Relative simplicial volume

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Chapter 1

Introduction

Topology studies topological spaces up to homeomorphism or up to homotopy equivalence. Algebraic topology associates algebraic objects to spaces such that for homeomorphic (or even homotopy equivalent) spaces the associated objects are isomorphic. Algebra often allows to draw conclusions which would be hard to get by topological means, e.g., about non-existence of maps with certain properties between two given spaces.

For manifolds, one often has more structure: smooth structures, Riemannian metrics, ... , which sometimes allow to draw global (topological) information. A kind of geometric structures, which seem to be particularly useful for the *topological* study of manifolds, at least in low dimensions, are (G, X)-structures in the sense of Thurston, i.e., locally homogeneous metrics.

In dimensions 2 and 3, the most complicated and interesting manifolds admit hyperbolic structures, i.e., (G, X)-structures with $X = H^n$ (the hyperbolic space) and $G = Isom(H^n)$ (its isometry group). (More precisely: all surfaces of genus ≥ 2 are hyperbolic, and conjecturally all 3-manifolds can be decomposed as connected sum and cut along π_1-injective tori into pieces which are either hyperbolic or are quite simple, namely finitely covered by an S^1-bundle.)

In dimensions ≥ 3, hyperbolic structures on a manifold are unique up to isometry, by Mostow's rigidity theorem. Therefore, geometric invariants arising from the hyperbolic metric, such as its volume, are topological invariants. It follows actually from the Chern-Gauß-Bonnet theorem that in even dimensions (including surfaces) hyperbolic volume is proportional to the Euler characteristic χ. In odd dimensions, χ vanishes by Poincare duality, and one might consider hyperbolic volume as a good replacement. Of course, there are plenty of topological invariants, but according to [61] "one gets a feeling that volume is a very good measure for the complexity" of a 3-manifold, and that the ordinal structure (of the set of hyperbolic volumes as a subset of R_+) "is really inherent in 3-manifolds."

Hyperbolic volume is a homotopy invariant and one might ask whether it is definable in terms of algebraic topology. Such a homotopy invariant was indeed defined by Gromov for all (compact, orientable, connected) manifolds: the

$$\text{simplicial volume } \parallel M, \partial M \parallel := inf \left\{ \sum_{i=1}^{r} \mid a_i \mid : \sum_{i=1}^{r} a_i \sigma_i \text{ repres. } [M, \partial M] \right\},$$

where $[M, \partial M]$ is the image of the relative fundamental cycle in $H_{dim(M)}(M, \partial M; R)$. The definition extends to arbitrary compact manifolds, see page 11.

The Gromov-Thurston theorem states: if $int\,(M)$ admits a hyperbolic metric of finite volume $Vol\,(M)$, then $\parallel M, \partial M \parallel = \frac{1}{V_n} Vol\,(M)$, where V_n is the volume of a regular ideal simplex in H^n, i.e., a constant depending only on dimension n. More generally, for any (G, X) one has a constant $V\,(G, X)$ such that $\parallel M, \partial M \parallel = V\,(G, X)\, Vol\,(M)$ whenever M has a (G, X)-structure. For many "simple" structures this constant is actually zero, e.g., if G is solvable.

The simplicial volume quantifies the topological complexity of manifolds. Indeed, define a partial order on the set of n-manifolds by: $M_1 \geq M_2$ if there exists a degree 1 map from M_1 to M_2. Then the simplicial volume is an order-preserving map from the set of n-manifolds to the nonnegative reals. More generally, if there is a degree d map from M_1 to M_2, then $\parallel M_1, \partial M_1 \parallel \geq d \parallel M_2, \partial M_2 \parallel$. As mentioned above, algebraic topology is often useful for finding restrictions on mappings between given spaces. However, it is hard to get such quantitative restrictions from non-numerical algebraic invariants.

Another use of the simplicial volume is that it relates to rigidity questions and somehow clarifies how a manifold's topology determines its hyperbolic geometry. It was used for Thurston's version of Mostow rigidity: any map $f : M_1 \rightarrow M_2$ between finite-volume hyperbolic manifolds satisfying $vol\,(M_1) = deg\,(f)\, vol\,(M_1)$ can be homotoped into a normal form, namely a locally isometric covering.

In spite of its relatively unassuming definition, the simplicial volume is quite hard to calculate. Gromov developped the theory of bounded cohomology to prove various vanishing results for the simplicial volume. He proved that:

- if ∂M is connected and $\pi_1 M, \pi_1 \partial M$ are amenable (e.g., virtually solvable), then $\parallel M, \partial M \parallel = 0$,

- if M admits a nontrivial (not necessarily free) S^1-action, then $\parallel M, \partial M \parallel = 0$,

- if M_1, M_2 are closed manifolds of dimension ≥ 3, then $\parallel M_1 \sharp M_2 \parallel = \parallel M_1 \parallel + \parallel M_2 \parallel$.

On the other hand, he proved nontriviality of $\parallel M, \partial M \parallel$ if $int\,(M)$ admits a complete metric with $-b^2 \leq$ sectional curvature $\leq -a^2 < 0$ and finite volume, and gave the exact formula for finite-volume hyperbolic manifolds mentioned above.
To describe results about the simplicial volume obtained in the last 20 years, there are triviality results such as $\parallel M^n \parallel = 0$ if M^n admits an amenable cover with n-dimensional nerve ([26],[35]) and a generalization for complex varieties ([64]), and nontriviality results as [55] for compact quotients of $SL_n R/SO_n$, [56] for bases of flat bundles with nontrivial Euler class and [33] for surface bundles with fiber genus ≥ 2. In a different direction, bounded cohomology has shown more applications than the simplicial volume, in particular in dynamics of group actions (see the relevant chapters of [45] for an overview), the most striking application being that the second bounded cohomology of a group classifies its representations in $Homeo^+\,(S^1)$ up to topological semi-conjugacy ([22]).
In section 2.3. we discuss bounded cohomology and simplicial volume of Lefschetz fibrations. The results are not related to the rest of this thesis, but we think that they should be of some independent interest. We show
Theorem 1: *If $\pi : M \rightarrow B$ is a Lefschetz fibration with fiber F_g, vanishing cycles $v_1, \ldots, v_r \subset F_g$, regular values $B' \subset B$ and monodromy $\rho : \pi_1 B' \rightarrow Map_{g,*}$, then the real Euler class $\overline{e}$ is bounded if and only if $\{(g)\,(v_i) : g \in \pi_1 B', i = 1, \ldots, r\}$ is an incomplete curve system.*
This gives a generalization to the results of [47], [22], [33], that surface bundles have bounded Euler class and, hence, positive simplicial volume.
Boundedness of the Euler class is a sufficient condition for a Lefschetz fibration, with base and fiber of genus ≥ 2, to have positive simplicial volume.
To generally determine triviality/nontriviality of the simplicial volume for Lefschetz fibrations (with base and fiber of genus ≥ 2) it would be necessary to have a criterion for (un)boundedness of $e_R \cup \pi^* \omega_B$, the cup-product of the real Euler class with the pulled-back volume form of the base.

Generally spoken, we study in this dissertation simplicial volume relative to codimension 1 objects.

On the one hand we study the behaviour of simplicial volume with respect to cut and paste, i.e., we wish to compare $\parallel M_F, \partial M_F \parallel$ to $\parallel M, \partial M \parallel$, where $F \subset M$ is a properly embedded (n-1)-submanifold and $M_F := \overline{M - N(F)}$ for a regular neighborhood $N(F)$.

On the other hand, we will study the foliated Gromov norm of codimension 1 foliations, which seems to be a good invariant to quantify the branching of foliations.

Cut and paste.
It is not hard to see that simplicial volume of surfaces is additive w.r.t. glueing along boundaries. For 3-manifolds, things become much more complicated, and there doesn't seem to exist a general formula for the behaviour of simplicial volume of 3-manifolds w.r.t. glueing along surfaces of genus ≥ 2. As a special case of lemma 11 and 12 cited below, we will get that simplicial volume of 3-manifolds is additive w.r.t. glueing along π_1-injective tori and superadditive w.r.t. glueing along π_1-injective annuli. (In the special case that **all** boundary components of the 3-manifolds are tori, a stronger statement was proved by Soma in [57]. Anyway, his proof relies in an essential way on a statement of Thurston which seems not so easy to prove.)

We want to remind what the glueing problem is about.
The inequality $\parallel M_F, \partial M_F \parallel \leq \parallel M, \partial M \parallel$ translates to the following statement: there exist fundamental cycles of M, with l^1-norm arbitrarily close to $\parallel M, \partial M \parallel$, which can be split into fundmamental cycles for the components of M_F, i.e., which don't invoke simplices cut into pieces by F.
In turn, the inequality $\parallel M_F, \partial M_F \parallel \geq \parallel M, \partial M \parallel$ has the following meaning: there exist fundamental cycles for the components of M_F, with l^1-norm arbitrarily close to the simplicial volume, which fit together at the boundary components of M_F, i.e., their boundaries cancel against each other.

Tori and annuli are distinguished from surfaces of genus ≥ 2 by the property that they have amenable fundamental groups. In fact, Gromov already showed in [26] that simplicial volume of **closed** manifolds is additive w.r.t. "amenable glueings" (see the introduction to chapter 3 for a precise definition), and he indicated that there are analogous results for glueing non-closed manifolds along parts of their boundaries. We use methods introduced by Gromov to write proofs of the following lemmata 11-12 (put together in theorem 2), which imply in particular: simplicial volume of manifolds with boundary is additive w.r.t. glueing along amenable π_1-injective closed (n-1)-manifolds and superadditive w.r.t. glue-

ing along amenable π_1-injective (n-1)-manifolds with boundary.
Lemma 11(i): *Let M_1, M_2 be two compact, connected n-manifolds, A_1, A_2 (n-1)-dimensional submanifolds of ∂M_1 resp. ∂M_2, $f : A_1 \rightarrow A_2$ a homeomorphism and $M = M_1 \cup_f M_2$ the glued manifold.*
If f_* maps $ker(\pi_1 A_1 \rightarrow \pi_1 M_1)$ isomorphically to $ker(\pi_1 A_2 \rightarrow \pi_1 M_2)$, and if $im(\pi_1 A_1 \rightarrow \pi_1 M_1)$ is amenable, then $\| M, \partial M \| \geq \| M_1, \partial M_1 \| + \| M_2, \partial M_2 \|$.
Lemma 11(ii): *Let M_1 be a compact, connected n-manifold, no component of which is a 1-dimensional closed intervall, A_1, A_2 disjoint (n-1)-dimensional submanifolds of ∂M_1, $f : A_1 \rightarrow A_2$ a homeomorphism and $M = M_1/f$ the glued manifold.*
If $im(\pi_1 A_1 \rightarrow \pi_1 M_1)$ is amenable, then $\| M, \partial M \| \geq \| M_1, \partial M_1 \|$.
Lemma 12: *Let M_1 be a (possibly disconnected) compact manifold, A_1, A_2 connected components of ∂M_1, $f : A_1 \rightarrow A_2$ a homeomorphism, $M = M_1/f$ the glued manifold. Assume that one has, for $i = 1, 2$, connected sets $A_i \subset A_i' \subset M$ such that $\pi_1 A_i'$ are amenable, then $\| M, \partial M \| \leq \| M_1, \partial M_1 \|$.*

It is not possible to give general formulae or just inequalities for glueing along non-amenable boundaries without restricting to special assumptions. In the case of 3-manifolds, it follows easily from the geometrisation of manifolds with non-spherical boundary, that all questions reduce to hyperbolic manifolds (with possibly infinite volume). We will consider the special case that the hyperbolic manifolds admit a hyperbolic metric with totally geodesic boundary and cusps. (See the introduction to chapter 5 for a precise definition. In dimension 3, these are the manifolds admitting a hyperbolic metric such that the boundary components of genus ≥ 2 are totally geodesic and the ends corresponding to torus boundary components are complete and of finite volume, i.e. cusps.) One motivation to study this special case is that any hyperbolic 3-manifold with π_1-injective boundary can be cut along π_1-injective annuli into pieces which admit such a hyperbolic metric with totally geodesic boundary and cusps.

In the case of no cups, the following theorem 4 is equivalent to the theorem of Jungreis in [36]. Our proof of the general case builds on similar basic ideas, but is technically much more involved. We will give some rough explanations to the proof at the end of the introduction.
Theorem 4: *Let $n \geq 3$ and let M_1, M_2 be compact n-manifolds with boundaries $\partial M_i = \partial_0 M_i \cup \partial_1 M_i$, such that $M_i - \partial_0 M_i$ admit incomplete hyperbolic metrics of finite volume such that $\partial_1 M_i$ are totally geodesic boundaries and the ends corresponding to $\partial_0 M_i$ are complete. If $\partial_1 M_i$ are not empty, $f : \partial_1 M_1 \rightarrow \partial_1 M_2$ is an isometry and $M = M_1 \cup_f M_2$, then*

$$\| M, \partial M \| < \| M_1, \partial M_1 \| + \| M_2, \partial M_2 \| .$$

The same statement holds if one glues only along some connected components

of $\partial_1 M_i$. One also has an analogous statement if two totally geodesic boundary components of the same hyperbolic manifold are glued by an isometry.
A reformulation of theorem 4 says that $\| M_1, \partial M_1 \| > \frac{1}{V_n} Vol(M_1)$ holds when M_1 is a hyperbolic manifold of dimension ≥ 3 with non-empty totally geodesic boundary and cusps. It remains open whether this lower bound is optimal.

In the case of 3-manifolds, theorem 4 serves as main step for the more general:
Theorem 5: *For a compact 3-manifold M, $\| DM \| < 2 \| M, \partial M \|$ holds if and only if $\| \partial M \| > 0$, i.e., if ∂M consists not only of spheres and tori.*

Here, DM is the manifold obtained by glueing two differently oriented copies of M via the identity of ∂M. Note that $\| DM \| \leq 2 \| M, \partial M \|$ trivially holds. (One should read the inequality $\| DM \| < 2 \| M, \partial M \|$ as follows: a fundamental cycle of DM with l^1-norm sufficiently close to $\| DM \|$ necessarily has to invoke simplices which are sort of transversal to the surface $\partial M \subset DM$.)

Another (direct) corollary from theorem 4 and Mostow rigidity is that (under the assumptions of theorem 4), in dimensions ≥ 4, we get the same inequality for *any* homeomorphisms f. This theorem seems to be hardly available by topological methods. The same statement in dimension 3 is unlikely to hold:
Question: If M_1, M_2 are hyperbolic 3-manifolds with totally geodesic boundary and $f : \partial M_1 \rightarrow \partial M_2$ is pseudo-Anosov, then is $\lim_{n\to\infty} \| M_1 \cup_{f^n} M_2 \| = \infty$?

Foliated Gromov norm.
The leaf space of a codimension 1 foliation $\mathcal{F}$ is a (non-Hausdorff) 1-manifold. We consider the leaf space of the induced foliation $\tilde{\mathcal{F}}$ on the universal cover $\tilde{M}$. According to [20], the leaf space of $\tilde{\mathcal{F}}$ is an order tree (with $\pi_1 M$ acting upon), if $\mathcal{F}$ is a taut foliation on a 3-manifold. This motivates the study of group actions on order trees. The following notion was introduced by Calegari in [11].

The Gromov norm of a foliation/lamination $\mathcal{F}$ on a manifold M is

$$\| M, \partial M \|_{\mathcal{F}} := inf \left\{ \sum_{i=1}^{r} | a_i | : \sum_{i=1}^{r} a_i \sigma_i \text{ represents } [M, \partial M], \sigma_i \text{ transversal to } \mathcal{F} \right\}.$$

One major motivation is that the difference $\| M, \partial M \|_{\mathcal{F}} - \| M, \partial M \|$ seems to quantify the amount of branching of the leaf space of $\tilde{\mathcal{F}}$. (There had been another kind of foliated Gromov norm defined by Connes, cf., [27], [28]. His definition worked only for foliations with transverse measures. However, most foliations don't admit a transversal measure, i.e., the leaf space of $\tilde{\mathcal{F}}$ is just an order tree, without metric structure.)

Calegari proved:
- $\| M \|_{\mathcal{F}} = \| M \|$, if M is a closed 3-manifold, $\mathcal{F}$ is taut and the leaf space of $\tilde{\mathcal{F}}$ is branched in at most one direction, and
- $\| M \|_{\mathcal{F}} > \| M \|$, if M is a closed hyperbolic 3-manifold and $\tilde{\mathcal{F}}$ is an asymptotically separated lamination.

Here, a lamination of H^3 is called asymptotically separated if there exists two geodesic planes on distinct sides of some leaf of $\tilde{\mathcal{F}}$. This is, for example, satisfied for one quasigeodesic leaf.

The first statement generalizes easily to manifolds with boundary (lemma 34). The second statement, which in the closed case follows with a relatively short argument from Jungreis theorem, becomes more technical if one has to control the foliation in the cusps; we discuss the argument at the end of the introduction. We prove the extension to the cusped case in the following theorem, where a (quite small) class of finite-volume hyperbolic 3-manifolds has to be excluded.

Theorem 6: *If the interior of M is a hyperbolic 3-manifold of finite volume which is not Gieseking-like, and if $\mathcal{F}$ is an asymptotically separated lamination, then*

$$\parallel M, \partial M \parallel < \parallel M, \partial M \parallel_{\mathcal{F}} .$$

Here, M is called Gieseking-like if it has a hyperbolic structure of finite volume such that the cusp set of M contains the cusp set of the Gieseking manifold, i.e., $Q\left(\sqrt{-3}\right) \cup \{\infty\}$ in the ideal boundary of the upper half-space model.

A conjecture of Fenley would imply that all foliations of finite-volume hyperbolic 3-manifolds with branching in both directions are asymptotically separated. Hence, theorem 6 suggests a conjectural **branching criterion** for foliations $\mathcal{F}$ on finite-volume hyperbolic 3-manifolds M: $\mathcal{F}$ branches in both directions iff $\parallel M, \partial M \parallel < \parallel M, \partial M \parallel_{\mathcal{F}}$.

To show the strength of theorem 6, we mention the following special case, which gives a topological criterion to decide whether a surface in a hyperbolic manifold is a virtual fiber:

Corollary 11: If $int\left(M\right)$ is a finite-volume hyperbolic 3-manifold which is not Gieseking-like and $F \subset M$ is a compact, properly embedded π_1-injective surface, then F is a virtual fiber if and only if $\parallel M, \partial M \parallel_{\mathcal{F}} = \parallel M, \partial M \parallel$.

This corollary is a reflection of the Thurston-Bonahon dichotomy: F is either a virtual fiber or quasigeodesic.

We want to give an **overview to the proofs of theorems 4 and 6**. These theorems can actually be regarded as statements about properties of "efficient fundamental cycles" on finite-volume hyperbolic manifolds. Namely, they mean that a relative fundamental cycle of M with l^1-norm sufficiently close to $\| M, \partial M \|$ can not fulfill any of the following two conditions:

- respect a given totally geodesic surface F in M, in the sense that it splits into relative fundamental cycles for the components of M_F,
- be transversal to an asymptotically separated lamination.

Our approach is to study limits of sequences c_ϵ of fundamental cycles with l^1-norm smaller than $\| M, \partial M \| + \epsilon$. Such sequences can degenerate in two ways:
- simplices degenerate to ideal simplices,
- singular chains (i.e., finite linear combinations of simplices) degenerate to signed measures on the space of simplices.

The natural notion of convergence is then weak-*-convergence of signed measures. Unfortunately, the space of ideal simplices (even of straight ideal simplices) is not Hausdorff. However, we show that we can restrict to consider chains consisting of nondegenerate straight simplices, and that this space is metrisable, hence, weak-*-limits of bounded sequences exist.

The limits are actually supported on the set of regular ideal simplices, which is the same as $\Gamma \backslash Isom(H^n)$, Γ being the deck group of the covering $H^n \rightarrow M$. In dimensions ≥ 3, the cycle property forces invariance under a reflection group, thus making these limits treatable by ergodic theory.

The reader will recognise Jungreis approach in [36] (up to the technicality that he does not restrict to nondegenerate simplices, which turns his proof of lemma 26 more involved). In the case of cusped hyperbolic manifolds, our approach is parallel, but the details become more technical.

For example, the fact that the limiting objects are cycles (hence, invariant under a reflection group), which comes for free in the closed case, is not obvious in the finite-volume case.

In fact, we will need an appropriate definition of the sequences c_ϵ : we exhaust $int(M)$ by the ϵ-thick parts $M_{[\epsilon,\infty]}$, consider relative fundamental cycles of M as relative fundamental cycles of $M_{[\epsilon,\infty]}$, straighten them and consider the limits μ. It is convincing (and we prove it in lemmata 22-24) that the boundaries of the straightened relative fundamental cycles "escape to infinity" and therefore disappear in the limit.

The outcome in the closed case was that the limiting signed measure μ has to be the "smearing cycle" smr (i.e., equidistribution of regular ideal simplices with signs according to orientation, see the introduction to chapter 4). In the finite-volume case, one also has the possibility of measures supported on sets of simplices with all vertices in cusps. We show that μ "decomposes" (in the sense of ergodic decomposition) into these two kinds of signed measures.

The proof of theorem 4 can then be described as follows: assume, $\mu^{\pm}$ would vanish on $S_F^n = \{$simplices cut into pieces by $F\}$. Then the ergodic decomposition of μ can not invoke *smr* because *smr* does not vanish on S_F^n. Hence, $\mu^{\pm}$ should be supported on the set of simplices with all vertices in cusps of M. But, as a limit of cycles respecting F, μ has to have simplices with boundary faces in F in its support. However, F does not have cusps. (This explanation is quite convincing, but we should mention that it will need some work to make a proof out of it. We do not want to discuss the arising technical problems here, but refer to section 4.4.)

Also the proof of theorem 6 gets rather technical. However, the basic idea is again quite simple: it follows from well-known facts about finite-covolume groups $\Gamma \subset Isom(H^3)$ that Γ-invariant, asymptotically separated laminations $\mathcal{F}$ can't be transversal to all regular ideal regular ideal simplices in H^n and, actually, there are 3 half-spaces H_0, H_1, H_2 such that 3-simplices with vertices $v_0 \in H_0, v_1 \in H_1, v_2 \in H_2, v_3$ arbitrary, can't be transversal to $\mathcal{F}$. This, together with Jungreis theorem, gives a proof of Calegari's result, i.e., theorem 6 for closed manifolds. The problem arising in the cusped case is, roughly, that, starting with a relative fundamental cycle transversal to $\mathcal{F}$, we know the chains c_ϵ to be transversal to $\mathcal{F}$ only on the ϵ'-thick part for some ϵ' slightly larger than ϵ, and this makes some annoying technicalities unavoidable.

Convention: If M is nonorientable, define $\parallel M, \partial M \parallel := \frac{1}{2} \parallel \overline{M}, \partial\overline{M} \parallel$, where $\overline{M}$ is the orientable double cover of M. If M is disconnected, define $\parallel M, \partial M \parallel := \sum_{i=1}^{r} \parallel M_i, \partial M_i \parallel$, where $M_1, \ldots, M_r$ are the connected components of M. Everything we discuss will easily reduce to orientable, connected manifolds, and we will do this reduction without mentioning. Moreover, if not stated differently, hyperbolic manifolds are supposed to be complete, that is, to be quotients $\Gamma \backslash H^n$ for some discrete subgroup $\Gamma \subset Isom(H^n)$.

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Chapter 2

Simplicial volume and bounded cohomology

The first two sections of this chapter serve to introduce definitions and known results and to provide the reader with some background knowledge which might be helpful for reading the following chapters. (They contain nothing new.) The third section contains a study of Lefschetz fibrations. We included it, although it is only superficially related to the rest of this thesis, since it should be of independent interest.

2.1 Definitions and examples

Simplicial volume. For a topological space X, denote

$$C_n(X;R) = \left\{ \sum_{i=1}^{k} r_i \sigma_i : r_i \in R, \sigma_i : \Delta_n \to X \right\}$$

its n-th chain group, the real vector space generated by singular n-simplices in X. We will consider the l^1-norm

$$\| \sum_{i=1}^{k} r_i \sigma_i \| = \sum_{i=1}^{k} | r_i | .$$

There is the boundary operator $\partial_n : C_n(X;R) \to C_{n-1}(X;R)$ defined by mapping each singular simplex to its boundary and linear extension. The singular homology is defined as

$$H_n(X;R) = ker(\partial_n) / im(\partial_{n+1}) .$$

For an element $c \in C_n(X;R) \cap kern(\partial_n)$ we denote its equivalence class $[c] \in H_n(X;R)$. We use the l^1-norm on $C_n(X;R)$ to define a pseudonorm on $H_n(X;R)$ as

$$\| h \| = inf \{ \| c \| : [c] = h \},$$

the so-called Gromov norm.
To get an invariant of closed, oriented, connected n-manifolds M, consider its fundamental class $[M]$, that is, the image of a generator of $H_n(M;Z) \simeq Z$ under the canonical morphism $H_n(M;Z) \to H_n(M;R)$, and define the simplicial volume

$$\| M \| = \| [M] \|$$

as the Gromov-norm of the fundamental class.
For a closed, connected, non-oriented manifold M define $\| M \| = \frac{1}{2} \| \tilde{M} \|$, where $\tilde{M}$ is the orientation cover of M. For any closed manifold M define $\| M \| = \sum_{i=1}^{k} \| M_i \|$, where $M_1, \ldots, M_k$ are the connected components of M.
Relative simplicial volume. There is a relative version of the above construction. For a pair (X, Y) of topological spaces consider $C_n(X,Y;R) = C_n(X;R)/C_n(Y;R)$ with the quotient norm. The boundary operator ∂_n maps $C_n(X,Y;R)$ to $C_{n-1}(X,Y;R)$ and one defines $H_n(X,Y;R) = ker(\partial_n)/im(\partial_{n+1})$ and $\| h \| = inf \{ \| c \| : [c] = h \}$ for $h \in H_n(X,Y;R)$.
For a compact, oriented, connected n-manifolds M with boundary ∂M, consider its fundamental class $[M, \partial M]$, that is, the image of a generator of $H_n(M, \partial M; Z) \simeq Z$ under the canonical morphism $H_n(M, \partial M; Z) \to H_n(M, \partial M; R)$, and define the simplicial volume $\| M, \partial M \| = \| [M, \partial M] \|$ as the Gromov-norm of the relative fundamental class. Again this definition extends to non-orientable, disconnected, compact manifolds.

Bounded cohomology. For a topological space X, define its n-th cochain group

$$C^n(X;R) = Hom(C_n(X,R);R)$$

and the subgroup of *bounded* cochains

$$C_b^n(X) = \{ f \in C^n(X;R) : sup \{ | f(\sigma) | : \sigma : \Delta_n \to X \} < \infty \}.$$

The norm $\| f \|_\infty = sup \{ | f(\sigma) | : \sigma : \Delta_n \to X \}$ is defined on $C_b^n(X)$. Let the coboundary $\delta_n : C^n(X;R) \to C^{n+1}(X;R)$ be the dual operator to ∂_{n+1}. It maps $C_b^n(X)$ to $C_b^{n+1}(X)$. We define the bounded cohomology

$$H_b^n(X) = ker\left(\delta_n |_{C_b^n(X)}\right) / im\left(\delta_{n-1} |_{C_b^{n-1}(X)}\right)$$

with the pseudonorm induced by $\| \cdot \|_\infty$.
The inclusion $C_b^n(X) \to C^n(X;R)$ induces the **canonical homomorphism**

$H_b^n(X) \to H_b^n(X;R)$.
Relative bounded cohomology. For a pair (X,Y) of topological spaces consider $C_b^n(X,Y) = C_b^n(X)/C_b^n(Y)$ with the quotient norm. The boundary operator δ_n maps $C_b^n(X,Y)$ to $C_b^{n-1}(X,Y)$ and one defines
$H_b^n(X,Y) = ker\left(\delta_n\mid_{C_b^n(X,Y)}\right)/im\left(\delta_{n-1}\mid_{C_b^{n-1}(X,Y)}\right)$.

Duality between homology and cohomology. We have, by definition, a duality between $C_b^n(X)$ and the completion of $C_n(X;R)$ with respect to the l^1-norm. This duality descends to the homology level as follows.
Theorem : *The pseudonorm on bounded cohomology is dual to the Gromov norm on homology: if $\beta \in H_b^*(M;R)$ and $h \in H_*(M;R)$ satisfy $<\beta,h>=1$, then $\| h \| = \frac{1}{\|\beta\|}$.*
Proof: $\frac{1}{\|\beta\|} \leq \| h \|$ is obvious. We prove the opposite inequality.
Recall that the value of a cohomology class β on a cycle is well defined, i.e. does not depend on the representative of β. Hence we may define $f : ker(\partial) \cap C_*(M;R) \to R$ by $f(z) := \beta(z)$. By the Hahn-Banach theorem, there is $\omega : C_*(M;R) \to R$ such that ω restricts to f on $ker(\partial)$ and that $\| \omega \| = \| f \|_\infty = sup\{\beta(z) : \| z \| = 1\}$. We claim that ω is a representative of β in $H_b^*(M;R)$.
Since the cohomology class of a cocycle is determined by its values on all cycles, we get that $[\omega]-\beta$ is in the kernel of $H_b^*(M;R) \to H^*(M;R)$. To show that $[\omega]-\beta = 0$, we consider the decomposition $C_n(M;R) = ker(\partial_n) \oplus C_n(M;R)/ker(\partial_n)$. For any representative $b \in \beta \in H_b^*(M;R)$ we have that $\omega - b$ vanishes on the first direct summand, hence corresponds to a *bounded* morphism $g : C_n(M;R)/ker(\partial_n) \to R$. Using the canonical isomorphism $C_n(M;R)/ker(\partial_n) \simeq im(\partial_n)$, and extending trivially on $C_{n-1}(M;R)/im(\partial_n)$, we get $g \in C^{n-1}(M;R)$ with $\delta g = \omega - b$.
□

Corollary: *Let M be a closed oriented connected n-dimensional manifold, and define its cohomological fundamental class as the (unique) class $\beta \in H^n(M;R)$, which satisfies $\beta([M]) = 1$. Then $\| M \| = \frac{1}{\|\beta\|}$.*
We will mainly use the following two special cases.
Let M be an n-dimensional closed, oriented, connected manifold.
Vanishing simplicial volume.

If $H_b^n(M) = 0$, then $\| M \| = 0$.

Positive simplicial volume.

If $H_b^n(M) \to H^n(M;R)$ is surjective, then $\| M \| > 0$.

Properties of simplicial volume and bounded cohomology - an overview.
For a topological space X let $f : X \to K(\pi_1 X, 1)$ be the classifying map of the

fundamental group. It induces an isometric isomorphism $H_b^*(\pi_1 X) \simeq H_b^*(X)$. This shows that the simplicial volume of simply connected manifolds vanishes. More generally, Gromov and Ivanov show
Theorem: *If $\pi_1 M$ is amenable, then $H_b^*(M) = 0$ for $* \geq 1$. Hence, $\| M \| = 0$.*

On the other hand, Gromov and Mineyev show
Theorem: *If G is word-hyperbolic, then $H_b^*(G) \to H^*(G; R)$ is surjective for $* \geq 2$. Hence, if M is aspherical and $\pi_1 M$ word-hyperbolic, then $H_b^*(M) \to H^*(M; R)$ is surjective for $* \geq 2$ and $\| M^{n \geq 2} \| > 0$.*

More information can be deduced from the following two theorems of Gromov:
Theorem: *If M_1 and M_2 have dimension ≥ 3, and $M_1 \sharp M_2$ is their connected sum, then $\| M_1 \sharp M_2 \| = \| M_1 \| + \| M_2 \|$.*
Theorem: *For any $m, n \in N$ there is a constant $C_{m,n}$ such that, if M_1 and M_2 have dimension m and n, then the inequalities $\| M_1 \| \| M_2 \| \leq \| M_1 \times M_2 \| \leq C_{m,n} \| M_1 \| \| M_2 \|$ hold.*
Proof of the product inequality: The upper bound is due to the fact that there exists a special triangulation for the product of simplices such that two products equipped with this triangulation always fit together at the corresponding boundary faces. The lower bound follows from the inequality $\| \alpha \cup \beta \| \leq \| \alpha \| \| \beta \|$ for the cup product of two bounded cohomology class. One should note that the theorem still holds true if M_1 has boundary and M_2 is closed, but that in general there is no lower bound on $\| M_1 \times M_2, \partial (M_1 \times M_2) \|$ if both M_1 and M_2 have nonempty boundary.

2.2 Volume and nonpositive curvature

A Riemannian manifold M is called a symmetric space if, for any $x \in M$, exists an isometry $I : M \to M$ such that $I(x) = x$ and $DI_x = -Id$. A symmetric space is termed irreducible if it is not a product of two symmetric spaces. It is well-known that irreducible symmetric spaces are of one of the following 3 types:
- symmetric spaces of compact type,
- euclidean spaces,
- symmetric spaces of noncompact type.

In terms of the sectional curvature K, these types are distinguished as follows: euclidean spaces satisfy $K = 0$, symmetric spaces of compact type satisfy $K \geq 0$, symmetric spaces of noncompact type satisfy $K \leq 0$.
It is well-known that, for M a compact manifold, $\| M \| = 0$ holds if the universal cover $\widetilde{M}$ is a symmetric space of compact type or an euclidean space.

Conjecture 1 *(Gromov): Let M be a compact Riemannian manifold such that its universal cover $\tilde{M}$ is an irreducible symmetric space of noncompact type. Then* $\parallel M \parallel > 0$.

This conjecture is known to be correct for $rk\left(\tilde{M}\right) = 1$ (i.e. $K < 0$) by [26], [34], and for $\tilde{M} = SL_nR/SO_n$ by [55].
For an ordered finite set of vertices in a nonpositively curved aspherical space $\tilde{M}$ one may define the **straight simplex** with vertices $\{v_0, \ldots, v_n\}$ by succesively forming the geodesic cone. That is, we succesively define $\sigma_0, \sigma_1, \ldots, \sigma_n$ by $\sigma_0 = v_0$ and $\sigma_{r+1} : \Delta^{r+1} \rightarrow \tilde{M}$ is defined on the standard simplex $\Delta^{r+1} \supset \Delta^r$ by the condition that $\sigma_{r+1}(t_0, \ldots, t_{r+1})$ is the point on the (unique) geodesic from $\sigma_r\left(\frac{t_0}{1-t_{r+1}}, \ldots, \frac{t_r}{1-t_{r+1}}\right)$ to v_{r+1} which has distance $t_{r+1} dist\left(v_{r+1}, \sigma_r\left(\Delta_r\right)\right)$ from v_{r+1}. (Note that this construction depends on the order of vertices, if $\tilde{M}$ has nonconstant sectional curvature.) Gromov's conjecture reformulates as follows:

Conjecture 2 *:Let $\tilde{M}$ be an n-dimensional irreducible symmetric space of noncompact type. Then there is a constant $C = C\left(\tilde{M}\right)$ sucht that $vol\left(\Delta\right) < C$ holds for any straight n-simplex Δ in $\tilde{M}$.*

Proof of "conjecture 2 ⇒ conjecture 1": Let $\sum_{i=1}^k a_i\sigma_i$ represent the fundamental class of M. For each σ_i let G_i be the symmetric group on the vertices of σ_i. For each $g \in G_i$ we get a straight simplex $str_g\left(\sigma_i\right)$ of volume smaller than C. The cycle $\sum_{i=1}^k \sum_{g \in G_i} \frac{a_i}{(n+1)!} str_g\left(\sigma_i\right)$ represents the fundamental class of M. Hence, $Vol\left(M\right) = \sum_{i=1}^k \sum_{g \in G_i} \frac{a_i}{(n+1)!} vol\left(str_g\left(\sigma_i\right)\right) < C \sum_{i=1}^k \mid a_i \mid$. Since this is true for any representative of the fundamental class, we get $\parallel M \parallel \geq \frac{1}{C} Vol\left(M\right)$. □

Simplices in spaces of nonpositive curvature. Let M be a Riemannian manifold of nonpositive curvature. Let Δ be a straight simplex in X with vertices $(v_0, \ldots, v_l, v_{l+1}, \ldots, v_k)$. Let $B_{l+1}, \ldots, B_k$ be the Busemann functions associated to the geodesics joining v_l to $v_{l+1}, \ldots, v_k$. The vector fields $Z_i = -gradB_i$ generate flows Ψ_t^i. We certainly have

$$\Delta \subset \cup_{t_{l+1} \geq 0, \ldots, t_k \geq 0} \Psi_{t_k} \ldots \Psi_{t_{l+1}}\left(\Delta^l\right),$$

where Δ^l is the straight simplex spanned by $v_0, \ldots, v_l$ in this order. Define

$$\tau : \Delta^l \times [0, \infty)^{k-l} \rightarrow M$$

by $\tau\left(y, t_{l+1}, \ldots, t_k\right) = \Psi_{t_k} \ldots \Psi_{t_{l+1}}\left(y\right)$.
Let $X_1, \ldots, X_l$ be an ON-reper in $< Z_{l+1}, \ldots, Z_k >^{\perp} \subset T\Delta^l$. Extend it to an ON-reper $\{X_1, \ldots, X_n\}$ with $X_{l+1} = Z_{l+1}$ and $X_{l+i} = Z_{l+i} + \sum j = 0^{i-1} \alpha_{l+i,l+j} Z_{l+j}$

for suitable coefficients. Choose coordinates $y_1,\ldots,y_l$ on Δ such that $\frac{\partial}{\partial y_i} = X_i + \sum_{j>l} b_{ij} Z_l$. For the volume form ω on M we get
$\tau^*\omega = f\left(y_1,\ldots,y_l,t_{l+1},\ldots,t_k\right) dy_1\ldots dy_l dt_{l+1}\ldots dt_k$ with

$$f = \omega\left(\tau_*\frac{\partial}{\partial y_1},\ldots,\tau_*\frac{\partial}{\partial t_k}\right) = det\left(A\right),$$

where A is the matrix with entries $a_{ij} = g\left(\tau_*\frac{\partial}{\partial y_i}, X_j\right)$ for $i \leq l$ and $a_{ij} = g\left(\tau_*\frac{\partial}{\partial t_i}, X_j\right)$ for $i \geq l+1$. Note that the volume form on the base simplex satisfies $dvol_{\Delta^l} = \frac{1}{B} dy_1\ldots dy_l$ with $B \geq 1$, ($B \geq 1$ is an elementary, but nontrivial, exercise), implying that $vol\left(\Delta\right) \leq \int\ldots\int\int_{\Delta^l} det\left(A\right) dvol_{\Delta^l} l_{\Delta^l} dt_{l+1}\ldots dt_k$.

Negatively curved manifolds. If there is a negative upper bound on the sectional curvature, the above argument can be used to give an upper bound on the volume of straight simplices in $\tilde{M}$. Namely, one can use the Jacobi equations to bound $det\left(A\right)$ in terms of the curvature bound. (Essentially this argument, up to the use of Busemann functions, was given in [34].) To consider nonpositively curved symmetric spaces of higher rank, we indicate a proof of the fact that the simplicial volume of the product of negatvely curved closed manifolds is positive. (This is, of course, well-known: it was proved by a different argument in [34], and clearly the shortest proof uses the cup product as in the last theorem of section 2.1.) For simplicity, we restrict to a special straight simplex in $H^2\times H^2$, but note that the argument, of course, can be generalized to straight simplices in products of negatvely curved manifolds.

Toy example: $H^2\times H^2$.

For simplicity, we consider the following situation: v_0,v_1,v_2 are nonideal vertices contained in the same $H^2\times\{y\}$, v_3 and v_4 are arbitrary ideal vertices. Let c and c' be the geodesics from v_2 to v_3 resp. v_4. Note that any unit speed geodesic c in $H^2\times H^2$ can be written in the form $c\left(t\right) = \left(c_1\left(\alpha t\right), c_2\left(\beta t\right)\right)$ with $\alpha^2+\beta^2=1$, with unit speed geodesics c_1,c_2 in H^2.

By [4], p.30, the corresponding Busemann functions satisfy $B_c = \alpha B_{c_1} + \beta B_{c_2}$.

We denote $\Psi_c^t\left(x,y\right)$ the flow corresponding to c.

Note that $\frac{d}{dt}\Psi_c^t\left(x,y\right) = -gradB_c\left(\Psi_c^t\left(x,y\right)\right) =$
$-\alpha gradB_{c_1}\left(\Psi_c^t\left(x,y\right)_1\right) - \beta gradB_{c_2}\left(\Psi_c^t\left(x,y\right)_2\right) = \alpha\frac{d}{dt}\Psi_{c_1}^t\left(x\right) + \beta\frac{d}{dt}\Psi_{c_2}^t\left(y\right)$,
implying

$$\Psi_{c*}^t = \begin{pmatrix} \Psi_{c_1*}^{\alpha t} & 0 \\ 0 & \Psi_{c_2*}^{\beta t} \end{pmatrix}.$$

In H^2, the Jacobi equation has a particularly simple form, giving that Z tangent to c satisfies $\Psi_{c*}^t Z = Z, \Psi_{c*}^t Z^\perp = e^{-t} Z^\perp$.

In this special case of products, we can choose an ON-basis which is easier to work with: choose bases $\left\{X_x, X_x^\perp, Y_y, Y_y^\perp\right\}_{(x,y)\in H^2\times H^2}$ of $T_{(x,y)}H^2\times H^2$ such that X tangent to c_1, Y tangent to c_2. Let θ_i be the angle between c_i and c_i' in H^2.

Then we get

$$\Psi_{c'}^s\Psi_c^tX = \Psi_{c'}^sX = \left(cos^2\theta_1 + sin^2\theta_1 e^{-\alpha s}\right)X + sin\theta_1 cos\theta_1\left(e^{-\alpha s}-1\right)X^\perp,$$

$$\Psi_{c'}^s\Psi_c^tX^\perp = \Psi_{c'}^s e^{-\alpha t}X^\perp = e^{-\alpha t}\left(sin\theta_1 cos\theta_1\left(e^{-\alpha' s}-1\right)X + \left(sin^2\theta_1 + cos^2\theta_1 e^{-\alpha' s}\right)X^\perp\right).$$

Analogously for $Y, Y^\perp$.
Calculating the matrix A with respect to this easier ON-reper, we get that A is a block matrix, one block having determinant

$$e^{-\alpha t}\left\{cos^2\theta sin^2\theta\left(1+e^{-2\alpha' s}\right) + e^{-\alpha' s}\left(cos^4\theta + sin^4\theta\right) - sin^2\theta cos^2\theta\left(e^{-\alpha' s}-1\right)^2\right\}$$

$$= e^{-\alpha t}e^{-\alpha' s}\left(cos^2\theta + sin^2\theta\right) = e^{-\alpha t}e^{-\alpha' s},$$

and the other block by analogous calculations having determinant $e^{-\beta t}e^{-\beta' s}$.
Hence,

$$det\,(A) = e^{-(\alpha+\beta)t}e^{-(\alpha'+\beta')s}.$$

We conclude $vol\left(\Delta^4\right) \leq \int_{\Delta^2}\int\int e^{-(\alpha+\beta)t}e^{-(\alpha'+\beta')s}dsdtdvol_{\Delta^2} = \frac{1}{\alpha+\beta}\frac{1}{\alpha'+\beta'}vol\left(\Delta^2\right).$

(G,X)-manifolds. Let V_n be the volume of a regular ideal simplex in hyperbolic n-space H^n. By the Haagerup-Munkholm theorem, any straight simplex in H^n has volume smaller than V_n. As explained on page 15, this implies that $\parallel M \parallel \geq \frac{Vol(M)}{V_n}$ holds for all closed hyperbolic n-manifolds. This inequality is in fact an equality by the Gromov-Thurston theorem, whose proof we will outline in subsection 4.1.4: $\parallel M \parallel = \frac{Vol(M)}{V_n}$.
A conjecture of Gromov, which seems still to be open, states that in any Riemannian manifold of sectional curvature ≤ -1 straight simplices should have volume smaller than V_n. This would again imply $\parallel M \parallel \geq \frac{Vol(M)}{V_n}$ for such manifolds.
According to [61], for any model geometry geometry (G,X) there is a constant C such that $\parallel M \parallel = CVol\,(M)$ holds for all manifolds M modelled on (G,X). In many cases this constant is zero. For geometries with G-invariant metrics of sectional curvature ≤ -1, such as rank-1 symmetric spaces of noncompact type after suitable rescaling, $\frac{1}{C}$ is bounded above by the maximal volume of ideal simplices, with equality only for H^n.

2.3 Lefschetz fibrations

This section is devoted to the proof of

Theorem 1 *Let $\pi : M \rightarrow B$ be a Lefschetz fibration with fiber F_g, vanishing cycles $v_1, \ldots, v_r \subset F_g$, and monodromy $\rho : \pi_1 B' \rightarrow Map_{g,*}$. Then the following two statements are equivalent:*
(i) The real Euler class $\overline{e}$ is bounded.
(ii) $\{\rho(g)(v_i) : g \in \pi_1 B', i = 1, \ldots, r\}$ is an incomplete curve system.

We will give all the basic definitions concerning Lefschetz fibrations and Euler class below. Here, to explain the notions used in theorem 1, in particular the notion of incomplete curve system, we recall that a Lefschetz fibration $\pi : M \rightarrow B$ with regular values $B' \subset B$ and an identification $F_g \simeq \pi^{-1}(b)$ for some $b \in B'$, is given by the monodromy $\rho : \pi_1 B' \rightarrow Map_{g,*}$ which sends a fixed system of loops $c_1, \ldots, c_r$ to Dehn twists at so-called vanishing cycles $v_1, \ldots, v_r$.
Moreover recall that $Map_{g,*}$ acts on $\pi_1(F, *)$, and hence on the Gromov-boundary $\partial_\infty \pi_1(\Sigma, *)$.

Definition 1 *Let Σ be a closed surface. A (possibly infinite) set of curves $\{c_i\}_{i \in I}$ on Σ is called an incomplete curve system, if there exist two points $p \neq q \in \partial_\infty \pi_1(\Sigma, *)$ which are fixed points of the Dehn twist at c_i, for all $i \in I$.*

The notation 'incomplete curved system' is motivated by the following observation: If $\{c_i\}_{i \in I}$ is a set of curves on Σ and we have a curve $c \subset \Sigma$ which is not null-homotopic and which does not intersect any c_i, then $\{c_i\}_{i \in I}$ is an incomplete curve system in the sense of definition 1.

Concerning the simplicial volume we get the following corollary:
Corollary: *Let $\pi : M \rightarrow B$ is a Lefschetz fibration with fiber F_g, vanishing cycles $v_1, \ldots, v_r \subset F_g$ and monodromy $\rho : \pi_1 B' \rightarrow Map_{g,*}$. Assume that base and have fiber have genus ≥ 2, and that $\{\rho(g)(v_i) : g \in \pi_1 B', i = 1, \ldots, r\}$is an incomplete curve system. Then $\parallel M \parallel > 0$.*

To put theorem 1 into context, we mention that Gromov proved, among other results, that (real) characteristic classes in $H^*\left(BG^\delta; R\right)$, for G^δ an algebraic subgroup of $GL(n, R)$ equipped with the discrete topology, are bounded. This generalized the classical Milnor-Sullivan theorem that Euler classes of flat affine bundles are bounded. A well-known theorem of Morita says that the Euler class e of a surface bundle is bounded. Theorem 1 generalizes this to give a precise condition under which the Euler class of a Lefschetz fibration is bounded. Morita's theorem was applied by Hoster and Kotschick to prove that surface bundles with base and fiber of genus ≥ 2 have positive simplicial volume (In particular, this provided the first examples of manifolds with positive simplicial volume but not admitting negatively curved metrics.) The proof of the corollary is a straightforward generalization of their argument.

We start with recalling the necessary definitions.

Lefschetz fibrations. A smooth map $\pi : M \rightarrow B$ from a smooth (closed, oriented, connected) 4-manifold M to a smooth (closed, oriented, oriented) 2-manifold B is said to be a Lefschetz fibration, if it is surjective and $d\pi$ is surjective except at finitely many critical points $\{p_1, \ldots, p_k\} =: C \subset M$, having the property that there are complex coordinate charts (agreeing with the orientations of M and B), U_i around p_i and V_i around $\pi(p_i)$, such that in these charts f is of the form $f(z_1, z_2) = z_1^2 + z_2^2$, see [24]. After a small homotopy the critical points are in distinct fibers, we assume this to hold for the rest of the paper.

The preimages of points in $B - \pi(C)$ are called regular fibers. It follows from the definition that all regular fibers are diffeomorphic and that the restriction $\pi' := \pi \mid_{M'}: M' \rightarrow B'$ to $M' := \pi^{-1}\pi(M - C)$ is a smooth fiber bundle over $B' := B - \pi(C)$. Let Σ_g be the regular fiber, a closed surface of genus g, and let, for an arbitrary point $* \in \Sigma_g$, be $Map_{g,*}$ the group of diffeomorphisms $f : \Sigma_g \rightarrow \Sigma_g$ with $f(*) = *$ modulo homotopies fixing $*$. It is well-known, cf. [47], that for any surface bundle one gets a monodromy $\rho : \pi_1 M' \rightarrow Map_{g,*}$, which factors over $\pi_1 B'$. It follows from the local structure of Lefschetz fibrations that, for a simple loop c_i surrounding $\pi(p_i)$ in B, $\rho(c_i)$ is the Dehn twist at some closed curve $v_i \subset \Sigma_g$, the 'vanishing cycle'.

Euler class of Lefschetz fibrations. For a topological space X, and a rank-2-vector bundle ξ over X, one has an associated Euler class $e(\xi) \in H^2(X; Z)$.

If $\pi : M \rightarrow B$ is a Lefschetz fibration, we may consider the tangent bundle of the fibers, TF, except at points of C, where this is not well defined. We get a rank-2-vector bundle L' over $M - C$ with euler class $e' := e(TF) \in H^2(M - C; Z)$. By a standard application of the Mayer-Vietoris sequence, there is an isomorphism $i^* : H^2(M; Z) \rightarrow H^2(M - C; Z)$ induced by the inclusion. Hence, $e := (i^*)^{-1} e' \in H^2(M; Z)$ is well-defined. In what follows we will denote e as the Euler class of the Lefschetz fibration $\pi : M \rightarrow B$. It is actually true (but we will not need it) that there exists a rank-2-vector bundle ξ over M such that $\xi \mid_{M-C} \simeq TF$. It is the pull-back of the universal complex line bundle, pulled back via the map $f : M \rightarrow CP^\infty$ corresponding to $e \in H^2(M; Z)$ under the bijection $H^2(M; Z) \simeq [M, CP^\infty]$.

S^1-bundles associated to surface bundles. For any surface bundle $\pi' : M' \rightarrow B'$ we may, after fixing a Riemannian metric, consider UTF, the unit tangent bundle of the fibers. This S^1-bundle is, according to [47], equivalent to the flat $Homeo^+(S^1)$-bundle with monodromy $\partial_\infty \rho$, where $\partial_\infty : Map_{g,*} \rightarrow Homeo^+(S^1)$ is constructed as follows. For $f \in Map_{g,*}$ let $f_* : \pi_1(\Sigma_g, *) \rightarrow \pi_1(\Sigma_g, *)$ be the

induced map of fundamental groups, and $\partial_\infty f_*$ the extension of f_* to the Gromov boundary $\partial_\infty \pi_1(\Sigma_g, *)$. It is well-known that $\partial_\infty f_*$ is a homeomorphism and that there is a canonical homeomorphism $\partial_\infty \pi_1(\Sigma_g, *) \simeq S^1$. (This works if $\pi_1 \Sigma_g$ is Gromov-hyperbolic, that is, for $g \geq 2$. If $\Sigma = T^2$, we homotope f to a map $g : T^2 \to T^2$ which has a *linear* lift $\tilde{g} : R^2 \to R^2$ and consider its action on the space of rays starting in 0, which is homeomorphic to S^1. It is easy to see that Morita's argument carries over. If $\Sigma = S^2$, there is nothing to do.)
One should be aware that the extension of UTF to $M - C$ is not flat: a loop surrounding a singular fiber is trivial in $\pi_1(M - C)$ but its monodromy is a Dehn twist, giving a nontrivial homeomorphism of S^1.
Bounded Cohomology. It will be important for us to distinguish between bounded cohomology with integer coefficients, $H_b^2(X; Z)$, and bounded cohomology with real coefficients, $H_b^2(X; R)$. To avoid too complicated notation, we use the following convention: for $\beta \in H^*(X; Z)$, we denote $\overline{\beta} \in H^*(X; R)$ its image under the canonical homomorphism $H^*(X; Z) \to H^*(X; R)$. Also, we will not distinguish between $H_b^*(X; R)$ and $H_b^*(\pi_1 X; R)$.

A cohomology class $\beta \in H^*(X; Z)$ is called bounded if it belongs to the image of the canonical homomorphism $H_b^*(X; R) \to H^*(X; R)$.
We will use the following two facts. (A) is proved in Bouarich's thesis, see [10]. (B) is proved in [21].

(A): If $1 \to N \to \Gamma \to G \to 1$ is an exact sequence of groups, then there is an exact sequence

$$0 \to H_b^2(G; R) \to H_b^2(\Gamma; R) \to H_b^2(N; R)^G \to H_b^3(G; R).$$

(B): For any group Γ, there is an exact sequence, natural with respect to group homomorphisms,

$$H^1(\Gamma; R/Z) \to H_b^2(\Gamma; Z) \to H_b^2(\Gamma; R).$$

Universal Euler class ([22]). There is a class $\chi \in H^2(Homeo^+ S^1; Z)$ such that, for any representation $\rho : \pi_1 M \to Homeo^+ S^1$ associated to a surface bundle with Euler class e, one has $\rho^* \chi = e$. By the explicit construction in [47] or [22], χ is bounded. By the main result of [22], representations $\rho : \Gamma \to Homeo^+(S^1)$ are determined up to semi-conjugacy by their Euler class in $H_b^2(\Gamma; Z)$. In particular, $\rho^* \chi = 0 \in H_b^2(\Gamma; Z)$ implies that ρ is semi-conjugate to the trivial representation.

It follows from boundedness of the universal Euler class that surface bundles have bounded Euler class. The general statement, theorem 1, will follow from the next lemmas which will be proved in sections 2.3.1 and 2.3.2.
Lemma 1: *Let $\pi : M \to B$ be a Lefschetz fibration with monodromy ρ and Euler*

class e. Let $V := ker\left(\pi_1 B' \to \pi_1 B\right)$ *and* e_V *the Euler class of the restriction* $\rho\mid_V$. *Then* $\overline{e}$ *is bounded if and only if* $e_V \in ker\left(H_b^2\left(V;Z\right) \to H_b^2\left(V;R\right)\right)$.
Lemma 2: *Let* Γ *be a group,* $\mathcal{A}$ *a (possibly infinite) set of generators of* Γ*, and* $\rho : \Gamma \to Map_{g,*}$ *a representation such that all elements of* $\mathcal{A}$ *are mapped to Dehn twists. The the following two statements are equivalent:*
(i) the Euler class of ρ *belongs to the kernel of the canonical homomorphism* $H_b^2\left(\Gamma;Z\right) \to H_b^2\left(\Gamma;R\right)$,
(ii) $\sharp \cap_{\gamma\in\Gamma} Fix\left(\partial_\infty \rho\left(\gamma\right)\right) \geq 2$.

Proof of Theorem 1: Assume that the Lefschetz fibration π has at least one critical point. Then B' is a punctured surface, $\pi_1 B'$ is a free group, and $V = ker\left(\pi_1 B' \to \pi_1 B\right)$ is a subgroup, with a set of generators given by

$$\mathcal{A} = \left\{ g c_1^{\pm 1} g^{-1}, \ldots, g c_r^{\pm 1} g^{-1} : g \in \pi_1 B' \right\},$$

where $c_1, \ldots, c_r$ represent simple loops around the punctures. (V is actually a free group, but we will not need this fact.)

The monodromy $\rho : \pi_1 B' \to Map_{g,*}$ maps c_i to Dehn twists at the vanishing cycles v_i. It follows that all elements of $\mathcal{A}$ are mapped to Dehn twists, since $\rho\left(g c_i g^{-1}\right) = \rho\left(g\right)\rho\left(c_i\right)\rho\left(g\right)^{-1}$ is the Dehn twist at $\rho\left(g\right)\left(v_i\right)$.

Let $\rho\mid_V$ be the restriction of the monodromy to V and e_V be the Euler class of $\rho\mid_V$. According to lemma 1, $\overline{e}$ is bounded if and only if $e_V \in ker\left(H_b^2\left(V;Z\right) \to H_b^2\left(V;R\right)\right)$. We have just checked that $\Gamma := V$ satisfies the assumptions of lemma 2. Hence $\overline{e}$ is bounded if and only if $\sharp\cap_{\gamma\in V} Fix\left(\partial_\infty\rho\left(\gamma\right)\right) \geq 2$. Since $\mathcal{A}$ generates V, we have $\cap_{\gamma\in V} Fix\left(\partial_\infty\rho\left(\gamma\right)\right) = \cap_{\gamma\in\mathcal{A}} Fix\left(\partial_\infty\rho\left(\gamma\right)\right)$, implying theorem 1. □

2.3.1 Criteria for bounded Euler class

In this section, we derive necessary and sufficient conditions for the Euler class of a Lefschetz fibration to be bounded.
Recall that, for a Lefschetz fibration $\pi : M \to B$ with critical points C, $B' := B - \pi\left(C\right)$ and $M' := \pi^{-1}\left(B'\right)$, we have a monodromy map $\rho : \pi_1 B' \to Homeo^+\left(S^1\right)$ with Euler class $e' \in H_b^2\left(\pi_1 B';Z\right)$. We will consider the subgroup $V := ker\left(\pi_1 B' \to \pi_1 B\right)$ and will denote $e_V \in H_b^2\left(V;Z\right)$ the Euler class of $\rho\mid_V$.

Lemma 1 *Let* $\pi : M \to B$ *be a Lefschetz fibration with Euler class e. Then* $\overline{e}$ *is bounded if and only if* $e_V \in ker\left(H_b^2\left(V;Z\right) \to H_b^2\left(V;R\right)\right)$.

Proof: From boundedness of $i^*\overline{e}$ and the commutative diagram

$$\begin{array}{ccc} H_b^2\left(M;R\right) & \xrightarrow{i^*} & H_b^2\left(M';R\right) \\ \downarrow & & \downarrow \\ H^2\left(M;R\right) & \xrightarrow{i^*} & H^2\left(M';R\right) \end{array}$$

we see that $\overline{e}$ is bounded if and only if $\overline{e}_b' \in H_b^2\left(M';R\right)$ is in the image of $i^* : H_b^2\left(M;R\right) \rightarrow H_b^2\left(M';R\right)$.
We consider the exact sequence $1 \rightarrow N \rightarrow \pi_1 M' \rightarrow \pi_1 M \rightarrow 1$, with $N := ker i_*$. Bouarich's exact sequence (A) implies that $\overline{e}_b' \in im\left(i^*\right)$ if and only if the restriction of e_b' to N is trivial in the *bounded* cohomology of N.

We have a commutative diagram

$$\begin{array}{ccccccccc} & & 1 & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & ker & \longrightarrow & N & \longrightarrow & V & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & \longrightarrow & \pi_1 F & \longrightarrow & \pi_1 M' & \longrightarrow & \pi_1 B' & \longrightarrow & 1 \end{array}$$

with all rows and columns being exact sequences.
A few remarks are in order about well-definedness of the involved homomorphisms.The second line is the long exact homotopy sequences of the surface bundle $M' \rightarrow B'$. Inclusion maps $ker\left(N \rightarrow V\right)$ to $ker\left(\pi_1 M' \rightarrow \pi_1 B'\right)$, hence $ker\left(N \rightarrow V\right) \subset \pi_1 F$.Clearly, the projection maps N to $ker\left(\pi_1 B' \rightarrow \pi_1 B\right) = V$. Surjectivity of this homomorphism does not follow from the commutative diagram, but is easy to see geometrically. Indeed, each simple loop c_i surrounding a puncture can be lifted to an element $\hat{c_i} \in N$, just working in coordinate charts. For $g \in \pi_1 B$, we fix some lift $\hat{g} \in \pi_1 M$. Then $\hat{g}\hat{c_i}\hat{g}^{-1}$ is an element of N, projecting to gc_ig^{-1}. Since V is generated by elements of the form gc_ig^{-1}, we have surjectivity.

It is clear from the construction in [47] that the restriction of the representation $\pi_1 M' \rightarrow Homeo^+\left(S^1\right)$ to $\pi_1 F$ is trivial. In particular, the restriction of $\overline{e}_b'$ to

$ker\,(N \to V)$ is trivial. Applying Bouarich's exact sequence (A) to the first row, we get an exact sequence

$$0 \to H_b^2\,(V;R) \to H_b^2\,(N;R) \to H_b^2\,(ker;R)$$

and we conclude that $\overline{e}_b' \mid_N$ has a preimage $\overline{e}_b'' \in H_b^2\,(V;R)$ and that $\overline{e}_b' \mid_N = 0$ if and only if $\overline{e}_b'' = 0 \in H_b^2\,(V;R)$. □

2.3.2 Mapping class groups generated by Dehn twists

Lemma 2 *: Let Γ be a group, $\mathcal{A}$ a (possibly infinite) set of generators of Γ, and $\rho : \Gamma \to Map_{g,*}$ a representation such that all elements of $\mathcal{A}$ are mapped to Dehn twists. The the following two statements are equivalent:*
(i) the Euler class of ρ belongs to the kernel of the canonical homomorphism $H_b^2\,(\Gamma;Z) \to H_b^2\,(\Gamma;R)$,
(ii) $\sharp \cap_{\gamma\in\Gamma} Fix\,(\partial_\infty \rho\,(\gamma)) \geq 2$.

Proof: For $\gamma \in \mathcal{A}$ let $j_\gamma : Z \to \Gamma$ be the homomorphism such that $j_\gamma\,(1) = \gamma$. By (B) (section 1), we have a commutative diagram

$$\begin{array}{ccccc}
\Pi_{\gamma\in\mathcal{A}} \mathrm{H}^1\,(Z;R/Z) & \xrightarrow{\simeq} & \Pi_{\gamma\in\mathcal{A}} \mathrm{H}_b^2\,(Z;Z) & \longrightarrow & \Pi_{\gamma\in\mathcal{A}} \mathrm{H}_b^2\,(Z;R) \\
\Pi\, j_\gamma^* \big\uparrow & & \Pi\, j_\gamma^* \big\uparrow & & \Pi\, j_\gamma^* \big\uparrow \\
\mathrm{H}^1\,(\Gamma;R/Z) & \longrightarrow & \mathrm{H}_b^2\,(\Gamma;Z) & \longrightarrow & \mathrm{H}_b^2\,(\Gamma;R)\,,
\end{array}$$

where the isomorphism

$$H_b^2\,(Z;Z) \simeq R/Z \simeq H^1\,(Z;R/Z)$$

follows from prop. 3.1. in [22].

Let $e \in H_b^2\,(\Gamma;Z)$ be the Euler class of ρ. Its image $j_\gamma^* e \in H_b^2\,(Z;Z)$ is the Euler class of of the representation $\rho j_\gamma : Z \to Map_{g,*}$ mapping 1 to the Dehn twist $\rho\,(\gamma)$. By theorem A3 in [22], the isomorphism $H_b^2\,(Z;Z) \simeq R/Z$ maps $j_\gamma^* e$ to the rotation number of $\partial_\infty \rho\,(\gamma)$. The rotation number of a Dehn twist is zero, since it has fixed points on S^1, hence $j_\gamma^* e = 0$ for all $\gamma \in \mathcal{A}$.

Now assume that e belongs to the kernel of the canonical homomorphism $H_b^2\,(\Gamma;Z) \to H_b^2\,(\Gamma;R)$. It follows that $e \in H_b^2\,(\Gamma;Z)$ has a preimage

$$E \in H^1\,(\Gamma;R/Z)\,.$$

Since $\mathcal{A}$ generates Γ, the homomorphism $\Pi\, j_\gamma^* : H^1\,(\Gamma;R/Z) \to \Pi_{\gamma\in\mathcal{A}} \mathrm{H}^1\,(Z;R/Z)$ is injective. With the commutativity of the leftmost square and $\Pi\, j_\gamma^* e = 0$, this implies $E = 0$. Therefore, also $e = 0$. That means, we have shown that under the assumptions of lemma 2 the equivalence $e \in ker\left(H_b^2(\Gamma; Z) \rightarrow H_b^2(\Gamma; R)\right) \Leftrightarrow e = 0$ holds.

According to [22], $e = 0$ implies that ρ is semi-conjugate to the trivial representation, that is, there is a (not necessarily continuous) map $h : S^1 \rightarrow S^1$, lifting to an increasing degree-1 map $\overline{h} : R \rightarrow R$, such that

$$\rho(\gamma) h(x) = h(x)$$

holds for all $\gamma \in \Gamma$ and all $x \in S^1$. In particular, for any (!) $\gamma \in \Gamma$ we get that the image of h consists only of fixed points of $\partial_\infty \rho(\gamma)$.

Since h can not be constant, this implies that

$$\sharp \cap_{\gamma \in \Gamma} Fix\left(\partial_\infty \rho(\gamma)\right) \geq 2.$$

On the other hand, if $p \neq q$ are fixed points of $\partial_\infty \rho(\gamma)$ for all $\gamma \in \Gamma$, we denote by I_1 and I_2 the connected components of $S^1 - \{p, q\}$ and define $h : S^1 \rightarrow S^1$ by $h(p) = p, h(I_1) \equiv q, h(q) = q, h(I_2) \equiv p$. h semi-conjugates ρ to id, hence $e = 0$. $\square$

Fixed points of Dehn twists

Here we want to prove the observation after definition 1, to get a more workable criterion for bounded Euler class.

Let Σ be a closed surface, $* \in \Sigma$, and $f : \Sigma \rightarrow \Sigma$ a homeomorphism with $f(*) = *$. We denote $f_* : \pi_1(\Sigma, *) \rightarrow \pi_1(\Sigma, *)$ the induced homomorphism, and $\partial_\infty f_* : S^1 \rightarrow S^1$ the homeomorphism of the Gromov-boundary $\partial_\infty \pi_1(\Sigma, *) \simeq S^1$, as in chapter 1. Let $Fix(\partial_\infty f_*) = \{p \in S^1 : \partial_\infty f_*(p) = p\}$ be the set of fixed points on the Gromov-boundary.

Observation: *Let Σ e a closed, oriented, hyperbolic surface and $\{c_i\}_{i \in \mathcal{A}}$ a (possibly infinite) set of simple closed curves on Σ. Let t_i be the Dehn twist at c_i. Assume there exists a (not necessarily closed) nonconstant geodesic c on Σ which is not null-homotopic and which does not intersect any c_i. Then $\sharp \cap_{i \in \mathcal{A}} Fix(\partial_\infty t_i) \geq 2$.*

Proof: Fix $\tilde{*} \in \tilde{\Sigma}$ projecting to $* \in \Sigma$. For $f : \Sigma \rightarrow \Sigma$ with $f(*) = *$, the (unique) lift $\tilde{f}$ of f to the universal cover $\tilde{\Sigma} \simeq H^2$ (with $\tilde{f}(\tilde{*}) = \tilde{*}$) is a quasi-isometry of H^2 and induces a homeomorphism $\partial_\infty \tilde{f}$ of $\partial_\infty H^2 \simeq S^1$ which agrees with $\partial_\infty f_*$, as is well-known.

Assume $* \in c$. There is a unique geodesic $\tilde{c} \subset H^2$ projecting to c and passing through $\tilde{*}$. Let p and q be the ideal boundary points of $\tilde{c}$. Since c does not intersect c_i, we have $t_i \mid_c = id$, implying that $\tilde{t_i} \mid_{\tilde{c}} = id$ and therefore $\partial_\infty t_i(p) = p, \partial_\infty t_i(q) = q$ for all $i \in \mathcal{A}$.

2.3.3 Conclusions

It remains an open question which Lefschetz fibrations have positive simplicial volume. A sufficient condition is the following:

Lemma 3 *Let $\pi : M \rightarrow B$ be a Lefschetz fibration with regular fiber F such that*
- genus(B) ≥ 2, genus(F) ≥ 2, and
- the real Euler class $e \in H^2(M;R)$ is bounded,
Then $\parallel M \parallel$ is positive.

Proof: The proof is a minor generalisation of the argument in [33].
We work with de Rham-cohomology. Define $\pi_* : H^2(M) \rightarrow H^0(B)$ by $\pi_* = D_B^{-1}\pi_* D_M$, where D_B resp. D_M are the Poincare duality maps. One has $< \pi^*\alpha \cup \beta, c >=< \alpha \cup \pi_*\beta, \pi_* c >$ for any $\alpha, \beta \in H^*, c \in H_*$. Like in [33] one gets, with $\omega_B \in H^2(B;R)$ satisfying $\int_B \omega_B = 1$,
$\mid e \cup \pi^*\omega_B([M]) \mid \leq \parallel e \parallel \parallel \omega_B \parallel \parallel M \parallel = \parallel e \parallel \frac{1}{\parallel B \parallel} \parallel M \parallel$.
Note that $e \mid_{\pi^{-1}\pi(C)}$ is the Euler class of the tangent bundle to the regular fibers, hence $e([F]) = \chi(F)$ is the Euler characteristic of the regular fiber.
$e \cup \pi^*\omega_B$ is a multiple of the volume form. Therefore its value on $[M]$ doesn't depend on the zero-volume set $\pi^{-1}\pi(C)$. Hence,
$\mid e \cup \pi^*\omega_B([M]) \mid = \mid \int_{M-\pi^{-1}\pi(C)} e \cup \pi^*\omega_B \mid = \mid \int_{B-\pi(C)} \pi_* e \cup \omega_B \mid$.
Using, for $b \in B$, $< \pi_* e, [b] >=< \pi_* e, \pi_*[F] =< e, [F] >= \chi(F)$,
we get $\mid e \cup \pi^*\omega_B([M]) \mid = \mid \chi(F) \int_B \omega_B \mid$,
and we conclude $\parallel M \parallel \geq \mid \chi(F) \mid \parallel B \parallel \frac{1}{\parallel e \parallel}$. □

Corollary 1 *Let $\pi : M \rightarrow B$ is a Lefschetz fibration with fiber F_g, vanishing cycles $v_1, \ldots, v_r \subset F_g$, regular values $B' \subset B$ and monodromy $\rho : \pi_1 B' \rightarrow Map_{g,*}$. Assume that base and have fiber have genus ≥ 2, and that there exists a geodesic c on F_g, such that $c \cap \rho(\gamma)(v_i) = \emptyset$ for all $\gamma \in \pi_1 B'$ and all vanishing cycles v_i. Then $\parallel M \parallel > 0$.*

Proof: It follows from the proof of theorem 1 that $\overline{e}$, the image of e in $H^2(M;R)$, is bounded. By genus(B) ≥ 2, ω_B is bounded. Hence, $e \cup \pi^*\omega_B$ is bounded, and we conclude with lemma 3. □

Chapter 3

Bounded cohomology and amenable glueings

This chapter is devoted to the study of the behaviour of simplicial volume with respect to cut and paste. That means, we are given an (n-1)-submanifold $F \subset M$ with $\partial F \subset \partial M$, and we wish to compare $\parallel M_F, \partial M_F \parallel$ to $\parallel M, \partial M \parallel$. Here, M_F denotes the manifold obtained by cutting off F, that is $M_F := \overline{M - N(F)}$ for a regular neighborhood $N(F)$ of F.

$\parallel M, \partial M \parallel$ may be strictly smaller than $\parallel M_F, \partial M_F \parallel$, as there may be fundamental cycles of M which are not the images of fundamental cycles of M_F. For example, we showed in chapter 4 and 5 that $\parallel M_F, \partial M_F \parallel > \parallel M, \partial M \parallel$ if $int(M)$ is a hyperbolic n-manifold of finite volume, $n \geq 3$, and F is a closed geodesic hypersurface. On the other hand, somewhat counter-intuitively, $\parallel M, \partial M \parallel$ may be strictly larger than $\parallel M_F, \partial M_F \parallel$, as fundamental cycles for M_F need not fit together at the copies of F.

For theorem 2, we consider the case that F is amenable, and prove:
Theorem 2: *Let M_1, M_2 be two compact n-manifolds, $A_1 \subset \partial M_1$ resp.$A_2 \subset \partial M_2$ (n-1)-dimensional submanifolds, $f : A_1 \rightarrow A_2$ a homeomorphism, $M = M_1 \cup_f M_2$ the glued manifold. If $\pi_1 A_1$, $\pi_1 A_2$ are amenable and f_* restricts to an isomorphism $f_* : ker(\pi_1 A_1 \rightarrow \pi_1 M_1) \rightarrow ker(\pi_1 A_2 \rightarrow \pi_1 M_2)$,*
then $\parallel M, \partial M \parallel \geq \parallel M_1, \partial M_1 \parallel + \parallel M_2, \partial M_2 \parallel$.
If moreover A_1, A_2 are connected components of ∂M_1 resp. ∂M_2,
then $\parallel M, \partial M \parallel = \parallel M_1, \partial M_1 \parallel + \parallel M_2, \partial M_2 \parallel$.
We prove analogous facts if A_1, A_2 are in the boundary of the same manifold M_1. Applied to 3-manifolds, theorem 2 means that simplicial volume is additive with respect to glueing along incompressible tori and superadditive with respect to glueing along incompressible annuli. In the special case that the boundary of the 3-manifolds consists of tori, Soma proved in [57] that simplicial volume is additive

with respect to glueing incompressible tori or annuli. We think that our proof, apart from being a generalisation to manifolds with arbitrary boundary, should be of interest because the proof in [57] heavily relies on theorem 6.5.5. from Thurston's lecture notes, of which no published proof is available so far.

Later, in chapter 5, we consider the special case of doubling a manifold, that is of glueing two (differently oriented) copies of M by the identity of ∂M. Here, two fundamental cycles of M, corresponding to opposite orientations, fit together at ∂M to give a fundamental cycle of DM, hence $\parallel DM \parallel \leq 2 \parallel M, \partial M \parallel$ trivially holds. We give precise conditions for 3-manifolds to satisfy the strict inequality.
Theorem 5: *Let M be a manifold of dimension $n \leq 3$. Then*
$\parallel DM \parallel < 2 \parallel M, \partial M \parallel$ *if and only if* $\parallel \partial M \parallel > 0$.
Theorem 5 will, using geometrization of 3-manifolds, follow from theorem 4 together with application of theorem 2 to 3-manifolds with boundary.

The chapter is organized as follows. Section 3.1. gives the necessary facts about multicomplexes. We usually refer to [26] where it contains complete proofs and just fix notations in a way useful for later chapters. Section 3.2. discusses treelike multicomplexes. The proved results are the same which are needed in [26] to prove results about glueing manifolds without boundary, our contribution consisting in writing complete proofs for the ideas indicated in section 3.5. of [26]. Theorem 2 is finally proved in section 3.3.

It might be helpful for the reader that we give some non-rigorous motivation for the proof of theorem 2. Let us consider a toy example. We glue two manifolds M_1 and M_2 to get $M_1 \vee M_2$. (This is not a manifold but one may define a fundamental class in the obvious way.) We want to show $\parallel M_1 \vee M_2 \parallel \geq \parallel M_1 \parallel + \parallel M_2 \parallel$, thus we have to find an efficient way to map representatives of $[M_1 \vee M_2]$ to a sum of representatives of $[M_1]$ and of $[M_2]$. That means, we look for a chain map r, leftinverse to the inclusions, which maps simplices in $M_1 \vee M_2$ to simplices either in M_1 or in M_2.

The universal cover of $M_1 \vee M_2$ is a tree-like complex made from copies of $\tilde{M}_1$ and $\tilde{M}_2$. In a tree, any nondegenerate tripel of vertices has a unique central point, belonging to all three geodesics between the vertices. More generally, in a tree-like complex, one might try to construct a central simplex associated to any nondegenerate tupel of at least three points. For example, if M_1 and M_2 admitted hyperbolic metrics, one would have unique geodesics between two vertices in $\tilde{M}_1$ or $\tilde{M}_2$, hence also in the tree-like complex, and one can actually show that, for a nondegenerate tupel of vertices, the associated set of geodesics intersects the full 1-skeleton of exactly one top-dimensional simplex. This 'central' simplex belongs to a copy of $\tilde{M_1}$ or $\tilde{M_2}$. It is then easy to define r.

There is clearly no such construction for arbitrary manifolds. However, we show in section 3.2. that this construction can be done if M_1 and M_2 are aspherical, minimally complete multicomplexes. Such multicomplexes have in fact several features in common with hyperbolic spaces. The theorem that we actually prove in section 3.2. is a generalization of the above. We consider not only multicomplexes glued at one vertex, but glueing multicomplexes along an arbitrary submulticomplex with the additional condition that a suitable group G acts with certain transitivity properties on the submulticomplex along which the glueing is performed, and we do the above construction not for simplices but for G-orbits of simplices. (It might be tempting to consider the quotients with respect to the G-action to reduce the glueing to a generalized wedge. However, this would raise technical problems related to the fact that these quotients are not multicomplexes.)

It should be noted that in the case of glueing two closed manifolds, [26],3.5. avoids the use of multicomplexes by using the classifying spaces of the fundamental groups, where the corresponding constructions are easier. This construction generalizes to manifolds with boundary only if one were to consider manifolds with exactly one boundary component.

Technically, the line of argument is as follows. To any space X, one associates an aspherical, minimally complete multicomplex $K(X)$. Its simplicial bounded cohomology $H_b^*(K(X))$ is isometrically isomorphic to the singular bounded cohomology $H_b^*(X)$. Bounded cohomology is a device that admits to dualize problems about the simplicial volume. We show in 3.3.1., using the isometric isomorphisms between bounded cohomology groups, that the glueing problem for manifolds M_1, M_2 can be translated into an analogous problem for the aspherical, minimally complete multicomplexes $K(M_1), K(M_2)$.

If M is a manifold obtained by glueing along A, one lets act a large amenable group $\Pi_M(A)$, which consists of all homotopy classes in M of pathes in A. This group action satisfies the transitivity properties needed to apply the results of section 3.2., i.e. to solve the glueing problem for the associated multicomplexes, and hence also for the original manifolds. In the proofs we will always have to distinguish two cases: the "amalgamated" case, where *two* manifolds are glued along parts of their boundary, and the "HNN"-case, where the glueing is performed by identifying two boundary subsets of *one* manifold.
Convention: For simplicity, we assume all manifolds to be oriented.

3.1 Multicomplexes

3.1.1 Definitions

Definition 2 *: A multicomplex K consists of the following data:*
- Simplices*: a set V and, for any finite ordered subset $F = \{v_0, \ldots, v_n\} \subset V$ with $| F | \geq 2$, a (possibly empty) set I_F, such that for any permutation $\pi : F_1 \to F_2$ there is a bijection $I_\pi : I_{F_1} \to I_{F_2}$,*
- Face maps*: for any finite ordered set $F = \{v_0, \ldots, v_n\} \subset V$ a family of maps $\left\{ d_j : I_F \to I_{F-\{v_j\}} \right\}_{0 \leq j \leq n}$.*

The elements of V are the 0-simplices or vertices of K. The pairs $\sigma = (F, i)$ with $| F |= n$ and $i \in I_F$ are the n-simplices of K. The j-th face of an n-simplex (F, i) is given by $\partial_j (F, i) := (F - \{v_j\}, d_j (F))$.
Let K_j be the union of j-simplices and K^n the n-skeleton of K, that is the union $\cup_{0 \leq j \leq n} K_j$.

The **geometric realization** $| K |$ of K is defined as follows:
$| K |$ is the set of pairs $\{(\lambda, i) : i \in I_{F_\lambda}\}$, where $\lambda : V \to [0,1]$ are maps such that $\sum_{v \in V} \lambda (v) = 1$ and $F_\lambda = \{v \in V : \lambda (v) > 0\}$ is finite.
The set of all λ with a given F_λ and a given $i \in I_{F_\lambda}$ is canonically identified with a standard simplex and herits a topology via this identification. We consider the topology on $| K |$ defined such that a set is closed if its intersection with each simplex is closed.

Definition 3 *We call a multicomplex minimally complete, or m.c.m. for short, if the following holds: whenever $\sigma : \Delta \to | K |$ is a singular simplex, such that $\partial \sigma$ is a simplex of K, then σ is homotopic relative $\partial \Delta$ to a* unique *simplex in K.*

If σ is an n-simplex, its n-1-skeleton is the set $\{\partial_0 \sigma, \ldots, \partial_n \sigma\}$. By recursion, we define that the n-k-skeleton σ_{n-k} of an n-simplex σ is the union of the n-k-skeleta of the simplices belonging to the n-k+1-skeleton of σ.

Definition 4 *We call a multicomplex K aspherical if all simplices $\sigma \neq \tau$ in K satisfy $\sigma_1 \neq \tau_1$.*

Orientation: Let $\pi : F_1 \to F_2$ be a permutation of finite sets and $i \in I_{F_1}$. We say that (F_1, i) and $(F_2, I_\pi (i))$ have the same orientation if π is even, and that they have different orientation if π is odd.
A submulticomplex L of a multicomplex K consists of a subset of the set of simplices closed under face maps. (K, L) is a pair of multicomplexes if K is a multicomplex and L is a submulticomplex of K. A group G acts simplicially on a pair of multicomplexes (K, L) if it acts on the set of simplices of K, mapping simplices in L to simplices in L, such that the action commutes with all face maps.

3.1.2 Bounded Cohomology

For a multicomplex K, let $F_j(K)$ be the R-vector space with basis the set of j-simplices of K. Let $O_j(K)$ be the subspace generated by the set $\{\sigma - sign(\pi)\,\pi(\sigma)\}$ where σ runs over all j-simplices, π runs over all permutations, and $\pi(F,i) := (\pi(F), I_\pi(i))$. Define the j-th chain group $C_j(K) = F_j(K)/O_j(K)$.
Let $C^j(K) = Hom_R(C_j(K), R)$.
If (K,L) is a pair of multicomplexes, we get an inclusion $C_j(L) \to C_j(K)$ and we define $C^j(K,L) := \{\omega \in C^j(K) : \omega(c) = 0 \text{ for all } c \in C_j(L)\}$ with the norm $\parallel \omega \parallel_\infty := sup\{\omega(\sigma) : \sigma \text{ j-simplex}\}$ on $C^j(K,L)$. Let $C_b^j(K,L) := \{\omega \in C^j(K,L) : \parallel \omega \parallel_\infty < \infty\}$.
The coboundary operator preserves $C_b^*(K,L)$, hence induces maps $\delta_b^i : C_b^i(K,L) \to C_b^{i+1}(X,Y)$. Define the bounded cohomology of (K,L) by $H_b^i(K,L) := ker\delta_b^i / im\delta_b^{i-1}$.
$\parallel . \parallel_\infty$ induces a pseudo-norm $\parallel . \parallel$ on $H_b^*(K,L)$.
For the bounded cohomology of topological spaces, we refer to [35]. When dealing with a pair of multicomplexes (K,L), we will distinguish between $H_b^*(K,L)$ and $H_b^*(\mid K \mid, \mid L \mid)$. Since any bounded singular cochain is in particular a bounded simplicial cochain, we have an inclusion $h : C_b^*(\mid K \mid) \to C_b^*(K)$.

Proposition 1 *(Isometry lemma, [26], S.43): If K is a connected minimally complete multicomplex with infinitely many vertices, then* $h^* : H_b^*(\mid K \mid) \to H_b^*(K)$ *is an isometric isomorphism.*

For an n-dimensional compact, connected, orientable manifold let β_M be the unique class in $H^n(M, \partial M)$ such that $< \beta_M, [M, \partial M] >= 1$.
By duality (section 2.1.), $\parallel M, \partial M \parallel = \frac{1}{\parallel \beta_M \parallel}$. In particular, $\parallel M, \partial M \parallel = 0$ if and only if $im\left(H_b^n(M, \partial M) \to H^n(M, \partial M)\right) = 0$.

It can be shown ([26],[35]) that the bounded cohomology and its pseudonorm depend only on the fundamental group. This works also for pairs (X,Y), if $\pi_1 Y \to \pi_1 X$ is injective. In particular, one has:

Lemma 4 *: If M and N are compact manifolds with incompressible boundary of the same dimension and there exists a map* $f : (M, \partial M) \to (N, \partial N)$ *inducing an isomorphism of pairs of fundamental groups, then* $\parallel M, \partial M \parallel = deg(f) \parallel N, \partial N \parallel$.

3.1.3 Aspherical multicomplexes

Proposition 2 *([26], p.46): Let X be a topological space. Then there is an aspherical m.c.m.* $K(X)$ *such that:*
(i) X is the set of vertices of $K(X)$*. Hence, we have an inclusion* $i : X \to K(X)$.
(ii) There is an isometric isomorphism $I : H_b^*(K(X)) \to H_b^*(X)$.

Warning: It will be convenient for us to use notation different from Gromov's, since we will make use of a certain functoriality of $K(X)$. So one should be

aware that our $K(X)$ corresponds to K/Γ_1 in Gromov's notation, as well as that our $\tilde{K}(X)$ will correspond to Gromov's K or $K(X)$. According to [26], $\mid K(X) \mid$ is actually an aspherical topological space. We will only need that $K(X)$ is aspherical in the sense of definition 3, what will follow from the geometric description of $K(X)$.
Because it will be of importance in the proof of lemma 11 and 12, we mention that I is constructed as the following composition:

$$H_b^*(K(X)) \xrightarrow{p^*} H_b^*\left(\tilde{K}(X)\right) \xrightarrow{h^{*-1}} H_b^*\left(\mid \tilde{K}(X) \mid\right) \xrightarrow{j^*} H_b^*(X).$$

Here, h^* is the isomorphism from proposition 1, p and j are described in the geometrical description below.
A proof of proposition 2 is given in [26]. Because it will be crucial for the proof of theorem 2, we recall the geometrical description of $K(X)$ as it can be read off the constructions in [26].

Geometrical description of $\tilde{K}(X)$: For a topological space X, $\tilde{K}(X)$ is the multicomplex defined as follows. Its 0-skeleton is $\{x : x \in X\}$. Its 1-skeleton is $\left\{(\{x,y\}, i) : x \neq y \in X, i \in I_{\{x,y\}}\right\}$ where $I_{\{x,y\}}$ is the set of homotopy classes relative $\{0,1\}$ of maps from $[0,1]$ to X mapping 0 to x and 1 to y. Having defined the n-1-skeleton of $\tilde{K}(X)$, the n-simplices of $\tilde{K}(X)$ are the pairs $(\{x_0,\ldots,x_n\}, i)$ with $x_0,\ldots,x_n \in X$ and i a homotopy class relative $\partial\Delta^n$ of mappings from the standard simplex Δ^n to X, taking the i-th vertex of Δ^n into x_i for $i = 0,\ldots,n$ and the n-1-skeleton of Δ^n into the n-1-skeleton of $\tilde{K}(X)$.
In particular, we have a canonical inclusion $j : X \to \tilde{K}(X)$, identifying X with the 0-skeleton of $\tilde{K}(X)$.

Geometrical description of $K(X)$: The multicomplex $K(X)$ is obtained from $\tilde{K}(X)$ by identifying, via simplicial maps, all n-simplices with a common n-1-skeleton, for all $n \geq 2$, succesively in order of increasing dimension.
In particular, we have a canonical projection $p : \tilde{K}(X) \to K(X)$.

Proposition 3 *: Let $Y \subset X$ be a subspace, such that $\pi_1(Y,y) \to \pi_1(X,y)$ is injective for all $y \in Y$. Then $K(Y)$ is a submulticomplex of $K(X)$.*
$I : H_b^*(K(X), K(Y)) \to H_b^*(X,Y)$ is an isometric isomorphism.

Proof: If two distinct 1-simplices in Y mapped to the same 1-simplex in X, the corresponding pathes in Y would be in $ker(\pi_1 Y \to \pi_1 X)$. By asphericity, simplices are determined by their 1-skeleton and the first claim follows. From the five lemma, I is an isomorphism. Therefore, it must be an isometry, since I and I^{-1} are composed by maps of norm ≤ 1. □

3.1.4 Amenable Groups and Averaging

Definition 5 *: For a group* Γ *let* $B(\Gamma)$ *be the space of bounded real-valued functions on* Γ*.* Γ *is called amenable if there is a* Γ*-invariant linear functional* $Av : B(\Gamma) \to R$ *such that* $inf(f) \leq Av(f) \leq sup(f)$ *holds for all* $f \in B(\Gamma)$*.*

If a group G acts simplicially on a pair of multicomplexes (K, L), we denote
$C_b^*\left(K^G, L^G\right) := \{c \in C_b^*(K, L) : gc = c \text{ for all } g \in G\}$,
$\delta_i^G := \delta \mid_{C_b^i(K^G, L^G)}$ and
$H_b^*\left(K^G, L^G\right) := ker\delta_i^G / im\delta_{i-1}^G$.

Lemma 5 *: (i) If an amenable group* Γ *acts on a pair of multicomplexes* (K, L)*, and* $p : C_b^*\left(K^\Gamma, L^\Gamma\right) \to C_b^*(K, L)$ *is the inclusion, then there is a homomorphism* $Av : H_b^*(K, L) \to H_b^*\left(K^\Gamma, L^\Gamma\right)$ *such that* $Av \circ p^* = id$ *and* $\parallel Av \parallel = 1$*.*
(ii) If, moreover, all elements of Γ *are homotopic to the identity in the category of continuous maps of pairs of spaces* $(\mid K \mid, \mid L \mid)$*, then* $p^* \circ Av = id$*.*

Proof: : The proof of (i) works, for $L = \emptyset$, the same way as for singular bounded cohomology in [26],p.39. In the relative setting, if $L \neq \emptyset$, Av still is an isometry because of $\parallel Av \parallel \leq 1, \parallel p^* \parallel \leq 1$ and $Avp^* = id$, and it is an isomorphism as a consequence of the five lemma. Part (ii) follows from the homotopy lemma, [26],p.42 and is implicit in [26],p.46,cor.D. □

3.1.5 Group actions on multicomplexes

In the following, (X, Y) will be a pair of topological spaces. For a path $\gamma : [0, 1] \to X$, we denote by $[\gamma]$ its homotopy class *in* X relative $\{0, 1\}$.

Definition 6 *: Define* $\Pi_X(Y) :=$
$\{([\gamma_1], \ldots, [\gamma_n]) : n \in N, \gamma_1, \ldots, \gamma_n : [0, 1] \to Y, \{\gamma_1(0), \ldots, \gamma_n(0)\} = \{\gamma_1(1), \ldots, \gamma_n(1)\}\}$.

$\Pi_X(Y)$ is a group with respect to the following product:
$\{[\gamma_1], \ldots, [\gamma_m]\}\{[\gamma_1'], \ldots, [\gamma_n']\} := \{[\gamma_1 * \gamma_1'], \ldots, [\gamma_i * \gamma_i'], [\gamma_{i+1}], \ldots, [\gamma_m], [\gamma_1'], \ldots, [\gamma_n']\}$,
where $*$ denotes the concatenation of pathes and $i \geq 0$ is chosen such that we have: $\gamma_j(1) = \gamma_j'(0)$ for $1 \leq j \leq i$ and $\gamma_j(1) \neq \gamma_k'(0)$ for $j \geq i+1, k \geq i+1$. (Such an i exists for a unique reindexing of the elements in the unordered sets $\{[\gamma_1], \ldots, [\gamma_m]\}$ and $\{[\gamma_1'], \ldots, [\gamma_n']\}$.)

Action of $\Pi_X(Y)$ on $K(X)$:
To define an action of $\Pi_X(Y)$ on the 0-skeleton of $K(X)$, we recall that $i : X \to K(X)$ maps X bijectively to $K(X)_0$. For $g = \{[\gamma_1], \ldots, [\gamma_n]\} \in \Pi_X(Y)$,

we define $gi(\gamma_1(0)) = i(\gamma_1(1)), \ldots, gi(\gamma_n(0)) = i(\gamma_n(1))$ and $gi(v) = i(v)$ if $v \notin \{\gamma_1(0), \ldots, \gamma_n(0)\}$.
As a next step, we extend this to an action on the 1-skeleton of $K(X)$. Recall that 1-simplices $[\sigma]$ in $K(X)$ correspond to homotopy classes $[\sigma]$ of pathes $\sigma : [0,1] \to X$. Let $g = \{[\gamma_1], \ldots, [\gamma_n]\} \in \Pi_X(Y)$. For a 1-simplex $[\sigma]$ define $g[\sigma] = [\sigma]$ if $\sigma(0) \notin \{\gamma_1(0), \ldots, \gamma_n(0)\}$ and $\sigma(1) \notin \{\gamma_1(0), \ldots, \gamma_n(0)\}$. If $\sigma(0) = \gamma_i(0)$ and $\sigma(1)$ differs from all $\gamma_j(0)$, define $g[\sigma]$ to be the 1-simplex of $K(X)$ corresponding to the homotopy class of the concatenation $\sigma * i(\overline{\gamma_i})$, where $\overline{\gamma_i}(t) := \gamma_i(1-t)$ for $t \in [0,1]$. If $\sigma(1) = \gamma_i(0)$ and $\sigma(0)$ differs from all $\gamma_j(0)$, define $g[\sigma]$ as the 1-simplex corresponding to the homotopy class of $i(\gamma_i) * \sigma$. If $\sigma(0) = \gamma_i(0)$ and $\sigma(1) = \gamma_j(0)$, define $g\sigma$ to be the 1-simplex corresponding to the homotopy class of $i(\gamma_j) * \sigma * i(\overline{\gamma_i})$. All these definitions were independent of the choice of σ in its homotopy class relative $\{0,1\}$.
To define the action of $\Pi_X(Y)$ on all of $K(X)$, we claim that for a simplex $\sigma \in K(X)$ with 1-skeleton σ_1, and $g \in \Pi_X(Y)$, there exists *some* simplex in $K(X)$ with 1-skeleton $g\sigma_1$. Since $K(X)$ is aspherical, this will allow a unique extension of the group action from $K(X)_1$ to $K(X)$. To prove the claim, observe the following: if g is a path in X connecting v_0 to v_0' and if σ is a simplex in $K(X)$ represented by a singular simplex $\hat{\sigma}$ in X with 0-th vertex v_0', then $\hat{\sigma}$ can clearly be homotoped so that one gets a singular simplex in X with 0-th vertex v_0, leaving the other vertices fixed, so that we get a simplex whose 1-skeleton is $g\sigma_1$. Argueing succesively, we get the claim for general $g \in \Pi_X(Y)$.

Definition 7 *: Let (K,L) be a pair of multicomplexes. Let $\{e_1, \ldots, e_n\}$ and $\{e_1', \ldots, e_n'\}$ be two n-tuples of 1-simplices in K with vertices in L. We say that $\{e_1, \ldots, e_n\}$ and $\{e_1', \ldots, e_n'\}$ are L-related, if there are 1-simplices $f_1, \ldots, f_m$ in L such that*
- the vertices $f_1(0), f_1(1), \ldots, f_m(0), f_m(1)$ of $f_1, \ldots, f_m$ are all distinct and are in bijection with the set of vertices of $e_1, \ldots, e_n, e_1', \ldots, e_n'$, (note that $m \leq 2n$, we do not assume the vertices of the e_i's and e_i''s to be distinct)
- the concatenations $f_k e_i f_l e_i'^{-1}$ (with f_k, f_l uniquely selected such that the vertices match) represent the identity in $\pi_1 K$.

The following observation is obvious from the construction.

Lemma 6 *: Let (X,Y) be a pair of spaces such that $\pi_1 Y \to \pi_1 X$ is injective. Then the action of $\Pi_X(Y)$ on $K(X)$ is transitive on $K(Y)$-related 1-simplices with vertices in $K(Y)$. That is, if e_1 and e_2 are $K(Y)$-related 1-simplices in $K(X)$, then exists $g \in \Pi_X(Y)$ with $ge_1 = e_2$.*

3.1.6 An application of averaging

Lemma 7 *: If $A \subset X$ is a subspace such that $im\left(\pi_1(A,x) \to \pi_1(X,x)\right)$ is amenable for all $x \in A$, then $\Pi_X(A)$ is amenable.*

Proof: There is an exact sequence
$1 \to \bigoplus_{y\in A} im\left(\pi_1(A,y) \to \pi_1(X,y)\right) \to \Pi_X(A) \to Perm_{fin}(A) \to 1$, where $Perm_{fin}$ are the permutations with finite support.
It is well known that a group is amenable if any finitely generated subgroup is amenable. All finitely supported permutations have finite order. It follows that any finitely generated subgroup of $Perm_{fin}(A)$ is finite and therefore amenable. Also any finitely generated subgroup of $\bigoplus_{y\in Y} im\left(\pi_1(A,y) \to \pi_1(X,y)\right)$ is contained in a finite sum of amenable groups and is therefore amenable. Thus $\Pi_X(A)$ is an amenable extension of an amenable group and, hence, is amenable. □

For $\epsilon \in R$ define a norm on $C_*(X,Y)$ by $\parallel z \parallel_\epsilon = \parallel z \parallel + \epsilon \parallel \partial z \parallel$. We get an induced pseudonorm $\parallel . \parallel_\epsilon$ on $H_*(X,Y)$.
More generally, if A is a union of connected components of Y, we define a norm on relative cycles of $C_*(X,Y)$ by $\parallel z \parallel_\epsilon^A = \parallel z \parallel + \epsilon \parallel \partial z \mid_A \parallel$ and consider the induced pseudonorm $\parallel . \parallel_\epsilon^A$ on $H_*(X,Y)$.

Proposition 4 *: If $im\left(\pi_1(A,y) \to \pi_1(X,y)\right)$ is amenable for all $y \in A$, then $\parallel h \parallel = \parallel h \parallel_\epsilon^A$ for all $h \in H_*(X,Y)$.*

Proof: : With the additional assumption $A = Y$, proposition 4 becomes the equivalence theorem in [26], p.57. To get the general claim, we give a straigtforward modification of Gromov's proof.

We consider the dual norm on bounded cohomology, which, by the Hahn-Banach theorem, is induced from the dual norm on the relative cocycles. We will show that $\parallel c \parallel_\epsilon^A = \parallel c \parallel$ for relative cocycles c.

By propositions 1 and 2, we may assume that we are working with the complex of antisymmetric simplicial cochains of $K(X)$. By lemma 2, we may assume the cochains to be invariant under the action of the amenable group $\Pi_X(A)$. Hence, we may assume that the relative cocycle c factors over Q, where Q is the quotient of $F_*(K(X))/F_*(K(Y))$ under the relations $\overline{\sigma} = -\sigma$ and $a\sigma = \sigma$ for all $a \in \Pi_X(A)$ and all simplices σ, where $\overline{\sigma}$ is σ with the opposite orientation. We can define in an obvious way analogs of our norms on the dual of Q and we get then $\parallel c \parallel = \parallel c^Q \parallel$ and $\parallel c \parallel_\epsilon^A = \parallel c^Q \parallel_\epsilon^A$, where c^Q is c considered as a map from Q to R.

But in Q, any simplex σ with an edge in A becomes trivial, because there is some element of $\Pi_X(A)$ mapping σ to $\overline{\sigma}$. Hence, for any relative cycle $z \in C_*(X,Y)$, the image of $\partial z \mid_A$ in Q is trivial. Hence, $\parallel c^Q \parallel$ and $\parallel c^Q \parallel_\epsilon^A$ agree. □

Corollary 2 *: If M is a compact manifold, A a union of connected components of ∂M, and $im\left(\pi_1(A,x)\rightarrow\pi_1(M,x)\right)$ is amenable for all $x\in A$, then for any $\epsilon>0$ exists a representative z of $[M,\partial M]$ with $\| z \|\leq\| M,\partial M \| +\epsilon$ and $\| \partial z \mid_A\|\leq\epsilon$.*

3.2 Retraction in aspherical treelike complexes

If a group G acts simplicially on a multicomplex M, then $C_*(M)/G$ are abelian groups with well-defined boundary operator, even though M/G may not be a multicomplex. (An instructive example for the latter phenomenon is the action of $G=\Pi_X(X)$ on $K(X)$, for a topological space X.)

3.2.1 The 'amalgamated' case

Lemma 8 *: Assume that $(M,M'),(K,K'),(L,L')$ are pairs of path-connected, minimally complete submulticomplexes, such that*
(i) K,L are submulticomplexes of M, with inclusions $i_K:K\rightarrow M, i_L:L\rightarrow M$,
(ii) $M_0=K_0\cup L_0$,
(iii) $A:=K\cap L$ is a submulticomplex, $\pi_1 A\rightarrow\pi_1 K$ and $\pi_1 A\rightarrow\pi_1 L$ are injective,
(iv) the inclusion $K\cup L\rightarrow M$ induces an isomorphism $\pi_1(K\cup L)\rightarrow\pi_1(M)$,
(v) K and L are aspherical in the sense of definition 3,
(vi) $K'=M'\cap K, L'=M'\cap L$.
Assume moreover that a group G acts simplicially on (M,M') such that
(vii) G maps (K,K') to (K,K') and (L,L') to (L,L'),
(viii) G acts transitively on A-related tuples of 1-simplices.
Then there is a relative chain map $r:G\backslash C_*(M,M')\rightarrow G\backslash C_*(K,K')\oplus G\backslash C_*(L,L')$ in degrees $*\geq 2$ such that
- if $G\sigma$ is the orbit of a simplex in M, then $r(G\sigma)$ either is the G-orbit of a simplex in K or the G-orbit of a simplex in L,
- $ri_{K*}=id_{G\backslash C_*(K,K')}$, $ri_{L*}=id_{G\backslash C_*(L,L')}$.

Proof: We consider first the case $M'=\emptyset$.

The plan of the proof is as follows: let σ be a simplex in M, let $\tilde{\sigma}$ be a lift to the universal cover $\tilde{M}$, and let $v_0,\ldots,v_n\in\tilde{M}_0$ be the vertices of $\tilde{\sigma}$. To each pair $\{v_i,v_j\}$ we associate a family of 'minimizing' pathes $\left\{p\left(\left\{a_l^{ij}\right\};\left\{h_k^{ij}\right\}\right)\right\}$ parametrised by vertices $a_0^{ij},\ldots,a_{m_{ij}}^{ij}\in A_0$ and by elements h_k^{ij} of $\pi_1 K$ or $\pi_1 L$ satisfying conditions described below. Associated to $\{v_0,\ldots,v_n\}$ and these families of 'minimizing' pathes, we construct a family of 'central' simplices $\left\{\tilde{\tau}\left(\left\{a_l^{ij}\right\};\left\{h_k^{ij}\right\}\right)\subset\tilde{M}\right\}$ and their projections $\left\{\tau\left(\left\{a_l^{ij}\right\};\left\{h_k^{ij}\right\}\right)\subset M\right\}$, which actually lie in K or L. We show then that all $\tau\left(\left\{a_l^{ij}\right\};\left\{h_k^{ij}\right\}\right)$, associated to a fixed $\tilde{\sigma}$, belong to the same G-orbit, and that also the simplices associated to either $g\tilde{\sigma}$ with $g \in \pi_1 M$ or to $\widetilde{g\sigma}$ with $g \in G$ belong to the same G-orbit. We define then $r(G\sigma) = G\tau$.

Assumption (iv) implies that the universal cover $\widetilde{K \cup L}$ is a submulticomplex of the universal cover $\tilde{M}$. Assumption (ii) together with assumption (iv) gives that the 0-skeleton of $\widetilde{K \cup L}$ is the whole 0-skeleton of $\widetilde{M}$.
Let $\pi : \widetilde{K \cup L} \to K \cup L$ be the projection. We will need a specific section s of π on the 1-skeleton

$$s : (K \cup L)_1 \to \left(\widetilde{K \cup L}\right)_1$$

$$\sigma \to s(\sigma) = \tilde{\sigma}$$

defined as follows:

Fix a vertex $p \in A_0$ and some lift $\tilde{p} \in \tilde{A}_0 \subset \tilde{M}_0$. For any vertex v of $K \cup L$, there is some edge $e \in K_1 \cup L_1$ connecting p to v, because K and L are complete. There are unique lifts $\tilde{e}$ and $\tilde{v}$ such that $\tilde{e}$ has boundary points $\tilde{p}$ and $\tilde{v}$. This defines $\sim$ on the 0-skeleton, and also on some 1-simplices. Now, for all other 1-simplices $e \in K_1 \cup L_1$ with boundary points v and w, possibly $v = p$, we fix the unique lift $\tilde{e}$ in $\widetilde{K \cup L}$ with 0-th vertex $\tilde{v}$. (Note that <u>in $K \cup L$</u> there are no edges with one vertex in $K_0 - A_0$, the other vertex in $L_0 - A_0$.)
It should be noted: if e has vertices v_0 and v_1, then $\tilde{e}$ has vertices $\tilde{v_0}$ and $h_1\tilde{v_1}$ with, a priori, $h_1 \in \pi_1(K \cup L)$. Assume that e is an edge in K. We have a unique edge connecting $\tilde{v_1}$ to $h_1\tilde{v_1}$. This edge projects to an edge f with both vertices v_1. Since v_1, as a vertex of e, belongs to K, we conclude that $f \in K_1$. This implies $h_1 \in \pi_1 K \subset \pi_1(K \cup L)$. In a similar way, if e is an edge in L, we conclude that $h_1 \in \pi_1 L$.
As a consequence, we get the following observation.

(A): *if $g\tilde{e}$ is an edge with boundary points $h_0\tilde{v_0}$ and $h_1\tilde{v_1}$, then $g = h_0$ and $h_1 h_0^{-1}$ is either* 1, *or an element of $\pi_1 K$, or an element of $\pi_1 L$.*

Indeed, we have just seen this for $g = 1$. The general case follows after applying g^{-1} to $g\tilde{e}$.
Moreover, if $g_1\tilde{e_1}$ is a 1-simplex with boundary points $h_{01}\tilde{v}_{01}$ and $h_{11}\tilde{v}_{11}$, and $g_2\tilde{e_2}$ is a 1-simplex with boundary points $h_{02}\tilde{v}_{02}$ and $h_{12}\tilde{v}_{12}$, then:

(B): *if $g_1\tilde{e_1}$ and $g_2\tilde{e_2}$ have a common boundary point $h\tilde{v} = h_{11}\tilde{v}_{11} = h_{02}\tilde{v}_{02}$, then one of the following two possibilities holds:*

- $v \in A$ or

- $h_{01}^{-1}h_{11}$ and $h_{02}^{-1}h_{12}$ either belong both to $\pi_1 K$ or belong both to $\pi_1 L$.

Indeed, if $v \notin A$, then v is not adjacent to both, edges of K and edges of L.

Minimizing pathes:

Let v_0, v_1 be vertices of $\widetilde{K \cup L}$. By a **path** from v_0 to v_1 we mean a sequence of 1-simplices $e_1, \ldots, e_r$ such that v_0 is a vertex of e_1, e_i and e_{i+1} have a vertex in common for $1 \leq i \leq r-1$ and, v_1 is a vertex of e_r.

Given two vertices $v_1, v_2 \in \tilde{M}_0$, they belong to $\left(\widetilde{K \cup L}\right)_0$ because of (iii) and (v), and we may represent them as $v_1 = g_1 \tilde{w}_1, v_2 = g_2 \tilde{w}_2$ with $g_i \in \pi_1 (K \cup L)$ and $w_i \in (K \cup L)_0$ for $i = 1, 2$.

$\pi_1 (K \cup L) = \pi_1 K *_{\pi_1 A} \pi_1 L$ is an amalgamated product, hence $g_1 g_2^{-1}$ either belongs to $\pi_1 A$ or it can be decomposed as $g_1 g_2^{-1} = h_1 \ldots h_m$, where h_i are elements of $\pi_1 K - \pi_1 A$ or $\pi_1 L - \pi_1 A$ and, $h_i \in \pi_1 K$ iff $h_{i+1} \in \pi_1 L$. Such an expression is called a **normal form** of $g_1 g_2^{-1}$. If $h_1 \ldots h_m$ and $h_1' \ldots h_l'$ are two normal forms of $g_1 g_2^{-1}$, then necessarily $l = m$ and for $i = 1, \ldots, m$ belong h_i and h_i' to the same equivalence class modulo $\pi_1 A$.

We call a path $e_0, \ldots, e_{m-1}$ from $g_1 \tilde{w}_1$ to $g_2 \tilde{w}_2$ **minimizing** if it satisfies the following:

there is a *normal form* $g_1 g_2^{-1} = h_1 \ldots h_m$ and a *set of vertices* $a_0, \ldots, a_m \in A_0$, with $a_i \neq a_{i+1}$ for $i = 0, \ldots, m-1$, such that:
- e_0 has vertices $g_2 \tilde{w}_2$ and $g_2 \tilde{a}_m$
- e_i has vertices $h_{m-i+1} \ldots h_m g_2 \tilde{a}_{m-i}$ and $h_{m-i+2} \ldots h_m g_2 \tilde{a}_{m-i+1}$ for $i = 1, \ldots, m$
- e_{m+1} has vertices $g_1 \tilde{w}_1$ and $g_1 \tilde{a}_0$.

One should note that all these edges exist in $\widetilde{K \cup L}$ because all neighboring points project to distinct points in K resp. L and can therefore be joined by an edge in K resp. L, by completeness. Moreover, the construction should be understood such that we skip e_0 resp. e_{m+1} if $w_2 \in A$ resp. $w_1 \in A$.

It follows from (A) and (B), that these pathes are length-minimizing in the sense of being exactly the pathes with a minimum number of edges between v_1 and v_2. Since this latter characterisation depends only on v_1 and v_2, we conclude: for different sections s_1 and s_2, there is a bijection between the corresponding sets of minimizing pathes from v_1 to v_2.

Since $\tilde{K}$ and $\tilde{L}$ are universal covers of *minimally complete* multicomplexes, there is at most one edge between two vertices, and therefore a path of length m becomes uniquely determined after fixing its $m+1$ vertices. Hence, after fixing $a_0, \ldots, a_m \in A_0$ and a normal form $g_1 g_2^{-1} = h_1 \ldots h_m$, we get a unique path, to be denoted $p(a_0, \ldots, a_m; h_1, \ldots, h_m)$.

We note for later reference the following obvious observations:

(C1) *Subpathes of minimizing pathes are minimizing.*

(C2) *If $e_1, \ldots, e_k$ is a minimizing path, e_k projects to an edge in K, e_{k+1} projects to an edge in L, e_k and e_{k+1} have a common vertex, then $e_1, \ldots, e_k, e_{k+1}$ is a minimizing path.*

Intersection with simplices:

Given a set of vertices $\{v_i\}_{1\leq i\leq n}$ we consider the set

$$\bigcup_{0\leq i,j\leq n} P(i,j)=\left\{r_{ij}^k : 1\leq i<j\leq n, r_{ij}^k\in P(i,j)\right\},$$

where $P(i,j)$ is the set of minimizing paths from v_i to v_j.

Let $\tilde{\tau}$ be any simplex in $\widetilde{K\cup L}$. We claim that $\left\{\tilde{\tau}\cap r_{ij}^k : r_{ij}^k\in P(i,j)\right\}$ is the full 1-skeleton of a subsimplex of $\tilde{\tau}$. We have to check the following claim:

if $[x,y],[z,w]\in\tilde{\tau}_1\bigcap\cup_{ij}P(i,j)$*, then also* $[x,z],[x,w]$*,* $[y,z]$ *and* $[y,w]$ *belong to* $\tilde{\tau}_1\bigcap\cup_{ij}P(i,j)$*.*

Indeed, assume that there is a minimizing path $\{e_0,\ldots,e_{m+1}\}$ with e_i having vertices $h_{m-i+1}\ldots h_m g_2\tilde{a}_{m-i}$ and $h_{m-i+2}\ldots h_m g_2\tilde{a}_{m-i+1}$ for $i=1,\ldots,m$ such that $[x,y]=e_l$, that is, $x=h_{m-l+1}\ldots h_m g_2\tilde{a}_{m-l}$ and $y=h_{m-l+2}\ldots h_m g_2\tilde{a}_{m-l+1}$. Assume also that there is a minimizing path $\{e_0',\ldots,e_{m'}'\}$ with analogous notations such that $[z,w]=e_l'$, that is $z=h_{m'-l'+1}'\ldots h_{m'}'g_2'\tilde{a}_{m'-l'}'$ and $w=h_{m'-l'+2}'\ldots h_{m'}'g_2'\tilde{a}_{m'-l'+1}'$.

Note that all simplices in $\widetilde{K\cup L}$ project to simplices in K or in L. Assume that $\tilde{\tau}$ projects to K. By the discussion preceding observation (A), this means that h_{m-l+1} and $h_{m'-l'+1}'$ belong both to $\pi_1K-\pi_1A$. It follows that $h_{m-l},h_{m-l+2},h_{m'-l'}',h_{m'-l'+2}'$ belong to $\pi_1L-\pi_1A$. But this implies, for example, that $[x,z]$ is part of some minimizing path, namely the path

$$\{e_0,\ldots,e_{l-1},[x,z],e_{l'-1}',\ldots,e_0'\}.$$

In a similar way, we see that [x,w], [y,z] and [y,w] belong to the intersection of $\tilde{\tau}$ with minimizing pathes between suitable pairs of v_i's.
The same argument works if τ projects to L.

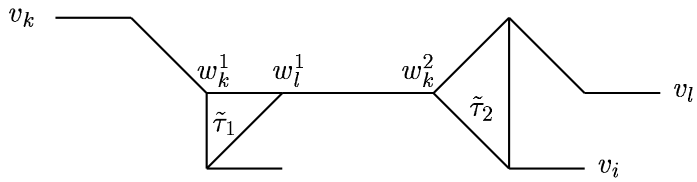
'Central' simplices:

We are given vertices $v_0=g_0\tilde{w_0},\ldots,v_n=g_n\tilde{w_n}\in\tilde{M_0}$, $n\geq 2$.
We claim: if we fix, for each index pair $\{i,j\}$, a normal form $g_ig_j^{-1}=h_1^{ij}\ldots h_{m_{ij}}^{ij}$, vertices $a_0^{ij},\ldots,a_{m_{ij}}^{ij}\in A_0$, and the minimizing path $p_{ij}\in P(i,j)$, then there is *at most one* **n-dimensional** simplex $\tilde{\tau}\in\widetilde{K\cup L}$ such that the intersection of $\tilde{\tau}$ with $\cup_{0\leq i,j\leq n}p_{ij}$ is the 1-skeleton of an **n-dimensional** simplex, i.e., is the full 1-skeleton of $\tilde{\tau}$.
(In fact, for such an n-simplex to exist, the h_{ij}'s as well as the a_{ij}'s have to satisfy obvious compatibilty conditions. We will not make use of these explicit conditions in our proof.)

We prove the claim. Assume there are two such simplices $\tilde{\tau}_1\neq\tilde{\tau}_2$ with $dim(\tilde{\tau}_1)=dim(\tilde{\tau}_2)=n$. By assumption (vi), it suffices to show that $\tilde{\tau}_1$ and

$\tilde{\tau}_2$ have the same 1-skeleta. We have to distinguish the cases that $\tilde{\tau}_1$ and $\tilde{\tau}_2$ have no common vertex or that they have a common subsimplex.

Assume first that $\tilde{\tau}_1$ and $\tilde{\tau}_2$ have no vertex in common. In the following, 'minimizing path' will mean the unique minimizing path with respect to our fixed choice of normal forms and of vertices in A. We will frequently use the following fact: each edge of $\tilde{\tau}_1$ (resp. $\tilde{\tau}_2$) is contained in the minimizing path from v_i to v_j for a unique pair $\{i,j\}$ of indices. This is true by a counting argument: there are $\frac{n(n+1)}{2}$ minimizing pathes and $\frac{n(n+1)}{2}$ edges of $\tilde{\tau}_1$, each edge belongs to some minimizing path by assumption, and no minimizing path can have two consecutive edges projecting both to K or both to L, by the definition of normal forms.
For each k and l, the minimizing path from v_k to v_l passes through $\tilde{\tau}_1$ as well as through $\tilde{\tau}_2$. Let $[w_k^1, w_l^1]$ resp. $[w_k^2, w_l^2]$ be the intersections of this minimizing path with $\tilde{\tau}_1$ resp. $\tilde{\tau}_2$.
We claim: all minimizing pathes from v_k to some v_i with $i \neq k$ pass through w_k^1 *and* w_k^2. To prove the claim, note that, by (C1), the subpath from v_k to w_k^1 is minimizing, that is, the corresponding sequence $h_1, \ldots, h_m$ is a normal form (for $\Pi_{i=1}^m h_i$) with $h_m \in \pi_1 L$ if τ_1 projects to K or vice versa. It follows from the definition of normal forms that also $\Pi_{i=1}^{m+1} h_i$ is a normal form if $h_{m+1} \in \pi_1 K$ is the element corresponding to an edge of K having w_k^1 as a vertex. Since normal forms are unique up to multiplication of the h_i's with elements of $\pi_1 A$, we conclude that there is no minimizing path from v_k to some vertex of τ_1 which does not pass through w_k^1. By the same argument, all minimizing pathes from v_k to some vertex of τ_2 have to pass through w_k^2. In particular, they have to contain the unique minimizing path from w_k^1 to w_k^2 as a subpath, by (C1). This, in turn, implies that all minimizing pathes from v_k to any $v_i, i \neq k$, contain the minimizing path from w_k^1 to w_k^2 and, in particular, contain the same edge of $\tilde{\tau}_1$. But, by the above counting argument, it may not happen that several minimizing pathes pass through the same edge of $\tilde{\tau}_1$. This gives the contradiction.



It remains to discuss the case that $\tilde{\tau}_1$ and $\tilde{\tau}_2$ have a proper boundary face in common. Let w_k be a vertex of τ_1 which is not a vertex of τ_2 and w_l a vertex of both, τ_1 and τ_2. After reindexing, there are, by the above counting argument, v_k and v_l such that the minimizing path from v_k to v_l contains the edge $[w_k, w_l] \subset \tau_1$. By the same argument as above, all minimizing pathes from v_k to any v_i pass through w_k as well as through w_l, that is, they all contain the edge $[w_k, w_l]$ giving a contradiction. This finishes the proof of the claim. Since it will be used again in

the proof that r is a chain map, we write down the following observation, which we have just proved.
(D): *If $v_0, \ldots, v_n \in \widetilde{M_0}$ are vertices of M, then their central simplex $\tilde{\tau}$ to some choice of $\left\{h_k^{ij}\right\}, \left\{a_l^{ij}\right\}$, if it exists, has vertices $w_0, \ldots, w_n$ such that for any i and j the minimizing path from v_i to v_j passes through w_i and w_j.*

We denote by $\tau\left(\left\{a_l^{ij}\right\};\left\{h_k^{ij}\right\}\right)$ the projection of $\tilde{\tau}$ to $K \cup L$.
Next we claim: if we still are given $v_0 = g_0\tilde{w_0}, \ldots, v_n = g_n\tilde{w_n} \in \tilde{M_0}$, but vary $a_0, \ldots, a_m \in A_0$ and the normal forms $g_i g_j^{-1} = h_1^{ij} \ldots h_{m_{ij}}^{ij}$, then all $\tau\left(\left\{a_l^{ij}\right\};\left\{h_k^{ij}\right\}\right)$ belong to the same G-orbit.

Let us consider $\tau\left(\{a_l\};\left\{h_k^{ij}\right\}\right)$ and $\tau\left(\{a_l'\};\left\{h_k^{ij}\right\}\right)$. The same argument which showed uniqueness of $\tilde{\tau}\left(\{a_l\};\left\{h_k^{ij}\right\}\right)$ lets us conclude that, representing the vertices of $\tilde{\tau}\left(\{a_l\};\left\{h_k^{ij}\right\}\right)$ as $\gamma_0\tilde{a}_0, \ldots, \gamma_n\tilde{a}_n$ and the vertices of $\tilde{\tau}\left(\{a_l'\};\left\{h_k^{ij}\right\}\right)$ as $\gamma_0'\tilde{a_0}', \ldots, \gamma_n'\tilde{a_n}'$, we must have $\gamma_0 = \gamma_0', \ldots, \gamma_n = \gamma_n'$. Hence, corresponding edges of τ are A-related in the sense of definition 4 and belong, by assumption (viii), to the same G-orbit.

Now we consider $\tau\left(\{a_l\};\left\{h_k^{ij}\right\}\right)$ and $\tau\left(\{a_l\};\left\{h_k^{ij\prime}\right\}\right)$, where $g_i g_j^{-1} = h_1^{ij} \ldots h_m^{ij} = h_1^{ij\prime} \ldots h_m^{ij\prime}$ are different normal forms. Since we may argue succesively, it suffices to consider the case that different normal forms occur for only one index pair i, j. For the same reason, it suffices to consider the case that there is $1 \leq s \leq m-1$ such that $h_s^{ij\prime} = h_s^{ij} a^{-1}, h_{s+1}^{ij\prime} = a h_{s+1}^{ij}$ and $h_l^{ij\prime} = h_l^{ij}$ otherwise. Now, from the above construction, it follows that one of the following two possibilities takes place:
- if $h_l^{ij} \ldots h_m^{ij} g_2^{ij} \tilde{a}_{l-1}^{ij}$ and $h_{l+1}^{ij} \ldots h_m^{ij} g_2^{ij} \tilde{a}_l^{ij}$ are not vertices of $\tilde{\tau}\left(\left\{a_l^{ij\prime}\right\};\left\{h_k^{ij}\right\}\right)$, then $\tilde{\tau}\left(\left\{a_l^{ij\prime}\right\};\left\{h_k^{ij\prime}\right\}\right) = \tilde{\tau}\left(\left\{a_l^{ij\prime}\right\};\left\{h_k^{ij}\right\}\right)$.
- if one resp. both of $h_l^{ij} \ldots h_m^{ij} g_2^{ij} \tilde{a}_{l-1}^{ij}$ and $h_{l+1}^{ij} \ldots h_m^{ij} g_2^{ij} \tilde{a}_l^{ij}$ are vertices of $\tilde{\tau}\left(\left\{a_l^{ij}\right\};\left\{h_k^{ij}\right\}\right)$, then $\tilde{\tau}\left(\left\{a_l^{ij}\right\};\left\{h_k^{ij\prime}\right\}\right)$ has n resp. $n-1$ vertices in common with $\tilde{\tau}\left(\left\{a_l^{ij}\right\};\left\{h_k^{ij}\right\}\right)$, and has moreover as remaining vertices one or both of $h_l^{ij} \ldots h_m^{ij} g_2^{ij} \tilde{a}_{l-1}^{ij}$ and $h_{l+1}^{ij} \ldots h_m^{ij} g_2^{ij} \tilde{a}_l^{ij}$.
It follows that the 1-skeleta of $\tilde{\tau}\left(\left\{a_l^{ij}\right\};\left\{h_k^{ij}\right\}\right)$ and $\tilde{\tau}\left(\left\{a_l^{ij}\right\};\left\{h_k^{ij\prime}\right\}\right)$, considered as tuples of 1-simplices, are A-related in the sense of definition 4. Indeed, that the concatenations are trivial in $\pi_1(K \cup L)$ follows from the assumption that K and L are aspherical in the sense of definition 3, which forces the concatenations to bound a 2-simplex resp. a union of two 2-simplices. By assumption (viii), we get that $\tau\left(\{a_l\};\left\{h_k^{ij}\right\}\right)$ and $\tau\left(\{a_l\};\left\{h_k^{ij\prime}\right\}\right)$ belong to the same G-orbit.

Retraction in treelike complex:

We are going to define r. Given an n-simplex $\sigma \in M$, $n \geq 2$, we consider a lift $\tilde{\sigma} \in \tilde{M}$ which projects to σ. Its vertices $v_0, \ldots v_n$ belong to $\widetilde{M}_0 = \left(\widetilde{K \cup L}\right)_0$.
Now we run the above constructions with $v_0, \ldots, v_n$ and look for simplices τ with $dim\,(\tau) = n$.

If there is no simplex τ with $dim\,(\tau) = n$, we define $r\,(\sigma) = 0$. Otherwise, there is a unique G-orbit of simplices $\{g\tau : g \in G\}$ with $dim\,(g\tau) = n$ for all $g \in G$, and we define $r\,(G\sigma) := G\tau$.

We have to check that the definition of r does not depend on the choice of $\tilde{\sigma}$ neither on the choice of σ in its G-orbit.

Observe the following: if $\tilde{f} : \tilde{M} \to \tilde{M}$ is a simplicial self-map of the universal cover, such that $\pi\tilde{f}$ maps K to K and L to L, then $\tilde{f}$ maps minimizing pathes from v_i to v_j to minimizing pathes from $\tilde{f}\,(v_i)$ to $\tilde{f}\,(v_j)$. Hence, simplices $\tilde{\tau}$ intersecting the family of minimizing pathes associated to $v_0, \ldots, v_n$ in the full 1-skeleton of an n-simplex are mapped by $\tilde{f}$ to simplices $\tilde{f}\,(\tilde{\tau})$ intersecting the family of minimizing pathes associated to $\tilde{f}\,(v_0)\,, \ldots, \tilde{f}\,(v_n)$ in the full 1-skeleton of an n-simplex. Thus, if $\tilde{\tau}$ belongs to the family of 'central' simplices associated to some simplex $\tilde{\sigma}$, then $\tilde{f}\,(\tilde{\tau})$ belongs to the family of 'central' simplices associated to $\tilde{f}\,(\tilde{\sigma})$.
We conclude: $r\,(G\sigma)$ does not depend on the choice of σ in its G-orbit, neither on the choice of $\tilde{\sigma}$, for fixed σ, in the orbit of the deck group.

Finally, the two desired conditions are clearly satisfied:
we have constructed $\tilde{\tau}$ with the help of the condition that it is a simplex in $\widetilde{K \cup L}$, hence $r\,(\sigma) = \tau$ is a simplex either in K or in L. If σ is a simplex in K or L, then clearly $r\,(\sigma) = \sigma$, hence r is leftinverse to i_{K*} and i_{L*}.

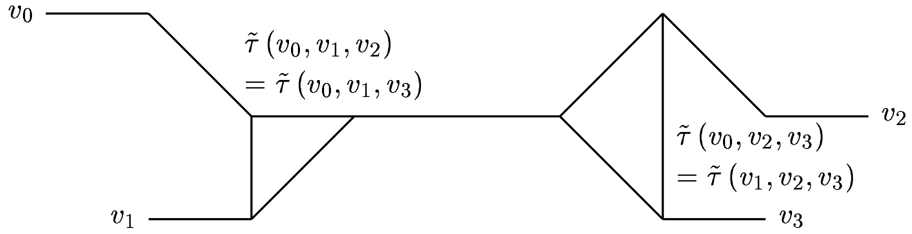
Compatibility with ∂-operator:

It remains to show that r is a chain map, i.e., that $\partial r\,(G\sigma) = r\,(G\partial\sigma)$ holds for all simplices σ in M.

First we consider the case $r\,(G\sigma) \neq 0$.
Let $v_0, \ldots, v_n$ be the vertices of a lift $\tilde{\sigma}$, and $\left\{h_k^{ij}\right\}$, $\left\{a_l^{ij}\right\}$ such that a central simplex $\tilde{\tau}$ exists. It is obvious, if we consider the set of vertices without v_k and the corresponding restricted sets of normal forms and vertices, that we get the k-th face of $\tilde{\tau}$ as a central simplex. That implies $r\,(G\partial_k\sigma) = \partial_k r\,(G\sigma)$ for all k, and hence $r\,(G\partial\sigma) = \partial r\,(G\sigma)$.

We consider now the case $r\,(G\sigma) = 0$.
If $r\,(G\partial_k\sigma) = 0$ for all faces $\partial_k\sigma$ of σ, we conclude $r\,(G\partial\sigma) = 0$.
So assume that for some face $\partial_k\sigma$ of σ we have $r\,(G\partial_k\sigma) = G\tau$ for some (n-1)-simplex τ. That means that, for the vertices $v_0, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n$ and some choice of $\{h_m^{ij}\}\,, \left\{a_l^{ij}\right\}$ we have the central simplex $\tilde{\tau}$. Here, the h_m^{ij} and a_l^{ij} are only chosen for $i \neq k, j \neq k$. By observation (D), $\tilde{\tau}$ has vertices $w_0, \ldots, w_{k-1}, w_k, \ldots, w_n$

such that the minimizing path from v_i to w_j passes through w_i for all $i \neq k \neq j$. For $i \neq k$ consider the set P'_{ik} of the minimizing pathes from v_k to w_i. There exists some vertex v, with the property that for each i exists a path in P'_{ik} containing $[v, w_i]$ as last edge. If v were not one of $w_0, \ldots, w_{k-1}, w_{k+1}, \ldots, w_n$, then this would give us a central simplex to the vertices $v_0, \ldots, v_{k-1}, v_k, v_{k+1}, \ldots, v_n$, contradicting the assumption. Hence $v = w_j$ for some j. But this implies that the central simplices to $v_0, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n$ are exactly the central simplices to $v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n$, and the corresponding simplices have the same orientation if and only if $j-k$ is *odd*. Therefore, $r(\partial_k \sigma)$ cancels against $r(\partial_j \sigma)$. As we find such a j for any k with $r(\partial_k \sigma) \neq 0$, we get $r(\partial \sigma) = 0$.



Relative construction:

Finally, we consider the general case $M' \neq \emptyset$. For the universal cover $\pi : \tilde{M} \to M$ denote the subcomplex $\tilde{M}' := \pi^{-1}(M')$. Since $A \subset M'$, the definition of 'minimizing pathes' implies that, for $x, y \in \tilde{M}'_0$, all minimizing pathes from x to y are subsets of $\tilde{M}'$. As a consequence, for $v_0, \ldots, v_n \in \tilde{M}'_0$, any central simplex, if it exists, must belong to $\tilde{M}'$. Hence, $\tau \subset M' \cap K = K'$ or $\tau \subset M' \cap L = L'$. This proves that r induces a chain map $r : C_*(M, M') \to C_*(K, K') \oplus C_*(L, L')$, finishing the proof of lemma 8. □

Corollary 3 *: Let $(M, M'), (K, K'), (L, L')$ satisfy all assumptions of lemma 8. Let $\gamma_1 \in H_b^p(K, K'), \gamma_2 \in H_b^p(L, L')$ be bounded cohomology classes, $p \geq 3$, such that the action of G fixes γ_1 and γ_2. Then exists a class $\gamma \in H_b^p(M, M')$ satisfying $\| \gamma \| \leq max\{\| \gamma_1 \|, \| \gamma_2 \|\}$, such that the restrictions to K and L give back γ_1 and γ_2 and that γ is fixed by G.*

Proof: Let c_1 resp. c_2 be bounded cocycles representing γ_1 resp. γ_2. To define a bounded cocycle c, it suffices to define its value on simplices. So let σ be a simplex in M. Define:
- $c(\sigma) := c_1(r(\sigma))$ if $r(G\sigma) \in G \backslash C_*(K)$,
- $c(\sigma) := c_2(r(\sigma))$ if $r(G\sigma) \in G \backslash C_*(L)$,
- $c(\sigma) := 0$ else.

If $p \geq 3$, we get an induced map in bounded cohomology. Indeed, let $c_1 - \tilde{c_1} = \delta b_1$, $c_2 - \tilde{c_2} = \delta b_2$ for (bounded) relative (p-1)-cochains b_1, b_2 and define again $b(\tau)$ as $b_1(r(\tau)), b_2(r(\tau))$ or 0 according to whether $r(\tau)$ is in $G \backslash C_*(K)$, in $G \backslash C_*(L)$

or 0. (This definition would not work for $p = 2$.) An obvious calculation yields $c - \tilde{c} = \delta b$.
All claims of the corollary are obvious except possibly that c is indeed a relative cocycle, i.e., that δc vanishes on (p+1)-simplices in M'. Assume w.l.o.g. that $r(\sigma) \in C_*(K')$. Then $c_1(\partial r(\sigma)) = 0$ because c_1 is a relative cocycle and, therefore $\delta c(\sigma) = c(\partial\sigma) = c_1(r(\partial\sigma)) = c_1(\partial r(\sigma)) = 0$. □

3.2.2 The 'HNN'-case

Lemma 9 : *Assume that (K, K') is a pair of path-connected, minimally complete multicomplexes, and that A_1, A_2 are disjoint minimally complete submulticomplexes of K' such that there exists a simplicial isomorphism $F : A_1 \rightarrow A_2$. Let (L, L') be the pair of multicomplexes obtained from (K, K') by identifying σ and $F(\sigma)$ for all simplices σ in A_1 and let $P : (K, K') \rightarrow (L, L')$ be the canonical projection.*
Moreover let (M, M') be a pair of multicomplexes. Assume that
(i) L is a submulticomplex of M with inclusion $i_L : L \rightarrow M$,
(ii) $M_0 = L_0$,
(iii) $\pi_1 A_1 \rightarrow \pi_1 K$ and $\pi_1 A_2 \rightarrow \pi_1 K$ are injective,
(iv) the inclusion $L \rightarrow M$ induces an isomorphism $\pi_1 L \rightarrow \pi_1 M$,
(v) K and L are aspherical in the sense of definition 3,
(vi)$L' = M' \cap L$.
Assume moreover that a group G acts simplicially on (M, M') as well as on (K, K') such that
(vii) the action of G commutes with $i_L P$, as well as with F,
(viii) G acts transitively on j-simplices of A_1, for any $j \geq 0$.
Then there is a relative chain map $r : G \backslash C_*(M, M') \rightarrow G \backslash C_*(K, K')$ in degrees $* \geq 2$ such that
- if $G\sigma$ is the G-orbit of a simplex in M, then $r(G\sigma)$ is the G-orbit of a simplex in K,
- $r i_{L*} P_* = id_{G \backslash C_*(K,K')}$.

Proof: The proof of lemma 9 parallels in several aspects the proof of lemma 8. We will then be somewhat briefer in the explanations.
By (iv), the universal covering $\tilde{L}$ is a submulticomplex of $\tilde{M}$. By (ii) and (iv), $\tilde{L}_0 = \tilde{M}_0$.
Denote A the image of A_1 (or A_2) in $L \subset M$. Fix a vertex $p \in A$ and some lift $\tilde{p}$. A vertex $v \in L_0$ is the image of some vertex of K, to be denoted v by abuse of notation. There is some edge in K with boundary points p and v, because K is complete. Its image under the projection P is an edge in L with boundary points p and v. There are unique lifts $\tilde{e} \in \tilde{L}_1$ and $\tilde{v} \in \tilde{L}_0$ such that $\tilde{e}$ has boundary

points $\tilde{p}$ and $\tilde{v}$. If v happens to be in A, we construct $\tilde{e}$ and $\tilde{v}$ by choosing an edge e which remains in A. This is possible because A is complete.
Recall that $\pi_1 L$ is an HNN-extension of $\pi_1 K$. We consider $\pi_1 K$ as subgroup of $\pi_1 L$ and denote by t the extending element of

$$\pi_1 L = < \pi_1 K, t \mid t^{-1} a t = F_* a \quad \forall a \in \pi_1 A > .$$

If $e \in L_1$ is an edge with boundary points v and w, we fix the unique lift $\tilde{e} \in \tilde{L}_1$ with 0-th vertex $\tilde{v}$. It will be crucial that the 1-th vertex of $\tilde{e}$ is then necessarily of the form $g\tilde{w}$ for some $w \in L_0$ with either $g \in \pi_1 K$ or $g = t$. This is true because: the edge f with vertices $\tilde{w}$ and $g\tilde{w}$ projects to a closed edge in L, which is either the image of a closed edge in K or the image of an edge in K with endpoints $b \in A_1, c \in A_2$ such that $F(b) = c$. In the first case, $g \in \pi_1 K$, in the second case $g = t$.

We have defined a map $\sim : L^1 \to \tilde{L}^1$, satisfying
(A): *if $g\tilde{e}$ is an edge with boundary points $h_0\tilde{v_0}$ and $h_1\tilde{v_1}$, then $g = h_0$ and $h_1 h_0^{-1}$ is either 1 or is an element of $\pi_1 K$ or equals t.*
Moreover, if $g_1\tilde{e_1}$ is a 1-simplex with boundary points $h_{01}\tilde{v}_{01}$ and $h_{11}\tilde{v}_{11}$, and $g_2\tilde{e_2}$ is a 1-simplex with boundary points $h_{02}\tilde{v}_{02}$ and $h_{12}\tilde{v}_{12}$, then:
(B): *if $g_1\tilde{e_1}$ and $g_2\tilde{e_2}$ have a common boundary point $h\tilde{v} = h_{11}\tilde{v}_{11} = h_{02}\tilde{v}_{02}$, then one of the following two possibilities holds:*
- $v \in A$ or
- $h_{01}^{-1}h_{11}$ and $h_{02}^{-1}h_{12}$ belong both to $\pi_1 K$ or equal both t.

Minimizing pathes:
Given two vertices $v_0, v_1 \in \tilde{M}_0 = \tilde{L}_0$, we may represent them as $v_1 = g_1\tilde{w}_1, v_2 = g_2\tilde{w}_2$ with $g_i \in \pi_1 L$ and $w_i \in M_0$ for $i = 1, 2$. $\pi_1 L$ is an HNN-extension of $\pi_1 K$, hence $g_1 g_2^{-1}$ has an expression $g_1 g_2^{-1} = h_1 \ldots h_m$ with $h_i \in \pi_1 K$ or $h_i \in t$ such that $h_i \in \pi_1 K$ implies $h_{i+1} \notin \pi_1 K$. (But we allow $h_i = h_{i+1} = t$.) This expression, which we will call a **normal form**, is unique up to compatible changes of the $h_i \in \pi_1 K$ in their equivalence class modulo $\pi_1 A$.

We call a path $e_1, \ldots, e_r$ **minimizing** if there are $a_0, \ldots, a_m \in A$ and a normal form $g_1 g_2^{-1} = h_1 \ldots h_m$ such that
- e_0 has vertices $g_2\tilde{w}_2$ and $g_2\tilde{a}$,
- e_i has vertices $h_{m-i+1} \ldots h_m g_2\tilde{a}$ and $h_{m-i+2} \ldots h_m g_2\tilde{a}$ for $i = 1, \ldots, m$,
- e_{m+1} has vertices $g_1\tilde{w}_1$ and $g_1\tilde{a}$.
One should note that the above edges exist in $\tilde{L}$, since K is complete. The construction should be understood that we skip e_0 resp. e_{m+1} if $w_2 \in A$ resp. $w_1 \in A$.
It follows from (A) and (B), that these pathes are length-minimizing in the sense

of being exactly the pathes between v_1 and v_2 with a minimum number of edges. Since this latter characterisation depends only on v_1 and v_2, we conclude: for different sections, there is a bijection between the corresponding sets of minimizing pathes from v_1 to v_2.

Since there is at most one edge between two vertices, a minimizing path becomes uniquely determined after fixing its vertices. So the only freedom in the choice of the minimizing path consists
- in the choice of $a_0, \ldots, a_m$, and
- in the choice of the h_i in their equivalence class modulo $\pi_1 A$.

The unique path corresponding to such a choice will be denoted $p\left(a_0, \ldots, a_m; h_1, \ldots, h_m\right)$.

Intersection with simplices:

Let $v_0, \ldots, v_n$ be vertices of $\tilde{L}$. Defining $P\left(i, j\right)$ like in the proof of lemma 5, we want to check that, for any simplex τ in $\tilde{L}$, $\left\{\tau \cap r_{ij}^k : r_{ij}^k \in P\left(i, j\right)\right\}$ is the full 1-skeleton of some subsimplex of τ.

Assume that $[x, y]$ and $[z, w]$ are edges of τ, with $x = h_{m-l+1} \ldots h_m g_2 \tilde{a}$, $y = h_{m-l+2} \ldots h_m g_2 \tilde{a}$, $z = h'_{m'-l'+1} \ldots h'_{m'} g'_2 \tilde{a}'$ and $w = h'_{m'-l'+2} \ldots h'_{m'} g'_2 \tilde{a}'$. By the discussion preceding observation (A), we get that h_{m-l+2} and $h'_{m'-l'+2}$ either belong both to $\pi_1 K - \pi_1 A$ or are both equal to t, and that the other of these two posibilities must hold true for $h_{m-l+1}, h_{m-l+3}, h'_{m'-l'+1}, h'_{m'-l'+3}$. This implies that $[x, z]$ is part of the minimizing path $e_0, \ldots, e_l, [x, z], e'_{l'}, \ldots, e'_0$, similarly for the other edges.

'Central' simplices:

We are given vertices $v_0 = g_0 \tilde{w}_0, \ldots, v_n = g_n \tilde{w}_n \in \tilde{M}_0$, $n \geq 2$.

We fix $a_0, \ldots, a_m \in A_0$ and normal forms $g_i g_j^{-1} = h_1^{ij} \ldots h_{m_{ij}}^{ij}$. Hence, we have unique minimizing pathes p_{ij} from v_i to v_j. Then there is at most one **n-dimensional** simplex $\tilde{\tau} \in \tilde{L}$ such that the intersection of $\tilde{\tau}$ with $\cup_{0 \leq i,j \leq n} p_{ij}$ is the 1-skeleton of an **n-dimensional** simplex, i.e., is the full 1-skeleton of $\tilde{\tau}$. This is proved by literally the same argument as in the corresponding part of the proof of lemma 5, to which we refer.

We point out that the edges of $\tilde{\tau}$ are of the form $[h_0 \tilde{w}_0, h_1 \tilde{w}_1]$ with $h_1 h_0^{-1} \in \pi_1 K$. Indeed, the definition of minimizing pathes implies that either $h_1 h_0^{-1} \in \pi_1 K$ or $h_1 h_0^{-1} = t$. But the latter case would contradict the assumption $n \geq 2$, because then $[h_0 \tilde{w}_0, t h_0 \tilde{w}_0]$ would be the only edge in $\tilde{\tau}$ having $h_0 \tilde{w}_0$ as a vertex.

We consider the projection τ' of $\tilde{\tau}$ to L. By construction, the edges of τ' are projections of 1-simplices of K. Assumption (v) implies then that τ' is the projection of some simplex $\tau \in K$.

The resulting simplex in K will be denoted as $\tau\left(\left\{a_l^{ij}\right\}; \left\{h_k^{ij}\right\}\right)$. Literally the same argument as in the proof of lemma 5 shows: if we fix $v_0, \ldots, v_n$, but vary

$a_0, \ldots, a_m \in A_0$ and the normal forms $g_i g_j^{-1} = h_1^{ij} \ldots h_{m_{ij}}^{ij}$,then all $\tau\left(\left\{a_l^{ij}\right\}; \left\{h_k^{ij}\right\}\right)$ belong to the same G-orbit.

Retraction in treelike complex:

Given an n-simplex $\sigma \in M$, we consider a lift $\tilde{\sigma} \in \tilde{M}$ with vertices $v_0, \ldots, v_n$, and we let $r(G\sigma) = G\tau$ if the above construction gives the G-orbit of a simplex τ with $dim(\tau) = n$, otherwise we define $r(G\sigma) = 0$. The definition of r does neither depend on the choice of the lift $\tilde{\sigma}$ nor on the choice of σ in its G-orbit, and it satisfies the two desired conditions.
The same argument as in the proof of lemma 5 shows that $r(G\partial\sigma) = G\partial r(G\sigma)$.
From the assumptions follows $A \subset M'$. It is then clear from the definition that minimizing pathes between points of M' remain in M'. Thus r maps $C_*(M')$ to $C_*(K \cap P^{-1}M') = C_*(K')$. □

In an analogous manner to corollary 5, we conclude

Corollary 4 *Let $(M, M'), (K, K'), (L, L')$ satisfy all assumptions of lemma 9. Denote P the composition of the projection $K \to L$ with the inclusion $L \to M$. Let $\gamma_1 \in H_b^p(K, K')$ be a bounded cohomology class, $p \geq 3$, such that the action of G fixes γ_1. Then exists a class $\gamma \in H_b^p(M, M')$, satisfying $\| \gamma \| \leq \| \gamma_1 \|$, such that $P^*\gamma = \gamma_1$ and that γ is fixed by G.*

3.3 Glueing along amenable boundaries

3.3.1 Dualizing the problem

Lemma 10 *: (i): Let M_1, M_2 be two compact n-manifolds with boundary, A_1, A_2 connected (n-1)-dimensional submanifolds of ∂M_1 resp. ∂M_2, $f : A_1 \to A_2$ a homeomorphism, $M = M_1 \cup_f M_2$ the glued manifold, $A \subset M$ the image of the A_i, and $j_1 : (M_1, \partial M_1) \to (M, \partial M \cup A), j_2 : (M_2, \partial M_2) \to (M, \partial M \cup A)$ the inclusions.*
Assume that the following holds: For all $\gamma_1 \in H_b^n(M_1, \partial M_1), \gamma_2 \in H_b^n(M_2, \partial M_2)$ one can find $\gamma \in H_b^n(M, \partial M \cup A)$ such that $j_1^*\gamma = \gamma_1, j_2^*\gamma = \gamma_2$ and $\| \gamma \| \leq max\{\| \gamma_1 \|, \| \gamma_2 \|\}$.
Then $\| M, \partial M \| \geq \| M_1, \partial M_1 \| + \| M_2, \partial M_2 \|$.
(ii): Let M_1 be a compact n-manifold with boundary, A_1, A_2 disjoint connected (n-1)-dimensional submanifolds of ∂M_1, $f : A_1 \to A_2$ a homeomorphism, $M = M_1/f$ the glued manifold, $A \subset M$ the image of the A_i, and $P : (M_1, \partial M_1) \to (M, \partial M \cup A)$ the canonical projection.
Assume that the following holds: For all $\gamma_1 \in H_b^n(M_1, \partial M_1)$ one can find $\gamma \in$

$H_b^n(M, \partial M \cup A)$ such that $P^*\gamma = \gamma_1$ and $\| \gamma \| \leq \| \gamma_1 \|$.
Then $\| M, \partial M \| \geq \| M_1, \partial M_1 \|$.

Proof: (i): First consider the case that M_1 and M_2 have nontrivial simplicial volume. Then, by 3.1.2., the relative fundamental cocycles have preimages $\beta_1 \in H_b^n(M_1, \partial M_1)$ and $\beta_2 \in H_b^n(M_2, \partial M_2)$. Consider for $i = 1, 2$

$$\gamma_i := \| M_i, \partial M_i \| \beta_i$$

By 3.1.2., we have $\| \gamma_i \| = 1$.

By assumption, we get $\gamma \in H_b^n(M, \partial M \cup A)$ satisfying $\| \gamma \| \leq max\{\| \gamma_1 \|, \| \gamma_2 \|\} = 1$ and $j_1^*\gamma = \gamma_1, j_2^*\gamma = \gamma_2$.

Let $i : (M, \partial M) \to (M, \partial M \cup A)$ be the inclusion. In $H_n(M, \partial M \cup A)$ we have $i_*[M, \partial M] = j_{1*}[M_1, \partial M_1] + j_{2*}[M_2, \partial M_2]$.

Hence,

$$i^*\gamma([M, \partial M]) = \gamma_1([M_1, \partial M_1]) + \gamma_2([M_2, \partial M_2]) = \| M_1, \partial M_1 \| + \| M_2, \partial M_2 \|$$

Thus, $\beta = \frac{1}{\|M_1, \partial M_1\| + \|M_2, \partial M_2\|} i^*\gamma$ is the relative fundamental cocycle of $(M, \partial M)$ and, by $\| \beta \| \leq \frac{1}{\|M_1, \partial M_1\| + \|M_2, \partial M_2\|}$ and duality follows $\| M, \partial M \| \geq \| M_1, \partial M_1 \| + \| M_2, \partial M_2 \|$.

Now consider the case $\| M_1, \partial M_1 \| \neq 0$ and $\| M_2, \partial M_2 \| = 0$. Consider $\gamma_2 = 0$ and again $\gamma_1 = \| M_1, \partial M_1 \| \beta_1 \in H_b^n(M_1, \partial M_1)$. Then we find $\gamma \in H_b^n(M, \partial M \cup A)$ with $j_1^*\gamma = \gamma_1, j_2^*\gamma = 0$ and $\| \gamma \| \leq \| \gamma_1 \|$. The same way as in the first case we get that $\frac{1}{\|M_1, \partial M_1\|} i^*\gamma$ represents β and thus $\| M, \partial M \| \geq \| M_1, \partial M_1 \|$. Finally the case $\| M_1, \partial M_1 \| = \| M_2, \partial M_2 \| = 0$ is trivial anyway.
(ii): We suppose $\| M_1, \partial M_1 \| \neq 0$, since otherwise the claim is trivially true. Then the relative fundamental cocycle has preimage $\beta_1 \in H_b^n(M_1, \partial M_1)$. Consider $\gamma_1 := \| M_1, \partial M_1 \| \beta_1$. By 1.2, $\| \gamma_1 \| = 1$.

We find $\gamma \in H_b^n(M, \partial M \cup A)$ satisfying $\| \gamma \| \leq \| \gamma_1 \| = 1$ and $P^*\gamma = \gamma_1$.

Let $i : (M, \partial M) \to (M, \partial M \cup A)$ be the inclusion. In $H_n(M, \partial M \cup A)$ we have $i_*[M, \partial M] = P_*[M_1, \partial M_1]$. Hence, $i^*\gamma[M, \partial M] = \gamma P_*[M_1, \partial M_1] = P^*\gamma[M_1, \partial M_1] = \gamma_1[M_1, \partial M_1] = \| M_1, \partial M_1 \|$.
Thus, $\beta = \frac{1}{\|M_1, \partial M_1\|} i^*\gamma$ is the relative fundamental class of $(M, \partial M)$ and by $\| \beta \| \leq \frac{1}{\|M_1, \partial M_1\|}$ and duality follows $\| M, \partial M \| \geq \| M_1, \partial M_1 \|$. □

3.3.2 Multicomplexes associated to glueings

The 'amalgamated' case.
We are going to consider the following situation: X_1, X_2 are topological spaces, $A_1 \subset X_1, A_2 \subset X_2$ path-connected subspaces, $f : A_1 \to A_2$ is a homeomorphism such that $f_* : \pi_1 A_1 \to \pi_1 A_2$ restricts to an isomorphism from $ker(\pi_1 A_1 \to \pi_1 X_1)$

to $ker\left(\pi_1 A_2 \rightarrow \pi_1 X_2\right)$. Let $X = X_1 \cup_f X_2$.

The assumption on f_* implies that $\pi_1 X_1, \pi_1 X_2$ inject into $\pi_1 X$. By proposition 3, it follows that
- (i) $K(X_1)$ and $K(X_2)$ are submulticomplexes of $K(X)$.
Concerning the 0-skeleta, we have
- (ii) $K(X)_0 = X = X_1 \cup X_2 = K(X_1)_0 \cup K(X_2)_0$.

Let $A = X_1 \cap X_2$ be the intersection of X_1 and X_2 as subspaces of X. There is an obvious homomorphism $\pi_1 A \rightarrow \pi_1 X$. Hence, there is a map from $K(A)$ to $K(X_1) \cap K(X_2)$, the intersection of $K(X_1)$ and $K(X_2)$ in $K(X)$, which identifies pathes in A whose composition gives an element of $ker\left(\pi_1 A \rightarrow \pi_1 X\right)$. Since this projection kills exactly the kernel of $\pi_1 K(A_i) \rightarrow \pi_1 K(X_i)$ we get that
- (iii) $\pi_1\left(K(X_1) \cap K(X_2)\right)$ injects into $\pi_1 K(X_1)$ resp. $\pi_1 K(X_2)$.
In particular, $\pi_1\left(K(X_1) \cup K(X_2)\right)$ is the amalgamated product of $\pi_1 K(X_1) \simeq \pi_1 X_1$ and $\pi_1 K(X_2) \simeq \pi_1 X_2$, amalgamated over $\pi_1\left(K(X_1) \cap K(X_2)\right) \simeq im\left(\pi_1 A_i \rightarrow \pi_1 X_i\right)$. But this amalgamated product is isomorphic to $\pi_1 X \simeq \pi_1 K(X)$ and we conclude
- (iv) the inclusion $K(X_1) \cup K(X_2) \rightarrow K(X)$ induces an isomorphism of fundamental groups.

Moreover, it follows from proposition 2 that
- (v) $K(X_1)$ and $K(X_2)$ are aspherical,
and we clearly have
- (vi) $K_i' = K(X_i) \cap K' \subset K(X)$ for $i = 1, 2$,
where K_i' is the image of $K(A_i)$ in $K(X_i)$ and K' is the image of $K(A)$ in $K(X)$.

Finally, from injectivity of $\pi_1 X_i \rightarrow \pi_1 X$, one gets that the canonical map $\Pi_{X_i}(A_i) \rightarrow \Pi_X(A)$ is an isomorphism for $i = 1, 2$. Consider the action of $G = \Pi_X(A)$ on $K(X)$.
- (vii) G maps $K(X_i)$ to $K(X_i)$ and K_i' to K_i' for $i = 1, 2$.
As a consequence, the action of $\Pi_X(A)$ on $K(X)$ preserves $K(X_1) \cap K(X_2) \subset K(X)$. Even though $\pi_1\left(X_1 \cap X_2\right)$ may not inject into $\pi_1 X$, analogously to lemma 6, we get:
- (viii) G acts transitively on $\left(K(X_1) \cap K(X_2)\right)$-related tuples of 1-simplices in $K(X)$.

The 'HNN'-case.
Let X_1 be a topological space, A_1, A_2 path-connected subspaces of X_1, $f : A_1 \rightarrow A_2$ a homeomorphism such that f_* restricts to an isomorphism from $ker\left(\pi_1 A_1 \rightarrow \pi_1 X_1\right)$ to $ker\left(\pi_1 A_2 \rightarrow \pi_1 X_1\right)$. Let $X = X_1/_\sim$, where $x_1 \sim x_2$ if $x_1 \in A_1, x_2 \in A_2$ and $x_2 = f(x_1)$.

We have canonical, not necessarily injective, maps from $K(A_1)$ and $K(A_2)$ to $K(X_1)$. For brevity, let us denote $K_i' \subset K(X_1)$ the image of the map from $K(A_i)$, for $i = 1, 2$.
f induces a simplicial isomorphism $F : K_1' \to K_2'$. Let L be the multicomplex obtained from $K(X_1)$ by identifying σ and $F(\sigma)$ for any simplex σ in K_1'.
We have then
(i) a canonical embedding $i_L : L \to K(X)$.
Indeed, the map $K(X_1) \to K(X)$, induced by the projection, factors over L.
One checks easily:
(ii) $K(X)_0 = L_0$,
(iii) $\pi_1 K_1' \to \pi_1 K(X_1)$ and $\pi_1 K_2' \to \pi_1 K(X_1)$ are injective,
(iv) the inclusion $L \to K(X)$ induces an isomorphism of π_1's,
(v) $K(X)$ and L are aspherical in the sense of definition 3.
Let A be the image of A_1 in X and K' the image of $K(A)$ in $K(X)$. Then
(vi) the projection from $K(X_1)$ to L maps K_i' to K'.

Finally, $\pi_1 X$ is an HNN-extension of $\pi_1 X_1$, i.e. $\pi_1 X_1 \to \pi_1 X$ is injective, and one gets that the canonical map $\Pi_{X_i}(A_1) \to \Pi_X(A)$ is an isomorphism for $i = 1, 2$. Consider the action of $G = \Pi_X(A)$ on $K(X)$. We use the isomorphisms $\Pi_{X_1}(A_1) \simeq \Pi_X(A) \simeq \Pi_{X_1}(A_2)$ to get identifications with G and observe that after these identifications the action of G commutes with F. Moreover,
(vii) the action of G commutes with $i_L P$, where $P : K(X_1) \to L$ is the canonical projection.
Similarly to lemma 6, we get
(viii) G acts transitively on K'-related tuples of 1-simplices in $K(X)$.

3.3.3 Proof of Theorem 2

In this section, we prove superadditivity for simplicial volume of manifolds with boundary with respect to glueing along amenable subsets of the boundary. A similar result for open manifolds is the Cutting-off-theorem in [26]. One should note that, at least for manifolds with boundary, the opposite inequality need not hold. As a counterexample one may glue solid tori along disks to get a handlebody.

Lemma 11 *: (i): Let M_1, M_2 be two compact, connected n-manifolds, A_1, A_2 (n-1)-dimensional submanifolds of ∂M_1 resp.∂M_2, $f : A_1 \to A_2$ a homeomorphism and $M = M_1 \cup_f M_2$ the glued manifold.*
If f_* maps $ker(\pi_1 A_1 \to \pi_1 M_1)$ isomorphically to $ker(\pi_1 A_2 \to \pi_1 M_2)$, and if $im(\pi_1 A_1 \to \pi_1 M_1)$ is amenable, then $\| M, \partial M \| \geq \| M_1, \partial M_1 \| + \| M_2, \partial M_2 \|$.
(ii): Let M_1 be a compact, connected n-manifold, no component of which is a 1-dimensional closed intervall, A_1, A_2 disjoint (n-1)-dimensional submanifolds of

∂M_1, $f : A_1 \to A_2$ a homeomorphism and $M = M_1/f$ the glued manifold.
If $im\,(\pi_1 A_1 \to \pi_1 M_1)$ is amenable, and $f_* : ker\,(\pi_1 A_1 \to \pi_1 M_1) \to ker\,(\pi_1 A_2 \to \pi_1 M_1)$ is an isomorphism, then $\| M, \partial M \| \geq \| M_1, \partial M_1 \|$.

Proof: (i): For manifolds of dimensions ≤ 2 one checks easily that there is no counterexample. So we are going to assume that $n \geq 3$.
We want to check the assumption of lemma 10. We can restrict to the case that A_1 and A_2 are path-connected, since we may argue succesively for their path-connected components.

First, we make the **restrictive assumption** that $\pi_1 \partial M_i$ **and** $\pi_1 A_i$ **should inject into** $\pi_1 M_i$ for $i = 1, 2$. We will show afterwards how to handle the general case. The advantage of this assumption is that, by proposition 3, we may assume $K\,(\partial M_i)$ to be a submulticomplex of $K\,(M_i)$ and $K\,(\partial M \cup A)$ to be a submulticomplex of $K\,(M)$. Denoting by j_1, j_2, k_1, k_2 the embeddings and by I, I_1, I_2 the isometric isomorphisms from prop. 2(ii), we claim that the following diagram commutes.

$$\begin{array}{ccc}
\bigoplus_{i=1}^{2} H_b^n\left(M_i, \partial M_i\right) & \xleftarrow{\;j_1^* \oplus j_2^*\;} & H_b^n\left(M, \partial M \cup A\right) \\
\Big\uparrow{\scriptstyle I_1 \oplus I_2} & & \Big\uparrow{\scriptstyle I} \\
\bigoplus_{i=1}^{2} H_b^n\left(K\left(M_i\right), K\left(\partial M_i\right)\right) & \xleftarrow{\;k_1^* \oplus k_2^*\;} & H_b^n\left(K\left(M\right), K\left(\partial M \cup A\right)\right)
\end{array}$$

To see that the diagram commutes, recall that I was constructed as a composition $I = S^* h^* p^*$. S^* was induced by an embedding $S : X \to \tilde{K}\,(X)$, hence $S_i^* j_i^* = j^* S^*$ follows from the obvious embedding relation $j_i S_i = S j$. Also, $h_i^* j_i^* = j^* h^*$ follows from the fact that h^* is induced by an inclusion of chain complexes. Finally, p^* is induced by the projection $p : \tilde{K}\,(X) \to K\,(X)$ described in 3.1.3. which clearly commutes with the inclusions coming from proposition 3.

Consider the following commutative diagram:

$$\begin{array}{ccc}
\bigoplus_{i=1}^{2} H_b^n\left(K\left(M_i\right), K\left(\partial M_i\right)\right) & \xleftarrow{\;k_1^* \oplus k_2^*\;} & H_b^n\left(K\left(M\right), K\left(\partial M \cup A\right)\right) \\
\Big\uparrow{\scriptstyle p_1 \oplus p_2} & & \Big\uparrow{\scriptstyle p} \\
\bigoplus_{i=1}^{2} H_b^n\left(K\left(M_i\right)^{\Pi_{M_i}(A_i)}, K\left(\partial M_i\right)^{\Pi_{M_i}(A_i)}\right) & \xleftarrow{\;k_1^* \oplus k_2^*\;} & H_b^n\left(K\left(M\right)^{\Pi_M(A)}, K\left(\partial M \cup A\right)^{\Pi_M(A)}\right)
\end{array}$$

By lemma 6, $\Pi_{M_1}(A_1), \Pi_{M_2}(A_2), \Pi_M(A)$ are amenable. Hence, by lemma 5(i), p_1, p_2, p have left inverses Av_1, Av_2, Av of norm = 1. For i=1,2 define

$$\gamma_i' := Av_i (I_i)^{-1} \gamma_i \in H_b^n \left(K(M_i)^{\Pi_{M_i}(A_i)}, K(\partial M_i)^{\Pi_{M_i}(A_i)} \right).$$

They satisfy $\| \gamma_i' \| = \| \gamma_i \|$.
It follows from the discussion in 3.3.2. that we can apply lemma 8 and corollary 5. Hence, we get

$$\gamma' \in H_b^n \left(K(M)^{\Pi_M(A)}, K_A(\partial M)^{\Pi_M(A)} \right),$$

satisfying $\| \gamma' \| \leq max \{ \| \gamma_1' \|, \| \gamma_2' \| \} = max \{ \| \gamma_1 \|, \| \gamma_2 \| \}$ and
$k_i^* \gamma' = \gamma_i'$ for i=1,2.
Let

$$\gamma := Ip^* \gamma' \in H_b^* (M, \partial M)$$

It satisfies $\| \gamma \| = \| \gamma' \| \leq max \{ \| \gamma_1 \|, \| \gamma_2 \| \} = 1$ and
$j_1^* \gamma = j_1^* I p^* \gamma' = I_1 k_1^* p^* \gamma' = I_1 p_1^* k_1^* \gamma' = I_1 p_1^* Av_1 (h_1^*)^{-1} \gamma_1$.

It is easy to see that $\Pi_{M_i}(A_i)$ are connected and that the actions of $\Pi_{M_i}(A_i)$ on $K(M_i)$ are continuous. Hence, all elements of $\Pi_{M_i}(A_i)$ are, as mappings from $K(M_i)$ to itself, homotopic to the identity. From part (ii) of lemma 5, we get that $p_i^* \circ Av_i = id$. Hence, we obtain $j_1^* \gamma = \gamma_1$. The same way, $j_2^* \gamma = \gamma_2$.

Thus, we have checked the assumptions of lemma 10.

We are now going to consider the **general case**, i.e., we do not assume any longer injectivity of fundamental groups.

Let $Y \subset X$ with $ker(\pi_1 Y \to \pi_1 X) \neq 0$. In this case, $\tilde{K}(Y)$ doesn't embed into $\tilde{K}(X)$. However, $\tilde{K}(.)$ is clearly functorial in the sense that continuous mappings $Y \to X$ induce simplicial maps $\tilde{K}(Y) \to \tilde{K}(X)$. In particular, the embedding induces simplicial maps $\tilde{f} : \tilde{K}(Y) \to \tilde{K}(X)$ and $f : K(Y) \to K(X)$. We consider its image $f(K(Y))$ as a submulticomplex of $K(X)$.

We want to show that there are maps

$$H : H_b^n (K(X), f(K(Y))) \to H_b^n (X, Y)$$

$$\overline{H} : H_b^n (X, Y) \to H_b^n (K(X), f(K(Y)))$$

of norm ≤ 1. (They are not isomorphisms, even though we will have $H\overline{H} = id$.)

To this behalf, we explain more in detail the construction of $I^{-1} = Av \circ h^* \circ S^*$ in proposition 2.
S^* is induced by a weak homotopy equivalence $S : \tilde{K}_X \to X$, which is homotopy inverse to the inclusion j. From the definition of S on [26], p.42, it is clear that S maps $\tilde{f}\left(\tilde{K}_Y\right)$ to Y. Hence S induces a map S^* of norm ≤ 1 from $H_b^n(X, Y)$

to $H_b^n\left(\mid \tilde{K}_X \mid, \mid \tilde{f}\left(\tilde{K}_Y\right) \mid\right)$.
The isomorphism $Av : H_b^n\left(\tilde{K}_X\right) \rightarrow H_b^n\left(K\left(X\right)\right)$ is induced by averaging over the amenable group Γ_1/Γ_n, where Γ_i is the group of simplicial automorphisms which are the identity on the i-skeleton, cf. [26], p.46. We get a map $Av : H_b^n\left(\tilde{K}_X, \tilde{f}\left(\tilde{K}_Y\right)\right) \rightarrow H_b^n\left(K_X, f\left(K_Y\right)\right)$ which has norm ≤ 1 by definition of the averaging.
Finally, by proposition 1 we get an isometry $h^* : H_b^n\left(\mid \tilde{K}_X \mid, \mid \tilde{f}\left(\tilde{K}_Y\right) \mid\right) \rightarrow H_b^n\left(\tilde{K}_X, \tilde{f}\left(\tilde{K}_Y\right)\right)$. Hence, we may define $\overline{H} = Av \circ h^* \circ S^*$ and $H = j^* \circ \left(h^*\right)^{-1} \circ p^*$.

We will call H_1, H_2, H and $\overline{H_1}, \overline{H_2}, \overline{H}$ the maps corresponding to $(X, Y) = \left(M_1, \partial M_1\right), \left(M_2, \partial M_2\right)$ resp. $\left(M, \partial M \cup A\right)$.

The action of $\Pi_{A_i}\left(M_i\right)$ on $K\left(M_i\right)$ preserves $f\left(K\left(\partial M_i\right)\right)$. We get the commutative diagram

$$\begin{array}{ccc}
\bigoplus_{i=1}^{2} H_b^n\left(M_i, \partial M_i\right) & \xleftarrow{j_1^* \oplus j_2^*} & H_b^n\left(M, \partial M \cup A\right) \\
\downarrow{\scriptstyle \overline{H_1} \oplus \overline{H_2}} & & \uparrow{\scriptstyle H} \\
\bigoplus_{i=1}^{2} H_b^n\left(K\left(M_i\right), f\left(K\left(\partial M_i\right)\right)\right) & \xleftarrow{k_1^* \oplus k_2^*} & H_b^n\left(K\left(M\right), f\left(K\left(\partial M \cup A\right)\right)\right) \\
\uparrow{\scriptstyle p_1^* \oplus p_2^*} & & \uparrow{\scriptstyle p^*} \\
\bigoplus_{i=1}^{2} H_b^n\left(K\left(M_i\right)^{\Pi_{M_i}(A_i)}, K\left(\partial M_i\right)^{\Pi_{M_i}(A_i)}\right) & \xleftarrow{k_1^* \oplus k_2^*} & H_b^n\left(K\left(M\right)^{\Pi_M(A)}, K\left(\partial M \cup A\right)^{\Pi_M(A)}\right).
\end{array}$$

One checks easily that all arguments in the first part of the proof go through, finishing the proof of part (i).

(ii): In dimensions ≤ 2 we check that the closed interval is the only connected counterexample. Assume then $n \geq 3$. Again we may suppose A_1, A_2 connected. We will again assume that $\pi_1\left(\partial M_1\right) \rightarrow \pi_1\left(M_1\right)$ and $\pi_1\left(\partial M \cup A\right) \rightarrow \pi_1\left(M\right)$ are injective. The generalisation to the case of compressible boundary follows then by arguments completely analogous to those in the proof of part (i).
Like in part (i), we get a commutative diagram, where P, Q, R are the obvious

projections.

$$\begin{array}{ccc}
H_b^n\left(M_1,\partial M_1\right) & \xleftarrow{P^*} & H_b^n\left(M,\partial M\cup A\right)\\
\uparrow I_1 & & \uparrow I\\
H_b^n\left(K\left(M_1\right),K\left(\partial M_1\right)\right) & \xleftarrow{R^*} & H_b^n\left(K\left(M\right),K\left(\partial M\cup A\right)\right)\\
\uparrow p_1 & & \uparrow p\\
H_b^n\left(K\left(M_1\right)^{\Pi_{M_1}(A_1\cup A_2)},K\left(\partial M_1\right)^{\Pi_{M_1}(A_1\cup A_2)}\right) & \xleftarrow{Q^*} & H_b^n\left(K\left(M\right)^{\Pi_M(A)},K\left(\partial M\right)^{\Pi_M(A)}\right)
\end{array}$$

Given $\gamma_1 \in H_b^*\left(M_1,\partial M_1\right)$, define

$$\gamma_1' := Av_1 I_1^{-1}\gamma_1 \in H_b^*\left(\left(K\left(M_1\right)^{\Pi_{M_1}(A_1\cup A_2)},K\left(\partial M_1\right)^{\Pi_{M_1}(A_1\cup A_2)}\right)\right).$$

We check that the assumptions of corollary 6 are satisfied and get

$$\gamma' \in H_b^n\left(K\left(M\right)^{\Pi_M(A)},K\left(\partial M\right)^{\Pi_M(A)}\right).$$

Then define $\gamma := Ip\gamma' \in H_b^n\left(M,\partial M\cup A\right)$. Analogously to the proof of part (i) we get that $P^*\gamma = \gamma_1$ and $\parallel \gamma \parallel \leq \parallel \gamma_1 \parallel$ holds.

Thus, we can apply lemma 10 to finish the proof. □

Lemma 12 *:*
i) Let M_1, M_2 be compact manifolds, A_1 resp. A_2 connected components of ∂M_1 resp. ∂M_2 and assume that there exist connected sets $A_i' \subset M_i$ with $A_i' \supset A_i$ and $\pi_1 A_i'$ amenable. Let $f : A_1 \to A_2$ be a homeomorphism and $M = M_1 \cup M_2/f$ the glued manifold. Then $\parallel M,\partial M \parallel \leq \parallel M_1,\partial M_1 \parallel + \parallel M_2,\partial M_2 \parallel$.
ii) Let M' be a compact manifold, A_1, A_2 connected components of $\partial M'$ with $\pi_1 A_i$ amenable. Let $f : A_1 \to A_2$ be a homeomorphism and $M = M'/f$ the glued manifold. Then $\parallel M,\partial M \parallel \leq \parallel M',\partial M' \parallel$.

Proof: ii) is reduced to i) via the homeomorphism $M = M' \cup_{(id,0)+(f,1)} (A_1 \times I)$. (Note that $\parallel A_1 \times I, A_1 \times \{0,1\} \parallel = 0$, since $\pi_1 A_1$ is amenable.)

To prove i), we need the following reformulation of a theorem of Matsumoto-Morita. For a space X and $q \in N$ let $C_q(X)$ be the group of singular chains and $B_q(X)$ the subgroup of boundaries. By theorem 2.8. of [43] the following two statements are equivalent:
a) there exists a number $K > 0$ such that for any boundary $z \in B_q(X)$ there is a

chain $c \in C_{q+1}(X)$ satisfying $\partial c = z$ and $\parallel c \parallel < K \parallel z \parallel$,
b) the homomorphism $H_b^{q+1}(X) \rightarrow H^{q+1}(X)$ is injective.

Now let $\sum_{i=1}^m a_i \sigma_i$ and $\sum_{j=1}^n b_j \tau_j$ be representatives of $[M_1, \partial M_1]$ and $[M_2, \partial M_2]$ with

$$\sum_{i=1}^m \mid a_i \mid \leq \parallel M_1, \partial M_1 \parallel + \epsilon$$

and

$$\sum_{j=1}^n \mid b_j \mid \leq \parallel M_2, \partial M_2 \parallel + \epsilon.$$

By proposition 4 we may suppose that $\partial \left(\sum_{i=1}^m a_i \sigma_i\right) \mid_{A_1} \in C_*(\partial M_1)$ and $\partial \left(\sum_{j=1}^n b_j \tau_j\right) \mid_{A_2} \in C_*(\partial M_2)$ have norm smaller than $\frac{\epsilon}{2K}$. (Note that $\pi_1 A_i \rightarrow \pi_1 M_i$ factors over $\pi_1 A_i'$, hence has amenable image.)

Let A' be the image of A_1' in M. As $\pi_1 A'$ is amenable, $H_b^{q+1}(A') = 0$ for $q \geq 0$ ([26],[35]), hence, $H_b^{q+1}(A') \rightarrow H^{q+1}(A_i')$ is clearly injective and we get a constant K with the property in a).
Therefore, we find $c \in C_*(A') \subset C_*(M)$ with $\parallel c \parallel \leq \epsilon$ and

$$\partial c = \partial \left(\sum_{i=1}^m a_i \sigma_i + \sum_{j=1}^n b_j \tau_j \right).$$

Then $z = \sum_{i=1}^m a_i \sigma_i + \sum_{j=1}^n b_j \tau_j - c \in C_*(M_1 \cup_f M_2)$ is a fundamental cycle of norm smaller than $\parallel M_1, \partial M_1 \parallel + \parallel M_2, \partial M_2 \parallel + 3\epsilon$. □

Remark: The assumption of lemma 12 is in particular satisfied if $im(\pi_1 \partial M_1 \rightarrow \pi_1 M_1)$ and $im(\pi_1 \partial M_2 \rightarrow \pi_1 M_1)$ are amenable and the (singular) compression disks can be chosen to be disjoint. For 3-manifolds M_i, by a theorem of Jaco, cf. [4], there is $A_i' \subset M_i$ with $\pi_1 A_i' = im(\pi_1 A_i \rightarrow \pi_1 M_i)$ if $im(\pi_1 A_i \rightarrow \pi_1 M_i)$ is finitely presented.

Theorem 2 *:*
(i):Let M_1, M_2 be two compact n-manifolds, A_1 resp.A_2 connected components of ∂M_1 resp.∂M_2, $f : A_1 \rightarrow A_2$ a homeomorphism, $M = M_1 \cup_f M_2$ the glued manifold.
If $\pi_1 A_1$ and $\pi_1 A_2$ are amenable
and $f_* : ker(\pi_1 A_1 \rightarrow \pi_1 M_1) \rightarrow ker(\pi_1 A_2 \rightarrow \pi_1 M_2)$ is an isomorphism,
then $\parallel M, \partial M \parallel = \parallel M_1, \partial M_1 \parallel + \parallel M_2, \partial M_2 \parallel$.
(ii): Let M_1 be a compact n-manifold, no connected component of M_1 a 1-dimensional closed intervall, A_1, A_2 connected components of ∂M_1, $f : A_1 \rightarrow A_2$ a homeomorphism, $M = M_1/f$ the glued manifold.

If $\pi_1 A_1$ is amenable and $f_* : ker\,(\pi_1 A_1 \rightarrow \pi_1 M_1) \rightarrow ker\,(\pi_1 A_2 \rightarrow \pi_1 M_1)$ is an isomorphism, then $\parallel M, \partial M \parallel = \parallel M_1, \partial M_1 \parallel$.

Proof: Theorem 2 follows from lemma 11 and 12. □

Corollary 5 *: (i) Let A be a properly embedded annulus in a compact 3-manifold M. If $im\,(\pi_1 A_1 \rightarrow \pi_1 M_A) \approx im\,(\pi_1 A_2 \rightarrow \pi_1 M_A)$ for the two images A_1, A_2 of A in M_A, then $\parallel M_A, \partial M_A \parallel \leq \parallel M, \partial M \parallel$.*
(ii) Let T be an embedded torus in a compact 3-manifold M. If $im\,(\pi_1 T_1 \rightarrow \pi_1 M_T) \approx im\,(\pi_1 T_2 \rightarrow \pi_1 M_T)$ for the two images T_1, T_2 of T in M_T, then
$\parallel M_T, \partial M_T \parallel = \parallel M, \partial M \parallel$.

Remark: If ∂M consists of tori and A is an incompressible annulus, then even $\parallel M_A, \partial M_A \parallel = \parallel M \parallel$ holds by a theorem of [57].

In [57], a version of corollary 7 has been proved for the special case that ∂M consists of tori. The principal ingredient in the proof is the following statement:
Proposition 5 ([57], Lemma 1): *Let M be a compact 3-manifold whose boundary ∂M consists of tori and H be a 3-dimensional compact submanifold of $int\,(M)$. Suppose $int\,(H)$ is a hyperbolic 3-manifold and ∂H is incompressible in M. Then we have $\parallel M, \partial M \parallel \geq \parallel H, \partial H \parallel$.*
In [61], this proposition is stated for closed M as theorem 6.5.5. without writing a proof. In [57], it is then derived for M with toral boundary, using the doubling argument, see the proof of lemma 1 in [57].
Hence, our proof seems to be the first written proof of corollary 7 and proposition 5. (It is easy to see that corollary 7 implies proposition 5.)
We want to mention that, according to Agol, an alternative proof of proposition 5 (hence, corollary 7) should be possible using the methods of [1].
We refer to [57] to see that corollary 7 actually allows to compute the simplicial volumes of all Haken 3-manifolds with (possibly empty) toral boundary.

3.3.4 Counterexamples

The first example shows that the condition $ker\,(\pi_1 A_1 \rightarrow \pi_1 M_1) \approx ker\,(\pi_1 A_2 \rightarrow \pi_1 M_2)$ in lemma 11 can not be weakened.

Example 1: *Dehn fillings*
Let K be a knot in S^3 such that $S^3 - K$ admits a hyperbolic metric of finite volume. Let $V \subset S^3$ be a regular neighborhood of K. ∂V is a torus, hence, $\pi_1 \partial V$ is amenable. But

$$\parallel S^3 \parallel = 0 < \parallel S^3 - V, \partial V \parallel + \parallel V, \partial V \parallel .$$

More generally, it is known([1]) that

$$\| M, \partial M \| < \| M - V, \partial V \cup \partial M \| = \| M - V, \partial V \cup \partial M \| + \| V, \partial V \|$$

holds if both $int\,(M)$ and $int\,(M-V)$ admit a hyperbolic metric of finite volume, i.e., if M is obtained by performing hyperbolic Dehn filling at $M - V$.
Lemma 11 does not apply because the meridian of ∂V maps to zero in $\pi_1 V$, but it doesn't so in $\pi_1\,(S^3 - V)$ resp. $\pi_1\,(M-V)$.

The second example shows that the assumption 'A_i connected' in lemma 12 can not be weakened to assume only $A_i \subset A_i'$. (In particular, one can not just glue along an amenable subset $A_i \subset \partial M_i$.)

Example 2: *Heegard splittings*
Any 3-manifold can be decomposed into two handlebodies H and H', to be identified along their boundaries. Let g be the genus of H and H' and consider a set of properly embedded disks $D_1, \ldots, D_g \subset H'$ such that $H' - \cup_{i=1}^{g} V_i$ is a 3-ball B, where V_i are disjoint open regular neighborhoods of D_i. Denote $A_i = V_i \cap \partial H'$.
M is then obtained as follows: $V_1, \ldots, V_g$ are glued to H along the annuli A_i, afterwards B is glued along its whole boundary.
Of course, $\| V_i, \partial V_i \| = 0$ and $\| B, \partial B \| = 0$. Thus, if lemma 12 were applicable to the annuli A_i, we would get that $\| M, \partial M \| < \| H_g, \partial H_g \|$.
But there are 3-manifolds of arbitrarily large simplicial volume which admit Heegard splittings of a given genus. To give an explicit example, let f be a pseudo-Anosov diffeomorphism on a surface of genus g, and let M_n be the mapping tori of the iterates f^n. By Thurstons hyperbolization theorem, M_1 is hyperbolic. Hence, $\| M_1 \| > 0$ and $\| M_n \| = n \| M_1 \|$ becomes arbitrarily large. On the other hand, all M_n admit a Heegardsplitting of genus 2g+1.

Chapter 4

Fundamental cycles of hyperbolic manifolds

The simplicial volume of finite-volume hyperbolic manifolds can be calculated by the Gromov-Thurston theorem.

Proposition 5 *If the interior of a manifold M admits a (complete) hyperbolic metric of finite volume, then $\| M, \partial M \| = \frac{1}{V_n} Vol(M)$. Here, V_n is the volume of a regular ideal simplex in H^n.*

However, there does not exist a fundamental cycle with l^1-norm equal to $\frac{1}{V_n} Vol(M)$, i.e., realizing the infimum of the l^1-norm over all cycles representing the fundamental class. To construct a "cycle" which has exactly this norm, one has to:
- admit cycles in the measure homology,
- admit ideal simplices.

In this setting, Gromov constructed a (signed) measure cycle $smr := \frac{1}{2V_n}(\mu^+ - \mu^-)$, the so-called smearing cycle, where μ^+ and μ^- are the equidistributions on the set of positively resp. negatively oriented regular ideal simplices. (Precisely, the set of ordered regular ideal simplices has to be identified with $Isom(H^n) = Isom^+(H^n) \cup Isom^-(H^n)$, and μ^+ corresponds after this identification to the Haar measure $Haar$, whereas μ^- corresponds to $r^* Haar$, where r is an orientation-reversing isometry.) This measure cycle has "l^1-norm" (total variation) equal to $\frac{1}{V_n} Vol(M)$.

It is not hard to approximate smr by measure cycles on authentic (non-ideal) simplices: the set of all regular simplices of fixed edgelength R can be identified with $Isom(H^n)$, and then consider $\frac{1}{2V_n}(Haar - r^* Haar)$ after this identification. For $R \to \infty$, we approach smr.

It seems, however, not to be proved, that measure homology is isometric to singular homology. Hence, to prove the Gromov-Thurston theorem, one has to approximate *smr* by authentic singular chains, i.e., finite linear combinations of (nonideal) simplices. This was done in [26], a detailled proof can be found in [6].

Technically, the main part of this chapter is devoted to the question whether there exist sequences of fundamental cycles with l^1-norms converging to $\frac{1}{V_n}Vol\,(N)$ which do not approximate Gromov's smearing cycle.

In dimension 2, it is actually easy to see that there are very many different possible limits of such sequences. This is not the case in dimensions ≥ 3. For closed manifolds of dimension ≥ 3, it was shown in [36] by Jungreis that *any* such sequence must converge to Gromov's smearing cycle. For finite-volume manifolds, there are slightly more possibilities, e.g., finite covers of the Gieseking manifold can be triangulated by ideal simplices of volume V_3 but, as a result of our analysis, we will also obtain severe restrictions on the possible limits for finite-volume manifolds of dimensions ≥ 3. The reason behind this dichotomy between dimension 2 and dimensions ≥ 3 is the elementary fact that in hyperbolic space of dimensions $n \geq 3$, a regular ideal (n-1)-simplex is the boundary face of only two regular ideal n-simplices.

4.1 Preliminaries

This section is organized as follows. In subsections 4.1.1, 4.1.2 and 4.1.3 we just collect definitions and facts needed later. Subsection 4.1.4 is of some importance: there we discuss the exact setting in which we will discuss convergence of fundamental cycles and give some motivation why this is the right setting to get meaningful results. Subsection 4.1.5 just explains the proof of the Gromov-Thurston theorem (for closed manifolds) and thereby introduces some notation. The actual proof of theorem 3 will be given in sections 4.2 up to 4.4.

4.1.1 Hyperbolic manifolds

Hyperbolic geometry

We recall some basic facts ([39], [6], [4]).

Hyperbolic space. We consider the Poincare model of the n-dimensional hyperbolic space H^n. This is the open unit ball $D^n := \{x \in R^n : d_{Eucl}\,(x,0) < 1\}$ with the Riemannian metric

$$g\,(v,w) := \frac{4}{\left(1 - d_{Eucl}\,(x,0)^2\right)^2} g_{Eucl}\,(v,w)$$

for all $v, w \in T_x D^n$, where g_{Eucl} denotes the Euclidean scalar product on $T_x D^n$. We denote $Isom\,(H^n)$ the group of isometries of H^n, $Isom^+\,(H^n)$ the subgroup of orientation-preserving isometries. There is a unique geodesic between any two points of H^n. Let $d(x,y)$ be the length of the geodesic from $x \in H^n$ to $y \in H^n$. d defines a metric on H^n. $g : [a,b] \to H^n$ is a geodesic if and only if it is subset of an Euclidean circle orthogonal to $S^{n-1} = \{x \in R^n : d_{Eucl}(x,0) = 1\}$. At some point (the definition of straightening of simplices), it will be more convenient to work with the projective model of H^n: it is D^n with the Riemannian metric g defined by the condition that the map $\phi : (D^n, Eucl) \to (D^n, g)$ defined by

$$\phi(x) := x\left(\frac{2d_{Eucl}(x,0)}{[d_{Eucl}(x,0)]^2 + 1}\right)$$

is an isometry. (D^n, g) is isometric to H^n, and it has the convenient property that its geodesics are exactly the geodesics of $(D^n, Eucl)$.

Ideal boundary. A geodesic $g : [0,\omega) \to H^n$ is called a geodesic ray if $lim_{t\to\infty} g(t)$ doesn't exist. Two geodesic rays g_1 and g_2 are said to be equivalent if there exists some constant C such that to any point $x \in g_1$ there is some $y \in g_2$ with $d(x,y) < C$ and vice versa. The set of equivalence classes is denoted by $\partial_\infty H^n$, it is called the ideal boundary of H^n. In other words, each equivalence class of geodesic rays is a point in $\partial_\infty H^n$. The union $\overline{H^n} := H^n \cup \partial_\infty H^n$ is given a topology such that H^n is open and inherits its own topology, and neighborhoods of $p \in \partial_\infty H^n$ are obtained in the following way: choose g in the class p, V a neighborhood of $g'(0)$ in the unit sphere of $T_{g(0)}H^n$ and $r > 0$. The sets

$$U(g,V,r) := \{g_1(t) : g_1 \text{ geodesic ray}, g_1(0) = g(0), g_1'(0) \in V, t > r\} \cup$$

$$\cup \{p \in \partial_\infty H^n : p \text{ is represented by a geodesic ray } g_1 \text{ with } g_1(0) = g(0), g_1'(0) \in V\}$$

for varying g, V, r form a fundaental system of neighborhoods of p.

Any two points $x, y \in \overline{H^n}$ can be joined by a unique geodesic. In both models of hyperbolic space, $S^{n-1} := \{x \in R^n : d_{Eucl}(x,0) = 1\}$ can in a canonical way be identified with $\partial_\infty H^n$, and this identification is a homeomorphism. Moreover, $\overline{D^n} := \{x \in R^n : d_{Eucl}(x,0) \leq 1\}$ is homeomorphic to $\overline{H^n}$.

An ideal simplex is a geodesic simplex with vertices in $\partial_\infty H^n$.

Straight simplices. We use the projective model of H^n to define what the straight simplex with vertices $v_o, \ldots, v_i \in \overline{H^n}$ is: it is the singular simplex $\tau : \Delta_i \to H^n$ defined by $\tau\left(\sum_{j=0}^i x_j e_j\right) := \sum_{j=0}^i x_j v_j$, where $e_0, \ldots, e_i$ are the vertieces of the standard simplex Δ^i. Note that all faces of τ are geodesic faces.

Isometry group. We denote $Isom(H^n)$ the group of isometries of H^n, $Isom^+(H^n)$ the subgroup of orientation-preserving isometries.

The Iwasawa decomposition $G = KAN$ of $G = Isom^+(H^n)$ can be constructed as follows: fix some $v_\infty \in \partial_\infty H^n$ and some $p \in H^n$. Then we may take K to be the group of orientation-preserving isometries fixing p, A the group of translations along the geodesic through p and v_∞, and N the group of translations along the horosphere through p and v_∞.

Hyperbolic manifolds

We call a manifold M hyperbolic if it is homeomorphic to $\Gamma\backslash H^n$ for a discrete, torsion-free subgroup $\Gamma \subset Isom\left(H^n\right)$. This is equivalent to the condition that M admits a complete metric of sectional curvature constantly -1. (If M is orientable, we actually have $\Gamma \subset Isom^+\left(H^n\right)$.) In chapter 5, we will also talk about incomplete hyperbolic manifolds with totally geodesic boundary. We give the definition of this notion in section 5.1.

For a Riemannian manifold N, and $a, b \in R \cup \infty$, one defines $N_{[a,b]} := \{x \in N : a \leq inj\left(x\right) \leq b\}$.
It follows from the Margulis lemma, see chapter D of [6], that for a finite-volume hyperbolic manifold N there exists some ϵ_0 s.t. one has a homeomorphism $h_\epsilon : (N, \partial N) \rightarrow \left(N_{[\epsilon,\infty]}, \partial N_{[\epsilon,\infty]}\right)$ for any $\epsilon < \epsilon_0$.
Moreover, for all $\epsilon < \epsilon_0$ one has that $N_{[0,\epsilon]}$ is convex in the following sense: if $\kappa : \Delta^i \rightarrow \partial N_{[\epsilon,\infty]} \subset N$ is a singular simplex, then $str\left(\kappa\right)$ maps Δ^i to $N_{[0,\epsilon]}$.

Note: if $n = dim\left(N\right)$, then $H_n\left(N, N_{[0,\epsilon]}; R\right) \approx R$ (for any ϵ such that $N_{[0,\epsilon]}$ is not empty). In fact, the isomorphism is induced by the map $algvol : C_n\left(N, N_{[0,\epsilon]}\right) \rightarrow R$, where $algvol\left(\sigma\right)$ for a singular simplex σ is its algebraic volume w.r.t the hyperbolic metric, i.e. the integral of $\sigma^* dvol$ over the standard simplex.
We will need the following fact: If $vol\left(N\right) < \infty$, then $\lim_{\epsilon\rightarrow 0} Vol\left(N_{[0,\epsilon]}\right) = 0$.

Volume of simplices

Definition 8 *: For a hyperbolic manifold N, denote $S_\infty^{reg}\left(N\right)$ the set of ordered regular ideal simplices in N, equipped with the well-defined action of $Isom\left(H^n\right)$.*

To see that the $Isom\left(H^n\right)$-action is well-defined, note that $\sigma \in S_\infty\left(N\right)$ has lifts $\gamma\tilde{\sigma} \in S_\infty\left(H^n\right)$ with $\gamma \in \pi_1 N$ and some fixed lift $\tilde{\sigma}$ and that, for $g \in Isom\left(H^n\right)$, σg can be defined as the projection of $\gamma\tilde{\sigma}g$ to N, which does not depend on γ.

Given two regular ideal n-simplices Δ_0 and Δ in H^n, with fixed orderings of their vertices, there is a unique $g \in Isom\left(H^n\right)$ mapping Δ_0 to Δ.
Hence, fixing a reference simplex Δ_0, we have an $Isom\left(H^n\right)$-equivariant bijection

$$\tilde{I} : S_\infty^{reg}\left(H^n\right) \rightarrow Isom\left(H^n\right)$$

between the set of ordered regular ideal n-simplices and $Isom\left(H^n\right)$, this bijection being unique up to the choice of Δ_0, i.e., up to multiplication with a fixed element of $Isom\left(H^n\right)$.

As another consequence, all regular ideal n-simplices in H^n have the same volume, to be denoted V_n.

By [29], any straight n-simplex σ in H^n satisfies $Vol(\sigma) \leq V_n$ and equality is achieved only for regular ideal simplices, i.e., $\sigma \in S_\infty^{reg}$.

If $N = \Gamma \backslash H^n$ is a hyperbolic manifold, $\tilde{I}$ descends to an $Isom(H^n)$-equivariant bijection

$$I : S_\infty^{reg}(N) \to \Gamma \backslash Isom(H^n)$$

between the set of ordered regular ideal n-simplices and $\Gamma \backslash Isom(H^n)$, this bijection being unique up to the choice of the reference simplex Δ_0, i.e., up to multiplication with a fixed element of $Isom(H^n)$.

4.1.2 Ergodic theory

Unipotent actions

Let G be a simple Lie group. (The only examples we need are $Isom^+(H^n)$.) It is well-known that G can be decomposed as KAN, for a compact group K, an abelian group A and a nilpotent group N. (This means that K, A, N are subgroups of G, and each $g \in G$ uniquely decomposes as $g = kan$ with $k \in K, a \in A, n \in N$.) We have given an explicit dexcription of this Iwasawa decomposition for $G = Isom^+(H^n)$ in 4.1.1.

Given a simple Lie group G with an Iwasawa decomposition $G = KAN$, there is a right hand action of N on G, defined by

$(kan)\, n' := ka\,(nn')$ for $k \in K, a \in A, n, n' \in N$.

The next lemma follows from [13]. It is nowadays a special case of the Ragunathan conjecture, which was proved by Ratner. (We will only need $G = Isom^+(H^n)$).

Lemma 13 : *Let G=KAN be the Iwasawa decomposition of a simple Lie group of R-rank 1, and $\Gamma \subset G$ a discrete subgroup of finite covolume. If μ is a finite N-invariant ergodic measure on $\Gamma \backslash G$, then μ is either a multiple of the Haar measure or it is determined on a compact N-orbit.*

For completeness, we give the proof of the following lemma, which is similar to theorem 4.4. of [14]:

Lemma 14 : *Let G=KAN be the Iwasawa decomposition of a simple Lie group of R-rank 1, and $\Gamma \subset G$ a discrete subgroup of finite covolume. Let $N' \subset N$ be a subgroup such that N/N' is compact. Then any N'-invariant ergodic measure on $\Gamma \backslash G$ is either a multiple of the Haar measure or is determined on a compact N-orbit.*

Proof: By Moore-equivalence, ergodic measures for the N'-action on $\Gamma \backslash G$ correspond to ergodic measures for the action of Γ on G/N'. Consider, therefore, μ as a measure on G/N', ergodic with respect to the Γ-action. Let $pr : G/N' \to G/N$ be the projection. Since N/N' is compact, we have a locally finite measure $pr_*\mu$

on G/N which is easily seen to be ergodic with respect to the Γ-action. By lemma 10 and Moore-equivalence, $pr_*\mu$ must either be the Haar measure or correspond to an N-invariant measure on $\Gamma\backslash G$ which is determined on a compact orbit $\Gamma\backslash\Gamma gN \subset \Gamma\backslash Isom\left(H^n\right)$.

If $pr_*\mu =$ Haar measure, it follows easily that μ is absolutely continuous with respect to the Haar measure and then one gets, from ergodicity of the Γ-action (theorem 7 in [46]), that μ is a multiple of the Haar measure.

In the second case, $pr_*\mu$ must be determined on the Γ-orbit of some $gN \in G/N$. Therefore, μ is determined on the $\Gamma \times N$-orbit of $gN' \in G/N'$. By Moore-equivalence we get a measure determined on the compact N-orbit. □

Ergodic decomposition

Let a group G act on a topological space X. A probability measure μ is called ergodic if any G-invariant set has measure 0 or 1. Denote $\mathcal{E}$ the set ergodic G-invariant measures on X.

We define a σ-algebra $\mathcal{A}$ on$\mathcal{E}$ as the smallest σ-algebra with the following property:

for all Borel sets $A \subset X$ is $\begin{array}{c} f_A : \mathcal{E} \to \mathcal{R} \\ \mu \to \mu(A) \end{array}$ measurable.

Lemma 15 *: Let a group G act on a complete separable metric space X. If there exists a G-invariant probability measure on X, then the set $\mathcal{E}$ of ergodic G-invariant measures on X is not empty and there is a decomposition map $\beta : X \to \mathcal{E}$.*

A decomposition map is a G-invariant map $\beta : X \to \mathcal{E}$, which is
- measurable with respect to $\mathcal{A}$,
- satisfies $e\left(\{x \in X : \beta(x) = e\}\right) = 1$ for all $e \in \mathcal{E}$ and,
- for all G-invariant probability measures μ and Borel sets $A \subset X$ holds

$$\mu(A) = \int_X \beta(x)(A)\, d\mu(x).$$

For a proof of lemma 15, see theorem 4.2. in [63].

For later reference we state the following lemma, part (i) of which is known as Alaoğlu's theorem, whereas a proof of part (ii) can be found in lemma 3.2. of [14].

Lemma 16 *: (i) Any weak-*-bounded sequence of signed regular finite measures on a locally compact metric space has an accumulation point in the weak-*-topology.*
(ii) If μ is the weak--limit of a sequence μ_n of measures on a space X, and $U \subset X$ is an open subset, then $\mu(U) \leq \liminf \mu_n(U)$.*

Moreover, we recall that the support of a measure μ on X is defined as the complement of the largest open set $U \subset X$ with the property $\mu(U) = 0$.

Regular ideal reflection groups

Proposition 6 *: Let $n \geq 4$, and let T be a regular ideal n-simplex in hyperbolic n-space H^n. Let Γ be the subgroup of $Isom\,(H^n)$ generated by the reflections in the faces of T. Then Γ is dense in $Isom\,(H^n)$.*

Proof: As a first step we prove that Γ **is not discrete**.
Let v_∞ be an ideal vertex of T and Γ' the intersection of Γ with the stabiliser of v_∞. Γ' stabilises horospheres centered at v_∞. The induced Riemannian metrics on horospheres are euclidean, hence, Γ' can be considered as a subgroup of $Isom\,(E^{n-1})$, the isometry group of euclidean n-1-space. Γ' is generated by the reflections in the codimension 1 faces of T', where T' is a regular n-1-simplex in euclidean n-1-space E^{n-1}. We show that Γ' can't be discrete in $Isom\,(E^{n-1})$.

Let v_0 be a vertex of T' and $\Gamma_0 \subset \Gamma'$ the stabiliser of v_0. We consider Γ_0 as a subgroup of $Isom\,(S^{n-2})$, S^{n-2} being the sphere with center v_0 and radius 1. T' intersects S^{n-2} in a regular spherical n-2-simplex T'' of edgelengths $\frac{\pi}{3}$. $\Gamma_0 \subset Isom\,(S^{n-2})$ is the subgroup of $Isom\,(S^{n-2})$ generated by the reflections in the faces of T''. We shall prove that Γ_0 can't be discrete in $Isom\,(S^{n-2})$.

Let α be the dihedral angle of T''. It is easy to see, using the spherical cosine law, that $cos\,(\alpha) = \frac{1}{n-1}$. Consider two faces b and c. Letting R_c, R_b be the reflections at c and b, we want to show first that one can find k such that the angle between c and $(R_bR_c)^k\,(c)$ is smaller than $\frac{\pi}{5}$.

Let $n = 4$. Then $cos\,(\alpha) = \frac{1}{3}$. By computation, it follows

$$cos\,(5\alpha) = \frac{241}{243} > cos\left(\frac{\pi}{5}\right).$$

The same way, if $n > 4$, one uses again $cos\,(\alpha) = \frac{1}{n-1}$. For n=5, we get $cos\,(5\alpha) = \frac{976}{1024} > cos\left(\frac{\pi}{5}\right)$, and for n=6: $cos\,(5\alpha) = \frac{2641}{3125} > cos\left(\frac{\pi}{5}\right)$. For n=7, $cos\,(5\alpha)$ as well as $cos\,(4\alpha)$ are smaller than $cos\left(\frac{\pi}{5}\right)$, but we get $cos\,(9\alpha) = \frac{10057216}{10077696} > cos\left(\frac{\pi}{5}\right)$. Finally, for all $n \geq 8$, we get $cos\,(4\alpha) = \frac{n^4-8n^2+8}{n^4} > 1 - \frac{8}{n^2} \geq \frac{41}{49} > cos\left(\frac{\pi}{5}\right)$.
Assume Γ_0 were discrete. Let R be the union of all γe, where $\gamma \in \Gamma_0$ and e is a codimension 1 hyperplane through one of the faces of T''. Each closure of a connected component of $S^{n-2} - R$ forms a fundamental domain for Γ_0. It is clear that none of the codimension1 hyperplanes passing through the faces of T'' can intersect the interior of a fundamental domain (because otherwise two interior points of the fundamental domain would be mapped to each other under the reflection). That means that T'' is actually composed by components of $S^{n-2} - R$. Considering a fundamental domain which has $b \cap c$ as a codimension 2 face, we get that the dihedral angle (of the fundamental simplex) at $b \cap c$ has to be smaller than $\frac{\pi}{5}$.

Recall from the classification of spherical reflection groups, [12]: if $\Gamma_0 \subset Isom\left(S^{n-2}\right)$ is a discrete group generated by reflections, then there is a fundamental simplex Δ, such that Γ_0 is generated by reflections in the faces of Δ. One can enumerate the fundamental simplices actually giving rise to discrete groups. In particular, the only simplex giving rise to a discrete reflection group with some angle smaller than $\frac{\pi}{5}$ is the 2-simplex with angles $\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{m}$. Hence, looking at the tesselation of S^2 obtained from the (2,2,m)-reflection group, we have to check whether one finds an equilateral triangle with edgelength $\frac{\pi}{3}$ composed by fundamental triangles.

However, the tesselation of S^2, obtained from the (2,2,m)-reflection group looks as follows: it has 2m+2 vertices $a, b, v_0, \ldots, v_{2m}$ and 2m+1 lines (great circles), one of them passing through all v_i, the other ones passing through a, b and exactly one of the v_i. It is then clear, that any nondegenerate triangle invokes as vertices a, b and one of the v_i, hence, has interior angles $\frac{\pi}{2}, \frac{\pi}{2}, \frac{l\pi}{m}$ for some $l \in \{1, \ldots, m\}$. Thus we don't find the equilateral triangle whose angle α satisfies $cos\,(\alpha) = \frac{1}{3}$.

We conclude now that Γ_0 **is dense in** $Isom\left(S^{n-2}\right)$.
The closure $\overline{\Gamma_0}$ is a closed subgroup, hence, a Lie subgroup of dimension ≥ 1. In particular, it contains some connected 1-dimensional Lie group S. All connected 1-dimensional subgroups of $Isom\left(S^{n-2}\right)$ are of the form $S_s = \left(\Theta_s\left(\phi\right) : \phi \in [0, 2\pi]\right)$, where s is some codimension 2 subspace of R^{n-1} and $\Theta_s\left(\phi\right)$ is the rotation of angle ϕ which fixes s.

We call a set $\{E_i\}_{1 \leq i \leq n-2}$ of codimension 2 subspaces of R^{n-1} in general position if there exists a basis $e_1, \ldots, e_{n-1}$ of R^{n-1} such that each E_i is spanned by e_i and e_{i+1}. It is well known and easy to prove that, if $E_1, \ldots, E_{n-2}$ are in general position, then each element $g \in Isom\left(S^{n-2}\right) = O_{n-1}$ is a product $g = g_1 \ldots g_k$ for some $k \in N$, where each g_j is of the form $g_j = \Theta_{E_{i_j}^{\perp}}\left(\phi_j\right)$ with $i_j \in \{1, \ldots, n-2\}$ and $\phi_j \in [0, 2\pi]$.

Let s be a codimension 2 subspace such that $S_s \subset \overline{\Gamma_0}$. For any $\gamma \in \Gamma_0$ we have $S_{\gamma s} = \gamma S_s \gamma^{-1} \subset \overline{\Gamma_0}$. Choosing n-3 elements $\gamma_i \in \Gamma_0$ such that the $\gamma_i s$ (and s) are in general position, we conclude that $\overline{\Gamma_0} = O_{n-1}$.

We show that Γ' **is dense in** $Isom\left(R^{n-1}\right)$.
Recall that $Isom\left(R^n - 1\right)$ is the semidirect product of O_{n-1} and R^{n-1}, where multiplication is defined by $(A, b)\left(A', b'\right) = \left(AA', Ab' + b\right)$ for $A, A' \in O_{n-1}$ and $b, b' \in R^{n-1}$.

We just proved $\overline{\Gamma'} \supset \left(O_{n-1}, 0\right)$. Now, we take some $\gamma \in \Gamma'$ with $\gamma\,(0) \neq 0$, i.e., $\gamma = (A, b)$ with $b \neq 0$.
Then $\overline{\Gamma'} \supset \gamma \overline{\Gamma'} \gamma^{-1} \supset \gamma\left(O_{n-1}, 0\right)\gamma^{-1} = \left(O_{n-1}, b\right)$.
On the other hand, $\overline{\Gamma'} \supset \overline{\Gamma'}\gamma \supset \left(O_{n-1}, 0\right)(A, b)$.

For any b' with $\parallel b' \parallel = \parallel b \parallel$, we find $(A, 0) \in \left(O_{n-1}, 0\right) \subset \overline{\Gamma'}$ with $(A, 0)\, b = b'$.

Hence, we have $\gamma' := (A, 0)\,\gamma \in \overline{\Gamma'}$ with $\gamma'(0) = b'$. By the argument before we conclude $\overline{\Gamma'} \supset (O_{n-1}, b')$, whenever $\| b' \| = \| b \|$.

But, by the same reasoning (replacing 0 by b), we can argue that $\overline{\Gamma'} \supset (O_{n-1}, b'')$, whenever $\| b'' - b \| = \| 0 - b \|$. In particular, for any positive number $r < 2 \| b \|$, we find some b'' with $\| b'' \| = r$, such that $\overline{\Gamma'} \supset (O_{n-1}, b'')$. But then we have just shown that $\overline{\Gamma'} \supset (O_{n-1}, b')$ actually holds for any b' with $\| b' \| = r < 2 \| b \|$.

Continuing this reasoning, we show inductively for any $k \in N$ that $\overline{\Gamma'} \supset (O_{n-1}, b')$, whenever $\| b' \| < 2^k \| b \|$.Hence, Γ' is dense in $Isom^{R^n - 1}$.

Finally, we show that Γ **is dense in** $Isom\,(H^n)$.
Using the identification of $Isom\,(H^n)$ with the set of ON-repers in H^n, our claim is: given two repers $t_1 = (p, u_1, \ldots, u_n)$ and $t_2 = (q, w_1, \ldots, w_n)$ we find an element of $\overline{\Gamma}$ mapping t_1 to t_2.

Let v_1 and v_2 be two ideal vertices of T. When H is a horosphere centered at an ideal vertex of T, we have just proved that $\overline{\Gamma} \supset Isom\,(H)$. One easily finds horospheres H_1, H_2, H_3 such that H_1 and H_3 are centered at v_1, H_2 is centered at v_2, $p \in H_1$, $q \in H_3$, H_1 intersects H_2 and H_2 intersects H_3. Denote $p_1 := H_1 \cap H_2$ and $p_2 := H_2 \cap H_3$. We find $h_i \in Isom\,(H_i)$, $i = 1, 2, 3$, such that $h_1(p) = p_1, h_2(p_1) = p_2, h_3(p_2) = q$. Recall that $Stab\,(p_2)$ is generated by $Stab\,(p_2) \cap Isom\,(H_1)$ and $Stab\,(p_2) \cap Isom\,(H_2)$. Hence, we find $g \in \overline{\Gamma} \cap Stab\,(p_2)$ with $gh_2h_1(t_1) = h_3^{-1}(t_2)$. This means that $h_3gh_2h_1$ maps t_1 to t_2 and this finishes the proof.

□

4.1.3 Algebraic topology

Measure homology

For a manifold M, let $C^0\left(\Delta^k, M\right)$ be the space of singular simplices in M, topologized by the compact-open-topology. Let $\mathcal{C}_k(M)$ be the vector space of all signed Borel measures μ on $C^0\left(\Delta^k, M\right)$ which have compact support and finite total variation. Let $\eta_i : \Delta^k \to \Delta^{k-1}$ be the i-th face map. It induces a map $\partial_i = (\eta_i^*)_* : \mathcal{C}_k(M) \to \mathcal{C}_{k-1}(M)$. We define the boundary operator $\partial := \sum_{i=0}^k \partial_i$, to make $\mathcal{C}_*(M)$ a chain complex. We denote the homology groups of this chain complex by $\mathcal{H}_*(M)$.

We have an obvious inclusion $j : C_*(M) \to \mathcal{C}_*(M)$, where $C_*(M)$ are the singular chains, considered as finite linear combination of atomic measures. Clearly, j is a chain map. Zastrow's theorem 3.4. in [68] says that we get an isomorphism $j_* : H_*(M) \to \mathcal{H}_*(M)$.

The l^1-norm on $C_*(M)$ extends to a norm on $\mathcal{C}_*(M)$, and we get an induced pseudonorm on $\mathcal{H}_*(M)$. Thurston conjectured in [62] that the isomorphism j_*

should be an isometry. There seems not to exist a proof of this conjecture so far. However, if M is a closed hyperbolic n-manifold, it follows from the proof of the Gromov-Thurston theorem that $j_n : H_n(M) \to \mathcal{H}_n(M)$ is an isometry.

Intersection numbers

Let M be a connected oriented n-manifold and M' an n-submanifold with boundary. Assume that $M_0 := M - M'$ is compact and homotopy equivalent to M, hence, that $H_i(M) \approx H_i(M_0)$.
By excision, we have isomorphisms $H_i(M, M') \approx H_i(M_0, \partial M_0)$.
Poincare duality gives us isomorphisms $PD_1 : H_i(M, M'; R) \to H^{n-i}(M; R)$ and $PD_2 : H_i(M; R) \to H^{n-i}(M, M'; R)$ for $i = 0, \ldots, n$. We use the cup product $\cup : H^i(M, M'; R) \otimes H^{n-i}(M; R) \to H^0(M; R) \equiv R$ to define an intersection product $i : H_i(M, M') \otimes H_{n-i}(M) \to R$ via the equality

$$i(a_1, a_2) := PD_1(a_1) \cup PD_2(a_2)$$

for $a_1 \in H_i(M, M')$ and $a_2 \in H_{n-i}(M)$.

On the other hand, there is an intersection number defined in differential topology (for M, M' smooth) as follows: Let N_1 be an i-dimensional compact oriented smooth submanifold of M, such that $\partial N_1 \subset M'$, and let N_2 be an (n-i)-dimensional closed oriented smooth submanifold of M. Assume that N_1 and N_2 are transversal, i.e., for any $x \in N_1 \cap N_2$ is $T_xM = T_xN_1 \oplus T_xN_2$. It follows that $N_1 \cap N_2$ is a finite number of points $x_1, \ldots, x_k$. For any $x \in N_1 \cap N_2$, let $e_1, \ldots, e_i$ be a positively oriented basis of T_xN_1 and $e_{i+1}, \ldots, e_n$ a positively oriented basis of T_xN_2 and define the local intersection number at x by

$$li_x(N_1, N_2) := \left\{ \begin{array}{c} 1 : e_1 \ldots, e_n \text{ positively oriented} \\ -1 : e_1, \ldots, e_n \text{ negatively oriented} \end{array} \right\},$$

where orientation of $\{e_1, \ldots, e_i, e_{i+1}, \ldots, e_n\}$ is meant w.r.t. the given orientation of T_xM. The intersection number of N_1 and N_2 is then defined as

$$i(N_1, N_2) = \sum_{x \in N_1 \cap N_2} li_x(N_1, N_2).$$

N_1 represents a relative homology class $n_1 \in H_i(M, M')$, and N_2 represents a homology class $n_2 \in H_{n-i}(M)$. It is well known that $i(n_1, n_2) = i(N_1, N_2)$. In particular, if $[M, M']$ is the (real) relative fundamental cycle, i.e., the image of the orientation class $[M] \in H_n(M; Z)$ under $H_n(M; Z) \to H_n(M, M'; Z) \to H_n(M, M'; R)$, we get for any point $x \in M$:

$$i([M, M'], [x]) = i(M, x) = 1,$$

where $[x] \in H_0(M;R)$ is the homology class of the point x.

One should generalise the differential-topological definition of intersection number as follows: if σ_1, σ_2 are transversal smooth singular simplices, then we get with the same definition a local intersection number at all $x \in im(\sigma_1) \cup im(\sigma_2)$ and, hence, a global intersection number. By linear extension, we get an intersection number i' of singular chains. One should note, however, that for singular chains $c_1 = \sum_{i=1}^k a_{1i}\sigma_{1i}$ and $c_1 = \sum_{i=1}^k a_{1i}\sigma_{1i}$ the equality $i'(c_1, c_2) = i([c_1], [c_2])$ holds only if for all i, j all intersection points $x \in im(\sigma_{1i}) \cup im(\sigma_{2j})$ are in the images of the interiors of σ_{1i} and σ_{2j}.

We are interested in the special case of $c_2 = x$, where x means the 0-simplex mapped to the point $x \in M$. Transversality of an n-simplex σ to x means just that $d\sigma$ is an isomorphism at all $y \in \sigma^{-1}(x)$, and the local intersection number is $\sum_{y\in\sigma^{-1}(x)} sign(d\sigma(y))$. If σ is not transversal to x, we have $d\sigma = 0$ at all $y \in \sigma^{-1}(x)$, so we can make the following definition:

Definition 9 *: Let N be an oriented differentiable n-manifold. For a differentiable simplex $\sigma : \Delta^n \to N$, and $x \in N$, define*

$$\Phi_x(\sigma) = \sum_{y\in\sigma^{-1}(x)} sign\ d\sigma(y).$$

For a singular chain $c = \sum_{i=1}^r a_i\sigma_i$, let $\Phi_x(c) = \sum_{i=1}^r a_i\Phi_x(\sigma_i)$.

It is probably well-known that the generalised differential-topological intersection number coincides with the algebraic-topological one, i.e., that $i(a_1, a_2) = i'(a_1, a_2)$ holds for all $a_1 \in H_i(M, M'), a_2 \in H_{n-i}(M)$, where one has to define i' in an appropriate way, namely admitting only representatives such that all intersection points belong to the interiors of the corresponding simplices. However, we are not aware of any reference, so we give a proof, stating actually only the case $a_2 = [x] \in H_0(M)$, since this is the case we are interested in. The reader will convince himself that the same proof works in general.

Lemma 17 *: Let M be a connected, oriented n-manifold, M' an n-submanifold with boundary, such that $M - M'$ is compact. Let $c = \sum_{i=1}^r a_i\tau_i$ be a singular n-chain representing the relative fundamental class $[M, M']$. Assume that all τ_i are immersed smooth n-simplices. Then $\Phi_x(c) = 1$ holds for almost all $x \in M - M'$.*

Proof: Let $K = \cup_{i=0}^r \mathrm{im}(\partial\tau_i)$. K is of measure zero, by Sard's lemma.

We want to show that $\Phi_x(c)$, as a function of x, is constant on $M - (M' \cup K)$. It is obvious that it is locally constant on $M - (M' \cup K)$, since all τ_i are either locally diffeomorphic. It remains to prove: for all $x \in K \cap \mathrm{int}(M - M')$, there is a neighborhood U of x in M such that $\Phi_.(c)$ is constant on $U \cap (M - K)$.

The point x is contained in the image of finitely many (n-1)-simplices $\kappa_1, \ldots, \kappa_k$, which are boundary faces of some $\tau_{i_1}, \ldots, \tau_{i_k}$. (Note that the τ_{i_j}'s needn't be distinct and that there might be further τ_i's containing x in the interior of their image.) Since $\partial \sum_{i=1}^{r} a_i \tau_i$ invokes only simplices whose image is contained in $N_{[0,\epsilon]}$, we necessarily have that all $\tau_{i_1}, \ldots, \tau_{i_k}$ cancel each other, i.e., there is a partition of $\{i_1, \ldots, i_k\}$ in some subsets, such that for each of these subsets of indices the sum of the corresponding coefficients a_{i_j}, multiplied with a sign according to orientation of τ_{i_j}, adds up to zero.

This clearly implies that $\Phi_.$ is constant in the intersection of a small neighborhood of x with the complement K and, hence, also constant on all of $M - (M' \cup K)$.

We now prove that this constant doesn't depend on the representative of the relative fundamental class. This implies that the constant must be 1, since one can choose a triangulation as representative of the relative fundamental class. (By Whitehead's theorem, smooth manifolds admit triangulations.)

If c and c' are different representatives of $[M, M']$, we have that $c - c' = \partial w + t$ for some $w \in C_{n+1}(M - M')$ and $t \in C_n(M')$. Because ∂w is a cycle, the same argument as above gives that $\Phi_.(\partial w)$ is a.e. constant on *all* of M. The constant must be zero, since ∂w has compact support in the noncompact manifold M. That means that $\Phi_x(c) - \Phi_x(c') = \Phi_x(t)$ for almost all $x \in M$. But $\Phi_x(t) = 0$ for all $x \in int(M - M')$. □

4.1.4 Fundamental cycles

Convergence of fundamental cycles - some motivating remarks

A major point of this chapter will be to consider limiting objects of sequences of relative fundamental cycles of a finite-volume hyperbolic manifold N with l^1-norms approximating the simplicial volume. It is quite clear that there do not exist relative fundamental cycles actually having l^1-norm equal to $\frac{1}{V_n} Vol(N)$. Hence, the limits of such sequences can't be just singular chains. What we are going to do is to embed the singular chain complex into a larger space, where any bounded sequence has accumulation points. A straightforward idea would be to use the inclusion $j : C_n(N) \to \mathcal{C}_n(N)$ and to consider weak-* accumulation points in $\mathcal{C}_n(N)$. This works perfectly well, however it is easy to see that the weak-* limits are just trivial measures. The reason is roughly the following: a singular chain with l^1-norm close to $\frac{1}{V_n} Vol(N)$ has to have a very large part of its mass on simplices σ with $vol(str(\sigma))$ quite close to V_n. If we consider a compact set of simplices, it will have some upper bound (better than V_n) on $vol(str(.))$. Hence, it will contribute very few to an almost efficient fundamental cycle, and the limiting measure will actually vanish on this set of simplices.

Therefore, to get nontrivial accumulation points, we are obliged to consider the larger space of simplices which might be ideal, i.e., whose lifts to H^n might have vertices in $\partial_\infty H^n$. This, however, raises another problem: the space of ideal simplices in $N = \Gamma\backslash H^n$ is not Hausdorff, and there is no theorem guaranteeing existence of weak-* accumulation points for signed measures on non-Hausdorff spaces.

Straightening chains

Let $p_0, \ldots, p_i$ be points in H^n. The straight simplex $(p_0, \ldots, p_i)$ is defined as the barycentric parametrization of the geodesic simplex having vertices $p_0, \ldots, p_i$. For a simplex σ in H^n, we denote by $\mathrm{Str}(\sigma)$ the straight simplex with the same vertices as σ. A straight simplex in a hyperbolic manifold $N = \Gamma\backslash H^n$ is the image of a straight simplex in H^n under the projection $p : H^n \to \Gamma\backslash H^n = N$. For a simplex σ in N, its straightening $Str\,(\sigma)$ is defined as $p\,(Str\,(\tilde{\sigma}))$, where $\tilde{\sigma}$ is a simplex in H^n projecting to σ. Since straightening in H^n commutes with isometries, the definition of $Str\,(\sigma)$ doesn't depend on the choice of $\tilde{\sigma}$.

Finally, the straightening of a singular chain c=$\sum_{j=1}^r a_j\sigma_j$ is defined as $Str\,(c) = \sum_{j=1}^r a_j Str\,(\sigma_j)$. $Str\,(c)$ is homologous to c, and clearly $\parallel Str\,(c) \parallel \leq \parallel c \parallel$ for any $c \in C_*\,(N)$. ($Str\,(c)$ may possibly have smaller norm than c, since different simplices may have the same straightenings.)

Alternating chains

The symmetric group S_{n+1} acts on the standard n-simplex Δ^n: any permutation π of vertices can be realised by an affine map $f_\pi : \Delta^n \to \Delta^n$. For a singular simplex $\sigma : \Delta^n \to N$ let $alt\,(\sigma) := \sum_{\pi\in S_{n+1}} sgn\,(\pi)\,\sigma f_\pi$, and for a singular chain $c = \sum_{i=1}^r a_i\sigma_i$ define $alt\,(c) := \sum_{i=1}^r a_i alt\,(\sigma_i)$. Clearly, $\parallel alt\,(c) \parallel \leq \parallel c \parallel$.

Nondegenerate chains

Let N be a hyperbolic manifold. We call a straight i-simplex $\sigma : \Delta^i \to N$ degenerate if two of its vertices are mapped to the same point, nondegenerate otherwise.

Lemma 18 *: Let N be a hyperbolic n-manifold, N' a convex subset. Let $\sum_{i\in I} a_i\sigma_i \in C_n\,(N, N'; R)$ be a relative n-cycle. Then there is a subset of indices $J \subset I$ such that all σ_j with $j \in J$ are non-degenerate and $\sum_{j\in J} a_j\sigma_j$ is relatively homologous to $\sum_{i\in I} a_i\sigma_i$.*

Proof: Let $K \subset I$ be the subset of indices such that $\{\sigma_k : k \in K\}$ are all degenerate simplices. We claim that $\sum_{k\in K} a_k\sigma_k$ is relatively 0-homologous. For this, it is sufficient to show that it is a relative cycle, since it is obvious that $vol(\sigma) = 0$ for degenerate simplices σ.

The degenerate faces of $\sum_{k\in K} a_k\sigma_k$ cancel each other (relatively), since they cancel in $\partial\left(\sum_{i\in I} a_i\sigma_i\right)$ and they can't cancel against faces of nondegenerate simplices.

The nondegenerate faces of degenerate simplices cancel anyway: if $(a, v_1, \ldots, v_n)$ and $(b, v_1, \ldots, v_n)$ are nondegenerate faces of a degenerate simplex, then necessarily $a = b$. Thus this face contributes twice to the boundary, with opposite signs.□

Hence, to any relative n-cycle $c \in C_n(N, N'; R)$ with $N' \subset N$ convex and $n = dim(N)$, we find $c' \in C_n(N, N'; R)$ homologous to c in $C_*(N, N'; R)$, such that $\parallel c' \parallel \leq \parallel c \parallel$ and c' is an alternating linear combination of nondegenerate straight simplices.

Straight chains as measures

We explained in 4.1.3. that singular chains may be considered as atomic measures on the space of singular simplices, thus getting a homomorphism $C_*(M) \rightarrow \mathcal{C}_*(M)$. As we said, to get nontrivial results, we should consider not only $\mathcal{C}_*(M)$, but measures on the space of possibly ideal simplices. Since it is hard to prove existence of accumulation points in this measure space, we will consider measures on smaller sets of simplices.

Let N be a hyperbolic manifold. The set of nondegenerate, possibly ideal, straight i-simplices in $N = \Gamma\backslash H^n$ is

$$SS_i(N) := \Gamma\backslash\left\{(p_0, \ldots, p_i) : p_0, \ldots, p_i \in \overline{H^n}, p_j \neq p_k \text{ if } j \neq k\right\},$$

where $g \in \Gamma$ acts by $g(p_0, \ldots, p_n) = (gp_0, \ldots, gp_n)$.

Denote $\mathcal{M}(SS_i(N))$ the space of signed regular measures on $SS_i(N)$. Straight singular chains $c = \sum_{j=1}^{r} a_j\sigma_j \in C_i(N; R)$, with all σ_j nondegenerate, can be considered as discrete signed measures on $SS_i(N)$ defined by

$$c(B) = \sum_{\{j:\sigma_j\subset B\}} | a_j |$$

for any Borel set $B \subset SS_i(N)$.

Let $n = dim(N)$. To apply Alaoǧlu's theorem to $\mathcal{M}(SS_n(N))$, we need to know that $SS_n(N)$ is locally compact (which is obvious) and metrizable.

Lemma 19 *: Let N be a hyperbolic manifold. Then $SS_n(N)$ is metrizable.*

Proof: We have to show that Γ-orbits on $\Pi_{j=0}^{n}\overline{H^n} - D$ are closed, D being the set of degenerate straight simplices. On the complement of $\Pi_{j=0}^{n}\partial_\infty\overline{H^n}$ this follows from proper discontinuity of the Γ-action on H^n.

To any n-tupel $(v_0, \ldots, v_{n-1}) \in \Pi_{j=0}^{n-1}\partial_\infty\overline{H^n}$ of *distinct* points that Γ-orbits on $\Pi_{j=0}^{n}\overline{H^n} - D$ are closed, D being the set of degenerate straight simplices. On the complement of $\Pi_{j=0}^{n}\partial_\infty\overline{H^n}$ this follows from proper discontinuity of the Γ-action on H^n.

To any n-tupel $(v_0, \ldots, v_{n-1}) \in \Pi_{j=0}^{n-1}\partial_\infty\overline{H^n}$ of *distinct* points corresponds a unique $v_n \in \partial_\infty\overline{H^n}$ such that $(v_0, \ldots, v_n)$ is a positively oriented regular ideal n-simplex. Together with the identification in 3.1.1.3, we get a Γ-equivariant homeomorphism
$\Pi_{j=0}^{n-1}\partial_\infty\overline{H^n} - D \to Isom^+(H^n)$. Γ acts properly discontinuously on $Isom^+(H^n)$, as well as on $\Pi_{j=0}^{n-1}\partial_\infty\overline{H^n} - D$, even more on $\Pi_{j=0}^{n}\partial_\infty\overline{H^n} - D$. □

4.1.5 Gromov-Thurston theorem

We outline the proof of the Gromov-Thurston theorem, for closed hyperbolic manifolds. This should be helpful as a motivation for the following sections, and serves in particular to introduce several notions which will show up in the proof of theorem 3. The presentation follows in parts that of [40].

Proposition 6: *Let M be a closed hyperbolic manifold. Then $\| M, \partial M \| = \frac{1}{V_n} Vol(M)$. Here, V_n is the volume of a regular ideal simplex in H^n.*

Proof: Let $C_*^{str}(M; R) \subset C_*(M; R)$ be the subcomplex generated by straight simplices. Straightening of simplices,

$$str : C_*(M; R) \to C_*^{str}(M; R)$$

gives a chain homotopy inverse of the inclusion. Hence, it induces an isomorphism of homology groups, of norm 1.

Let $n = dim(M)$. The composed isomorphism

$$H_n(M, R) \to H_n^{str}(M; R) \to R$$

is given by integrating $\frac{1}{vol(M)} dvol$ over straight cycles representing homology classes. Every straight cycle representing the fundamental class $[M]$ must cover all of M. Since each of its simplices covers volume $< V_n$ (by the Haagerup-Munkholm theorem in section 4.1.1), such a straight cycle representing the fundamental class has l^1-norm larger than $\frac{vol(M)}{V_n}$. This shows $\| M \| \geq \frac{vol(M)}{V_n}$.

Let $C_*^{meas,str}(M;R)$ be the complex of measure chains, that is, of compactly supported signed measures on the space of straight simplices of M. Note that a signed measure on the space of straight simplices is compactly supported iff it is supported on simplices with a common diameter bound. The total variation of signed measures generalizes the l^1-norm on singular chains.

Let D be a measurable fundamental domain for the action of $\Gamma = \pi_1 M$ on $\tilde{M} = H^n$. Since M is compact, D may be choosen to have finite diameter. Choose a Γ-orbit Γx and let the (non-continuous) retraction

$$r : H^n \to \Gamma x$$

be the equivariant measurable map which, for each $\gamma \in \Gamma$, collapses each translate γD to the unique orbit point γx contained in it. We use this map on the level of vertices to define a map on the level of straight simplices inducing a map

$$wiggle : C_*^{meas,str}(M;R) \to C_*^{str}(M;R)$$

which has norm ≤ 1 and is a chain homotopy inverse to the inclusion. One should note that $wiggle$ indeed maps compactly supported signed measures to finite linear combinations of straight simplices because there are only finitely many straight simplices with a given diameter bound and all vertices in Γx.

Consider the space $S_L^{reg}(M)$ of ordered regular geodesic n-simplices with side length L in (M). There is a $Isom(H^n)$-equivariant bijection

$$\tilde{I}_L : S_L^{reg}(H^n) \to Isom(H^n),$$

descending to a bijection

$$I_L : S_L^{reg}(M) \to \Gamma \backslash Isom(H^n).$$

It is well-known that $Isom^+(H^n)$ is unimodular, i.e. that it admits a biinvariant Haar measure, descending to a finite measure μ^+ on $\Gamma \backslash Isom^+(H^n)$. To get a signed measure on allo of $Isom(H^n)$, consider

$$\mu := \frac{1}{2}\left(\mu^+ - \mu^-\right)$$

with $\mu^- := r^* \mu^+$ for an arbitrary fixed reflection $r \in Isom^-(H^n)$. Note that μ is preserved by $Isom^+(H^n)$, but changes its sign under the action of $Isom^-(H^n)$.

We use the bijection I to define a signed measures

$$\mu_L = \frac{1}{2}\left(\mu_L^+ - \mu_L^-\right)$$

with

$$\mu_L^{\pm} := I^* \mu^{\pm}$$

on $S_L^{reg}(M)$. μ_L is a cycle of norm 1 in $C_n^{meas,str}(M;R)$, representing $\frac{v(L)}{vol(M)}[M]$, where $v(L)$ is the volume of a regular geodesic n-simplices with edge-length L in H^n. It follows that

$$\frac{vol(M)}{v(L)} wiggle(\mu_L)$$

represents the fundamental class in $H_n(M;R)$. Since it has norm $\leq \frac{vol(M)}{v(L)}$, and $lim_{L\to\infty} v(L) = V_n$, we get $\| M \| \leq \frac{vol(M)}{v(L)}$.

It is worth pointing out that $\frac{vol(M)}{v(L)} wiggle(\mu_L)$ weak-*-converges to

$$\frac{vol(M)}{V_n} wiggle\left(\frac{1}{2}\left(\mu_\infty^+ - \mu_\infty^-\right)\right),$$

where $\mu_\infty^\pm$ are the measures defined on $S_\infty^{reg}(M)$, the space of ordered regular ideal n-simplices in M, by pulling back the Haar measure via the $Isom(H^n)$-equivariant bijection $H_\infty : S_\infty^{reg}(H^n) \to Isom(H^n)$.

4.2 Degeneration

The aim of this section is to give a precise definition of efficient fundamental chains in the case of compact manifolds with boundary admitting a complete finite-volume hyperbolic on its interior, and to show, through a series of lemmata, that efficient fundamental chains are signed measures on the set of regular ideal simplices, invariant under the action of a certain group $R^+ \subset Isom(H^n)$.

4.2.1 Efficient fundamental cycles

For a closed hyperbolic manifold, the Gromov-Thurston theorem gives $\| N \| = \frac{1}{V_n} Vol(N)$. In particular, for any $\epsilon > 0$, there is some fundamental cycle d_ϵ satisfying

$$\| d_\epsilon \| \leq \| N \| + \frac{\epsilon}{V_n}.$$

By 4.1.4, we can replace d_ϵ by a homologous **alternating** chain c_ϵ consisting of **nondegenerate straight** simplices, without increasing the l^1-norm. To speak about limits of sequences of c_ϵ, it will be convenient to regard them as elements of some space with compact balls, namely the space of signed measures on $SS_n(N) = \Gamma \backslash \left(\Pi_{j=0}^n \overline{H^n} - D\right)$ with the weak-*-topology, as in 4.1.4. (The reader might wonder why we don't consider them as signed measures on $\Gamma \backslash \left(\Pi_{j=0}^n H^n - D\right)$, where balls are still weak-*-compact. The point is that in this space, the weak-*-limits of the c_ϵ would just be trivial measures, what does not imply much.)

Jungreis results from [36], for closed hyperbolic manifolds of dimension ≥ 3, can be phrased as follows:
- any sequence of c_ϵ with $\epsilon \rightarrow 0$ converges,
- the limit μ is supported on the set of regular ideal simplices, to be identified with $Isom\left(H^n\right)$, and
- up to a multiplicative factor one has $\mu = \mu^+ - \mu^-$ with μ^+ the Haar measure on $Isom^+\left(H^n\right)$ and $\mu^- = r^*\mu^+$ for an arbitrary orientation reversing $r \in Isom\left(H^n\right)$.

The aim of this chapter is to generalize these results to finite-volume hyperbolic manifolds. For these cusped hyperbolic manifolds, there arises a technical problem: we wish to consider chains representing the relative fundamental class of a manifold with boundary, but we have a hyperbolic metric (and a notion of straightening) only on the interior. In the following, we will get around this problem and analyse the possible limits.

Recall from 4.1.1 that there is some ϵ_0 s.t. for any $\epsilon < \epsilon_0$ there is a homeomorphism $h_\epsilon : (N, \partial N) \rightarrow \left(N_{[\epsilon,\infty]}, \partial N_{[\epsilon,\infty]}\right)$. Let, for $\epsilon < \epsilon_0$, $d_\epsilon \in C_n\left(N, \partial N; R\right)$ be some relative fundamental cycle satisfying

$$\parallel d_\epsilon \parallel \leq \parallel N, \partial N \parallel + \frac{\epsilon}{V_n}.$$

and consider $h_{\epsilon *} d_\epsilon \in C_n\left(N_{[\epsilon,\infty]}, \partial N_{[\epsilon,\infty]}; R\right)$. Let

$$exc : C_n\left(N_{[\epsilon,\infty]}, \partial N_{[\epsilon,\infty]}; R\right) \rightarrow C_n\left(N, N_{[0,\epsilon]}; R\right)$$

be the excision morphism, and let

$$Str : C_n\left(N, N_{[0,\epsilon]}\right) \rightarrow C_n\left(N, N_{[0,\epsilon]}\right)$$

be the morphism induced by str, straightening of chains, which is well defined because $N_{[0,\epsilon]}$ is convex. Then consider

$$c_\epsilon := Str\left(exc\left(h_{\epsilon *} d_\epsilon\right)\right) \in C_n\left(int\left(N\right), N_{[0,\epsilon]}; R\right).$$

The following notion of efficient fundamental chains will be the topic of interest in this chapter.

Definition 10 : *Let N be a compact manifold with a given hyperbolic metric on its interior. A signed measure μ on $SS_n\left(N\right)$ is called an* **efficient fundamental chain** *if there exists a sequence of ϵ with $\epsilon \rightarrow 0$ and a sequence of $d_\epsilon \in C_n\left(N, \partial N; R\right)$ representing the relative fundamental cycle, such that*

(i) d_ϵ are alternating chains invoking only nondegenerate simplices,
(ii) they satisfy the inequality $\| d_\epsilon \| \leq \| N, \partial N \| + \frac{\epsilon}{V_n}$, and
(iii) the sequence $c_\epsilon := Str\left(exc\left(h_{\epsilon *} d_\epsilon\right)\right) \in C_n\left(int\left(N\right), N_{[0,\epsilon]}; R\right)$ converges to μ in the weak--topology of the space of signed measures on $SS_n\left(int\left(N\right)\right)$.*

Lemma 20 *: Let N be a compact manifold supporting a complete hyperbolic metric on its interior. Then there is at least one efficient fundamental chain.*

Proof: By definition of the simplicial volume there exists a sequence of representatives of the relative fundamental class, satisfying the inequality (ii) in definition 9, for $\epsilon \to 0$. By the arguments in 4.1.4 we may assume that also condition (i) is satisfied for this sequence d_ϵ. Let $c_\epsilon = Str\left(exc\left(h_{\epsilon *} d_\epsilon\right)\right.$ and regard this sequence of singular straight chains as a sequence c_ϵ of signed measures on the locally compact metric space $SS_n\left(N\right)$ as in 4.1.4. The sequence c_ϵ is bounded by its definition and, hence, lemma 16(i) guarantees the existence of a weak-*-accumulation point μ. □

We recall that excision and straightening, as well as the homeomorphism h_ϵ induce isomorphisms in relative homology. Hence, any c_ϵ represents the relative fundamental class in $H_n\left(int\left(N\right), N_{[0,\epsilon]}; R\right)$. From lemma 17 follows (see definition 8 for Φ_x):

Lemma 21 *: Let N be a manifold and $N_0 \subset N$ be a codimenson 0 submanifold, such that $N - N_0$ is compact, and let c_ϵ be a representative of the relative fundamental class $\left[int\left(N\right), N_{[0,\epsilon]}\right]$. Then $\Phi_x\left(c_\epsilon\right) = 1$ holds for almost all $x \in N_{[\epsilon,\infty]}$.*

Let, for $\delta \geq 0$,

$$S_\delta := \left\{\sigma \in SS_n\left(N\right) : vol\left(\sigma\right) < V_n - \delta\right\}.$$

A priori, efficient fundamental chains μ are signed measures on $SS_n\left(N\right) = \bigcup_{V_n > \delta \geq 0} S_\delta^c$. The following lemma 22 shows that they are actually supported on S_0^c.

Lemma 22 *: Let N be a compact manifold admitting a complete finite-volume hyperbolic metric on its interior. Then any efficient fundamental chain is supported on the set of straight simplices of volume V_n (= the set of regular ideal simplices).*

Proof: Any subset $A \subset S_0$ can be written as a countable union $A = \cup_{i \in N} A_i$ such that each A_i is contained in S_{ϵ_i} for a suitable ϵ_i. It suffices therefore to show that $\mu^{\pm}\left(S_{\epsilon'}\right) = 0$ holds for any $\epsilon' > 0$. This, in turn, follows with lemma 16(ii) if we can prove $lim\epsilon \to 0 c_\epsilon^{\pm}\left(S_{\epsilon'}\right) = 0$ for any $\epsilon' > 0$, because $S_{\epsilon'}$ is open.

From lemma 21, we conclude $\int_N \Phi_x(c_\epsilon)\, dvol(x) \geq Vol\left(N_{[\epsilon,\infty]}\right)$.
But $\int_N \Phi_x(c_\epsilon)\, dvol(x) = \sum_{i=1}^r a_i \int_N \Phi_x(\sigma_i)\, dvol(x) = \sum_{i=1}^r a_i algvol(\sigma_i)$, where $algvol(\sigma_i)$ is $Vol(\sigma_i)$ with a sign according to orientation. In particular,

$$\sum \mid a_i \mid Vol(\sigma_i) \geq Vol\left(N_{[\epsilon,\infty]}\right).$$

On the other hand, we want $c_\epsilon = \sum a_i \sigma_i$ to satisfy $V_n \sum \mid a_i \mid \leq Vol(N) + \epsilon$. Substracting the two inequalities yields

$$\sum \mid a_i \mid (V_n - Vol(\sigma_i)) \leq \epsilon + Vol\left(N_{[0,\epsilon]}\right).$$

We get $\epsilon + Vol\left(N_{[0,\epsilon]}\right) a \geq \sum \mid a_i \mid (V_n - Vol(\sigma_i))$
$= \sum_{i:Vol(\sigma_i)\geq V_n-\epsilon'} \mid a_i \mid (V_n - Vol(\sigma_i)) + \sum_{i:Vol(\sigma_i)<V_n-\epsilon'} \mid a_i \mid (V_n - Vol(\sigma_i))$
$\geq \sum_{i:Vol(\sigma_i)<V_n-\epsilon'} \mid a_i \mid (V_n - Vol(\sigma_i))$
$\geq \epsilon' \sum_{i:Vol(\sigma_i)<V_n-\epsilon'} \mid a_i \mid$
$= \epsilon' \left(c_\epsilon^+(S_{\epsilon'}) + c_\epsilon^-(S_{\epsilon'})\right)$.
From $\lim_{\epsilon\to 0} Vol\left(N_{[0,\epsilon]}\right) = 0$, we conclude $lim\epsilon \to 0 c_\epsilon(S_{\epsilon'}) = 0$. □

We will use lemma 22 to consider efficient fundamental chains as signed measures on $S_\infty^{reg}(N)$ (definition 7).

The following lemma 23 states that efficient fundamental chains are not trivial (what is of course important to make nontrivial use of them). It is at this point where we use that we admit also ideal simplices.

Lemma 23 *: If N is a compact manifold admitting a complete finite-volume hyperbolic metric on its interior, and μ an efficient fundamental chain on N, then $\mu \neq 0$.*

Proof: Choose $f : SS_n(N) \to [0,1]$, which is zero on some $S_{\delta'}$ and is one on the complement of some S_δ. As a function on $SS_n(N)$, f is compactly supported. Hence, $\mu(f) = lim_{\epsilon\to} c_\epsilon(f)$ which does not vanish by the arguments in the proof of lemma 22. □

Although efficient fundamental chains are constructed as limits of *relative* cycles, the following lemma shows that they are actual cycles. Intuitively spoken, their boundary escapes to infinity.

Lemma 24 *: Let N be a compact manifold whose interior admits acomplete hyperbolic metric of finite volume. If μ is an efficient fundamental chain, then $(\partial\mu)^+ = (\partial\mu)^- = 0$.*

Proof: Denote by $SS_\epsilon^i(N)$ the set of (possibly ideal) i-simplices intersecting the interior of $N_{[\epsilon,\infty]}$. Since $h_{\epsilon *}d_\epsilon$ is a relative fundamental cycle, we clearly have, with $n = dim(N)$:

$$B \subset SS_\epsilon^{n-1}(N) \text{ measurable } \Rightarrow \partial (h_{\epsilon *}d_\epsilon)^\pm (B) = 0.$$

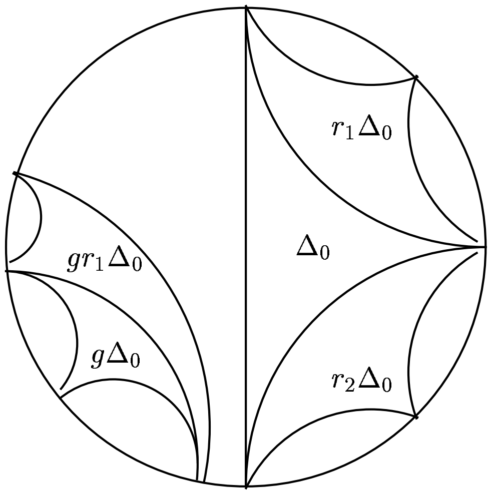
If $\epsilon' \leq \epsilon$, we have $N_{[0,\epsilon']} \supset N_{[0,\epsilon]}$, implying $\partial (h_{\epsilon'} * d'_\epsilon)^\pm (B) = 0$ for all $B \subset SS_\epsilon^{n-1}(N)$.
Since $N_{[0,\epsilon']}$ is convex and $c_{\epsilon'}$ is defined by straightening, we get that, for all $\epsilon' < \epsilon$,

$$B \subset SS_\epsilon^{n-1}(N) \text{ measurable } \Rightarrow \partial c_{\epsilon'}^\pm (B) = 0.$$

When $\partial\mu^\pm$ is a weak-* accumulation point of a sequence $\partial c_\epsilon^\pm$, we conclude $\partial\mu^\pm(B) = 0$ for all measurable sets B contained in *some* $SS_\epsilon^{n-1}(N)$ by part (ii) of lemma 16, since we may consider them as subsets of an open set still contained in some slightly larger $SS_\epsilon^{n-1}(N)$.

But clearly, $\cup_{k=1}^\infty SS_{\frac{1}{k}}^{n-1}(N)$ is the set of all (even ideal) (n-1)-simplices, hence the claim of the lemma. □

Remark: In the case of closed manifolds, lemma 24, of course, an immediate consequence of the fact that ∂ is a bounded operator.



4.2.2 Invariance under ideal reflection group

We have shown that efficient fundamental chains are measure cycles on the set of regular ideal simplices. In this subsection we sue the observation that regular ideal simplices in $\Gamma\backslash H^n$ are in bijection with $\Gamma\backslash Isom(H^n)$ to traduce the cycle condition $\partial\mu^\pm = 0$ into the condition that the corresponding measures on $\Gamma\backslash Isom(H^n)$

are invariant under the right-hand action of some group $R^{=} \subset Isom(H^n)$. (Of course, there is no difference between left-hand and right-hand actions. The point is that Γ and R^+ commute, i.e., act form different sides.)

Since we have an ordering of the vertices of a simplex Δ, we can speak of the i-th face of Δ, the codimension 1-face not containing the i-th vertex.

Definition 11 : *Fix a regular ideal simplex Δ_0 and, for $i = 0, \ldots, n$, let r_i be the reflection in the i-th face of Δ_0. Let $R \subset Isom(H^n)$ be the subgroup generated by $r_0, \ldots, r_n$ and let $R^+ = R \cap Isom^+(H^n)$.*

We have got that $\mu^{\pm}$ are measure cycles supported on $S_\infty^{reg}(N)$, the set of regular ideal simplices. As explained in 4.1.1, after fixing some regular ideal simplex Δ_0 in H^n, we have an $Isom(H^n)$-equivariant bijection $I : S_\infty^{reg}(N) \rightarrow \Gamma \backslash Isom(H^n)$. We use this bijection to consider $\mu^{\pm}$ as measures on $\Gamma \backslash Isom(H^n)$.

We will use the convention that $g \in Isom(H^n)$ corresponds to the simplex $g\Delta_0$, i.e., we let $Isom(H^n)$, and in particular Γ, act from the **left**. It will be important to note that, after this identification, the **right**-hand action of R corresponds to the following operation on the set of regular ideal simplices: r_i maps a simplex to the simplex obtained by reflection in its i-th face. This is clear from the picture on page 81.

Lemma 25 : *For $n \geq 3$, efficient fundamental chains are invariant under the right-hand action of R^+ on $\Gamma \backslash Isom(H^n)$.*

Note: If $\Delta = g\Delta_0$ for some $g \in \Gamma \backslash Isom(H^n)$, then the reflection s_i in the i-th face of Δ maps $\Delta = g\Delta_0$ to $gr_i(\Delta_0)$. In other words, the choice of another reference simplex changes the identification with $Isom(H^n)$ by **left** multiplication with $g \in Isom(H^n)$, but doesn't alter the **right**-hand action of R^+ on $Isom(H^n)$. This implies that the truth of lemma 25 is independent of the choice of Δ_0.

Lemma 25 follows from

Lemma 26 : *In dimensions $n \geq 3$, a signed alternating measure μ on the set of maximal volume simplices is a cycle iff $r_i^*(\mu) = -\mu$ for all $i = 0, \ldots, n$.*

Proof: If $n \geq 3$, then for any ordered regular ideal (n-1)-simplex τ, there are exactly two ordered regular ideal n-simplices, τ_i^+ and τ_i^-, having τ as i-th face. (By the way, this is the only point entering the proofs of our theorems which uses $n \geq 3$.) We fix them such that τ_i^+ is positively oriented. For a measurable set $B \subset \{$ordered regular ideal (n-1)-simplices$\}$ define
$B_i^+ = \left\{\tau_i^+ : \tau \in B\right\}$ and $B_i^- = \left\{\tau_i^- : \tau \in B\right\}$.

Since μ is determined on the set of regular ideal n-simplices, we have that

$$\partial\mu\left(B\right)=\sum_{i=0}^{n}\left(-1\right)^{i}\mu\left(\partial_{i}^{-1}\left(B\right)\right)$$

$$=\sum_{i=0}^{n}\left(-1\right)^{i}\left(\mu\left(B_{i}^{+}\right)+\mu\left(B_{i}^{-}\right)\right).$$

We may assume that μ is alternating, in particular $\pi_{ik}^{*}\mu=\left(-1\right)^{i-k}\mu$, where π_{ik} is induced by the affine map realizing the transposition of the i-th and k-th vertex. If $i-k$ is even, π_{ik} maps B_i^+ to B_k^+ and B_i^- to B_k^-. If $i-k$ is odd, π_{ik} maps B_i^+ to B_k^- and B_i^- to B_k^+. Therefore, we get

$$\partial\mu\left(B\right)=\left(n+1\right)\left(\mu\left(B_{i}^{+}\right)+\mu\left(B_{i}^{-}\right)\right)$$

for all $i\in\{0,\ldots,n\}$.
In particular $\partial\mu\left(B\right)=0$ holds if and only if $\mu\left(B_{i}^{+}\right)=-\mu\left(B_{i}^{-}\right)$ for $i=0,\ldots,n$.

The action of r_i maps B_i^+ bijectively to B_i^- and vice versa. Thus, $\partial\mu=0$ implies that $r_i^*\mu=-\mu$ holds, at least for sets of the form B_i^+ or B_i^-. But, clearly, any measurable set of ordered regular ideal n-simplices is the union of two sets having this form for suitable measurable sets, so the claim follows. □

Remark: A different, but in our opinion considerably more involved, proof of the same fact is given in lemma 2.2. of [36].

4.3 Decomposition of efficient fundamental cycles

The aim of this section is to give a decomposition of efficient fundamental chains into measures which can be explicitly described. Such a decomposition exists in dimensions ≥ 3, and in dimensions ≥ 4 it will actually be trivial.

If $n\geq 4$, then the group generated by reflections in the faces of a regular ideal n-simplex in H^n is dense in $Isom\left(H^n\right)$. We get from lemma 25 that efficient fundamental cycles are invariant under the right-hand action of $Isom^+\left(H^n\right)$. This implies that they are a multiple of $Haar-r^*Haar$, where $Haar$ is the Haar measure on $Isom^+\left(H^n\right)$.

In the following we will discuss the case $n=3$.
Back to the situation of section 4.2. Let v be an ideal vertex of the reference simplex Δ_0. Let $N_v\subset Isom^+\left(H^n\right)$ be the subgroup of parabolic isometries fixing v. As in the 4.1.2, we may consider N_v as the N-factor in the Iwasawa decomposition $Isom^+\left(H^n\right)=K_vA_vN_v$. (That means we use $v\in\partial_\infty H^n$ and

some arbitrary $p \in H^n$ to construct the Iwasawa decomposition. In the following, we will fix some arbitrary $p \in H^n$ but consider various $v \in \partial_\infty H^n$, therefore the labelling of the Iwasawa decompositions.)

Instead of R^+, we consider only the subgroup T_v' generated by products of even numbers of reflections in those faces of Δ_0 which contain v. μ is, of course, also invariant under the smaller group T_v'. If n=3, then T_v' contains a subgroup T_v which is a cocompact subgroup of N_v (this is easy to see, cf. [36]). Thus, in any case, we have proved that μ is invariant under some cocompact lattice $T_v \subset N_v$.

The signed measure μ decomposes as a difference of two measures μ^+ and μ^-. We rescale $\mu^\pm$ to probability measures $\overline{\mu}^\pm$, to be able to apply the ergodic decomposition from subsection 4.1.2.

Ergodic decomposition. $\overline{\mu}^+$ and $\overline{\mu}^-$ are invariant under the right-hand action of T_v. From lemma 15, we get that the restrictions of $\overline{\mu}^\pm$ to $\Gamma\backslash Isom^+(H^n)$, have decomposition maps with respect to the action of T_v,

$$\beta_v^\pm : \Gamma\backslash Isom^+(H^n) \to \mathcal{E}.$$

Here, $\mathcal{E}$ is the set of ergodic T_v-invariant measures on $\Gamma\backslash Isom^+(H^n)$. From lemma 14, we get that $\mathcal{E}$ consists of $Haar$ (the Haar measure, rescaled to a probability measure) and measures determined on compact N_v-orbits. The latter class can be better characterized by help of the following well-known lemma.

Lemma 27 *: An orbit gN_v is compact in $\Gamma\backslash Isom(H^n)$ iff all simplices $gn\Delta_0$ with $n \in N_v$ have its ideal vertex gv in a parabolic fixed point of Γ.*

Proof: Parametrise elements of N_v as $u(s), s \in R^{n-1}$ (identifying a stabilized horosphere with euclidean (n-1)-space). The N_v-orbit of g on $\Gamma\backslash Isom(H^n)$ is compact if and only if, for all $s \in R^{n-1}$, one finds $\gamma \in \Gamma$ and $t \in R$ such that $gu(ts) = \gamma g$. This γ is then conjugated to $u(ts)$ and, in particular, is parabolic, i.e., has only one fixed point. The fixed point of γ must be $g(v)$, since $\gamma g(v) = gu(ts)(v) = g(v)$. The other implication is straightforward. □

To summarize, we have the following statement:

For any vertex v of the reference simplex Δ_0, the **ergodic decomposition** of the rescaled $\overline{\mu}^\pm$ with respect to the right-hand action of T_v uses the **Haar measure** and **measures** determined **on the set of those simplices** $g\Delta_0$ **which have the vertex** gv **in a parabolic fixed point** of Γ.

4.4 Non-transversal fundamental cycles

Definition 12 *: For a hyperbolic manifold N and a two-sided codimension-1 submanifold $F \subset N$ call*

- S^i_{cusp} the set of positively oriented ideal i-simplices with all vertices in parabolic fixed points of N, and
- S^i_F the set of (possibly ideal) positively oriented i-simplices that intersect F transversally.

Here, a simplex σ is said to intersect F transversally if it intersects both components of any regular neighborhood of F.

Lemma 28 *: If F is a two-sided totally geodesic codimension-1-submanifold, then $S^n_F \cap \{regular\ ideal\ simplices\} \subset \{regular\ ideal\ simplices\}$ has positive Haar measure.*

Proof: It is easy to see that $S^n_F \cap \{regular\ ideal\ simplices\}$ is an open, non-empty subset of $\{regular\ ideal\ simplices\}$. □

Theorem 3 *: Let N be a compact manifold of dimension $n \geq 3$ such that $int(N)$ admits a hyperbolic metric of finite volume, and let $F \subset N$ be a closed totally geodesic codimension-1-submanifold.*
If μ is an efficient fundamental cycle (with $\mu^+ \mid_{Isom^+(H^n)} \neq 0$), then $\mu^+ (S^n_F) \neq 0$.

Proof: Very roughly, the idea is the following: If $\mu^+ (S^n_F)$ vanishes, then the Haar measure can only give a zero contribution to the ergodic decomposition of μ^+, hence, μ^+ is supported on S^n_{cusp}. In particular, μ^+ vanishes on the set of simplices with boundary faces in F, and this will give a contradiction.

Rescale $\mu^+ \mid_{Isom^+(H^n)}$ to a probability measure $\overline{\mu}^+$.
Assume for some totally geodesic surface F we had $\overline{\mu}^+ (S^n_F) = \mu^+ (S^n_F) = 0$.

Let v be a vertex of the reference simplex Δ_0. Using the ergodic decomposition with respect to the T_v-action on $\Gamma\backslash G = \Gamma\backslash Isom^+ (H^n)$ yields

$$0 = \overline{\mu}^+ (S^n_F) = \int_{\Gamma\backslash G} \beta_v (g) (S^n_F) \, d\overline{\mu}^+ (g) \geq \int_{g\in\Gamma\backslash G:\beta_v(g)=Haar} \beta_v (g) (S^n_F) \, d\overline{\mu}^+ (g)$$

$$= \int_{g\in\Gamma\backslash G:\beta_v(g)=Haar} Haar (S^n_F) \, d\overline{\mu}^+ (g) = Haar (S^n_F) \int_{g\in\Gamma\backslash G:\beta_v(g)=Haar} d\overline{\mu}^+ (g)$$

By lemma 28, $Haar (S^n_F) \neq 0$ and, thus,

$$\int_{g\in\Gamma\backslash G:\beta_v(g)=Haar} d\overline{\mu}^+ (g) = 0.$$

We will conclude that μ^+ is determined on S^n_{cusp} by means of lemma 29, which we state separately because it will be of independent use in chapter 6.

Definition 13 *: Let $\Gamma \subset G = Isom^+(H^n)$ be a cocompact discrete subgroup, $v \in \partial_\infty H^n$, $T_v \subset Isom^+(H^n)$ the subgroup defined in section 4.3 and β a decomposition map for the right-hand action of T_v, as defined in 4.3. Let*

$$H_v = \{g \in \Gamma\backslash G : \beta_v(g) = Haar\}.$$

Lemma 29 *: Let $v_0, \ldots, v_n$ be the vertices of a regular ideal simplex in H^n and $\overline{\mu}^+$ a probability measure on $\Gamma\backslash G := \Gamma\backslash Isom^+(H^n)$, invariant with respect to the right-hand action of R^+. If $\overline{\mu}^+(H_{v_i}) = 0$ for $i = 0, \ldots, n$, then $\overline{\mu}^+$ is supported on S^n_{cusp}.*

Proof: : Let

$$A_i = \{g \in \Gamma\backslash G : gv_i \text{ is cusp of } \Gamma\}$$

and

$$B_i = \{g \in \Gamma\backslash G : \Gamma\backslash\Gamma g N_{v_i} \text{ is compact }\}.$$

We have

$\Gamma\backslash G - S^n_{cusp} = \Gamma\backslash G - \cap_{i=0}^n A_i = \Gamma\backslash G - \cap_{i=0}^n B_i = \cup_{i=0}^n \Gamma\backslash G - B_i$,
where the second equality holds by lemma 27.

If e is a T_{v_i}- ergodic measure supported on a compact N_{v_i}-orbit, then

$$e(\Gamma\backslash G - B_i) = 0.$$

Thus (abbreviating $\beta_g := \beta_{v_i}(g)$),

$$\overline{\mu}^+(\Gamma\backslash G - B_i) = \int_{\Gamma\backslash G} \beta_g(\Gamma\backslash G - B_i)\, d\overline{\mu}^+(g)$$

$$= \int_{H_{v_i}} \beta_g(\Gamma\backslash G - B_i)\, d\overline{\mu}^+(g) + \int_{\Gamma\backslash G - H_{v_i}} \beta_g(\Gamma\backslash G - B_i)\, d\overline{\mu}^+(g)$$

$$= Haar(\Gamma\backslash G - B_i)\, \overline{\mu}^+(H_{v_i}) + \int_{\Gamma\backslash G - H_{v_i}} \beta_g(\Gamma\backslash G - B_i)\, d\overline{\mu}^+(g)$$

$$= Haar(\Gamma\backslash G - B_i)\ \mathbf{0}\ + \int_{\Gamma\backslash G - H_{v_i}} \mathbf{0}\ d\overline{\mu}^+(g) = \mathbf{0}$$

and, therefore,

$$\overline{\mu}^+\left(\Gamma\backslash G - S^n_{cusp}\right) = \overline{\mu}^+(\cup_{i=0}^n \Gamma\backslash G - B_i) \leq \sum_{i=0}^n \overline{\mu}^+(\Gamma\backslash G - B_i) = 0.$$

□

We are now going to finish the proof of theorem 3:

We know (from the proof of lemma 21) that $\Phi_x\left(c_\epsilon^+\right) \geq \Phi_x\left(c_\epsilon\right) \geq 1$ for all $x \in N_{[\epsilon,\infty]}$. F is a closed totally geodesic hypersurface. Therefore $F \subset N_{[\epsilon,\infty]}$ for sufficiently small ϵ. We conclude $\Phi_x\left(c_\epsilon^+\right) \geq 1$ for all $x \in F$.

For $x \in N$ let S_x^n be the set of straight n-simplices Δ containing x in their image. Define $R := \frac{V_n}{inj(N)} > 0$. A straight n-simplex can't cover x more than R times, hence $c_\epsilon^+\left(S_x^n\right) \geq \frac{1}{R}$ for all $x \in F$. We claim that this implies $\mu^+\left(S_x^n\right) \geq \frac{1}{R}$ for all $x \in F$.

Namely, since the complement of S_x^n is open, we can apply part (ii) of lemma 16 to get
$\mu^+\left(\Gamma\backslash\left(\Pi_{j=0}^n\overline{H^n}-D\right)-S_x^n\right) \leq c_\epsilon^+\left(\Gamma\backslash\left(\Pi_{j=0}^n\overline{H^n}-D\right)-S_x^n\right)$. Hence,
$\mu^+\left(S_x^n\right) \geq \mu^+\left(\Gamma\backslash\left(\Pi_{j=0}^n\overline{H^n}-D\right)\right)-c_\epsilon^+\left(\Gamma\backslash\left(\Pi_{j=0}^n\overline{H^n}-D\right)\right)+\frac{1}{R}$.
To control $\mu^+\left(\Gamma\backslash\left(\Pi_{j=0}^n\overline{H^n}-D\right)\right)-c_\epsilon^+\left(\Gamma\backslash\left(\Pi_{j=0}^n\overline{H^n}-D\right)\right)$, choose (for some fixed $\epsilon_1 < \epsilon_2$) a continuous function f with values in $[0,1]$ which is zero on S_{ϵ_1} and is one on the complement of S_{ϵ_2}, where S_{ϵ_i} is the set of simplices of volume smaller than $V_n - \epsilon_i$ as in the proof of lemma 22.
f has compact support (this is by the way the point where we use that we are working on $\Pi_{j=0}^n\overline{H^n}-D$ rather then $\Pi_{j=0}^n H^n - D$). Hence, $\mu^+(f) - c_\epsilon^+(f)$ tends to zero by the definition of weak-*-convergence. But $\mu^+\left(\Gamma\backslash\left(\Pi_{j=0}^n\overline{H^n}-D\right)\right)$ equals $\mu^+(f)$ and the difference between $c_\epsilon^+\left(\Gamma\backslash\left(\Pi_{j=0}^n\overline{H^n}-D\right)\right)$ and $c_\epsilon^+(f)$ is certainly smaller than $c_\epsilon^+\left(S_{\epsilon_2}\right)$, which tends to zero by the argument in the proof of lemma 22. Thus, for sufficiently small ϵ, the difference $\mu^+\left(\Gamma\backslash\left(\Pi_{j=0}^n\overline{H^n}-D\right)\right) - c_\epsilon^+\left(\Gamma\backslash\left(\Pi_{j=0}^n\overline{H^n}-D\right)\right)$ becomes as small as one wishes. Hence $\mu^+\left(S_x^n\right) \geq \frac{1}{R}$.

But F is totally geodesic and so the set of straight simplices containing some $x \in F$ consists of two kinds of simplices:

-simplices intersecting F transversally, and

-simplices with a vertex in F.

μ^+ vanishes on the second set, since it is determined on S_{cusp}^n and the closed totally geodesic hypersurface F can't have cusps. Thus, we obtain

$$\mu^+\left(S_F^n \cap S_x^n\right) \geq \frac{1}{R}$$

and, consequently, $\mu^+\left(S_F^n\right) \geq \frac{1}{R} > 0$. □

Remark: If $\mu^+ |_{Isom^+(H^n)} = 0$, then $\mu^- |_{Isom^+(H^n)} \neq 0$ by lemma 26 and lemma 23, and we get with an analogous proof $\mu^-\left(S_F^n\right) \neq 0$.

4.5 A remark on rigidity

Call a hyperbolic manifold $M = \Gamma\backslash H^n$ Gieseking-like, if there is a regular ideal triangulation of H^n s.t. all ideal vertices of this triangulation are parabolic fixed points of Γ, see page 107. One deduces from the results of this chapter:

Rigidity of efficient fundamental cycles: If M has a hyperbolic structure of finite volume which is not Gieseking-like, then the only efficient fundamental cycles are Gromov's smearing cycles.

We will provide the proof of this fact in the course of the proof of theorem 6, see page 109.

We want to end this chapter with a remark about why this is a restatement of rigidity of hyperbolic structures.

Mostow's rigidity theorem: *Let M_1, M_2 be compact manifolds, of the same dimension $n \geq 3$, such that their interiors admits hyperbolic metrics of finite volume. If $f : M_1 \rightarrow M_2$ is a homotopy equivalence, then there is an isometry $g : M_1 \rightarrow M_2$ which is homotopic to f.*

Mostow's proof used ergodic theory and analysis of quasiconformal mappings, cf., [61], and it actually applied not only to H^n, but to all rank-1 symmetric spaces of noncompact type. One of Gromov's motivations to consider the simplicial volume was to give a more topological proof of this theorem for the special case of real hyperbolic manifolds.
The following facts are (in the case of closed manifolds) not too hard to show, cf., [61],[6], (in the non-closed case one needs for the first fact also [51]):
- if f is a homotopy equivalence, then its lift $\tilde{f} : H^n \rightarrow H^n$ extends to a continuous map $\partial_\infty H^n \rightarrow \partial_\infty H^n$, satisfying $f_*(\gamma)\tilde{f} = \tilde{f}\gamma$ for all $\gamma \in \pi_1 M_1$,
- assume that there exists an isometry $q \in Isom(H^n)$ such that $q\mid_{\partial_\infty H^n} = \tilde{f}\mid_{\partial_\infty H^n}$, then there is an isometry $F : M_1 \rightarrow M_2$ with $\tilde{F} = q$ such that F and f are homotopic.

In view of these two facts, the proof of Mostow's rigidity reduces to the proof of the following statement: If $f : M_1 \rightarrow M_2$ is a homotopy equivalence, then exists an isometry $q \in Isom(H^n)$ such that $q\mid_{\partial_\infty H^n} = \tilde{f}\mid_{\partial_\infty H^n}$.

Gromov observed, cf., [61] that (in dimensions $n \geq 3$) a map $g : \partial_\infty H^n \rightarrow \partial_\infty H^n$ is an extension of an isometry $q \in Isom(H^n)$ iff it maps (the vertices of) regular ideal simplices to (the vertices of) regular ideal simplices.

Hence, a statement equivalent to Mostow rigidity is the following: If f is a homotopy equivalence between finite-volume hyperbolic manifolds of dimension ≥ 3, then f maps regular ideal simplices to regular ideal simplices.

M_1 and M_2 have equal simplicial volume, since they are homotopy equivalent. If c_ϵ is a sequence of a (straight, nondegenerate) cycles representing $[M_1, \partial M_1]$ with l^1-norm converging to $\| M_1, \partial M_1 \|$, then the l^1-norms of $f_* (c_\epsilon)$ converge to $\| M_2, \partial M_2 \|$, because of $| f_* (c_\epsilon) | \leq | c_\epsilon |$. Hence, f_* maps efficient fundamental cycles to efficient fundamental cycles.

Hence, if M_1 and M_2 are not Gieseking-like, we get as a consequence of the "rigidity of efficient fundamental cycles": f maps regular ideal simplices to regular ideal simplices and it preserves the "equidistribution" on the space of regular ideal simplices. (This would, of course, also follow from Mostow rigidity, since an isometry clearly preserves the equidistribution.)

Chapter 5

3-manifolds of higher genus boundary

We recollect the structure theory of 3-manifolds, which will be needed in section 5.2. We assume to have a compact orientable 3-manifold M.

By Kneser's theorem, any compact 3-manifold is a connected sum of finitely many "prime factors", c.f., [30], 3.15. If M is orientable, then the prime factors are irreducible or $S^2 \times S^1$, by [30], 3.13., where a 3-manifold is termed irreducible if each embedded 2-sphere bounds an embedded 3-ball in M.

Any 3-manifold can be cut along finitely many disks to get pieces which have incompressible (i.e., π_1-injective) boundary. Indeed if ∂M is not incompressible, then by Papakyriakopoulos theorem, c.f., [30], 4.2., there is a properly embedded disk in M such that its boundary represents a nontrivial element in $\pi_1 \partial M$. We cut M along this disk. If the obtained manifold still does not have incompressible boundary, we find another disk and cut again. Cutting 3-manifolds along disks increases the Euler characteristic of the boundary. Hence, the processus of cutting along disks has to stop after finitely many steps.

Now let M be a compact irreducible 3-manifold with incompressible boundary. We use the Jaco-Shalen-Johannson theorem to decompose M, following [48].
In section 2 of [48], there is defined a so-called W-decomposition, which is a family of disjoint properly embedded incompressible tori and annuli. Denoting M_i the connected manifolds obtained after cutting M along the tori and annuli from the W-decomposition, proposition 3.2. of [48] asserts that the M_i are either Seifert fibered (i.e., finitely covered by S^1-bundles), I-bundles or "simple", where simple in the terminology of [48] means that any properly *immersed* incompressible torus or annulus can be isotoped into ∂M_i.

By Thurston's hyperbolization theorem, the simple pieces admit a hyperbolic metric. Moreover, we have

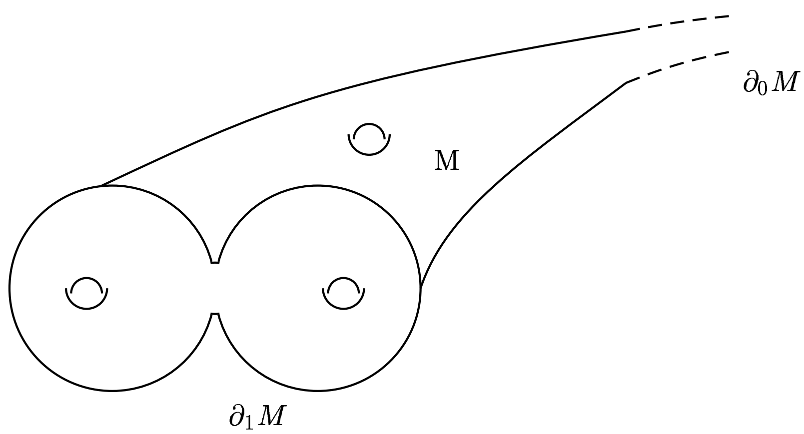
Proposition 8 ([62], Theorem 3): *Let M be a hyperbolic manifold such that $int(M)$ admits a complete hyperbolic metric. There is a totally geodesic surface in $int(M)$ for each non-torus component of ∂M if and only if any incompressible, properly embedded annulus is boundary-parallel and ∂M is incompressible.*
(Note that the condition "∂M incompressible", missing in [62], is necessary to exclude handlebodies. It is needed in the proof to guarantee that DM is irreducible.)

Therefore the "simple" pieces admit an (incomplete) hyperbolic metric such that the toral boundary components correspond to cusps and the boundary components of higher genus are totally geodesic.

5.1 Hyperbolic manifolds with geodesic boundary

If M is a hyperbolic manifold, define its convex core to be the minimal closed convex subset of M whose embedding induces a homotopy equivalence. The boundary of the convex core is a hyperbolic surface which, in general, will be pleated. M is said to "have totally geodesic boundary" if the convex core is homeomorphic to M and its boundary is totally geodesic. Note that we admit that the convex core may have cusps. In dimension 3, the totally geodesic boundary (as well as the boundary of the convex core of any geometrically finite hyperbolic 3-manifold) consists of all non-torus components of the topological boundary ∂M.

Although hyperbolic structures of infinite volume are not rigid, it follows easily from Mostow's rigidity theorem that on a manifold of dimension ≥ 3, there can be at most one hyperbolic metric g_0 admitting totally geodesic boundary. In particular, the volume of the convex core with respect to the metric g_0 is a topological invariant. Actually, it was shown in [9] that g_0 minimizes the volume of the convex core among all hyperbolic metrics on M.



Lemma 30 *: Let M be a compact 2-manifold with boundary $\partial M = \partial_0 M \cup \partial_1 M$, such that $M - \partial_0 M$ admits an incomplete hyperbolic metric of finite volume with $\partial_1 M$ totally geodesic and the ends corresponding to $\partial_0 M$ complete. Then*

$$\| M, \partial M \| = \frac{1}{V_2} Vol(M).$$

Proof: It is well-known that any (possibly bounded) surface of non-positive Euler characteristic satisfies $\| M, \partial M \| = -2\chi(M)$. By the Gauß-Bonnet-formula, this is the same as $\frac{1}{\pi} Vol(M)$. □

Corollary 6 *: Let $n \geq 3$ and let M be a compact n-manifold with boundary $\partial M = \partial_0 M \cup \partial_1 M$, such that $M - \partial_0 M$ admits an incomplete hyperbolic metric of finite volume with $\partial_1 M$ totally geodesic and the ends corresponding to $\partial_0 M$ complete. Then,*

$$\| M, \partial M \| > \frac{1}{V_n} Vol(M).$$

Proof: $\| M, \partial M \| \geq \frac{1}{V_n} Vol(M)$ follows from the familiar argument that fundamental cycles can be straightened to invoke only simplices of volume smaller than V_n or, equivalently, from the trivial inequality $\| DM \| \leq 2 \| M, \partial M \|$.

Suppose we had equality. Glue two differently oriented copies of M via $id|_{\partial M}$ to get $N = DM$. The incomplete metrics can be glued along the totally geodesic boundary and, hence, we have that N is a complete hyperbolic manifold of finite volume $Vol(N) = 2Vol(M)$. A relative fundamental cycle for M of norm smaller than $\frac{1}{V_n} Vol(M) + \frac{\epsilon}{2}$ fits together with its reflection to give a relative fundamental cycle $c_\epsilon^\prime$ on N of l^1-norm smaller than $2\frac{1}{V_n} Vol(M) + \epsilon = \frac{1}{V_n} Vol(N) + \epsilon$, consisting of simplices which do not intersect transversally the totally geodesic surface $\partial M \subset N$. That means $c_\epsilon^{\prime\pm}(S_{\partial M}^n) = 0$, what implies $c_\epsilon^\pm(S_{\partial M}^n) = 0$, since h_ϵ may be the identity close to $\partial M \subset N$ and because straightening in DM preserves the set of simplices not intersecting transversally the totally geodesic surface ∂M.

By lemma 20, we have some accumulation point μ of $\{c_\epsilon\}$ for a sequence of ϵ tending to zero. Similarly to lemma 28, it is easy to see that $S_{\partial M}^n$ is open in $SS_n(N)$. Hence, we can apply part (ii) of lemma 16 to get $\mu^+(S_{\partial M}^n) = 0$. But this contradicts theorem 3. □

Theorem 4 *: (a) Let $n \geq 3$ and let $M_i, i = 1, 2$ be compact n-manifolds with boundaries $\partial M_i = \partial_0 M_i \cup \partial_1 M_i$, such that $M_i - \partial_0 M_i$ admit incomplete hyperbolic metrics of finite volume with $\partial_1 M_i$ totally geodesic and the ends corresponding to $\partial_0 M_i$ complete. If $\partial_1^\prime M_i \subset \partial_1 M_i$ are non-empty sets of connected components of $\partial_1 M_i$, $f : \partial_1^\prime M_1 \to \partial_1^\prime M_2$ is an isometry, and $M = M_1 \cup_f M_2$, then*

$$\| M, \partial M \| < \| M_1, \partial M_1 \| + \| M_2, \partial M_2 \|.$$

(b) Let $n \geq 3$ and let M_0 be a compact n-manifold with boundary $\partial M_0 = \partial_0 M_0 \cup \partial_1 M_0$, such that $M_0 - \partial_0 M_0$ admits an incomplete hyperbolic metric of finite volume with $\partial_1 M_0$ totally geodesic and the ends corresponding to $\partial_0 M_i$ complete. If $\partial_1' M_0 \subset \partial_1 M_0$ is a non-empty set of connected components of $\partial_1 M_0$, and $f : \partial_1 M_0 \to \partial_1 M_0$ is an orientation-reversing involutive isometry of $\partial_1' M_0$ exchanging the connected components by pairs, then, letting $M = M_0/f$,

$$\| M, \partial M \| < \| M_0, \partial M_0 \| .$$

Proof: (a) The incomplete hyperbolic metrics on M_1 and M_2 glue together to give a complete hyperbolic metric on M of volume $Vol\,(M) = Vol\,(M_1) + Vol\,(M_2)$. By the Gromov-Thurston theorem, we know that $\| M, \partial M \| = \frac{1}{V_n} Vol\,(M)$ and, by corollary 8, we have $\| M_i, \partial M_i \| > \frac{1}{V_n} Vol\,(M_i)$. The claim follows.
The proof of (b) is similar. □

Corollary 7 *: Let $n \geq 4$, and let M_1, M_2, M_0 and $\partial_1' M_i$ satisfy all assumptions of theorem 3.*
If $f : \partial_1 M_1 \to \partial_1 M_2$ is a homeomorphism, then
$\| M_1 \cup_f M_2, \partial\,(M_1 \cup_f M_2) \| < \| M_1, \partial M_1 \| + \| M_2, \partial M_2 \|$.
If $f : \partial_1 M_0 \to \partial_1 M_0$ is an orientation-reversing involutive homeomorphism of $\partial_1 M_0$ exchanging the boundary components by pairs, then
$\| M_0/f, \partial\,(M_0/f) \| < \| M_0, \partial M_0 \|$.

Proof: Since the totally geodesic boundary is a hyperbolic manifold of dimension ≥ 3, f is homotopic to an isometry g by Mostow rigidity. By homotopy equivalence, $\| M_1 \cup_f M_2, \partial\,(M_1 \cup_f M_2) \| = \| M_1 \cup_g M_2, \partial\,(M_1 \cup_g M_2) \|$, resp. $\| M_0/f, \partial\,(M_0/f) \| = \| M_0/g, \partial\,(M_0/g) \|$. Then apply theorem 4. □

5.2 Doubling 3-manifolds

For an oriented manifold M, let DM denote the double of M, defined by glueing two differently oriented copies of M via the identity of ∂M. It is trivial that $\| DM \| \leq 2 \| M, \partial M \|$. Theorem 2 implies: if M is a compact 3-manifold with $\| \partial M \| = 0$, then $\| DM \| = 2 \| M, \partial M \|$. We will show that this is actually an if-and-only-if condition.

Theorem 5 *: If M is a compact 3-manifold with $\| \partial M \| > 0$, then*
$\| DM \| < 2 \| M, \partial M \|$.

Proof: First, we want to reduce the claim to compact irreducible manifolds. For this purpose, note that $\| M_1 \sharp M_2, \partial\,(M_1 \sharp M_2) \| = \| M_1, \partial M_1 \| + \| M_2, \partial M_2 \|$ holds also for manifolds with boundary (of dimension ≥ 3). Indeed, defining a fundamental class of the wedge $M_1 \vee M_2$ as $[M_1 \vee M_2, \partial(M_1 \vee M_2)] := i_{1*}[M_1, \partial M_1] + i_{2*}[M_2, \partial M_2]$, for the inclusions $i_1 : M_1 \to M_1 \vee M_2$ and $i_2 : M_2 \to M_1 \vee M_2$, we can define the simplicial volume $\parallel M_1 \vee M_2, \partial(M_1 \vee M_2) \parallel$ as the infimum over the l^1-norms of relative cycles representing $[M_1 \vee M_2, \partial(M_1 \vee M_2)]$ and it is implicit in the proof of theorem 2 that with this definition $\parallel M_1 \vee M_2, \partial(M_1 \vee M_2) \parallel = \parallel M_1, \partial M_1 \parallel + \parallel M_2, \partial M_2 \parallel$ holds. Consider the projection from $M_1 \sharp M_2$ to $M_1 \vee M_2$ which pinches the connecting sphere to a point. It induces an isomorphism of fundamental groups (this is the point, where dimension ≥ 3 is needed) and has degree 1. By the same argument as in the proof of lemma 1, we get $\parallel M_1 \sharp M_2, \partial(M_1 \sharp M_2) \parallel = \parallel M_1 \vee M_2, \partial(M_1 \vee M_2) \parallel$.

In the same way we get more generally that identification of pairs of points in manifolds does not change the simplicial volume. In particular one has $\parallel D(M_1 \sharp M_2) \parallel = \parallel DM_1 \parallel + \parallel DM_2 \parallel$.

Any compact 3-manifold is a connected sum of 3-manifolds which are either irreducible or $S^1 \times S^2$. Since $\parallel S^1 \times S^2 \parallel = 0$, we can reduce the claim to irreducible 3-manifolds because of $\parallel D(M_1 \sharp M_2) \parallel = \parallel DM_1 \parallel + \parallel DM_2 \parallel$ and $\parallel M_1 \sharp M_2, \partial(M_1 \sharp M_2) \parallel = \parallel M_1, \partial M_1 \parallel + \parallel M_2, \partial M_2 \parallel$.

By the discussion at the beginning of this chapter, we can cut M along properly embedded disks and incompressible properly embedded annuli and tori such that the obtained pieces are either Seifert fibered, I-bundles or "simple", where the "simple" pieces admit an (incomplete) hyperbolic metric such that the toral boundary components correspond to cusps and the boundary components of higher genus are totally geodesic.

We argue that these pieces M_i satisfy the claim of theorem 5. For a Seifert fibration, the boundary consists of tori, hence, there is nothing to prove. If M_i is an I-bundle, then DM is an S^1-bundle and $\parallel DM_i \parallel = 0$ holds by corollary 6.5.3 of [61]. (But, if $\parallel \partial M_i \parallel > 0$, then $\parallel M_i \parallel \geq \frac{1}{3} \parallel \partial M_i \parallel > 0$ by the argument on page 97.) Finally, if M_i is "simple" and $\parallel \partial M_i \parallel > 0$, then the totally geodesic boundary of the hyperbolic structure is non-empty (not all boundary components can correspond to cusps), and $\parallel DM_i \parallel < 2 \parallel M_i, \partial M_i \parallel$ holds by theorem 4.

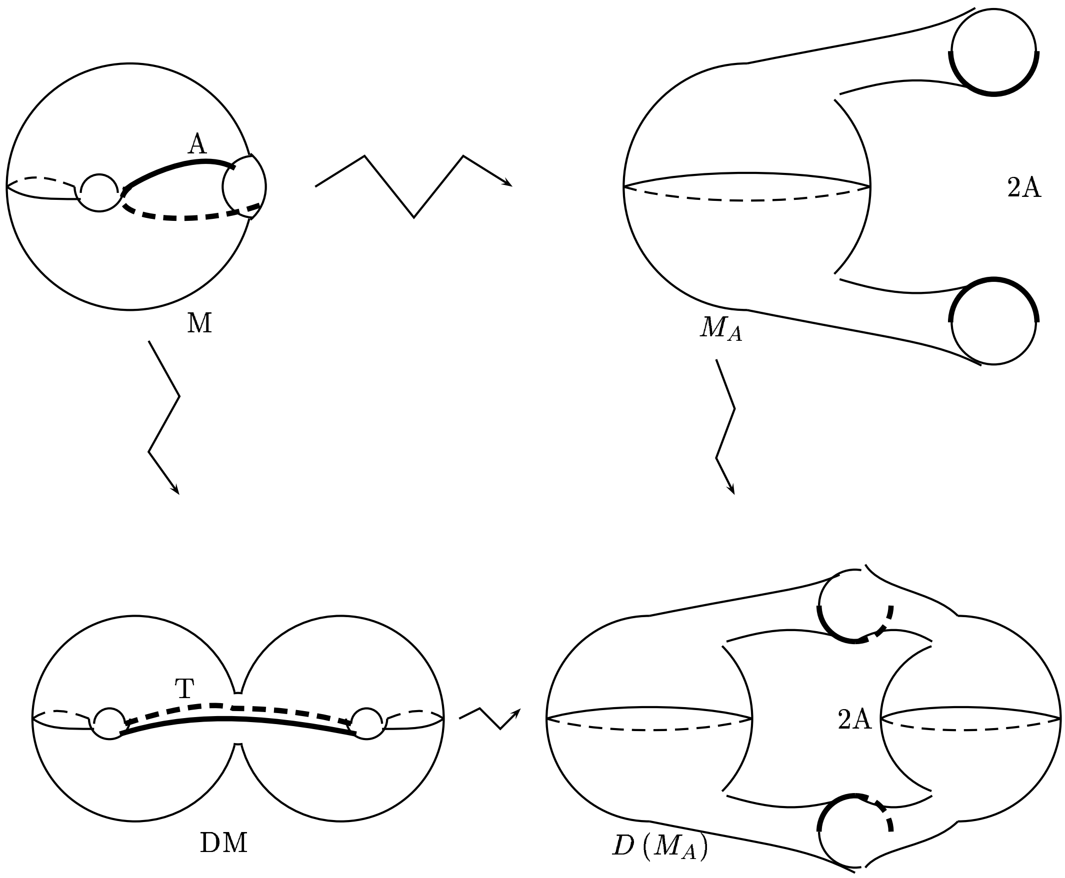
If M is a compact irreducible 3-manifold with $\parallel \partial M \parallel > 0$, then clearly $\parallel \partial M_i \parallel > 0$ holds for at least one of the pieces in its Jaco-Shalen-Johannson decomposition. To finish the proof of theorem 5, we still need the following lemma 31, where M_F is defined as in the introduction.

Lemma 31 *: Let M be a compact 3-manifold and F an incompressible, properly embedded annulus, torus or disk.*
If $\parallel D(M_F) \parallel < 2 \parallel M_F, \partial M_F \parallel$, then $\parallel DM \parallel < 2 \parallel M, \partial M \parallel$.

Proof: The claim follows from corollary 7 if F is a torus.
From now on, consider $F = A$ an annulus.

The claim will follow from the somewhat paradoxical observation that
$\| M_A, \partial M_A \| \leq \| M, \partial M \|$, but $\| D(M_A) \| \geq \| DM \|$.

DM_A is obtained from DM by cutting off an incompressible torus $T = DA$ and identifying afterwards in a different way the pairs of incompressible annuli which are "halves" of the same copy of T. In other words $(D(M_A))_{2A} = (DM)_T$. Look at the following picture, where dimension has been lowered by one.



Hence, we get $\| D(M_A) \| \geq \| (DM)_T \| = \| DM \|$, what implies the claim of lemma 31 because of $\| M_A, \partial M_A \| \leq \| M, \partial M \|$.
If F is a disk, the argument is the same □
This finishes the proof of theorem 5. □

Handlebodies.

Note that for any n-dimensional compact manifold holds:
$\| M, \partial M \| \geq \frac{1}{n} \| \partial M \|$. Namely, the boundary operator $\partial : H_n(M, \partial M) \rightarrow H_{n-1}(\partial M)$ maps the relative fundamental class $[M, \partial M]$ to the fundamental class $[\partial M]$. It is obvious that $\| \partial \| \leq n+1$. For a representative $\sum_{i=1}^{r} a_i \sigma_i$ of $[M, \partial M]$ one even gets $\| \partial \sum_{i=1}^{r} a_i \sigma_i \| \leq n \| \sum_{i=1}^{r} a_i \sigma_i \|$, because each σ_i has to have at least one face not in ∂M, cancelling against some other face.
Let H_g denote the 3-dimensional handlebody of genus g. H_g is a (g-1)-fold covering of H_2, hence, $\| H_g, \partial H_g \| = C(g-1)$. From the above argument follows $C \geq \frac{4}{3}$. (In fact, we showed in [38], by constructing triangulations of handlebodies, that $\frac{4}{3} \leq C \leq 3$.)

The double of H_g is the g-fold connected sum $S^2 \times S^1 \sharp \ldots \sharp S^2 \times S^1$, whose simplicial volume vanishes. This shows that there may be manifolds M of arbitrarily large simplicial volume with $\| DM \| = 0$.

In fact, we can give a precise condition when $\| DM \| = 0$ holds for a compact 3-manifold M. Recall from the introduction of this chapter that any compact 3-manifold can be cut along disks into finitely many pieces M_j which have incompressible boundary. We claim that $\| DM \| = 0$ if and only if all these M_j have a Jaco-Shalen-Johannson-decomposition without "simple" pieces in the sense of [48].

Namely, if M_j has a JSJ-decomposition without simple pieces, one easily gets $\| DM_j \| = 0$, thus, DM_j is a graph manifold by [57]. DM is obtained from $\cup DM_j$ by cutting off some 3-balls and identifying their boundaries in pairs. By the same argument as in the proof of lemma 4 in [57], DM is then a graph manifold and $\| DM \| = 0$.

To prove the other implication, assume that M_j had a "simple" piece H, on which we put a hyperbolic metric with totally geodesic boundary. Inside DM this gives us a submanifold H', obtained from two copies of H by glueing via the identity on *a submanifold of* ∂H. Clearly, H' admits a hyperbolic metric with totally geodesic boundary. From proposition 8, one can conclude $\| DM \| > 0$.

Chapter 6

Gromov norm and branching of laminations

In this chapter, we always consider foliations/laminations of codimension 1.
For more background on the Gromov norm of foliations (and foliations in general), we refer to [11].

Definition 14 *: Let M be a manifold, possibly with boundary, and $\mathcal{F}$ a lamination of M. Define*

$$\| M, \partial M \|_{\mathcal{F}} := inf \left\{ \sum_{i=1}^{r} | a_i | : \sum_{i=1}^{r} a_i \sigma_i \in [M, \partial M], \sigma_i \ transversal \ to \ \mathcal{F} \right\}.$$

Here, a simplex σ is said to be transversal to the lamination $\mathcal{F}$, if the induced lamination $\mathcal{F} |_\sigma$ is topologically conjugate to the subset of a foliation of σ by level sets of an affine map $f : \sigma \to R$.

A typical example for non-transversality of a tetrahedron Δ to a lamination $\mathcal{F}$ is the following: let e_1, e_2, e_3 be the three edges of a face $\tau \subset \Delta$. If $\mathcal{F} |_\tau$ contains three lines which connect respectively e_1 to e_2, e_2 to e_3 and e_3 to e_1, then Δ can't be transversal to $\mathcal{F}$.

Remark: If $\mathcal{F}$ is not transversal to ∂M nor contains ∂M as a leaf, then $\| M, \partial M \|_{\mathcal{F}} = \infty$. Otherwise the foliated Gromov norm is finite. In the following, we will always assume that $\mathcal{F}$ is transversal to ∂M or that ∂M is a leaf of $\mathcal{F}$.

6.1 Gromov norm of confoliations

This section will not be used in the following two sections. Its content is to generalize the notion of foliated Gromov norm to contact structures and confoliations, as it was suggested in one of the concluding remarks in [11]. We define a Gromov norm for contact structures and a Gromov norm for confoliations (which does not necessarily agree with the foliated Gromov norm resp. the Gromov norm of contact structures if the confoliation happens to be a foliation resp. a contact structure). We calculate this confoliated Gromov norm in some examples. Examples of confoliations with non-trivial Gromov norm can be obtained by perturbing asymptotically separated foliations on hyperbolic manifolds (see section 6.3) via the Eliashberg-Thurston theorem (example 4). In principle, we think that, once a classification of contact structures on hyperbolic manifolds will be at hand (a first step being done in [31]), it should be possible to use the results of section 4 to decide which contact structures have trivial or nontrivial Gromov norm, in a similar spirit as it will be done for foliations in section 6.3.

Let M be a smooth closed oriented 2n-1-manifold with volume form $dvol$. A plane field is a 2n-2-dimensional subbundle ξ of TM. It may be represented as $\xi = ker\alpha$ for a 1-form α. ξ is a positive confoliation if $\alpha \wedge (d\alpha)^{n-1} = f dvol$ with $f \geq 0$ everywhere. It is a positive contact structure if $f > 0$ everywhere. The term 'confoliation' resp. 'contact structure' will in the following mean positive confoliations resp. positive contact structures.

In what follows we restrict to the case of 3-manifolds, that is $n = 2$.
If σ is a smooth singular simplex in M and ξ a confoliation, then $\xi \cap T\partial\sigma$ is a vector field, hence integrable. We say that σ is in general position to ξ if
- its 1-skeleton is transverse to ξ,
- $\xi \cap T\partial\sigma$ vanishes exactly in two vertices of σ.

If ξ happens to be a contact structure, we say that σ is transversal to ξ if it is in general position and any point in $\partial\sigma$ belongs to a flowline connecting the zeroes of $\xi \cap T\partial\sigma$.
Remark: If ξ is a contact structure on M and $x \in M$, there exists a coordinate neighborhood of x in which $\xi = dx_{2n-1} - \sum_{i=1}^{n-1} dx_{2i-1}dx_{2i}$, by the Darboux

lemma. Any *straight* simplex contained in this neighborhood is transversal to ξ.

Definition 15 *For a confoliation ξ on a smooth closed oriented manifold M let $C_*^\xi(M;R)$ be the subspace of the singular chain complex $C_*(M;R)$ generated by singular simplices in general position to ξ.*

The inclusion $C_*^\xi(M;R) \to C_*(M;R)$ commutes with the boundary operator and induces hence a morphism of homology groups.

Definition 16 *For a confoliation ξ on a smooth closed oriented manifold M and a homology class $h \in H_*(M;R)$ define the confoliated Gromov-norm $\| h \|_\xi$ as the infimum of $\sum_{i=1}^k | a_i |$ over all cycles $\sum a_i \sigma_i \in C_\xi(M;R)$, representing h in $H_*(M;R)$.*

In particular, $\| h \|_\xi = \infty$ if $h \notin im\left(H_*^\xi(M;R) \to H_*(M;R)\right)$.

Definition 17 *For a confoliation ξ on a smooth closed oriented manifold M define $\| M \|_\xi$ as the confoliated Gromov-norm of the fundamental class $[M] \in H_3(M;R)$.*

It is maybe worth pointing out that, if ξ happens to be tangential to a foliation $\mathcal{F}$, one has $\| M \|_\mathcal{F} \geq \| M \|_\xi$ (but not necessarily equality). Hence, $\| M \|_\xi$ is a weaker invariant in this case. In particular, the following lemma 32 holds, for this weaker invariant, without the additional assumption on tautness in lemma 2.2.3. of [11].

Lemma 32 *Let ξ_j be a sequence of confoliations converging to ξ.*
Then $\| h \|_\xi \geq limsup \| h \|_{\xi_j}$.

Proof: Morally the lemma is due to the fact that the singular foliation, induced by a confoliation on a codimension-1 face, can only have elliptic singularities. The picture that flow-lines flow from one singularity to another can only happen for flow-lines connecting vertices of the simplex.
Let $\sum_i a_i \sigma_i$ with $\sum | a_i | \leq \| h \|_\xi + \epsilon$ and σ_i in general position to ξ. It suffices to show that σ_i is in general position to ξ_j, if ξ_j is sufficiently close to ξ.
The first condition is clearly satisfied: if the 1-skeleton is transversal to ξ, there is a positive lower bound on the angle formed with ξ, hence a slightly weaker positive bound on the angle formed with ξ_j.
To check the second condition it suffices to note that a foliation of the boundary of a 3-simplex having elliptic singularities other than the vertices could not be transversal to the 1-skeleton (and that a small deformation of the standard foliation on S^2 necessarily has two eliptic singularities close to the north and south

pole.) □

Gromov norm of contact structures. As for foliations, one can also for contact structures define an invariant that is finer as the Gromov norm of con- foliations. (This definition was suggested in one of the final remarks of [11].) Namely, let $C_*^{\xi,ct}(M;R) \subset C_*^{\xi}(M;R) \subset C_*(M;R)$ be the subsace generated by all simplices **transversal** to ξ and carry along the obvious analoga of the above definitions to define $\| M \|_\xi^{ct}$. Clearly, $\| M \|_\xi \leq \| M \|_\xi^{ct}$.

Examples

Example 1: *Tight contact structures on T^3.*
Any tight contact structure ξ on the 3-torus satisfies

$$\| T^3 \|_\xi^{ct} = \| T^3 \|_\xi = 0.$$

Indeed, by a theorem of Giroux, for any tight contact structure ξ on $T^3 = R^2/Z^2 \times R/2\pi Z$ exists an integer n such that ξ is isotopic to the contact structure ξ_n defined as kernel of $\alpha_n = cos(n\theta)\,dx + sin(n\theta)\,dy$. Consider $f : T^3 \to T^3$ defined by $f(x,y,\theta) = (2x, 2y, \theta)$. One checks that $f^*\alpha_n = 2\alpha_n$. Hence, for all $v \in \xi_n$ we get $\alpha_n(f_*v) = 2\alpha_n(v) = 0$, i.e., $f_*v \in \xi_n$.
Consider any fundamental cycle $\sum_{i=1}^r a_i\sigma_i$ transversal to ξ_n. (Since straight simplices contained in a Darboux neighborhood are transversal to ξ_n, such a fundamental cycle can be produced by subdividing a given fundamental cycle sufficiently often and straightening, by a standard argument invoking the Lebesgue number of a finite Darboux cover.) All $f(\sigma_i)$ are transversal to $f_*\xi_n = \xi_n$. On the other hand, $deg(f) = 4$, hence $\frac{1}{4^k}\sum_{i=1}^r a_i f^k(\sigma_i)$ is a sequence of fundamental cycles, transversal to ξ_n, with l^1-norm tending to zero if k goes to infinity.

Example 2: *Contact structures on S^3, tight and overtwisted.*
All tight contact structures on S^3 are isotopic to the standard contact structure ξ. Introducing polar coordinates on $S^3 \subset R^2 \times R^2$, we can write
$\xi = ker(r_1^2 d\phi_1 + r_2^2 d\phi_2)$. The map $f : S^3 \to S^3$ defined by $f(r_1, \phi_1, r_2, \phi_2) = (r_1, 2\phi_1, r_2, 2\phi_2)$ preserves ξ and has degree 4. The same argument as in example 1 allows to conclude that

$$\| S^3 \|_\xi = 0.$$

The same argument works also for the family of overtwisted contact structures ξ_n, considered in [25], which is obtained from ξ by Lutz modification (Dehn chirurgie) at a Hopf circle. This exhibits a family ξ_n of *overtwisted* contact structures with trivial Gromov norm $\| S^3 \|_{\xi_n} = 0$.

Example 3: *Extremal contact structures on hyperbolic surface bundles.*
Let M^3 be a Σ^2-bundle over S^1 admitting a hyperbolic metric (i.e. the monodromy $f : \Sigma^2 \to \Sigma^2$ is a pseudo-Anosov surface diffeomorphism). It is well known that for the Euler class $e(\xi)$ of any contact structure ξ one has the inequality $| e(\xi) | \leq | \chi(\Sigma) |$. A contact structure is called extremal if equality holds. We claim: if ξ is extremal, then $\| M \|_\xi = \| M \|$.
Indeed, according to [31], there exist isotopies ψ_j such that $\psi_j(\xi)$ converges to $\mathcal{F}$, the foliation by fibers. By lemma 32 and lemma 34 this implies

$$\| M \| = \| M \|_{\mathcal{F}} = lim \| M \|_{\psi_j \xi} = \| M \|_\xi .$$

Example 4: *Contact structures with nontrivial Gromov norm.*
Let M be a closed hyperbolic 3-manifold and $\mathcal{F}$ an asymptotically separated foliation on M. According to [16], there exists a sequence of contact structures converging geometrically to $\mathcal{F}$. We claim that $\| M \|_\xi > \| M \|$ if ξ is sufficiently close to $\mathcal{F}$.
Indeed, if a fundamental cycle has l^1-norm sufficiently close to $\| M \|$, then it contains some simplex one of whose 2-faces T is a triangle (singularly) foliated in such a way that to any two edges of T there exist leaves joining them as in the remark after definition 12. (This will follow from the arguments in section 6.3.) If ξ is close to $\mathcal{F}$, one can control the distance between an orbit of $\xi |_T$ and the orbit (with the same initial point) of $\mathcal{F} |_T$ (because the orbit remains a finite time in T). In particular, if ξ is sufficiently close to $\mathcal{F}$, then we have, for any pair of edges of T, a leaf of $\xi |_T$ joining them. Hence, the simplex with 2-face T is not transversal to ξ.

Confoliated bounded cohomology

Let M be a smooth closed oriented manifold and ξ a confoliation on M. Define a (not necessarily finite) norm $\| \beta \|_\xi$ for singular cochains $\beta \in C^*(M;R)$ as the supremum of $\beta(\sigma)$ over all singular simplices σ which are in general position to ξ. Define $C_\xi^*(M;R) = \{\beta \in C^*(M;R) : \| \beta \|_\xi < \infty\}$. The coboundary operator δ preserves $C_\xi^*(M;R)$, hence we may define

$$H_\xi^*(M;R) = \left(ker\delta \cap C_\xi^*(M;R)\right) / \left(im\delta \cap C_\xi^*(M;R)\right).$$

The norm $\| . \|_\xi$ induces a pseudonorm on $H_\xi^*(M;R)$.

Lemma 33 *Let $\beta \in H_\xi^*(M;R)$ and $h \in H_*(M;R)$ satisfy $< \beta, h >= 1$. Then $\frac{1}{\|\beta\|_\xi} = \| h \|_\xi$.*

Proof: $\frac{1}{\|\beta\|_\xi} \leq \| h \|_\xi$ is obvious. We prove the opposite inequality.
Recall that the value of a cohomology class β on a cycle is well defined, i.e. does not depend on the representative of β. Hence we may define $f : ker(\partial) \cap C_\xi(M;R) \rightarrow R$ by $f(z) := \beta(z)$. By the Hahn-Banach theorem, there is $\omega : C_\xi(M;R) \rightarrow R$ such that ω restricts to f on $ker(\partial)$ and that $\| \omega \|_\xi = \| f \|_\infty = sup\{\beta(z) : \| z \| = 1\}$. We claim that ω is a representative of β in $H_\xi(M;R)$.
Since the cohomology class of a cocycle is determined by its values on all cycles, we get that $[\omega] - \beta$ is in the kernel of $H_\xi(M;R) \rightarrow H^*(M;R)$. To show that $[\omega] - \beta = 0$, we consider the decomposition $C_n^\xi(M;R) = ker(\partial_n) \oplus C_n^\xi(M;R)/ker(\partial_n)$. For any representative $b \in \beta \in H_\xi(M;R)$ we have that $\omega - b$ vanishes on the first direct summand, hence corresponds to a *bounded* morphism $g : C_n^\xi(M;R)/ker(\partial_n) \rightarrow R$. Using the canonical isomorphism $C_n^\xi(M;R)/ker(\partial_n) \simeq im(\partial_n)$, and extending trivially on $C_{n-1}^\xi(M;R)/im(\partial_n)$, we get $g \in C_\xi^{n-1}(M;R)$ with $\delta g = \omega - b$.
□

Corollary 8 *: If $H_\xi^{2n-1}(M;R) = 0$, then $\| M \|_\xi = 0$.*
If $H_\xi^{2n-1}(M;R) \rightarrow H^{2n-1}(M;R)$ is surjective, then $\| M \|_\xi > 0$.

If the confoliation ξ happens to be either a contact structure or the tangent field of a foliation $\mathcal{F}$, one can modify the above definition in an obvious way to construct contact bounded cohomology $H_\xi^{*,ct}$ or foliated bounded cohomology $H_{\mathcal{F}}^*$ (cf., [11]). The above statements and their proof carry literally over.

6.2 One-sided branching

Let M be a compact, orientable 3-manifold with incompressible boundary.
Call a foliation (of a manifold with boundary) taut if it contains a circle or an arc, transversal to ∂M, which intersects every leaf transversally. That means simply that the glued foliation of the double DM is taut in the usual sense.
Leaves of taut foliations are π_1-injective. This follows for closed manifolds from Novikovs theorem, since taut foliations have no Reeb component and, for manifolds with boundary it is easily deduced by doubling (using the injectivity of $\pi_1\partial M \rightarrow \pi_1 M$).
For a foliation finitely covered by the product foliation of $S^2 \times S^1$, the leaf space of the pull-back foliation $\tilde{\mathcal{F}}$ on the universal cover $\tilde{M}$ is clearly the real line R.
Otherwise, by the Reeb stability theorem applied to the double DM, no leaf is a sphere. Hence, π_1-injectivity of the leaves implies that (the interior of) $\tilde{M}$ is foliated by planes.
By Palmeira's theorem in [49], we conclude that int$(\tilde{M})$ is homeomorphic to R^3 and that, up to homeomorphism, $\tilde{\mathcal{F}}$ is a foliation of R^3 by planes, where every

plane is properly embedded and separates R^3 into two half-spaces.
Hence, we can apply [20] to equip the leaf space of $\tilde{\mathcal{F}}$ with the structure of an order tree, where the vertices correspond to the leaves and two vertices are joined by a segment if there is a transversal arc joining the corresponding leaves. (Compare [20] for the definition of "order tree".)

$\mathcal{F}$ is then called R-covered, one-sided branched, or two-sided branched according to whether the leaf space of $\tilde{\mathcal{F}}$, considered as an order tree, is R, branched in one direction, or branched in both directions. For example, perturbations of surface bundles over S^1 are R-covered.

Since an order tree is orientable, we get a partial order on the set of leaves. Two leaves are called comparable if they are comparable with respect to this partial order, i.e., if there is a transversal arc in M joining them.

Lemma 34 *: If $\mathcal{F}$ is a sublamination of an R-covered or one-sided branched taut foliation on a 3-manifold M, then*

$$\| M, \partial M \| = \| M, \partial M \|_{\mathcal{F}} .$$

Proof: This is shown in theorems 2.2.10 and 2.5.9 of [11], assuming that M is closed. However, the proof works also for manifolds with boundary.

Indeed, since ∂M is either transversal to $\mathcal{F}$ or is a leaf of $\mathcal{F}$, the straightening defined in lemma 2.2.8 of [11], for chains with vertices on comparable leaves, is the identity on $C_* (\partial M)$. This implies, in particular, the claim for R-covered foliations. In the case of one-sided branching (say in positive direction), the argument in 2.5.9 of [11] was then to isotope a chosen lift of the finite singular chain in $\tilde{M}$ in the negative direction until its vertices are on comparable leaves. (This has to be done $\pi_1 M$-equivariantly in the sense that the projection to M stays a relative cycle.) If ∂M is a leaf of $\mathcal{F}$, then one can leave all vertices on ∂M fixed and only isotope the other vertices. If ∂M is transversal to $\mathcal{F}$, the isotopy can clearly be performed in such a way that vertices on ∂M are isotoped inside ∂M.

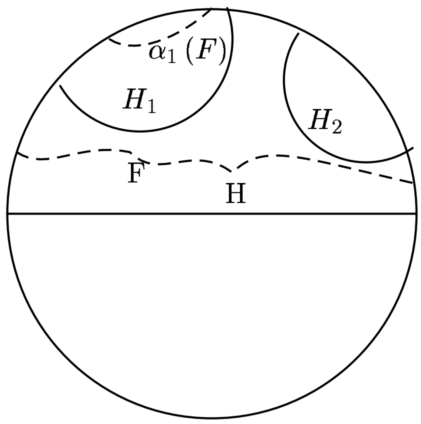
Hence, in any case, the straightening maps $C_* (\partial M)$ to $C_* (\partial M)$ and, by the five lemma, it induces the identity map in relative homology. Thus, it maps relative fundamental cycles to relative fundamental cycles transversal to $\mathcal{F}$, not increasing the l^1-norm. □

In particular, the foliated Gromov norm is a stronger invariant than the Godbillon-Vey invariant. For example, the stable foliation $\mathcal{F}$ of the geodesic flow on the unit tangent bundle $T^1 (\Gamma \backslash H^2)$ of some hyperbolic surface $M = \Gamma \backslash H^2$ is R-covered, hence $\| M \|_{\mathcal{F}} = 0$, but the Godbillon-Vey invariant is proportional to the volume of $\Gamma \backslash H^2$, hence can be arbitrarily large.

6.3 Asymptotically separated laminations

Definition 18 *: Let int (M) be hyperbolic and let $\mathcal{F}$ be a lamination of M. Let $\tilde{\mathcal{F}}\mid_{int(\tilde{M})}$ be the covering lamination of H^3. $\mathcal{F}$ is called asymptotically separated if, for some leaf $F \in \tilde{\mathcal{F}}$, there are two geodesic 2-planes on distinct sides of F.*

We include a proof of the following lemma, implicit in [11], for lack of an explicit reference and because it might help to understand the idea behind theorem 6.



Lemma 35 *: If $\mathcal{F}$ is an asymptotically separated lamination of a finite-volume hyperbolic manifold $M = \Gamma\backslash H^n$, then $\mathcal{F}$ is two-sided branched.*

Proof: Let F be the leaf of $\tilde{\mathcal{F}}$, satisfying that there exist half-spaces U_1 and U_2 in its complement. Let H be the complement of U_1 (i.e. $F \subset H$) and let H_1 and H_2 be disjoint half-spaces in U_2.

If $\Gamma \subset Isom^+(H^n)$ has finite covolume, then it is well-known that the Γ-orbits on the space of pairs of *distinct* points in $\partial_\infty H^n$ are dense.

In particular, fixing some arbitrary $\gamma \in \Gamma$ with fixed points p_1, p_2, one finds conjugates of γ in Γ, such that their fixed points come arbitrarily close to two given points $q_1 \neq q_2$ in $\partial_\infty H^n$. (Namely, conjugate with elements of Γ which map p_1 close to q_1 and p_2 close to q_2.)

It follows that, in a finite-covolume subgroup $\Gamma \subset Isom^+(H^n)$, to any given disk $D \subset \partial_\infty H^n$, one finds loxodromic isometries with both fixed points in this disk. Let α_1 resp. α_2 be such loxodromic isometries with both fixed points in $\partial_\infty H_1$ resp. both fixed points in $\partial_\infty H_2$. Loxodromic isometries map any set in the complement of a neighborhood of the repelling fixed point, after sufficiently many iterations, inside any neighborhood of the attracting fixed point. Hence, replacing α_1 and α_2 by sufficiently large powers, we get that $\alpha_1(F) \subset H_1$ and $\alpha_2(F) \subset H_2$.

Since $\tilde{\mathcal{F}}$ is Γ-invariant, we have found incomparable leaves $\alpha_1(F)$ and $\alpha_2(F)$ above F and, by analogous arguments, we also get incomparable leaves below F. $\square$

Remark: A conjecture of Fenley would imply that a foliation (of a finite-volume hyperbolic 3-manifold $int(M)$) is two-sided branched if and only if it is asymptotically separated, see the discussion in chapter 2.5. of [11]. Namely, Calegari proves that a two-sided branched foliation (on an arbitrary hyperbolic manifold) either is asymptotically separated or the leaves have as limit sets all of $\partial_\infty H^3$. On the other hand, Fenley conjectures that for foliations of finite-volume hyperbolic manifolds (which are transversal to the boundary ∂M), the limit set of a leaf can be all of $\partial_\infty H^3$ only if $\mathcal{F}$ is R-covered.

The following definition is to describe the exceptional case in theorem 6:

Definition 19 *: A 3-manifold is* **Gieseking-like** *if it has a hyperbolic structure $M = \Gamma\backslash H^3$ of finite volume such that $Q(\omega) \cup \{\infty\} \subset \partial_\infty H^3$ are parabolic fixed points of* Γ.

Here, we have used the upper half space model of H^3, and identified the ideal boundary with $C \cup \{\infty\}$. $\omega = \frac{1}{2} + \frac{\sqrt{-3}}{2}$ is the 4th vertex of a regular ideal simplex with vertices $0, 1, \infty$. The condition is, of course, equivalent to the condition that Γ is conjugate to a discrete subgroup of $PSL_2Q(\omega)$ after the identification of $Isom^+(H^3)$ with PSL_2C. One doesn't know any example of a Gieseking-like manifold which is not a finite cover of the Gieseking manifold (communicated to the author by Alan Reid, see also [41]).

The following theorem 5 is the extension of Theorem 2.4.5 in [11] to the cusped case. The restriction to not Gieseking-like manifolds is necessary as shown by the following example: finite covers of the Gieseking manifold are surface bundles over S^1 with pseudo-Anosov monodromy. Take an invariant lamination for the pseudo-Anosov map and suspend it to a lamination of the surface bundle. (It is well-known that such a lamination can actually be completed to a foliation of the surface bundle.) The suspended lamination is asymptotically separated and it is transversal to the ideal triangulation by simplices of volume V_3.

Theorem 6 : *If the interior of M is a hyperbolic n-manifold of finite volume which is not Gieseking-like, $n \geq 3$, and if $\mathcal{F}$ is an asymptotically separated lamination, then*

$$\| M, \partial M \| < \| M, \partial M \|_{\mathcal{F}} .$$

Proof:
We want to give an outline of the proof. We will show that there exist three half-spaces D_0, D_1, D_2 such that the following holds: whenever a straight simplex has at least one vertex in each of D_0, D_1, D_2, it can't be transversal to $\mathcal{F}$. Assuming $\| M, \partial M \|_{\mathcal{F}} = \| M, \partial M \|$, we had an efficient fundamental cycle μ which actually comes from a sequence of fundamental cycles transversal to $\mathcal{F}$. If M is closed, one gets easily that $\mu^{\pm}$ have to vanish on the set of those ideal simplices with at least one vertex in each of $\partial_\infty D_0, \partial_\infty D_1, \partial_\infty D_2$. If M has cusps, we still get the slightly weaker statement that $\mu^{\pm}$ have to vanish on the set of those ideal simplices with at least one vertex in each of $\partial_\infty D_0 - P, \partial_\infty D_1 - P, \partial_\infty D_2 - P$, where P is the set of parabolic fixed points of Γ. We can then use our knowledge of μ to derive a contradiction.

Let F be a leaf which has the property in the definition of "asymptotically separated", i.e., there are half-spaces U_1 and U_2 in disjoint components of $H^3 - F$. We choose in U_2 two smaller disjoint half-spaces H_1 and H_2. Like in the proof of lemma 35, one finds loxodromic isometries $\alpha_1 \in \Gamma$ with both fixed points in H_1 and $\alpha_2 \in \Gamma$ with both fixed points in H_2. Replacing, if necessary, α_1 and α_2 by sufficiently large powers, we arrange that $\alpha_1(U_1) \subset H_1$ and $\alpha_2(U_1) \subset H_2$, and that $F, \alpha_1(F), \alpha_2(F)$ are disjoint. Letting $D_0 = U_2, D_1 = \alpha_1(U_1)$, and, $D_2 = \alpha_2(U_1)$, the remark after definition 13 tells us that there is no tetrahedron transversal to $\tilde{\mathcal{F}}$ with one vertex in each of D_0, D_1 and D_2.

For the convenience of the reader, we first explain the proof for **closed manifolds**. Assume that we have straight fundamental cycles c_ϵ, transversal to $\mathcal{F}$, with $\| c_\epsilon \| < \| M \| + \epsilon$, and that μ is the weak-*-limit of c_ϵ. Denoting by V the open set of straight (possibly ideal) simplices with one vertex in each of D_0, D_1 and D_2, we have just seen that transversality to $\mathcal{F}$ implies $c_\epsilon^{\pm}(V) = 0$. This implies $\mu^{\pm}(V) = 0$, contradicting the fact that μ^{+} is the Haar measure. (A similar argument was given by Calegari in 2.4.5 of [11].)

Now we are going to consider **hyperbolic manifolds of finite volume**. Let $P \subset \partial_\infty H^3$ be the parabolic fixed points of Γ and $H_\epsilon = p^{-1}\left(M_{[0,\epsilon]}\right) \subset H^3$ the preimage of the ϵ-thin part. It is the union of horoballs centered at the points of P. For δ sufficiently small, $D_0 - \overline{H_\delta}, D_1 - \overline{H_\delta}$ and $D_2 - \overline{H_\delta}$ are nonempty. Fix such a δ. Let

$$V = \left\{ \text{simplices which have vertices } v_0 \in D_0 - \overline{H_\delta}, v_1 \in D_1 - \overline{H_\delta}, v_2 \in D_2 - \overline{H_\delta} \right\},$$

where we admit ideal simplices.
We have seen that simplices in V are not transversal to $\tilde{\mathcal{F}}$. Moreover, we define

$$W = \{str(\sigma) ; \sigma \in V\}$$

and

$$U = \{ \text{ positive regular ideal simplices } (v_0, v_1, v_2, v_3) : v_i \in \partial_\infty D_i - P \text{ for } i = 0, 1, 2\} .$$

Now suppose we had the equality $\parallel M, \partial M \parallel = \parallel M, \partial M \parallel_{\mathcal{F}}$. We will stick to the notations of chapter 4. Take some transversal relative fundamental cycle c'_ϵ of norm smaller than $\parallel M, \partial M \parallel + \epsilon$ and make it, via the homeomorphism h_ϵ, to a relative fundamental cycle $d_\epsilon := h_{\epsilon *}(c'_\epsilon)$ of the ϵ-thick part, which is transversal to the foliation $h_\epsilon(\mathcal{F})$. We may arrange h_ϵ to be the identity on the ϵ'-thick part for ϵ' close to ϵ. Then, the lift of d_ϵ to H^n is transversal to $\tilde{\mathcal{F}}$ outside $H_{\epsilon'}$. By choosing ϵ sufficiently small, one may make this exceptional set $H_{\epsilon'}$ as small as one wishes.

Decompose V as a countable union $V = \cup_{i=1}^\infty V_i$, where $V_i \subset V$ is the open subset of (possibly ideal) positively oriented simplices σ satisfying $\sigma \cap H_{\frac{1}{i}} = \emptyset$. (The union is all of V because any ideal or non-ideal simplex with vertices outside H_δ must remain outside some $H_{\frac{1}{i}}$ for sufficiently large i.) Let $W_i = \{str(\sigma) : \sigma \in V_i\}$. For ϵ sufficiently small (such that $\epsilon' < \frac{1}{i}$), we have $d_\epsilon^\pm(V_i) = 0$, since d_ϵ is transversal to $\mathcal{F}$ outside $H_{\frac{1}{i}}$ and V_i consists of simplices which do not intersect $H_{\frac{1}{i}}$ and which are not transversal to $\mathcal{F}$. As a consequence, $c_\epsilon^\pm(W_i) = 0$, with $c_\epsilon := str(exc(d_\epsilon))$. If μ is the weak-* limit of the sequence c_ϵ with $\epsilon \to 0$, we get $\mu^\pm(W_i) = 0$ by the openness of W_i and part (ii) of lemma 16.

$W = \{str(\sigma) ; \sigma \in V\}$ is a countable increasing union $W = \cup_{i=1}^\infty W_i$. Hence $\mu^\pm(W) = 0$. $U \subset W$ implies

$$\mu^\pm(U) = 0.$$

On the other hand, U has nontrivial Haar measure. Indeed, $Isom^+(H^3)$ corresponds to ordered triples of points in $\partial_\infty H^3$, because any such ordered triple is the set of first three vertices for a unique ordered regular ideal simplex. Hence, the set of positive regular ideal simplices, with $v_i \in \partial_\infty D_i$ for $i = 0, 1, 2$, corresponds to an open set of positive Haar measure in $Isom^+(H^3)$. Clearly, a discrete subgroup of $Isom^+(H^3)$ has a countable number of parabolic fixed points. Thus, U has positive Haar measure.

Recall the notation from section 4.3: $v \in \partial_\infty H^3$ is an arbitrary vertex of the reference simplex Δ_0 and $\beta_v(g)$ is the ergodic component of $g \in \Gamma\backslash G$ with respect to the T_v-action. We define

$$H_v = \{g \in \Gamma\backslash G : \beta_v(g) = Haar\}.$$

$Haar(U) \neq 0$ implies $\mu^\pm(H_v) = 0$. Indeed, from lemma 14 and lemma 27 we know that the complement of H_v in $S_\infty^{reg}(M)$ is the set of simplices $g\Delta_0$ with the vertex gv in a parabolic fixed point of Γ. Γ has a countable number of parabolic fixed points and, therefore, this complement is a set of trivial Haar measure. Thus,

$$Haar(U \cap H_v) = Haar(U) > 0$$

and we apply the ergodic decomposition from section 4.3 to get

$$0 = \mu^\pm(U \cap H_v) = Haar(U \cap H_v)\,\mu^\pm(H_v)$$

which implies

$$\mu^\pm(H_v) = 0.$$

This discussion applies to all vertices v_i of the reference simplex Δ_0. By lemma 29, we can conclude that $\mu^\pm$ are determined on S_{cusp}^3.

In particular, since $\mu \neq 0$, there necessarily *are* regular simplices with all vertices in parabolic fixed points. By lemma 26, μ is invariant up to sign under the right-hand action of the regular ideal reflection group R (defined in section 4.3). Hence, there must even be a R-invariant family of regular ideal simplices with vertices in parabolic fixed points. That means, after conjugating with an isometry, $Q(\omega) \cup \{\infty\}$ must be parabolic fixed points of Γ. □

A surface F in a 3-manifold M is called a virtual fiber if there is some finite cover $p : \overline{M} \rightarrow M$ and some fibration $\overline{F} \rightarrow \overline{M} \rightarrow S^1$ with $\overline{F}$ isotopic to $p^{-1}(F)$. A theorem of Thurston and Bonahon asserts that a properly embedded compact π_1-injective surface in a finite-volume hyperbolic 3-manifold is either quasigeodesic or a virtual fiber.

Corollary 9 *: If the interior of M is a hyperbolic 3-manifold of finite volume which is not Gieseking-like and $F \subset M$ is a properly embedded compact π_1-injective surface, then F is a virtual fiber if and only if $\| M, \partial M \|_{\mathcal{F}} = \| M, \partial M \|$.*

Proof: Again, the case of closed M is due to Calegari, cf. theorem 4.1.4 in [11]. If F is (virtually) the fiber of a fibration over S^1, the claim follows from lemma 34. If not, $F \subset M$ must be a quasigeodesic surface in virtue of the Thurston-Bonahon theorem. In particular, it remains in bounded distance from some totally geodesic surface. Hence, F forms an asymptotically separated lamination and we can apply theorem 6. □

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Chapter 7

Zusammenfassung

Wir betrachten, wie sich das simpliziale Volumen $\| M, \partial M \|$ einer Mannigfaltigkeit (mit Rand) M relativ zu Kodimension 1-Objekten verhält.

In Kapitel 3 diskutieren wir, wie sich das simpliziale Volumen ändert,wenn man Mannigfaltigkeiten entlang amenabler Untermannigfaltigkeiten des Randes verklebt. Wir zeigen:
Satz 2: *Seien M_1, M_2 kompakte n-Mannigfaltigkeiten, A_1 bzw. A_2 (n-1)-dimensionale Untermannigfaltigkeiten von ∂M_1 bzw. ∂M_2, $f : A_1 \rightarrow A_2$ ein Homöomorphismus und $M = M_1 \cup_f M_2$ die durch Verkleben mit f erhaltene Manngfaltigkeit.*
Wenn $\pi_1 A_1, \pi_1 A_2$ mittelbar sind, und $f_* : ker\,(\pi_1 A_1 \rightarrow \pi_1 M_1) \rightarrow ker\,(\pi_1 A_2 \rightarrow \pi_1 M_2)$ ein Isomorphismus ist, dann ist $\| M, \partial M \| \geq \| M_1 \partial M_1 \| + \| M_2, \partial M_2 \|$.
Wenn außerdem A_1, A_2 Zusammenhangskomponenten von ∂M_1 bzw. ∂M_2 sind, dann ist $\| M, \partial M \| = \| M_1, \partial M_1 \| + \| M_2, \partial M_2 \|$.
Wir beweisen die analoge Ungleichung/Gleichung für den Fall, daß A_1 und A_2 im Rand derselbe Mannigfaltigkeit M_1 liegen.
Insbesondere ist simpliziales Volumen von 3-Mannigfaltigkeiten additiv für Verkleben inkompressibler Tori und superadditiv für Verkleben inkompressibler Zylinder.

Kapitel 4 diskutiert Mannigfaltigkeiten, die eine hyperbolische Metrik von endlichem Volumen tragen. Wir betrachten Folgen von Fundamentalzykeln, deren l^1-Norm gegen $\| M, \partial M \|$ konvergiert. Im Grenzfall degenerieren diese zu signierten Maßen, getragen auf der Menge der regulären idealen Simplizes. Wir bezeichnen diese Grenzwerte als "effiziente Fundamentalzykel" und beweisen:
Satz 3: *Sei M eine kompakte Mannigfaltigkeit der Dimension ≥ 3, deren Inneres eine hyperbolische Metrik von endlichem Volumen trägt. Sei $F \subset M$ eine geschlossene totalgeodätische Kodimension 1-Untermannigfaltigkeit. Wenn μ ein effizienter Fundamentalzykel ist, dann ist $\mu^+ \left(S_F^n\right) \neq 0$ oder $\mu^- \left(S_F^n\right) \neq 0$.*

Hierbei bezeichnet S_F^n die Menge derjenigen Simplizes, die von F in verschiedene Stücke zerschnitten werden.
Tatsächlich beweisen wir wesentlich stärkere Aussagen über effiziente Fundamentalzykel. Insbesondere erhalten wir, wenn M *nicht* Gieseking-ähnlich ist, daß es nur einen effizienten Fundamentalzykel gibt: die signierte Gleichverteilung auf regulären idealen Simplizes. Das ist eine interessante Starrheitseigenschaft und wird eine große Rolle beim Beweis von Satz 6 spielen. (Eine hyperbolische Mannigfaltigkeit M heißt Gieseking-ähnlich, wenn alle Ecken einer regulären idealen Triangulierung des H^n Spitzen von M sind; die einzigen bekannten Beispiele sind kommensurabel zum Komplement des Achterknotens in S^3.)

In Kapitel 5 benutzen wir die Resultate aus Kapitel 3 und 4, sowie Geometrisierung von 3-Mannigfaltigkeiten, um zu beweisen:
Satz 5: *Sei M eine Mannigfaltigkeit der Dimension ≤ 3. Dann gilt $\parallel DM \parallel < 2 \parallel M, \partial M \parallel$ genau dann, wenn $\parallel \partial M \parallel > 0$.*
Im Fall hyperbolischer Mannigfaltigkeiten mit totalgeodätischem Rand haben wir den folgenden allgemeineren
Satz 4: *Sei $n \geq 3$ und M_1, M_2 kompakte n-Mannigfaltigkeiten mit Rändern $\partial M_i = \partial_0 M_i \cup \partial_1 M_i$, so daß $M_i - \partial_0 M_i$ hyperbolische Metriken von endlichem Volumen mit totalgeodätischem Rand $\partial_1 M_i$ tragen. Seien $\partial_1' M_i$ nichtleere Mengen von Zusammenhangskomponenten von $\partial_1 M_i$, $f : \partial_1' M_1 \rightarrow \partial_1' M_2$ eine Isometrie, und $M = M_1 \cup_f M_2$. Dann ist $\parallel M, \partial M \parallel < \parallel M_1, \partial M_1 \parallel + \parallel M_2, \partial M_2 \parallel$.*
Auch hier beweisen wir eine analoge Aussage für den Fall, daß die beiden zu verklebenden Randkomponenten zum Rand derselben Mannigfaltigkeit M_1 gehören.

Kapitel 6 behandelt die Gromov-Norm von Blätterungen und Laminierungen.
Satz 6: *Sei M eine 3-Mannigfaltigkeit, deren Inneres eine hyperbolische Metrik von endlichem Volumen trägt. M sei nicht Gieseking-ähnlich. Wenn $\mathcal{F}$ eine asymptotisch separierte Laminierung ist, dann ist $\parallel M, \partial M \parallel_{\mathcal{F}} > \parallel M, \partial M \parallel$.*
Dieser Satz bestätigt die Calegari-Vermutung für eine weitere große Klasse von Blätterungen. Diese besagt: wenn $\mathcal{F}$ eine Blätterung einer 3-Mannigfaltigkeit M ist, deren Inneres eine hyperbolische Metrik von endlichem Volumen trägt, dann verzweigt der Blattraum von $\mathcal{F}$ in beiden Richtungen genau dann, wenn $\parallel M, \partial M \parallel_{\mathcal{F}} > \parallel M, \partial M \parallel$ gilt.

Nur oberflächlich mit den anderen Kapiteln verbunden ist Sektion 2.3. Dort betrachten wir die Euler-Klasse von Lefschetzfaserungen und geben eine äquivalente Bedingung dafür an, dass sie ein Urbild in der (reellen) beschränkten Kohomologie hat. Als Korollar bekommen wir eine hinreichende Bedingung für positives simpliziales Volumen von Lefschetzfaserungen.