_k)$	type of P
1	1	$-v_i$ AS-pt: case I(a)
2	1	$-v_i$ AS-pt: case I(d)
1	2	$-v_i$ AS-pt: case I(b)
2	2	10.2

Subcase (f): $v_l + v_r = v_k$. (P_i is of type 5b.) Then by 4.2.15(2) $m_i(v_k) = 2$, $m_i(v_l) = 1 = m_i(v_r - v_j)$. By 4.3.8(1,3) it follows:

$m_i(v_j)$	$m_i(v_r)$	type of P
$\cdot$	1	$v_k - v_i$ AS-pt: deg. 4
1	2	9.7
2	2	v_j AS-pt: case I(e)

Subcase (g): $v_l + v_r = v_k - v_i$ is a quasi-primitive relation. This is a contradiction to 4.3.8(4).

Subcase (h): $v_l + v_r = v_j - v_i$ is the parallelogram relation of a facet. (P_i is of type 5a.) Especially $m_i(v_l) = m_i(v_j) = m_i(v_r) = 1$. Then $v_l + (v_r - v_j) = -v_i$, so also by 4.2.15(3) $m_i(v_r - v_j) = 1$. So the following two cases are left:

$m_i(v_k)$	type of P
1	7.10
2	$v_k - v_i$ AS-pt: deg. 4

Subcase (i): $v_l + v_r = v_j + (v_j - v_i)$ is the parallelogram relation of a facet. (P_i is of type 5b.) By 4.2.15(2) especially $m_i(v_j) = 2$, $m_i(v_l) = m_i(v_r) = 1$. Then $v_l + (v_r - v_j) = v_j - v_i$, so also $v_r - v_j \in \partial_M(-v_i)$ and $m_i(v_r - v_j) = 1$ by 4.2.15(1). It follows:

$m_i(v_k)$	type of P
1	8.6
2	$-v_i$ AS-pt: case I(f)

CASE II: $[v_i, v_j]$ is an edge of P and $\text{conv}(v_i, v_r, v_k, v_l)$ is a parallelogram facet with diagonals $[v_i, v_k]$ and $[v_l, v_r]$. (P_i is of type 5a.)

By 4.2.16 let without restriction $\text{conv}(v_i, v_j, v_r - v_k, v_r)$ be a parallelogram facet of P. Then all vertices in $\partial_M(v_i)$ are determined. By symmetry (along $[v_i, v_r]$) let $m_i(v_j) \geq m_i(v_l)$, and $m_i(v_r - v_k) \geq m_i(v_k)$, if $m_i(v_j) = m_i(v_l)$. If $m_i(v_j) = 2 = m_i(v_k)$ or $m_i(v_l) = 2 = m_i(v_r - v_k)$, then $-v_i$ is an AS-point of case I because of 4.3.8(3). So by 4.3.8(1,3) the following possibilities are left:

$m_i(v_j)$	$m_i(v_r - v_k)$	$m_i(v_r)$	$m_i(v_k)$	$m_i(v_l)$	type of P
1	1	1	1	1	7.6
1	1	2	1	1	8.16
1	2	·	1	1	$v_r - v_k - v_i$ AS-pt: case I
1	2	2	2	1	10.3
2	1	1	1	1	$v_j - v_i$ AS-pt: deg. 4
2	1	2	1	1	$v_j - v_i$ AS-pt: case I
2	2	·	1	1	$v_j - v_i$ AS-pt: case I
2	1	1	1	2	$v_j - v_i$ AS-pt: case I
2	1	2	1	2	10.8

□

Lemma 4.3.11. *There exist exactly* 12 *quasi-smooth Fano polytopes having an AS-point of minimal degree* 6*. They are all singular.*

Proof. Let $v_i = v_j + v_k$ be an AS-point of degree 6 with associated quasi-primitive relation, and $\{v_l, v_r\}$ the unique quasi-primitive collection with $[v_i, v_l]$ and $[v_i, v_r]$ edges of P. By 4.2.16 we will distinguish the following cases:

CASE I: There are 3 parallelogram facets of P containing v_i.

Let $v_j, v_r - v_k, v_r, v_k, v_l = v_i + v_k - v_r, v_i - v_r$ be an ordering of $\partial_M(v_i)$ according to 4.2.16(2). (P_i is of type 6a.) Let $W := \{\pi_i(v_r - v_k), \pi_i(v_k), \pi_i(v_i - v_r)\}$.

Subcase (a): There exists an element $w \in W$ with w and $-w$ double points. Without restriction let $m_i(v_k) = 2$ and $m_i(v_j) = 2$. Then v_k is an AS-point, having necessarily degree ≥ 6, it follows $m_i(v_l) = 2 = m_i(v_r)$. Without restriction there are now three cases:

$m_i(v_r - v_k)$	$m_i(v_i - v_r)$	type of P
1	1	12.2
2	1	13.1
2	2	14.1

Subcase (b): There exists an element $w \in W$ with w double point and $-w$ simple point, but no element in W as in subcase (a). Without restriction let $m_i(v_k) = 2$ and $m_i(v_j) = 1$. Also let $m_i(v_r) \geq m_i(v_l)$ and $m_i(v_r - v_k) \geq m_i(v_i - v_r)$, if $m_i(v_r) = m_i(v_l)$. By 4.3.8(1) there are now the following cases:

$m_i(v_r - v_k)$	$m_i(v_r)$	$m_i(v_l)$	$m_i(v_i - v_r)$	type of P
1	1	1	1	9.1
1	2	·	1	$v_r - v_i$ AS-pt: deg. 5
2	2	1	1	11.5

Subcase (c): All elements in W are simple points, i.e. $m_i(v_r - v_k) = 1$, $m_i(v_k) = 1$, $m_i(v_i - v_r) = 1$. Without restriction there are the following cases:

$m_i(v_j)$	$m_i(v_r)$	$m_i(v_l)$	type of P
1	1	1	8.19
2	·	1	$v_j - v_i$ AS-pt: deg. ≤ 5
2	2	2	11.6

CASE II: There are exactly 2 parallelogram facets of P containing v_i and there is no AS-point of the type as in case I.

Here the following sublemma is useful:

Sublemma 4.3.12. *In this situation let there be nonzero lattice points $w, w' \in M_i \cap P_i$ such that $w + w' = 0$. Then w is not the middle point of a facet of P_i with three lattice points.*

Let $v, v' \in \partial_M(v_i)$ with $\pi_i(v) = w$, $\pi_i(v') = w'$. If additionally w is a double point and $[v_i, v]$ a diagonal of a parallelogram facet of P, then v is not symmetric, $v + v' = v_i$, w' is a simple point and $v - v_i$ is an AS-point.

Proof. First three remarks:

(i) Assume there were an AS-point $v_t \in \mathcal{V}(P)$ such that $v_t = v_p + v_q$ with $[v_t, v_p]$ a diagonal of a parallelogram facet of P. By 4.2.16(2) then v_t would have degree ≤ 5 or degree 6 with 3 parallelogram facets containing v_t as in case I ; a contradiction.

(ii) Let now w be a double point and assume v were symmetric. Then $v = v_i + (v - v_i)$ were an AS-point. So by (i) $[v_i, v]$ or $[v, v - v_i]$ must not be diagonals.

(iii) Finally if w is a double point, then by 4.2.15(3) $v + v' \in \{0, v_i\}$.

Now put these observations together:

Assume w is the middle point of a facet of P_i with three lattice points. By 4.2.15(2) w is a double point and $[v, v - v_i]$ a diagonal. So by (ii) v must not be symmetric, therefore by (iii) $v + v' = v_i$. Then $v - v_i$ is an AS-point with $[v - v_i, v]$ a diagonal; a contradiction to (i). Now let w be a double point and $[v, v_i]$ a diagonal, so by (ii) v is not symmetric, especially by (iii) $v + v' = v_i$ and $v' - v_i \notin \mathcal{V}(P)$, i.e. w' simple.

□

As an immediate corollary we get:

Sublemma 4.3.13. *P_i does not contain facets with three lattice points, especially P_i is of type 6a.*

Proof. By 4.2.13 P_i contains facets with three lattice points only if P_i is of type 6b or 6c. But in these cases P_i also has symmetric middle points of facets with three lattice points; this is not possible by the previous sublemma.

□

There are now two cases:

CASE IIA: The two parallelogram facets containing v_i are contained in one halfspace of the plane $\mathbb{R}v_j + \mathbb{R}v_i$.

Let also v_r be element of this halfspace. Then $v_j, v_r - v_k, v_r, v_r - v_j, v_k, v_l$ is an ordering of $\partial_M(v_i)$ as in 4.2.11(3) and $\pi_i(v_r)$ is the middle point on a facet of P_i with three lattice points; a contradiction to Sublemma 4.3.13.

CASE IIB: The two parallelogram facets containing v_i are not contained in one halfspace of the plane $\mathbb{R}v_j + \mathbb{R}v_i$.

Without restriction let $\text{conv}(v_i, v_j, v_r - v_k, v_r)$ be a parallogram facet. There are now two further cases to consider:

Case IIB1: $\text{conv}(v_i, v_l, v_l - v_k, v_j)$ is a parallellogram facet of P.

Then $v_j, v_r - v_k, v_r, v_k, v_l, v_l - v_k$ is an ordering of $\partial_M(v_i)$ as in 4.2.11(3). By Sublemma 4.3.13 we must have $\pi_i(v_l + v_r) = \pi_i(v_k)$.

Subcase (a): $v_l + v_r = v_k$. Then $(v_l - v_k) + v_r = 0$, so by Sublemma 4.3.12 $m_i(v_l - v_k) = 1$. Analogously $m_i(v_r - v_k) = 1$. Let without restriction $m_i(v_r) \leq m_i(v_l)$. By 4.3.8(1) there are the following cases:

$m_i(v_j)$	$m_i(v_r)$	$m_i(v_k)$	$m_i(v_l)$	type of P
$\cdot$	$\cdot$	1	$\cdot$	v_l AS-pt: deg. 5
1	1	2	1	9.2
1	1	2	2	10.13
$\cdot$	2	2	2	$v_k - v_i$ AS-pt: case IIA
2	1	2	$\cdot$	v_k AS-pt: deg. ≤ 5

Subcase (b): $v_l + v_r = v_k - v_i$. Then $(v_l - v_k) + (v_r - v_k) = (v_j - v_i) - v_i$ is a quasi-primitive relation, so by 4.2.15(1) $m_i(v_l - v_k) = 1$, $m_i(v_j) = 2$, $m_i(v_r - v_k) = 1$ and $v_j - v_k$ is an AS-point of degree 4.

Subcase (c): $v_l + v_r = (v_k - v_i) - v_i$ is the parallelogram relation of a facet of P. Then $(v_l - v_k) + (v_r - v_k) = (v_j - v_i) - 2v_i$ is a quasi-primitive relation; a contradiction.

Case IIB2: $\operatorname{conv}(v_k, v_l - v_j, v_j, v_l, v_j)$ is a parallellogram facet of P.

Then $v_j, v_r - v_k, v_r, v_k, v_l - v_j, v_l$ is an ordering of $\partial_M(v_i)$ as in 4.2.11(3). By Sublemma 4.3.13 we must have $\pi_i(v_l + v_r) = 0$.

Subcase (a): $v_l + v_r = 0$. By 4.2.15(3) without restriction $m_i(v_r) = 1$. It is $(v_l - v_j) + (v_r - v_k) = -v_i$, so $m_i(v_l - v_j) = 1 = m_i(v_r - v_k)$ by 4.2.15(3). Then by 4.3.8(1,2) there are only the following cases:

$m_i(v_j)$	$m_i(v_k)$	$m_i(v_l)$	type of P
1	1	1	8.20
2	1	$\cdot$	$v_j - v_i$ AS-pt: deg. 5
$\cdot$	2	1	$v_k - v_i$ AS-pt: deg. 5
$\cdot$	2	2	$v_k - v_i$ AS-pt: case IIA

Subcase (b): $v_l + v_r = v_i$. Then $(v_l - v_j) + (v_r - v_k) = 0$, so $m_i(v_l - v_j) = 1 = m_i(v_r - v_k)$ by Sublemma 4.3.12. By symmetry there are only the following cases to consider:

$m_i(v_j)$	$m_i(v_r)$	$m_i(v_k)$	$m_i(v_l)$	type of P
1	1	1	1	8.10
2	1	$\cdot$	$\cdot$	$v_j - v_i$ AS-pt: deg. 5
2	2	$\cdot$	1	$v_j - v_i$ AS-pt: case IIB1
2	2	2	2	12.3

Subcase (c): $v_l + v_r = -v_i$. Then $(v_l - v_j) + (v_r - v_k) = -2v_i$; a contradiction. $\square$

Lemma 4.3.14. *There exist no quasi-smooth Fano polytopes having an AS-point of minimal degree ≥ 7.*

Proof. Let $v_i \in \mathcal{V}(P)$ be an AS-point of degree ≥ 7. Because of 4.2.13 and 4.2.15(2) there exists a symmetric double point $w \in P_i$ on the middle of a facet of P_i. Because of 4.2.15(3) there exists an AS-point $v \in \mathcal{V}(P)$ with $\pi_i(v) = w$. By 4.2.15(2) and 4.2.16(2) therefore v is an AS-point of degree ≤ 6. $\square$

4.4 Table of quasi-smooth Fano polytopes

Throughout the section let P be a quasi-smooth Fano polytope with toric variety $X = X(M, \Sigma_P)$.

In the tables below all three-dimensional Gorenstein toric Fano varieties with terminal singularities are listed by their corresponding quasi-smooth Fano polytopes. Also shown are their interesting numerical characteristics.

An obvious combinatorial invariant of a quasi-smooth Fano polytope P is the *f-vector* (f_0, f_1, f_2) giving the number of vertices, edges and facets. In the following let always $n = f_0$ be the number of vertices and p the number of parallelogram facets of P.

Proposition 4.4.1. *Let P be a quasi-smooth Fano polytope with n vertices and p parallelogram facets. Then*

$$f_1 = 3n - 6 - p, \quad f_2 = 2n - 4 - p, \quad \mathrm{vol}(P) = \frac{1}{3}n - \frac{2}{3}.$$

Proof. Let s be the number of simplicial facets of P. Then the statements follow from solving these three equations for s, f_1, f_2:

$$f_1 = \frac{3s + 4p}{2},\ f_2 = s + p,\ n - f_1 + f_2 = 2,$$

and 3.7.5. Alternatively use 3.7.6.

□

Next it shall be described how to determine the Picard number ρ_X of these varieties by calculating the combinatorial Picard number of the associated quasi-smooth Fano polytope P, see [Ewa96, V.5]. In this special case this leads to the following:

Proposition 4.4.2. *Let P be a quasi-smooth Fano polytope with n vertices $\{v_1, \ldots, v_n\}$ and p parallelogram facets $\{F_1, \ldots, F_p\}$. Then*

$$\mathrm{Pic}(X) \cong \mathbb{Z}^{n-3-\lambda(P)},\ \ \rho_X = n - 3 - \lambda(P),$$

for

$$\lambda(P) := \mathrm{rank}(M(P)),$$

where $M(P)$ is a matrix with n columns and p rows, where each row consists exactly of two entries 1, two entries -1 and the remaining entries 0, such that $[v_{j_1}, v_{j_2}]$ is a diagonal of F_i iff $M(P)_{i,i_1} = M(P)_{i,i_2} \neq 0$.

Especially the Picard number of a Gorenstein toric Fano variety with terminal singularities does only depend on the combinatorial *structure of its associated quasi-smooth Fano polytope.*

Proof. Define for a facet F of P the affine relation space

$$AR(F) := \{\alpha \in \mathbb{Q}^n \ \mid \ \sum_{v_j \in F} \alpha_j v_j = 0,\ \sum_{v_j \in F} \alpha_j = 0, \ \text{ and } \alpha_j = 0 \text{ if } v_j \notin F\}.$$

If F is simplicial, $AR(F) = 0$ obviously. If F is a parallelogram facet with diagonals $[v_{j_1}, v_{j_3}]$, $[v_{j_2}, v_{j_4}]$, then $AR(F) = \mathbb{Z}(e_{j_1} + e_{j_3} - e_{j_2} - e_{j_4}) = \mathbb{Z}(e_{j_2} + e_{j_4} - e_{j_1} - e_{j_3})$ as is easily seen from the definition.

Now the proposition follows immediately from [Eik93, Thm. 4.1], which states:

$$\mathrm{Pic}(X) \cong \mathbb{Z}^{n-3-\dim \sum_{\text{facets } F \text{ of } P} AR(F)}.$$

□

Remark 4.4.3. One should remark that instead of mechanically computing the rank of the matrix $M(P)$ it is very often possible to directly derive the value of $\lambda(P)$ from the combinatorial structure of the parallelogram facets of the polytope P *by identifying rows with parallelograms.* This means that certain operations on the rows correspond to some on the parallelograms, starting with the set of parallelogram facets of P. The following guidelines (i) to (iii) give, when recursively used, in most cases a basis of parallelograms, i.e. the corresponding rows in $M(P)$ form a basis of the row space of $M(P)$, thereby giving $\lambda(P)$.

(i) If there is a vertex contained in only one parallelogram F, calculate a basis of the row space without F and append F to get a basis of the whole row space.

(ii) If there is a cycle of parallelograms, i.e. a closed sequence of parallelograms such that each pair of adjacent ones has exactly one common edge and any two of these common edges have empty intersection, then removing any parallelogram gives a basis of the corresponding row space.

(iii) If there are parallelograms $F = \mathrm{conv}(a, b, c, d)$ and $F' = \mathrm{conv}(a', b', c', d')$ with the common edge $[b, c] = [a', d']$ for $b = a'$ and $c = d'$, then the associated row space is also generated by F and $F'' := \mathrm{conv}(a, b', c', d)$. This has the advantage that F, F' can be substituted by F, F'', where for instance b is contained in one parallelogram less. It is important to note that F'' is in general only a lattice parallelogram but no facet of P any more. But this is irrelevant for the continuing calculation of $\lambda(P)$.

Example 4.4.4. Let P as usual a quasi-smooth Fano polytope.

1. If $p = 0$, $\rho_X = n - 3$ as is expected for a nonsingular toric variety.

2. If $p \leq 2$, then $\lambda(P) = p$. If $p = 3$, then $\lambda(P) = 3$ except when the parallelogram facets of P form a prisma. In this case $\lambda(P) = 2$ by 4.4.3(ii).

3. The cube 8.21 has $\lambda(P) = 4$. This follows from applying 4.4.3(ii) two times to get a basis of four parallelogram facets, as is immediately checked by 4.4.3(i). Therefore $\rho_X = 1$. The cube is simple, so this could have also been concluded from [Eik93, Thm. 4.6]. As can be seen from the classification 8.21 is the only quasi-smooth Fano polytope with parallelogram facets that is simple, because it is the only one to satisfy the equivalent equation $n = 2f_2 - 4$ (see [Eik93, Lemma 3.6]).

In the tables below now all quasi-smooth Fano polytopes P are listed with the following invariants: $n = f_0$ ist the number of vertices, p is the number of parallelograms facets and f_2 the number of facets of P, ρ_X the Picard number and $\deg(X) = (-K_X)^3$ the anticanonical degree of the Gorenstein toric Fano variety $X = X(M, \Sigma_P)$ with terminal singularities, here we have by (3.7)

$$\deg(X) = 3!\,\mathrm{vol}(P^*).$$

First the list of all *18 smooth Fano polytopes* (i.e. $p = 0$) in the notation of [Bat99]:

n^0	n	f_2	ρ_X	$\deg(X)$
$\mathbb{P}^3$	4	4	1	64
$\mathcal{B}_1$	5	6	2	62
$\mathcal{B}_2$	5	6	2	56
$\mathcal{B}_3$	5	6	2	54
$\mathcal{B}_4$	5	6	2	54
$\mathcal{C}_1$	6	8	3	52
$\mathcal{C}_2$	6	8	3	50
$\mathcal{C}_3$	6	8	3	48
$\mathcal{C}_4$	6	8	3	48
$\mathcal{C}_5$	6	8	3	44
$\mathcal{D}_1$	6	8	3	50
$\mathcal{D}_2$	6	8	3	46
$\mathcal{E}_1$	7	10	4	46
$\mathcal{E}_2$	7	10	4	44
$\mathcal{E}_3$	7	10	4	42
$\mathcal{E}_4$	7	10	4	40
$\mathcal{F}_1$	8	12	5	36
$\mathcal{F}_2$	8	12	5	36

Now the list of all *82 quasi-smooth Fano polytopes with* $p > 0$, here the last column gives a reference to Proposition 4.3.3, if there are no AS-points, or otherwise to the proof of one of the Lemmas 4.3.9, 4.3.10, 4.3.11, where the isomorphism type according to Prop. 4.2.17 has been described.

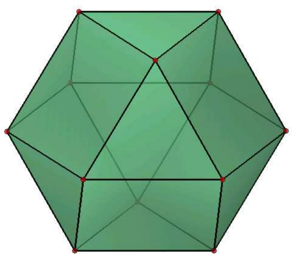
n^0	n	p	f_2	ρ_X	$\deg(X)$	type of P
5.1	5	1	5	1	54	4.3.3
6.1	6	1	7	2	54	4.3.9(II)
6.2	6	1	7	2	46	4.3.3
6.3	6	1	7	2	46	4.3.9(I(g))
6.4	6	1	7	2	48	4.3.3
7.1	7	1	9	3	48	4.3.9(I(f))
7.2	7	1	9	3	40	4.3.3
7.3	7	1	9	3	38	4.3.3
7.4	7	1	9	3	42	4.3.10(I(d))
7.5	7	1	9	3	46	4.3.9(II)
7.6	7	2	8	2	46	4.3.10(II)
7.7	7	1	9	3	42	4.3.9(I(h))

n^0	n	p	f_2	ρ_X	deg(X)	type of P
7.8	7	1	9	3	42	4.3.9(I(d))
7.9	7	1	9	3	44	4.3.10(I(b))
7.10	7	2	8	2	38	4.3.10(I(h))
7.11	7	3	7	1	40	4.3.3
7.12	7	2	8	2	40	4.3.10(I(a))
7.13	7	3	7	2	40	4.3.3
7.14	7	1	9	3	38	4.3.9(I(g))
8.1	8	1	11	4	36	4.3.9(I(f))
8.2	8	3	9	3	32	4.3.3
8.3	8	1	11	4	38	4.3.9(I(b))
8.4	8	1	11	4	36	4.3.9(I(a))
8.5	8	2	10	3	42	4.3.9(II)
8.6	8	3	9	2	40	4.3.10(I(i))
8.7	8	2	10	3	36	4.3.9(II)
8.8	8	2	10	3	34	4.3.9(I(d))
8.9	8	2	10	3	38	4.3.9(II)
8.10	8	2	10	3	36	4.3.11(IIB2(b))
8.11	8	2	10	3	38	4.3.9(I(e))
8.12	8	3	9	2	34	4.3.10(I(a))
8.13	8	2	10	3	36	4.3.10(I(a))
8.14	8	1	11	4	40	4.3.9(I(d))
8.15	8	2	10	3	38	4.3.10(I(d))
8.16	8	2	10	3	42	4.3.10(II)
8.17	8	2	10	3	38	4.3.10(I(d))
8.18	8	3	9	2	38	4.3.10(I(b))
8.19	8	3	9	2	38	4.3.11(I(c))
8.20	8	4	8	2	32	4.3.11(IIB2(a))
8.21	8	6	6	1	32	4.3.3
9.1	9	5	9	2	32	4.3.11(I(b))
9.2	9	5	9	2	30	4.3.11(IIB1(a))
9.3	9	4	10	2	30	4.3.9(II)
9.4	9	4	10	2	34	4.3.10(I(a))
9.5	9	4	10	2	30	4.3.10(I(b))
9.6	9	3	11	3	36	4.3.9(II)
9.7	9	3	11	3	38	4.3.10(I(f))
9.8	9	3	11	3	34	4.3.9(I(d))
9.9	9	3	11	3	34	4.3.10(I(d))
9.10	9	3	11	3	36	4.3.10(I(b))
9.11	9	3	11	3	32	4.3.10(I(a))
9.12	9	3	11	3	30	4.3.10(I(d))
9.13	9	3	11	3	32	4.3.10(I(a))
9.14	9	3	11	3	32	4.3.9(I(e))
9.15	9	2	12	4	34	4.3.10(I(a))
9.16	9	2	12	4	36	4.3.10(I(b))
9.17	9	2	12	4	32	4.3.9(I(a))
9.18	9	2	12	4	32	4.3.9(I(d))

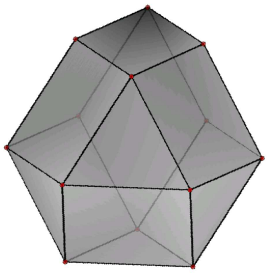
n^0	n	p	f_2	ρ_X	deg(X)	type of P
10.1	10	4	12	4	28	4.3.9(I(b))
10.2	10	4	12	3	30	4.3.10(I(e))
10.3	10	5	11	2	34	4.3.10(II)
10.4	10	5	11	3	28	4.3.9(I(d))
10.5	10	4	12	3	32	4.3.10(I(a))
10.6	10	4	12	3	30	4.3.10(I(d))
10.7	10	5	11	2	30	4.3.10(I(b))
10.8	10	5	11	3	28	4.3.10(II)
10.9	10	3	13	4	34	4.3.10(I(b))
10.10	10	5	11	2	28	4.3.10(I(a))
10.11	10	4	12	3	28	4.3.10(I(b))
10.12	10	3	13	4	30	4.3.10(I(a))
10.13	10	5	11	2	28	4.3.11(IIB1(a))
11.1	11	6	12	2	28	4.3.10(I(a))
11.2	11	5	13	3	28	4.3.10(I(b))
11.3	11	6	12	2	26	4.3.10(I(d))
11.4	11	5	13	3	26	4.3.10(I(b))
11.5	11	8	10	2	26	4.3.11(I(b))
11.6	11	6	12	2	26	4.3.11(I(c))
12.1	12	8	12	2	24	4.3.10(I(b))
12.2	12	7	13	3	24	4.3.11(I(a))
12.3	12	6	14	4	24	4.3.11(IIB2(b))
13.1	13	9	13	1	22	4.3.11(I(a))
14.1	14	12	12	2	20	4.3.11(I(a))

Remark 4.4.5. As will be seen from the list the only quasi-smooth Fano polytopes where also the polar polytope is quasi-smooth are the self-dual polytope 13.1 and the dual pair 12.3, 14.1. These polytopes are also the only one that satisfy the obstruction $\deg(X) = 2f_2 - 4$ as given in Prop. 4.4.1. By the way 14.1 is exactly $\mathcal{Z}_3$ (see 3.5.1) and 12.3 is hence $\mathcal{Z}_3^*$, that is, the inner polytope on the third page of this thesis. Here are the figures of these polytopes, visualized by the software package `polymake` [GJ00, GJ05]:

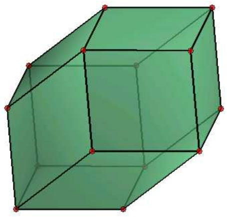
The quasi-smooth Fano polytope 12.3 ($\cong \mathcal{Z}_3^*$):



The quasi-smooth Fano polytope 13.1 (self-dual):



The quasi-smooth Fano polytope 14.1 ($\cong \mathcal{Z}_3$):



Chapter 5

The set of roots

Introduction

In this paper we study the set of roots that is essential for determining the automorphism group of a complete toric variety. Using these results we can give criteria for the automorphism group of a complete toric variety to be reductive.

Here one source of motivation comes from the following result (for the definition of an Einstein-Kähler metric see [BS99] or [WZ04]):

Theorem (Matsushima 1957). *If a nonsingular Fano variety X admits an Einstein-Kähler metric, then* $\mathrm{Aut}(X)$ *is a reductive algebraic group.*

In 1983 Futaki introduced the so called *Futaki character*, whose vanishing is another important necessary condition for the existence of an Einstein-Kähler metric. For a nonsingular toric Fano variety with reductive automorphism group there is an explicit criterion (see [Mab87, Cor. 5.5]):

Theorem (Mabuchi 1987). *Let X be a nonsingular toric Fano variety with* $\mathrm{Aut}(X)$ *reductive.*

The Futaki character of X vanishes if and only if the barycenter of P is zero, where P is the reflexive polytope with $X \cong X_P$ (see 3.1.5).

In [BS99, Thm. 1.1] Batyrev and Selivanova were able to give a sufficient criterion for the existence of an Einstein-Kähler metric:

Theorem (Batyrev/Selivanova 1999). *Let X be a nonsingular toric Fano variety. Let P be the reflexive polytope with $X \cong X_P$.*

If X is symmetric*, i.e., the group of lattice automorphisms leaving P invariant has no non-zero fixpoints, then X admits an Einstein-Kähler metric.*

In particular they got as a Corollary [BS99, Cor. 1.2] that the automorphism group of such a symmetric X is reductive. Expressed in combinatorial terms this just means that the set of lattice points in the relative interiors of facets of P is centrally symmetric. So they asked whether a direct proof of this result exists. Indeed there is even a generalization to complete toric varieties with a simple combinatorial proof (see Theorem 5.3.1(1) and Prop. 5.4.2):

Theorem. *Let X be a complete toric variety.*

If the group of automorphisms of the associated fan has no non-zero fixpoints, then $\mathrm{Aut}(X)$ *is reductive.*

Motivated by above results it was conjectured by Batyrev that in the case of a nonsingular toric Fano variety already the vanishing of the barycenter of the associated reflexive polytope were sufficient for the automorphism group to be reductive. Indeed there is even the following more general result that has a purely convex-geometric proof (see Theorem 5.3.1(2i)):

Theorem. *Let X be a Gorenstein toric Fano variety. Let P be the reflexive polytope with $X \cong X_P$.*

If the barycenter of P is zero, then $\mathrm{Aut}(X)$ *is reductive.*

Only very recently Xu-Jia Wang and Xiaohua Zhu could prove that the vanishing of the Futaki character is even sufficient for the existence of an Einstein-Kähler metric in the toric case (see [WZ04, Cor. 1.3]):

Theorem (Wang/Zhu 2004). *Let X be a nonsingular toric Fano variety.*

Then X admits an Einstein-Kähler metric if and only the Futaki character of X vanishes.

Combined with the previous results this yields a generalization of the above theorem of Mabuchi that is also implicit in [WZ04, Lemma 2.2]:

Corollary. *Let X be a nonsingular toric Fano variety.*

Then X admits an Einstein-Kähler metric if and only if the barycenter of P is zero, where P is the reflexive polytope with $X \cong X_P$.

It is now conjectured by Batyrev that this result may also hold in the singular case of a Gorenstein toric Fano variety.

Another source of motivation that orginated this research was the aim to give mathematical explanations for observations made by Batyrev, Kreuzer and the author in the computer database [KS04b] of 3- and 4-dimensional reflexive polytopes. Here one of the main results is a necessary condition for the automorphism group of a complete toric variety to be reductive that is given by the following sharp upper bound on the dimension (see Theorem 5.1.25):

Theorem. *Let X be a d-dimensional complete toric variety that is not a product of projective spaces.*

If $\mathrm{Aut}(X)$ *is reductive, then* $\dim \mathrm{Aut}(X) \begin{cases} = 2 & , \quad \textit{for } d = 2 \\ \leq d^2 - 2d + 4 & , \quad \textit{for } d \geq 3 \end{cases}$

This chapter is organized as follows:

The first section deals with the automorphism group $\mathrm{Aut}(X)$ of a d-dimensional complete toric variety X. Here the set of roots $\mathcal{R}$ plays a crucial part in determining the dimension and whether the group is reductive (see Prop. 5.1.3). Using results of Cox in [Cox95] we construct families of roots that parametrize the set of semisimple roots $\mathcal{S} := \mathcal{R} \cap -\mathcal{R}$ in a geometrically convenient way, these are called $\mathcal{S}$-root bases. As an application we show in Prop. 5.1.19 that X is isomorphic to a product of projective spaces if and only if there are d linearly independent semisimple roots. When $\mathrm{Aut}(X)$ is reductive, we obtain the bound $\dim \mathrm{Aut}(X) \leq d^2 + 2d$, with equality iff $X \cong \mathbb{P}^d$ (see 5.1.20). Moreover studying this approach in more detail we get in Prop. 5.1.22 the existence of some special families of roots that yields several restrictions on the set $\mathcal{R}$ (see 5.1.23 and 5.1.24). From this we can derive the above bound on $\dim \mathrm{Aut}(X)$ in Theorem 5.1.25.

In the second section we more closely examine the case of a d-dimensional Gorenstein toric Fano variety $X = X_P$ associated to a reflexive polytope P. Here a root of X is just a lattice point in the relative interior of a facet of P, so the results of the previous section have a direct geometric interpretation. For instance we obtain that P has at most $2d$ facets containing roots of P, with equality if and only if X is the product of d projective lines (see Corollary 5.2.4). Furthermore the intersection of P with the space spanned by all semisimple roots is a reflexive polytope associated to a product of projective spaces (see Theorem 5.2.12).

In the third section we present and discuss several equivalent and sufficient combinatorial criteria for the automorphism group of a complete toric variety, respectively a Gorenstein toric Fano variety, to be reductive (see Theorem 5.3.1).

In the fourth section we deal with the above mentioned notion of a symmetric toric variety, and sketch the proof of the classification of all three-dimensional symmetric reflexive polytopes (see Theorem 5.4.5).

The fifth section is concerned with an analogue of the notion of the Ehrhart polynomial. Here we do not count lattice points in multiples of lattice polytopes but we sum them. This yields a vector-polynomial and we determine the two heighest coefficients (see Prop. 5.5.2). Their vanishing is another strong sufficient condition for the automorphism group to be reductive (see Cor. 5.5.5).

In the last section we compare all these combinatorial conditions and give several examples.

The work in the last two sections was done in collaboration with M. Kreuzer.

Summary of most important new results of this chapter:

- A characterization of products of projective spaces (Prop. 5.1.19, p. 127)
- A sharp upper bound on the dimension of the reductive automorphism group of a complete toric variety (Thm. 5.1.25, p. 129)
- A d-dimensional reflexive polytope has at most $2d$ facets containing lattice points in their interior, with equality if and only if isomorphic to $[-1,1]^d$ (Corollary 5.2.4, p. 130)
- The intersection of a reflexive polytope with the space spanned by all semisimple roots is a reflexive polytope associated to a product of projective spaces (Theorem 5.2.12, p. 133).
- We give equivalent and sufficient combinatorial criteria for the automorphism group of a complete toric variety, respectively a Gorenstein toric Fano variety to be reductive; including the vanishing of the barycenter (Theorem 5.3.1, p. 134)
- There are up to isomorphism 31 three-dimensional symmetric reflexive polytopes (Theorem 5.4.5, p. 139).

5.1 The set of roots of a complete toric variety

In this section the set of roots of a complete toric variety is investigated, and some classification results and bounds on the dimension of the automorphism group are achieved.

Throughout the section let $\triangle$ be a complete fan in $N_{\mathbb{R}}$ with associated complete toric variety $X = X(N, \triangle)$.

Definition 5.1.1. Let $\mathcal{R}$ be the set of *Demazure roots* of $\triangle$, i.e.,

$$\mathcal{R} := \{m \in M \mid \exists\, \tau \in \triangle(1) : \langle v_\tau, m\rangle = -1 \text{ and } \forall\, \tau' \in \triangle(1)\backslash\{\tau\} : \langle v_{\tau'}, m\rangle \geq 0\}.$$

For $m \in \mathcal{R}$ we denote by η_m the unique primitive generator v_τ of the unique ray τ with $\langle v_\tau, m\rangle = -1$. For a subset $A \subseteq \mathcal{R}$ we define $\eta(A) := \{\eta_m \,:\, m \in A\}$.

Let $\mathcal{S} := \mathcal{R} \cap (-\mathcal{R}) = \{m \in \mathcal{R} \,:\, -m \in \mathcal{R}\}$ be the set of *semisimple* roots and $\mathcal{U} := \mathcal{R}\backslash\mathcal{S} = \{m \in \mathcal{R} \,:\, -m \notin \mathcal{R}\}$ the set of *unipotent* roots. We say that $\triangle$ is *semisimple*, if $\mathcal{R} = \mathcal{S}$, or equivalently $\mathcal{U} = \emptyset$.

Furthermore we define $\mathcal{S}_1 := \{x \in \mathcal{S} \,:\, \eta_x \notin \eta(\mathcal{U})\}$ and $\mathcal{S}_2 := \mathcal{S}\backslash\mathcal{S}_1$, analogously $\mathcal{U}_1 := \{x \in \mathcal{U} \,:\, \eta_x \notin \eta(\mathcal{S})\}$ and $\mathcal{U}_2 := \mathcal{U}\backslash\mathcal{U}_1$. In particular $\eta(\mathcal{S}_1) \cap \eta(\mathcal{S}_2) = \emptyset$ and $\eta(\mathcal{S}_2) = \eta(\mathcal{U}_2)$.

Usually the set $-\mathcal{R}$ is denoted as the set of Demazure roots (see [Oda88, Prop. 3.13]), however the sign convention here will turn out to be more convenient when considering normal fans of polytopes. Note that $\mathcal{R}$ only depends on the set of rays $\triangle(1)$.

Here is a direct combinatorical proof of a well-known fact:

Proposition 5.1.2. $|\mathcal{R}| < \infty$.

Proof. Since $\triangle$ is complete, the origin is in the interior of the convex hull of $\{v_\tau\}_{\tau \in \triangle(1)}$. By Steinitz's theorem 1.5.2 we find a subset $I \subseteq \triangle(1)$ containing an $\mathbb{R}$-basis of $N_{\mathbb{R}}$ such that $0 = \sum_{\tau \in I} k_\tau v_\tau$ for positive integers $\{k_\tau\}_{\tau \in I}$. For $m \in \mathcal{R}$ this yields

$$\sum_{\mathrm{pos}(\eta_m) \neq \tau \in I} k_\tau \langle v_\tau, m\rangle = k_{\mathrm{pos}(\eta_m)},$$

hence $\langle v_\tau, m\rangle \in \{-1, 0, \ldots, k_{\mathrm{pos}(\eta_m)}\}$ for all $\tau \in I$. Since I contains an $\mathbb{R}$-basis of $N_{\mathbb{R}}$, there are only finitely many choices for the coordinates of $m \in \mathcal{R}$ in a dual $\mathbb{R}$-basis of $M_{\mathbb{R}}$.

$\square$

For a root $m \in \mathcal{R}$ we get a one-parameter subgroup $x_m : \mathbb{C} \to \mathrm{Aut}(X)$. Then the identity component $\mathrm{Aut}^\circ(X)$ is a semidirect product of a reductive algebraic subgroup containing the big torus $(\mathbb{C}^*)^d$ and having $\mathcal{S}$ as a root system and the unipotent radical that is generated by $\{x_m(\mathbb{C}) \,:\, m \in \mathcal{U}\}$. Furthermore $\mathrm{Aut}(X)$ is generated by $\mathrm{Aut}^\circ(X)$ and the finite number of automorphisms that are induced by lattice automorphisms of the fan $\triangle$. These results are due to Demazure (see [Oda88, p. 140]) in the nonsingular complete case, and were generalized by Cox [Cox95, Cor. 4.7] and Bühler [Büh96]. Bruns and Gubeladze considered the case of a projective toric variety in [BG99, Thm. 5.4]. In particular there is the following result (recall that an algebraic group is *reductive*, if the unipotent radical is trivial).

Proposition 5.1.3.

1. $\mathrm{Aut}(X)$ *is reductive if and only if* $\triangle$ *is semisimple.*
2. $\dim \mathrm{Aut}^\circ(X) = |\mathcal{R}| + d$.

When X is nonsingular, it is well-known (see [Oda88, p. 140]) that each irreducible component of the root system $\mathcal{S}$ is of type **A**. Here we will give an explicit description of $\mathcal{S}$ and $\eta(\mathcal{R})$ by orthogonal families of roots that will turn out to be useful for geometric applications.

When considering roots there is an algebraic Ansatz due to Cox that will be discussed below. It is very convenient for proofs however has very little geometric intuition attached. On the other hand there is an approach due to Bruns and Gubeladze that is especially in the Gorenstein case very close to convex geometry. We proceed here in a kind of combination of these ideas, essentially the definitions are close to geometry (and were introduced by the author before having seen [BG02] and [Cox95]), most results however are proved in the simplest way using the paper of Cox.

Definition 5.1.4. A pair of roots $v, w \in \mathcal{R}$ is called *orthogonal*, in symbols $v \perp w$, if $\langle \eta_v, w \rangle = 0 = \langle \eta_w, v \rangle$. In particular $\eta_{-v} \neq \eta_w \neq \eta_v \neq \eta_{-w}$.

We remark that the term 'orthogonal' may be misleading, because most standard properties do not hold, e.g., $v \perp w$ does not necessarily imply $(-v) \perp w$.

Lemma 5.1.5. *Let $B = \{b_1, \ldots, b_l\}$ be a non-empty set of roots such that $\langle \eta_{b_i}, b_j \rangle = 0$ for $1 \leq j < i \leq l$. Then B is a $\mathbb{Z}$-basis of* $\mathrm{lin}(B) \cap M$.

Proof. We prove the base property by induction on l. Let $x := \lambda_1 b_1 + \cdots + \lambda_l b_l \in M$ with $\lambda_1, \ldots, \lambda_l \in \mathbb{R}$. Then $\lambda_l = -\langle \eta_{b_l}, x \rangle \in \mathbb{Z}$. So $x - \lambda_l b_l = \lambda_1 b_1 + \cdots + \lambda_{l-1} b_{l-1} \in M$. Now proceed by induction.

□

We define two special pairwise orthogonal families of roots:

Definition 5.1.6. Let $A \subseteq \mathcal{R}$.
A pairwise orthogonal family $B \subseteq A$ is called

- *A-facet basis*, if $\eta(A) = \{\eta_b \,:\, b \in B\} \cup \{\eta_{-b} \,:\, b \in B, -b \in A\}$.
- *A-root basis*, if $A = \mathcal{R} \cap \mathrm{lin}(B)$.

Remark 5.1.7. When B is an A-root basis, we have $\mathrm{lin}(A) = \mathrm{lin}(B)$, hence $\dim_{\mathbb{R}} \mathrm{lin}(A) = |B|$ by 5.1.5. If furthermore $B \subseteq \mathcal{S}$, then Prop. 5.1.12 below implies that $A \subseteq \mathcal{S}$, A can be easily described by B, and B is also an A-facet basis. Note however that in general an $\mathcal{S}$-root basis is *not* a fundamental system for the root system $\mathcal{S}$ in the usual sense.

For arbitrary $A \subseteq \mathcal{R}$ we cannot expect the existence of an A-root basis. However it is one of the goals of this section to show that there are always $\mathcal{R}$-facet bases (5.1.22(2)) and $\mathcal{S}$-root bases (5.1.17). To explicitly construct these families an algebraic approach due to Cox shall now be discussed:

In [Cox95] Cox described $\mathcal{R}$ as a set of *ordered* pairs of monomials in the homogeneous coordinate ring of the toric variety. For this we denote by $S := \mathbb{C}[x_\tau \,:\, \tau \in \triangle(1)]$ the homogeneous coordinate ring of X, i.e., S is just a polynomial ring where any monomial in S is naturally graded by the class group $\mathrm{Cl}(X)$, i.e., the degree of a monomial $\prod_\tau x_\tau^{k_\tau}$ is the class of the Weil divisor $\sum_\tau k_\tau \mathcal{V}_\tau$, where $\mathcal{V}_\tau$ is the torus-invariant prime divisor corresponding to the ray τ. Recall that each $\tau \cap N$ is generated by v_τ.

We let Y denote the set of indeterminates $\{x_\tau \,:\, \tau \in \triangle(1)\}$ and $\mathcal{M}$ the set of monomials in S. For any root $m \in \mathcal{R}$ we define $\tau_m := \mathrm{pos}(\eta_m) \in \triangle(1)$ and $x_m := x_{\tau_m} \in Y$. Now there is the following fundamental result [Cox95, Lemma 4.4] (with a different sign convention):

Lemma 5.1.8 (Cox 95). *In this notation there is a well-defined bijection*

$$h \;:\; \mathcal{R} \to \{(x_\tau, f) \in Y \times \mathcal{M}, \,:\, x_\tau \neq f,\, \deg(x_\tau) = \deg(f)\},$$

$$m \mapsto (x_m, \prod_{\tau' \neq \tau_m} x_{\tau'}^{\langle v_{\tau'}, m\rangle}).$$

For $m \in \mathcal{R}$ we have

$$m \in \mathcal{S} \Longleftrightarrow h(m) \in Y \times Y,$$

in this case $h(m) = (x_m, x_{-m})$.

The next result can be used to 'orthogonalize' pairs of roots:

Lemma 5.1.9. *Let $v, w \in \mathcal{R}$, $v \neq -w$, $\langle \eta_v, w\rangle > 0$.*
Then $\langle \eta_w, v\rangle = 0$ and $v + w \in \mathcal{R}$.
Moreover $p(v,w) := \langle \eta_v, w\rangle v + w \in \mathcal{R}$, $v \perp p(v,w)$, $\eta_{p(v,w)} = \eta_{v+w} = \eta_w$.

$$p(v,w) \in \mathcal{S} \text{ iff } v + w \in \mathcal{S} \text{ iff } v \in \mathcal{S} \text{ and } w \in \mathcal{S}.$$

Proof. Let v correspond to (x_v, f) and w to (x_w, g) as in Lemma 5.1.8. It is $x_v \neq x_w$. The assumption implies that x_v appears in the monomial g. Assume $\langle \eta_w, v\rangle > 0$. Then x_w would appear in the monomial f. However since $v \neq -w$ this is a contradiction to the antisymmetry of the order relation defined in [Cox95, Lemma 1.3]. The remaining statements are easy to see.

□

Corollary 5.1.10. *$v \in \mathcal{U}$ and $w \in \mathcal{S}_1$ implies $\langle \eta_v, w\rangle = 0$.*

Lemma 5.1.9 defines a partial addition on $\mathcal{R}$ and generalizes parts of [BG02, Prop. 3.3] in a paper on polytopal linear groups due to Bruns and Gubeladze. The setting there is that of so called 'column structures' of polytopes where 'column vectors' correspond to roots. Most parts of this lemma were however already independently known and proven by the author as an application of Corollary 5.2.8 below in the case of a reflexive polytope.

For an unambiguous description of $\mathcal{S}$ it is now convenient to define an equivalence relation on the set of semisimple roots.

Definition 5.1.11. Let $v \equiv w$ (v is *equivalent* to w), if $v, w \in \mathcal{S}$, $v \neq w$ and $\eta_{-v} = \eta_{-w}$. In particular this yields $\langle \eta_{-v}, w\rangle = -\langle \eta_{-v}, -w\rangle = 1$.

Proposition 5.1.12. *Let $A \subseteq \mathcal{R}$ and $B \subseteq \mathcal{S}$ an A-root basis partitioned into t equivalence classes of order $c_1, \ldots, c_t$. Then:*
$A = \{\pm b \,:\, b \in B\} \cup \{b - b' \,:\, b, b' \in B,\, b \neq b',\, b \equiv b'\} \subseteq \mathcal{S}$,
$|A| = |B| + \sum_{i=1}^t c_i^2 \leq |B| + |B|^2$,
$\eta(A) = \{\eta_{\pm b} \,:\, b \in B\}$, $|\eta(A)| = |B| + t \leq 2|B|$.

Proof. Only the first equation has to be proven: Let $m \in A$, by 5.1.5 we have $m = \sum_{b \in B} \lambda_b b$ for $\lambda_b \in \mathbb{Z}$. Let $l := \sum_{b \in B} |\lambda_b|$. Proceed by induction on l, let $l > 1$. By orthogonality we have $-1 \leq \langle \eta_b, m \rangle = -\lambda_b$, hence $\lambda_b \leq 1$ for all $b \in B$. Assume there is an element $b \in B$ with $\lambda_b < 0$. Lemma 5.1.9 implies $b + m \in \operatorname{lin}(B) \cap \mathcal{R} = A$, so $b + m \in \mathcal{S}$ by induction hypothesis. Now Lemma 5.1.9 yields $-m \in A$. Hence $\lambda_b = -1$. Therefore $\lambda_b \in \{1, 0, -1\}$ for all $b \in B$. Assume $l > 2$. By possibly replacing m with $-m$ we can achieve that there are two elements $b, b' \in B$ with $\lambda_b = 1 = \lambda_{b'}$, hence $\eta_b = \eta_m = \eta_{b'}$, a contradiction. Therefore $l = 2$, and there are two elements $b, b' \in B$ with $m = b - b'$. Assume $b \not\equiv b'$. Then necessarily $\langle \eta_{-b'}, b \rangle = 0$, so $\eta_b = \eta_m = \eta_{-b'}$, a contradiction.

□

Definition 5.1.13. The grading of the polynomial ring $S := \mathbb{C}[x_\tau : \tau \in \triangle(1)]$ by the class group $\mathrm{Cl}(X)$ induces a partition of Y into equivalence classes:

1. Let $Y_1, \ldots, Y_p$ be the equivalence classes of order at least two such that there exists no monomial in $\mathcal{M} \backslash Y$ of the same degree.
2. Let $Y_{p+1}, \ldots, Y_q$ be the remaining classes of order at least two.
3. Let $Y_{q+1}, \ldots, Y_r$ be the the equivalence classes of order one such that there exists an monomial in $\mathcal{M} \backslash Y$ of the same degree.
4. Let $Y_{r+1}, \ldots, Y_s$ be the remaining classes of order one.

By Lemma 5.1.8 ordered pairs of indeterminates contained in one of the equivalence classes Y_1, …, Y_p correspond to roots in $\mathcal{S}_1$, ordered pairs in Y_{p+1}, …, Y_q correspond to roots in $\mathcal{S}_2$. As changing $m \leftrightarrow -m$ for $m \in \mathcal{S}$ just means to reverse the corresponding pair of monomials, we immediately see that $-\mathcal{S}_1 = \mathcal{S}_1$ and $-\mathcal{S}_2 = \mathcal{S}_2$. Moreover Lemma 5.1.8 yields that *any root in $\mathcal{S}_1$ is orthogonal, and not equivalent, to any root in $\mathcal{S}_2$.*

We have:

$$p = |\,\eta(\mathcal{S}_1)\,|,\ q - p = |\,\eta(\mathcal{S}_2)\,| = |\,\eta(\mathcal{U}_2)\,|,\ r - q = |\,\eta(\mathcal{U}_1)\,|,\ r = |\,\eta(\mathcal{R})\,|.$$

We get from Lemma 5.1.8:

$$|\,\mathcal{S}_1\,| = \sum_{i=1}^{p} |\,Y_i\,|(|\,Y_i\,| - 1), \quad |\,\mathcal{S}_2\,| = \sum_{i=p+1}^{q} |\,Y_i\,|(|\,Y_i\,| - 1).$$

Moreover if we define for $i = p+1, \ldots, r$ the equivalence class $\mathcal{M}_i$ consisting of monomials in $\mathcal{M} \backslash Y$ having the same degree as an element in Y_i, then we get:

$$|\mathcal{U}_1\,| = \sum_{i=q+1}^{r} |\,\mathcal{M}_i\,|, \quad |\,\mathcal{U}_2\,| = \sum_{i=p+1}^{q} |\,Y_i\,||\,\mathcal{M}_i\,|.$$

In particular $|\mathcal{U}_2| \neq \emptyset$ implies $|\mathcal{U}_2| \geq 2$. Since by Lemma 5.1.8 for $i = p+1, \ldots, r$ no indeterminate in Y_i can appear in an monomial in $\mathcal{M}_i$, we obtain that $v, w \in \mathcal{U}$ with $\eta_v \neq \eta_w$ and $\deg(x_v) = \deg(x_w)$ are orthogonal. See Example 5.1.15 for an illustration.

Example 5.1.14. Let's look at $X = \mathbb{P}^d$: We let E_d denote the d-dimensional simplex $\operatorname{conv}(e_1, \ldots, e_d, -e_1 - \cdots - e_d)$, where $e_1, \ldots, e_d$ is a $\mathbb{Z}$-basis of N. Hence $P := E_d^*$ is the reflexive polytope corresponding to d-dimensional projective space $X_P = \mathbb{P}^d$. The homogeneous coordinate ring $S = \mathbb{C}[x_0, \ldots, x_n]$ is trivially graded. $\operatorname{Aut}(X)$ is reductive with $d^2 + d$ roots. For $e_1^*, \ldots, e_d^*$ the dual basis of M the family $b_1 := e_1^*$, $b_2 := e_1^* - e_2^*$, $\ldots$, $b_d := e_1^* - e_d^*$ forms an $\mathcal{S}$-root basis, where all elements are mutually equivalent.

Example 5.1.15. For another example with $X = X_P$ we consider the three-dimensional reflexive simplex $P := \operatorname{conv}((1,0,0),(1,3,0),(1,0,3),(-5,-6,-3))$ with $\mathcal{V}(P^*) = \{(-1,0,0),(-1,0,2),(2,-1,-1),(-1,1,0)\}$. We have $\dim_{\mathbb{R}} \mathcal{S} = 2$, $|\,\mathcal{S}\,| = 4$. F_1 and F_2 contain one antipodal pair of semisimple roots, while F_3 and F_4 contain the other pair. F_3, F_4 each contain three unipotent roots, pairs of unipotent roots in different facets are orthogonal. We can read this off the data $S = \mathbb{C}[x_0, x_1, x_2, x_3]$, $\operatorname{Cl}(X_P) \cong \mathbb{Z}$, $\deg(x_1) = \deg(x_2) = 1$ and $\deg(x_3) = \deg(x_4) = 2$. Hence $Y_1 = \{x_1, x_2\}$, $Y_2 = \{x_3, x_4\}$, $p = 1$, $q = r = s = 2$, $\{x_1^2, x_1x_2, x_2^2\}$ are the elements in $\mathcal{M}\backslash Y$ of degree 2. X_P is just the weighted projective space with weights $(1,1,2,2)$.

The next proposition shows how to construct root bases:

Proposition 5.1.16. *Let a subset $I \subseteq \{1, \ldots, q\}$ be given. Choose for any element $i \in I$ a family of subset $K_{i,j} \subseteq Y_i$ of cardinality $c_{i,j} + 1$ for $1 \leq j \leq i_r$. Denote by $R_{i,j}$ the set of $c_{i,j}$ semisimple roots corresponding to ordered pairs in $K_{i,j}$ with the same fixed second element. Define $B := \cup_{i \in I, 1 \leq j \leq i_r} R_{i,j}$ and $A := \operatorname{lin}(B) \cap \mathcal{R}$.*

Then B is an A-root basis partitioned into equivalence classes $\{R_{i,j}\}$, and any root in A corresponds exactly to an ordered pair in $K_{i,j}$ for some $i \in I$ and $1 \leq j \leq i_r$.

Moreover any A-root basis is given by this construction.

Proof. By construction and Lemma 5.1.8 $\langle \eta_v, w \rangle = 0 = \langle \eta_w, v \rangle$ for $v, w \in B$, $v \neq w$, hence B is an A-root basis with given equivalence classes. Using Lemma 5.1.8 and the description of A in Prop. 5.1.12 the remaining statements are easy to see.

□

For $A \subseteq \mathcal{S}$ and $v \in A$ we also see that

$$|\,\{w \in A \,:\, \eta_w = \eta_v\}\,| = |\,\{w \in A \,:\, \eta_w = \eta_{-v}\}\,|\,.$$

Choosing $I = \{1, \ldots, q\}$, $i_r = 1$ for all i, and $K_i = Y_i$, we get (see also Remark 5.1.7):

Corollary 5.1.17. *$\mathcal{S}$-root bases exist, in particular $\mathcal{R} \cap \operatorname{lin}(\mathcal{S}) = \mathcal{S}$. Moreover*

$$\dim_{\mathbb{R}} \operatorname{lin}(\mathcal{S}) = \sum_{i=1}^{q} (|\,Y_i\,| - 1).$$

Remark 5.1.18. $P' := \operatorname{conv}(\mathcal{S})$ is a centrally symmetric terminal reflexive polytope with $\mathcal{V}(P') = \partial P' \cap M = \mathcal{S}$.

More precisely due to 5.1.12 there is an isomorphism of lattice polytopes (with respect to lattices $\mathrm{lin}(\mathcal{S}) \cap M$ and $\mathbb{Z}^{c_1+\cdots+c_q}$)

$$\mathrm{conv}(\mathcal{S}) \cong (\mathcal{Z}_{c_1} \oplus \cdots \oplus \mathcal{Z}_{c_q})^*,$$

where $c_i := |Y_i| - 1$ for $i = 1, \ldots, q$, and $\mathcal{Z}_n = \mathrm{conv}(\pm[0,1]^n)$ is the n-dimensional standard lattice zonotope (see 3.5.1). For a stronger statement see Theorem 5.2.12.

The existence of an $\mathcal{S}$-root basis yields:

Proposition 5.1.19. *A d-dimensional complete toric variety is isomorphic to a product of projective spaces iff there are d linearly independent semisimple roots.*

In this case

$$X \cong \mathbb{P}^{|Y_1|-1} \times \cdots \times \mathbb{P}^{|Y_q|-1}.$$

Proof. Let $q = 1$, so there is an $\mathcal{S}$-root basis $b_1, \ldots, b_d$ with $\eta_{-b_1} = \cdots = \eta_{-b_d}$. Assume there exists $\tau \in \triangle(1)$ with $\tau \notin \{\tau_{b_1}, \ldots, \tau_{b_d}, \tau_{-b_1}\}$. Then $\langle v_\tau, b_i \rangle = 0$ for $i = 1, \ldots, d$, since $b_i \in \mathcal{S}$. This implies $v_\tau = 0$, a contradiction. Therefore $\triangle(1)$ is determined. Since no cone in $\triangle$ contains a linear subspace, this already implies $X \cong \mathbb{P}^d$. The general case is treated similarly and left to the reader. □

As a corollary we get from the existence of an $\mathcal{S}$-root basis and Prop. 5.1.12:

Corollary 5.1.20. *$|\mathcal{S}| \leq d^2 + d$, with equality iff $X \cong \mathbb{P}^d$.*

Moreover using Propositions 5.1.12 and 5.1.16 we can now characterize the subsets of $\mathcal{S}$ that admit root bases:

Corollary 5.1.21. *Let $A \subseteq \mathcal{S}$. The following conditions are equivalent:*

1. *There exists an A-root basis*
2. *$A = \mathcal{S} \cap V$ for an $\mathbb{R}$-subvectorspace V of $M_\mathbb{R}$*
3. *$\mathcal{R} \cap \mathrm{lin}(x, y) \subseteq A$ for any $x, y \in A$*
4. *$A = -A$, and if $x, y \in A$ with $x \neq \pm y$ and $\langle \eta_x, y \rangle > 0$, then $p(x, y) \in A$*

The details of the proof are left to the reader (only 4. to 1. has to be proven).

Above results yield now the following existence theorem:

Proposition 5.1.22.

1. *There exists an $\mathbb{R}$-linearly independent family B of roots that can be partitioned into three pairwise disjoint subsets B_1, B_2, B_3 such that B_1 is an $\mathcal{S}_1$-root basis, B_2 is an $\mathcal{S}_2$-root basis, $B_1 \cup B_2$ is an $\mathcal{S}$-root basis and B_3 is a $\mathcal{U}_1$-facet basis such that $\langle \eta_b, b' \rangle = 0$ for all $b \in B_1 \cup B_2$ and $b' \in B_3$.*

   Hence $\dim_\mathbb{R} \mathrm{lin}(\mathcal{S}) + |\eta(\mathcal{U}_1)| = |B| \leq d$.

2. *There exists an $\mathcal{R}$-facet basis D that can be partitioned into three pairwise disjoint subsets D_1, D_2, D_3 such that D_1 is a $\mathcal{U}_1$-facet basis, D_2 is a $\mathcal{U}_2$-facet basis, $D_1 \cup D_2$ is a $\mathcal{U}$-facet basis and D_3 is an $\mathcal{S}_1$-root basis.*

   Hence $|\,\eta(\mathcal{U}_1)\,| + |\,\eta(\mathcal{U}_2)\,| + \dim_{\mathbb{R}} \operatorname{lin}(\mathcal{S}_1) = |\,D\,| \le d$.

Proof. 1. Applying 5.1.16 to $Y_1, \ldots, Y_p$, respectively $Y_{p+1}, \ldots, Y_q$, gives the existence of an $\mathcal{S}_1$-root basis B_1, respectively an $\mathcal{S}_2$-root basis B_2. The union of these two families gives an $\mathcal{S}$-root basis.

Let $x \in \mathcal{U}_1$. For any $b \in B_1 \cup B_2$ we have $\langle \eta_b, x\rangle \ge 0$. If $\langle \eta_b, x\rangle \ge 1$, then by 5.1.9 we can substitute $p(b,x) \in \mathcal{U}_1$ for x because of $\eta_{p(b,x)} = \eta_x$. Since the elements in $B_1 \cup B_2$ are pairwise orthogonal, this process finally gives an element $x \in \mathcal{U}_1$ with $\langle \eta_b, x\rangle = 0$ for all $b \in B_1 \cup B_2$.

Assume now we already have $x_1, \ldots, x_l \in \mathcal{U}_1$ pairwise orthogonal such that any one satisfies the previous condition. Let $x \in \mathcal{U}_1$ with $\eta_x \ne \eta_{x_i}$ for $i = 1, \ldots, l$. First we can assume as before that $\langle \eta_b, x\rangle = 0$ for all $b \in B_1 \cup B_2 \cup \{x_1, \ldots, x_l\}$. Now all we have to do is to "orthogonalize" the set $x_1, \ldots, x_l, x$. Let's do this by induction on $i = 1, \ldots, l$: We assume $x_j \perp x$ for $j = 1, \ldots, i-1$. Since $\langle \eta_{x_i}, x\rangle = 0$, we substitute $p(x, x_i)$ for x_i because of $\eta_{p(x,x_i)} = \eta_{x_i}$. Then $x_1, \ldots, x_l$ is still a pairwise orthogonal family, and we additionally get $x_j \perp x$ for $j = 1, \ldots, i$.

2. As before it is not difficult to get the existence of a $\mathcal{U}$-facet basis $D_1 \cup D_2$ such that D_1 is a $\mathcal{U}_1$-facet basis and D_2 is a $\mathcal{U}_2$-facet basis. Let D_3 be an $\mathcal{S}_1$-root basis. By 5.1.10 $\langle \eta_y, x\rangle = 0$ for all $y \in D_1 \cup D_2$ and $x \in \mathcal{S}_1$. For any element in D_3 we now just have to successively modify $D_1 \cup D_2$ in the same way as at the end of 1.

□

Corollary 5.1.23.

1. *$|\,\eta(\mathcal{U})\,| \le d$, where equality implies that $\eta(\mathcal{R}) = \eta(\mathcal{U})$.*

2. *$|\,\eta(\mathcal{U})\backslash\eta(\mathcal{S})\,| \le \operatorname{codim}_{\mathbb{R}}\operatorname{lin}(\mathcal{S})$.*

3. *$|\,\eta(\mathcal{R})\,| \le 2d$, with equality iff $X \cong \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$.*

Proof. 1. Follows from 5.1.22(2). 2. Follows from 5.1.22(1).

3. Let D be the $\mathcal{R}$-facet basis from 5.1.22(2), we have $|\,D\,| \le d$. By definition $\eta(\mathcal{R}) = \{\eta_x \,:\, x \in D_1 \cup D_2\} \cup \{\eta_{\pm x} \,:\, x \in D_3\}$, this gives the upper bound. Equality implies $D = D_3$, i.e., $\mathcal{R} = \mathcal{S}$, with no element in D equivalent to any other. Applying the previous proposition we get the desired result.

□

While the case when $M_{\mathbb{R}}$ is spanned by semisimple roots is completely classified, there are at least some partial results in the case of codimension one. This research was motivated by the observations of the author that there were no semisimple reflexive polygons with only one pair of roots (see Corollary 5.2.5) and that were there no semisimple three-dimensional reflexive polytopes with 6 roots apart from $[-1,1]^3$ in a list given to the author by Kreuzer [Kre03a].

Proposition 5.1.24. *Let $\dim_{\mathbb{R}} \operatorname{lin}(\mathcal{S}) = d - 1$.*

1. *If $|\,\triangle(1)\,| \ne \eta(\mathcal{S})$, then there exists $\tau \in \triangle(1)\backslash\eta(\mathcal{S})$ such that $\triangle(1)\backslash\eta(\mathcal{S}) \subseteq \{\pm\tau\}$, and we have $\mathcal{V}_\tau \cong \mathbb{P}^{|Y_1|-1} \times \cdots \times \mathbb{P}^{|Y_q|-1}$.*

2. If $q = 1$, i.e., $|\mathcal{S}| = d^2 - d$, then $|\eta(\mathcal{U})| = 1$ and $\eta(\mathcal{S}) \cap \eta(\mathcal{U}) = \emptyset$.

Proof. Let $b_1, \ldots, b_{d-1}$ be an $\mathcal{S}$-root basis. By 5.1.5 we can find a lattice point $b_d \in M$ such that $b_1, \ldots, b_d$ is an $\mathbb{Z}$-basis of M. Let $e_1, \ldots, e_d$ denote the dual $\mathbb{Z}$-basis of N.

1. Let $\tau \in \triangle(1)\backslash\eta(\mathcal{S})$. Then $\langle v_\tau, b_i\rangle = 0$ for all $i = 1, \ldots, d-1$, hence $v_\tau \in \{\pm e_d\}$. The set $\mathcal{S}$ is by construction canonically the set of roots of $\mathcal{V}_\tau$, so we can apply Prop. 5.1.19.

2. Let $q = 1$. By 5.1.12 this is equivalent to $|\mathcal{S}| = (d-1)^2 + d - 1 = d^2 - d$. For $i = 1, \ldots, d-1$ there exist $k_i \in \mathbb{Z}$ such that $\eta_i := \eta_{b_i} = -e_i + k_i e_d$. There exists $k_d \in \mathbb{Z}$ such that $\eta_d := \eta_{-b_1} = e_1 + \cdots + e_{d-1} + k_d e_d$.

Since $|\eta(\mathcal{S})| = d$, there exists $\tau \in \triangle(1)\backslash\eta(\mathcal{S})$, we may assume $v_\tau = e_d$. Let $x = \lambda_1 b_1 + \cdots + \lambda_d b_d \in M$. We have $x \in \mathcal{R}$ with $\eta_x = e_d$ iff $\langle x, e_d\rangle = -1$ and $\langle x, \eta_i\rangle \geq 0$ for $i = 1, \ldots, d$. This is equivalent to $\lambda_d = -1$, $\lambda_i \leq -k_i$ for $i = 1, \ldots, d-1$ and $\lambda_1 + \cdots + \lambda_{d-1} \geq k_d$. Hence there exists a root $x \in \mathcal{R}$ with $\eta_x = e_d$ if and only if $k_1 + \cdots + k_d \leq 0$.

On the other hand let $u := k_1 b_1 + \cdots + k_{d-1} b_{d-1} + b_d \in M$. Then $u^\perp$ is a hyperplane spanned by $\eta_1, \ldots, \eta_{d-1}$. We have $\langle u, e_d\rangle = 1$ and $\langle u, \eta_d\rangle = k_1 + \cdots + k_d$. Therefore when $|\triangle(1)| = d+1$, we get $\langle u, \eta_d\rangle < 0$, so there exists $x \in \mathcal{R}$ with $\eta_x = e_d$, necessarily $e_d \in \eta(\mathcal{U})$. Otherwise for $\triangle(1)\backslash\eta(\mathcal{S}) = \{\pm e_d\}$, the analogous computation for $-e_d$ yields that either e_d or $-e_d$ is in $\eta(\mathcal{U})$.

Assume $\eta(\mathcal{S}) \cap \eta(\mathcal{U}) \neq \emptyset$, so $\mathcal{S}_2 \neq \emptyset$. Use the family B in Prop. 5.1.22(1): Since by assumption all elements in $B_1 \cup B_2$ are mutually equivalent, however no element in $\mathcal{S}_1$ is equivalent to one in $\mathcal{S}_2$, we have $B = B_2$, i.e., $\mathcal{S} = \mathcal{S}_2$. This yields $|\eta(\mathcal{U}_2)| = d$. Since $|\eta(\mathcal{U}_1)| = 1$, we get a contradiction to 5.1.23(1). □

For Gorenstein toric Fano varieties the second point cannot simply be improved as can be seen from Example 5.1.15.

This result yields sharp upper bounds on $\dim \mathrm{Aut}(X)$ in the reductive case:

Theorem 5.1.25. *Let X be a d-dimensional complete toric variety with reductive automorphism group. Let $n := \dim \mathrm{Aut}^\circ(X)$. Then*

$$n \leq d^2 + 2d, \quad \textit{with equality only in the case of projective space.}$$

If $d = 2$ and X is not a product of projective spaces, then $n = 2$.
If $d \geq 3$ and X is not a product of projective spaces, then

$$n \leq d^2 - 2d + 4,$$

where equality holds iff $q = 2$ with $|Y_1| = 2$ and $|Y_2| = d - 1$.

Proof. Let $c_i := |Y_i| - 1$ for $i = 1, \ldots, q$. By 5.1.12 and 5.1.17, we have $l := c_1 + \cdots + c_q = \dim_{\mathbb{R}} \mathcal{S}$ and $|\mathcal{S}| = c_1^2 + \cdots + c_q^2 + l \leq l^2 + l$. Recall from 5.1.3 that $n = |\mathcal{S}| + d$. From 5.1.19 we get the first statement for $l = d$ (or see 5.1.20). Moreover for the second statement we can assume $l = d - 1$, since $(d-2)^2 + (d-2) < d^2 - 3d + 4$.

By 5.1.24(2) we have $q > 1$, since $\triangle$ is semisimple; in particular $d > 2$.

We may assume $c_1 \leq \ldots \leq c_q$.

If $q = 2$, then $c_1 + c_2 = d - 1$, hence either $c_1 = 1$ and $c_2 = d - 2$ (this yields $c_1 c_2 = d - 2$), or $c_1 \geq 2$ and $c_2 \geq (d-1)/2$ (this yields $c_1 c_2 \geq d - 1$).

If $q \geq 3$, then $\sum_{i<j} c_i c_j \geq c_1(c_2 + \cdots + c_q) + c_2 c_3 = c_1(c_1 + \cdots + c_q) + c_2 c_3 - c_1 c_1 \geq c_1(d-1) \geq d-1$.

In any case $|\mathcal{S}| = c_1^2 + \cdots c_q^2 + d - 1 = (c_1 + \ldots + c_q)^2 + d - 1 - 2\sum_{i<j} c_i c_j \leq (d-1)^2 + d - 1 + 2(2-d) = d^2 - 3d + 4$, with equality only for $q = 2$ with $c_1 = 1$ and $c_2 = d - 2$.

□

The following example shows that the last bound is sharp for any $d \geq 3$:

Example 5.1.26. *(due to C. Haase, Duke University)*
Let $P \subseteq \mathbb{R}^d$ be the d-dimensional reflexive polytope defined as the convex hull of $(2E_1^*) \times \{0\} \times \{1\}$ and $\{0\} \times (2E_{d-2}^*) \times \{-1\}$, where $E_k^* \subseteq \mathbb{R}^k$ denotes as in 5.1.14 the k-dimensional reflexive polytope corresponding to $\mathbb{P}^k$. This implies that $P \cap (\mathbb{R}^1 \times \mathbb{R}^{d-2} \times \{0\}) \cong E_1^* \times E_{d-2}^*$, $\mathcal{N}_P$ is semisimple with $\dim_{\mathbb{R}} \mathcal{S} = d-1$, and the last upper bound in the previous theorem is attained by X_P.

5.2 The set of roots of a reflexive polytope

Throughout the section let P be a d-dimensional reflexive polytope in $M_{\mathbb{R}}$.

In this section we will focus on Gorenstein toric Fano varieties, these varieties correspond to reflexive polytopes as described in the third chapter (see Cor. 3.1.5). When P is reflexive, we have by definition that *the set of roots $\mathcal{R}$ of the normal fan $\mathcal{N}_P$ is exactly the set of lattice points in the relative interior of facets of P.*

Definition 5.2.1. The set $\mathcal{R}$ of *roots* of P is defined as the set of roots of $\mathcal{N}_P$. For $m \in \mathcal{R}$ we denote by $\mathcal{F}_m$ the unique facet of P that contains m, and we again define $\eta_m = \eta_{\mathcal{F}_m}$ to be the unique primitive inner normal with $\langle \eta_m, \mathcal{F}_m \rangle = -1$. For a subset $A \subseteq \mathcal{R}$ it is convenient to define $\mathcal{F}(A) := \{\mathcal{F}_m \,:\, m \in A\}$. We say P is *semisimple*, if $\mathcal{N}_P$ is semisimple, i.e., $\mathcal{R} = -\mathcal{R}$.

Most results of the previous section have now a direct geometric interpretation. Here three examples shall be explicitly stated (just use Corollary 5.1.23(1), the basic fact $-\mathcal{S}_1 = \mathcal{S}_1$, and Corollary 5.1.23(3)):

Corollary 5.2.2. *There are at most d facets of P containing unipotent roots.*

Corollary 5.2.3. *If a facet of P contains an unipotent root and a semisimple root x, then the facet containing $-x$ also contains an unipotent root.*

Corollary 5.2.4. *There are at most $2d$ facets containing roots; equality holds if and only if $P \cong [-1,1]^d$ (isomorphic as lattice polytopes).*

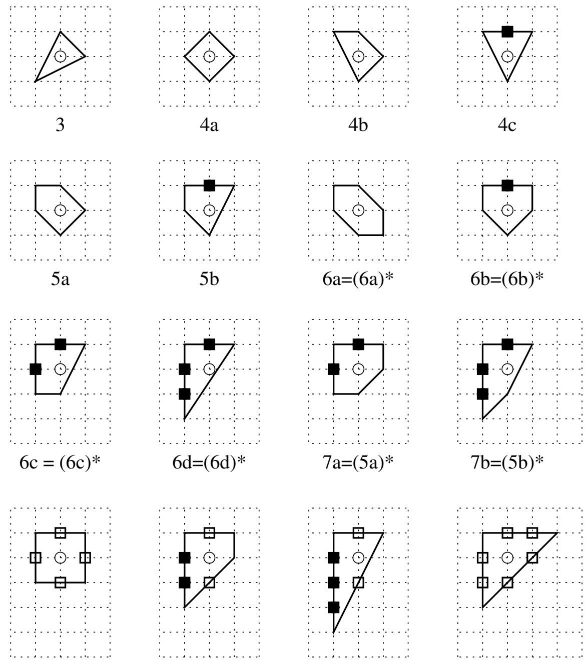
For another example we apply Prop. 5.1.19 and Prop. 5.1.24(2) to $d = 2$ to get a characterization of semisimple reflexive polygons without using the existing classification 3.4.1. The proof relies on the well-known fact that a two-dimensional terminal Fano polytope is a smooth Fano polytope, e.g., 3.1.8(1).

Corollary 5.2.5. *Let P be a two-dimensional reflexive polytope. For $k \in \mathbb{N}_{>0}$ let the reflexive polytope E_k be defined as in 5.1.14, i.e., $X_{E_k^*} \cong \mathbb{P}^k$.*

Then P is semisimple iff P is a smooth Fano polytope or $P \cong E_2^*$ or $P \cong E_1^2$.

P or P^* is semisimple iff P or P^* is a smooth Fano polytope.

As an illustration of these results we give the list of reflexive polygons, where filled squares are unipotent roots and empty squares are semisimple roots:



3 4a 4b 4c

5a 5b 6a=(6a)* 6b=(6b)*

6c = (6c)* 6d=(6d)* 7a=(5a)* 7b=(5b)*

8a=(4a)* 8b=(4b)* 8c=(4c)* 9=3*

Especially we see that type 9, where X_P corresponds to $\mathbb{P}^2$, has the maximal number 6 of semisimple roots; type 8a, i.e., $[-1,1]^2$, has the maximal number 4 of facets containing roots; and there is no semisimple polygon with precisely one pair of roots, as proven in Prop. 5.1.24(2), respectively Thm. 5.1.25.

In general there is a nice property of pairwise orthogonal families of roots:

Proposition 5.2.6. *Let B be a non-empty set of pairwise orthogonal roots.*

Then $F := \bigcap_{b \in B} \mathcal{F}_b$ is a non-empty face of P of codimension $|B|$, and the sum over all elements in B is a lattice point in the relative interior of F.

Proof. Let $B = \{b_1, \ldots, b_l\}$ with $|B| = l$. For $i \in \{1, \ldots, l\}$ we define $s_i := \sum_{j=1}^{i} b_j$ and $F_i := \cap_{j=1}^{i} \mathcal{F}_{b_j}$. Orthogonality implies that $\{\mathcal{F}_{b_1}, \ldots, \mathcal{F}_{b_l}\}$ is exactly the set of facets containing s_l. Therefore $s_l \in \mathrm{relint} F_l$, and since any l-codimensional face of P is contained in at least l facets, we must have $\mathrm{codim} F_l \leq l$. On the other hand $s_i \notin F_{i+1}$ for all $i = 1, \ldots, l$, so $F_1 \supsetneq \cdots \supsetneq F_l$, hence we obtain $\mathrm{codim} F_l = l$.

□

This proposition can be applied to a $\mathcal{U}$-facet basis (see 5.1.22(2)):

Corollary 5.2.7. *If $\mathcal{U} \neq \emptyset$, then $\bigcap_{F \in \mathcal{F}(\mathcal{U})} F$ is a face of codimension $|\mathcal{F}(\mathcal{U})| \leq d$.*

In particular if P is not semisimple, then the sum over all lattice points in the non-empty face $\bigcap_{F \in \mathcal{F}(\mathcal{U})} F$ is a non-zero fixpoint of $\mathrm{Aut}_M(P)$.

To sharpen the results of the previous section we use the elementary but fundamental property of pairs of lattice points on the boundary of a reflexive polytope as described in Prop. 3.3.1. This partial addition extends the partial addition of roots in 5.1.9 (see also [BG02, Def. 3.2]).

Now we easily get:

Lemma 5.2.8. *Let $v \in \mathcal{R}$, $w \in \partial P \cap M$ with $w \notin \mathcal{F}_v$ and $w \neq -v$.*
Then $v + w \in \partial P \cap M$ and $z(v, w) \in \mathcal{F}_v$. Moreover

$$\langle \eta_v, w \rangle > 0 \text{ iff } z(v, w) = av + w \text{ for } a \geq 2.$$

In this case $z(v, w) = (\langle \eta_v, w \rangle + 1)v + w$ and $v \perp \langle \eta_v, w \rangle v + w = p(v, w)$.

Extending Definition 5.1.4 we may also more generally define an intrinsic notion of orthogonality:

Definition 5.2.9. $v \perp w$ for $v, w \in \partial P \cap M$, if $v + w \in \partial P$ and $z(v, w) = v + w$.

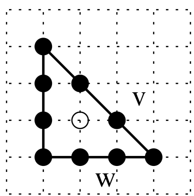
As a corollary of 5.2.6 we get:

Corollary 5.2.10. *Let $v, w \in \partial P \cap M$ with $v + w \in \partial P$. Then $v \perp (z(v, w) - v)$ or $w \perp (z(v, w) - w)$.*
Moreover if $v, w \in \mathcal{R}$, then $z(v, w)$ is in the relative interior of the face $\mathcal{F}(v) \cap \mathcal{F}(w)$ of codimension two.

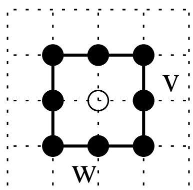
Remark 5.2.11. As can be immediately seen from the results of the previous section (or by elementary observations), there are essentially two possibilities for the set of roots in the span of two linearly independent semisimple roots. In the case of a reflexive polytope this shall be clearly illustrated:

So let $v, w \in \mathcal{S}$ with $v \neq \pm w$. By orthogonalizing with Lemma 5.1.9 we can assume $v \perp w$. Let $A := \mathcal{R} \cap \mathrm{lin}(v, w)$. By 5.1.12 $\{v, w\}$ is an A-root basis, and there are two cases:

1. $v \equiv w$. Hence $A = \{\pm v, \pm w, \pm(v - w)\}$.
The previous lemma implies for $P \cap \mathrm{lin}(v, w)$:



2. $v \not\equiv w$. Hence $A = \{\pm v, \pm w\}$.
The previous lemma implies for $P \cap \mathrm{lin}(v, w)$:



Note that $v \equiv w$ if and only if $v - w \in \mathcal{F}(v) \cap \mathcal{S}$.

Now we can improve Prop. 5.1.19 by taking the ambient space of semisimple roots into account (recall the definition of E_d in 5.1.14).

Theorem 5.2.12. *Let $B \subseteq \mathcal{S}$ be an A-root basis for a subset $A \subseteq \mathcal{R}$, and $R_1, \ldots, R_t$ the partition of B into equivalence classes of order $c_1, \ldots, c_t$. Then there are isomorphisms of lattice polytopes (with respect to lattices* $\operatorname{lin}(A) \cap M$ *and* $\mathbb{Z}^{c_1+\cdots+c_t}$*)*

$$P \cap \operatorname{lin}(A) \;\cong\; \bigoplus_{i=1}^{t} P \cap \operatorname{lin}(R_i) \;\cong\; \bigoplus_{i=1}^{t} E^*_{c_i}.$$

In particular the intersection of P with the space spanned by all semisimple roots is again a reflexive polytope associated to a product of projective spaces.

Proof. Let $t = 1$, i.e., all elements in B are mutually equivalent. Let $l = |\,B\,| \geq 2$, $B = \{b_1, \ldots, b_l\}$, $b := b_1 + \cdots + b_l$.

Claim: $P \cap \operatorname{lin}(b_1, \ldots, b_l) = \operatorname{conv}(b, b - (l+1)b_i \,:\, i = 1, \ldots, l) \cong E^*_l$.

Denote by Q the simplex on the right hand side of the claim, so $Q \cong E^*_l$.

By 5.2.6 $b \in \bigcap_{i=1}^{l} \mathcal{F}_{b_i}$. Since by assumption $\langle \eta_{-b_i}, b\rangle = \sum_{j=1}^{l} \langle \eta_{-b_i}, b_j\rangle = \sum_{j=1}^{l} \langle \eta_{-b_j}, b_j\rangle = l$, it follows from 5.2.8 that $z(-b_i, b) = b - (l+1)b_i \in \mathcal{F}_{-b_i}$ for $i = 1, \ldots, l$. Hence $Q \subseteq P \cap \operatorname{lin}(b_1, \ldots, b_l)$. On the other hand the previous calculation and orthogonality also implies that $Q \cap \mathcal{F}_{b_1}, \ldots, Q \cap \mathcal{F}_{b_l}, Q \cap \mathcal{F}_{-b_1}$ are exactly the facets of the simplex Q. This proves the claim.

Let $t > 1$.

We define for $i = 1, \ldots, t$ the reflexive polytopes $Q_i := P \cap \operatorname{lin}(R_i) \cong E^*_{c_i}$.

Let $x_i \in Q_i \cap M$ for $i = 1, \ldots, t$. We show by induction that $x_1 + \ldots + x_s \in P$ for $s = 1, \ldots, t$. Assume $s \geq 2$ and $x' := x_1 + \ldots + x_{s-1} \in P$. Without restriction $0 \neq x_s \neq -x'$. Assume $x' \sim x_s$. Let $F \in \mathcal{F}(P)$ be a facet containing x' and x_s. Since $x_s \in \operatorname{lin}(R_s) \cap M$, by 5.1.5 x_s is an integral linear combination of elements in R_s, hence there has to exist some $b_s \in R_s$ such that $b = b_s$ or $b = -b_s$ satisfies $\langle \eta_F, b\rangle = -1$. This implies $F = \mathcal{F}(b)$. Since $x' \in F$ in the same way there exists $j \in \{1, \ldots, s-1\}$ such that $x_j \in \mathcal{F}(b)$. This implies $\langle \eta_b, x_j\rangle = -1$, hence again, since $x_j \in \operatorname{lin}(R_j)$, by 5.1.5 there exists $b' \in \pm R_j$ such that $\langle \eta_b, b'\rangle = -1$, a contradiction. Hence by 3.3.1 we get $x' + x_s \in P$ as desired.

Therefore the polytope $Q := \bigoplus_{i=1}^{t} P \cap \operatorname{lin}(R_i)$ is contained in $P \cap \operatorname{lin}(A)$. However since the facets of Q are exactly $Q \cap \mathcal{F}_{\pm b_i}$ for $i = 1, \ldots, t$, both polytopes have to be equal.

□

Remark 5.2.13. The figure on the title of this work shows exactly the situation for $P \cong E^*_3$, where the inner polytope is the convex hull of all roots (isomorphic to $\mathcal{Z}_3$, see Remark 5.1.18) and the outer simplex is E^*_3. The visualization was done using the program `polymake` [GJ00, GJ05].

5.3 Criteria for a reductive automorphism group

In this section we give several criteria for the automorphism group of a complete toric variety, respectively a Gorenstein toric Fano variety, to be reductive.

Theorem 5.3.1.

1. *Let $X = X(N, \triangle)$ be a complete toric variety.*

   The following conditions are equivalent*:*

   (a) $\triangle$ *is semisimple, i.e.,* $\mathrm{Aut}(X)$ *is reductive*
   (b) $\mathrm{conv}(\mathcal{R})$ *is centrally symmetric*
   (c) $\mathrm{conv}(\mathcal{R})$ *is a centrally symmetric terminal reflexive polytope (with respect to the lattice* $M \cap \mathrm{lin}(\mathcal{R})$*) with vertices* $\mathcal{R}$
   (d) $\sum_{x \in \mathcal{R}} x = 0$
   (e) $\sum_{\tau \in \triangle(1)} \langle v_\tau, x \rangle = 0$ *for all* $x \in \mathcal{R}$

   If $\sum_{\tau \in \triangle(1)} v_\tau = 0$*, then* $\triangle$ *is semisimple.*

2. *Let* X_P *be a Gorenstein toric Fano variety for* $P \subseteq M_{\mathbb{R}}$ *reflexive.*

   The following conditions are equivalent*:*

   (a) P *is semisimple, i.e.,* $\mathrm{Aut}(X_P)$ *is reductive*
   (b) $\sum_{x \in \mathcal{R}} x = 0$
   (c) $\sum_{v \in \mathcal{V}(P^*)} \langle v, x \rangle = 0$ *for all* $x \in \mathcal{R}$
   (d) $\sum_{y \in P^* \cap N} \langle y, x \rangle = 0$ *for all* $x \in \mathcal{R}$
   (e) $\langle b_{P^*}, x \rangle = 0$ *for all* $x \in \mathcal{R}$
   (f) $\mathrm{rvol}(F') = \mathrm{rvol}(\mathcal{F}_x)$ *for all* $x \in \mathcal{R}$*,* $F' \in \mathcal{F}(P)$ *with* $\langle \eta_{F'}, x \rangle > 0$
   (g) $|\, F' \cap M \,| = |\, \mathcal{F}_x \cap M \,|$ *for all* $x \in \mathcal{R}$*,* $F' \in \mathcal{F}(P)$ *with* $\langle \eta_{F'}, x \rangle > 0$

   Any one of the following conditions is sufficient *for* P *to be semisimple:*

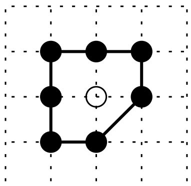
   i. $b_P = 0$
   ii. $\sum_{m \in P \cap M} m = 0$
   iii. $b_{P^*} = 0$
   iv. $\sum_{y \in P^* \cap N} y = 0$
   v. $\sum_{v \in \mathcal{V}(P^*)} v = 0$
   vi. *All facets of* P *have the same relative lattice volume*
   vii. *All facets of* P *have the same number of lattice points*

   Condition vi. *implies* v.*, e.g., if* P *is a smooth Fano polytope.*

Remark 5.3.2. Using the list of d-dimensional reflexive polytopes for $d \leq 4$ and the computer program PALP due to Kreuzer and Skarke (see [KS04a, KS04b]) we found examples showing that in the second part of the theorem the sufficient conditions i. – v. are pairwise independent, i.e., in general no condition implies any other. These examples can be found in the next section. For instance the following seven column vectors are the vertices of a four-dimensional reflexive polytope P that satisfies $b_P = 0$, $\sum_{m \in P \cap M} m = 0$, $\sum_{v \in \mathcal{V}(P)} v = 0$, however P^* does not satisfy *any* of these three conditions.

$$\begin{array}{ccccccc} 1 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{array}$$

Example 5.3.3. The "dual" of condition v. is not a sufficient condition: The following polygon is not semisimple, however the sum of the five vertices is zero.



For the proof of Theorem 5.3.1 we need some lemmas. The first is just a simple observation:

Lemma 5.3.4. *Let $\triangle$ be a complete fan in $N_\mathbb{R}$.*

$$m \in \mathcal{R} \Longrightarrow \sum_{\tau \in \triangle(1)} \langle v_\tau, m \rangle \in \mathbb{N},$$

in this case

$$m \in \mathcal{S} \Longleftrightarrow \sum_{\tau \in \triangle(1)} \langle v_\tau, m \rangle = 0.$$

Lemma 5.3.5. *Let $\triangle$ be a complete fan in $N_\mathbb{R}$.*
Let $A \subseteq \mathcal{R}$ be a subset such that

$$\sum_{m \in A} k_m m = 0$$

for some positive integers $\{k_m\}_{m \in A}$. Then $A \subseteq \mathcal{S}$.

Proof. Assume $A \cap \mathcal{U} \neq \emptyset$. Then by 5.3.4
$0 = \sum_{\tau \in \triangle(1)} \langle v_\tau, \sum_{m \in A} k_m m \rangle = \sum_{m \in A \cap \mathcal{U}} k_m \sum_{\tau \in \triangle(1)} \langle v_\tau, m \rangle \geq 1$, a contradiction. □

In the case of a reflexive polytope the following result is fundamental:

Lemma 5.3.6. *Let P be a d-dimensional reflexive polytope in $M_\mathbb{R}$.*
Let $m \in \mathcal{R}$. Define the canonical projection map along m

$$\pi_m : M_\mathbb{R} \to M_\mathbb{R}/\mathbb{R}m.$$

Then π_m induces an isomorphism of lattice polytopes

$$\mathcal{F}_m \to \pi_m(P),$$

with respect to the lattices $\operatorname{aff}(F) \cap M$ *and $M/\mathbb{Z}m$.*

Proof. Prop. 3.2.2 immediately implies that $\pi_m : \mathcal{F}_m \to \pi_m(P)$ is a bijection. It is even an isomorphism of lattice polytopes by 3.1.4(11).

Another proof can be easily done using only the definition of a root.

□

Using this lemma we get a reformulation of 5.3.4. Note that $A - B := \{a-b : a \in A, b \in B\}$ for arbitrary sets $A, B \subseteq \mathbb{R}^d$; a facet F of a d-dimensional polytope $Q \subseteq M_{\mathbb{R}}$ is said to be *parallel* to $\mathbb{R}x$ for some $x \in M_{\mathbb{R}}$, if $\langle \eta_F, x\rangle = 0$.

Lemma 5.3.7. *Let P be a d-dimensional reflexive polytope in $M_{\mathbb{R}}$.*

For $m \in \mathcal{R}$ with $F := \mathcal{F}_m$ we have:

1. *$P \subseteq F - \mathbb{R}_{\geq 0}x$, $P \cap M \subseteq (F \cap M) - \mathbb{N}x$, $\{n \in P^* \cap N : \langle n, m\rangle < 0\} = \{\eta_m\}$.*
2. *$P = \operatorname{conv}(F, F')$ iff there is only one facet F' with $\langle \eta_{F'}, m\rangle > 0$.*
3. *$m \in \mathcal{S}$ iff the previous condition is satisfied and $\langle \eta_{F'}, m\rangle = 1$.*

   In this case $F' = \mathcal{F}_{-m}$. Furthermore F and F' are naturally isomorphic as lattice polytopes and $\{n \in P^* \cap N : \langle n, m\rangle \neq 0\} = \{\eta_m, \eta_{-m}\}$.

Lemma 5.3.8. *Let P be a d-dimensional reflexive polytope in $M_{\mathbb{R}}$.*

For $v \in \mathcal{V}(P^*)$ we denote by $v^* \in \mathcal{F}(P)$ the corresponding facet of P. Then

$$\sum_{v \in \mathcal{V}(P^*)} \operatorname{rvol}(v^*)\, v = 0.$$

Proof. Having chosen a fixed lattice basis of M we denote by vol the associated differential-geometric volume in $M_{\mathbb{R}} \cong \mathbb{R}^d$. Let $F \in \mathcal{F}(P)$ arbitrary. Since η_F is primitive, it is a well-known fact that the determinant of the lattice $\operatorname{aff}(F) \cap M$, i.e., the volume of a fundamental paralleloped, is exactly $\|\eta_F\|$, hence we get $\operatorname{vol}(F) = \operatorname{rvol}(F) \cdot \|\eta_F\|$. The easy direction of the so called existence theorem of Minkowski (see [BF71, no. 60]) yields $\sum_{F \in \mathcal{F}(P)} \operatorname{rvol}(F)\, \eta_F = 0$.

□

The approximation approach in the next proof is based upon an idea of Batyrev.

Lemma 5.3.9. *Let $Q \subseteq M_{\mathbb{R}}$ be a d-dimensional polytope with a facet F and $x \in \operatorname{aff}(F)$ such that $Q \subseteq F - \mathbb{R}_{\geq 0}x$. For $q \in Q$ with $q = y - \lambda x$, where $y \in F$ and $\lambda \in \mathbb{R}_{\geq 0}$, we define $A(q) := y - 2\lambda x$. This definition extends uniquely to an affine map A of $M_{\mathbb{R}}$.*

Then $A(b_Q)$ is either in the interior of Q or in the relative interior of a facet of Q not parallel to $\mathbb{R}x$. The last case happens exactly iff there exists only one facet $F' \neq F$ not parallel to $\mathbb{R}x$.

Proof. First assume there is exactly one facet $F' \neq F$ not parallel to $\mathbb{R}x$. This implies $Q = \operatorname{conv}(F, F')$. Choose an $\mathbb{R}$-basis $e_1, \ldots, e_d$ of $M_{\mathbb{R}}$ such that $e_d = x$ and $\mathbb{R}e_1, \ldots, \mathbb{R}e_{d-1}$ are parallel to F. Now let $y \in F$ and define $h(y) \in \mathbb{R}_{\geq 0}$ such that $y - h(y)x \in F'$. For $k \in \mathbb{N}_{>0}$ let $F_k(y) := y + \cup_{i=1}^{d-1} [-1/(2k), 1/(2k)]e_i$ and $Q_k(y) := F_k(y) - [0, h(y)]x$. Then $b_{Q_k(y)} = y - h(y)/2x$ and $A(b_{Q_k(y)}) = y - h(y)x \in F'$. Let $M' :=< e_1, \ldots, e_{d-1} >_{\mathbb{Z}}$ and $z \in \operatorname{relint} F$. For any $k \in \mathbb{N}_{>0}$

we define $G_k := (z + M'/k) \cap F$ and $F_k := \cup_{y \in G_k} F_k(y)$. For $k \to \infty$ the sets F_k converge uniformly to F. Therefore also $Q_k := \cup_{y \in G_k} Q_k(y)$ converges uniformly to Q for $k \to \infty$. This implies that b_{Q_k} converges to b_Q for $k \to \infty$. Since A is affine and b_{Q_k} is a finite convex combination of $\{b_{Q_k(y)} : y \in G_k\}$ for any $k \in \mathbb{N}_{>0}$, also $A(b_{Q_k})$ is a finite convex combination of $\{A(b_{Q_k(y)} : y \in G_k\} \subseteq F'$ for any $k \in \mathbb{N}_{>0}$. This implies $A(b_{Q_k}) \in F'$ for any $k \in \mathbb{N}_{>0}$. Since A is continuous and F' is closed, this yields $A(b_Q) \in F'$. Moreover obviously $A(b_Q) \in \operatorname{relint} F'$.

Now let there be more than one facet different from F and not parallel to $\mathbb{R}x$. Then we can choose a polyhedral subdivision of Q into finitely many polytopes $\{K_j\}$ such that any K_j satifies the assumption of the previous case. Hence, since b_Q is a proper convex combination of $\{b_{K_j}\}$, also $A(b_Q)$ is a proper convex combination of $\{A(b_{K_j})\} \subseteq \partial Q$. However since not all $A(b_{K_j})$ are contained in one facet, $A(b_Q)$ is in the interior of Q.

□

Proof of Theorem 5.3.1. The first part of the theorem, when X is a complete toric variety, follows from 5.1.18, 5.3.4 and 5.3.5. So let $X = X_P$ for $P \subseteq M_{\mathbb{R}}$ a d-dimensional reflexive polytope, and we consider the second part of the theorem.

(a) and (b) are equivalent by 5.3.5. The equivalences of $(a), (c), (d), (e)$ and the sufficiency of iii., iv., v. follow from 5.3.4 and 5.3.7.

(f) and (g) are necessary conditions for semisimplicity due to 5.3.7.

Let (f) be satisfied and $x \in \mathcal{R}$. By 5.3.7(1) and 5.3.8 we have

$$\operatorname{rvol}(\mathcal{F}_x) = \sum_{v \in \mathcal{V}(P^*),\, \langle v,x\rangle > 0} \operatorname{rvol}(v^*)\langle v, x\rangle.$$

By assumption there is only one vertex $v \in \mathcal{V}(P^*)$ with $\langle v, x\rangle > 0$, furthermore $\langle v, x\rangle = 1$. Hence 5.3.7 implies $x \in \mathcal{S}$.

Let (g) be satisfied. Let $x \in \mathcal{R}$, $F := \mathcal{F}_x$ and $F' \in \mathcal{F}(P)$ with $\langle \eta_{F'}, x\rangle > 0$. Due to 5.3.7(1) and by assumption there is a bijective map $h : F' \to F$ of lattice polytopes, i.e., $h(F' \cap M) = F \cap M$. Now there exists a lattice point $x' \in F'$ with $h(x') = x$, necessarily $x' = -x \in \operatorname{relint} F'$, so $x \in \mathcal{S}$.

The sufficiency of vi., vii. is now trivial, 5.3.8 shows that vi. implies v.

From now on let $x \in \mathcal{R}$ and A the affine map defined as in 5.3.9 for $Q := P$ and $F := \mathcal{F}_x$.

Let i. be satisfied. By 5.3.7(1) we can apply Lemma 5.3.9 to get $-x = x - 2x = A(0) = A(b_P) \in \mathcal{R}$, since P is a canonical Fano polytope.

Finally let ii. be satisfied. For any $y \in F \cap M$ define $x_y \in P \cap M$ with $x_y := y - kx$ for $k \in \mathbb{N}$ maximal, and let $T_y := [y, x_y]$. Then 5.3.7(1) implies that

$$\begin{aligned} -x &= A(0) = A\left(\tfrac{1}{|P\cap M|}\sum_{m\in P\cap M} m\right) = A\left(\sum_{y\in F\cap M}\tfrac{|T_y\cap M|}{|P\cap M|}\tfrac{1}{|T_y\cap M|}\sum_{m\in T_y\cap M} m\right)\\ &= \sum_{y\in F\cap M}\tfrac{|T_y\cap M|}{|P\cap M|}A\left(\tfrac{1}{|T_y\cap M|}\sum_{m\in T_y\cap M} m\right) = \sum_{y\in F\cap M}\tfrac{|T_y\cap M|}{|P\cap M|}x_y.\end{aligned}$$

Hence $-x$ is a proper convex combination of $\{x_y\}_{y\in F\cap M}$, so $-x \in \mathcal{R}$.

□

5.4 Symmetric toric varieties

Definition 5.4.1. A toric variety $X = X(N, \triangle)$ is called *symmetric*, if the linear automorphism group of the fan $\triangle$ has only the origin as a fixpoint.

A lattice polytope $Q \subseteq N_{\mathbb{R}}$ is called *symmetric*, if the group of linear automorphisms $\mathrm{Aut}_N(Q)$ of Q has only the origin as a fixpoint.

When X is a Fano variety, i.e., $\triangle = \Sigma_Q$ for a Fano polytope $Q \subseteq N_{\mathbb{R}}$, then X is symmetric iff Q is symmetric.

In particular centrally symmetric lattice polytopes are symmetric and products of projective spaces are symmetric. Moreover symmetry is invariant under dualizing as observed by Batyrev, hence the previous definition equals the one given in the introduction:

Proposition 5.4.2. *Let $P \subseteq M_{\mathbb{R}}$ a reflexive polytope. Then*

$$P \text{ is symmetric if and only if } P^* \text{ is symmetric.}$$

In particular a Gorenstein toric Fano variety X_P is symmetric if and only if P is symmetric.

Proof. Let $G := \mathrm{Aut}_M(P)$. We denote by $\mathrm{Fix}(G)$ the set of elements in $M_{\mathbb{R}}$ that are fixed by the action of G. Since G is a finite group, the theorem of Maschke yields an G-invariant $\mathbb{R}$-subvectorspace U of $M_{\mathbb{R}}$ such that $\mathrm{Fix}(G) \oplus_G U = M_{\mathbb{R}}$. Dualizing yields $\mathrm{Fix}(G)^* \oplus_{G^*} U^* = N_{\mathbb{R}}$. Hence $\dim_{\mathbb{R}}(\mathrm{Fix}(G)) \leq \dim_{\mathbb{R}}(\mathrm{Fix}(G^*))$. Symmetry yields $\dim_{\mathbb{R}}(\mathrm{Fix}(G)) = \dim_{\mathbb{R}}(\mathrm{Fix}(G^*))$. So G has non-trivial fixpoints iff G^* does. Since $\mathrm{Aut}_N(P^*) = \mathrm{Aut}_M(P)^*$, the proof is finished.

□

As explained in the introduction Batyrev and Selivanova obtained in [BS99] from the existence of an Einstein-Kähler metric, that if X_P is a nonsingular symmetric toric Fano variety, then P has to be semisimple. They asked whether a direct proof for this combinatorial result exists. Indeed there are even at least five essentially different proofs:

Corollary 5.4.3. *Let $P \subseteq M_{\mathbb{R}}$ be a reflexive polytope.*
If P is symmetric, then P and P^* are semisimple.

Proof. We easily see that the second equivalent and even the first five sufficient conditions in the second part of Theorem 5.3.1 are satisfied. For yet another proof we could use Corollary 5.2.7.

□

Furthermore the first part of Theorem 5.3.1 immediately yields a generalization to complete toric varieties:

Corollary 5.4.4. *Let X be a complete toric variety.*
If X is symmetric, then $\mathrm{Aut}(X)$ is reductive.

Since symmetric toric Fano varieties were of interest as natural examples of the existence of an Einstein-Kähler metric, Kreuzer gave in [Kre03a] a list of these polytopes up to dimension four using the computer database (finding 527 polytopes for $d = 4$). The author verified the list for $d \leq 3$:

Theorem 5.4.5. $E_2, E_2^*, E_1^2, (E_1^2)^*, \mathcal{Z}_2$ *are precisely the two-dimensional symmetric reflexive polytopes. There are* 31 *three-dimensional symmetric reflexive polytopes.*

Sketch of proof. For $d = 2$ by 5.4.3 and 5.2.5 we only have to look at the reflexive polytopes E_2^* and E_1^2 and the five del Pezzo-surfaces. Now the result is obvious.

For $d = 3$ we can assume that P is not centrally symmetric, since in this case there already exists a classification due to Wagner [Wag95] (for a straightforward approach by putting P in a box see Proposition 6.4.1).

Let's look at $G := \mathrm{Aut}_M(P) \subseteq \mathrm{GL}(M) \cong \mathrm{GL}_3(\mathbb{Z})$. In [Tah71] Tahara classified all finite subgroups of $\mathrm{GL}_3(\mathbb{Z})$. From this list we get that there are exactly 10 non-conjugated finite subgroups G of $\mathrm{GL}_3(\mathbb{Z})$ such $-\mathrm{id} \notin G$ and G is minimal with $\mathrm{Fix}(G) = \{0\}$. Six of these have order 4 and four have order 6.

The idea of the classification is to distinguish these 10 cases and to use the symmetries in G to determine P.

Here we will show how to proceed in two typical situations:

We observe in the list that except when $G \cong \mathbb{Z}/6\mathbb{Z}$ (which is a special and rather easy case) there is an element $g \in G$ that is conjugated by $\mathrm{GL}_3(\mathbb{Z})$ to $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, hence P has always a rotational symmetry.

The simplest situation occurs, when g is exactly the first matrix before, in the $\mathbb{Z}$-basis e_1, e_2, e_3. Let $v \in \mathcal{V}(P)$ with $v_3 \neq 0$. Then $(v + v^g)/2 = (0, 0, v_3) \in P \cap M$. As P is canonical, $(0, 0, v_3) \in \partial P \cap M$, and since any lattice point on the boundary is primitive, we get $v_3 = \pm 1$. By assumption the sum of all vertices of P is zero, P is canonical but not centrally symmetric, so we can easily prove that $P_{\pm 1} := P \cap \{x \in M_{\mathbb{R}} \,:\, x_3 = \pm 1\}$ are either both one- or both two-dimensional faces of P with $\pm e_3 \in P_{\pm 1}$. However if both were two-dimensional, then by projecting along $\pm e_3$ Prop. 3.2.2 implied that P were just a prisma over centrally symmetric facets, so itself centrally symmetric, a contradiction. Hence $P_1 = [v, v^g]$ and $P_{-1} = [w, w^g]$. By looking at the list we can assume that $|\,G\,| = 4$, hence G is abelian. Now since P is symmetric, it is easy to see that there exists $h \in G$ that does not fix e_3. So $(e_3)^h = (v{+}v^g)^h/2 = (v^h{+}(v^h)^g)/2 = (0, 0, (v^h)_3)$, hence h has to exchange $e_3 \leftrightarrow -e_3$. We set $w := v^h$. Let $v \sim -e_3$, then $\mathrm{conv}(P_{-1}, v)$ are contained in a facet. By applying g, h, hg we get that also $\mathrm{conv}(P_{-1}, v^g)$, $\mathrm{conv}(P_1, w)$, $\mathrm{conv}(P_1, w^g)$ are contained in facets, so P is just a simplex. Therefore we can assume by 3.3.1 that also $v{-}e_3, v^g{-}e_3, w{+}e_3, w^g{+}e_3$ are contained in $P_0 := P \cap \{x \in M_{\mathbb{R}} \,:\, x_3 = 0\}$, and furthermore any lattice point in P_0 different from these four has to be in $\mathrm{st}(e_3) \cap \mathrm{st}(-e_3)$. This yields that $\pi_{e_3}(P) = P_0 = \pi_{-e_3}(P)$ is a canonical, hence reflexive polygon, so using the classification in 3.4.1 it is a straightforward calculation to determine P.

Another typical argument can be explained in the case $G = <\sigma> \cong \mathbb{Z}/4\mathbb{Z}$ for $\sigma = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & -1 \end{pmatrix}$. Let $v \in \partial P \cap M$. Here σ^2 is conjugated to the second matrix above. We have $m := \frac{v + v^{\sigma^2}}{2} = (0, 0, \frac{v_1 + v_2 + 2v_3}{2})$. Since P is canonical, this implies

$$v_1 + v_2 + 2v_3 \in \{0, \pm 1, \pm 2\}.$$

On the other hand $y := \frac{v+v^\sigma}{2} = (\frac{v_1+v_2}{2}, \frac{-v_1+v_2}{2}, \frac{-v_2}{2}) \in P_0 := P \cap \{x \in M_\mathbb{R} : \langle (1,1,2), x\rangle = 0\}$. Furthermore if $(v_1, v_2) \neq (0,0)$, then $C := \operatorname{conv}(y, y^\sigma, y^{\sigma^2} = -y, y^{\sigma^3} = -y^\sigma)$ is a two-dimensional centrally symmetric polytope contained in P_0 that does not contain interior lattice points. Now the lattice point theorem of Minkowski implies that the volume of C (measured with respect to a $\mathbb{Z}$-basis of $P_0 \cap M$) is at most 4. From this we can calculate

$$v_1^2 + v_2^2 \leq 8.$$

Using these two equations we can put P in a (capped) cylinder. Now we have to calculate the G-orbits of the lattice points in this container and check out in a rather tedious calculation, how they can be united to get a reflexive polytope.

Most other cases are treated in an analogous manner.

□

5.5 Successive sums of lattice points

Here we continue the discussion on lattice points from section 1.5 and subsection 3.7.1 by looking not at the *number* of lattice points in multiples of a reflexive polytope but at their *sum*. This leads to a variant of the Erhart-polynomial and another condition for semisimplicity. The most part of this section was done in collaboration with V. V. Batyrev and M. Kreuzer.

Let $Q \subseteq M_\mathbb{R}$ be an n-dimensional lattice polytope.

Definition 5.5.1. We define for $k \in \mathbb{N}$ the *k-th lattice point sum* $s_Q(k)$ of Q as

$$s_Q(k) := \sum_{v \in kP \cap M} v \in M.$$

Furthermore the *weighted barycenter* of Q is defined as

$$wb_Q := b_Q \operatorname{rvol}(Q) \in M_\mathbb{R}.$$

There is now the following result, that must be seen in total similarity to the theorem of Ehrhart 1.5.3:

Proposition 5.5.2 (Kreuzer, Nill). *For any $i \in \{1, \ldots, d\}$ there exists a unique polynomial $s_i(X) \in \mathbb{Q}[X]$ such that $s_i(k) = s_Q(k)_i$ for all $k \in \mathbb{N}$. Any s_i has degree $\leq n+1$.*

We define $s_Q := (s_1, \ldots, s_d)$, and let $\operatorname{coeff}_l(s_Q) := (\operatorname{coeff}_l(s_1), \ldots, \operatorname{coeff}_l(s_d))$ denote the (vector-)coefficient of degree $l \in \mathbb{N}$. Then one has

$$\operatorname{coeff}_{n+1}(s_Q) = wb_Q,$$

$$\operatorname{coeff}_n(s_Q) = \frac{1}{2} \sum_{F \in \mathcal{F}(Q)} wb_F,$$

$$\operatorname{coeff}_0(s_Q) = 0.$$

The following reciprocity law *holds:*

$$\sum_{v \in \operatorname{relint}(kQ) \cap M} v = (-1)^{n+1} s_Q(-k) \ \forall k \in \mathbb{N}_{>0}.$$

This result shall be illustrated by the example of the standard smooth simplex $C_n := \mathrm{conv}(0, e_1, \ldots, e_n)$. Calculations as explained in [Sta86] show that

$$G(X_1, \ldots, X_{n+1}) := \sum_{\mu \in \mathrm{pos}(C_n \times \{1\})} X^\mu = \frac{1}{1 - X_1 X_{n+1}} \cdots \frac{1}{1 - X_n X_{n+1}} \frac{1}{1 - X_{n+1}}$$

Therefore one has $\sum\limits_{k \in \mathbb{N}} e(k) t^k = G(1, \ldots, 1, t) = \frac{1}{1-t^{n+1}}$. This gives

$$e_{C_n}(k) = \binom{k+n}{n}$$

Fix now $i \in \{1, \ldots, n\}$. Now $\sum\limits_{k \in \mathbb{N}} s_i(k) t^k = (\partial_i G)(1, \ldots, 1, t) = \frac{t}{1-t^{n+2}}$. This gives

$$s_{C_n}(k)_i = \binom{k+n}{n+1}$$

Obviously $b_{C_n} = (1/(n+1))(e_1 + \ldots + e_n)$. Hence

$$(wb_{C_n})_i = (b_{C_n})_i \mathrm{rvol}(C_n) = \frac{1}{n+1} \frac{1}{n!} = \frac{1}{(n+1)!} = \mathrm{coeff}_{n+1}((s_{C_n})_i)$$

And

$$\begin{aligned} \frac{1}{2} \sum_{F \in \mathcal{F}(C_n)} (wb_F)_i &= \frac{1}{2} \left((wb_{\mathrm{conv}(e_1, \ldots, e_n)})_i + \sum_{j=1}^n (wb_{\mathrm{conv}(0, e_l \,:\, l=1,\ldots,n; l \neq j)})_i \right) \\ &= \frac{1}{2} \left(\frac{1}{n!} + (n-1) \frac{1}{n!} \right) = \frac{1}{2(n-1)!} = \mathrm{coeff}_n((s_{C_n})_i) \end{aligned}$$

Proof. The proof that $s_i(k)$ gives a polynomial of degree $\leq n+1$ and the reciprocity law follows analogously as the proof of the theorem of Ehrhart given in the book of Stanley [Sta86], relying on the result that $G(X_1, \ldots, X_{n+1}) := \sum\limits_{\mu \in \mathrm{pos}(Q \times \{1\})} X^\mu$ is a special rational function satisfying a reciprocity law and the observation $\sum\limits_{k \in \mathbb{N}} s_i(k) t^k = (\partial_i G(X_1, \ldots, X_{n+1}))(1, \ldots, 1, t)$. This is explained in detail by Kreuzer in the manuscript [Kre03b].

However this result can also be derived from the general result Thm. 1.5.4 due to Brion and Vergne (let ϕ be the identity map).

For calculating the highest coefficient we refine a covering of the relative interior of Q by n-dimensional cubes of length $1/k$ (in the subspace $\mathrm{aff}(Q) \cap M_\mathbb{R}$) centered at the points in $Q \cap \frac{1}{k} M$. This yields

$$wb_Q = \frac{\int_Q x \, dx}{\det \mathrm{aff} Q} = \lim_{k \to \infty} \sum_{w \in Q \cap \frac{1}{k} M} w \frac{1}{k^n} = \lim_{k \to \infty} \frac{s_Q(k)}{k} \frac{1}{k^n} = \mathrm{coeff}_{n+1}(s_Q).$$

The formula for determining the second-highest coefficient was done by the author (inspired by the proof of the original Erhart theorem due to Betke and Kneser [BK85]).

It is straightforward to prove the following observation:

Transformation formula: Let $T := m \mapsto Lm + v$ be an affine unimodular transformation, so $L \in \mathrm{GL}(M)$ and $v \in M$. Then

$$s_{T(Q)}(x) = Ls_Q(x) + e_Q(x)vx.$$

Now we define an auxiliary function where $Q' \subseteq M_\mathbb{R}$ is a lattice polytope of dimension $\leq n$:

$$w(Q') := \begin{cases} \frac{1}{2}\sum_{F \in \mathcal{F}(Q')} wb_F & \text{, if } \dim(Q') = n \\ wb_{Q'} & \text{, if } \dim(Q') = n-1 \\ 0 & \text{, if } \dim(Q') < n-1 \end{cases}$$

In the above example it was shown that $w(C_j) = \mathrm{coeff}_n(s_{C_j})$ for the standard smooth simplex $C_j := \mathrm{conv}(0, e_1, \ldots, e_j)$ for $j = 1, \ldots, n$. Now take any *primitive* lattice simplex C of dimension j, i.e., a lattice simplex where the differences of its vertices generate the affine lattice, then there exists an affine unimodular transformation from C_j to C. Using the above transformation rule one verifys that also in this case $w(C) = \mathrm{coeff}_n(s_C)$ holds.

Now there is a well-known theorem [KKMS73] stating that there exists a natural number $l \in \mathbb{N}_{>0}$ such that lQ can be triangulated into primitive simplices of dimension n.

If Q', Q'' are two lattice polytopes of the dimension n intersecting in a lattice polytope contained in mutual facets, then obviously $s_{Q'+Q''} = s_{Q'} + s_{Q''} - s_{Q' \cap Q''}$, so also $\mathrm{coeff}_n(s_{Q'+Q''}) = \mathrm{coeff}_n(s_{Q'}) + \mathrm{coeff}_n(s_{Q''}) - \mathrm{coeff}_n(s_{Q' \cap Q''})$. On the other hand by looking at coeff_{n+1} one gets the formula $wb_{Q'+Q''} = wb_{Q'} + wb_{Q''}$. Therefore it follows easily $w(Q' + Q'') = w(Q') + w(Q'') - w(Q' \cap Q'')$.

Using these results we get $l^n 1/2 \sum_{F \in \mathcal{F}(Q)} wb_F = 1/2 \sum_{F \in \mathcal{F}(Q)} wb_{lF} = w(lQ) = \mathrm{coeff}_n(s_{lQ}(x)) = \mathrm{coeff}_n(s_Q(lx)) = l^n \mathrm{coeff}_n(s_Q)$.

□

Corollary 5.5.3 (Batyrev,Kreuzer). *Let $P \subseteq M_\mathbb{R}$ be a reflexive polytope. Then the (vector-)polynomial s_P is determined by the values for $k = 1, \ldots, \lfloor \frac{d+1}{2} \rfloor$. Moreover*

$$wb_P = \frac{1}{d+1} \sum_{F \in \mathcal{F}(P)} wb_F, \quad \mathrm{coeff}_d(s_P) = \frac{d+1}{2} \mathrm{coeff}_{d+1}(s_P).$$

Proof. By Proposition 3.7.2 we get

$$s_P(k) = \sum_{v \in \mathrm{relint}((k+1)Q) \cap M} v = (-1)^{d+1} s_P(-k-1) \; \forall k \in \mathbb{N}$$

Therefore $s_P(k)$ for $k = 0, \ldots, \lfloor \frac{d+1}{2} \rfloor$ determines $\geq d+2$ values of s_P, hence the (vector-)polynomial s_P.

Also by 3.1.4(6) one can embed any facet $F \in \mathcal{F}(Q)$ in a hyperplane of lattice distance one from the origin. Then one calculates the formula $wb_{\mathrm{conv}(0,F)} = \frac{1}{d+1} wb_F$.

□

Now back to the set of roots:

Here Condition (2b) of Theorem 5.3.1 leads one to consider sums of roots. From the reciprocity law in Prop. 5.5.2 we get:

Corollary 5.5.4. *Let Q be an n-dimensional lattice polytope in $M_{\mathbb{R}}$.*

For any $i \in \{1, \ldots, d\}$ there exists a unique polynomial $r_i(X) \in \mathbb{Q}[X]$ such that

$$r_i(k) = \sum_{x \in \mathcal{R}(kQ)} x_i \text{ for all } k \in \mathbb{N},$$

where $\mathcal{R}$ is the set of lattice points in the relative interior of facets.

For any i the polynomial $r_i(X) = (-1)^n \sum_{F \in \mathcal{F}(Q)} (s_F)_i(-X)$ has degree $\leq n$.

We define $r_Q := (r_1, \ldots, r_d)$. Then

$$\operatorname{coeff}_n(r_Q) = \sum_{F \in \mathcal{F}(Q)} wb_F = 2 \operatorname{coeff}_n(s_Q).$$

The following corollary gives a summary (use also 5.3.1, 5.5.3):

Corollary 5.5.5. *Let $P \subseteq M_{\mathbb{R}}$ be a reflexive polytope.*

Then $\operatorname{coeff}_d(r_P) = (d+1)wb_P$.

In particular we get by 5.3.1 the following implications:

1. *P centrally symmetric $\Rightarrow$ P symmetric $\Rightarrow$ $s_P = 0$ and $r_P = 0$*
2. *$s_P = 0$ or $r_P = 0$ $\Rightarrow$ $b_P = 0$ $\Rightarrow$ P, P^* semisimple*
3. *$s_P = 0 \iff s_P(k) = 0$ for $k = 1, \ldots, \lfloor \frac{d+1}{2} \rfloor$*
4. *$s_P(1) = 0$ $\Rightarrow$ P, P^* semisimple*
5. *P semisimple $\iff r_P(1) = 0$*

5.6 Examples

We discuss possible implications among above observed conditions for a reflexive polytope $P \subseteq M_{\mathbb{R}}$ by presenting several examples.

Here we are interested in the following seven conditions:

0: P symmetric

1: $s_P = 0$

2: $b_P = 0$

3: the sum of the lattice points of P is 0 (i.e., $s_P(1) = 0$)

4: the sum of the vertices of P is 0

5: $r_P = 0$

6: P is semisimple (i.e., $r_P(1) = 0$)

Together with the corresponding statements for P^* that will be enumerated as $0^*, 1^*, \ldots, 6^*$ these are 14 conditions.

By Proposition 5.4.2 0 holds iff 0^*, and obviously 0 implies any other condition $1, \ldots, 6, 1^*, \ldots, 6^*$. No reverse implication holds as can be immediately seen by below table (e.g., 1 does not imply 4, hence not 0).

The following tables summarize possible implications:

$\Rightarrow$	1	2	3	4	5	6
1	y	y	y	n4c	n4a	y
2	n3b	y	n3b	n4b	n3b	y
3	n3a	n3a	y	n3c	n3a	y
4	n3a	n3a	n2	y	n3a	n2
5	n4b	y	n4b	n4b	y	y
6	n2	n2	n2	n2	n2	y

$\Rightarrow$	1^*	2^*	3^*	4^*	5^*	6^*
1	n4a	n4a	n4a	n4a	n4a	y
2	n3b	n4a	n3b	n4a	n3b	y
3	n3a	n3a	n3a	n3c	n3a	y
4	n3a	n3a	n3a	n2	n3a	y
5	n4b	n4b	n4c	n4b	n4b	y
6	n2	n2	n2	n3c	n2	n2

Note that by duality, since P is reflexive, e.g., $3 \Rightarrow 6^*$ holds iff $3^* \Rightarrow 6$ holds. In the table "y" means that the implication holds (see Theorem 5.3.1 and Corollary 5.5.5), however "n3a" for instance means that example 3a below gives a counterexample of minimal dimension 3. "n2" means that one finds a simple example by looking at the 16 reflexive polygons in Proposition 3.4.1.

In particular we see that we have found all possible pairwise implications among these conditions. Furthermore no condition "dualizes"!

Moreover Kreuzer (and the author) observed that $0 \Leftrightarrow 1 \Leftrightarrow 2 \Leftrightarrow 5$ for $d = 2$, and $0 \Leftrightarrow 1 \Leftrightarrow 5$ for $d = 3$. For a complete list of reflexive polytopes satisfying 0, 1 or 2 with $d \leq 4$ we refer to [Kre03a].

Next we give all the examples. They were found using the classification of d-dimensional reflexive polytopes for $d \leq 4$ and the computer program PALP due to Kreuzer and Skarke (see [KS04a, KS04b]).

Here the column vectors are the vertices of the reflexive polytope:

$$\begin{array}{cc} \text{3a} & \begin{array}{cccccc} 1 & -1 & 0 & -2 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array} \end{array}$$

$$\begin{array}{cc} \text{3b} & \begin{array}{cccc} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 4 & -4 \end{array} \end{array}$$

$$\begin{array}{cc} \text{3c} & \begin{array}{cccccc} 1 & 0 & 1 & -2 & 0 & 1 \\ 0 & 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array} \end{array}$$

$$\text{4a} \quad \begin{array}{ccccccc} 1 & 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{array}$$

$$\text{4b} \quad \begin{array}{ccccccccc} -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & -1 & 1 & -1 \\ -1 & -1 & -1 & -1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & -1 & -1 & -1 & -1 \end{array}$$

$$\begin{array}{cccccccccccc} 1 & 0 & 1 & 1 & 0 & -1 & -1 & -1 & 0 & 0 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 1 & 0 \end{array}$$

$$\text{4c} \quad \begin{array}{ccccccc} 1 & 0 & 0 & 1 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{array}$$

Chapter 6

Centrally symmetric reflexive polytopes

Introduction

In this chapter we give as a final application some insight in centrally symmetric reflexive polytopes; they correspond to Gorenstein toric Fano varieties with a toric morphism of order two fixing only the neutral element of the torus. Here the set of roots and the maximal number of vertices and lattice points are examined. It is further investigated, whether centrally symmetric polytopes can be embedded into the unit lattice cube. Moreover a complete classification in the simplicial case is given, even if only a centrally symmetric pair of facets exists. This generalizes results of Ewald [Ewa88] and his students [Wag95, Wir97].

Summary of most important new results of this chapter:

- Two different proofs that a centrally symmetric d-dimensional reflexive polytopes has at most 2d lattice points in interior of facets, with equality only for $[-1,1]^d$ (Thm. 6.1.1, p. 148; Prop. 6.1.4, p. 149)
- Proof of Conjecture 3.5.2 on the maximal number of vertices of a centrally symmetric simple reflexive polytope (Thm. 6.2.2, p. 149)
- Characterization of cases of maximal number of vertices of a simplicial reflexive polytope with a centrally symmetric pair of facets (Thm. 6.2.4, p. 150)
- Classification of simplicial reflexive polytopes with a centrally symmetric pair of facets (Thm. 6.3.1, p. 151; Cor. 6.3.3, p. 153)
- There are $4, 5, 15, 20$ isomorphism classes of 2,3,4,5-dimensional simplicial reflexive polytopes with a centrally symmetric pair of facets (Thm. 6.3.12, p. 156)
- Any d-dimensional simplicial reflexive polytope with a centrally symmetric pair of facets can be embedded into $[-1,1]^d$. (Cor. 6.4.2, p. 158)
- Any d-dimensional centrally symmetric simple reflexive polytope can be embedded into $\lfloor d/2 \rfloor [-1,1]^d$. (Cor. 6.4.3, p. 158)

- A general result on embedding a reflexive polytope into a small multiple of $[-1,1]^d$ (Thm. 6.4.4, p. 159; Cor. 6.4.8, p. 160)
- A d-dimensional centrally symmetric reflexive polytope has at most 3^d lattice points, with equality only for $[-1,1]^d$ (Thm. 6.5.1, p. 161)
- A d-dimensional simplicial reflexive polytope with a centrally symmetric pair of facets has at most $2d^2+1$ lattice points, with uniqueness of the equality case (Thm. 6.5.3, p. 163)

6.1 Roots

In this section the following result is going to be proved (recall the definition of the lattice polytope $E_1 := [-1,1]$).

Theorem 6.1.1. *Let $P \subseteq M_{\mathbb{R}}$ be a centrally symmetric reflexive polytope with X_P the toric variety associated to $\mathcal{N}_P$.*

1. *$P \cong E_1^{\frac{|\mathcal{R}|}{2}} \times G$ for a $\frac{|\mathcal{R}|}{2}$-codimensional face G of P that is a centrally symmetric reflexive polytope (with respect to* $\operatorname{aff}(G) \cap M$ *and a unique lattice point in* $\operatorname{relint} G$*) and has no roots itself.*
2. *Any facet has at most one root of P. P contains at most $2d$ roots. Hence*
$$\dim \operatorname{Aut}^\circ(X_P) \leq 3d.$$
3. *The following statements are equivalent:*
   (a) *P has $2d$ roots, i.e.,* $\dim \operatorname{Aut}^\circ(X_P) = 3d$
   (b) *Every facet of P contains a root of P*
   (c) *$P \cong E_1^d$, i.e., $X_P \cong \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$*

The first property immediately implies (see 5.1.3):

Corollary 6.1.2. *Let P be a centrally symmetric reflexive polytope with X_P the toric variety associated to $\mathcal{N}_P$.*

If P contains no facet that is centrally symmetric with respect to a root of P, or there are at most $d-1$ facets of P that can be decomposed as a product of lattice polytopes $E_1 \times F'$, then P has no roots.

Hence if $d \geq 3$ and P is simplicial, or $d \geq 4$ and any facet of P is simplicial, then
$$\dim \operatorname{Aut}^\circ(X_P) = d.$$

For the proof of Theorem 6.1.1 we need the following lemma that is an easy corollary of 5.3.7 and 3.1.4(11):

Lemma 6.1.3. *Let P be a centrally symmetric reflexive polytope.*
Let $F \in \mathcal{F}(P)$. Then
$$P \cong E_1 \times F \text{ iff } F \text{ contains a root } x \text{ of } P.$$
In this case F is a centrally symmetric reflexive polytope (with respect to the lattice $\operatorname{aff}(F) \cap M$ *with origin x).*

Proof of Theorem 6.1.1. 1. Apply the previous lemma inductively.

2. Since as just seen any facet of P containing a root is reflexive, hence a canonical Fano polytope, it contains only one root of P. Now we apply 5.1.3(2) and 1.

3. Since P as a centrally symmetric polytope contains at least $2d$ facets, we derive the equivalences from 1.

□

An alternative proof can be derived from the following algebraic-geometric proposition that uses the results of the previous chapter:

Proposition 6.1.4. *Let $X = X(N, \triangle)$ be a complete toric variety with centrally symmetric $\triangle(1)$. Then*

$$\dim \operatorname{Aut}^\circ(X) \leq 3d, \text{ with equality if and only if } X \cong (\mathbb{P}^1)^d.$$

Proof. By 5.1.3 and 5.1.23(3) it remains to show that for $\tau \in \triangle(1)$ there is at most one root $m \in \mathcal{R}$ such that $\eta_m = \tau$. However central symmetry yields that such a root is characterized by $\langle \pm v_\tau, m \rangle = \mp 1$ and $\langle v_{\tau'}, m \rangle = 0$ for $\tau' \neq \pm\tau$. Since $\triangle$ is complete, this gives the uniqueness of such a root.

□

6.2 Vertices

Let $P \subseteq M_\mathbb{R}$ be a d-dimensional reflexive polytope.

By Conjecture 3.5.2 the maximal number of vertices P can have is obtained only for the simple centrally symmetric reflexive polytope $(\mathcal{Z}_2)^{\frac{d}{2}}$ for d even. For $d = 2$ a direct proof is given in 3.3.3, for $d = 3$ we have the following result:

Proposition 6.2.1. *Any three-dimensional polytope with a centrally symmetric pair of facets has at most* 14 *vertices.*

Proof. By the classification of two-dimensional reflexive polytopes and Lemma 6.1.3 we can assume that there are no lattice points in the interior of facets. However any two-dimensional lattice polytope with no interior points can have at most four vertices as is easily seen. So we can assume there are two facets F and $-F$ with normal vector u having at most four vertices. Then the convex hull Q of vertices in $u^\perp$ has either no interior lattice points, so again there are only at most four vertices of Q, or it has the origin of the lattice as its only interior lattice point, so Q is a canonical, hence reflexive polygon and has therefore by 3.3.3 at most six vertices. So overall there are at most $4 + 4 + 6 = 14$ vertices.

□

The only non-trivial result valid in higher dimensions is the following theorem, it will be proven in the next section on page 155:

Theorem 6.2.2. *Conjecture 3.5.2 holds for duals of simplicial reflexive polytopes having a centrally symmetric pair of facets. In particular it holds for centrally symmetric simple reflexive polytopes.*

In the simplicial case we have a direct proof of Conjecture 3.5.9 in the case of a centrally symmetric pair of facets.

For this we need the following important observation:

Lemma 6.2.3. *Let P be a simplicial reflexive polytope. Let $F \in \mathcal{F}(P)$ such that also $-F \in \mathcal{F}(P)$. We set $u := \eta_F \in \mathcal{V}(P^*)$.*

Let $\mathcal{V}(F) = \{e_1, \ldots, e_d\}$, and $e_1^*, \ldots, e_d^*$ be the dual $\mathbb{R}$-basis of $N_{\mathbb{R}}$. For $i = 1, \ldots, d$ we denote by F_i the unique facet of P such that $F_i \cap F = \operatorname{conv}(e_j : j \neq i)$.

Let $v \in \mathcal{V}(P) \cap u^\perp$. We write $v = \sum_{j=1}^d q_j e_j$ as a rational linear combination. Then for $i = 1, \ldots, d$:

$$q_i < 0 \iff q_i = -1 \iff v \in F_i.$$

In particular there are $I, J \subseteq \{1, \ldots, d\}$ with $I \cap J = \emptyset$ and $|I| = |J|$ such that

$$v = \sum_{j \in J} e_j - \sum_{i \in I} e_i.$$

Moreover $e_i^* = \eta_{F_i} - u \in P^* \cap N$ for any $i \in I$.

Proof. As in the proof of Lemma 3.5.6 we have for $i = 1, \ldots, d$ that $\eta_{F_i} = u + \alpha_i e_i^*$ for a positive natural number α_i. Moreover by 3.5.6(1) we already know that $q_i = \langle e_i^*, m \rangle < 0 \Leftrightarrow v \in F_i$.

So let $q_i = \langle e_i^*, m \rangle < 0$, $v \in F_i$. Since therefore $F_i \cap (-F) = \emptyset$, by 3.3.1 $\eta_{F_i} - u = \alpha_i e_i^* \in P^* \cap N$. Hence $-1 \leq \langle \alpha_i e_i^*, -e_i \rangle = -\alpha_i \leq -1$, so $\alpha_i = 1$ and $q_i = \langle e_i^*, m \rangle = \langle \eta_{F_i} - u, m \rangle = -1$.

Since $v = \sum_{j=1}^d (-q_j)(-e_j)$ and $u = \sum_{j=1}^d (-e_j^*)$, applying the previous statement to $-F$ finishes the proof.

□

As an immediate application we determine in any dimension, not only in even dimension, the simplicial reflexive polytopes with the maximal number of vertices in the case of a centrally symmetric pair of facets (an alternative proof can be derived from the classification result Cor. 6.3.3):

Theorem 6.2.4. *Let $P \subseteq M_{\mathbb{R}}$ be a simplicial reflexive polytope that contains a centrally symmetric pair of facets, e.g., P is centrally symmetric.*
Then $|\mathcal{V}(P)| \leq 3d$. Moreover:

$|\mathcal{V}(P)| = 3d$ if and only if d is even and $P^* \cong (\mathcal{Z}_2)^{\frac{d}{2}}$.

$|\mathcal{V}(P)| = 3d - 1$ if and only if d is odd and $P^* \cong [-1, 1] \times (\mathcal{Z}_2)^{\frac{d-1}{2}}$.

Proof. By Theorem 3.5.11 we only have to consider the following situation:

Let d be odd and $|\mathcal{V}(P)| = 3d-1$. Let $F \in \mathcal{F}(P)$ such that also $-F \in \mathcal{F}(P)$. We set $u := \eta_F \in \mathcal{V}(P^*)$. Let $\{v^1, \ldots, v^d\}$ denote the set of vertices not in F but in facets intersecting F in a codimension two face.

By Lemma 3.5.6 we get that $\mathcal{V}(P) \backslash (\mathcal{V}(F) \cup \mathcal{V}(F')) = \{v \in \mathcal{V}(P) : \langle u, v \rangle = 0\} \subseteq \{v^1, \ldots, v^d\}$. We may assume that

$$\{v \in \mathcal{V}(P) : \langle u, v \rangle = 0\} = \{v^1, \ldots, v^{d-1}\}$$

is a set of cardinality $d - 1$. We can enumerate the vertices of F as $e_1, \ldots, e_d$ such that v^i is a facet of $F_i := \operatorname{conv}(v^i, e_j : j \neq i)$ for $i = 1, \ldots, d-1$. We also define for $i = 1, \dots, d$ in the same way $F'_i := \operatorname{conv}(w^i, e_j \,:\, j \neq i)$ for a unique vertex w^i.

Let $I := \{i \in \{1, \dots, d\} \,:\, v^i \notin F_j \,\forall\, j \in \{1, \dots, d\} \backslash \{i\}\}$ and $I' := \{i' \in \{1, \dots, d\} \,:\, w^{i'} \notin F'_j \,\forall\, j \in \{1, \dots, d\} \backslash \{i'\}\}$. Let $i \in I$, then 6.2.3 yields $v^i = e_{\sigma(i)} - e_i = w^{\sigma(i)}$ for $\sigma(i) \in I'$. This defines a map $\sigma : I \to I'$. Moreover by symmetry we have a map $\psi : I' \to I$ defined by $w^{i'} = e_{i'} - e_{\psi(i')} = v^{\psi(i')}$. Obviously $\psi = \sigma^{-1}$, so σ is a fixpointfree permutation. Assume σ were not an involution. Then there exist $i \in I$, $j := \sigma(i)$, $k := \sigma(j) \neq i$, so $v^i = e_j - e_i$ and $v^j = e_k - e_j$. Hence $-e_i \not\sim e_j$ and $v^j \not\sim e_j$, so Lemma 3.5.15 yields a contradiction.

Therefore σ is just a product of disjoint transpositions. In particular $|I|$ is even. Assume $v^d \in u^\perp$. Then we may assume $v^d = v^{d-1}$, hence $I = \{1, \dots, d-2\}$, a contradiction to d odd. So $v^d \in -F$. Therefore $v^d = -e_d$. Since by 3.5.6(3) $e_1, \dots, e_d$ is a $\mathbb{Z}$-basis, P is uniquely determined up to isomorphism.

□

6.3 Classification theorem

In the case of a smooth Fano polytope where the centrally symmetric pairs of vertices span $M_\mathbb{R}$ there exists a complete explicit classification that is due to Casagrande (see [Cas03b]). However we cannot expect such a result for general centrally symmetric reflexive polytopes, since by the classification of Kreuzer and Skarke there are 150 centrally symmetric reflexive polytopes already in dimension four. For $d = 2$ we have 3 (see 3.4.1) and for $d = 3$ there are 13 (see [Wag95]) d-dimensional centrally symmetric reflexive polytopes.

However in [Wir97, Satz 3.3] there was a characterization of centrally symmetric reflexive polytopes presented that have the minimal number of $2d$ vertices. It is the goal of this section to generalize this, the main result will be presented in Corollary 6.3.3.

Theorem 6.3.1. *Let P be a simplicial reflexive polytope with facets $F, -F$.*

There exists a $\mathbb{Z}$-basis $m_1, \dots, m_d$ of M such that in this basis the matrix A consisting of the vertices $e_1, \dots, e_d$ of F as columns has the following properties: $A = \begin{pmatrix} 2\mathrm{id}_f & 0 \\ C & \mathrm{id}_{d-f} \end{pmatrix} \in \mathrm{Mat}_d(\mathbb{N})$ where $f \in \{0, \dots, d-1\}$ and $C \in \mathrm{Mat}_{(d-f)\times f}(\{0,1\})$ such that any column of C has an odd number of 1's.

For $i = 1, \dots, d$ we let v^i denote the unique vertex of P contained in a facet that intersects F in the codimension two face $\operatorname{conv}(e_j \,:\, j \neq i)$; obviously $v^i \notin \eta_F^\perp$ iff $v^i = -e_i$. Any vertex of P is in $\{\pm e_1, \dots, \pm e_d, v^1, \dots, v^d\}$.

There exist pairwise disjoint subsets $I_1, \dots, I_l \subseteq \{1, \dots, d\}$ and pairwise disjoint subsets $J_1, \dots, J_l \subseteq \{1, \dots, d\}$ with $I_k \cap J_k = \emptyset$ and $|I_k| = |J_k|$ for all $k = 1, \dots, l$ such that for $i \in \{1, \dots, d\}$ we have

$$v^i \in \eta_F^\perp \iff i \in \bigcup_{k=1}^{l} I_k \iff v^i = \sum_{j \in J_k} e_j - \sum_{i' \in I_k} e_{i'} \text{ for } i \in I_k.$$

In this case the ith row for $i \in I_1 \cup \cdots I_l \cup J_1 \cup \cdots \cup J_l$ is of the form $(0, \dots, 0, 1, 0, \dots, 0)$.

If for $k, k' \in \{1, \dots, l\}$ the sets I_k and $J_{k'}$ intersect, then $k \neq k'$, $I_k = J_{k'}$ and $J_k = I_{k'}$.

Proof. Let $\mathcal{V}(F) = \{e_1, \ldots, e_d\}$, u, F_i defined as in 6.2.3. Consider the following steps for the construction of A and $m_1, \ldots, m_d$:

1. Let $i \in \{1, \ldots, d\}$.
   If $v^i \in u^\perp$, by 6.2.3 $e_i^* = \eta_{F_i} - u \in N$.
   If $v^i \notin u^\perp$, obviously $v^i = -e_i$, so $e_i^* = \frac{\eta_{F_i} - u}{2} \in \frac{1}{2}N$.
   So in any case $e_1^*, \ldots, e_d^* \in \frac{1}{2}N$. In particular this yields
   $$2M \subseteq < e_1, \ldots, e_d >_{\mathbb{Z}} \subseteq M.$$

2. We define for an arbitrary Z-basis of M the matrix $B \in \mathrm{Mat}_d(\mathbb{Z}) \cap \mathrm{GL}_d(\mathbb{Q})$ consisting of the columns $e_1, \ldots, e_d$ in this basis. By Theorem 3.6.6 there is an unimodular transformation L of M such that $A := LB \in \mathrm{Mat}_d(\mathbb{N})$ is a lower triangular matrix with $A_{i,j} \in \{0, \ldots, A_{j,j} - 1\}$ for $i > j$. Then there is a $\mathbb{Z}$-basis $m_1, \ldots, m_d$ of M such that the columns of A are $e_1, \ldots, e_d$ in this basis.

3. The first point yields that $2m_i$ is contained in the column space of A, any diagonal element of A is contained in $\{1, 2\}$.

4. If $A_{i,j} = 1$ for $i > j$, then necessarily $A_{j,j} = 2$ and $A_{i,i} = 1$ (again since $2m_j$ is a $\mathbb{Z}$-linear combination of $e_j, \ldots, e_d$.)

5. Using the previous point we can easily get by possibly permutating the columns and the rows the desired form of A as a blockmatrix (since any vertex of a reflexive polytope is primitive, obviously $f \neq d$). It remains to show that any column has an odd number of 1's: By the previous point we get $e_j = 2m_j + \sum_{k=1}^s e_{i_k}$, where $i_k > j$. We get $2\langle \eta_F, m_j \rangle = \langle \eta_F, e_j - \sum_{k=1}^s e_{i_k} \rangle = -1 + s$, so s has to be odd.

Now using Cramer's rule we can calculate that $e_i^* \in N$ if and only if the ith row of A is of the form $(0, \ldots, 0, 1, 0, \ldots, 0)$.

By this observation, Lemma 6.2.3 (applied to F *and* $-F$) and the first point in the proof we get that it only remains to show the last remark:

So let $k, k' \in \{1, \ldots, l\}$ with $I_k \cap J_{k'} \neq \emptyset$. By construction $I_k \cap J_k = \emptyset$, hence $k \neq k'$.

Assume $I_{k'} \not\subseteq J_k$. Then there exists $i \in I_{k'}$, $i \notin J_k$. Let $j \in I_k \cap J_{k'}$. Now we define in the dual $\mathbb{R}$-basis $e_1^*, \ldots, e_d^*$ of $N_\mathbb{R}$ the vector $w := e_i^* - \sum_{s \neq i,j} e_s^*$. By construction it is easy to check that w is the inner normal of a face of P containing as vertices e_s for $s = 1, \ldots, d$ with $s \neq i, j$, as well as $-e_i$ and v^i and v^j. This is a contradiction to P being simplicial.

Hence $I_{k'} \subseteq J_k$. In particular $I_{k'} \cap J_k \neq \emptyset$, so also $I_k \subseteq J_{k'}$. Since therefore $|\, I_{k'} \,| \leq |\, J_k \,| = |\, I_k \,| \leq |\, J_{k'} \,| = |\, I_{k'} \,|$, we have $I_{k'} = J_k$ and $I_k = J_{k'}$.

□

In [VK85] Voskresenskij and Klyachko completely classified centrally symmetric smooth Fano polytopes, this was extended by Ewald in [Ewa88] to smooth Fano polytopes having a centrally symmetric pair of facets. For this they defined two special classes of polytopes:

Definition 6.3.2. Let $e_1, \ldots, e_d$ be a $\mathbb{Z}$-basis of M. Let d be even.

1. $\mathcal{F}_d := \mathrm{conv}(\pm e_1, \ldots, \pm e_d, \pm(e_1 + \cdots + e_d))$ is called a *del Pezzo polytope*. It is a centrally symmetric smooth Fano polytope. The toric variety $X(M, \Sigma_{\mathcal{F}_d})$ is denoted by W_d, it is $(\mathbb{P}^1)^d$ blown-up in two torus-invariant points. We have $\mathcal{F}_2 = \mathcal{Z}_2$ and $W_2 = S_3$.

2. $\tilde{\mathcal{F}}_d := \mathrm{conv}(\pm e_1, \ldots, \pm e_d, -e_1 - \cdots - e_d)$ is called a *pseudo del Pezzo polytope*. It is smooth Fano polytope that is not centrally symmetric, but has a centrally symmetric pair of facets. The toric variety $X(M, \Sigma_{\tilde{\mathcal{F}}_d})$ is denoted by $\tilde{W}_d$, it is $(\mathbb{P}^1)^d$ blown-up in only one torus-invariant point.

Now we get as a corollary of the previous theorem a further generalization of the classification results of Ewald and his students:

Corollary 6.3.3. *Let $P \subseteq M_{\mathbb{R}}$ be a simplicial reflexive polytope with a centrally symmetric pair of facets. Then*

$$X(M, \Sigma_P) \cong X(M, \Sigma_{P'}) \times (\mathbb{P}^1)^r \times W_{k_1} \times \cdots \times W_{k_s} \times \tilde{W}_{p_1} \times \cdots \times \tilde{W}_{p_t},$$

where $k_1, \ldots, k_s, p_1, \ldots, p_t$ are even, r is chosen to be maximal, and P' is an n-dimensional centrally symmetric simplicial reflexive polytope with $2n$ vertices for $n := d - r - k_1 - \cdots - k_s - p_1 - \cdots - p_t$.

Let $f := \log_2[M :< \mathcal{V}(P) >_{\mathbb{Z}}]$. P is a smooth Fano polytope iff $P' = \{0\}$ iff $n = 0$ iff $f = 0$.

Let $P' \neq \{0\}$, i.e., $n \geq 2$, and $V(P') = \{\pm e_i \,:\, i = 1, \ldots, n\}$. Then there exists a lattice basis of M such that the matrix A' consisting of the columns $e_1, \ldots, e_n$ is of the form $A' = \begin{pmatrix} 2\mathrm{id}_f & 0 \\ C' & \mathrm{id}_{n-f} \end{pmatrix}$ where $f \in \{0, \ldots, n-1\}$ and $C' \in \mathrm{Mat}_{(n-f) \times f}(\{0,1\})$ such that any column of C has an odd number of 1's and any row of C contains some 1.

We can say something about the uniqueness of this structure theorem:

Corollary 6.3.4. *Let P be a simplicial reflexive polytope that has a centrally symmetric pair of facets.*

Then in any product representation as given in the previous corollary the numbers $k_1, \ldots, k_s, p_1, \ldots, p_t$ are uniquely determined by the combinatorial type of the polytope P. The numbers f, n and r are uniquely determined by the isomorphism class of the lattice polytope P.

Moreover the matrix A' is independent (up to multiplication by a matrix in $\mathrm{GL}_n(\mathbb{Z})$ and permutation of columns) of the chosen pair of facets.

In particular any two facets of P are isomorphic as lattice polytopes, i.e., there is a lattice automorphism of M mapping one facet onto the other. Especially any two facets of P have the same number of lattice points.

Proof. First looking at the primitive collections (see section 4.1) of $\mathcal{F}_{k_i}$ and $\tilde{\mathcal{F}}_{p_j}$ as described in [Cas03b] it is easy to see that the numbers $k_1, \ldots, k_s, p_1, \ldots, p_t$ are combinatorial invariants.

Now let $F_1, F_2 \in \mathcal{F}(P)$ with $F_1 \neq F_2$; they define two isomorphic product representations of P, where we denote P_1 and P_2 for the respective polytopes defining $X(M, \Sigma_{P'}) \times (\mathbb{P}^1)^r$, where it is not a priori obvious that r is the same for P_1 and P_2. Again by looking at the primitive collections we see that necessarily P_1 and P_2 have to be isomorphic. Let C_1 be the matrix consisting of the columns of P_1. The facet of P_1 corresponding to the (restriction of the) facet F_2 has as vertices just plus or minus the columns of C_1. After possibly permutating the columns we get a lower triangular matrix. Now we modify the lattice basis by exchanging e_i with $-e_i$, if the corresponding column has -2 on the diagonal; then we just have to add the ith row to any row that has -1 as an entry in ith column. Now there has to be a unimodular matrix such that the multiplication with this matrix yields the matrix C_2. Finally it is easy to see that by multiplicating a "normal form matrix" C_1 the number of rows having only one 1 can at most decrease; especially it is left invariant, if the result is again a normal form matrix, as here C_2. Hence the number r (and therefore also n) is an invariant of P.

□

Remark 6.3.5. Hence Corollary 6.3.3 gives a well-defined matrix normal form. Furthermore one can check that any of these matrices indeed defines a centrally symmetric simplicial reflexive polytope with the minimal number of vertices, and there is an explicit criterion when two of these normal forms are equivalent, this is examined and discussed in [Wir97, Satz 3.3, Satz 3.9].

We get the original result of Ewald in [Ewa88] under milder assumptions:

Corollary 6.3.6. *Let $P \subseteq M_{\mathbb{R}}$ be a simplicial reflexive polytope where the vertices span the lattice M, and assume that P has a centrally symmetric pair of facets.*

Then the corresponding toric variety $X(M, \Sigma_P)$ is just a product of projective lines, del Pezzo varieties and pseudo del Pezzo varieties, in particular it is nonsingular.

We also get the following result (we just have to apply [Oda88, Cor. 1.16]):

Corollary 6.3.7. *Any $\mathbb{Q}$-factorial Gorenstein toric Fano variety where the associated fan has a centrally symmetric pair of d-dimensional cones is the projection for the quotient of a product of projective lines, del Pezzo varieties and pseudo del Pezzo varieties with respect to the action of a finite group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^f$ for $f \leq d - 1$.*

The combinatorial statement sounds rather surprising:

Corollary 6.3.8. *Any simplicial reflexive polytope that has a centrally symmetric pair of facets is combinatorially isomorphic to a smooth Fano polytope having a centrally symmetric pair of facets.*

One should not be misled by this result: Without the symmetry assumption the combinatorics of simplicial reflexive polytopes can be much more complicated than the one of smooth Fano polytopes.

Now we are going to prove Theorem 6.2.2. For this we need the following remark:

Remark 6.3.9. By setting $e_0 := -e_1 - \cdots - e_d$ it is a straightforward calculation (see also [VK85] or [Cas03b]) that the facets of $\mathcal{F}_d$ have as vertices precisely $\{\pm e_j \ : \ j = 0, \ldots, d, \, j \neq i\}$ for fixed $i \in \{0, \ldots, d\}$ where exactly half of the signs are equal to $+1$ and the others are equal to -1. Hence we get:

$$|\,\mathcal{F}(\mathcal{F}_d)\,| = (d+1)\binom{d}{\frac{d}{2}}.$$

In just the same way (see [Cas03b]) we can calculate

$$|\,\mathcal{F}(\tilde{\mathcal{F}}_d)\,| = d\binom{d-1}{\frac{d}{2}} + \sum_{i=\frac{d}{2}}^{d} \binom{d}{i}.$$

Proof of Theorem 6.2.2. By Corollary 6.3.3 we can assume that the simple reflexive polytope $P \subseteq M_\mathbb{R}$ is combinatorially isomorphic to $[-1,1]^l \times (\mathcal{F}_{k_1})^* \times \cdots \times (\mathcal{F}_{k_s})^* \times (\tilde{\mathcal{F}}_{p_1})^* \times \cdots \times (\tilde{\mathcal{F}}_{p_t})^*$, where $l + k_1 + \cdots + k_s + p_1 + \cdots + p_t = d$ for $k_1, \ldots, k_s, p_1, \ldots, p_t$ even. Now a standard induction argument shows that

$$2n\binom{2n-1}{n} + \sum_{i=n}^{2n} \binom{2n}{i} < (2n+1)\binom{2n}{n} \leq 6^n$$

for $n \in \mathbb{N}_{\geq 1}$, with equality at the right only for $n = 1$. Hence the previous remark and the fact that also $|\,\mathcal{V}([-1,1])\,| = 2 < 6^{1/2}$ yields $|\,\mathcal{V}(P)\,| \leq 6^{l/2} 6^{k_1/2} \cdots 6^{k_s/2} 6^{p_1/2} \cdots 6^{p_t/2} = 6^{d/2}$, where equality implies that d is even and P is combinatorially isomorphic to $\mathcal{Z}_2^{d/2}$. In this case however P^* is a simplicial reflexive polytope with $3d$ vertices and a centrally symmetric pair of facets, hence $P^* \cong (\mathcal{Z}_2^{d/2})^*$ by 6.2.4 (or again directly by Cor. 6.3.3 and Cor. 6.3.4).

□

There is one special centrally symmetric reflexive polytope that is extreme with respect to the number of lattice points (see Theorem 6.5.3):

Definition 6.3.10. We define for a $\mathbb{Z}$-basis $m_1, \ldots, m_d$ of M the reflexive polytope $D_d := \operatorname{conv}(\pm(2m_1 + m_d), \ldots, \pm(2m_{d-1} + m_d), \pm m_d)$. The toric variety $X(M, \Sigma_{D_d})$ is denoted by $\mathcal{D}_d$.

The polytopes D_d have the following basic properties that are straightforward to verify:

Remark 6.3.11. D_d is a centrally symmetric simplicial reflexive polytope. The dual polytope $D_d^* = \{m_d^* - x \ : \ x \in \sum_{i=1}^{d-1} \{0,1\} m_i^*\}$ is a terminal reflexive polytope (in the dual $\mathbb{Z}$-basis of $m_1, \ldots, m_d$).

If a centrally symmetric simplicial reflexive polytope P is given such that

$$A = \begin{pmatrix} 2\mathrm{id}_{d-1} & 0 \\ 1 \cdots 1 & 1 \end{pmatrix},$$

in the notation of Theorem 6.3.1, then $P \cong D_d$.

The remainder of this section will apply Cor. 6.3.3 to lower dimensions. Here we will prove:

Theorem 6.3.12. *For $d = 2, 3, 4, 5$ there are exactly $3, 4, 10, 14$ isomorphism classes of d-dimensional centrally symmetric simplicial reflexive polytopes, and $4, 5, 15, 20$ isomorphism classes of d-dimensional simplicial reflexive polytopes with a centrally symmetric pair of facets.*

A rigorous proof of the previous result was in the centrally symmetric case up to now only known for $d \leq 3$. For $d = 4$ the centrally symmetric case of up to 10 vertices was dealt with by rather complicated and long calculations in [Wir97, Satz 5.11].

The proof is split in four parts (here we always denote $X = X(M, \Sigma_P)$). It is important to note that by Cor. 6.3.3 we actually only have to deal with the minimal case of $2n$ vertices!

Example 6.3.13. We classify all *two*-dimensional reflexive polytopes P with a centrally symmetric pair of facets, P is necessarily simplicial.

By Cor. 6.3.3 we have $X \cong \tilde{W}_2$ or $X \cong W_2$ or $X \cong \mathbb{P}^1 \times \mathbb{P}^1$, if $n = 0$.

Finally let $n = 2$ and $A' = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$. By 6.3.11 we have $P \cong D_2 \cong E_1^2$, i.e., $X \cong \mathcal{D}_2$.

Example 6.3.14. We classify all *three*-dimensional simplicial reflexive polytopes P with a centrally symmetric pair of facets.

By Cor. 6.3.3 we have $X \cong \mathbb{P}^1 \times \tilde{W}_2$ or $X \cong \mathbb{P}^1 \times W_2$ or $X \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, if $n = 0$. If $n = 2$, then $X \cong \mathcal{D}_2 \times \mathbb{P}^1$ by 6.3.13.

Finally let $n = 3$. Then $A' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$. By 6.3.11 we have $X \cong \mathcal{D}_3$.

Example 6.3.15. We classify all *four*-dimensional simplicial reflexive polytopes P with a centrally symmetric pair of facets.

By Cor. 6.3.3 we have $X \cong \tilde{W}_4$ or $X \cong W_4$ or $X \cong (\mathbb{P}^1)^2 \times \tilde{W}_2$ or $X \cong W_2 \times \tilde{W}_2$ or $X \cong \tilde{W}_2 \times \tilde{W}_2$ or $X \cong (\mathbb{P}^1)^2 \times W_2$ or $X \cong W_2 \times W_2$ or $X \cong (\mathbb{P}^1)^4$, if $n = 0$. If $n = 2$, then $X \cong \mathcal{D}_2 \times (\mathbb{P}^1)^2$ or $X \cong \mathcal{D}_2 \times W_2$ or $X \cong \mathcal{D}_2 \times \tilde{W}_2$ by 6.3.13. If $n = 3$, then $X \cong \mathcal{D}_3 \times \mathbb{P}^1$ by 6.3.14.

Finally let $n = 4$ and $A = A'$, in particular $|\mathcal{V}(P)| = 8$. Here there are three (non-unimodularly equivalent) possibilities corresponding to $f = 1, 2, 3$:

1. $A' = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$. We denote the toric variety X by X_4.

2. $A' = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$. We have $X \cong \mathcal{D}_2 \times \mathcal{D}_2$.

3. $A' = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$. By 6.3.11 we have $X \cong \mathcal{D}_4$.

Example 6.3.16. We classify all *five*-dimensional centrally symmetric simplicial reflexive polytopes P with a centrally symmetric pair of facets.

By Cor. 6.3.3 we have $X \cong \mathbb{P}^1 \times \tilde{W}_4$ or $X \cong \mathbb{P}^1 \times W_4$ or $X \cong (\mathbb{P}^1)^3 \times \tilde{W}_2$ or $X \cong \mathbb{P}^1 \times W_2 \times \tilde{W}_2$ or $X \cong \mathbb{P}^1 \times \tilde{W}_2 \times \tilde{W}_2$ or $X \cong (\mathbb{P}^1)^3 \times W_2$ or $X \cong \mathbb{P}^1 \times W_2 \times W_2$ or $X \cong (\mathbb{P}^1)^5$, if $n = 0$. If $n = 2$, then $X \cong \mathcal{D}_2 \times (\mathbb{P}^1)^3$ or $X \cong \mathcal{D}_2 \times \mathbb{P}^1 \times W_2$ or $X \cong \mathcal{D}_2 \times \mathbb{P}^1 \times \tilde{W}_2$ by 6.3.13. If $n = 3$, then $X \cong \mathcal{D}_3 \times (\mathbb{P}^1)^2$ or $X \cong \mathcal{D}_3 \times \tilde{W}_2$ or $X \cong \mathcal{D}_3 \times W_2$ by 6.3.14. If $n = 4$, then $X \cong X_4 \times \mathbb{P}^1$ or $X \cong \mathcal{D}_2 \times \mathcal{D}_2 \times \mathbb{P}^1$ or $X \cong \mathcal{D}_4 \times \mathbb{P}^1$ by 6.3.15.

Finally let $n = 5$ and $A = A'$, in particular $|\mathcal{V}(P)| = 10$. Here there are three (non-unimodularly equivalent) possibilities corresponding to $f = 2, 3, 4$ ($f = 1$ is not possible due to 6.3.3, since there would be an even number of 1s in the first column):

1. $A' = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ & & 1 & 0 & 0 \\ & C' & 0 & 1 & 0 \\ & & 0 & 0 & 1 \end{pmatrix}$. Up to permutation of rows and columns C' has the following form:

   (a) $C' = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$. We denote the toric variety X by X_5.

   (b) $C' = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$. As was observed in [Wir97, Bemerkung 3.8] this matrix is equivalent to the previous one, so $X \cong X_5$.

2. $A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}$. We have $X \cong \mathcal{D}_2 \times \mathcal{D}_3$.

3. $A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$. By 6.3.11 we have $X \cong \mathcal{D}_5$.

6.4 Embedding theorems

We are interested in finding an *embedding* of a reflexive polytope $P \subseteq M_{\mathbb{R}}$ into kE_1^d for $k \in \mathbb{N}$ small, i.e., in finding a lattice automorphism of M that maps P in a lattice polytope isomorphic to $[-k, k]^d$, or more simply put, in finding a lattice basis of M such that any vertex of P has coordinates in $[-k, k]$.

There is a conjecture due to Ewald (see [Ewa88]) that *any d-dimensional smooth Fano polytope can be embedded in the unit lattice cube* $[-1, 1]^d$. It is proven for $d \leq 4$ by the classification or under additional symmetries. It is wrong for simplicial reflexive polytopes, e.g., type 9 in Prop. 3.4.1 contains 10 lattice points. However we still have the following well-known result:

Proposition 6.4.1. *For $d \leq 3$ we can always embedd a centrally symmetric reflexive polytope P into E_1^d.*

Proof. For this we choose by 3.1.8(1,2) a $\mathbb{Z}$-basis $b_1, \ldots, b_d$ of lattice points in ∂P^*, so $P \subseteq \{\sum \lambda_i b_i^* \,:\, \lambda_i \in \{-1,0,1\}\}$ for the dual $\mathbb{Z}$-basis $b_1^*, \ldots, b_d^*$.

□

There is no such result for $d \geq 4$. For instance let P be the 4-dimensional centrally symmetric reflexive polytope P in Example 3.2.6 (this is also X_4 in 6.3.15). Assume P^* could be embedded as a lattice polytope in $[-1,1]^4$. Then $([-1,1]^4)^*$ would be a lattice subpolytope of P with the same number of vertices. Since P is terminal, this would be an equality, a contradiction. This example is taken from [Wir97, Kapitel 4] where this topic is thoroughly dicussed. There it is shown in [Wir97, Satz 4.4] that centrally symmetric polytopes with the minimal number of vertices can always be embedded in the unit lattice cube.

Using Theorem 6.3.1 we can prove a generalization:

Corollary 6.4.2. *Let P be a simplicial reflexive polytope with a centrally symmetric pair of facets. Then P can be embedded into E_1^d.*

Proof. Since the (pseudo) del Pezzo polytopes are by definition contained in E_1^d, by Corollary 6.3.3 we just have to show that we get by row operations on A' a matrix containing only $\{-1,0,1\}$. For this we assume $A'_{j,j} = 2$, then there is a $i > j$ (minimal) such that $A'_{i,j} = 1$. Now we subtract the ith row from the jth. We proceed by induction on j.

□

Due to the structure theorem of the last section we can also prove the following result (note that by 6.4.1 and the previous example this bound is sharp for $d \leq 4$):

Corollary 6.4.3. *Let P be a simplicial reflexive polytope with a centrally symmetric pair of facets. Then P^* can be embedded into $\lfloor \frac{d}{2} \rfloor E_1^d$.*

Proof. Since the duals of the (pseudo) del Pezzo polytopes are always contained in E_1^d (for this use 6.3.9 and [Cas03b]), by Cor. 6.3.3 we can assume that $P = \operatorname{conv}(\pm e_1, \ldots, \pm e_d) \subseteq M_{\mathbb{R}}$. Let $m_1, \ldots, m_d$ be the $\mathbb{Z}$-basis of M in Theorem 6.3.1 such that $A = \begin{pmatrix} 2\mathrm{id}_f & 0 \\ C & \mathrm{id}_{d-f} \end{pmatrix} \in \mathrm{Mat}_d(\mathbb{N})$. Then $A^{-1} = \begin{pmatrix} 1/2\mathrm{id}_f & 0 \\ -C/2 & \mathrm{id}_{d-f} \end{pmatrix}$. Now the rows are precisely the coordinates of the dual $\mathbb{R}$-basis $e_1^*, \ldots, e_d^*$ (in the dual $\mathbb{Z}$-basis $m_1^*, \ldots, m_d^*$). Furthermore for any facet $F \in \mathcal{F}(P)$ we have $\eta_F = \pm e_1^* \pm \cdots \pm e_d^* \in N$ for some signs $\pm$. Hence the vertices of P^* have coordinates in $[\lfloor -d/2 \rfloor, \lfloor d/2 \rfloor]$ with respect to the lattice basis $m_1^*, \ldots, m_d^*$.

□

The main result of this section is the following coarse but more general embedding theorem:

Theorem 6.4.4. *Let P be a reflexive polytope of dimension $d \geq 3$ with a facet $F \in \mathcal{F}(P)$ having a vertex that is only contained in $(d-1)$ facets of F (e.g., F is simple). We set*

$$w := \operatorname{lcm}\{\langle u, v \rangle + 1 \,:\, u \in \mathcal{V}(P^*),\, v \in \mathcal{V}(P)\}.$$

Then P can be embedded in cE_1^d, where $c \in O(dw^2)$ is a constant that depends only on d and w.

More precisely: c is a positive natural number with

$$c \leq \min\{w((d-2)(w-1)+z),\ \left\lceil \frac{w}{2} \right\rceil ((d-1)w-1)\},$$

where z denotes the greatest proper divisor of w.

Proof. Let $e_d \in \mathcal{V}(F)$ have the assumed property. There exist $e_1, \ldots, e_{d-1} \in \mathcal{V}(F)$ such that for $i = 1, \ldots, d-1$ the set $\mathrm{conv}(e_j : j \neq i)$ is contained in a facet H_i of F. For $i = 1, \ldots, d-1$ we define $F_i \in \mathcal{F}(P)$ such that $F_i \cap F = H_i$.

Again as in the proof of Lemma 3.5.6 we have for $i = 1, \ldots, d-1$ that $\eta_{F_i} = u + \alpha_i e_i^*$ for a positive natural number α_i; moreover $\alpha_i = \langle \eta_{F_i}, e_i \rangle + 1$, this implies $e_i^* = \frac{\eta_{F_i} - u}{\alpha_i} \in \frac{N}{w}$. Since $u = -\sum_{i=1}^d e_i^*$ this implies also $e_d^* \in \frac{N}{w}$. In particular $wM \subseteq < e_1, \ldots, e_d >_{\mathbb{Z}} \subseteq M$.

We define $M_u := M \cap u^{\perp}$, and for a fixed Z-basis $m_1, \ldots, m_{d-1}$ of M_u the matrix $B' \in \mathrm{Mat}_{d-1}(\mathbb{Z}) \cap \mathrm{GL}_{d-1}(\mathbb{Q})$ consisting of the columns $e_1 - e_d, \ldots, e_{d-1} - e_d$. By Theorem 3.6.6 there is an unimodular transformation L' of M_u such that $A' := L'B' \in \mathrm{Mat}_{d-1}(\mathbb{N})$ is a lower triangular matrix with $A'_{i,j} \in \{0, \ldots, A'_{j,j} - 1\}$ for $i > j$.

Moreover since wM_u is contained in the column space of A', any diagonal element of A' is a positive divisor of w. Now we define the matrix $A \in \mathrm{Mat}_d(\mathbb{Z})$ as the blockmatrix where in the upper left A' is put and the dth row has ones everywhere. Since $M = M_u \oplus \mathbb{Z}e_d$ obviously in the $\mathbb{Z}$-basis $m_1, \ldots, m_{d-1}, m_d = e_d$ of M the columns $a_1, \ldots, a_d$ of A are the images of $e_1, \ldots, e_d$ under a unimodular transformation of M leaving the sets M_u and $\{e_d\}$ invariant.

Without restriction we identify now P and $L(P)$, in particular $u = -m_d^*$.

Let $v \in \mathcal{V}(P)$ and write $v = \sum_{j=1}^d q_j a_j = \sum_{j=1}^d q'_j m_j$ as rational linear combinations. We set $t := \langle u, v \rangle \in \{-1, \ldots, w-1\}$.

For $j = 1, \ldots, d-1$ we have $q_j = \frac{\langle \eta_{F_j}, v \rangle - t}{\alpha_j} \in [-w, w]$.

We get the two inequalities for c in two different ways:

1. To bound the coefficients of vertices of P in the $\mathbb{Z}$-basis $m_1, \ldots, m_d$ of M we can assume that the coefficient q'_j is either minimal or maximal among $q'_1, \ldots, q'_{d-1}$.

   (a) $A_{j,j} = w$. Assume $j > 1$ and there is $k < j$ maximal with $A_{j,k} > 0$. Since wm_k is contained in the column space, there exists $n \in \mathbb{N}$ such that $\frac{w}{A_{k,k}} A_{j,k} = nw$, hence $0 \leq A_{j,k} = nA_{k,k} < A_{k,k}$, so $n = 0$ and $A_{j,k} = 0$, a contradiction. This yields $q'_j = wq_j \in [-w^2, w^2]$, hence $|q'_j| \leq w((d-2)(w-1)+z)$, since $d \geq 3$.

   (b) $A_{j,j} < w$, i.e., $A_{j,j} \leq z$. We have $q'_j = \sum_{k=1}^j A_{j,k} q_k$. When q'_j is maximal, we get $q'_j \leq (d-2)(w-1)w + zw = w((w-1)(d-2)+z)$. When q'_j is minimal, we get $q'_j \geq (d-2)(w-1)(-w) + z(-w) = (-w)((w-1)(d-2)+z)$. This gives the first bound.

2. First we subtract for $i = 1, \ldots, d-1$ from the ith row of A $\lfloor \frac{A_{i,i}}{2} \rfloor$-times the dth row of A. This gives a matrix having as entries only integers in $\{-\left\lceil \frac{w}{2} \right\rceil, \ldots, \left\lceil \frac{w}{2} \right\rceil\}$. By this procedure we find a $\mathbb{Z}$-basis of M such that $a_1, \ldots, a_d \in \left\lceil \frac{w}{2} \right\rceil E_1^d$.

Second we can assume that $q_d = -t - \sum_{j=1}^{d-1} q_j$ is chosen to be maximal or minimal. Then $q_d = -t - \sum_{j=1}^{d-1} \frac{\langle \eta_{F_j}, v\rangle - t}{\alpha_j}$. If q_d is maximal, this yields $q_d \leq (d-2)(w-1) + (d-1) = (d-2)w + 1$. If q_d is minimal, we get $q_d \geq (d-2)(-1) - (d-1)(w-1) = -((d-1)w-1)$.

In any case we obtain $q_1, \ldots, q_d \in [-((d-1)w-1), (d-1)w-1]$. This gives the second bound.

□

For $w = 2$ the second bound is always better than the first, it yields that centrally symmetric reflexive polytopes having such a vertex can be embedded into the $(2d-3)$th multiple of the unit lattice cube. We got significant improvements of this result in the special cases of Propositions 6.4.1, 6.4.2, 6.4.3. Also note that for $w = 3$ the first bound in the theorem is always better than the second ($c \leq 6d - 9$).

Usually one does not deal with the invariant w but with the so called *width* w' as defined in the following proposition:

Proposition 6.4.5 ([Deb03]). *Let P be a canonical Fano polytope.*

We set

$$w' := \max\{\langle u, v\rangle + 1 \; : \; u \in \mathcal{V}(P^*),\, v \in \mathcal{V}(P)\}.$$

Then

$$\mathrm{vol}(P) \leq (w')^d.$$

Especially we have a result, that would be trivial, if any centrally symmetric canonical Fano polytope could be embedded into the unit lattice cube.

Corollary 6.4.6. *Any centrally symmetric canonical Fano polytope has volume at most 2^d.*

Hence Theorem 3.7.14 yields:

Corollary 6.4.7. *Any d-dimensional canonical Fano polytope P can be embedded in the lattice cube of side length at most $dd!(w')^d$, where w' is defined as in 6.4.5.*

Now this should be compared with the following corollary to Theorem 6.4.4:

Corollary 6.4.8. *Any d-dimensional reflexive polytope with a facet F having a vertex that is only contained in $(d-1)$ facets of F can be embedded in the lattice cube of side length at most $d(w'!)^2$, where w' is defined as in 6.4.5.*

We see that especially for small values of w' this bound is a sharpening of the previous one.

6.5 Lattice points

By an embedding we trivially get that the number of lattice points in the polytope is bounded by 3^d with equality only in the case of the unit lattice cube. However this is even true in general:

Theorem 6.5.1. *Let $P \subseteq M_{\mathbb{R}}$ be a centrally symmetric canonical Fano polytope. Then*

$$|\,P \cap M\,| \leq 3^d.$$

Any facet of P has at most 3^{d-1} lattice points.

If P is additionally reflexive, then the following statements are equivalent:

1. $|\,P \cap M\,| = 3^d$
2. *Every facet of P has 3^{d-1} lattice points*
3. $P \cong E_1^d$ *as lattice polytopes*

We even have the following algebraic-geometric formulation:

Proposition 6.5.2. *Let $X = X(N, \triangle)$ be a complete Gorenstein toric variety with centrally symmetric $\triangle(1)$. Then*

$$h^0(X, -K_X) \leq 3^d.$$

The proofs of these bounds rely on the method of counting modulo 3. So we define the map

$$\alpha\,:\,P \cap M \to M/3M \cong (\mathbb{Z}/3\mathbb{Z})^d.$$

Proof of Proposition 6.5.2. Let $h \in \mathrm{SF}(N, \triangle)$ define the Cartier divisor $-K_X$, hence $P := P_h := \{x \in M_{\mathbb{R}}\,:\,\langle v_\tau, x\rangle \geq -1\,\forall\,\tau \in \triangle(1)\}$. Since $\triangle(1)$ is centrally symmetric, we get that P is a centrally symmetric rational polytope and $|\,P \cap M\,| = h^0(X, -K_X)$ by (1.7). Hence we only have to show that α is injective. So suppose there are $x, y \in P \cap M$ such that $\alpha(x) = \alpha(y)$. This implies $(x - y)/3 \in M$. For arbitrary $\tau \in \triangle(1)$, we get

$$\frac{\langle v_\tau, x\rangle - \langle v_\tau, y\rangle}{3} = \langle v_\tau, (x - y)/3\rangle \in \mathbb{Z}.$$

Since by assumption $\langle v_\tau, x\rangle, \langle v_\tau, y\rangle \in \{-1, 0, 1\}$, this yields $\langle v_\tau, x\rangle = \langle v_\tau, y\rangle$ for any $\tau \in \triangle(1)$. Therefore $x = y$, because $\triangle$ is complete.

□

Proof of theorem 6.5.1. Let there be $x, y \in P \cap M$ such that $\alpha(x) = \alpha(y)$, hence $(x - y)/3 \in \mathrm{int}P \cap M = \{0\}$, so $x = y$. Therefore α is injective.

Let $F \in \mathcal{F}(P)$ be arbitrary but fixed. Define $u := \eta_F \in \mathcal{V}(P^*)$ and also the $\mathbb{Z}/3\mathbb{Z}$-extended map $\alpha(u)\,:\,M/3M \to \mathbb{Z}/3\mathbb{Z}$. For $m \in P \cap M$ we have $\langle u, m\rangle \in \{-1, 0, 1\}$, in particular

$$m \in F \iff \langle \alpha(u), \alpha(m)\rangle = -1 \in \mathbb{Z}/3\mathbb{Z}.$$

3. $\Rightarrow$ 1.: Trivial.

1. $\Rightarrow$ 2.: If P contains 3^d lattice points, then α is a bijection, and therefore $|\,F \cap M\,| = |\,\{z \in M/3M\,:\,\langle \alpha(\eta_F), z\rangle = -1\}\,| = 3^{d-1}$.

2. $\Rightarrow$ 3.: The assumption implies that for any facet $F' \in \mathcal{F}(P)$ the map

$$\alpha|_{F'} : F' \cap M \to \{z \in M/3M \,:\, \langle \alpha(\eta_{F'}), z\rangle = -1\}$$

is a bijection. Define $x := (1/3^{d-1}) \sum_{m \in F \cap M} m \in \mathrm{relint} F$.

By Theorem 6.1.1(3) it remains to prove that x is a root, i.e., $x \in M$.

Choose a facet $G \in \mathcal{F}(P^*)$ and an $\mathbb{R}$-linearly independent family $w_1, \ldots, w_d$ of vertices of G such that $w_1 = u$ and $w_2, \ldots, w_d$ are contained in a $(d-2)$-dimensional face of P^*.

Denote the corresponding facets of P by $F_1, F_2, \ldots, F_d$ with $\eta_{F_j} = w_j$ for $j = 1, \ldots, d$, so $F_1 = F$. Then $Q := \cap_{j=2}^d F_j$ is a one-dimensional face of P. Therefore also the affine span of $\alpha(Q \cap M)$ is a one-dimensional affine subspace of $M/3M$. Since $|\,F \cap Q\,| = 1$ there exists an element $b \in M/3M$ such that $\langle \alpha(u), b\rangle = 0$ and $\langle \alpha(w_j), b\rangle = -1$ for all $j = 2, \ldots, d$. Applying the assumption to F_2 yields a lattice point $v \in P \cap M$ with $\alpha(v) = b$. Hence also $\langle u, v\rangle = 0$ and $\langle w_j, v\rangle = -1$ for $j = 2, \ldots, d$.

By 3.1.4(11) we find a $\mathbb{Z}$-basis $e_1^* = u, e_2^*, \ldots, e_d^*$ of N such that for any $j = 2, \ldots, d$ there exist $\lambda_{j,k} \in \mathbb{R}$ with $e_j^* = \lambda_{j,2}(w_2 - u) + \cdots + \lambda_{j,d}(w_d - u)$.
· *Fact 1:* $\langle w_k, \sum_{m \in F \cap M} m\rangle = 0$ for $k = 2, \ldots, d$.
(*Proof:* Since $F \cap F_k \neq \emptyset$, the assumption implies for $i = -1, 0, 1 \in \mathbb{Z}/3\mathbb{Z}$: $|\,\{z \in M/3M \,:\, \langle \alpha(u), z\rangle = -1, \langle \alpha(w_k), z\rangle = i\}\,| = 3^{d-2}$.)
· *Fact 2:* $\sum_{k=2}^d \lambda_{j,k} \in \mathbb{Z}$ for $j = 2, \ldots, d$.
(*Proof:* $\langle e_j^*, v\rangle = (-\sum_{k=2}^d \lambda_{j,k})\langle u, v\rangle + \sum_{k=2}^d \lambda_{j,k} \langle w_k, v\rangle = -\sum_{k=2}^d \lambda_{j,k}$ by the choice of v.)

Using these two facts we can finish the proof:

$$\begin{aligned}
\langle e_1^*, x\rangle &= \langle u, x\rangle = -1 \in \mathbb{Z},\\
\langle e_j^*, x\rangle &= (1/3^{d-1}) \left((-\sum_{k=2}^d \lambda_{j,k})\langle u, \sum_{m \in F \cap M} m\rangle + \sum_{k=2}^d \lambda_{j,k} \langle w_k, \sum_{m \in F \cap M} m\rangle \right)\\
&= \sum_{k=2}^d \lambda_{j,k} \in \mathbb{Z} \ \text{ for } j = 2, \ldots, d.
\end{aligned}$$

Hence $x \in M$.

□

In the simplicial case the last theorem of this thesis shows that D_d has the maximal number of lattice points among all simplicial reflexive polytopes that have a centrally symmetric pair of facets:

Theorem 6.5.3. *Let $P \subseteq M_{\mathbb{R}}$ be a simplicial reflexive polytope with a centrally symmetric pair of facets. Then*

$$|\,P \cap M\,| \leq 2d^2 + 1.$$

Any facet of P has at most $\frac{(d+1)d}{2}$ lattice points.

The following statements are equivalent:

1. $|\,P \cap M\,| = 2d^2 + 1$
2. *Some facet has $\frac{(d+1)d}{2}$ lattice points*

3. *Any facet has* $\frac{(d+1)d}{2}$ *lattice points*

4. $P \cong D_d$ *as lattice polytopes*

Proof. Let $F, -F \in \mathcal{F}(P)$. Applying Theorem 6.3.1 to P and F we can assume that $\mathcal{V}(F) = \{e_1, \ldots, e_d\}$ are the columns of a matrix A of the form given in the theorem. Hence we get:

$$m \in (F \cap M) \backslash \mathcal{V}(P) \implies m = \frac{e_i + e_j}{2} \text{ for } (e_i)_i = 2. \tag{6.1}$$

In particular

$$|\, F \cap M \,| \leq \binom{d}{2} + d = \frac{(d+1)d}{2},$$

where equality implies

$$A = \begin{pmatrix} 2\mathrm{id}_{d-1} & 0 \\ 1 \cdots 1 & 1 \end{pmatrix},$$

hence Remark 6.3.11 implies $P \cong \mathcal{D}_d$. In particular this proves $2. \Rightarrow 3. \Rightarrow 4.$, since by Cor. 6.3.4 all facets of P have the same number of lattice points.

Let $u := \eta_F$ and $m \in \partial P \cap M \cap u^\perp$. By 3.5.6 we get $m \in F_i$ for some $i \in \{1, \ldots, d\}$. If $m \in \mathcal{V}(P)$, then we have $m \in \{v^1, \ldots, v^d\}$. So let $m \notin \mathcal{V}(P)$. In particular $v^i \notin u^\perp$, so $v^i = -e_i$. Since m is not away from v^i, we easily obtain only by looking at $Q := \mathrm{lin}(e_i, m) \cap P$ that $Q \cong E_1^2$ as lattice polytopes (in the lattice $< e_i, m >_{\mathbb{Z}} = \mathrm{lin}(e_i, m) \cap M$). Let $z := z(m, e_i) \in \mathcal{V}(Q) \cap F \cap M$. Since $\frac{e_i + z}{2} \in F \cap M$, equation (6.1) implies that $z = e_j$ for some $j \neq i$. Hence $m = \frac{-e_i + e_j}{2}$.

In particular this yields

$$|\, \partial P \cap M \cap u^\perp \,| \leq d(d-1).$$

where equality holds for $P \cong D_d$.

Hence we have

$$|\, P \cap M \,| \leq 1 + 2|\, F \cap M \,| + d(d-1) = 2d^2 + 1,$$

where equality implies $|\, F \cap M \,|$ to be maximal. This proves the implications $4. \Rightarrow 1. \Rightarrow 2.$.

□

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Appendix A - Zusammenfassung in deutscher Sprache

Diese Arbeit befasst sich mit torischen Gorenstein-Fano-Varietäten. Dabei handelt es sich um vollständige normale torische Varietäten, deren antikanonischer Divisor ein ampler Cartierdivisor ist.

Diese algebraisch-geometrischen Objekte haben ihre Entsprechung in der konvexen Geometrie in der Form so genannter reflexiver Polytope. Dabei ist ein reflexives Polytop ein Gitterpolytop, das in seinem Inneren den Ursprung enthält, mit der Eigenschaft, dass das duale Polytop wiederum ein Gitterpolytop ist. Insbesondere treten reflexive Polytope immer in Paaren auf. Diese Begriffsbildung wurde erstmals von Batyrev in [Bat94] eingeführt, als er zeigte, dass generische antikanonische Hyperflächen torischer Gorenstein-Fano-Varietäten Calabi-Yau sind und sich nach Auflösung der Singularitäten aufgrund der natürlichen Dualität reflexiver Polytope Kandidaten für Mirror-Symmetrie ergeben. Daraufhin wurde angestrebt, sämtliche reflexive Polytope im physikalisch relevanten vierdimensionalen Fall zu klassifizieren. Kreuzer und Skarke fanden schließlich mit Hilfe ihres Computerprogramms PALP [KS04a] 16, 4319 bzw. 473800776 nichtisomorphe zwei-, drei- bzw. vierdimensionale reflexive Polytope [KS98, KS00, KS04b]. Weiter ist bekannt, dass es in jeder Dimension nur endlich viele Isomorphieklassen gibt. Auf der Suche nach allgemeineren Calabi-Yau-Varietäten haben Kreuzer und Skarke in letzter Zeit auch begonnen, fünf und sechs Dimensionen in Angriff zu nehmen [KS04b].

Während es schon einige mathematische Arbeiten gibt, die sich mit glatten torischen Fano-Varietäten beschäftigen [WW82, Bat82a, Bat82b, Bat99, Sat00, Deb03, Cas03a, Cas03b], wurde der singuläre Fall noch nicht so intensiv untersucht, insbesondere in höheren Dimensionen. Dies hängt damit zusammen, dass hier einige fundamentale Schwierigkeiten auftreten. Zunächst sind manche algebraisch-geometrische Methoden wie Riemann-Roch oder Schnitttheorie nicht ohne Weiteres anwendbar, insbesondere da in höheren Dimensionen keine torische krepante Auflösung existieren muss. Zum Zweiten verwenden viele konvex-geometrische Beweise die Voraussetzung, dass die Ecken einer Facette im glatten Fall eine Gitterbasis bilden, wogegen reflexive Polytope im Allgemeinen sogar Gitterpunkte im Inneren einer Facette enthalten können. Schließlich verhindert allein die extrem große Anzahl an Isomorphieklassen selbst in niedrigen Dimensionen in den meisten Fällen eine vollständige rigorose Klassifikation, die ohne die Hilfe eines Computers auskommt.

Das Ziel dieser Dissertation ist eine erste systematische Untersuchung torischer Gorenstein-Fano-Varietäten. Hierfür werden zunächst Methoden und Resultate über glatte torische Fano-Varietäten auf torische Fano-Varietäten mit milden Singularitäten verallgemeinert. Dabei lässt die Konzentration auf Methoden der konvexen Geometrie auch schon im glatten Fall bekannte Resultate transparenter erscheinen. Überraschend ist in diesem Zusammenhang die Gutartigkeit $\mathbb{Q}$-faktorieller torischer Gorenstein-Fano-Varietäten. Weiter ist es das Ziel, in wichtigen Fällen vollständige Klassifikationsresultate auch in höheren Dimensionen zu gewinnen. Dabei stehen durchweg die kombinatorischen und geometrischen Eigenschaften und Invarianten reflexiver Polytope im Vordergrund, für welche Einschränkungen, Abschätzungen und Vermutungen bewiesen und formuliert werden. Diese Ergebnisse liefern insbesondere auch Erklärungen für interessante Beobachtungen in der Datenbank.

Diese Arbeit besteht aus einer Einleitung, einer Liste der verwendeten Notationen, sechs Kapiteln sowie einem Index und einer ausführlichen Bibliographie. Jedes größere Kapitel besitzt eine eigene Einleitung, der eine Übersichtsliste mit Referenzen auf die wichtigsten Ergebnisse angefügt ist.

Im Folgenden sollen nun die einzelnen Kapitel dieser Arbeit zusammenfassend beschrieben werden.

In den ersten beiden Kapiteln werden Grundlagen besprochen. Kapitel 1 enthält die fundamentalen Aussagen der torischen Geometrie. Kapitel 2 beschäftigt sich zunächst mit der Auflösung und Hierarchie von Singularitäten und deren Beschreibung im torischen Fall. Schließlich werden torische Fano-Varietäten genauer untersucht. Diese algebraisch-geometrischen Objekte haben eine Eins-zu-eins-Entsprechung in der konvexen Geometrie in Form so genannter Fano-Polytope. Torische Fano-Varietäten mit kanonischen bzw. terminalen Singularitäten korrespondieren dabei mit kanonischen bzw. terminalen Fano-Polytopen. Glatte bzw. $\mathbb{Q}$-faktorielle torische Fano-Varietäten entsprechen glatten bzw. simplizialen torischen Fano-Varietäten.

Kapitel 3 ist das Herzstück dieser Arbeit. Hier werden reflexive Polytope definiert und untersucht, also die konvex-geometrischen Gegenstücke zu torischen Fano-Varietäten mit Gorenstein-Singularitäten, kurz, torischen Gorenstein-Fano-Varietäten.

Das Kapitel beginnt mit der Verallgemeinerung zweier wichtiger Hilfsmittel, die schon erfolgreich zur Untersuchung glatter torischer Fano-Varietäten eingesetzt wurden. Zum Einen ist dies die Projektion reflexiver Polytope entlang von Ecken oder, allgemeiner, entlang von Gitterpunkten auf dem Rand. Diese Abbildung hat einige wichtige Einschränkungen, die in Proposition 3.2.2 genau beschrieben werden. Hier soll nur eine unmittelbare Anwendung erwähnt werden: Batyrev bewies in [Bat99], dass die antikanonische Klasse eines Torus-invarianten Primdivisors einer glatten torischen Fano-Varietät numerisch effektiv ist. In konvex-geometrischer Sprache besagt dies, dass die Projektion eines glatten Fano-Polytops reflexiv ist. Hier wird nun ein einfacherer Beweis dieser Aussage angegeben (Korollar 3.2.8) und darüber hinaus verallgemeinernd in Proposition 3.2.4 gezeigt, dass die Projektion eines terminalen reflexiven Polytops entlang einer Ecke ein kanonisches Fano-Polytop ist.

Bei dem zweiten wichtigen Hilfsmittel handelt es sich um primitive Kollektionen und Relationen. Dies sind spezielle Mengen und Relationen von Ecken, die von Batyrev in [Bat91] eingeführt wurden, um glatte Fano-Polytope eindeutig zu beschreiben. Sie waren essentiell für seine Klassifikation vierdimensionaler glatter torischer Fano-Varietäten in [Bat99]. Diese Begriffsbildung benötigt jedoch die Existenz von Gitterbasen auf den Facetten, ist also im Allgemeinen nicht auf reflexive Polytope anwendbar. Beschränkt man sich aber auf den Fall einer primitiven Kollektion der Länge zwei, was einfach einem Paar von Ecken entspricht, die nicht in einer gemeinsamen Facette liegen, so kann in Proposition 3.3.1 gezeigt werden, dass eine sinnvolle Verallgemeinerung des Begriffes einer primitiven Relation existiert. Als eine direkte Anwendung ergibt sich in Korollar 3.3.2, dass je zwei Ecken eines simplizialen reflexiven Polytops durch maximal drei Kanten verbunden werden können, wobei nur im Fall eines zentralsymmetrischen Paares möglicherweise nicht schon zwei oder weniger Kanten ausreichen. Der Kantengraph hat also höchstens Durchmesser drei. Mit Hilfe dieser kombinatorischen Beschränkungen kann in den Korollaren 3.3.4 und 3.3.5 erklärt werden, welche Platonischen Körper überhaupt als reflexive Polytope auftreten können. Primitive Kollektionen und Relationen der Länge zwei sind in vielerlei Hinsicht fundamental und wurden daher im glatten Fall von Casagrande in [Cas03a] genauer untersucht. Sie konnte dort zeigen, dass die Picardzahl einer glatten torischen Fano-Varietät die Picardzahl eines Torusinvarianten Primdivisors höchstens um drei überschreiten kann. Mit Hilfe obiger Verallgemeinerungen kann nun in Korollar 3.5.17 bewiesen werden, dass die entsprechende Aussage auch für $\mathbb{Q}$-faktorielle torische Gorenstein-Fano-Varietäten gilt.

Der Hauptteil des dritten Kapitels beschäftigt sich damit, obere Schranken für Invarianten reflexiver Polytope zu finden, für das Volumen, die Anzahl der Ecken und der Gitterpunkte.

Für die Eckenanzahl eines reflexiven Polytops stellen wir, motiviert durch Beobachtungen in der Datenbank, die Vermutung auf, dass ein d-dimensionales reflexives Polytop höchstens $6^{d/2}$ Ecken hat, wobei der Maximalfall eindeutig bestimmt ist. Diese Vermutung wird im letzten Kapitel dieser Arbeit für zentralsymmetrische einfache reflexive Polytope beliebiger Dimension bewiesen. Hier dagegen liegt der Schwerpunkt auf der maximalen Eckenanzahl simplizialer reflexiver Polytope. Für glatte d-dimensionale Fano-Polytope wurde schon seit längerer Zeit vermutet, dass diese höchstens $3d$ Ecken besitzen, wobei die Gleichheit nur in geraden Dimensionen möglich sein sollte. In 3.5.7 wird nun die Vermutung aufgestellt, dass dies auch für simpliziale reflexive Polytope gilt und der Fall von $3d$ Ecken darüber hinaus eindeutig bestimmt ist. Unter der Voraussetzung, dass das duale Polytop ein zentralsymmetrisches Eckenpaar besitzt, wird diese Vermutung in Satz 3.5.11 erstmals bewiesen. Basierend auf der Veröffentlichung dieser Ergebnisse als Preprint in [Nil04a] ist diese Vermutung dann von Casagrande vollständig in [Cas04] gezeigt worden.

Des Weiteren wird in diesem Kapitel angestrebt, eine gute Abschätzung für die Anzahl der Gitterpunkte eines reflexiven Polytops zu finden. Dies würde durch eine obere Schranke für das Volumen erreicht. Da in der Datenbank die reflexiven Polytope mit dem größten Volumen allesamt Simplexe sind, wurde in dieser Arbeit versucht, eine scharfe Abschätzung für das Volumen eines reflexiven Simplex zu finden. Dies ist der Inhalt von Satz 3.7.13. Die entsprechende algebraisch-geometrische Aussage besagt, dass jede torische Gorenstein-Fano-Varietät mit Klassenzahl eins, also z.B. jeder gewichtete projektive Raum mit Gorenstein-Singularitäten, höchstens einen antikanonischen Grad von 9 in Dimension zwei, 72 in Dimension drei und $2(y_{d-1} - 1)^2$ ab Dimension vier besitzt, wobei die Maximalfälle genau bestimmt werden können. Die hier auftretende Zahlenfolge $y_0, y_1, y_2, \ldots$ wird dabei durch $y_0 = 2$ und $y_n = 1 + y_0 \cdots y_{n-1}$ definiert. Es kann nun vermutet werden, dass diese obere Abschätzung auch für sämtliche Fano-Varietäten mit kanonischen Gorenstein-Singularitäten gültig ist. Im dreidimensionalen Fall ist eine solche Vermutung, bekannt unter dem Namen Fano-Iskovskikh-Vermutung, vor Kurzem in [Pro04] bewiesen worden. Die in diesem Kapitel verwendeten Methoden gründen darauf, dass jedem reflexiven Simplex eine Menge von Stammbrüchen der Summe eins (z.B. $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$) zugeordnet werden kann, siehe [Bat94] und [Con02]. Konvex-geometrische Fragestellungen führen somit zu nicht-trivialen Problemen in der elementaren Zahlentheorie, die eng mit der Darstellung rationaler Zahlen als ägyptische Brüche verwandt sind. Weiterhin wird in Satz 3.7.19 bewiesen, dass es in jeder Dimension einen eindeutigen reflexiven Simplex mit der maximalen Anzahl an Gitterpunkten auf einer Kante gibt. Dies war zuvor in [HM04] von Haase und Melnikov in der Datenbank beobachtet worden.

In Kapitel 4 klassifizieren wir dreidimensionale torische Fano-Varietäten mit terminalen Gorenstein-Singularitäten. Diese Varietäten entsprechen dreidimensionalen reflexiven Polytopen, die auf dem Rand außer den Ecken keine weiteren Gitterpunkte besitzen. Im Hauptsatz 4.3.2 dieses Kapitels wird gezeigt, dass es genau 100 Isomorphieklassen solcher so genannter quasi-glatter Fano-Polytope gibt. Die Idee des Beweises ist, mit den Methoden des vorigen Kapitels zu zeigen, dass quasi-glatte Fano-Polytope schon durch so genannte quasi-primitive Relationen bestimmt sind. Die Klassifikation selbst wird explizit ausgeführt.

In Kapitel 5 untersuchen wir eine spezielle Menge von Gitterpunkten, genannt Wurzelmenge, die zu einem vollständigen Fächer assoziiert werden kann. Im Falle eines Fächers, der von den Normalen eines reflexiven Polytops aufgespannt wird, handelt es sich bei den Wurzeln gerade um die Gitterpunkte im Inneren der Facetten des Polytops. Die Relevanz der Wurzelmenge rührt daher, dass sie essentiell für die Bestimmung der Automorphismengruppe einer torischen Varietät ist. So bestimmt die Anzahl der Wurzeln die Dimension der Automorphismengruppe. Besonders wichtig sind diejenigen Wurzeln, deren Negative auch wieder Wurzeln sind; sie werden als halbeinfach bezeichnet. Die Automorphismengruppe ist genau dann reduktiv, wenn jede Wurzel halbeinfach ist.

In diesem Kapitel führen wir so genannte Facetten-Basen bzw. Wurzel-Basen ein, die auf geometrisch befriedigende Weise die Menge der Facetten, die Wurzeln enthalten, bzw. die Menge der halbeinfachen Wurzeln parametrisieren. Für den Beweis für deren Existenz ist im Fall reflexiver Polytope der im vorigen Kapitel verallgemeinerte Begriff einer primitiven Relation der Länge zwei von Nutzen. Wie sich im Nachhinein herausstellte, entsprach dieser in diesem speziellen Zusammenhang Überlegungen von Bruns und Gubeladze in [BG99]. Im allgemeineren Fall vollständiger Varietäten werden Resultate von Cox über den homogenen Koordinatenring in [Cox95] herangezogen. Als erste Anwendung können wir in Proposition 5.1.19 Produkte projektiver Räume als genau diejenigen vollständigen torischen Varietäten charakterisieren, deren halbeinfache Wurzeln den ganzen Raum aufspannen. Als weiteres konvex-geometrisches Ergebnis besagt Korollar 5.2.4, dass d-dimensionale reflexive Polytope höchstens $2d$ Facetten mit Wurzeln im Inneren besitzen, wobei Gleichheit nur für $[-1, 1]^d$

gilt. Das erste Hauptresultat dieses Kapitels, Satz 5.1.25, liefert eine Erklärung für Beobachtungen in der Datenbank über die Anzahl an Wurzeln reflexiver Polytope. Es wird gezeigt, dass die reduktive Automorphismengruppe einer d-dimensionalen vollständigen torischen Varietät, die kein Produkt projektiver Räume ist, höchstens Dimension 2 für $d = 2$ bzw. $d^2 - 2d + 4$ für $d \geq 3$ besitzen kann.

Eine entscheidende Motivation, sich mit der Wurzelmenge auseinanderzusetzen, rührt von einem Ergebnis von Batyrev und Selivanova in [BS99] her, welches besagt, dass eine glatte torische Fano-Varietät eine Einstein-Kähler-Metrik besitzt, falls die linearen Automorphismen des Dualen des entsprechenden glatten Fano-Polytops keinen gemeinsamen nicht-trivialen Fixpunkt besitzen. Da nach einem bekannten Satz von Matsushima die Existenz einer Einstein-Kähler-Metrik impliziert, dass die Automorphismengruppe der Varietät reduktiv ist, ergibt sich als unmittelbares Korollar, dass jedes reflexive Polytop, dessen Duales ein glattes Fano-Polytop ist und dessen lineare Automorphismen keinen gemeinsamen nicht-trivialen Fixpunkt haben, nur halbeinfache Wurzeln besitzt. Die Autoren fragten daher nach einem rein konvex-geometrischen Beweis für dieses Resultat. Darüber hinaus vermutete Batyrev, dass die Halbeinfachheit aller Wurzeln eines reflexiven Polytops schon aus dem Verschwinden seines Schwerpunktes folgt. Im eben beschriebenen glatten Fall ist kürzlich von Wang und Zhu in [WZ04] sogar gezeigt worden, dass diese Bedingung äquivalent zur Existenz einer Einstein-Kähler-Metrik ist.

Im zweiten Hauptsatz dieses Kapitels, Satz 5.3.1, geben wir nun eine Reihe kombinatorischer Kriterien an, die hinreichend oder äquivalent zur Halbeinfachheit aller Wurzeln einer vollständigen torischen Varietät oder eines reflexiven Polytops sind. Diese Bedingungen beinhalten obige Vermutungen; dabei sind sämtliche Beweise rein konvex-geometrisch.

Kapitel 6 beschäftigt sich mit zentralsymmetrischen reflexiven Polytopen. Ein Hauptresultat, Satz 6.5.1, besagt, dass $[-1,1]^d$ bis auf Isomorphie das einzige d-dimensionale zentralsymmetrische reflexive Polytop mit der Maximalzahl von 3^d Gitterpunkten ist. Im Beweis wird dabei benutzt, dass außer $[-1,1]^d$ jedes d-dimensionale zentralsymmetrische reflexive Polytop weniger als $2d$ Wurzeln hat. Was die maximale Anzahl der Ecken angeht, so können wir in Satz 6.2.2 die allgemeine Vermutung im Fall eines einfachen zentralsymmetrischen reflexiven Polytops beweisen. Dies ist eine Anwendung eines weiteren Hauptergebnisses dieses Kapitels, Korollar 6.3.3 zu Satz 6.3.1, nämlich einer vollständigen Klassifikation beliebigdimensionaler simplizialer reflexiver Polytope, die ein zentralsymmetrisches Paar von Facetten besitzen. Mit Hilfe dieses Satzes, der eine Verallgemeinerung und Vereinfachung von Resultaten von Ewald im glatten und von Wirth im kombinatorisch einfachsten Fall darstellt, wird in Satz 6.3.12 die explizite Klassifikation bis zu Dimension fünf durchgeführt. Weiter zeigen wir, dass in jeder Dimension d ein solches Polytop die gleiche kombinatorische Struktur wie im glatten Fall hat, höchstens $2d^2 + 1$ Gitterpunkte besitzt, in $[-1,1]^d$ einbettbar ist und dass dies für das Duale noch in $\lfloor d/2 \rfloor [-1,1]^d$ möglich ist. Ein letztes Hauptresultat, Satz 6.4.4, gibt eine milde Bedingung an, unter der ein allgemeines d-dimensionales reflexives Polytop in ein Vielfaches von $[-1,1]^d$ eingebettet werden kann. Besitzt demnach ein zentralsymmetrisches Polytop eine einfache Facette, so ist eine Einbettung in ein $(2d-3)$faches möglich.

Appendix B - Lebenslauf

06.02.1976	geboren in Reutlingen
1982 – 1986	Besuch der Grundschule in Entringen
1986 – 1995	Besuch des Uhland-Gymnasiums Tübingen
1995	Abitur am Uhland-Gymnasium Tübingen
1995 – 1996	Zivildienst am Institut für medizinische Informationsverarbeitung Tübingen
1995 – 2001	Mathematikdiplomstudium mit Nebenfach Informatik an der Universität Tübingen
WiSe 1999/2000	Auslandsaufenthalt an der University of Sussex im Rahmen von „International Studies in Mathematics"
WiSe 2000/2001	Diplomarbeit über „Berechnung des Eliminationsgrades", betreut von Prof. Dr. G. Scheja
WiSe 2000/2001, SoSe 2001	Wissenschaftliche Hilfskraft am Mathematischen Institut der Universität Tübingen
Oktober 2001	Diplom in Mathematik an der Universität Tübingen, Beginn der Promotion, betreut von Prof. Dr. V. V. Batyrev
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