MODALITY WITHOUT REFERENCE
An Alternative Semantics for Substitutional Quantified Modal Logic and its Philosophical Significance

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Abstract

This dissertation develops a substitutional semantics for first-order (modal) logic which, unlike truth-value semantics, allows a fine-grained analysis of the semantical behaviour of the terms and predicates from which atomic formulae are composed. Moreover, it proposes a nondenotational philosophical foundation for the semantics of substitutional quantified (modal) logic.

Keywords: modality, predication, quantified modal logic, substitutional quantification, substitutional semantics.

For Mariola & Jurek

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Preface

This text is the final version of my doctoral dissertation. At the stage of making it accessible to the public, I would like to thank a number of people and institutions for their support.

Above all I wish to express my gratitude to my supervisors Peter Schroeder-Heister and Ede Zimmermann for their help and encouragement over the years, and for their model. Without their numerous corrections and suggestions during the final stages of writing many passages of the thesis would have remained in a rather muddled state. (I should mention that, in view of the plenitude of penetrating comments I have received from Ede, I feel somewhat uneasy about having to call him a 'Mitberichterstatter'.)

I am also greatly indebited to Kit Fine for being my advisor at New York University during the academic year 2001/02, at an early stage of my research. Much of what I have to say is influenced in some way or other by his work. Moreover, I gratefully recall helpful discussions on modality and related issues with Fabrice Correia, Christopher Peacocke, Joshua Schechter, and Zsofia Zvolenszky at various places around Washington Square Park.

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Of course, mistakes and all the desiderata of my English are my responsibility.

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Moringen
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B.J.P.W.

Introduction

The standard semantics for first-order languages is denotational semantics. The basic intuition by which this semantics is underlain is captured by the claim that language is about the world. Denotational semantics reflects this idea as follows: names are taken to denote objects, predicates are viewed as being satisfied by objects, or as being true of them, and true sentences are regarded as providing accurate descriptions of the level of denotata of the language. Tarski once gave the following rough characterization of the term 'semantics' which expresses that very intuition.

> "Semantics is a discipline which, speaking loosely, deals with certain relations between expressions of a language and the objects (or "states of affairs") "referred to" by those expressions."[1]

The intuition that language is about the world, the *intuition of designation* as I shall call it, and denotational semantics are relatively natural (in particular with respect to atomic predications) when the portion of the object

[1] [Tarski, 1944] p. 345. (Charcterizations of this lax sort are rather frequent in Tarski's writings: "A characteristic feature of the semantical concepts is that they give expression to certain relations between the expressions of language and the objects about which these expressions speak". (At page 252 of the English translation of the German translation.) [Cechę charakterystyczną pojęć semantycznych stanowi ta okoliczność, że wyrażają one pewne zależności między wyrażeniami języka a przedmiotami «o których w tych wyrażeniach mowa» [Tarski, 1995 (1933)] p. 139.] Similarly: "We shall understand by semantics the totality of considerations concerning those concepts which, roughly speaking, express certain connexions between expressions of a language and the objects and states of affairs referred to by these expressions." (At page 401 of the English translation of the German translation.) [[P]rzez semantykę będziemy rozumieli ogół rozważań, dotyczących tego rodzaju pojęć, które – mówiąc ogólnikowo i niezbyt ściśle – wyrażają pewne związki między wyrażeniami języka a przedmiotami [i stanami rzeczy]$^+$, «o których w tych wyrażeniach mowa». [Tarski, 1995 (1936)] p. 173.] The text enclosed in [ ]$^+$ indicates that, interestingly, the fragment "and states of affairs"—whose equivalent also occurs in the German version—does not occur in the Polish original.)

language whose semantics is to be elucidated is denoting and transparent. But they lose much of their intuitive appeal when the semantic analysis of statements is asked for which contain occurrences of nondenoting singular terms, occurences of general terms which do not have a denotational extension, or occurrences of singular and general terms in intensional contexts, irrespective of whether they are denotationless or not.

Let me discern, for the purpose of motivating my project, but three kinds of philosophical problem which pertain to the transference of denotational semantics to nondenoting (e.g., fictional) and intensional (e.g., modal) language—the problems are ontological, semantical, and logical in nature.

First and foremost, the applied semantical denotational analysis of fictional and modal truths such as 'Ulysses is a man' and 'Tarski could have been a violinist', respectively, gives rise to an inflation of ontology—with fictional objects (e.g., Ulysses), on the one hand, and possible objects of some sort (for example, a *possibile*, say, a flesh-and-blood counterpart of Tarski, or an actualistically acceptable surrogate, e.g., the haecceity of Tarski), on the other hand, over which the quantifiers of the object language are taken to range. Accordingly, applied denotational semanticists construe fictional and modal truths of the object language as describing a fictional or modal reality which is represented or constituted by more or less bizarre entities. An obvious consequence of ontological inflation is the inflation of the applied denotational semanticist's metaphysical agenda, since many questions concerning the existence and the nature of these entities need then to be addressed.

Second, the applied denotational analysis gives rise, as it seems, to problems concerning fictional and modal reference. In the former case the applied denotational semanticist construes—perplexingly enough—nondenoting singular terms as denoting or, alternatively, as make-believedly denoting fictional objects. (We shall return to these issues in Subsection 1.1.1.e.) In the latter case he construes singular terms which occur in modal claims (e.g., 'Tarski could have been a violinist') not as denoting the intuitively correct denotata, that is, the denotata the speaker of the modal language naturally takes himself to be talking about (e.g., Tarski), but as referring to things of a different sort instead—to individual representatives, i.e., to possible objects (e.g., of the sort already mentioned) which are contained in the domains of the intended Kripke-models and over which the quantifiers of the modal language are taken to range. By contrast, according to the applied denotational account of nonmodal and denoting discourse the intuition of designation is not violated in this way, since there is no such shift of referents. For example, on the denotational account of the truth of 'Tarski is a logician', 'Tarski' does denote Tarski—and it is Tarski who is contained in the intended domain of discourse. (We shall discuss the problem of modal reference more closely in Subsection 1.1.2.h and 1.1.2.i.)

Thirdly, on the canonical interpretation of Kripke-models, which takes the items in the intended domains of modal discourse to be possible *individuals*, denotational semantics does not only give rise to ontological inflation and reference problems. It also renders straightforward theorems of constant domain systems of quantified modal logic problematic as the philosophical debates concerning such well-known formulae like the Barcan Formula (BF), i.e., $(\forall x)\Box A \rightarrow \Box(\forall x)A$, the Converse Barcan Formula (CBF), i.e., $\Box(\forall x)A \rightarrow (\forall x)\Box A$, and the necessitarian formula (NE), i.e., $(\forall x)\Box(\exists y)(y = x)$, show. These formulae—or, more exactly, their dual equivalents—are problematic as they require the applied denotational semanticist to consider the issues of *possibilia*, antiessentialist *actualia*, or necessitarianism. (These issues will be discussed in Subsection 1.1.2.d.)

Problems of this sort have motivated the rejection of constant domain systems and semantics and supported the development and acceptability of varying domain and free quantified modal logics. As is well-known, though, such logics involve a certain amount of technical complication (concerning, e.g., the axiomatics or the completeness proofs). On the other hand, philosophers who wish to retain the appealing technical simplicity of constant domain semantics have found themselves forced to provide that semantics with applied interpretations which involve metaphysically problematic entities (e.g., contingently abstract objects).

Luckily, as one might think, there is an alternative to denotational semantics: substitutional semantics. This sort of semantics is underlain by the intuition of designation, too. However, it does not appeal to denotation and satisfaction. Instead, it proceedes in terms of truth-value assignments to atomic sentences. This has many attractive consequences: no ontological inflation, at least, with respect to the quantifiers of the object language; no problems which pertain to the metaphysical nature of these *inflata*; no problems with fictional or modal reference; no philosophical problems with constant domain systems and no technical complications of modal logic. Additionally, as is well-known, substitutional semantics does also have its advantages when quantification into quotational contexts is at stake. All these advantages, I take it, are worth having.

But substitutional semantics has, as can be observed, a couple of disadvantages which are peculiar to it. As a consequence of the lack of satisfaction and reference substitutional languages are not adequate for clarifying intuitions of *de re* modality. It is mainly for this reason—not for some argument to the effect that there are only denumerably many names, but nondenumerably many objects—that leading theorists of substitutional quantification like Saul Kripke and Ruth Marcus have questioned the very philosophical utility of substitutional semantics.

> "[T]he intelligibility of substitutional quantification into a belief or modal context is guaranteed provided the belief or modality is intelligible when applied to a closed sentence. The reason is that, in the theory of substitutional quantification [...], the truth conditions of closed sentences always reduce to conditions on other closed sentences. As Quine has pointed out, even for a context as opaque as quotation, where no one thinks that satisfaction for referential variables makes any immediate sense, substitutional quantification is immediately intelligible. [...] it is the ability to avoid all *de re* considerations and interpret quantifying into such contexts solely in terms of *de dicto* [...] modality which has been a principal motivation for interest in the use of substitutional quantification in such contexts. The present writer, being primarily interested in the question of *de re* modality, is less attracted to substitutional quantification here, though of course I grant its intelligibility."[2]

[2] [Kripke, 1976] p. 375.

For Ruth Marcus substitutional quantification

> "has certain interesting uses in a semantics for fictional or mythological discourse and for discourse about putative possibilia freed from commitment to mythical or possible objects. But it misses a metaphysical point. Identity which is a feature of *objects*, cannot be defined in such a semantics. Intersubstitutivity of *syntactical* items *salva veritate* does not generate objects, which must be *given* if identity is to hold."[3]

Clear enough, a philosopher who opts for substitutional semantics as the right and proper tool for the semantical analysis of modal discourse won't find such theses like, say, the doctrines of rigid designation, the necessity of identity or distinctness, essentialism, metaphysical necessity, or the necessary a posteriori appealing. Similarly, he will view modal arguments for the existence of necessary beings (e.g., God) or the identity or distinctness of certain phenomena (e.g., mental and physical phenomena) with suspicion. It is, as it seems to me, for *de re* intuitions, that denotational semanticists reject substitutional semantics and *vice versa*.

There is a further, and related, reason why denotational and substitutional semanticists alike might view substitutional semantics with dissatisfaction: the lack of compositionality. Atomic sentences such as 'Tarski is a logician' certainly feel to be analysable in such a way that their truth-values be determined by the semantic values of their constituting terms. Moreover, it is natural to expect that the truthmaker of that sentence changes when 'Tarski' is replaced by 'Frege'. But on the traditional substitutional account the truthmaker of all truths is, on the most straightforward applied interpretation, a single undiscerning thing, the True. The problems concerning the nature of the True and the False and other metaphysical problems notwithstanding, the feeling is rather strong that doing semantics the substitutional way too much is lost or left unexplained. Presumably, it is the lack of compositionality with respect to atoms why denotational semantics has—in view of the lack of suitably established alternatives—become standard among philosophers. (These issues will be discussed in more detail in Section 1.2.)

[3][Marcus, 1993 (1985/86)] p. 213.

In effect, one might argue, as I think, that the advantages of substitutional semantics (no ontological inflation with respect to the quantifiers of the object language, no problems with fictional and modal reference and so on) are rooted in a drawback, the lack of denotational compositionality; and that, by contrast, the drawbacks of denotational semantics (ontological inflation with respect to object language quantifiers, problems with fictional and modal reference and so on) have their source in its advantage, the presence of denotational compositionality. Chapter 1 will develop this "dialectic" in more depth.

What is common to denotational and substitutional semantics is that the assumption is made that language is being about the world and, moreover, the assumption that denotational models and their substitutional counterparts represent the world as a level of reference. As I have indicated above and as I shall argue in more detail below, these assumptions are intuitively appealing—in particular with respect to atomic predications—when the portion of the object language under consideration is wholly denotational and nonintensional, but they become problematic, when this is not so.

In the case of fictional discourse both assumptions are clearly objectionable. As far as the intuition of designation is concerned, a fictional truth like 'Ulysses is a man' is, intuitively, not about anything. Nor is it appealing, in view of what has been already said, to assume that it is about the denotationalist's fictional reality. In the case of modal discourse the situation is slightly different. It is natural to assume that a modal truth like 'Tarski could have been a violinist' is about something, i.e., about Tarski. However, in view of the reasons mentioned above, it is objectionable to assume that it is about the denotationalist's modal reality. Of course, it will be also objectionable, for the reasons already given, to assume that these statements appeal, in some sense, to the True.

The basic intuition which guides the project of this dissertation is the intuition that a statement can be true without being about something. In my oppinion, the usual denotationalist generalization of the intuition of designation to all portions of language is mistaken. In particular, I believe that the semantics of fictional and modal (or, perhaps, intensional) language should not be construed as being about some level of reference. Accordingly, I reject the suitability of intended denotational or substitutional models (in any sense of 'intended') for the semantics of fictional and modal discourse.

The project has two main objectives, first, to explain how sentences can be true without being true of something and, second, to provide this idea with a formal underpinning. To achieve the letter objective, I shall propose—as an alternative to standard substitutional semantics—an alternative substitutional semantics for quantified (modal) logic, "associative substitutional semantics" as I shall call it. This semantics retains, as we shall see, the strengths of traditional substitutional semantics and—being compositional also with respect to atomic formulae—avoids the intuitive weaknesses that arise from the lack of compositionality. To meet the philosophical task, I shall then propose an applied semantical interpretation of the associative framework which dovetails with the intuition that guides the project.

Since the notion of compositionality involved in the formal and in the applied associative framework to be presented will not be developed in terms of denotation the reader who—unlike the author—believes in modality *de re* and adheres to the doctrines which come with it (e.g., essentialism or the necessary a posteriori) is not likely to find it satisfactory. At any rate, it will be part of my project, though, to elucidate intutitions of modality *de re* and the escorting doctrines from the perspective of the nondenotational theory to be developed.

But how should fictional and modal language be understood, when it is not viewed as being about a level of reference? Certainly, the intuition of designation and the corresponding "descriptive" way of understanding the relation between language and world is deeply entrenched in our minds; perhaps so deeply that a suggestion to the effect that one may talk without talking about something will be viewed as misguided to the degree to which the very claim expressing this suggestion violates grammar. On the other hand, it appears that "a robust sense of reality" (in more or less the Russellian sense) simply dictates that a fictional truth like 'Ulysses is a man' is in no way about anything. And it may seem that it is this sense of reality which is also responsible for the received philosophical terminology which calls such names like 'Ulysses' "nondenoting" or "denotation-less" terms. My proposal is intended to be in accordance with this sense of reality.

I wish to suggest that a fictional truth like 'Ulysses is a man' is better understood as accurately "reflecting" the "meanings" of the terms which occur in it rather than being descriptive of some ontologically dubious level of reference. Less roughly, on the theory which I shall attempt to develop, the meanings of names are viewed as being specified by (partial or total) nominal definitions and the meanings of predicates are regarded as being specified by meaning postulates of a certain sort. These definitions and postulates serve to determine what I shall call the "sense-extensions" of the corresponding names and predicates, respectively. Sense-extensions are, roughly, sets of atomic sentences (of some natural language or, more generally, of some language-like system). The sense-extension of a name will contain all the atomic sentences that are built up out of that name in accordance with the definitions and postulates; and, similarly, the sense-extension of a predicate will contain all the atomic sentences that are built up out of that predicate in accordance with these semantic rules as well. In this way, sense-extensions encode, so to speak, the meaning (or the semantical essence) of names and predicates. In fictional and in modal (or, maybe, intensional) discourse the sense-extensions of the nonlogical vocabulary do the job denotata do in denoting and nonintensional discourse. The totality of the sense-extensions of the names and predicates is, what I shall call "the level of sense".

In the simplest case, an atomic sentence (e.g., 'Ulysses is a man') will be said to be true just in case it is encoded in both the sense-extensions of the names (here 'Ulysses') which are contained in that sentence and in the sense-extension of that sentence's predicate ('... is a man'). Sentences which are true in virtue of the sense-extensions of their consituting terms will be said to be "true with respect to the level of sense". Truth in fiction and modal truth is, as I wish to suggest, truth with respect to the level of sense.

So far I have characterized the framework of sense-extensions only for the case of fictional language. But what about modal language? Consider, the modal statement 'Tarski could have been a violinist' again. Unlike in the fictional case, it will be natural to think that this claim is about Tarski. However, as mentioned above, this modal claim is, as denotational semanticists construe it, implicitly about some representative of Tarski, rather than about Tarski himself—like in the case of the nonmodal sentence 'Tarski is a logician'. For, as I have already indicated above, on the denotational account of the truth of this modal claim, the *prima facie* impression notwithstanding, the intuitively correct referent of 'Tarski' (i.e., Tarski) and the ultimate referent of the claim (i.e., an individual representative of Tarski) come apart. In my oppinion, such observations encourage a different semantical perspective on the relation between modal language and reality, one which is not prone to the perplexities of the denotationalist view of modal discourse. As the reader might expect, my suggestion will be that we should view modal discourse as being reflective, rather than descriptive. Rougly, when we talk modally about Tarski, so my suggestion goes, we do not describe a modal reality of any sort, rather we reflect the name 'Tarski' with respect to the level of sense.

The text is divided into two parts. A part which motivates the theory to be proposed (Chapter 1) and a part which presents it (the rest). The presentation is splitted into two subparts, a technical portion (Chapters 2 and 3) which provides the formal framework of the theory and a philosophical portion (Chapters 4 and 5) which develops a nondenotational theory of modality consonant with that framework.

Chapter 1, "Motivation", discusses three ways of setting up the semantics of quantified modal logic in truth-conditional terms: the denotational, the substitutional, and the mixed way. It discusses, in more detail than has been done in this introduction, the intuitions by which these approaches are underlain and a couple of philosophical problems to which they give rise. The discussion of that chapter helps to motivate both the formal framework to be developed and its applied semantical interpretation.

Chapter 2, "Associative Substitutional Semantics", proposes, as an alternative to standard substitutional semantics, a model-theoretic semantics for a modal substitutionally quantified first-order language with "substitutional identity" which allows to explain the contribution of the terms and predicates to the truth value of atomic formulae and which does not suggest that the modal object language might be viewed as being about objects. This framework modifies and extends the account of substitutional quantification given in [Kripke, 1976]. The semantic values of terms and predicates (the "associates" as I shall call them) are sets of certain atoms of the substitutional base language. (These associates are the formal counterparts of "sense-extensions".) The chapter provides soundness and completeness results for various constant and varying substitution class axiom systems.

Chapter 3, "Constrained Associative Semantics", imposes formal constraints on the admissibility of associative models. Also it introduces "aspectualized" 2-ary predicates and suggests a solution to the problem of asymmetrical essential relations which has been recently presented in [Fine, 1994] in terms of the constrained associative framework.

Chapter 4, "Applied Associative Semantics", provides an applied interpretation of associative semantics developing it as a semantics for the level of sense. The chapter introduces two modes of evaluating sentences— the referential and the nonreferential mode—suggesting that denotational semantics is at best an adequate semantics for truth with respect to the level of reference —but only for the rather small portion of wholly referential and transparent discourse. It explains the central notions of the applied associative framework: the notion of sense-extension, the notion of the level of sense, the principles which govern that level, the notion of truth with respect of the level of sense, and the notion of definitional necessity. Moreover, the chapter links ordinary modal truth with truth-in-an-associative-model proving a connecting theorem along the lines of [Chihara, 1998]. As a further result, the chapter connects up the notion of truth with respect to the level of sense with the notion of truth with respect to the level of reference by proving the straightforward Level Connecting Theorem. The chapter closes with a discussion of a couple of issues in the philosophy of quantified modal logic and in the philosophy of language (e.g., the problem of primitive modality, ontological commitments, actualism and *possibilia*, necessitarianism, the problem of transworld identity, truth in fiction, negative existentials, and the prospects of modal existence and distinctness arguments in philosophical theology and in the philosophy of mind, respectively) from the perspective of the nonreferential theory.

Finally, Chapter 5, "Integrated Associative Semantics", connects—taking up a motif from Christopher Peacocke's recent work on modality—the truth conditions for statements of definitional necessity with the conditions for modal knowledge by exploiting the principles which govern the level of sense.

A brief discussion of how the nonreferential semanticist of the present approach can make sense of Kripke's notion of a posteriori necessity concludes the chapter.

Chapter 1

Motivation

The purpose of this chapter is to motivate the framework of associative substitutional semantics (see Chapters 2 and 3) by isolating a couple of problems which pertain to the standard formulations of denotational and substitutional semantics of (modal) first-order logic. We shall discuss denotational semantics first (Section 1.1), and then turn to substitutional semantics (Section 1.2) and to mixtures of the denotational and substitutional account (Section 1.3.). In the final section we shall highlight the main semantical features of the associative framework against the background of these discussions.

1.1 Denotational Semantics

1.1.1 Nonmodal Denotational Semantics

1.1.1.a Formal Nonmodal Denotational Semantics

A nonmodal denotational first-order formal language $\mathcal{L}^d$ is a pair $\langle Alph, Gram \rangle$ where the first element is an uninterpreted first-order alphabet of a denumerable stock of individual terms (i.e., individual constants $\underline{\alpha}, \underline{\beta}, ...$ and individual variables $\underline{x}, \underline{y}, ...$), n-ary predicates $\varphi^n, \chi^n, ...$ logical constants ($\neg$ for negation, $\wedge$ for conjunction, $\vee$ for disjunction, $\rightarrow$ for the material conditional, and $\leftrightarrow$ for the material biconditional), referential quantifiers, the universal referential quantifier, $(\forall \underline{x})$, and the particular referential quantifier, $(\exists \underline{x})$, and parentheses (, and). The second element of $\mathcal{L}^d$ is a grammar which generates the well-formed formulae of that language from *Alph* in the standard way. A language whith identity, $\mathcal{L}^{d=}$ will be obtained, when the alphabet of $\mathcal{L}^d$ is supplemented with the identity predicate $=$ and the grammar with the usual formation rule. $\mathcal{L}^d$ is called "denotational" (hence the superscript), since it will be later distinguished from a "substitutional" language $\mathcal{L}^s$ which is syntactically identical to $\mathcal{L}^d$, but which differs from that language in that its quantifier symbols (Π instead of $\forall$, and Σ instead of $\exists$) and the symbols for the variables ($x, y, ...$ instead of the underlined denotational counterparts) differ typographically. This typographical distinction indicates that the referential and substitutional quantifiers (written (Πx) and (Σx), respectively[1]) are intended to receive a different semantics and it also helps to avoid ambiguities when languages are considered which contain these two kinds of quantifier (see Section 1.3). Analogous remarks apply to substitutional counterparts of $\mathcal{L}^{d=}$ which use different symbols for identity predicates.

The truth of the formulae of $\mathcal{L}^d$ is defined in terms of ordinary first-order denotational models (or "Tarskian models" as we shall also say). A denotational model $\mathcal{T}$ is a pair $\langle D, v\rangle$ whose first element, the domain, is a non-empty (possibly nondenumerable) set of objects (notation: $\mathtt{d}_1$, $\mathtt{d}_2$, ...) and whose second element is a valuation function which assigns elements of D to the individual constants of $\mathcal{L}^d$ and which assigns to the n-ary predicates of that language n-tuples of elements of D . The valuation function v provides, so to speak, a denotational link between the language and the objects in the domain which are the denotata of the individual terms and sets of such denotata which are extensions of the predicates of $\mathcal{L}^d$. The notion of a variable assignment δ and its variants for a model is defined in the usual way. Truth in $\mathcal{T}$ for formulae of $\mathcal{L}^d$ is defined by the usual recursion. In the simplest case, an atomic formula composed from an 1-ary predicate and a constant, is true in $\mathcal{T}$ just in case the semantic value of the constant is an

[1]This way of writing the substiutional quantifiers is taken from [Kripke, 1976]. The symbols are not exactly the ones used by Kripke since he uses unitalicized letters as substitutional variables. (Since substitutional variables will play the more important role in the present setting, we shall reserve the usual italic way of writing variables for the substitutional case.)

element of the value of the predicate.

It is one thing to use a model-theoretic semantics as a formal device to model a formal language. It is another to use a model-theoretic semantics to interpret a meaningful language (or language-like system). In the first case we are concerned with formal semantics just in the way in which we provided the formal language $\mathcal{L}^d$ with models. This way of doing semantics is usually taken to be appropriate to explain what the validity of a formula comes to, but it is typically regarded as inappropriate if the meanings of sentences of interpreted languages are to be elucidated. What is crucial for that purpose is applied semantics.[2]

1.1.1.b Applied Nonmodal Denotational Semantics

Unlike truth in $\mathcal{T}$, truth *simpliciter* is usually regarded as a semantic property of declarative sentences of interpreted or meaningful languages (e.g., natural languages like German or idealizations of such languages). Whether a sentence of a natural language is true or false is naturally taken to depend upon first, the meaning of the expressions from which that sentence is composed and, second, upon the world. We may view all the sentences of a natural and thus meaningful ordinary language as being generated from an alphabet of that language by its grammar in a way analogous to $\mathcal{L}^d$; and we may take it, for simplicity, that for each sentence of the ordinary language there is exactly one formula of $\mathcal{L}^d$ which symbolizes it accurately capturing its logical form.

How is truth *simpliciter* related to truth in $\mathcal{T}$? According to the fundamental intuition by which denotational semantics is driven, meaningful languages are *about* the world: the names of such languages are taken to refer to objects in the world, the predicates are taken to be satisfied by them and the sentences are taken to be true of them.

According to Tarski's own philosophical remarks on the notion of satisfaction, this intuition seems to be almost defining of the term 'semantics'. "Semantics is a discipline which, speaking loosely, deals with certain relations

[2]The distinction between pure (or formal) and applied semantics has been emphasized most notably in [Plantinga, 1974]; see, in particular, section 7.4.

between expressions of a language and the objects (or "states of affairs") "referred to" by those expressions."[3] Tarski's "semantic" concepts of denotation and satisfaction correlate, so to speak, the object language and the objects that language is about. Thus "the expression 'the father of his country' designates (denotes) George Washington" and "snow satisfies the sentential function (the condition) 'x is white'." And, as Tarski points out, his definitions of Convention T, satisfaction and truth *simpliciter* "refer not only to sentences themselves, but also to objects «talked about» by these sentences, or possibly to «states of affairs» described by them."[4]

The intuition of designation, that is, the intuition that language is about the world is mirrored in the way in which the connection between truth *simpliciter* and truth in $\mathcal{T}$ is established. It is usually established in terms of the notion of an intended model $\mathcal{T}^* = \langle D^*, v^* \rangle$. The first element D^* is the set of the real world objects over which ordinary language quantifies (understood referentially) range and the second element v^* is a reference function which assigns referents, that is, objects from D^*, to the singular terms of the natural language, and which assigns to the predicates of that language tuples of referents as their actual extensions. The link between truth *simpliciter* and truth in $\mathcal{T}$ is then captured by the following biconditional. A natural language sentence is true *simpliciter* if and only if the formula A of $\mathcal{L}^d$ which symbolizes that sentence and is suitably provided with its meaning is true in the intended model $\mathcal{T}^*$.

The applied semantical interpretation of the nonmodal denotational framework gives rise to a couple of problems, though.

1.1.1.c The Problem of Ontological Inflation

A central problem of denotational semantics is *the problem of ontological inflation.* The denotationalist's quantifier, of course, is the referential or objectual quantifier. This conception of quantification works in an intuitively correct way when referring terms are under consideration. But it loses much of its intuitive appeal when nondenoting terms are involved even in so uncompromising nonfictional and nonmodal truths like

[3] [Tarski, 1944] p. 345.

[4] *Ibid.* (The variable in the quoted fragment is, of course, referential.)

(1) ‘A statue of Venus is in the Louvre’.

For, on the straightforward construal, the denotational semanticist incurrs an ontological commitment to a Greek goddess when he paraphrases (1) as $(\exists \underline{x})$(A statue of $\underline{x}$ is in the Louvre). As Ruth Marcus puts it:

> “The standard [i.e., denotational] semantics *inflates* the meanings of sentences it paraphrases, those, for example, that did not *originally* have the existential import they acquire on such paraphrase.”[5]

Part of the problem of ontological inflation then is to account for the nature of the *inflata*.

1.1.1.d The Problem of Nonmodal Reference

A corollary of the problem of ontological inflation in nonficitional and nonmodal discourse is *the problem of nonmodal reference*. The problem consists in the fact that, on an accurate paraphrase, the denotational semanticist must construe occurrences of nondenoting names like ‘Venus’ in such outright truths like (1) as denoting some *prima facie* not intended referent.

1.1.1.e Denotational Semantics of Fiction

Related problems arise for the denotational semantics of fictional discourse. It seems that a claim like, for example,

(2) ‘Laertes is the father of Ulysses’

is intuitively true; rather than false, or undetermined, or of no truth value.

Currently there seem to be basically two ways of explaining the truth of such claims like (2): the “object theoretic” and the “pretence theoretic”

[5] [Marcus, 1993 (1972)] p. 82. The above example is Marcus’s. As Ede Zimmermann suggested to me ‘The Louvre contains a statue of a Greek goddess’ would be a happier example, since ‘Venus’ is obviously not a quantifier.

approach.[6] (Of course, both approaches may be mixed in some way, but we shall be content to summarize the pure views.) Both approaches appeal to the intuition of designation, but they do so in very different ways.

On the object theoretic approach, the intuition of designation is preserved without modifications: fictional discourse is viewed as being descriptive of some reality of fictional objects.[7] Accordingly, (some version of) denotational semantics is accepted as the proper tool of analysis for the semantics of fictional discourse. Thus, for an object theorist, to accept the truth of (2) is to incurr an ontological commitment to fictional objects of some sort to which 'Laertes' and 'Ulysses' are taken to refer. Usually such fictional objects will be either, *possibila* of some sort or some abstract actually existing surrogates for them. In this way the problem of ontological inflation finds its way into fictional discourse.

Moreover, *the problem of fictional reference*, as we shall call it, arises. It can be put as follows. The denotational semanticist of fiction construes fictional names that occur in fictional stories as referring even though these names do intuitively not refer. To put it paradoxically—nondenoting names are viewed as being denoting. This problem has a connotation which is slightly different from the problem of nonmodal reference. For the latter problem arises in the context of intuitively referential discourse about ordinary things like statues, wheras the latter arises in the context of intuitively nonreferential discourse.

Pretence theoretic approaches to fictional discourse[8] appeal to the intuition of designation as well, but they do so only in an oblique way, namely make-believedly. Very roughly, for the pretence theorists stories like, e.g., the Homeric epics (i.e., the *Iliad* and the *Odyssey*) count as fiction if there is a prescription in a game of make-believe (of pretence, or of imagination) which requires the appreciators to pretend that there are objects such that names

[6]The terminology is taken from [Everett & Hofweber, 2000] p. xv. Their recommendable anthology [Everett & Hofweber, 2000] includes recent papers on the topic and an extensive bibliography.

[7]Among the most prominent object theorists, more specifically, among those who have provided book-length contributions are Terence Parsons (see his [Parsons, 1980]) and Edward Zalta (see [Zalta, 1988]).

[8]See, in particular, [Currie, 1990] and [Walton, 1990].

like 'Laertes' and 'Ulysses' that occur in the texts of the epics directly refer to those objects and that sentences like (2) describe them. A sentence like (2) is then said to be true in a fiction just in case there is a prescription that it is to be made-believe by the appreciators. It is save to claim, I think, that pretence theorists are primarily concerned with what speakers say in uttering such sentences like (2) rather than with what those sentences themselves mean or which propositions they express. So the pretence theorist's primary subject is the fictional use of language, rather than the semantics of fictional statements.

Edward Zalta, an object theorist, insists that sentences like (2) should be viewed as being false, and that they are true only when prefixed with an appropriate 'in-the-fiction'-operator (like 'In the epic *Odyssey* ...'). He argues that pretence theorists should be able to account for the semantics of such operators and seems to presuppose that an adequate account would treat them as intensional operators along the usual denotational lines of possible worlds semantics. But this would require a commitment to fictional objects.[9] For in case a denotational account of the logical behaviour of fictional operators is called for to make them respectable, the pretence theorist of fiction will have to face the problems of ontological inflation and fictional reference as well.

It would be desirable, I believe, to have a theory of fictional discourse, which, in contrast to pretence theory, accounts for the semantics of fictional statements and which, in contrast to object theory, does so without appeal to fictional objects. (I shall make suggestions—much later—in Chapter 4.)

1.1.2 Modal Denotational Semantics

1.1.2.a Formal Modal Denotational Semantics

To obtain a denotational first-order modal language $\mathcal{L}^{d^{=\Box}}$ we add the intensional box-operator $\Box$ to the alphabet of $\mathcal{L}^{d^{=}}$ and the corresponding formation rule to the grammar of that language.

[9]See [Kroon, 1992] p. 516, note 22.

A model of first-order modal denotational semantics (or possible worlds semantics) is the usual kind of tuple $\mathcal{K} = \langle S, R, D, d, v \rangle$, where S is a non-empty set, $R \subseteq S \times S$, D is a non-empty set, d is a function from S to subsets of D, where $D = \bigcup_{s \in S} d(s)$. In view of the influential [Kripke, 1963], we shall call such models, following common practice, "Kripke-models".[10] A Kripke-model which satisfies the constancy condition (i.e., $d(s) = D$ for every $s \in S$) is a constant domain Kripke-model; otherwise it is a varying domain model. v is a function which takes each individual constant of $\mathcal{L}^{d=\Box}$ to a member of D. Moreover, v takes pairs containing an n-place predicate as their first member and an element of S as second member to n-tuples of elements of D. The function δ takes each individual variable $\underline{x}, \underline{y}, ...$ of $\mathcal{L}^{d=\Box}$ to a member from D of the model $\mathcal{K}$. Variants of δ are defined in the usual way. Let j be a metavariable ranging over the individual terms of $\mathcal{L}^{d=\Box}$ and let the notion of a value of an individual term j with respect to the valuation v of the model and the assignment δ, that is $v_\delta(j)$ be defined as usual. Then $\mathcal{K}_\delta \models_s \varphi^n j_1 ... j_n$ just in case $\langle v_\delta(j_1), ..., v_\delta(j_n), s \rangle \in v(\varphi^n)$ and so on in the usual recursive way. In particular, $\mathcal{K}_\delta \models_s \Box A$ if and only if $\mathcal{K}_\delta \models_t A$ for every $t \in S$ such that sRt. Among the philosophically prominent theorems of constant domain systems are the following formulae:

BF $\quad (\forall \underline{x})\Box A \rightarrow \Box(\forall \underline{x})A;$
CBF $\quad \Box(\forall \underline{x})A \rightarrow (\forall \underline{x})\Box A;$
NE $\quad (\forall \underline{x})\Box(\exists \underline{y})\underline{y} = \underline{x}.$

(We shall discuss these formulae shortly in Subsection 1.1.2.d.)

There are various alternatives to constant domain Kripke-models. If we drop the constancy condition we obtain varying domain Kripke-models. We may distinguish two subspecies of such models: monotonic and antimonotonic. The former satisfy the requirement that for all $s, t \in S$, if sRt then $d(s) \subseteq d(t)$, whereas the latter satisfy the condition that for all $s, t \in S$, if sRt then $d(t) \subseteq d(s)$. The choice of the model may lead to changes in the evaluation clauses for the quantifiers and to the invalidation of the Barcan

[10]The standard reference for quantified modal logic is the textbook [Hughes & Cresswell, 1996]. See also [Fitting & Mendelsohn, 1998].

Formula (BF), its converse (CBF), and the formula NE formula. BF is invalidated in case the model is not antimonotonic, CBF is invalidated when the model is not monotonic, and when CBF is invalidated so is NE.

A further alternative to constant domain semantics does not consist in modifying the interpretations of the members of the Kripke-model, but in supplementing it with additional sets. A well-known alternative of this kind is the semantics of disjunctive domain Kripke-models which takes up elements from David Lewis's counterpart theory.[11] Counterpart-theoretic Kripke-models reflect the view that every index is equipped with its own domain of objects and that these domains never overlap. (Kripke-models of this sort form the other extreme of constant domain Kripke-models which say, in effect, that every index is equipped with exactly the same objects.) In such models the function d assigns disjoint subsets of D to each world; more exactly, d satisfies the condition that when $s \neq t$, then $d(s) \cap d(t) = \emptyset$. Crucially, counterpart-theoretic models involve a relation, C, which is called "counterpart relation" as a further element. The counterpart relation C is a relation on D. xCy says that object x is a counterpart of object y at another index. C satisfies the constraint that if x and y are elements of $d(s)$, then xCy just in case x = y. Like R, C may be taken to exemplify various logical properties. Independently of its logical properties, C is usually taken to satisfy the condition that every object has exactly one counterpart at every index. The valuation clause for formulae which have $\Box$ as their main logical connective says that $\mathcal{K}_\delta \models_s \Box A$ just in case $\mathcal{K}_\zeta \models_t A$ for every $t \in S$ such that sRt and for every assignment ζ such that for every referential variable $\underline{x}$ free in A, $\delta(\underline{x})\mathrm{C}\zeta(\underline{x})$ and $\zeta(\underline{x}) \in d(t)$.[12]

[11]See [Lewis, 1983 (1968)].

[12]Hughes and Cresswell have shown that given, first, that C is an equivalence relation and, second, that every individual has a counterpart at every index the quantified modal logic of counterpart theory coincides with the logic which validates BF. If C is taken to be reflexive only, then simple K-theorems can be invalidated. See the discussion in [Hughes & Cresswell, 1996] pp. 353-358 and the references therein. David Lewis, however, prefers—in particular, because of considerations of expressive power—to use the resources of counterpart theory and his possibilist doctrine of modal realism as developed in [Lewis, 1986] directly without appeal to modal operators; see *ibid.* pp. 12-13.

1.1.2.b Applied Modal Denotational Semantics: "Canonical"

But what is the connection between truth in $\mathcal{K}$ and modal truth? Just like in the nonmodal case the truth of modal sentences of natural languages is taken to be determined by two factors: the meanings of the expressions and reality. But now the reality is "modal reality".[13]

Usually this reality is taken to be represented by an intended Kripke-model $\mathcal{K}^*$ which is a tuple $\langle S^*, s^@, R^*, D^*, d^*, v^* \rangle$. According to the "canonical" articulation,[14] such a model consists out of the following elements: S^* is the set of all possible worlds; $s^@$ is the actual world, R^* is a relation of accessibility on the set of all possible worlds; D^* is the set of all possible individuals; d^* is a domain function which assigns to each possible world a domain of individuals which exist at that world, v^* is a valuation function that assigns to each name an individual from the set of all possible individuals and to each predicate an extension at each possible world. The function δ assigns to each individual variable an object from D^* of the model $\mathcal{K}^*$. A sentence A is said to be true in $\mathcal{K}^*$ just in case it is true in $\mathcal{K}^*$ at $s^@$ under δ. The link between the absolute notion of truth *simpliciter* and the relative notion of truth in $\mathcal{K}$ is then captured by the following claim: A sentence of regimented ordinary modal language is true *simpliciter* if and only if the formula A of $\mathcal{L}^{d=\Box}$ which accurately symbolizes the sentence and is suitably provided with its meaning is true in the intended Kripke-model $\mathcal{K}^*$. In this way, the relationship between $\mathcal{L}^{d=\Box}$ and the intended Kripke-model $\mathcal{K}^*$ corresponds to the relation which obtains between ordinary modal language and modal reality which is represented by $\mathcal{K}^*$.

The denotationalist preservation of the intuition of designation with respect to modal language is nicely reflected in the following passage:

> "Modal language, as such, is directly about an independent reality (...), and the relationship between the formal language and the intended model exactly mirrors the relationship between ordinary modal language and the reality that grounds modal truth."[15]

[13] See, for example, [Menzel, 1990] p. 359.

[14] See [Plantinga, 1976] pp. 139-142 for the *locus classicus*.

[15] [Linsky & Zalta, 1994] p. 450. This quotation seems to reflect common sense. See also

So whereas the applied denotational semanticist takes nonmodal language to describe the actual world and the actual objects, he conceives of modal language—according to the "canonical" account—as describing a reality as represented (or constituted) by possible worlds and possible individuals.

1.1.2.c The Problem of Ontological Inflation: the Canonical Case

According to a plausible and widely shared assumption, the proponent of a semantic theory is committed to the existence of those entities which are involved in his account of truth.[16] Taking it that S^* contains other worlds than the actual one and that D^* need not contain only the actual individuals, the canonical applied semanticst is committed to the existence of alternative possible worlds and individuals.

Moreover, on the assumption that the speaker of the object language is committed to the existence of the things over which the quantifiers of the object language range, the speaker of the modal language is committed to the existence of the items which are contained in the intended domain of discourse of that language.

1.1.2.d Philosophical Issues in Denotational Quantified Modal Logic

The canonical articulation of first-order denotational modal semantics which preserves the intuition of designation poses several well-known problems for the philosophical interpretation of quantified modal logic. We shall first consider constant domain semantics, in which the Barcan Formula (BF), its converse (CBF), and the NE formula are all valid.

According to the canonical account, to endorse the truth of the formula $\Diamond(\exists \underline{x})\varphi\underline{x} \rightarrow (\exists \underline{x})\Diamond\varphi\underline{x}$, which is equivalent to an instance of BF, and its antecedent seems to involve an ontological commitment to the actual existence of *antiessentialist actualia* as we shall call them. So, for example, when φ

[Stalnaker, 2003 (1997)] p. 169 ("Whatever one's metaphysical beliefs about the reality that modal discourse purports to describe, ...").

[16]See, for instance, [Menzel, 1990] p. 363, [Linsky & Zalta, 1994] p. 38, or [Chihara, 1998] pp. 2-3.

is taken to have the meaning of '... is a harpy', then, accepting that formula and its antecedent, one will be committed to the view that among the actually existing individuals there is one which is possibly a harpy. If this individual is taken to be concrete, that view will be problematic, at least on the assumption that every individual, has its fundamental kind properties essentially.[17]

A further problem with the Barcan Formula is that its acceptance, for example in the form of $\Diamond(\exists \underline{x})A \rightarrow (\exists \underline{x})\Diamond A$, seems to violate the intuition that there might have been some possible individual which is distinct from every actual individual (e.g., Prince William's older sister).

Finally, the converse of the Barcan Formula, CBF, does allow for the derivation of NE, i.e. $(\forall \underline{x})\Box(\exists \underline{y})(\underline{y} = \underline{x})$. According to the canonical reading, this formula says, in effect, that everything does necessarily exist. The truth of this claim—which amounts to a sort of *necessitarianism*, a view usually attributed to Leibniz—violates the intuition that there are individuals (e.g., Prince William) which might not have existed.[18]

It is, in part, philosophical perplexities of this sort which have motivated the development of varying domain axiom systems and semantics. However, on the canonical reading such systems are problematic as well. Thus, for example, to endorse the truth of $\Diamond(\exists \underline{x})\varphi\underline{x} \wedge \neg(\exists \underline{x})\Diamond\varphi\underline{x}$, which is equivalent to the negation of BF, leads to the acceptance of the claim that there is an individual in the domain of some possible world which is not contained in the domain of the actual world. If, for instance, φ is taken to mean '... is made out of matter', then one seems to be committed to the view that there is a merely possible material individual which does not actually exist. On the canonical reading, then, the rejection of BF seems to involve a commitment to the existence of *possibilia*, that is, roughly, to things which there are, but which are not actual. Clearly, since the intended domains of counterpart-theoretic Kripke-semantics are disjoint, it will be obvious how *possibilia* can sneak into the canonical interpretation of that semantics. (We shall deal with *possibilia* and possibilism in more detail in Subsection 1.1.2.g.)

[17]A view to this effect seems to be endorsed in [Marcus, 1993 (1985/86)].

[18]For an illuminating discussion of the philosophical issues raised by the constant domain theorems BF, CBF and NE see [Linsky & Zalta, 1994].

No matter which sort of Kripke-semantics is chosen, there are problems with the interpretation of certain kinds of modal formulae. For example, to subscribe to the truth of a formula like $(\exists \underline{x})\Box\varphi\underline{x}$ where the predicate is taken to have the meaning of '... is a man', amounts, according to the canonical interpretation, to the endorsement of an essentialist claim to the effect that there is at least one actual individual which necessarily has the property of being a man. The proponent of quantified modal logic, then, seems to be committed to *essentialism*, the doctrine that things can have essential as opposed to merely contingent properties.

As a corollary to the endorsement of essentialism, the advocates of such *de re* necessities like $(\exists \underline{x})\Box\varphi\underline{x}$, seem to be committed—in view of the usual natural language interpretation of the box-operator—to the assumption that there are statements or propositions which are true of *metaphysical necessity*, i.e., propositions which are true—as we might put it following [Fine, 1994]—in virtue of the natures (or essence) of the objects they are about. (The notion of metaphysical necessity will be discussed also in Subsections 4.2.13 and 4.5.3.)

Further problems which arise on the canonical interpretation of intended Kripke-models in connection with *de re* modal claims (i.e., claims in which the modal operators occur in the scope of a quantifier, such as $(\exists \underline{x})\Box\varphi\underline{x}$, or atomic sentences which are prefixed with these operators), are the problems of the *transworld identity* of indivduals (i.e., the metaphysical problem of explaining how one and the same individual can be contained in the domain of different possible worlds and thereby exist at more than one world) and of the *transworld reidentification* of individuals (i.e., the epistemological problem of reidentifying an individual at various possible worlds).[19] Needless to say, such transworld problems do not arise when the Kripke-models are counterpart-theoretic. (We shall return to transworld problems later in Chapter 4.)

1.1.2.e Applied Modal Denotational Semantics: "Paradigmatic"

The "paradigmatic" articulation of intended Kripke-models, as I shall call it, differs from the canonical account only in that it is neutral on what the

[19]See, in particular, [Divers, 2002] ch. 16 (and the references therein) for a discussion of the problems of transworld identity and reidentification.

items contained in S^* and D^* are. Crucially, the elements of the latter set are now possible objects, and these need not be individuals.

1.1.2.f The Problem of Ontological Inflation: the Paradigmatic Case

On the assumption that a semantic theory is committed to the existence of those entities which are involved in his account of truth, the paradigmatic applied semanticist of intended Kripke-models is committed to the existence of possible worlds of some (i.e., possibilist or actualist) sort and to possible objects (i.e., to *possibilia* or, alternatively, to actualistically acceptable individual representatives of some brand). (We shall say more on these entities in the next Subsection.)

Moreover, on the assumption that the speaker of the object language is committed to the existence of the things over which the quantifiers of the object language range, the speaker incurrs an ontological commitment to the items which are contained in the intended domain for the quantifiers of that language—that is, to possible objects.

1.1.2.g Actualism and Possible Worlds

As I wish to understand the paradigmatic account, it does (i) not speak to the question of whether the models should be construed in accordance with possibilism or not, it is (ii) neutral on the question of whether the domains should be constant; and it does (iii) not take a stance on the exact metaphysical nature of possible worlds and possible objects contained in the intended index sets and domains, respectively. Let me explain these points one after another.

Possibilism. The metaphysical details of the paradigmatic articulation of applied Kripke semantics can be spelled out in a possibilist or in an actualist fashion. There seem to be at least three ways in the literature in which *the thesis of possibilism*, that there are things that are not actual, can be understood.[20]

[20] See [Linsky & Zalta, 1994] and [Menzel, 2000] for a survey.

According to one of them, the "existence view", as we might call it, we need to distinguish between the things which *there are* and the things which *exist*. On this way of understanding possibilism, then, to endorse possibilism is to accept the thesis that there are things that do not exist. Possibilists of this brand insist that the existential quantifier, $(\exists \underline{x})$, does not have existential import and is to be read in the sense of 'there is'. In order to be in a position to express existence claims proper, they introduce an extra existence predicate, $E!$. This predicate is primitive, has the meaning of 'exists', and its extension may vary across possible worlds just like the extension of a usual predicate does.

Unlike the advocates of the existence view of possibilism, the proponents of the "actuality view", as we might call it, construe the thesis of possibilism as claiming that the existential quantifier, $(\exists \underline{x})$, does indeed have existential import and thus allows for the expression of existence claims. However, they take it that there are two groups of objects which *exist*. There are, on the one hand, the existing objects which are *actual*, and the existing objects which are not actual, on the other hand. Possibilists who endorse the actuality view thus construe the thesis of possibilism as claiming that there are existent things which are not actual.

There is a further way of understanding the thesis of possibilism, the "indexical view" for short. On this view, to endorse that thesis, is to claim that there are things that are parts of worlds other than the one we inhabit. A view of this sort has been famously held by David Lewis. His worlds are, roughly, universes of the same robust kind as ours which, however, are spatiotemporally isolated from our universe. This view may be called "indexical", because its proponents take the word 'actually' to be an indexical expression, taking its reference to be determined by the Lewisian world in which it is uttered.

Given these three ways of construing the thesis of possibilism, we may distinguish three kinds of *possibilia*: first, *possibilia qua* things that are but do not exist; second, *possibilia qua* things that exist but are not actual; and, thirdly, *possibilia qua* other-worldly things. The actualist refuses a commitment to the first kind of *possibilia*, because—as I wish to understand him—he does not find the distinction between 'there is' and 'exists' intelligible; he rejects the second kind, because he takes existence to be just actual existence, and he rejects Lewisian *possibilia*, because he thinks—for some reason—that there is just one world of the Lewisian sort, to wit, ours.

Actualism may then be defined as the view which (unlike the existence view of possibilism) takes the quantifiers to have existential import and which (unlike the actuality view) takes existence to be actual existence, and which (unlike the indexical view) takes it that unrestricted quantification just is quantification over what is actual. Thus *the thesis of actualism* claims that unrestrictedly everything that exists (or, equivalently, everything there is) is actual.[21]

There is a certain distinction that actualists like to draw which might be confused with a distinction made by possibilists who endorse the actuality view. The distinction is the one between the things which exist and the things which are *actualized* (in some sense or other). This actualist distinction does not coincide with the possibilist's distinction between the things which *exist* and the things which are *actual*, for the former unlike the latter presumes, as one might expect in view of the above definition of the actualist thesis, that the word 'exists' is to be understood in the sense of actual existence. So the actualist is free to claim that not every actually existing thing is actualized.[22]

Domains. On the paradigmatic account the choice of possibilism or actualism is independent of one's choice as to the variability of the domain of the intended Kripke-model.

Thus, for instance, constant domain applied semantics may be used by possibilists and actualists alike. For example, an actuality view possibilist might take the domain D^* of the constant model to contain both the things which exist and the things which do actually exist and read NE, the formula which is characteristic of constant domain axioms systems, as saying that everything exists necessarily, but not as asserting that everything is

[21] Our formulation of the thesis of actualism is a slight modification of the definition offered in [Linsky & Zalta, 1994] p. 434. It adds that 'everything' is to be understood in the unrestricted sense explained above.

[22] This claim is usually made by actualists who find BF problematic, because it seems to offend the intuition that there could be things distinct from the actual things.

actual necessarily. By contrast, an actualist, for instance, the actualist of [Linsky & Zalta, 1994],[23] will view the domain as containing only actually existing objects (more exactly, concrete objects, contingently nonconcrete objects, and necessarily nonconcrete objects). And he will say that NE claims that everything does necessarily exist (in the actualist sense), but that not everything is necessarily concrete.[24]

Similarly, varying domain intended Kripke-models may be deployed by possibilists and actualists alike. So a possibilist who endorses the actuality view might take D^* to contain all the objects which exist. And he might take $d^*(s^@)$ to contain all the objects which do actually exist. Given that $d^*(s^@)$ is a proper subset of D^*, there will be existing objects, which do not actually exist. By contrast, an actualist will avoid a commitment to nonactual objects. He will claim that D^* contains all the existing (i.e., actually existing) objects (e.g., Plantingan haecceities) whereas $d^*(s^@)$ contains the existing objects which are also actualized (e.g., the haecceities which are exemplified in the actual world).[25]

Analogous remarks apply to intended Kripke-models with disjoint domains. A possibilist might wish to say that only $d^*(s^@)$ contains existing, actually existing, or this-worldly objects, whereas the other domains contain *possibilia* of the corresponding sort. The actualist, on the other hand, might employ such a semantics making a claim to the effect that all domains contain actually existing objects but that only $d^*(s^@)$ contains the ones which are actualized in some sense.

Preferences may vary. However, it will be uncontroversial that from a purely technical point of view constant domain semantics and systems are most appealing, since such systems are straightforward extensions of standard first-order logic and do neither involve changes of the first-order axioms and rules, nor changes in modal rules and thus no complications of completeness proofs.[26]

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[23]Linsky and Zalta endorse possibilism; see [Linsky & Zalta, 1994] p. 431.

[24]See [Linsky & Zalta, 1994] p. 448.

[25]For Plantinga's account see [Plantinga, 1974] and [Plantinga, 1976]. A formal underpinning for this account which has been endorsed by Plantinga, is presented in [Jager, 1982].

[26]For Timothy Williamson such complications even are "a warning sign of philosophical

Possible Worlds and Possible Objects. As the previous discussion did already suggest, the paradigmatic articulation of intended Kripke-models is not only neutral on the issue of possibilism vs. actualism. It is also neutral on the question of what kind of thing the elements of S^*, the possible worlds, and D^*, the possible objects are. According to current metaphysics of possible worlds, possible worlds (or ways the world might have been) are taken to be either concrete worlds of the same kind as ours or, alternatively, abstract representations of ways our world might have been. Correspondingly, possible objects are taken to be either concrete individuals, or alternatively, abstract individual representatives of some sort. The ontological commitments which denotational modal semanticists who adhere to intended Kripke-models incur are reflected, more closely, in the truth conditions which they offer for modal sentences. Consider, for example, the following accounts of the truth conditions of a sentence like

(3) 'Tarski could have been a violinist':

(i). (3) is true just in case there is a maximal consistent state of affairs (i.e., a Plantingan world) in which the haecceity of Tarski, Tarski-eity, is coexemplified with the property of being a violinist.[27] (ii). (3) is true exactly if there is a maximal consistent set of propositions (i.e., an Adams-style world) which contains the proposition that Tarski is a violinist.[28] (iii). (3) is true if and only if there is a possible world such that Tarski is concrete at that world and is a violinist.[29] (iv). (3) is true just in case there is a complete complex property (e.g., a Stalnaker-style world) which would render 'Tarski is a violinist' true if that property where instantiated by the actual world.[30] (v). (3) is true just when there is at least one combinatorial rearrangement of our physical world, e.g. an Armstrongian world, i.e., a rearrangement of the actual individuals and their actually instantiated properties and relations

error" ([Williamson, 1998] p. 262).

[27]See [Plantinga, 1976].

[28]See [Adams, 1979 (1974)].

[29]See [Linsky & Zalta, 1994]. A view like this is also endorsed in [Williamson, 1998].

[30]See [Stalnaker, 2003 (1976)] p. 28.

which renders 'Tarski is a violinist' true—more exactly, Armstrongian worlds are rearrangements of *possible* atomic states of affairs.[31] (vi). (3) is true if and only if there is a concrete possible world which is of the same kind as our universe, a mereological sum of concrete individuals, which is spatiotemporally isolated from our world and which has as its part a flesh-and-blood individual which resembles Tarski in almost every detail and which is a violinist in that universe.[32]

Of course, the exact explications of the metaphysical issues involved in these accounts of the truth conditions of modal statements, e.g., concerning the nature of possible objects, the constitution of possible worlds (i.e., the problem of how possible worlds are built up by worldmaking elements, in case they aren't ontological simples) or, say, the way they represent (i.e., the problem of how a world represents that something is true at it) do involve much work indeed. However, for our purposes we need not go into these issues any further.[33]

1.1.2.h The Problem of Modal Reference

The intuition of designation is preserved in Kripke's transference of Tarskian semantics from nonmodal to modal formal languages. As mentioned earlier, the semanticist of intended Kripke-models views ordinary modal language as describing a modal reality which is represented by possible worlds and possible objects of some sort. Unlike in the nonmodal case, the intended domains of discourse for modal language do not contain the individuals singular terms as used in modal talk are naturally taken to refer to. They contain, at least on a paradigmatic construal (which does not take the items in the intended domains to be individuals, as is the case on the canonical account for Kripke-models which are not of the counterpart-theoretic sort) representatives of these individuals. In effect, the denotational semanticists of intended Kripke-models shift the referents of singular terms in modal discourse by letting the domains contain representatives of the natural referents

[31] See [Armstrong, 1989].

[32] See [Lewis, 1986].

[33] The reader might wish to consult [Divers, 2002] for an in-depth comparative study of major realist approaches to the metaphysics of possible worlds.

and not these referents themselves. We shall call this the *problem of modal reference*. Here is a brief illustration.

Due to the intuition of designation, the name ‘Tarski’ in (3) should be taken to refer outright to Tarski, however, as the above proposals concerning the truth conditions of that sentence suggest, on the denotationalist account, ‘Tarski’ does not denote Tarski, but something different instead. So, according to (i), ‘Tarski’ refers to Tarski-eity, not to Tarski. According to (ii), ‘Tarski’, presumably, refers to the corresponding component of the structured proposition. That component will be the concept TARSKI not the person, if the proposition is of a Fregean sort; in case the propositions is Russellian it will be a constitutent of that proposition. However, one might doubt, given the modal character of the sentence, whether that constitutent will be our Tarski. In any event, the Russellian construal loses its appeal, when we replace ‘Tarski’ in (3) by ‘Holmes’.[34] The referent of ‘Tarski’ would be the name «Tarski» of the worldmaking language —maybe ‘Tarski’ itself, given the worldmaking language is the language of (3)—in case the possible world were taken to be a maximal consistent set of sentences, that is, a linguistic ersatz possible world of some sort. According to the modal denotationalist, who, like Linsky and Zalta, endorses (iii), the name ‘Tarski’ refers to a Tarski who is concrete and a violinist in some other possible world. Interestingly though, the account of the truth of a statement like ‘Tarski could not have existed’ will construe ‘Tarski’ not as referring to our Tarski but a Tarski who is abstract at some possible world. (Unfortunately, Linsky and Zalta are less than entirely explicit on the nature of their worlds.) According to (iv), the name ‘Tarski’ will refer to some constituent of a complete complex property which would have been instantiated had the world been such that Tarski was a violinist. According to (v), the name ‘Tarski’ refers to some constituent of a rearrangement of the actual world which will be a Tarski which constitutes a *possible* atomic state of affairs. Since this state of affairs is a possible one, one might doubt whether this constituent is Tarski himself. This proposal, however, loses much of its appeal once we realize that the referent of ‘Holmes’ in ‘Holmes could have been a violinist’ is a rearrangement of actual

[34]For a critical discussion of structured—in particular, Fregean and Russellian—propositions see, for example, [Schiffer, 2003] ch. 1.

individuals and properties of some sort. Finally, on the possibilist view (vi), the name 'Tarski' does not refer to Tarski himself but to a flesh-and-blood counterpart of him which in some other Lewisian world plays the violine.

So much for the illustration of the problem of modal reference. By contrast, as has been already noted, the denotational proposal is intuitively correct for nonmodal claims like 'Tarski is a logician'. For here 'Tarski' is construed as referring to the intuitively correct referent, that is, to Tarski.

1.1.2.i Intended* Kripke-Models

As we have seen in the previous sections, the strategy of intended Kripke models involves an inflation of ontology and gives rise to the problem of modal reference.

However, luckily as one might think, there is a way of linking the truth of modal formulae as relativized to Kripke-models to the notion of truth *simpliciter* other than the one of intended Kripke-models. This way is the strategy of intended*, or Menzel-intended, Kripke-models which does not involve a commitment to possible worlds and possible objects at all.[35]

A common feature of the possible worlds realist interpretations of quantified modal logic is that their intended Kripke-models are taken to represent "modal reality" in virtue of the intrinsic properties of the items contained in the intended index sets (i.e., possible worlds) and domains (i.e., possible objects) contained in them. Intended* Kripke-models, by contrast, do not represent in this way. According to this approach, formal Kripke-models may represent outright without their domains containing the objects the

[35]This approach has been initiated, as far as I can see, in [Menzel, 1990]. It has been pursued further in [Ray, 1996] and [Chihara, 1998]. See also [Zimmermann, 1999] for a similar approach. Typically, adherents of intended* Kripke-models not only try to avoid a commitment to the existence of possible worlds and possible objects but to much of a contemporary metaphysician's ontological inventory as well. Christopher Menzel's ontology is the least parsimonious, since it does not only involve the model-theoretic inventory and individuals but also universals (i.e., n-adic properties). Greg Ray refuses the latter and admits only the model-theorist's set theoretic entities and individuals into his ontology whereas Charles Chihara seems to be even more parsimonious rejecting sets (and mathematical enitities in general).

modal language is implicitly construed as being about. Indeed, intended* Kripke-models are taken to be tuples of mathematical objects (e.g., pure sets). Roughly, for any index of the intended* Kripke-model there is a bijective function from the domain of that index to the objects of the real world, such that, given a natural language interpretation of the formal modal language, it maps (i) for every predicate of the formal language the elements of its extension for that index to the real world objects which have the property that is designated by the natural language counterpart of that predicate, and (ii) for every individual constant its referent which is contained in the domain for that index to the real world referent of the name which is symbolized by the constant. For any index, then, there is a purely set-theoretical tuple—the bijection-tuple—built up from three items: the domain of that index, the extensions of the predicates for that index, and the referents of the constants for that index. Given some natural language interpretation of the formal language, such a tuple represents via the bijection for that index how the world would have been had such-and-such been the case.

What is relevant to representation, according to the Menzelian approach, is thus the purely set-theoretic models, the mere tuples, not the intrinsic natures of the objects contained in an intended domain. No possible worlds and no possible objects are involved in this account. The only things appealed to are the things which exist in the actual world and the mere model-theoretic machinery.[36] The proponent of intended* Kripke-models, thus, may justifiably claim that he uses talk of possible worlds and possible objects as a mere *façon de parler* that aids imagination.

Given the fact that the bijection-tuple for the distinguished index $s^{@} \in S$ is a purely mathematical object, there can be no single (standardly) intended Kripke-model, because there will be infinitely many bijection-tuples which are structurally isomorphic to the tuple for $s^{@}$. In effect, a whole class of intended* Kripke-models fulfills the task of a single intended Kripke model. Moreover, given the fact that intended* Kripke-models are mathematical objects, they have no intrinsic bearing on the truth *simpliciter* of ordinary

[36]As has been already mentioned, the degree of ontological parsimony may differ from theorist to theorist and so will the constituents of the actual world.

modal statements such as (3), even though they deliver extensionally adequate assignments of truth or falsity for them.[37]

Roughly, on this kind of approach a modal statement like (3) is true just in case for some intended* Kripke-model there is an index $s \in S$ for which there is a bijection which results in a bijection-tuple for s which could be the tuple for the index $s^{@}$ and such that, if it had been, Tarski would have been a violinist. Modality is thus not explained in terms of quantification over possible worlds. According to accounts of this sort, an ordinary modal sentence is said to be true *simpliciter* just in case its symbolization A endowed with the meaning of the original sentence is true in the intended* Kripke-model. In effect, for the modal denotationalist who adheres to intended* models the truth of modal statements is not grounded in modal reality which is modeled by an intended model; for him it is grounded in the mere structure of modal reality which is *possibly* represented by intended* Kripke-models.[38]

Proponents of intended Kripke-models are dissatisfied with the Menzelian strategy for various reasons. Linsky and Zalta, for example, raise the following objections. First, the strategy of intended* Kripke-models gives up the intuitive extensional characterization of necessity as truth in all possible worlds. Second, it suggests that modal discourse is not about something besides the structure of modal reality. Thus Linsky and Zalta write:

> "Menzel [and so do his followers; B.W.] rejects the idea that there is an 'intended' Kripke-model, which represents pieces of the world itself as configured in a way which correctly reflects modal reality. He gets by instead with a notion of *intended* models*, that is those Kripke-models (constructed out of pure sets) that, roughly, would have been structurally isomorphic to the intended model had there been one. This suffice, he argues, since there is nothing more to modal truth than the structure they capture. But surely there is something more to modal truth than this; surely *necessity* and *possibility* are about something besides <u>the structure of intended* models</u>, something which *grounds* mo-

[37] See [Menzel, 1990] p. 381.

[38] Cf. [Linsky & Zalta, 1994] p. 456, note 38.

> dal truth and which is modeled by an intended model. Menzel suggests that modal semantics need not try to say what this something is."[39] Moreover, they add: "none of these intended* models are in fact genuine *models* of any*thing*. At best they have the property of being actual objects that *possibly* model the *structure* of modal reality. But a model of the pure structure of modal reality is not the same as a genuine model of modal reality."[40]

Thirdly, Linsky and Zalta object to an asymmetry in Menzel's account according to which intended Tarski-models are appropriate for the representation of the reality nonmodal discourse is about, but on which intended* Kripke-models—and not the intended ones—are considered to be adequate for representing the reality modal language is about.

> "With Menzel's defense of Kripke-models we cannot say that modal discourse is in part *about* the objects over which the quantifiers range, at least not in the same way that we can say that nonmodal language is about these objects."[41]

Menzel acknowledges this very criticism writing:

> "And if it is not, it is hard to see in what sense the [strategy of intended* Kripke-models] accounts for modal truth at all."[42]

Where does this criticism leave the applied semanticist of intended* models? The first objection is surely well-taken. But Menzel's answer to this[43] will be, prsumably, the usual modalist one. I.e., one which takes modal operators to be primitives and which claims—in agreement with the homophonic

[39] [Linsky & Zalta, 1994] p. 444; my emphasis (underlines).

[40] [Linsky & Zalta, 1994] p. 456, note 38; my emphasis (underlines).

[41] [Linsky & Zalta, 1994] p. 444. Let me note that Linsky and Zalta use 'Kripke-model' so as to refer to varying domain Kripke-models (cf. [Linsky & Zalta, 1994] p. 431). Our use is therefore more general.

[42] [Menzel, 2000] p. 6 of the subdocument 'Problems with the actualist accounts'.

[43] See [Menzel, 1990] p. 383 and p. 385.

account—that their truth conditions can be stated in a way which is analogous to the usual truth conditions for the truth functional connectives where, roughly, the connective appears on both sides of the biconditional—as part of the object language on the left-hand side and as part of the metalanguage on the other. (We shall return to modalism later in Subsection 4.5.1.)

The second objection seems to aim at the fact that Menzel does not view modal discourse as being about modal reality, but about something other instead, about a structure of some sort. Now, there seem to be two proposals in the passage as to what that structure could be (see the underlined fragments): the structure of the intended* models on the one hand, and the structure of modal reality on the other hand.

It will be natural, as it seems to me, to assume that according to Menzel's view, modal discourse is supposed to be outright about modal reality, but that that reality is represented (by the intended* models) in such a way that only its structure is relevant. So it might seem that the second kind of structure is what is intended.

However, textual evidence (in particular the passage surrounding the first underlined fragment and the quotation which states the third objection) and Menzel's agreement seem to suggest that the first sort of structure is the intended one. If this is so, we may reinstantiate the problem of modal reference for the case of intended* Kripke-models as follows. On this approach, even though there are no intended domains and no possible objects, 'Tarski' which occurrs in a modal claim like 'Tarski could have been a violinist' will not refer to Tarski but instead to a pure set, a constituent of the intended* model, which possibly represents Tarski via some bijection. This reinstantiation of the problem of modal reference might help to see what Linsky and Zalta's criticism aims at, for it seems that the level of denotation is shifted from modal reality to its representative.

As for the third doubt, there is certainly an asymmetry of this sort in Menzel's approach. (Chihara avoids an asymmetry of this sort, assuming that representation is intended* representation in both the modal *and* the nonmodal case; see [Chihara, 1998] sect. 5. However, because of the objection from the previous paragraph, this move generates a problem of reference in the nonmodal case. For then names which occur in nonmodal sentences will be construed as referring to pure sets.)

Ironically, as it seems to me, there is a certain asymmetry in Linsky and Zalta's approach of intended models as well. For given, first, the shift of referents in the domain of discourse for modal language (from the intuitive denotata to their representatives) and, second, the fact that no such shift takes place in the domain of nonmodal discourse, they do not seem to be entitled to saying that modal and nonmodal language alike are "in the same way" about the items over which the quantifiers range.

With regard to Menzel's own comment on Linsky and Zalta's objection, I think that we should distinguish between that what a modal statement is about (i.e., the subject of the modal claim) and the metaphysical ground of the truth of modal statements. The proponents of intended* Kripke-models seem to have an answer to the first point, but they owe us a response to the second. In my oppinion, it is not obvious how the second point is to be answered, since it is not clear at all what "the metaphysical ground of modal truth"[44] is intended to mean on that approach. Does it refer to the ultimate truthmakers of modal truths? Does it refer to the feature of the subject of a necessarily true proposition (e.g., its essence) in virtue of which that proposition is true (i.e., the source of the necessity)? Or is it something else? Whatever the ground of metaphysically modal truth is supposed to be, the proponents of intended* Kripke-models do not provide us with a response to this question. And it is hard to see, in view of their framework and, in particular, their slim metaphysical inventory, how they could.

To sum up, as far as ontological commitments are concerned, the central difference between the two denotationalist strategies of connecting up the relative notion of truth in a Kripke-model to the absolute notion of modal truth which have been proposed so far is that the strategy of intended* Kripke-models—unlike the account in terms of intended Kripke-models—does not involve an ontology of possible worlds and objects. By contrast, the adherents of intended Kripke-models are, perhaps, better off as regards the question of the metaphysical ground of modal truth, as they may draw on the intrinsic features of the objects contained in the intended domains.

[44] [Menzel, 1990] p. 385.

Of course, I do not claim that these points of difference are exhaustive. But they are the ones that are important for our purposes.

In any event, both accounts are taken to describe a modal reality as represented by Kripke-models and so both accounts incorporate the intuition of designation. This intuition and its denotationalist adaptation are reflected in a couple of philosophical theses of "Kripkean orthodoxy".

1.1.2.j Modal Denotational Semantics and Philosophy

Several technical phenomena of standard denotational quantified modal logic are mirrored in various well-known philosophical doctrines—largely due to Saul Kripke (see [Kripke, 1980 (1972)])—whose plausibility is dependent upon the intuition of designation as spelled out in terms of satisfaction and denotation.

The simple stipulation, for instance, that (in the simplest setting) the valuation function contained in a Kripke-model is to map the individual constants to the element contained in its domain irrespectvely of which elements are contained in the domain of an index, is mirrored in Kripke's doctrine that names are *rigid designators*, that is expressions whose referents do not vary across possible worlds.

The truth conditions of claims of necessity *de re* are mirrored in the philosophical doctrine of *essentialism*, that is, as I have mentioned above, the doctrine that things may have accidental as well as essential properties (e.g., a person's property of originating from certain gametes) and that, accordingly, 'necessity' means 'metaphysical necessity'.

The derivability of the formula $(\forall \underline{x})(\forall \underline{y})((\underline{x} = \underline{y}) \rightarrow \Box(\underline{x} = \underline{y}))$, in constant and varying domain axiom systems, mirrors the doctrine of the *necessity of identity* (see, in particular, [Marcus, 1993 (1961)] and [Kripke, 1980 (1972)]). This doctrine, together with the doctrine of rigid designation, is pivotal for the argument that there are *a posteriori necessities* like, for instance, the proposition that Hesperus is necessarily identical with Phosphorus.

Granted that conceivability does entail possibility, the derivability of $(\forall \underline{x})$ $(\forall \underline{y})$ $(\Diamond(\underline{x} = \underline{y}) \rightarrow (\underline{x} = \underline{y}))$ which is equivalent to the *necessity of distinctness*, i.e. $(\forall \underline{x})(\forall \underline{y})(\neg(\underline{x} = \underline{y}) \rightarrow \Box\neg(\underline{x} = \underline{y})))$, together with the doctrine of rigid designation formulated so as to apply to such terms like 'pain' and 'C-fibre stimulation' is essential for the success of antimaterialist arguments in the philosophy of mind.[45]

A final example. The validity of such S5 theorems like $\Diamond\Box A \rightarrow \Box A$ and $\Box A \rightarrow A$ had some impact on philosophical theology, since they allow, given some *prima facie* plausible assumption to the effect that there possibly is some necessary being which is God-like, to infer that such a being does actually exist.[46]

If sound, modal arguments of this sort can teach us—as their denotationalist proponents seem to assume—from the armchair, so to speak, what things are "out there" in the real world.

1.1.3 Summary: Doubts about Denotational Semantics

Denotational semantics incorporates the intuition of designation, i.e. the intuition that language is about the world, in terms of denotation and satisfaction. According to applied interpretations of the denotational framework, fictional and modal truths are viewed as being descriptive of fictional and

[45] Consider, for example, the following scheme of a two-dimensionalist argument against materialism which may be found in [Gendler & Hawthorne, 2002] at page 55:

1. A statement is superconceivable iff it is diagonally possible.
2. 'Pain is not C-fibre stimulation' is superconceivable.
3. 'Pain is not C-fibre stimulation' is diagonally possible iff it is possible.
4. Therefore 'Pain is not C-fibre stimulation' is possibly true.
5. Where a claim of disctinctness flanked by two rigid designators is possibly true, it is necessarily true.
6. Therefore 'Pain is not C-fibre stimulation' is actually true.

For a recent antimaterialist argument which proceedes along such lines, see e.g. [Chalmers, 2002]. See also the discussion in [Gendler & Hawthorne, 2002] and the references therein.

[46] A modal ontological argument of this sort has been proposed most notably by Kurt Gödel. For a recent rigorous discussion of this argument and references see, in particular, [Fitting, 2002]. See also [Löffler, 2000] where a rich bibligraphy of the literature on modal ontological arguments is provided that appeared until the year of its publication.

modal reality, respectively. It has been argued that this conception gives rise to a couple of problems.

The view that fictional truths are descriptive gives rise to ontological inflation and to the problem of fictional reference (see Subsection 1.1.1e). Even the intuition that nonmodal and nonfictional language is descriptive is problematic on the denotational construal as straightforward paraphrases of such sentences like 'There is a statue of Venus in the Louvre' suggest (see Subsections 1.1.1.c and 1.1.1.d).

The denotationalist view that modal language is descriptive of a modal reality is responsible for the problem of ontological inflation (see Subsections 1.1.2.c and 1.1.2.f), for several philosophical perplexities about quantified modal logic (see Subsection 1.1.2.d), and for the problem of modal reference (see Subsection 1.1.2.h). On the approach of intended* Kripke-models, a commitment to possible worlds and to possible objects is arguably avoided. However, the problem of modal reference can be reinstantiated and it is not clear how the problem of the ground of modal truth could be solved (see Subsection 1.1.2.i).

Finally, denotational modal semantics, i.e., "Kripke semantics", supports several substantial—"Kripkean"—philosophical doctrines and modal arguments, which to a certain extent are historically prepared by the success of Kripke semantics. These doctrines are intelligible only when the intuition of designation for modal language is preserved and spelled out in terms of satisfaction and denotation (see Subsection 1.1.2.j).

1.2 Substitutional Semantics

1.2.1 Nonmodal Substitutional Semantics

There is a well-established but less popular alternative way of interpreting quantification which does not construe the quantifiers of the object language as denotational and which, therefore, does not give rise to the aforementioned problems of ontological inflation and reference in modal (or more generally, intensional) and fictional contexts: the so called "substitutional" interpretation of the quantifiers.

Let $\mathcal{L}^s$ be a substitutional language which is just like $\mathcal{L}^d$ except that the variables and the quantifiers are written in a different way. As I have already mentioned in the beginning of the previous section, we use $x, y, ...$ as substitutional variables and write (Πx) and (Σx) for the substitutional universal and existential quantifier, respectively. Substitutional quantifiers receive a different interpretation.

The essentials of a substitutional semantics for a language like $\mathcal{L}^s$ are nicely captured by Ruth Marcus as follows:

> "On a substitutional semantics" of a first-order substitutional language "a domain of objects is not specified. Variables do not range over objects. They are place markers for substituends. Satisfaction relative to objects is not defined. Atomic sentences are assigned truth values. Truth for sentences built up out of the sentential connectives [is] defined in the usual way. The quantifier clauses in the truth definition say that
>
> [1] $(\Pi x)Ax$ is true just in case $A(t)$ is true for all names t.
>
> [2] $(\Sigma x)Ax$ is true just in case $A(t)$ is true for at least one name t."[47]

The names or substituends which replace the substitutional variables may be names in the sense of 'proper name', but they need not. They may be expressions which belong to any syntactic category of the language, e.g., predicates or sentences. If the class of substituends (i.e., the class of expressions which may be substituted for the variables which the substitutional quantifier binds) is a class of proper names or definite descriptions, the quantifier will be nominal; if they are predicates, the quantifier will be predicative and so on.

Let the class of substituends (or the substitution class) consist of the two proper names 'Pegasus' and 'Bucephalus', and let Ax be 'x is a winged horse'. Then the substitution instances of [1] and [2], that is the $A(t)$s, will

[47] [Marcus, 1993 (1978)] p. 119; cf. also [Marcus, 1976] p. 47. The numbering is mine. I have also replaced Marcus's original symbols for the quantifiers with the notation from [Kripke, 1976] which we shall use from now on.

be 'Pegasus is a winged horse' and 'Bucephalus is a winged horse'. On this interpretation, then, [1] will be true just in case all its substitution instances are true; and [2] will be true exactly if at least one substitution instance is true. On this interpretation then, [1] will be false and [2] will be true.

Suppose now, the class of substituends contains predicates like 'is a man', 'is a philosopher' and so on and suppose that the substitutional variables bound by the quantifiers are predicate variables, then a sentence like '(Σx) (Plato x and Aristotle x)' will be true just in case there is at least one true substitution instance of it. In view of the way in which the class of substituends is specified, that quantified sentence will be true, since among the true substitution instances will be such sentences like 'Plato is a man and Aristotle is a man' or 'Plato is a philosopher and Aristotle is a philosopher'.

If the class of substituends is taken to contain sentences and the variables bound by the quantifiers are sentential variables, then a quantified sentence like '(Σx)(Aristotle thinks that x)' will be true just in case it has at least one substitution instance, like for example, 'Aristotle thinks that Plato is a philosopher' which is true.

Finally, to take a somewhat artificial example which is traditionally attributed to Stanisław Leśniewski,[48] we may even assume that the class of substituends contains parantheses '(' and ')' if the xs are parentheses quantifiers, a sentence like '(Σx) x Plato is a man)' will be true exactly if that sentence has at least one true substitution instance, say '(Plato is a man)' where the x is replaced by the left-hand parenthesis.

Since substitutional quantifiers do not have objectual domains the semantics does not construe the truth of such a statement like '(Σx) (x is a winged horse)' as requiring a denotatum for the singular term which replaces the x. According to substitutional semantics, a singular term like 'Pegasus' does neither refer to some denotatum, nor does the account of the truth of the above statement involve a commitment to the existence of a fictional object. And, as the examples of nonnominal quantifications such as '(Σx)(Plato x and Aristotle x)' and '(Σx)(Aristotle thinks that x)' show, ontological commitments to universals (e.g., the property of being a philosopher) or such things

[48]See, for instance, [Quine, 1969 (1968)] p. 106 or [Kripke, 1976] p. 329.

like propositions (e.g., the proposition that Plato is a philosopher which is expressed by the sentence ‘Plato is a philosopher’ and its translations into other languages) are avoided.[49]

Finally, logical paraphrases like the one of ‘There is a statue of Venus in the Louvre’ to ‘(Σx)(A statue of x is in the Louvre)’ (see Subsection 1.1.1.c) are not compromising, since they do not ontologically inflate the meaning of the original sentence. According to the substitutional account of object language quantifiers, there is, therefore, no problem with fictional objects and no problem of fictional reference.

Let me add that the substitutional account does also prove helpful in cases of quantification into quotational contexts. When we take the variables which are bound by the substitutional quantifiers to be sentential ones and assume that the class of substituends for them contain sentences, a statement like, for exampe, the formula ‘(Πx)(‘x’ is true just in case x)’ will be true in case its substitution instances are all true.[50]

It is well-known that the substitution interpretation of the quantifiers is usually taken not only to be helpful in cases of nondenoting occurrences of singular terms (or, in cases in which some ontological commitment is preferably avoided) but also in cases in which singular terms occur in modal and other referentially opaque (or nontransparent) contexts. We shall take up the discussion of intensional, in particular, modal issues after the formal and the applied substitutional semantics for simple nonmodal first-order languages has been set up more thoroughly.

1.2.1.a Formal Nonmodal Substitutional Semantics

Ruth Barcan Marcus's paper [Marcus, 1993 (1961)] is commonly regarded as the widely read pioneering publication on substitutional quantification

[49] Examples like these may be found, for instance, in [Marcus, 1962] or [Marcus, 1993 (1978)].

[50] For discussions of truth definitions in terms of substitutional quantification into quotation contexts see, e.g. [Grover, 1973] and [Soames, 1999] pp. 86-92, where Tarski's remarks on this issue in [Tarski, 1995 (1933)] are discussed. For influential considerations on substitutional truth theories see [Kripke, 1976].

theory.[51] For some reason, though, she never gave a formal presentation of it.

A first formal semantics for a substitutional first-order language has been proposed, as far as I know, in [Dunn & Belnap, 1968]. On their account an interpretation, I, maps the atomic sentences of the language into $\{1, 0\}$, that is, the truth values 1 (= "true") and 0 (= "false"). The valuation for that interpretation, v_I, for a closed formula A is then given by the following clauses:

1. if A is an atomic sentence, $v_I(A) = I(A)$;
2. if A is $\neg B$, $v_I(A) = 1$ iff $v_I(B) = 0$;
3. if A is $B \wedge C$, $v_I(A) = 1$ iff $v_I(B) = 1$ and $v_I(C) = 1$;
4. if A is $(\Pi x)B(x)$, $v_I(A) = 1$ iff $v_I(B(t)) = 1$ for all names t.

Since the set of names is—as is standardly assumed—denumerably infinite, it is assured that the truth conditions for the universal quantifier are not equivalent with the truth conditions for conjunction. Otherwise, of course, the introduction of that quantifier would make no semantical sense. A closed formula is standardly *valid* just in case it is true in all interpretations I.[52] It

[51]Indeed, Marcus's paper seems to be the first one to promote the adoption of substitutional quantification in modal logic. The substitution interpretation of the quantifiers seems to have been in the air long before, though. [Quine, 1969 (1965)] p. 63 attributes it to Leśniewski and [Leblanc, 1971] p. 165, note 3, to Bertrand Russell. Neither, though, gives textual evidence. In [Leblanc, 1973a], at page 2, a substitutional understanding of the quantifiers is located (without page references) in Wittgenstein's *Tractatus* (1921) and attributed to Ramsey and Carnap. See also the preface of [Leblanc, 1976]. In [Leblanc, 2001], at page 124, the substitution interpretation of the quantifiers is located as early as in Frege's *Begriffsschrift* (1879). For further remarks on the "still unchronicled history" of substitutional semantics the reader is referred to the Appendix in [Leblanc, 2001].

[52]See [Dunn & Belnap, 1968] p. 179. Since I is a truth-value assignment to the atomic formulae, substitutional semantics—in the sense specified by Ruth Marcus in the quoted passage at the beginning of Section 1.2—is sometimes called "truth-value semantics". This terminology is preferred, e.g. by J. Michael Dunn, Hugues Leblanc and others. Leblanc's use of the term 'substitutional semantics', however, differs from Marcus's usage (in the initial characterization) as the following passage shows: "I shall study several [alternatives]

will be noted that truth and validity are defined only for closed formulae of the language.[53]

Dunn and Belnap's semantics makes Marcus's ideas somewhat more precise. In particular, it is more precise in that it explicitly states that the truth-values of the atomic formulae are determined by truth-value assignments. Marcus's original ideas might be taken to suggest, as Dunn and Belnap point out, that the truth-values of atomic sentences are determined in the ordinary denotational way—this, however, would be inadequate for atoms with nondenoting constants.[54] Hugues Leblanc later offered a version of the semantics which explicitly relativizes the truth-value assignment to a (possibly empty) substitution class of individual constants.[55]

The authoritative theory of substitutional quantification has been proposed by Saul Kripke in [Kripke, 1976]. Kripke suggests that substitutional quantifiers should be regarded as being introduced by extending a given language which is already interpreted to a new language. Roughly, Kripke takes

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here, among them: *substitutional semantics*, *truth-value semantics*, and *probabilistic semantics*. All three interpret the quantifiers *substitutionally*, i.e. all three rate a universal (an existential) quantification true if, and only if, every one (at least one) of its substitution instances is true. As a result, the first, which retains models, retains only those which are to be called *Henkin models*. The other two dispense with models entirely, truth-value semantics using instead truth-value assignments (...) and probabilistic semantics using probability functions. So reference, central to standard semantics, is no concern at all to truth-value and probabilistic semantics; and truth, also central to standard semantics, is but a marginal concern of probabilistic semantics." ([Leblanc, 2001] p. 53. A Henkin model is, in effect, a tuple $\langle D, v\rangle$ where for every element $\mathtt{d}$ of D there is an individual constant α of $\mathcal{L}^d$ such that $v(\alpha) = \mathtt{d}$. According to Leblanc, such models are countable by definition. See [Leblanc, 2001] p. 61.) In contrast to both Marcus's usage and Leblanc's usage, we shall here refer with 'substitutional semantics' to truth-conditional semantics which appeal exclusively to the substitution interpretation of the quantifiers. Thus, according to our terminology, substitutional semantics is the union of what Marcus and Leblanc, respectively, call by that term.

[53]Open formulae, thus, are standardly left without a semantic interpretation, since this would be "merely a distracting complication" ([Dunn & Belnap, 1968] p. 179). In particular, as Kripke suggests (see [Kripke, 1976] p. 330, note 4) an interpretation of open formulae would play no role in the inductive definition of semantical terms.

[54]See [Dunn & Belnap, 1968] p. 182.

[55]See, e.g., [Leblanc, 1973b] p. 250.

a base language and extends it into a new language by introducing substitutional variables for the substitutional quantifiers, defining a substitution class for the quantifiers in such a way that it contains only expressions of the base language. Kripke, then, takes the atomic sentences of the extended language to be all the sentences of the base language together with the formulae which result from replacing members of the substitution class in the base language with variables.

Given that truth has been defined for the sentences of the base language, Kripke shows that the truth conditions for the sentences of the base language together with the clauses for the formulae which contain the logical vocabulary of the extended language determine the truth conditions for the extended language.[56] Since the substitution class is constrained to expressions of the base language it does not contain expressions in which substitutional quantifiers occur. For suppose this were not so and assume, moreover, that A is $(\Pi x)B(x)$ and has complexity $n+1$, and that $B(t)$ is the result of replacing x by a term t in $B(x)$. In this case one cannot be sure that $B(t)$ has complexity $\leq n$, since it could be that t itself contained quantifiers and thus increased complexity.[57] By excluding such cases, Kripke's extension strategy allows an adequate inductive definition of truth for substitutional languages.[58]

The model-relative notions of truth, validity and logical consequence as explicated in terms of denotational semantics can be related to their substitutional conterparts as follows.

Let $\mathcal{L}^d$ be an ordinary denotational first-order language (without identity) and let $\mathcal{T} = \langle D, v\rangle$ be an ordinary denotational model constrained by the following condition: for every element $\mathtt{d}$ of D there is an individual constant

[56]See [Kripke, 1976] pp. 330-331.

[57]See [Kripke, 1976] p. 331. If the base language does already contain quantifiers then the quantifiers which are introduced to extend that language must be of a different variety and must differ notationally. Kripke considers such an extension in [Kripke, 1976] section 4.

[58]Kripke's proposal has become standard. For example, Marcus's preferred account of the substitutional quantifier is the "minimal substitutional semantics" as proposed by Dunn and Belnap with a base language that does not contain quantifiers (see [Marcus, 1993 (1978)] p. 120, note 9). Kripke's account is also endorsed in [Copeland, 1982] and [Copeland, 1985].

α of $\mathcal{L}^d$ such that $v(\alpha) = \mathtt{d}$. (This is what Hugues Leblanc calls "Henkin model".) Correspondingly, let $\mathcal{L}^s$ be as before, with $\mathcal{L}^s_0$ as base language whose formulae are just the atomic sentences of $\mathcal{L}^s$. Moreover, let $I^{\mathcal{T}}$ be the (minimal) substitutional analogue for $\mathcal{T}$ which is such that it assigns 1 to all the atomic sentences of $\mathcal{L}^s$ whose $\mathcal{L}^d$ analogues are true in $\mathcal{T}$ and 0 to the rest. Then a simple inductive proof will show that a sentence A of $\mathcal{L}^s$ is true in $I^{\mathcal{T}}$ just in case the $\mathcal{L}^d$-counterpart of A is true in $\mathcal{T}$.[59]

As for validity, let DC be a denotational and let SC be a substitutional version of an uninterpreted first-order system without identity, let $\mathcal{T} = \langle D, v\rangle$ be a denotational model and let I be a substitutional interpretation. Moreover, let validity for DC be defined as truth in all denotational models and let validity for SC be truth under all substitutional interpretations. Then a sentence of SC is valid just in case its DC analogue is valid. Similarly, for the consequence relation.[60] Moreover, weak and strong completeness results for SC can be provided (for various sorts of SC) as well.[61]

Now, there is a well-known problem which is almost defining of substitutional quantification: *the problem of nondenumerable domains* as we might call it. The set of nominal constants of $\mathcal{L}^s$ and the set of individual constants of $\mathcal{L}^d$ are supposed to be denumerably infinite whereas the domain D of $\mathcal{T}$ may contain nondenumerably many objects. Let this be so. Then, assuming that x is substitutional when it is bound by a substitutional quantifier and an individual variable when it is bound by a referential quantifier, $(\Pi x)A$ will hold in I but its denotational analogue $(\forall x)A$ will not since it will be faslified by a nameless object from D; and there must always be some such object, since there is only a denumerable infinity of names. The condition which we have imposed above, in order to relate the substitutional and the denotational account of truth (i.e., for every element $\mathtt{d}$ of D there is an individual constant a of $\mathcal{L}^d$ such that $v(a) = \mathtt{d}$) just assured that D is denumerable.

The usual—and in my oppinion acceptable—reaction of the proponents of

[59] A proof of to this effect can be found in [Leblanc, 1973b] p. 249. It is straightforwardly adapted to Kripke's extension strategy.

[60] See [Kripke, 1976] pp. 336-337.

[61] See, for example, [Dunn & Belnap, 1968], [Leblanc, 1971], [Leblanc, 1976], and the remarks in [Kripke, 1976] and [Davies, 1980].

substitutional quantification is to appeal to the Löwenheim-Skolem theorem. Thus, for example, Marcus suggests that

> "the fact that every *referential* first-order language that has a nondenumerable model must have a denumerable model gives little advantage to the referential view."[62]

On the assumption, then, that there are nondenumerably many things—something, as Marcus assumes a nominalist being "diffident about nondenumerable collections"[63] might wish to reject—the proponents of referential quantification are in advantage since they need not support their view by some such Löwenheim-Skolem argumentation.

There is a further, somewhat speculative, way suggested by W. V. Quine and Saul Kripke in which the proponent of substitutional quantification might wish to react to the problem of nondenumerable domains. He might allow that there be nondenumerably many things but hope at the same time that all the properties of the unnamed things that can be expressed in the denotational language are shared by the named objects. If this is so, every quantification which is expressible in that language will be true when referentially construed just in case it is true when it is interpreted substitutionally.[64]

So on the assumption that there are nondenumerably many things and that the unnamed objects differ in the properties which are expressible in the referential language from the named ones, the proponents of referential quantification do have an advantage. However, in view of the possibility of some Löwenheim-Skolem argumentation this advantage is, in effect, not decisive.

As a further alternative to a Löwenheim-Skolem argumentation one might simply permit that the substitutional language be nondenumerable by just

[62][Marcus, 1993 (1978)] p. 124. (The Downward Löwenheim-Skolem Theorem says, in effect, that if a set of sentences of a denotational first-order language has a denotational model at all, that is, an interpretation in which all the sentences in the set come out true, then it has a model with a denumerably infinite domain. The Upward Löwenheim-Skolem Theorem claims that if such a set of sentences has a model in any infinite cardinality, it has models in every infinite cardinality.)

[63]*Ibid.*

[64]See [Quine, 1969 (1965)] p. 65.

allowing, as is often done, for there to be nondenumerably many constants (and, correspondingly, names). So, if every object in D is named, and D is nondenumerable, the number of constants will be nondenumerable as well.[65] In accordance with this assumption, one may then assume that there are nondenumerably many names of natural language. Presumably, being infinite—even nondenumerable—these names won't be utterances of names or name-tokens, but types of names. One could then assume that there will be name-types of this sort which do not have tokens. Some, however, might find that idea counterintuitive.

1.2.1.b Applied Nonmodal Substitutional Semantics

Now, what kind of items does a formal substitutional interpretation involve? In view of the previous summary of the evolution of substitutional semantics we may construe the "standard" substitutional model for $\mathcal{L}^s$, taking C to be a class of substituends from the alphabet of the base language $\mathcal{L}^s_0$ and Atm to be the set of atomic sentences of $\mathcal{L}^s_0$, as a tuple $\mathcal{S} = \langle \{1, 0\}, I \rangle$ where I is a map from Atm to $\{1, 0\}$.[66] When A is a closed formula, then given $\mathcal{S}$ the valuation v as relativized to C, $v_{\mathcal{S}_C}$, is defined by the recursion:

1. if A is an atomic sentence, $v_{\mathcal{S}_C}(A) = I(A)$;
2. if A is $\neg B$, $v_{\mathcal{S}_C}(A) = 1$ iff $v_{\mathcal{S}_C}(B) = 0$;
3. if A is $B \wedge C$, $v_{\mathcal{S}_C}(A) = 1$ iff $v_{\mathcal{S}_C}(B) = 1$ and $v_{\mathcal{S}_C}(C) = 1$;
4. if A is $(\Pi x)B(x)$, $v_{\mathcal{S}_C}(A) = 1$ iff $v_{\mathcal{S}_C}(B(a)) = 1$ for all names a.

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[65] Cf., for example, the following passage form [Kripke, 1975] p. 705 which is suitable for our context: "If L contains a name for each object in D, and a denotation relation is defined (if D is nondenumerable, this means that L contains nondenumerably many constants), the notion of satisfaction can (for most purposes) effectively be replaced by that of truth: e.g., instead of talking of $A(x)$ being satisfied by an object a, we can talk of $A(x)$ becoming true when the variable is replaced by a name of a."(The italicized signs and sequences thereof refer to Kripke's use of them, but no confusion will arise from this of course.)

[66] We thus deviate from the tradition of reserving the term 'model' for denotational models.

What, then, is the intended standard substitutional model $\mathcal{S}^* = \langle\{1^*, 0^*\}, I^*\rangle$? The answer is, as one might expect in view of the previous intuitive remarks, that $\{1^*, 0^*\}$ is the set of truth-values, 1^* being the truth-value "true" (or the True) and 0^* being the truth-value "false" (or the False); and that I^* is a truth-value assignment (or a map from atomic sentences to the True and the False, respectively).

The notion of truth of a closed formula A of $\mathcal{L}^s$ in a formal standard substitutional model $\mathcal{S}$ is then linked to the notion of truth *simpliciter* of an ordinary nonmodal sentence as follows: the ordinary sentence is true *simpliciter* just in case its symbolization A, taken to have the meaning of the original sentence, is true in the intended substitutional model $\mathcal{S}^*$. But what does this interpretation teach us about the connection between $\mathcal{S}$ and reality?

The first question the applied semanticist will have to ask, so to speak *ex officio*, is surely this: What kind of entity are the True and the False? What is their nature? Moreover, what do the True and the False represent? Suppose the True just is reality or, alternatively, represents reality. What then is or represents the False?[67]

Second, taking it that a semantic theory is committed to the existence of whatever kind of entity is involved in its account of truth (see Subsection 1.1.2.c), the applied semanticist of $\mathcal{S}$-models will have to incur a commitment to whatever 1^* and 0^* are. On the approach of intended* standard substitutional models, there will be, presumably, no commitment to the True and the False, but the question of what 1 and 0 represent will remain.

Thirdly, all true sentences will have the same "truthmaker", that is for example, 1^* on the approach of intended models or, alternatively, 1 as representing something, which needs to be specified, via a bijection on the intended*

[67]Clearly, a theorist who argues as follows misses the point of applied semantics: "There are just two truth-values—true and false. What are they: mysterious Fregean objects, properties, relations of correspondence and noncorrespondence? The answer is that it does not matter what they are; there is nothing essential to them except that there are exactly two of them." ([Stalnaker, 1984] p. 2.) (The context of this passage is Stalnaker's discussion of the definition of propositions as functions from possible worlds into truth-values—a definition Stalnaker, being an adherent of unstructured propositions, accepts.)

account. Intuitively, however, the sentence ‘Plato is a philosopher’ changes its truthmaker, when ‘Plato’ is replaced by ‘Aristotle’. We shall call this *the problem of truthmaker monism*.

Furthermore, the undiscerning monolithic truthmaker will not only be insensitive with respect to the change of the subjects of the sentences but also with respect to the category of discourse in which the sentences occur. Thus, fictional sentences like ‘Pegasus flies’ and nonfictional statements like ‘Aristotle is a philosopher’ will be true of the same portion of reality.

Finally, the account of the truth of *atomic* sentences is unsatisfactory compared to the denotational approach. Thus the truth of, for instance, ‘Aristotle is a philosopher’ just cannot be explained further, for example, in terms of Aristotle’s satisfying the predicate ‘... is a philosopher’. But clearly, the sentence feels to be further analysable.

All these problems of standard substitutional semantics do have, as it seems to me, the same source: the fact that the semantic values of atomic sentences are not dertermined in a compositional way, i.e., the fact that the semantic value of an atomic sentence does not depend upon the semantic values of its constitutive terms. We shall call this problem, which subsumes the aforementioned ones, *the problem of noncompositionality*.

The applied semantical interpretation of standard substitutional semantics shares the problem of noncompositionality with the applied interpretation of the standard semantics of languages for propositional logic. It is this coarseness of substitutional semantics, I think, which is a chief reason for dissatisfaction with that kind of framework. In some sense, to accept the standard substitutional framework is to make a step backwards in the evolution of truth-conditional logical semantics, a step from the semantics of first-order logic back to the semantics of propositional logic.

Moreover, what might make the substitutional semanticist feel even less confident than the semanticist of propositional logic is the fact that the syntactical generation of a sentence of his first-order substitutional language is not mirrored semantically—an asymmetry of this sort does not arise in the case of propositional semantics.

It should be mentioned, though, that proponents of substitutional semantics might not perceive the lack of compositionality as a problem. Indeed, Huges Leblanc seems to view this as an advantage, since a substitution interpretation of the quantifier is—to his mind—more natural and since substitutional semantics is technically much simpler.[68] Indeed, it is simpler. But I doubt that this can outweigh the disadvantages we have discussed.[69]

A further and related reason for dissatisfaction with substitutional semantics is the lack of reference for denoting singular terms. A crucial consequence of this is *the problem of the inexpressibility of identity claims*. Since the relation of identity is one which requires individuals as relata, and since the models of the standard substitutional semantics do neither involve objectual domains nor functions which assign objects to constants, identity statements—more exactly, statements of referential identity—cannot be handled on this semantics. It is for this reason that the substitutional language $\mathcal{L}^s$ does not involve an identity predicate of this sort. The only notion of identity which may be introduced into that language is a notion of "nonreferential identity" (symbolically: 'I') which given a standard substitutional model, may be defined by the principle of substitutivity as follows.

> If A is $a\mathrm{I}b$, $v_{\mathcal{S}_C}(A) = 1$ iff for all sentences B_1 and B_2 where B_2 is like B_1 except for containing occurrences of the name b at one or more places where B_1 contains the name a, $v_{\mathcal{S}_C}(B_1) = 1$ iff $v_{\mathcal{S}_C}(B_2) = 1$.[70]

With this notion of nonreferential (or substitutional) identity we may claim that a sentence like 'Santa Claus is Father Christmas' (taking the 'is' to be nonreferential) is true even though there will be no individual which will serve as a subject for that sentence, that is, something it is about. In other words,

[68] See, e.g., [Leblanc, 1973a] p. 1 or [Leblanc, 1976] p. 1.

[69] Perhaps it should be also mentioned that Leblanc is mainly concerned with the technical development of truth-value semantics rather than with the philosophical issues associated with it. Indeed, in his monograph [Leblanc, 1976], which unifies much of his technical work, he refers the philosophically interested reader to the writings of Marcus, Quine and others (see page ix, note 2). No doubt, it is in part for this reason, that his work in substitutional semantics has, unfortunately, not received much attention in general philosophy.

[70] Cf. [Marcus, 1993 (1972)] p. 85. Here the Bs should, perhaps, better not contain the predicate 'I' as Marcus misses to note.

the substitutional identity of 'Santa Claus' and 'Father Christmas' does not give us a referent; nor does the substitutional identity of 'Ruth Marcus' with itself.[71]

In order to obtain a formal nonmodal language with nonreferential identity, we enrich $\mathcal{L}^s$ with the predicate 'I' and the appropriate formation rule so as to obtain the substitutional language $\mathcal{L}^{s^I}$. The semantics for that language is the semantics for $\mathcal{L}^s$ supplemented with the clause for 'I' displayed above.

The fact that all singular terms, whether denoting or not, are construed as nondenoting on substitutional semantics is often found counterintuitive, since—at least on the descriptive picture of the relation between language and reality—this construal appears to be correct for nondenoting names like 'James Bond' but not for denoting names like 'Roger Moore'. Analogous remarks apply to the substitutional semantics of predication and quantification. Consequently, the ontological deflation achieved by substitutional semantics is often felt much too sweeping.[72] Let me call this *the problem of ontological deflation*.

Finally, in view of the traditional Fregean assumption that all truths have the same referent, the True[73], one is naturally inclined to think that even if names do not denote in substitutional semantics, the true sentences of the language are intuitively understood as being about the world. What is puzzling about standard substitutional semantics, then, is —so it seems to me—that it construes sentences (in particular, the atomic ones) as being descriptive of reality (in some sense) without also construing the denoting singular terms and predicates which occur in them as being referring and as having an appropriate extension, respectively. In any case it is not clear what the substitutional semanticist's picture of the relation between language and world is, if there is one at all.

[71]Cf., [Marcus, 1993 (1985/86)] p. 213.

[72]See, for instance, [Parsons, 1980] p. 36: "My main reason for not taking substitutional quantification seriously here is that it can be used just as well to avoid commitment to *anything at all*. If I insist that there are cows, an anticowist can grant me the truth of what I say, but hold that this does not really commit him to there being cows, since 'there are' may be taken as merely substitutional."

[73]See, for example, [Frege, 1994 (1892)] pp. 48 and 50.

1.2.2 Modal Substitutional Semantics

1.2.2.a Formal Modal Substitutional Semantics

The literature on substitutional quantified modal logic does not abund. Currently available formal semantics for substitutional quantified modal languages may be found, e.g., in [Leblanc, 1973b], [Dunn, 1973], [Leblanc, 1976], as well as in [Copeland, 1982] and [Copeland, 1985] where also some philosophical issues pertaining to it are discussed. The contributions [Marcus, 1993 (1961)], [Marcus, 1976], [Marcus, 1993 (1985/86)], and [Kripke, 1976][74] deal mainly with the philosophical significance of modal substitutional semantics.

To obtain a formal modal language with nonreferential identity we enrich $\mathcal{L}^{s^I}$ with the box-operator $\Box$ (for "necessarily") and the appropriate formation rule so as to obtain the modal substitutional language $\mathcal{L}^{s^{I\Box}}$.

The standard modal substiutional model for $\mathcal{L}^{s^{I\Box}}$ will be a tuple $\mathcal{R} = \langle S, R, c, \{1,0\}, I\rangle$. We let C be a class of substituends constrained to the individual constants from the alphabet of the base language $\mathcal{L}_0^{s^{I\Box}}$ $(= \mathcal{L}_0^{s})$ so that the substitutional quantifiers will be nominal, and we let Atm be the set of atomic sentences of $\mathcal{L}_0^{s^{I\Box}}$. The model contains the following components. S is a set of indices $s, t, ...$; $R \subseteq S \times S$; $c : S \rightarrow \wp(C)$, where $c(s) = c(t)$ for every $s, t \in S$; $\{1, 0\}$ is a set; and $I : Atm \times S \rightarrow \{1, 0\}$. When A is a closed formula of $\mathcal{L}^{s^{I\Box}}$, then given some $\mathcal{R} = \langle S, R, c, \{1,0\}, I\rangle$ and an index $s \in S$ the valuation v as relativized to C, $v_{\mathcal{R}_C}$, is defined by the recursion:

1. if A is an atomic sentence, $v_{\mathcal{R}_C}(A, s) = I(A, s)$;

2. if A is $a\mathrm{I}b$, $v_{\mathcal{R}_C}(A) = 1$ iff for all sentences B_1 and B_2 (both without 'I') where B_2 is like B_1 except for containing occurrences of the name b at one or more places where B_1 contains the name a, $v_{\mathcal{R}_C}(B_1) = 1$ iff $v_{\mathcal{R}_C}(B_2) = 1$;

3. if A is $\neg B$, $v_{\mathcal{R}_C}(A, s) = 1$ iff $v_{\mathcal{R}_C}(B, s) = 0$;

4. if A is $B \wedge C$, $v_{\mathcal{R}_C}(A, s) = 1$ iff $v_{\mathcal{R}_C}(B, s) = 1$ and $v_{\mathcal{R}_C}(C, s) = 1$;

[74]See section 3, in particular, p. 350, note 20 and section 6, in particular, pp. 374-375.

5. if A is $(\Pi x)B(x)$, $v_{\mathcal{R}_C}(A, s) = 1$ iff $v_{\mathcal{R}_C}(B(a), s) = 1$ for all names a.

6. if A is $\Box B$, $v_{\mathcal{R}_C}(A, s) = 1$ iff for all $t \in S$, $v_{\mathcal{R}_C}(B, t) = 1$.

A closed formula A is true in $\mathcal{R}$ just in case $v_{\mathcal{R}_C}(A, s) = 1$ for every $s \in S$; and it is valid if it is true in all models.[75]

Among the theorems of constant substitution class systems will be the following substitutional versions of BF, CBF, and NE:

SBF $(\Pi x)\Box A \rightarrow \Box(\Pi x)A$;
CSBF $\Box(\Pi x)A \rightarrow (\Pi x)\Box A$;
NSE $(\Pi x)\Box(\Sigma y)y\mathrm{I}x$.

When we drop the constancy condition we obtain varying substitution class standard models in a way analogous to the denotational counterpart. SBF will be invalid when the model is not antimonotonic and CSBF will be invalid when the model is not monotonic. In the latter case NSE will be also invalid.

1.2.2.b Applied Modal Substitutional Semantics

Now, what is the connection between the notion of truth in $\mathcal{R}$ and the notion of modal truth according to the substitutional view? The answer, most likely, will be that a regimented sentence of ordinary modal language (with identity) read substitutionally will be true *simpliciter* just in case the closed formula A of $\mathcal{L}^{s^{I\Box}}$ which is interpreted to have the meaning of the sentence is true in the intended standard substitutional modal model $\mathcal{R}^* = \langle S^*, s^@, R^*, c^*, \{1^*, 0^*\}, I^*\rangle$ (where the latter will be the case exactly if the symbolization is true in $\mathcal{R}^*$ at $s^@$).

In a way analogous to the denotational case, the relationship between $\mathcal{L}^{s^{I\Box}}$ and $\mathcal{R}^*$ will be taken to mirror the relation which obtains between ordinary modal language (with identity) taken substitutionally and modal reality.

But what does an intended model $\mathcal{R}^* = \langle S^*, s^@, R^*, c^*, \{1^*, 0^*\}, I^*\rangle$ involve on the "paradigmatic" substitutional account? Standardly, $\{1, 0\}^*$ and

[75] This is essentially the framework given in [Copeland, 1985] pp. 1-2. It differs in the following respects: the language contains the predicate for nonreferential identity and it is left open whether the substitution classes are constant and whether $R = S \times S$.

I^* will be interpreted as in the nonmodal case (the former is the set of truth-values and the latter a truth-value assignment); and S^* and R^* will be interpreted just like in the denotational case as the set of possible worlds (which includes the actual world $s^{@}$) and a relation on that set. On the denotational account d^* has been interpreted as an assignment of possible objects from D^* to possible worlds. The substitutional counterpart of that assignment, c^*, by contrast, assigns names of an ordinary modal language to possible worlds. In this way the intended model $\mathcal{R}^*$ represents modal reality on the substitutional approach.[76] In effect, the models of standard (modal) substitutional semantics just are the usual "models" of ordinary (modal) sentential (or propositional) calculi.

The entities involved in intended modal standard substitutional models are possible worlds of some sort and besides this the True and the False (or some surrogates). By contrast, intended* modal standard substitutional models do not involve representatives of any such sort. Obviously, on either accout of representation, possible objects are not involved.

Since there are no possible objects, the problem of ontological inflation with respect to them and the problem of modal reference are avoided. Moreover, on the substitutional account, the philosophical issues concerning the denotational interpretation of formulae such as BF, CBF, NE, or *de re* necessary formulae (i.e., antiessentialist *actualia*, *possibilia*, necessitarianism, essentialism, metaphysical necessity, transworld identity, transworld reidentification and so on) do not arise.[77]

Furthermore, in view of the interpretation of the index-set, the substitutional modal semanticist of intended models ontologically inflates his theory with possible worlds. This inflation will force him to take a stance on the problem of possibilism and actualism and the details concerning the metaphysical nature of possible worlds. As the reader might expect, a suitable

[76]Hints at an informal interpretation of the set of indices in terms of possible worlds may be found, e.g. in [Leblanc, 1973b], [Copeland, 1982] and [Copeland, 1985]. Copeland seems to find this applied semantical interpretation acceptable whereas Leblanc rejects it as we shall see shortly.

[77]We shall address these issues more extensively in Chapter 4 from the perspective of the associative framework.

adaptation of the strategy of intended* Kripke-models (see Subsection 1.1.2.i above) to the case of modal standard substitutional models should help to avoid a commitment to possible worlds.

Another strategy to avoid a commitment to possible worlds, is the line taken by Hugues Leblanc in [Leblanc, 1973b]. According to his account of substitutional modal semantics, the formal models do not contain a set of indices and a relation on it as they do on the standard substitutional account. Instead, Leblanc's models include a set of truth-value assignments and a relation on them.[78] However, this strategy does not seem to avoid a commitment to possible worlds in and of itself. For it will be natural, as one might argue, to identify each truth-value assignment contained in Leblanc's set with a (maximal) consistent set of sentences which these assignments map to 1. An applied semanticist who adheres to the strategy of intended models, then, is likely to interpret such sets—in the terminology of [Lewis, 1986]—as linguistic ersatz possible worlds.

Let me close this discussion noting that a philosopher who, in spite of its difficulties opts for a pure standard substitutional quantified modal logic as the right and proper tool for the analysis of modal discourse will find neither the doctrines of rigid designation, the necessity of identity, essentialism, or the necessary a posteriori appealing, nor modal arguments for the existence of necessary beings or the identity or distinctness of natural kinds feasible.

1.2.3 Summary: Doubts about Substitutional Semantics

The aforementioned problems of applied standard substitutional semantics (e.g., the problem of the nature and existence of the True and the False (or their surrogates); the problem of what 1 and 0 represent on any account of representation; the problem of truthmaker monism; the problem of an insensitivity to the category of discourse; the problem of the unanalysed truth of atomic sentences) are, in the end, consequences of the problem of noncompositionality, which is due to the fact that in substitutional semantics

[78]See [Leblanc, 1973b] pp. 256-257. Cf. also [Dunn, 1973].

the semantic values of atomic sentences do not depend upon the semantic values of their constituents.

There are also problems which result from the referential inability of substitutional semantics. As a consequence, the denotational semanticist will find it problematic that substitutianal languages cannot express claims of referential identity and that the semantics for such languages leads to the problem of ontological deflation. Moreover, given the lack of modality *de re*, a denotationalist might miss the doctrines of Kripkean orthodoxy and the means to set up modal existence and distinctness arguments.

Furthermore, the substitutional language, whether fictional or not, is construed, in effect, as being descriptive of some reality which is represented in some way by intended (or maybe intended*) standard substitutional models, even if its singular terms do not refer and its quantifiers do not range over anything and thus do not have ontological import.

Finally, if there are only denumerably many names but nondenumerably many objects, substitutional semantics needs to be supported by a Löwenheim-Skolem argumentation.

We now turn to semantic frameworks which combine elements from denotational and substitutional semantics.

1.3 Mixed Semantics

There is a natural way in which substitutional semanticists tend to react to the problems of noncompositionality, the incapability of substitutional languages to express referential identity and to the problem of ontological deflation. The strategy is to supplement substitutional semantics with elements from denotational semantics.

According to the proposal made by Kripke in section 4 of [Kripke, 1976], the substitutional semanticist may introduce substitutional quantifiers into the referential language which already contains the referential quantifier, by taking the latter language as a base language and extending it with substitutional variables and quantifiers. In the resulting language both kinds of quantifier may occur without conflict.

By contrast, according to a less ecumenical account suggested by Ruth Marcus in [Marcus, 1993 (1978)], the referential quantifier is taken to be a special case of the substitutional. Here the semantics of substitutional quantification is enriched with a substitution class of denoting constants that define an objectual domain so as to achieve the effects of referential quantification. We shall deal with both accounts in the rest of this section.

1.3.1 Nonmodal Mixed Semantics

Let me first summarize Kripke's ideas concerning a mixed language which contains both the substitutional and the referential quantifier.

Let the base language $\mathcal{L}_0^{Kri}$ of the mixed Kripkean language $\mathcal{L}^{Kri}$ be just $\mathcal{L}^d$. However, in order to distinguish the two kinds of quantification we let the individual terms be underlined. Thus if φ^n is an n-ary predicate of $\mathcal{L}_0^{Kri}$ and $\underline{o}_1, ..., \underline{o}_n$ are the individual terms of that language (which can be either individual constants $\underline{\alpha}_1, \underline{\alpha}_2, ...$ or individual variables $\underline{x}_1, \underline{x}_2, ...$, then $\varphi^n \underline{o}_1 ... \underline{o}_n$ is an atomic formula of the base language. All atomic formulae of the base language are formulae of that language and if A and B are formulae of the base language, and $\underline{x}$ is an individual variable, then the following are also formulae of the base language: $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $(A \leftrightarrow B)$, $(\forall \underline{x})A$, $(\exists \underline{x})A$. Nothing else is a formula of $\mathcal{L}_0^{Kri}$. Bound and free individual variables are defined in the usual way.

We now extend the base language $\mathcal{L}_0^{Kri}$ so as to yield $\mathcal{L}^{Kri}$ by adding the substitutional quantifiers and the nominal substitutional variables $x, x_1, x_2, ...$, $y, ...$ bound by substitutional quantifiers.

Let $\widetilde{\varphi^n}$s be the n-ary predicates of $\mathcal{L}^{Kri}$ which just are the predicates of the base language and let $\widetilde{o}_1, \widetilde{o}_2, ...$ be the terms of $\mathcal{L}^{Kri}$ which can be either individual terms of the base language or nominal substitutional variables. So if $\widetilde{\varphi^n}$ is an n-ary predicate of $\mathcal{L}^{Kri}$ and if $\widetilde{o}_1, ..., \widetilde{o}_n$ are the singular terms of the specified sorts of that language, then $\widetilde{\varphi^n}\widetilde{o_1}...\widetilde{o_n}$ is an atomic formula of the language $\mathcal{L}^{Kri}$.

Then all atomic formulae of $\mathcal{L}^{Kri}$ are formulae of that language and if $\widetilde{A}$ and $\widetilde{B}$ are formulae of $\mathcal{L}^{Kri}$, $\underline{x}$ is an individual variable and x is a nominal substitutional variable, then the following are also formulae of the base language: $\neg\widetilde{A}$, $(\widetilde{A}\wedge\widetilde{B})$, $(\widetilde{A}\vee\widetilde{B})$, $(\widetilde{A}\rightarrow\widetilde{B})$, $(\widetilde{A}\leftrightarrow\widetilde{B})$, $(\forall\underline{x})\widetilde{A}$, $(\exists\underline{x})\widetilde{A}$, $(\Pi x)\widetilde{A}$, and $(\Sigma x)\widetilde{A}$. Nothing else counts as a formula of $\mathcal{L}^{Kri}$. Bound and free variables are defined in the usual way.

Now, we let $\mathcal{S}^{Kri}$ be a mixed standard model $\langle D, \{1,0\}, v, I\rangle$ for $\mathcal{L}^{Kri}$, where D is an ordinary non-empty domain of objects $\mathtt{d}_1$, $\mathtt{d}_2$, ...; v is a function which assigns semantic values to the individual constants from the substitution class C of individual constants $\underline{\alpha}$, $\underline{\beta}$, ... of the base language $\mathcal{L}_0^{Kri}$ and to the predicates of $\mathcal{L}^{Kri}$. (Recall that these are all contained in the alphabet of $\mathcal{L}_0^{Kri}$.) The assignments to predicates are slightly less ordinary since the predicates may be complex.

Let Prd_m be the set of all possibly complex predicates of $\mathcal{L}^{Kri}$ which are built up from a predicate and from individual terms of $\mathcal{L}_0^{Kri}$. Such a predicate will have the general form $\varphi^{n,m}\underline{o}_1;...;\underline{o}_n$, where the superscript n indicates the arity of the predicate and the superscript m (where $n \geq m \geq 1$) indicates the number of free (not necessarily distinct) individual variables occuring among $o_1;...;o_n$. The semicolons indicate that the order of the individual terms, in the general form, is arbitrary. Concrete instances of this general form will contain no semicolons for in them the order of occurrence is not arbitrary. The following predicates are, for instance, among the elements of Prd_m: $\varphi^{1,1}\underline{x}$, $\varphi^{2,1}\underline{x\alpha}$, and $\varphi^{3,2}\underline{x_1\alpha x_2}$. In effect, Prd_m is the set of the atomic formulae of $\mathcal{L}_0^{Kri}$ with at least one free individual variable. The second part of the definition of v then is:

$$v : Prd_m \rightarrow \wp(D^m).$$

The referential extensions of the three predicates listed above will be: $v(\varphi^{1,1}\underline{x}) \subseteq D$, $v(\varphi^{2,1}\underline{x\alpha}) \subseteq D$, and $v(\varphi^{3,2}\underline{x_1\alpha x_2}) \subseteq D \times D$. (Informally, we may take them to mean, for example, '... is a man', '... loves Mary', and '... has the choice between Mary and ...', respectively.) In general the extension of a predicate $v(\varphi^{n,m}\underline{o}_1;...;\underline{o}_n)$, will be a set of m-tuples of the form $\{\langle \mathtt{d}_1; \ldots ; \mathtt{d}_m\rangle, ...\}$. When the semicolons occur in tuples they are meant to indicate that their elements are ordered—in a way, though, which our semicolon notation does not capture. (This is not to say, of course, that the order of the elements contained in the tuples is arbitrary.)

As for I, we let Atm be the set of all atomic sentences $\widetilde{A}$ of $\mathcal{L}^{Kri}$ which contain exclusively individual constants that are assigned no object from D and stipulate that $I : Atm \rightarrow \{1, 0\}$.

Let δ be an ordinary denotational variable assignment which assigns to each individual variable $\underline{x}$ an object $\mathtt{d}$ from D. $\underline{x}$-variants ζ of δ are defined in the common way. Nominal substitutional variables are assigned no values.

Given some Kripke-style mixed model $\mathcal{S}^{Kri} = \langle D, \{1, 0\}, v, I\rangle$ and some assignment δ based on it, we let the $\tilde{o}$s be individual terms of $\mathcal{L}^{Kri}$ and define the notion of the value of some $\tilde{o}$ with respect to $\mathcal{S}^{Kri}$ under δ, $v_\delta(\tilde{o})$, as follows.

1. If $\tilde{o}$ is an individual variable $\underline{x}$ then $v_\delta(\tilde{o}) = \delta(\tilde{o})$.

2. If $\tilde{o}$ is an individual constant $\underline{\alpha}$ then $v_\delta(\tilde{o}) = v(\tilde{o})$.

When $\tilde{o}$ is a nominal substitutional variable it does not receive a value.

We may now define the truth conditions for the formulae of $\mathcal{L}^{Kri}$ with respect to $\mathcal{S}^{Kri} = \langle D, \{1, 0\}, v, I\rangle$ under δ as follows:

1. $\mathcal{S}^{Kri}_\delta \models \varphi^{n,m}\tilde{o}_1; ...; \tilde{o}_n$ iff $\langle v_\delta(\tilde{o}_1); ...; v_\delta(\tilde{o}_m)\rangle \in v_\delta(\varphi^{n,m}\tilde{o}_1; ...; \tilde{o}_n)$. (For example, $\mathcal{S}^{Kri}_\delta \models \varphi^{3,2}\underline{x}_1\underline{\alpha}\underline{x}_2$ iff $\langle v_\delta(\underline{x}_1), v_\delta(\underline{x}_2)\rangle \in v_\delta(\varphi^{3,2}\underline{x}_1\underline{\alpha}\underline{x}_2)$.)

   When $\widetilde{A}$ and $\widetilde{B}$ do not contain free nominal substitutional variables x then:

2. $\mathcal{S}^{Kri}_\delta \models \neg\widetilde{A}$ iff $\mathcal{S}^{Kri}_\delta \not\models \widetilde{A}$.

3. $\mathcal{S}^{Kri}_\delta \models \widetilde{A} \wedge \widetilde{B}$ iff $\mathcal{S}^{Kri}_\delta \models \widetilde{A}$ and $\mathcal{S}^{Kri}_\delta \models \widetilde{B}$.

4. $\mathcal{S}^{Kri}_\delta \models (\forall\underline{x})\widetilde{A}$ iff for all $\underline{x}$-variants ζ of δ: $\mathcal{S}^{Kri}_\zeta \models \widetilde{A}$.

5. If $\widetilde{A} \in Atm$ then $\mathcal{S}^{Kri}_\delta \models \widetilde{A}$ iff $I(\widetilde{A}) = 1$; otherwise $\mathcal{S}^{Kri}_\delta \not\models \widetilde{A}$.

   The last clause is reserved for the substitutional quantifier. If $\widetilde{A}$ is a formula of $\mathcal{L}^{Kri}$ with at most one free occurrence of the nominal substitutional variable x, then

6. $\mathcal{S}^{Kri}_\delta \models (\Pi x)\widetilde{A}$ iff for all individual constants $\underline{\alpha} \in C$: $\mathcal{S}^{Kri}_\delta \models \widetilde{A'}$, where $\widetilde{A'}$ comes from $\widetilde{A}$ by replacing any free occurrences of x by $\underline{\alpha}$.

The notion of truth in $\mathcal{S}^{Kri}$ for the closed formulae of $\mathcal{L}^{Kri}$ can be defined in the ordinary way as truth in $\mathcal{S}^{Kri}$ under all assignments to the individual variables.

Illustration: Consider, for instance, the mixed quantified formula $(\Sigma x)(\forall \underline{y})$ $\varphi^{2,1}x\underline{y}$, which will be true in the model exactly if there is an individual constant $\underline{\alpha}$ in C such that $(\forall \underline{y})\varphi^{2,1}\underline{\alpha y}$ is true in the model. And this will be so just in case the complex predicate $\varphi^{2,1}\underline{\alpha y}$ is true of every object from D under any assignment to the individual variable $\underline{y}$. For example, let $\varphi^{2,1}x\underline{y}$ have the meaning of the English predicate '... is admired by ...', let D be the set of all school boys and let C contain the names of comic heroes.

Consider next an even simpler formula like $(\Sigma x)\varphi^{1,0}x$? That formula will be true in $\mathcal{S}^{Kri}$ under δ just in case for at least one individual constant $\underline{\alpha} \in C$: $\mathcal{S}^{Kri}_{\delta} \models \varphi^{1,0}\underline{\alpha}$. And this will be so if something to the effect of the right-hand side of clause (1) will hold. But note that $(\Sigma x)\varphi^{1,0}x$, taken to mean 'There is at least one Hobbit' and read substitutionally, won't be true on the original proposal. For $\varphi^{1,0}\underline{\alpha}$ won't be true when $\underline{\alpha}$ is interpreted as 'Frodo' (or receives the meaning of any other Hobbit-name) and assigned no object from D. In such cases we take it that I maps $\varphi^{1,0}\underline{\alpha}$ to the True and apply clause 5.

To obtain a semantics for ordinary statements of referential identity like 'Bob Dylan is Robert Zimmerman' the base language is extended so as to contain the binary predicate '=' and the appropriate formation rule; and the semantics for this language, $\mathcal{L}^{Kri^{=}}$, is to be supplemented with the standard truth conditions for such identity statements.

Similarly, to accommodate such sentences like 'Santa Claus is Father Christmas' a nonreferential identity predicate, 'I', and the corresponding truth clause may be added so as to obtain a semantics for $\mathcal{L}^{Kri^{I}}$.

In a language $\mathcal{L}^{Kri^{=I}}$ with the appropriate semantics both kinds of identity predicate may, then, interact in one single sentence like, for instance, in $(\Sigma x)(\Sigma y)(\exists \underline{x})(\exists \underline{y})(x\mathrm{I}y \wedge \neg(\underline{x} = \underline{y}))$.

In [Marcus, 1993 (1978)] Ruth Marcus opposes Kripke's proposal and suggests a different view of the relation between substitutional and referential quantification. On her view the substitutional account is the more general one and may be construed so as to capture the referential quantifier as a special case. A formalization of Marcus's ideas[79] might be helpful as well. Here is a suggestion.

Let $\mathcal{L}^{Mar}$ be just the purely substitutional language $\mathcal{L}^{s}$ and let the quantifiers bind nominal variables. We stipulate that the base language $\mathcal{L}_0^{Mar}$ does only contain atomic sentences of $\mathcal{L}^{Mar}$ which in turn results from the base language when it is extended with the nominal variables and quantifiers. To distinguish the denoting from the nondenoting nominal constants we underline them.

A Marcus-style substitutional model $\mathcal{S}^{Mar} = \langle D, \{1, 0\}, v, I\rangle$ for $\mathcal{L}^{Mar}$ is just like $\mathcal{S}^{Kri}$ except for D being denumerable. In particular, C is a substitution class of nominal constants α, β, ... of $\mathcal{L}_0^{Mar}$, where C^d is the subset of C which contains the denoting constants $\underline{\alpha}$, $\underline{\beta}$, ... ; and v an ordinary denotational valuation function which is defined as usual, in particular, $v : C^d \rightarrow D$, where for every $\mathtt{d} \in D$ there is a nominal constant $\underline{\alpha} \in C^d$ such that $v(\underline{\alpha}) = \mathtt{d}$.

We let δ be an ordinary denotational variable assignment based on $\mathcal{S}^{Mar}$ which assigns to each nominal variable x an object $\mathtt{d}$ from D. x-variants ζ of δ are defined in the usual way and so is the notion of the value of a nominal term o of $\mathcal{L}^{Mar}$ (which may be either a denoting nominal constant

[79] "Now, suppose each of our denumerably infinite stock of names does refer to an object. Let those objects make up our reference class: a domain. Under those conditions, we can introduce a substitutional analogue of satisfaction of a formula relative to that domain. If all the contexts in the interpreted language are transparent, then the substitutional analogue of satisfaction converges with the referential definition of satisfaction. Under those conditions the quantifiers can be read with existential import. I see the referential quantifier as a limiting case. Substitutional quantification, together with a substitution class of names that define a reference class of objects, yields a referential quantifier. If our substitutional language allows wider substitution classes beyond the set of referring names, then of course it is important to distinguish with an alternative notation those cases of quantification where substituends are referring names, for it is *those* cases that can be read back into English as 'There is something such that' and 'Everything is such that'. They have existential import." ([Marcus, 1993 (1978)] pp. 120-121). In a footnote to this passage Marcus adds: "If our substitution class of names is extended to include nonreferring (syntactic) names as well, then it is for the denumerable subset of referring names that the quantifiers are read with existential import."

or a nominal variable) with respect to $\mathcal{S}^{Mar}$ under δ, $v_\delta(o)$.

We may then define truth for the formulae of $\mathcal{L}^{Mar}$ with respect to $\mathcal{S}^{Mar} = \langle D, \{1,0\}, v, I\rangle$ for $\mathcal{L}^{Mar}$ under δ as follows:

1. $\mathcal{S}_\delta^{Mar} \models \varphi^n o_1 ... o_n$ iff $\langle v_\delta(o_1), ..., v_\delta(o_n)\rangle \in v_\delta(\varphi^n)$.

   For the complex cases we let the subformulae be underlined so as to indicate that they exclusively contain occurrences of nominal terms which have received a denotation.

2. $\mathcal{S}_\delta^{Mar} \models \neg\underline{A}$ iff $\mathcal{S}_\delta^{Mar} \not\models \underline{A}$.

3. $\mathcal{S}_\delta^{Mar} \models \underline{A} \wedge \underline{B}$ iff $\mathcal{S}_\delta^{Mar} \models \underline{A}$ and $\mathcal{S}_\delta^{Mar} \models \underline{B}$.

4. $\mathcal{S}_\delta^{Mar} \models (\Pi x_i)\underline{A}$ iff for all x-variants ζ of δ: $\mathcal{S}_\zeta^{Mar} \models \underline{A}$.

   Formulae which contain nondenoting nominal constants receive the following truth conditions.

5. When $\alpha_1, ..., \alpha_n \in C$ then the following holds: $\mathcal{S}_\delta^{Mar} \models \varphi^n \alpha_1 ... \alpha_n$ iff $I(\varphi^n \alpha_1 ... \alpha_n) = 1$; otherwise $\mathcal{S}_\delta^{Mar} \not\models \varphi^n \alpha_1 ... \alpha_n$.

   For the complex cases we leave the subformulae without underlines in order to indicate that they may contain occurrences of nondenoting nominal terms.

6. $\mathcal{S}_\delta^{Mar} \models \neg A$ iff $\mathcal{S}_\delta^{Mar} \not\models A$.

7. $\mathcal{S}_\delta^{Mar} \models A \wedge B$ iff $\mathcal{S}_\delta^{Mar} \models A$ and $\mathcal{S}_\delta^{Mar} \models B$.

8. $\mathcal{S}_\delta^{Mar} \models (\Pi x_i)A$ iff for every α from C $\mathcal{S}_\delta^{Mar} \models A'$ where A' results when all the occurrences of x_i in A are replaced by α.

The notion of truth in $\mathcal{S}^{Mar}$ for the closed formulae of the denoting portion of $\mathcal{L}^{Mar}$ can be defined in the ordinary way as truth in $\mathcal{S}^{Mar}$ under all variable assignments. In case $C - C^d = \emptyset$, truth in $\mathcal{S}^{Mar}$ under all variable assignments and substitutional truth coincide for the closed formulae. Variable assignments, of course, do nothing to define truth in $\mathcal{S}^{Mar}$ for the whole of $\mathcal{L}^{Mar}$.

There are thus two ways of evaluating substitutionally quantified formulae depending on whether it contains nondenoting nominal constants or not. In the former case (e.g., 'All boys admire James Bond') is to be evaluated according to clause 8, in the latter case (e.g., 'All boys admire Sean Connery') clause 4 applies.

On this construal of Marcus's account, then, every atomic formula which contains at least one nondenoting nominal constant is to be evaluated by purely substitutional means. By contrast, on our previous construal of Kripke's approach, only atomic formulae which contained exclusively nondenoting terms were evaluated that way.

We may extend the Marcusian language to a language $\mathcal{L}^{Mar^I}$ with the nonreferential identity predicate 'I' and amend the semantics with two kinds of truth clause for identity statements. Formulae in which 'I' is flanked exclusively by denoting nominal terms receive the usual referential truth conditions; otherwise the truth conditions will be the substitutional ones.

The question to be asked now is which view on the relationship between substitutional and referential quantification should be preferred, the Kripkean or the Marcusian? The central difference between these accounts is, in my oppinion, that the latter account unlike the former requires support by some reasoning to the effect of a Löwenheim-Skolem argument, since it captures only referential quantifiers whose domains are denumerable. And this is an advantage of Kripke's view. At any rate, in [Marcus, 1993 (1985/86)] Ruth Marcus opts for an account with mixed quantifiers.[80]

1.3.2 Modal Mixed Semantics

To obtain modal versions of the languages $\mathcal{L}^{Kri}$ and $\mathcal{L}^{Mar}$ and their versions with identity, they have to be enriched with the box-operator and the appropriate formation rule. To obtain the semantics for the resulting modal languages $\mathcal{L}^{Kri^\Box}$ and $\mathcal{L}^{Mar^\Box}$ and their identity extensions, the models $\mathcal{S}^{Kri}$ and $\mathcal{S}^{Mar}$ are to be supplemented with the familiar sets. The resulting models $\mathcal{R}^{Kri}$ and $\mathcal{R}^{Mar}$ will both be tuples of the form $\langle S, R, D, c, d, \{1, 0\}, I, v \rangle$,

[80]See [Marcus, 1993 (1985/86)] p. 213. She thus seems to return to the view of [Marcus, 1976] (see, in particular, page 48).

but D will be denumerable in $\mathcal{R}^{Mar}$. The clauses for the formulae with the box as their main logical operator are the usual ones. As usual, there will be much room for choice, for instance, concerning the logical properties of R or the constancy of the model. We need not go more into this here. Instead, let us consider the question of how the mixed substitutional semanticist can make profit from his resources.

Here is a suggestion made by Ruth Marcus on how *possibilia* can be dispensed with in a mixed Kripkean framework with varying domains. (Recall that the problem of *possibilia* does not arise for semantics which employ models with constant domains but only for varying domain semantics whose models are set up in such a way as to invalidate the Barcan Formula.)

> "Where the substitution class for the quantifiers are the names assigned to the actual world, we can read the quantifiers objectually, or existentially. Indeed, we can reintroduce full fledged reference by associating a domain of objects with the actual world and view our quantifiers as mixed: referential for this world and substitutional otherwise. We can thus dispense with the artifice of domains of possibilia."[81]

First, let us assume that the modal language is $\mathcal{L}^{Kri^{=I\Box}}$ and the semantics for this language will involve models $\mathcal{R}^{Kri}$ whose objectual domains and nominal substitution classes co-vary and the appropriate truth clauses (which might also involve a clause for the existence predicate).

Now, how should Marcus's suggestion be understood? Presumably in a way like this. Whenever a formula of a mixed language which contains a quantifier that does not occur in the scope of a modal operator is to be evaluated at the index which represents the actual world, this quantifier is a referential one. Indeed, on this proposal the mixed formula $\Diamond(\Sigma x)\widetilde{A} \wedge \neg(\exists \underline{x})\Diamond\widetilde{A}$ of $\mathcal{L}^{Kri^{=I\Box}}$ which will be equivalent to the negation of the Barcan Formula will give no rise to the existence of some *possibile* or some actualistically acceptable surrogate. But notice that according to this account, such formulae like $(\exists \underline{x})\Diamond\widetilde{A}$ will also be admitted for evaluation at the index for the actual

[81] [Marcus, 1976] p. 48.

world. And since this is so, the proposal—on the canonical construal—must provide answers to concerns of transworld identity and thus to metaphysical questions concerning the nature of transworld individuals and epistemological issues concerning their reidentification. The proponent of the paradigmatic account of intended mixed models, by contrast, will have to address the issue of how some individual representative does manage to represent one and the same individual across different possible worlds. The intended* mixed semanticist, too, will have to answer this question. Such complications of transworld identity, I take it, will not encourage the advocate of pure substitutional (modal) semantics to embrace the mixed proposal.

There is a further, rather straightforward way, in which such problems can be avoided as well. We simply modify Marcus's proposal (or her proposal as understood above) so as to require that only those referential quantifiers which occur in nonmodal formulae of $\mathcal{L}^{Kri^{=I\Box}}$ are to be evaluated at the actual index in the denotational way and that these are the only referential quantifiers. An analogous cure will also work for $\mathcal{L}^{Mar^{I\Box}}$.

This crude therapy, though, causes a couple of unwanted side effects which will not come as a surprise. First, since the intended (or intended*) interpretations of $\mathcal{R}^{Kri}$ and $\mathcal{R}^{Mar}$ will involve something like the True and the False (or, in the intended* case, things represented by 1 and 0 via a bijection), all the intuitive problems of noncompositionality which we have discussed in Section 1.2 in the modal case do reappear (e.g., the problems of the nature of the True and the False or truthmaker monism.)

A second side effect of this therapy which surely makes it unbearable for Marcusians and Kripkeans alike, is that the doctrine of the necessity of referential identity can no longer be maintained in the usual denotational way. As a consequence, then, Kripke's necessary a posteriori and modal distinctness (or identitiy) arguments will lose their appeal. Analogous remarks apply to necessary existence and to modal existence arguments.

The mixed modal substitutional semanticist, therefore, has to face the following dilemma with respect to denoting constants: either accept the therapy and the problems of noncompositionality, necessary referential identity, and necessary existence; or reject it and incur a commitment to possible objects. An argument which will support the latter choice is that possible objects (and possible worlds) can be dispensed with in terms of intended* models (or some other strategy). An argument which might be put forward in support of the former option is that there is no obvious way to explain the ground of modal truth. Either way, the mixed semanticist will have to face the problems of noncompositionality for nondenoting constants.

Let me close this discussion of mixed semantics with a simple observation. It will have been noted that the work done by our Kripke-style and Marcus-style models $\langle D, \{1, 0\}, v, I\rangle$ may be done, equally well, by an ordinary denotational model $\mathcal{T} = \langle D, v\rangle$ letting, as is often done, v doing the job of I and suppressing the appearance of $\{1, 0\}$ in the model. The models of mixed (modal) semantics thus are, in effect, ordinary denotational (modal) models. As in the pure denotational and substitutional case the mixed (modal) language is construed as being about a reality which constitutes the level of denotation of the object language and which, in the mixed case, is represented by ordinary denotational models.

1.3.3 Summary: Doubts about Mixed Semantics

The strategy of mixed semantics provides a solution to the problems of noncompositionality, the problem of the inexpressibility of referential identity, the problem of ontological deflation, and the problem of nondenumerable domains for the denoting portion of the nonmodal mixed language. However, the problems of noncompositionality reappear for the nonreferential portion of the nonmodal mixed language.

When the semantics is extended so as to deal with a modal mixed language, one has to decide whether modal formulae are to be evaluated by substitutional or by denotational means. In the first case, the problems of the substitutional modal semantics will be inherited, in the second case the problems of denotational modal semantics will reappear.

1.4 Concluding Remarks: Semantic Intuitions

In view of the preceding discussion, it seems that the strengths of substitutional semantics (no ontological inflation with respect to the quantifier of the object language, no problems with fictional and modal reference and so on) are rooted in a drawback, the lack of compositionality and Tarskian satisfaction. And, on the other hand, it appears that the disadvantages of denotational semantics (ontological inflation, problems with fictional and modal reference, etc.) are rooted in its strength, the presence of compositionality and satisfaction.

My project will be to develop a substitutional semantics in which compositionality is present but Tarskian satisfaction and denotation is not. The framework of associative substitutional semantics to be presented will retain the positive aspects of substitutional semantics without involving the aforementioned problems of noncompositionality. The semantics to be proposed is primarily developed with fictional and modal language in mind. However, as I shall argue in Chapter 4, it is also suitable for the referentially transparent and denoting portion of language.

Associative substitutional semantics is a truth-conditional semantics. Theorists working within the tradition of truth-conditional semantics typically take the intuition of designation to be a necessary condition which must be fulfilled by any truth-conditional semantics for natural language. This is nicely expressed in an influential textbook on denotational (more specifically, Montague) semantics as follows—I quote extensively as the passage helps to see the characteristics of the present approach:

> "A truth-conditional theory of semantics is one which adheres to the following dictum: To know the meaning of a (declarative) sentence is to know what the world would have to be like for the sentence to be true. (...) It is clear that one of the central notions of the truth-conditional approach is the relationship which sometimes holds between a sentence and the world. (*«The world» is here simply intended to refer to the vast complex of things and situations that sentences can be «about»*.) Many philosophers of language—and many linguists also, for that matter— would contend that it is an essential requirement of any semantic theory that it specify the nature of this relationship. In support of this, they cite the fact that a fundamental characteristic of natural

> language is that it can be used by human beings to communicate about things in the world. Any theory which ignores this essential property, it is argued, cannot be an adequate theory of natural language. Examples would be theories which, in effect, give the meaning of a sentence by translating it into another language, such as a system of semantic markers or some sort of formal logic, where this language is not further interpreted by specifying its connection to the world. The approach of Katz and his co-workers seems to be of this sort ([Katz and Fodor, 1963]; [Katz and Postal, 1964]), as is that of [Jackendoff, 1972], and of the framework known as Generative Semantics ([Lakoff, 1972]; [McCawley, 1973]; [Postal, 1970]). The point is controversial, and we will not enter into a discussion of the issues and alternatives here. We merely wish to emphasize that truth-conditional semantics, in contrast to the other approaches mentioned, is based squarely on the assumption that *the proper business of semantics is to specify how language connects with the world—in other words, to explicate the inherent «aboutness» of language.*"[82]

I also take it that the business of semantics is—in particular with respect to elementary predication—to explain how language connects with the world—'world' in the sense of 'reality'. But—in the light of the discussion in this chapter—I do not accept the view that it is a defining feature of truth-conditional semantics that language must be viewed as being about something, that is, as being about the world as its level of reference, or equiva-

[82] [Dowty *et al.*, 1981] pp. 4-5. (The bibliographical references for the citations are (in alphabetical order): Ray Jackendoff. *Semantic Interpretation in Generative Grammar*, The MIT Press, Cambridge Mass., 1972; Jerrold Katz and Jerry A. Fodor. The structure of a semantic theory, *Language*, 39 (1963), pp. 170-210; Jerrold Katz and Jerry A. Fodor. *An Integrated Theory of Linguistic Descriptions*, The MIT Press, Cambridge Mass., 1964; George Lakoff. Linguistics and natural logic, in: D. Davidson & G. Harman (eds.) *Semantics for Natural Language*, D. Reidel, Dordrecht, 1972, pp. 545-665; James McCawley. Syntactic and logical arguments for semantic structures, in: O. Fujimura (ed.) *Three Dimensions of Linguistic Theory*, The TEC Corporation, Tokyo, 1973, pp. 259-376; and Paul Postal. On coreferential complement subject deletion, *Linguistic Inquiry*, 14 (1970), pp. 439-500.)

lently, as being "externally significant"[83], as is sometimes said.

The semantical framework I wish to present differs from standard truth-conditional semantics in that it rejects the view that truth-conditional semantics has to be based on some aboutness assumption. However, being truth-conditional, it differs from the alternative approaches mentioned above in that it rests on the conviction that—in particular with respect to predication—a semantic theory has to provide an account of the relation between language and the world, if it is to be satisfactory.

I have already mentioned towards the end of the Introduction that what is distinctive of the framework to be presented is that it rests on the *intuition of reflection* as I call it, the intuition which supports the view that language need not be about a level of denotata for its sentences to be true, and that a sentence, e.g., an atomic one, can be true solely in virtue of the fact that it accurately reflects—in a sense to be explained—the meanings of the name(s) and the predicate from which it is composed. According to the account I shall propose, the intution of reflection will be explicated in terms of the "sense-extensions" of names and predicates (whose meanings are determined by the semantic rules which govern them) and in terms of the notion of "truth with respect to the level of sense" rather than in terms of (their) designations (in case they have any) and "truth with respect to the level of reference" which capture the intuition of designation.

The substitutional semantics to be presented is not only truth-conditional, it is also model-theoretic. However, since it is not denotational, it is model-theoretic in a nonstandard sense. I therefore reject the usual equation of model-theoretic with denotational (or referential) semantics.[84]

It is now time to develop the semantics of this dissertation in more detail. I shall first present the formal framework of "associative substitutional

[83]See [Larson & Segal, 1995] p. 5.

[84]A nice instance of the identification of model-theoretic with referential semantics can be found, for example, in [Lewis, 1983 (1970)] at page 190: "My proposals are in the tradition of *referential*, or *model-theoretic*, semantics descended from Frege, Tarski, Carnap (in his later works), and recent work of Kripke and others on semantic foundations of intensional logic." (The emphasis is Lewis's.) Clearly, this is a passage most denotationalists are likely to endorse. Indeed, this equation can be attributed to standard substitutional semanticists as well. Cf., for instance, the writings of Leblanc listed in the bibliography.

semantics" (Chapters 2 and 3) and then provide it with an applied semantical interpretation (Chapter 4) which captures the intuitions by which that framework is underlain. Chapter 5, the final chapter, will put the semantics into the perspective of modal epistemology. The reader who wants to learn more about sense-extensions and the notion of truth with respect to the level of sense without going through the technical material first, is referred directly to Section 4.2.

Chapter 2

Associative Substitutional Semantics

2.1 Introduction

My aim in this chapter is to propose an alternative semantics for substitutional quantified modal logic and to provide soundness and completeness results for various constant and varying substitution class axiom systems with "substitutional identity".

The formal framework presented here extends and modifies the theory of substitutional first-order quantification as proposed in [Kripke, 1976]. It enriches Kripke's original substitutional first-order language with a box-operator and a binary predicate for substitutional identity. The substitution classes for the quantifiers and the variables are constrained to nominal constants (or names for short).

The substitutional models of the modal semantics which will be suggested here differ from the ones offered in [Copeland, 1985] primarily in that they do not contain a valuation function which assigns truth values to sentences (of a base language). Instead, the models contain a function which assigns certain sets of atomic sentences of the base language—"associates" as I shall call them—to both nominal constants and predicates as their semantic values.

The truth conditions for pure atomic sentences of the base language L_0 (i.e., atomic sentences built up out of nominal constants and pure n-ary predicates only) will be defined in terms of name and predicate associates. The semantics, therefore, explains how the semantic values of nominal constants and predicates, respectively, contribute to the semantic value of atomic sentences. It is mainly with respect to the compositionality of the truth conditions for atomic sentences that this semantics differs from the substitutional semantics which have been proposed hitherto. Truth conditions for the rest of the formulae of the extended modal language L are then given inductively in terms of the truth conditions of pure atomic sentences of L_0.

The chapter is organized as follows. Section 2.2 sets up the language L for first-order modal logic with substitutional identity. Section 2.3 introduces "associative substitutional models" and gives a semantics for the sentences of L. Section 2.4 extends this semantics to all formulae of that language. In sections 2.5 through 2.7 completeness results for various axiom systems (more exactly: systems with SFB, systems without SFB, and systems without CSBF) are presented along largely familiar lines.

2.2 Substitutional Language

The substitutional language presented here extends the substitutional first-order language discussed in [Kripke, 1976] by supplementing Kripke's base language with a predicate for substitutional identity and adding a box-operator to the extended language.

2.2.1 Basic Language L_0

Alphabet of L_0

The alphabet of L_0 comprises the following symbols:

1. substitutional nominal constants (or names): $a, b, c, ...$
2. n-ary predicates: $F^n, G^n, H^n, ...$ $(n \geq 1)$
3. substitutional identity predicate: $\ddot{=}$

Let $\alpha, \beta, ..., \alpha_1, \beta_1, ...$ be metavariables ranging over name constants and let $\varphi^n, \chi^n, \psi^n, ...$ be metavariables ranging over the "pure" n-ary predicates listed under 2.

C is the set of all nominal constants of L_0. P will be the set of all pure predicates. So P does not contain $\ddot{=}$. We let C be denumerably infinite and we let P be a finite set.[1]

Sentences of L_0

The notion of a sentence of L_0 is defined by the following clauses.

1. If $\alpha_1, ..., \alpha_n$ are any nominal constants and φ^n is any predicate, then $\varphi^n\alpha_1...\alpha_n$ is a sentence of L_0.

2. If α_1 and α_2 are any nominal constants, then $\alpha_1 \ddot{=} \alpha_2$ is a sentence of L_0.

3. Nothing else is a sentence of L_0.

Let $Snt(L_0)$ be the set of all sentences of L_0. And let Atm be the set of L_0-sentences of kind 1, the set of "pure atomic L_0-sentences". (So Atm does not contain the substitutional identity sentences of L_0). We have $Atm \subseteq Snt(L_0)$.

Moreover, we define the sets $Atm(\alpha)$ and $Atm(\varphi^n)$ as follows.

$Atm(\alpha) \quad =_{df} \quad \{A \in Atm$: A contains at least one occurrence of the nominal constant $\alpha\}$.

$Atm(\varphi^n) \quad =_{df} \quad \{A \in Atm$: A contains an occurrence of the predicate $\varphi^n\}$.

[1]The stipulation that P be finite is required by the axiomatization of substitutional identity to be given later (in Subsection 2.5.4), for axiom $\ddot{=}2$ involves every member of P.

2.2.2 Language L

Alphabet of L

The alphabet of L extends the alphabet of L_0 by adding the following symbols:

1. substitutional nominal variables: $x, y, z, ...$
2. substitutional universal quantifier: (Πx)
3. truth-functional connectives: $\neg$ (negation) and $\wedge$ (conjunction)
4. intensional connective: $\Box$ (necessity)
5. parentheses: (,)

V is the denumerable set of nominal variables. A *nominal term* of L is either a nominal constant or a nominal variable. Let $o, o_1, ..., o_n, p, p_1, ..., p_n, q, ...$ be metavariables ranging over nominal terms and let T be the denumerable set of nominal terms.

Atomic Formulae of L

1. If $o_1, ..., o_n$ are nominal terms and φ^n is a predicate, then $\varphi^n o_1 ... o_n$ is an atomic formula of L. We call such formulae "pure atomic formulae".
2. If o_1 and o_2 are nominal terms, then $o_1 \ddot{=} o_2$ is an atomic formula of L.
3. Nothing else is an atomic formula of L.

Formulae of L

1. All atomic formulae of L are formulae of L.
2. If A is a formula of L, then so is $\neg A$.
3. If A and B are formulae of L, then so is $A \wedge B$.
4. If A is a formula of L, and x any nominal variable, then $(\Pi x)A$ is a formula of L.

5. If A is a formula of L, then so is $\Box A$.

6. Nothing else is a formula of L.

Let $Fml(L)$ be the set of all formulae of L.

Sentences of L

A formula of L which does not contain free variables is a *sentence* of L. Let $Snt(L)$ be the set of all sentences of L. So $Atm \subseteq Snt(L_0) \subseteq Snt(L) \subseteq Fml(L)$. Hence the sentences of L_0 are the atomic sentences of L.

Other Connectives

1. $(A \vee B) \quad =_{df} \quad \neg(\neg A \wedge \neg B)$

2. $(A \rightarrow B) \quad =_{df} \quad \neg(A \wedge \neg B)$

3. $(A \leftrightarrow B) \quad =_{df} \quad (A \rightarrow B) \wedge (B \rightarrow A)$

4. $(\Sigma x)A \quad =_{df} \quad \neg(\Pi x)\neg A$

5. $\Diamond A \quad =_{df} \quad \neg\Box\neg A$

2.3 Associative Substitutional Semantics

On the present approach to the semantics of substitutionally quantified modal logic the semantical evaluation of sentences of L is construed as being concerned with names, predicates and their "associates", rather than with elements of objectual domains and referential extensions.

2.3.1 Associative Substitutional Models

An *associative substitutional model* $\mathcal{M}$ for L is a 6-tuple

$$\mathcal{M} = \langle S, R, C, c, P, v \rangle,$$

which is defined as follows.

1. S is a non-empty set of indices (notation: $s, t, ...$).

2. $R \subseteq S \times S$.

3. C is a non-empty substitution class of nominal constants of L_0 (and thus of L).

4. $c : S \to \wp(C)$. $c(s)$ is the substitution class for some $s \in S$ and $C = \bigcup_{s \in S} c(s)$.

5. P is the set of pure predicates of L_0. Recall that P does not contain $\ddot{=}$.

6. v is a (restricted) assignment which is defined as follows:

   $v : C \times S \to \wp(Atm)$ such that $v(\alpha, s) \subseteq Atm(\alpha)$;

   $v : P \times S \to \wp(Atm)$ such that $v(\varphi^n, s) \subseteq Atm(\varphi^n)$.

   Terminology: $v(\alpha, s)$ is a *local associate for a nominal constant* α (or its local name associate); $v(\varphi^n, s)$ is a *local associate for a predicate* φ^n (or its local predicate associate).

   We can rigidify the valuations of the nominal constants and pure predicates, respectively, by imposing the following *rigidity conditions* on v:

   (a) For any $\alpha \in C$ and any $s, t \in S$, $v(\alpha, s) = v(\alpha, t)$.

   (b) For any $\varphi^n \in P$ and any $s, t \in S$, $v(\varphi^n, s) = v(\varphi^n, t)$.

   In view of these conditions we can distinguish four *rigidity types* of models. We call models which satisfy the first condition *name rigid models* or simply *rigid models* and models which satisfy the second condition *predicate rigid models*. Models which satisfy both rigidity conditions are *strongly rigid models* and models which satisfy neither will be called *nonrigid models*.

2.3.2 Associative Substitutional Models: Comments

1. It will be obvious that the substitutional models of this semantical framework differ significantly from the substitutional models for (modal) first-order languages which have been offered hitherto. They do not contain truth-values and truth-value assignments to the atomic sentences of some base language. Rather, nominal constants and predicates are assigned semantic values of their own. The present semantics thus is not a truth-value (or valuational) semantics.

2. The way in which associative substitutional models for an extensional first-order language can be obtained is obvious. A model for such a language will contain neither a frame nor the c function and the assignments of associates won't be local. Thus a *first-order associative model* will be a 3-tuple $\mathcal{I} = \langle C, P, v \rangle$, where v is defined as follows

   $v : C \rightarrow \wp(Atm)$ such that $v(\alpha) \subseteq Atm(\alpha)$;

   $v : P \rightarrow \wp(Atm)$ such that $v(\varphi^n) \subseteq Atm(\varphi^n)$.

3. As usual, we distinguish two main kinds of substitutional model, *constant* substitution class models and *varying* substitution class models. Constant substitution class models satisfy the *constancy* condition that $c(s) = C$ for every $s \in S$ whereas varying substitution class models do not. As a consequence of constancy models of the former kind satisfy the *monotonicity* condition, that is the requirement that for all $s, t \in S$, if sRt, then $c(s) \subseteq c(t)$; however, varying substitution class models need not do this. The same applies, *mutatis mutandis*, to *anti-monotonicity* (i.e., for all $s, t \in S$, if sRt, then $c(t) \subseteq c(s)$). We shall later consider models of these kinds.

4. Since the domain of the valuation function v is a substitution class rather than an objectual domain, we cannot regard from an intuitive point of view the name associates as referents (or denotata) of the nominal constants and the predicates as the denotational extensions of predicates. On the present semantics there just are no denotata and denotational extensions at all. Intuitively, we may conceive of the associates as capturing the "senses" or, more exactly, the "sense-extensions"

of names and predicates, respectively. These sense-extensions are determined by the meanings of these nonlogical terms which in turn are captured by nominal definitions and meaning postulates. (This will be discussed in Chapters 3 and 4 in more detail.)

5. In what follows we shall deal almost exclusively with rigid models (as specified in clause 2.3.1(5)). The reason for this is that a semantics which is based on these models will resemble the usual denotational semantics which are commonly used in discussions of quantified modal logic and its philosophy most closely. In view of the considerations of Chapter 1, the associative counterpart of the denotational semantics for constant domain S5 systems with referential identity will be of particular interest, since it will highlight some metaphysically interesting aspects of the associative framework. Models of the other three rigidity types seem to be interesting for various reasons. In particular, nonrigid models could be interesting, in view of their dynamic potential, from an epistemic perspective on quantified modal logic. However, in the present context, we shall not pursue such issues further.

6. For the purposes of an applied semantical interpretation of the associative framework admissibility constraints can be imposed on the models, in particular on the valuation function v, so as to capture philosophical considerations concerning their applied semantical counterparts. A proposal to this effect will be made in Chapter 3.

7. It will be noted that, unlike in denotational semantics, the semantic values of distinct nominal constants and of distinct predicates, respectively, can never be the same. (This fact does, of course, not depend upon rigidity assumptions.) So the name associates of 'Hesperus' and 'Phosphorus' differ (whereas their denotata do not) and the predicate associates of '... has a heart' and '... has kidneys' do not coincide (although their usual referential extensions do). I shall return to this observation in Chapter 4.

8. There is much room for modification and adaptation of associative models. For example, they may be modified by letting the substitution

classes (or P) be non-nominal or adapted via supplementation of the (first-order) models with further sets and functions in order to study substitutional languages involving other intensional operators. In the present context, though, we shall be concerned solely with the modal language L and models of the sort specified above. Moreover, mixed associative semantics can be construed adapting the ideas which have been discussed in Section 1.3.

2.3.3 Truth at an Index in a Model

We now define the truth conditions for the sentences of L with respect to some index s in an associative substitutional model $\mathcal{M} = \langle S, R, C, c, P, v \rangle$. We begin with the truth conditions of the sentences of the basic language L_0 which are contained in Atm (clause (1)). We then define truth for the substitutional identity sentences of L_0 (clause (2)). Having defined the truth for all of L_0 we recursively define the extended truth conditions for the rest of the sentences of L (clauses (3)-(6)).

1. $\mathcal{M} \models_s \varphi^n \alpha_1 ... \alpha_n$ iff (i) $\alpha_1, ..., \alpha_n \in c(s)$ and (ii) $\varphi^n \alpha_1 ... \alpha_n \in v(\alpha_1, s) \cap ... \cap v(\alpha_n, s) \cap v(\varphi^n, s)$; otherwise $\mathcal{M} \not\models_s \varphi^n \alpha_1 ... \alpha_n$.

2. $\mathcal{M} \models_s \alpha_1 \ddot{=} \alpha_2$ iff for all sentences B_1 and B_2 in Atm where B_2 is like B_1 except for containing occurrences of the nominal constant α_2 at one or all places where B_1 contains the nominal constant α_1: $\mathcal{M} \models_s B_1$ iff $\mathcal{M} \models_s B_2$.

3. $\mathcal{M} \models_s \neg A$ iff $\mathcal{M} \not\models_s A$.

4. $\mathcal{M} \models_s A \wedge B$ iff $\mathcal{M} \models_s A$ and $\mathcal{M} \models_s B$.

5. $\mathcal{M} \models_s (\Pi x)A$ iff for all $\alpha \in c(s)$: $\mathcal{M} \models_s A[\alpha/x]$, where $A[\alpha/x]$ comes from A by replacing any free occurrences of x in A by α.

6. $\mathcal{M} \models_s \Box A$ iff for all $t \in S$ such that sRt: $\mathcal{M} \models_t A$.

Derivatively: 7. $\mathcal{M} \models_s A \vee B$ iff it is not the case that both $\mathcal{M} \not\models_s A$ and $\mathcal{M} \not\models_s B$; 8. $\mathcal{M} \models_s A \rightarrow B$ iff it is not the case that both $\mathcal{M} \models_s A$ and $\mathcal{M} \not\models_s B$; 9. $\mathcal{M} \models_s (\Sigma x)A$ iff for some $\alpha \in c(s)$: $\mathcal{M} \models_s A[\alpha/x]$; 10. $\mathcal{M} \models_s \Diamond A$ iff for some $t \in S$ such that sRt: $\mathcal{M} \models_t A$.

2.3.4 Truth at an Index in a Model: Comments

1. On the account of the truth conditions just given, the truth conditions of every sentence always reduce to truth conditions on (pure atomic) sentences. This puts the present semantics in an obvious contrast to satisfaction semantics on which the truth conditions of all formulae reduce to truth conditions on atomic formulae.

2. The first conjunct of the right hand side in clause 2.3.3(1) is trivially satisfied once the model has a constant substitution class, that is, once for any $s \in S$: $c(s) = C$. In particular, it will be satisfied trivially in models in which R is total (i.e., $R = S \times S$) and monotonicity holds (i.e. for all $s, t \in S$, if sRt, then $c(s) \subseteq c(t)$).

3. Clause 2.3.3(1) may be relaxed so as to provide the resources for a non-bivalent semantics. For example, we might say that an atomic sentence is undecided, when it is contained in only one of the associates it takes to make it true.

4. It will be noted that the account of the truth conditions of atomic sentences which contain predicates of arity $n \geq 2$ given in clause 2.3.3(1) does not appeal to set-theoretic relations (i.e., sets of n-tuples of objects). Instead, the order in the semantic values of the predicates is guaranteed by the syntactical order of their elements.

5. We illustrate the basic clause with two examples.

   (a) Let $\mathcal{M} = \langle S, R, C, c, P, v \rangle$ be a model based on any frame $\mathcal{F} = \langle S, R \rangle$, and let s be some index in this model with the substitution class $c(s) = \{a, b, d, e\}$. (Though every $\alpha \in C$ and every $\varphi^n \in P$ are assigned at any index in any model an association set, we consider merely the associates for the elements of $c(s)$ and for just one predicate from P.) Let the name associates be $v(a, s) = \{Fa\}$, $v(b, s) = \{Fb\}$, $v(c, s) = \{Fc\}$, $v(d, s) = \emptyset$, $v(e, s) = \{Fe\}$. And let the predicate associate for F be $v(F, s) = \{Fa, Fb, Fc, Fd\}$. Now, since both $a \in c(s)$ and $Fa \in v(a, s) \cap v(F, s)$, we have $\mathcal{M} \models_s Fa$. Similarly we get $\mathcal{M} \models_s Fb$. Since $Fc \in v(c, s) \cap v(F, s)$

and $c \notin c(s)$, we have $\mathcal{M} \not\models_s Fc$. Now although $d \in c(s)$ we have $Fd \notin v(d, s) \cap v(F, s)$, since $v(d, s) = \emptyset$. Thus $\mathcal{M} \not\models_s Fd$. And finally, though $e \in c(s)$ and $Fe \in v(e, s)$ we have $Fe \notin v(e, s) \cap v(F, s)$ for $v(e, s) \cap v(F, s) = \emptyset$. So $\mathcal{M} \not\models_s Fe$.

(b) Let $\mathcal{M} = \langle S, R, C, c, P, v\rangle$ be a rigid constant substitution class model with a frame $\mathcal{F} = \langle S, R\rangle$ where $S = \{s, t, u\}$, $R = S \times S$, $C = \{a, b\}$ and $c(s) = c(t) = c(u) = C$. Let the associates at s be $v(G, s) = \{Gab, Gba\}$ and $v(a, s) = v(b, s) = \{Gab, Gba\}$. Thus $\mathcal{M} \models_s Gab$ and $\mathcal{M} \models_s Gba$. Let the associates for t be $v(G, t) = \{Gab\}$ and $v(a, t) = v(b, t) = \{Gab, Gba\}$. So $\mathcal{M} \models_t Gab$. Let the associates for u be $v(G, u) = \{Gab\}$ and $v(a, u) = v(b, u) = \{Gab, Gba\}$. Thus $\mathcal{M} \models_u Gab$ and $\mathcal{M} \not\models_u Gba$. Consequently, by clause (6), for any index $w \in S$ we have $\mathcal{M} \models_w \Box Gab$ and $\mathcal{M} \not\models_w \Box Gba$. (An example to the same effect may be construed also for models that are nonrigid. To obtain such an example we might alter the associates for u in the above example as follows: $v(G, u) = \{Gab, Gba\}$, $v(a, u) = \{Gab, Gba\}$ and $v(b, u) = \{Gab\}$.)

6. Clause 2.3.3(2) can be stated directly in terms of associates. Let φ^n be an arbitrary pure predicate, let α_1 and α_2 be nominal constants (corresponding to α_1 and α_2 in that clause, respectively) which are not necessarily distinct, and let $\gamma_1, ..., \gamma_{n-1}$ be arbitrary nominal constants. Then the clause will be: $\mathcal{M} \models_s \alpha_1 \ddot{=} \alpha_2$ iff for all pure atomic sentences $\varphi^n(\alpha_1, \gamma_1, ..., \gamma_{n-1})$ and $\varphi^n(\alpha_2, \gamma_1, ..., \gamma_{n-1})$ (which, simplifying somewhat, we take to correspond to the B_1s and B_2s, respectively) the following holds: $\varphi^n(\alpha_1, \gamma_1, ..., \gamma_{n-1}) \in v(\alpha_1, s) \cap v(\gamma_1, s) \cap ... \cap v(\gamma_{n-1}, s) \cap v(\varphi^n, s)$ iff $\varphi^n(\alpha_2, \gamma_1, ..., \gamma_{n-1}) \in v(\alpha_2, s) \cap v(\gamma_1, s) \cap ... \cap v(\gamma_{n-1}, s) \cap v(\varphi^n, s)$. In effect, clause 2.3.3(2) is an abbreviation of this clause.

7. As an alternative to the clause for substitutional identity, we might have replaced clause 2.3.3(2) with the following condition:

2.3.3(2*) $\mathcal{M} \models_s \alpha_1 \ddot{=} \alpha_2$ iff for all sentences B_1 and B_2 in *Atm* where B_2 is like B_1 except for containing occurrences of the nominal constant α_2 at one or all places where B_1 contains the nominal constant α_1: for all $t \in S$ such that sRt, $\mathcal{M} \models_t B_1$ iff $\mathcal{M} \models_t B_2$.

The interpretation of substitutional identity defined by clause 2.3.3(2*) amounts to what we might call the *modal interpretation of substitutional identity*, since it requires the equivalence to hold with respect to the indices which are accessible to the index of evaluation. In this way the the interpretation of substitutional identity, when it is applied in modal contexts, can be made dependent upon the properties of the accessibility relation. (It is natural to assume that these properties are just the ones which are involved in the interpretation of the modal operators.) The following rigid model shows that our *nonmodal interpretation of substitutional identity*, as presented in clause 2.3.3(2) are not equivalent in general. Let $\mathcal{M} = \langle S, R, C, c, P, v \rangle$ be a rigid constant substitution class model with a frame $\mathcal{F} = \langle S, R \rangle$ where $S = \{s, t\}$, $R = \{\langle s, t \rangle\}$, $C = \{a, b\}$ and $c(s) = c(t) = C$. Let the associates at s be $v(a, s) = \{Fa\}$, $v(b, s) = \{Fb\}$, and $v(F, s) = \{Fa, Fb\}$. And let the associates at t be $v(a, t) = \{Fa\}$, $v(b, t) = \{Fb\}$, and $v(F, t) = \{Fa\}$. Now, according to the nonmodal interpretation of substitutional identity, that is, clause 2.3.3(3), we have $\mathcal{M} \models_s a \ddot{=} b$. But according to the modal interpretation (i.e., clause 2.3.3(2*)), $\mathcal{M} \not\models_s a \ddot{=} b$. (In order for that sentence to be true, R would have to be reflexive.) Accordingly, these two interpretations will give rise to two different axiomatizations of $\ddot{=}$. In what follows, we shall almost exclusively discuss the nonmodal interpretation.

8. Independently of which rigidity type of model is selected and which interpretation of substitutional identity is chosen, substitutional identity turns out to be a contingent notion of identity unless further constraints are imposed. We shall return to these issues in the discussion of the validitiy of NSI in 2.3.8(6) below.

9. The clause for substitutional identity statements of ordinary substitutional first-order languages interpreted in terms of first-order associa-

tive models $\mathcal{I} = \langle C, P, v \rangle$ receives the following shape:

> $\mathcal{I} \models \alpha_1 \ddot{=} \alpha_2$ iff for all sentences B_1 and B_2 in *Atm* where B_2 is like B_1 except for containing occurrences of the nominal constant α_2 at one or all places where B_1 contains the nominal constant α_1: $\mathcal{I} \models B_1$ iff $\mathcal{I} \models B_2$.

10. It is obvious from clause 2.3.3(2) that the predicate $\ddot{=}$ is governed by the (metalinguistic) principle of substitutivity. In this respect this condition is similar to the usual clause for substitutional identity discussed in the previous chapter. However, it is governed so only in a restricted sense, since the clause does not turn upon all sentences of the substitutional language, but exclusively upon its pure atomic sentences (and so only upon the pure predicates). It is also in this respect in which clause 2.3.3(2) differs from Marcus's characterization which turns (as it stands) on all sentences of the language and thus on sentences of any complexity.[2] Of course, substitutional identity (on any characterization) must not be confused with (referential) identity. The latter is a relation that obtains between objects, that is, the semantic values of individual terms which themselves are members of an objectual domain. By contrast, substitutional identity is a relation obtaining between names (or more generally, as we shall see later between nominal terms). Moreover, unlike $\ddot{=}$, the ordinary identity predicate $=$ is governed by Leibniz's law, that is the (metaphysical) second-order principle $\underline{\alpha}_1 = \underline{\alpha}_2 \leftrightarrow (\forall F)(F\underline{\alpha}_1 \rightarrow F\underline{\alpha}_2)$ (the underlines are meant to indicate that the constants are denoting). Finally, both predicates differ also in their formal behaviour as will become apparent from the axioms for $\ddot{=}$ given below in Subsection 2.5.3.

[2]See [Marcus, 1993 (1972)] p. 85. An alternative account of substitutional identity which involves "identity-normal" truth-value assignments is given in [Leblanc, 1976] ch. 5. We cannot go into this here.

2.3.5 Truth-Conditions Theorem for L-Sentences

We now prove, adapting Kripke's original proof of a corresponding theorem for a nonmodal language without substitutional identity,[3] that, granted that truth has been characterized for atomic L-sentences, the truth conditions for the atomic portion of L together with clauses (2)-(6) determine the truth conditions for all sentences of L. More precisely, we shall prove the following theorem:

> *Truth-Conditions Theorem: Sentences*. Let $\mathcal{M} = \langle S, R, C, c, P, v \rangle$ be a model. Given the family Γ mapping any $s \in S$ to the set Γ_s of pure atomic L-sentences true at s in $\mathcal{M}$, there is a unique family Γ' mapping any $s \in S$ to the set of L-sentences true at s in $\mathcal{M}$ satisfying clauses (2)-(6) and also coinciding with Γ on the atomic portion of L.

This theorem claims two things, first, that there exists a certain family Γ' satisfying certain conditions and, second, that this family is unique. We prove both claims via induction on the complexity of L-sentences, where the complexity of an L-sentence is defined as the number of the operators it contains.

Uniqueness of Γ'. We first show by induction on the complexity of a sentence of L that any families Γ' and Γ'' coinciding with Γ on the atomic portion of L and satisfying (2)-(6) coincide.

That is we prove by induction on n that if A has complexity n the following holds: $A \in \Gamma'_s$ iff $A \in \Gamma''_s$.

Let A be an L-sentence of complexity $= 0$. Then, since every sentence of complexity $= 0$ is in the atomic portion of L and Γ' and Γ'' coincide with Γ on the atomic portion of L we have: $A \in \Gamma'_s$ iff $A \in \Gamma''_s$ iff $A \in \Gamma_s$.

We now turn to the inductive cases. Let us assume for induction: if A has complexity $\leq n$, then $A \in \Gamma'_s$ iff $A \in \Gamma''_s$. For reasons of symmetry it suffices to prove only one direction.

[3] See [Kripke, 1976] pp. 330-331.

Let A be an L-sentence of complexity $n+1$. Since the complexity of A exceeds 0, A must contain an operator. (A is therefore not a sentence of the atomic portion of L.) We confine ourselves to the cases $A = (\Pi x)B$ and $A = \Box B$.

If $A = (\Pi x)B$, then for ($\rightarrow$) if $A \in \Gamma'_s$, i.e. $(\Pi x)B \in \Gamma'_s$, by clause (5) any sentence $B[\alpha/x]$ is in Γ'_s, where $B[\alpha/x]$ comes from B by replacing free occurrences of x by any name $\alpha \in c(s)$. Since any such $B[\alpha/x]$ has complexity $= n$, we have by induction hypothesis $B[\alpha/x] \in \Gamma''_s$. But then by (5) for Γ''_s, we get $(\Pi x)B \in \Gamma''_s$. Hence, if $A \in \Gamma'_s$ then $A \in \Gamma''_s$.

If $A = \Box B$, then for ($\rightarrow$) if $A \in \Gamma'_s$, i.e. $\Box B \in \Gamma'_s$, by clause (6), for every $t \in S$ such that sRt: $\mathcal{M} \models_t B$; that is to say, for every $t \in S$ such that sRt: $B \in \Gamma'_t$. Now, given that B is of complexity $= n$, the induction hypothesis applies and so for every $t \in S$ such that sRt: $B \in \Gamma''_t$. So, since Γ'' satisfies (6), $\Box B \in \Gamma''_s$ and hence $A \in \Gamma''_s$.

The cases $A = \neg B$ and $A = B \wedge C$ are proved in the same way. This completes the proof of the uniqueness of any Γ' coinciding with Γ on the atomic portion of L and satisfying clauses (2)-(6).

Existence of Γ'. We now have to prove the existence of such a Γ' coinciding with Γ on the atomic portion of L and satisfying clauses (2)-(6).

Let $\mathcal{M} = \langle S, R, C, c, P, v \rangle$ be a model and let $(\Gamma_s)_{s \in S}$ be the family of the atomic L-sentences true at each s in $\mathcal{M}$. Then, given $(\Gamma_s)_{s \in S}$, we construct a family $(\Gamma^n_s)^{n \in \omega}_{s \in S}$ ($= \Gamma'$) of the L-sentences A of complexity $\leq n \in \omega$ true at each s in $\mathcal{M}$ which is already unique at s.

So for each $s \in S$ and each $n \in \omega$ the family $(\Gamma^n_s)^{n \in \omega}_{s \in S}$ has to satisfy the following condition:

($*$) For every sentence A of complexity $\leq n$: $A \in \Gamma^n_s$ iff

1. $A = \varphi^n \alpha_1 ... \alpha_n$ and both (i) $\alpha_1, ..., \alpha_n \in c(s)$ and (ii) $\varphi^n \alpha_1 ... \alpha_n \in v(\alpha_1, s) \cap ... \cap v(\alpha_n, s) \cap v(\varphi^n, s)$; or
2. $A = \alpha_1 \ddot{=} \alpha_2$ and for all pure atomic L_0-sentences B_1 and B_2 where B_2 is like B_1 except for containing occurrences of the nominal constant α_2 at one or all places where B_1 contains the nominal constant α_1: $B_1 \in \Gamma^n_s$ iff $B_2 \in \Gamma^n_s$; or

3. $A = \neg B$ and $B \notin \Gamma_s^n$; or
4. $A = (B \wedge C)$ and both $B \in \Gamma_s^n$ and $C \in \Gamma_s^n$; or
5. $A = (\Pi x)B$ and for all $\alpha \in c(s)$, $B[\alpha/x] \in \Gamma_s^n$ where $B[\alpha/x]$ comes from B by replacing any free occurrences of x by α; or
6. $A = \Box B$ and for all $t \in S$, if sRt then $B \in \Gamma_t^n$.

To construct $(\Gamma_s^n)_{s\in S}^{n\in\omega}$ we first define Γ_s^n for each $s \in S$ inductively taking Γ_s to be the set of atomic L-sentences true at s in $\mathcal{M}$ and taking the displayed subformulae to be of complexity $\leq n$ as follows:

$\Gamma_s^0 = \Gamma_s$;

$\Gamma_s^{n+1} = \Gamma_s^n \cup \{\neg B : B \notin \Gamma_s^n$ where B is of complexity $\leq n\} \cup \{(B \wedge C) : B \in \Gamma_s^n$ and $C \in \Gamma_s^n\} \cup \{(\Pi x)B :$ for all $\alpha \in c(s)$, $B[\alpha/x] \in \Gamma_s^n\} \cup \{\Box B :$ for all $t \in S$, if sRt then $B \in \Gamma_t^n\}$.

We then define $(\Gamma_s^n)_{s\in S}^{n\in\omega}$ as follows:

$$(\Gamma_s^n)_{s\in S}^{n\in\omega} \quad = \quad \bigcup_{s\in S} \Gamma_s^n \text{ for each } n \in \omega.$$

To show that $(\Gamma_s^n)_{s\in S}^{n\in\omega}$ satisfies condition $(*)$ we show that each Γ_s^n for each $s \in S$ does. The proof is by induction on n.

For $n = 0$ suppose $A \in \Gamma_s^0$. Then A is either a pure atomic sentence or a $\ddot{=}$-sentence. Since $\Gamma_s^0 = \Gamma_s$, $A \in \Gamma_s$. But, by the definition of Γ_s, this is so just in case A is true at s in $\mathcal{M}$; and so the clauses (1) and (2) follow.

For the rest we assume as an inductive hypothesis that for every sentence A of complexity $< n$ with $n > 0$: $A \in \Gamma_s^{n+1}$. Then for any A of complexity $= n + 1$ the sentence A must be one of the following forms: $\neg B$, $(B \wedge C)$, $(\Pi x)B$, or $\Box B$. In each case the subformulae are of complexity $< n$ and the hypothesis will apply. We may thus conclude that each clause of the condition will be satisfied. This concludes the existence part.

The proofs of uniqueness and existence taken together establish the Truth-Conditions Theorem for L-sentences.

2.3.6 Truth in a Model

A sentence A of L is *true in an associative model* $\mathcal{M} = \langle S, R, C, c, P, v\rangle$ (formally, $\mathcal{M} \models A$) iff for every $s \in S$: $\mathcal{M} \models_s A$.

2.3.7 Validity in a Frame

As usual we say that a model $\langle \mathcal{F}, C, c, P, v\rangle$ is *based on the frame* $\mathcal{F}$, where $\mathcal{F} = \langle S, R\rangle$.

A formula A of L is *valid in a frame for associative models* (formally, $\mathcal{F} \models A$) iff it is true in all associative models based on that frame. Moreover, $\mathcal{F}$ *is a frame for* a system Λ iff every theorem of Λ is valid in $\mathcal{F}$.

Finally, a *class* $\mathcal{C}^{\mathcal{F}}$ *of frames determines* Λ iff for every formula A, A is valid in every frame in $\mathcal{C}^{\mathcal{F}}$ iff it is a theorem of Λ.

2.3.8 Validity in a Frame: Illustrations

We now illustrate the semantics by considering the validity and the invalidity of a couple of sentences with respect to the relevant kinds of associative substitutional model. The examples are the substitutional counterparts of the formulae BF, CBF, NE and NI which play a prominent role in philosophical discussions of the semantics of quantified modal logic.

1. Substitutional Barcan Formula SBF (i.e., $(\Pi x)\Box A \rightarrow \Box(\Pi x)A$) and its converse CSBF (i.e., $\Box(\Pi x)A \rightarrow (\Pi x)\Box A$). To prove the validity of both schemes with respect to rigid constant substitution class models let $\mathcal{M}$ be an arbitrary model of this kind and s any index in that model. Now we have (i) $\mathcal{M} \models_s (\Pi x)\Box A$ just in case, by 2.3.3(5), (ii) for all $\alpha \in c(s)$ $\mathcal{M} \models_s \Box A[\alpha/x]$, where $A[\alpha/x]$ comes from A by replacing any free occurrences of x in A by $\alpha \in c(s)$; this holds just in case, by 2.3.3(6), (iii) for all $\alpha \in c(s)$: for all $t \in S$, if sRt, then $\mathcal{M} \models_t A[\alpha/x]$. This in turn holds iff, by the constancy condition which guarantees that for all $s, t \in S$, $c(s) = c(t)$ and an obvious step, (iv) for all $t \in S$: for all $\alpha \in c(t)$, if sRt, then $\mathcal{M} \models_t A[\alpha/x]$. This is the case iff, by a first-order equivalence, (v) for all $t \in S$: if sRt, then for all $\alpha \in c(t)$

$\mathcal{M} \models_t A[\alpha/x]$. This holds just in case, by 2.3.3(5), (vi) for all $t \in S$: if sRt, then $\mathcal{M} \models_t (\Pi x)A$. Finally, this is the case iff, by 2.3.3(6), (vii) $\mathcal{M} \models_s \Box(\Pi x)A$.

2. Invalidity of SBF with respect to rigid varying substitution class models. Consider the following countermodel. Let $\mathcal{M} = \langle S, R, C, c, P, v\rangle$ be such a model based on a frame $\mathcal{F} = \langle S, R\rangle$ with $S = \{s, t\}$ and $R = \{\langle s, t\rangle\}$. Let $C = \{a, b\}$ and let $c(s) = \{a\}$ and $c(t) = C$. So this model is not antimonotonic. Let the associates be $v(a, s) = v(a, t) = \{Fa\}$, $v(b, t) = \{Fb\}$ and $v(F, s) = v(F, t) = \{Fa\}$. Since both $a \in c(s)$ and $Fa \in v(a, s) \cap v(F, s)$, we have $\mathcal{M} \models_s Fa$. So we have $\mathcal{M} \models_s (\Pi x)\Box Fx$. This is so because every name in $c(s)$ renders Fx true at any index which is accessible to s when it replaces the x in Fx. And since both $a \in c(t)$ and $Fa \in v(a, t) \cap v(F, t)$, we have $\mathcal{M} \models_t Fa$. But, although $b \in c(t)$ we have $Fb \notin v(F, t)$ and so $Fb \notin v(b, t) \cap v(F, t)$. So $\mathcal{M} \not\models_t Fb$. Consequently, $\mathcal{M} \not\models_s \Box(\Pi x)Fx$. Thus it is not the case that at every index every name in the substitution class for that index renders Fx true when it is substituted for x.

3. Invalidity of CSBF with respect to rigid varying substitution class models. Consider the following countermodel of this sort. Let $\mathcal{M} = \langle S, R, C, c, P, v\rangle$ be a model based on a frame $\mathcal{F} = \langle S, R\rangle$ with $S = \{s, t\}$ and $R = \{\langle s, t\rangle\}$. Let $C = \{a, b\}$ and let $c(s) = C$ and $c(t) = \{b\}$. This model is obviously not monotonic. Let the associates be $v(a, s) = \{Fa\}$, $v(b, s) = v(b, t) = \{Fb\}$, $v(F, s) = \{Fa, Fb\}$ and $v(F, t) = \{Fb\}$. Since both $a \in c(s)$ and $Fa \in v(a, s) \cap v(F, s)$, we have $\mathcal{M} \models_s Fa$. Similarly, we get $\mathcal{M} \models_s Fb$ and $\mathcal{M} \models_t Fb$. Thus we have $\mathcal{M} \models_s \Box(\Pi x)Fx$. However, since $\mathcal{M} \not\models_t Fa$, we have $\mathcal{M} \not\models_s (\Pi x)\Box Fx$.

4. We now show that the L-sentence $(\Pi x)\Box(\Sigma y)(y \ddot{=} x)$, NSE (Necessary Substitutional Existence), is valid with respect to rigid constant models. Let $\mathcal{M}$ be an arbitrarily chosen rigid constant substitution class model and let s be an arbitrary index in that model. To show that $\mathcal{M} \models_s (\Pi x)\Box(\Sigma y)(y \ddot{=} x)$ we assume for reductio that $\mathcal{M} \not\models_s (\Pi x)\Box(\Sigma y)$ $(y \ddot{=} x)$. Thus for some $\alpha_i \in c(s)$: $\mathcal{M} \not\models_s \Box(\Sigma y)(y \ddot{=} \alpha_i)$, where $\Box(\Sigma y)$

$(y \ddot{=} \alpha_i)$ comes from $\Box(\Sigma y)(y \ddot{=} x)$ by replacing any free occurrences of x in $(\Sigma y)(y \ddot{=} x)$ by α_i. So for all $\alpha_i \in c(s)$ there is some $t \in S$ such that sRt and $\mathcal{M} \not\models_t (\Sigma y)(y \ddot{=} \alpha_i)$. Now since $\mathcal{M}$ is a constant substitution class model, monotonicity holds. That is any name in $c(s)$ will be contained in the substitution class $c(t)$ of any $t \in S$ such that sRt. So since $c(s)$ is nonempty $c(t)$ must contain at least one name. Let α_i be that name. Thus for any $\alpha_i \in c(s)$ there is some $t \in S$ such that sRt and $\mathcal{M} \not\models_t \alpha_i \ddot{=} \alpha_i$, where $\alpha_i \ddot{=} \alpha_i$ results from $(\Sigma y)(y \ddot{=} \alpha_i)$ by replacing any free occurrence of y in $y \ddot{=} \alpha_i$ by α_i. But then, by clause 2.3.3(2), it is not the case that $\mathcal{M} \models_t B_i$ iff $\mathcal{M} \models_t B_i$, which gives us a contradiction. Since this holds for arbitrary constant rigid models and arbitrary indices in these models we have the desired result.

5. NSE is invalid in rigid varying substitution class models. Let $\mathcal{M} = \langle S, R, C, c, P, v\rangle$ be a model of this kind based on a frame $\mathcal{F} = \langle S, R\rangle$ with $S = \{s, t\}$ and $R = \{\langle s, t\rangle\}$. Let $C = \{a\}$ and let $c(s) = C$ and $c(t) = \emptyset$. So this model is not monotonic. This is all the information we need to see that this is a countermodel to $(\Pi x)\Box(\Sigma y)(y \ddot{=} x)$. Since $c(t) = \emptyset$, $\mathcal{M} \not\models_t (\Sigma y)(y \ddot{=} a)$. Thus $\mathcal{M} \not\models_s \Box(\Sigma y)(y \ddot{=} a)$, by the truth conditions for the box-operator. But then, by the truth clause for the substitutional universal quantifier, $\mathcal{M} \not\models_s (\Pi x)\Box(\Sigma y)(y \ddot{=} x)$.

6. Necessary Substitutional Identity (NSI), $\alpha \ddot{=} \beta \rightarrow \Box(\alpha \ddot{=} \beta)$, is invalid on any rigid model. To verify this consider the following rigid constant substitution class model $\mathcal{M} = \langle S, R, C, c, P, v\rangle$, where $S = \{s, t\}$, $R = \{\langle s, t\rangle\}$, $c(s) = c(t) = \{a, b\}$, $v(a, s) = v(a, t) = \{Fa\}$, $v(b, s) = v(b, t) = \{Fb\}$, $v(F, s) = \{Fa, Fb\}$, and $v(F, t) = \{Fa\}$. Obviously, by clause 2.3.3(2) $\mathcal{M} \models_s a \ddot{=} b$, but, by clause 2.3.3(10), $\mathcal{M} \models_s \Diamond\neg(a \ddot{=} b)$, since, by clause 2.3.3(1), even though $\mathcal{M} \models_s Fa$ iff $\mathcal{M} \models_s Fb$ we have $\mathcal{M} \models_t Fa$ and $\mathcal{M} \not\models_t Fb$. (A similar situation can be construed for rigid varying substitution class models. To see this consider a model which is just like the previous one except for letting $c(s) = \{a, b\}$, $c(t) = \{a, b, c\}$ and $v(c, t) = \{Fc\}$.) On the modal interpretation of substitutional identity as given in clause 2.3.3(2*) (see comment 2.3.4(7)) we obtain for the previous constant model a different result: $\mathcal{M} \not\models_s a \ddot{=} b$

and $\mathcal{M} \models_s \Diamond\neg(a \ddot{=} b)$. The former sentence is false at s in that model exactly because the latter is true at that index.

What would it take to ensure the validity of NSI? NSI will be valid if it will be true in all models which guarantee the truth of $\alpha \ddot{=} \beta$ at every index of evaluation and every index accessible from that index. The following kind of model is of this sort. Let a *static model* be a (constant or varying) associative model $\mathcal{M} = \langle S, R, C, c, P, v \rangle$ where

(a) R is (weakly) reflexive, that is, for all $s, t \in S$ (where s and t need not be distinct), if sRt then tRt;

(b) For any $s \in S$, for any two $\alpha_1, \alpha_2 \in c(s)$, for any $\varphi^n \in P$, and for all sentences B_1 and B_2 in Atm where φ^n is the predicate in these sentences and B_2 is like B_1 except for containing occurrences of the nominal constant α_2 at one or all places where B_1 contains the nominal constant α_1 the following holds: $B_1 \in v(\alpha_1, s)$ and $B_2 \in v(\alpha_2, s)$ and $B_1, B_2 \in v(\varphi^n, s)$.

To verify the validity of NSI with respect to static models, consider any (constant or varying) static model $\mathcal{M} = \langle S, R, C, c, P, v \rangle$ and any index s in that model. Suppose $\mathcal{M} \models_s \alpha \ddot{=} \beta$ and $\mathcal{M} \not\models_s \Box(\alpha \ddot{=} \beta)$. Then, by clause 2.3.3(6), there will be a $t \in S$ such that if sRt then $\mathcal{M} \models_t \neg(\alpha \ddot{=} \beta)$. But since the model is static and thus (weakly) reflexive, we also have, $\mathcal{M} \models_t \alpha \ddot{=} \beta$ and thus obtain a contradiction. Since the static model was selected arbitrarily we may conclude that $\alpha \ddot{=} \beta \to \Box(\alpha \ddot{=} \beta)$ is valid with respect to static models based on (weakly) reflexive frames.

Every static model is strongly rigid. However, the fact that a model is strongly rigid (see clause 2.3.1(6)) is not sufficient for the validity of NSI as the following model shows. Consider the strongly rigid constant model model $\mathcal{M} = \langle S, R, C, c, P, v \rangle$ where $S = \{s, t\}$, $R = \{\langle s, t \rangle\}$, $C = \{a, b\}$ and $c(s) = c(t) = C$. Let the associates at s be $v(a, s) = \{Fa\}$, $v(b, s) = \{Fb\}$, and $v(F, s) = \{Fa\}$. And let the associates at t be $v(a, t) = \{Fa\}$, $v(b, t) = \{Fb\}$, and $v(F, t) = \{Fa\}$. Although the model is strongly rigid we have, by clause 2.3.3(2), $\mathcal{M} \not\models_s a \ddot{=} b$ and $\mathcal{M} \not\models_t a \ddot{=} b$. And so, by clause 2.3.3(6), $\mathcal{M} \not\models_s \Box(a \ddot{=} b)$. Nor does the fact that a model is strongly rigid and satisfies condition (b) give us the validity of NSI. To obtain such a model we modify the previous one by adding Fb to $v(F, s) = v(F, t)$ so as to obtain $v(F, s) = v(F, t) = \{Fa, Fb\}$. Clearly, we have, by clause 2.3.3(2), $\mathcal{M} \models_s a\ddot{=}b$ and $\mathcal{M} \models_t a\ddot{=}b$. And so, by clause 2.3.3(6), $\mathcal{M} \models_s \Box(a\ddot{=}b)$. However, since $\mathcal{M} \not\models_t \Box(a\ddot{=}b)$, NSI will not be true in that model. But when we modify the previous model insisting that the accessibility relation be (weakly) reflexive NSI will be true in the resulting model as can be easily verified. $\mathcal{M} = \langle S, R, C, c, P, v\rangle$ where $S = \{s, t\}$, $R = \{\langle s, t\rangle, \langle t, t\rangle\}$, $C = \{a, b\}$ and $c(s) = c(t) = C$. Let the associates at s be $v(a, s) = v(a, t) = \{Fa\}$, $v(b, s) = v(b, t) = \{Fb\}$, and $v(F, s) = v(F, t) = \{Fa, Fb\}$. (From this we may obtain the simplest reflexive model in which NSI (with distinct nominal constants) is true by letting $s = t$.) Constructions to this effect can be also provided in terms of strongly rigid varying substitution class models.

Does it make a difference to the validity of NSI with respect to static models when substitutional identity is interpreted modally? No, but the evaluation is slightly less direct, since we have to look to other indices in order to see whether a substitutional identity sentence is true at a given index. Consider the previous static model. By clause 2.3.3(2^*), $\mathcal{M} \models_s a\ddot{=}b$, since for all $t \in S$ such that sRt it is the case that for all sentences B_1 and B_2 in Atm where B_2 is like B_1 except for containing occurrences of the nominal constant α_2 at one or all places where B_1 contains the nominal constant α_1: for all $u \in S$ such that sRu, $\mathcal{M} \models_u B_1$ iff $\mathcal{M} \models_u B_2$. In an exactly analogous way we obtain $\mathcal{M} \models_t a\ddot{=}b$. But then we see that, by clause 2.3.3(6), $\Box(a\ddot{=}b)$ will be true at every index in that model. And since this will be so for all static models, we may conclude that NSI is valid with respect to static models which are based on (weakly) reflexive frames.

Clearly, static models run counter to intuitions, for they make all nominal constants whatever substitutionally identical with each other, and, what is more, they are to the effect that this is so at all indices. However, the fact that NSI is invalid with respect to rigid models does not mean that there is no reasonable way to impose constraints upon them which guarantee that certain instances of NSI will hold. I shall propose such constraints in Chapter 3, comment 3.2.4(7). See also the discussion in the second half of Subsection 2.5.7 below (in particular the passages arround clause 2.4.2.($2^{\dagger}$)).

7. Observations corresponding to the ones just stated for NSI can be made also for necessity of substitutional distinctness (NSD), i.e. $\neg(\alpha \ddot{=} \beta) \rightarrow \Box\neg(\alpha \ddot{=} \beta)$.

2.4 Associative Substitutional Semantics with Variable Assignments

In this section a version of the semantics of the previous section is presented according to which the clause for the substitutional quantifier is given in terms of substitutional (x, s)-variants rather than (directly) in terms of the nominal constants contained in a substitution class for an index. It will become apparent that nothing of substance changes. However, it will be convenient to have an account of truth conditions of this sort when we come to discuss the soundness and completeness of axiomatic systems which allow for instances of axioms which contain open formule (e.g., $(\Pi x)Fx \rightarrow Fy$ or $x \ddot{=} x$).

The semantics to be presented in this section is one which is suitable for normal axiom systems with SBF. We shall discuss such systems in the next section. The semantics will need some modification, when we turn to varying substitution class systems. A system which invalidates SBF will be presented in Section 2.6 and a system which invalidates CSBF will be discussed in Section 2.7.

2.4.1 Assignments to Nominal Variables

To obtain a semantics which handles open formulae we introduce the notion of a *nominal variable assignment* σ based on a modal associative model which assigns names to nominal variables relative to indices. More exactly, given some model $\mathcal{M} = \langle S, R, C, c, P, v\rangle$ on which σ is based this assignment is defined as follows.

$$\sigma : V \times S \rightarrow C$$

Thus for any $x \in V$ and $s \in S$, $\sigma(x, s) = \alpha$ where α a name in C of that model.

It is obvious that the semantic values of substitutional variables are local in the same way in which the semantic values of names and predicates are local. Moreover, it will be noted that the semantic values of variables do not belong to the same "ontological category" as the ones assigned to names and pure predicates. The former are assigned names, the latter sets of pure atomic sentences. This puts the present semantics in contrast to denotational semantics according to which the semantic values of individual constants and individual variables belong to the same ontological category (of not necessarily linguistic members of objectual domains).

When we are dealing with rigid models (see clause 2.3.1(6)), we have to introduce *rigid nominal variable assignments* by requiring the following rigidity condition to hold for nominal variable assignments σ with respect to some rigid model $\mathcal{M} = \langle S, R, C, c, P, v\rangle$:

For all $x \in V$ and for all $s, t \in S$: $\sigma(x, s) = \sigma(x, t)$.

Let σ and τ be two nominal variable assignments for $s \in S$. σ and τ are (x, s)-*variants* just in case for all nominal variables y except at most x, $\tau(y, s) = \sigma(y, s)$. The variants will be rigidified in case the assignments are.

Let $\mathcal{M} = \langle S, R, C, c, P, v\rangle$ be a model, s an index in S, and σ an assignment in $c(s)$. Then for any term o the *term value* of o with respect to v and σ, $v_\sigma(o, s)$ is defined as follows.

$$v_\sigma(o, s) = \begin{cases} v(o, s) & \text{if } o \text{ is a nominal constant} \\ v(\sigma(o, s), s) & \text{if } o \text{ is a nominal variable.} \end{cases}$$

When we deal with rigid models the term values will be rigid as well. So if $\mathcal{M} = \langle S, R, C, c, P, v\rangle$ is a rigid model, s an index in S, and σ an assignment in $c(s)$. Then for any term o the *rigid term value* of o with respect to v and σ, $v_\sigma(o, s)$, satisfies the following condition:

For all terms $o \in T$ and for all $s, t \in S$: $v_\sigma(o, s) = v_\sigma(o, t)$.

From now on we indicate that a semantics is rigid merely by indicating that the models are rigid.

2.4.2 Truth at an Index in a Model with Variable Assignments

We define the conditions for truth at an index s in a model $\mathcal{M} = \langle S, R, C, c, P, v \rangle$ with respect to some variable assignment σ as follows.

1. $\mathcal{M}_\sigma \models_s \varphi^n o_1 ... o_n$ iff (i) if $o_1, ..., o_n$ are nominal constants, then $o_1, ..., o_n \in c(s)$ and if they are nominal variables, then $\sigma(o_1, s), ..., \sigma(o_n, s) \in c(s)$ and (ii) $\varphi^n o_1 ... o_n \in v_\sigma(o_1, s) \cap ... \cap v_\sigma(o_n, s) \cap v(\varphi^n, s)$; otherwise $\mathcal{M}_\sigma \not\models_s \varphi^n o_1 ... o_n$.

2. $\mathcal{M}_\sigma \models_s o_1 \ddot{=} o_2$ iff for all pure atomic formulae B_1 and B_2 where B_2 is like B_1 except for containing occurrences of the term o_2 at one or all places where B_1 contains the term o_1: $\mathcal{M}_\sigma \models_s B_1$ iff $\mathcal{M}_\sigma \models_s B_2$.

3. $\mathcal{M}_\sigma \models_s \neg A$ iff $\mathcal{M}_\sigma \not\models_s A$.

4. $\mathcal{M}_\sigma \models_s A \wedge B$ iff $\mathcal{M}_\sigma \models_s A$ and $\mathcal{M}_\sigma \models_s B$.

5. $\mathcal{M}_\sigma \models_s (\Pi x) A$ iff for every (x, s)-variant τ of σ: $\mathcal{M}_\tau \models_s A$.

6. $\mathcal{M}_\sigma \models_s \Box A$ iff for all $t \in S$ such that sRt: $\mathcal{M}_\sigma \models_t A$.

Derivatively: 7. $\mathcal{M}_\sigma \models_s A \vee B$ iff it is not the case that both $\mathcal{M}_\sigma \not\models_s A$ and $\mathcal{M}_\sigma \not\models_s B$; 8. $\mathcal{M}_\sigma \models_s A \rightarrow B$ iff it is not the case that both $\mathcal{M}_\sigma \models_s A$ and $\mathcal{M}_\sigma \not\models_s B$; 9. $\mathcal{M}_\sigma \models_s (\Sigma x) A$ iff for some (x, s)-variant τ of σ: $\mathcal{M}_\tau \models_s A[\alpha/x]$; 10. $\mathcal{M}_\sigma \models_s \Diamond A$ iff for some $t \in S$ such that sRt: $\mathcal{M}_\sigma \models_t A$.

2.4.3 Truth at an Index in a Model with Variable Assignments: Comments

The comments made in Subsection 2.3.4, except for 2.3.4(1), also apply, *mutatis mutandis*, to the notion of truth at an index in a model explained in terms of variable assignments. Here are some further remarks.

1. Unlike in Subsection 2.3.3, the truth conditions of all formulae do no longer reduce to truth conditions for closed formulae (more precisely, for closed pure atomic formulae). For closed formulae of L the notion of truth at an index with respect to a associative model may now be defined inductively, in a way which, in view of the underlying definition of term values, differs slightly from the usual denotational definition, as truth under all nominal variable assignments. Obviously, an account of this sort cannot be given for a formal language which lacks nominal constants. The way in which this kind of "nondenotational" definition of truth can be stated for a nonmodal substitutional language and nonmodal associative models $\mathcal{I} = \langle C, P, v\rangle$ is obvious.

2. We might have offered a different account of truth conditions for the pure atomic formulae by first giving a clause for pure atomic sentences and then a clause for pure atomic formulae. The way I have in mind would involve an account of term values along the following lines. Let $\mathcal{M} = \langle S, R, C, c, P, v\rangle$ be a rigid model, s an index in S, and σ a rigid assignment in $c(s)$. Then for any term o the rigid term value of o with respect to v and σ, $v^*_\sigma(o, s)$, will be defined as follows:

$$v^*_\sigma(o, s) = \begin{cases} v(o, s) & \text{if } o \text{ is a nominal constant} \\ \sigma(o, s) & \text{if } o \text{ is a nominal variable.} \end{cases}$$

The clauses for the pure atomic sentences and formulae would then take the following shape:

> (1′) Where $o_1, ..., o_n$ are nominal constants: $\mathcal{M}_\sigma \models_s \varphi^n o_1...o_n$ iff (i) $o_1, ..., o_n \in c(s)$ and (ii) $\varphi^n o_1, ..., o_n \in v^*_\sigma(o_1, s) \cap ... \cap v^*_\sigma(o_n, s) \cap v(\varphi^n, s)$; otherwise $\mathcal{M}_\sigma \not\models_s \varphi^n o_1...o_n$.
>
> (2′) Where $o_1, ..., o_n$ are nominal variables and $A(o_1...o_n)$ is an atomic formula of L possibly containing nominal constants with $o_1, ..., o_n$ free: $\mathcal{M}_\sigma \models_s A(o_1...o_n)$ iff $\mathcal{M}_\sigma \models_s A([v^*_\sigma(o_1, s)/o_1]...[v^*_\sigma(o_n, s)/o_n])$, where this formula is the result of replacing the free variables $o_1, ..., o_n$ in $A(o_1...o_n)$ by the term values specified above. Otherwise $\mathcal{M}_\sigma \not\models_s A(o_1...o_n)$.

On this account the truth conditions of all (open) formulae reduce to truth conditions on closed sentences (more precisely, on pure atomic sentences) just like in Subsection 2.3.3.[4] The account given in the previous section is more appealing than the one just given, not only because it allows to define truth in a way similar to denotational semantics, but also because it is less cumbersome, since it treats open and closed pure atomic formulae in a single clause.

2.4.4 Truth-Conditions Theorem for L-Formulae

The proof of the following analogue of the Truth-Conditions Theorem for sentences proceeds in essentially the same way as in Subsection 2.3.5.

> *Truth-Conditions Theorem: Formulae.* Let $\mathcal{M} = \langle S, R, C, c, P, v\rangle$ be an associative model. Given the family Γ mapping any $s \in S$ to the set Γ_s of atomic L-formulae true at s in $\mathcal{M}$, there is a unique family Γ' mapping any $s \in S$ to the set of L-formulae true at s in $\mathcal{M}$ satisfying clauses (2)-(6) and also coinciding with Γ on the atomic portion of L.

2.4.5 Truth in a Model with Variable Assignments

The introduction of nominal variable assignments allows us to state the conditions for truth in a model in a way that parallels the usual denotational definition.

A formula A of L is *true in an associative substitutional model* (formally, $\mathcal{M} \models A$) iff for every $s \in S$ and every nominal variable assignment σ: $\mathcal{M}_\sigma \models_s A$.

[4] A semantical interpretation of open formulae along these lines is suggested in [Kripke, 1976] p. 330, note 4. "Of course a formula with free variables can be interpreted semantically by observing that it becomes true if such and such terms replace the variables and false if other terms replace them. This, however, is merely a concept defined in terms of truth for sentences; unlike satisfaction in the case of referential quantifiers, it plays no role in the inductive definitions of semantical terms."

2.4.6 Validity in a Frame with Variable Assignments

The definition of validity (and the other terminology) remains the same as in Subsection 2.3.7.

A formula A of L is *valid in a frame for substitutional models* (formally, $\mathcal{F} \models A$) iff it is true in all associative models based on that frame. In the present setting we take these models to be rigid.

2.4.7 The Syntactical *de nomine*/*de dicto* Distinction

A formula A is said to be *de dicto* if no free nominal variable or nominal constant occurs within the scope of a modal operator. A formula A is *de nomine* if it is not *de dicto*.[5] So, for instance, $(\Sigma x)\Box Fx$, $\Diamond(\Pi x)\Box Fx$ and $\Diamond Fa$ will be *de nomine*, but $\Box(\Sigma x)Fx$ will not.

We call formulae of the first sort "*de nomine*", since it would be misleading to call them "*de re*". There just are no objectual domains of referents in the present framework and thus no *res* on this semantics for the language to be about. Indeed, the language is not even about names or associates. (It might therefore be more appropriate to call *de nomine* formulae "*ex nomine*", since unlike the preposition '*de*' the preposition '*ex*' hardly seemes to involve connotations of aboutness. However, we shall stick to the first terminological option, since it sounds more familiar.)

The intuitive motivation behind this shift of terminology will become more apparent in Chapter 4, when the applied semantical interpretation of the associative framework will be discussed. For the time being we shall say, anticipating the terminology of that chapter, that the names and the predicates do not denote anything but reflect the information included in their associates. Roughly, a *de nomine* sentence like $(\Sigma x)\Box Fx$ says that there is a nominal constant which is such that its associates and the associates of F are arranged in such a way that Fx is rendered true at all accessible indices once the constant is substituted for the variable. A *de dicto* sentence like $\Box(\Sigma x)Fx$ says, that the sentence $(\Sigma x)Fx$ is true at all accessible indices. We

[5] This characterization of *de dicto* is an adaptation of Kit Fine's "*de dicto* in the strict sense" (see [Fine, 1978] p. 143) to our substitutional language.

shall clarify this intuitive semantical distinction further in Subsection 2.5.7 after substitutional axiom systems with SFB have been specified.

2.5 Axiom Systems with SBF

Most of the discussion to follow in the rest of this chapter consists of stright-forward adaptations of soundness and completeness results from denotational modal semantics patterned along the exposition in [Hughes and Cresswell, 1996] to associative modal semantics. However, since the adaptations differ slightly from the denotational counterparts (for example, with respect to the presence of substitutional identity and the construction of the canonical models) the inclusion of the material to will be justified as I hope.

The present section discusses systems with $\ddot{=}$ and with SBF, Section 2.6 systems with $\ddot{=}$ but without SBF and Section 2.7 $\ddot{=}$-systems without CSBF. We shall study these systems in terms of rigid models. Only the material in which the differences mentioned above show up (with few exceptions) will be included in the body of this chapter. The material common to both the denotational and the associative framework are relegated to Appendix A where it is included only to let the present study be self-contained.

2.5.1 Substitutability

A term o is *substitutable for* a nominal variable x in A (or, free for x in A) provided there is no free occurrence of x in the scope of a substitutional quantifier (Πy) or (Σy) whose y occurs in o. Here to say that a variable occurs in a term is just to say that the variable and the term are identical. A consequence of this definition is that nominal constants are always substitutable for every nominal variable.

2.5.2 Relettering, Agreement, and Replacement

Formulae A and B are *bound alphabetic variants* just in case they differ only in that A contains a well formed subformula $(\Pi x)C$ where B has $(\Pi y)D$ and C and D differ only in that C has x free exactly where D has y free. As usual we then let $A[y/x]$ be the formula which results from taking a bound alphabetic variant of a formula A in which no substitutional y-quantifier is present and then replacing every free occurrence of x in the resulting variant by y.

We adapt two familiar first-order principles to the present framework, the modal principles of agreement (i.e. MSPA) and replacement (MSPR). The versions of these principles involve rigid models.

MSPA

> Let $\mathcal{M} = \langle S, R, C, c, P, v \rangle$ be any rigid associative model for the language L, let $s \in S$, and let σ and τ be rigid nominal variable assignments in $c(s)$ which agree on all the variables x free in a formula A of the language (i.e., $\sigma(x, s) = \tau(x, s)$), then the following holds: $\mathcal{M}_\sigma \models_s A$ iff $\mathcal{M}_\tau \models_s A$.

Proof of MSPA. The proof is rutine and proceedes by induction over the complexity of A.[6] We show that for all rigid variable assignments σ and τ in $c(s)$ of $\mathcal{M} = \langle S, R, C, c, P, v \rangle$ the following holds:

> If $\sigma(x, s) = \tau(x, s)$ for the variables x free in A, then: $\mathcal{M}_\sigma \models_s A$ iff $\mathcal{M}_\tau \models_s A$.

We give the cases for pure atomic formulae, for substitutional identities, substitutional universal quantifications, and necessities.

For $A = \varphi^n o_1 ... o_n$: $\mathcal{M}_\sigma \models_s \varphi^n o_1 ... o_n$ iff, by clause 2.4.2(1), $\varphi^n o_1 ... o_n \in v_\sigma(o_1, s) \cap ... \cap v_\sigma(o_n, s) \cap v(\varphi^n, s)$; iff, by the assumption that $\sigma(x, s) = \tau(x, s)$ for the variables x free in A, $\varphi^n o_1 ... o_n \in v_\tau(o_1, s) \cap ... \cap v_\tau(o_n, s) \cap v(\varphi^n, s)$; iff, by clause 2.4.2(1), $\mathcal{M}_\tau \models_s \varphi^n o_1 ... o_n$.

For $A = o_1 \ddot{=} o_2$: $\mathcal{M}_\sigma \models_s o_1 \ddot{=} o_2$ iff, by clause 2.4.2(2), for all pure atomic formulae B_1 and B_2 where B_2 is like B_1 except for containing occurrences of the term o_2 at one or all places where B_1 contains the term o_1: $\mathcal{M}_\sigma \models_s B_1$

[6] Our proof of MSPA and the proof of MSPR to be given shortly are adaptations of the proofs set out in [Friedrichsdorf, 1992] in terms of ordinary first-order denotational models.

iff $\mathcal{M}_\sigma \models_s B_2$; iff, in view of the considerations in 2.3.4(5) as generalized to terms $o, p \in T$, for all pure atomic formulae (which correspond to B_1 and B_2, respectively) $\varphi^n(o_1, p_1, ..., p_{n-1})$ and $\varphi^n(o_2, p_1, ..., p_{n-1})$ the following holds: $\varphi^n(o_1, p_1, ..., p_{n-1}) \in v_\sigma(o_1, s) \cap v_\sigma(p_1, s) \cap ... \cap v_\sigma(p_{n-1}, s) \cap v(\varphi^n, s)$ iff $\varphi^n(o_2, p_1, ..., p_{n-1}) \in v_\sigma(o_2, s) \cap v_\sigma(p_1, s) \cap ... \cap v_\sigma(p_{n-1}, s) \cap v(\varphi^n, s)$; iff, by the condition that $\sigma(x, s) = \tau(x, s)$ for the variables x free in A, for all pure atomic formulae $\varphi^n(o_1, p_1, ..., p_{n-1})$ and $\varphi^n(o_2, p_1, ..., p_{n-1})$ the following holds: $\varphi^n(o_1, p_1, ..., p_{n-1}) \in v_\tau(o_1, s) \cap v_\tau(p_1, s) \cap ... \cap v_\tau(p_{n-1}, s) \cap v(\varphi^n, s)$ iff $\varphi^n(o_2, p_1, ..., p_{n-1}) \in v_\tau(o_2, s) \cap v_\tau(p_1, s) \cap ... \cap v_\tau(p_{n-1}, s) \cap v(\varphi^n, s)$; iff, by clause 2.4.2(2), $\mathcal{M}_\tau \models_s o_1 \ddot{=} o_2$.

For $A = (\Pi x)B$. Since the rigid assignments σ and τ agree on the free variables in $(\Pi x)B$, their (x, s)-variants σ' and τ', respectively, they agree on the variables free in B, because $\sigma'(x, s) = \tau'(x, s)$. We then have using the induction hypothesis for B the following equivalences: $\mathcal{M}_\sigma \models_s (\Pi x)B$ iff, by clause 2.4.2(5), for every (x, s)-variant σ' of σ: $\mathcal{M}_{\sigma'} \models_s A$; iff, by induction hypothesis, for every (x, s)-variant τ' of τ: $\mathcal{M}_{\tau'} \models_s A$; iff, by clause 2.4.2(5), $\mathcal{M}_\tau \models_s (\Pi x)B$.

For $A = \Box B$ we obtain the following equivalences: $\mathcal{M}_\sigma \models_s \Box B$ iff, by clause 2.4.2(6), for every $t \in S$, if sRt then $\mathcal{M}_\sigma \models_t A$; iff, by induction hypothesis, for every $t \in S$, if sRt then $\mathcal{M}_\tau \models_t A$; iff, by clause 2.4.2(6), $\mathcal{M}_\tau \models_s \Box B$.

As a corollary to MSPA the following holds: where A contains no free variables and $\mathcal{M} = \langle S, R, C, c, P, v \rangle$ is a rigid model for L the following holds for all assignments σ and τ in $c(s)$: $\mathcal{M}_\sigma \models_s A$ iff $\mathcal{M}_\tau \models_s A$.

In case the models and the assignments under discussion are rigid, MSPA coincides with its nonmodal version SPA which can be obtained *mutatis mutandis*. When the models are not rigid, the nonrigid version of the MSPA will not so coincide, for then the assignments will be nonrigid and thus relativized to the indices. Note that this is a point of difference between associative and denotational modal semantics, since the latter does not need to introduce a modal version of this principle. The same applies to MSPR. Here is a version of this principle which appeals to rigid models:

MSPR

> Let A be any formula of the language L and let o be the nominal term of that language which is substitutable for the nominal variable x in A, then for every rigid associative model $\mathcal{M} = \langle S, R, C, c, P, v \rangle$, with s being an index in S, and for every rigid nominal variable assignment σ in $c(s)$ the following holds: $\mathcal{M}_\sigma \models_s A[o/x]$ iff $\mathcal{M}_{\tau[v_\sigma(o,s)/x]} \models_s A$.

In other words, the formula $A[o/x]$ is true under the assignment σ just in case A is true under its (x, s)-variant τ which assigns to x the term value of o under σ, that is, $v_\sigma(o, s)$. (Recall the convention, according to which assignments and term values are rigid, in case the models are.)

Proof of MSPR. The proof again is a simple adaptation of the standard result for denotational frameworks. It proceeds via induction on the complexity of A. We show that the following relationship holds:

> If o is a nominal term substitutable for x in A, then: $\mathcal{M}_\sigma \models_s A[o/x]$ iff $\mathcal{M}_{\tau[v_\sigma(o,s)/x]} \models_s A$. In this proof we shall abbreviate $\tau[v_\sigma(o, s)/x]$ by σ'.

We first need to verify that the following identity statement is true of all terms o': $v_\sigma(o'[o/x], s) = v_{\sigma'}(o', s)$, where $o'[o/x]$ is the result of substituting o for x in o'. (Recall from 2.5.2 that to say that a variable occurs in a term is just to say that the variable and the term are identical.)

The case in which o is a nominal constant α. Since the interpretation of nominal constants is independent of variable assignments and since for any nominal constant α we have it that $\alpha[o/x] = \alpha$ the above identity statement is true for constants.

The case in which o is a nominal variable y. We have to consider two situations, one in which y is syntactically identical with x and one in which it is not. In the first case, in which $y = x$, we have the following identities: $v_\sigma(y[o/x], s) = v_\sigma(o, s) = v_{\sigma'}(x, s) = \sigma'(x, s) = \sigma'(y, s) = v_{\sigma'}(y, s)$. In the second case, in which $y \neq x$, we get $y[x/o] = y$ and also $\sigma(y, s) = \sigma'(y, s)$, since σ' is an (x, s)-variant of σ. Thus we have verified the required identity statement. We may now turn to the induction on the complexity of A. We consider the cases where A is a pure atomic formula, a substitutional identity formula, and a substitutional universal quantification, or a necessity.

$A = \varphi^n o_1...o_n$. From the identity statement we obtain the following identity: $\varphi^n o_1...o_n[o/x] = \varphi^n o_1[o/x]...o_n[o/x]$ and then the following equivalences: $\mathcal{M}_\sigma \models_s \varphi^n o_1...o_n[o/x]$ iff, by clause 2.4.2(1), $\varphi^n o_1...o_n[o/x] \in v_\sigma(o_1[o/x], s) \cap ... \cap v_\sigma(o_n[o/x], s) \cap v(\varphi^n, s)$; iff, by the identity statement, $\varphi^n o_1...o_n[o/x] \in v_{\sigma'}(o_1, s) \cap ... \cap v_{\sigma'}(o_n, s) \cap v(\varphi^n, s)$; iff, by clause 2.4.2(1) $\mathcal{M}_{\sigma'} \models_s \varphi^n o_1...o_n$.

$A = o_1 \ddot{=} o_2$. In view of the identity statement we may claim that $o_1 \ddot{=} o_2[o/x] = o_1[o/x] \ddot{=} o_2[o/x]$. With this we obtain the following equivalences: $\mathcal{M}_\sigma \models_s o_1 \ddot{=} o_2[o/x]$ iff, by clause 2.4.2(2), for all pure atomic formulae $B_1[o/x]$ and $B_2[o/x]$ where $B_2[o/x]$ is like $B_1[o/x]$ except for containing occurrences of the term $o_2[o/x]$ at one or all places where $B_1[o/x]$ contains the term $o_1[o/x]$: $\mathcal{M}_\sigma \models_s B_1[o/x]$ iff $\mathcal{M}_\sigma \models_s B_2[o/x]$; iff, in view of 2.3.4(5) as generalized to terms $o, p \in T$, for all pure atomic formulae (which correspond to $B_1[o/x]$ and $B_2[o/x]$, respectively) $\varphi^n(o_1[o/x], p_1[o/x], ..., p_{n-1}[o/x])$ and $\varphi^n(o_2[o/x], p_1[o/x], ..., p_{n-1}[o/x])$ the following holds: $\varphi^n(o_1[o/x], p_1[o/x], ..., p_{n-1}[o/x]) \in v_\sigma(o_1[o/x], s) \cap v_\sigma(p_1[o/x], s) \cap ... \cap v_\sigma(p_{n-1}[o/x], s) \cap v(\varphi^n, s)$ iff $\varphi^n(o_2[o/x], p_1[o/x], ..., p_{n-1}[o/x]) \in v_\sigma(o_2[o/x], s) \cap v_\sigma(p_1[o/x], s) \cap ... \cap v_\sigma(p_{n-1}[o/x], s) \cap v(\varphi^n, s)$; iff, by the identity statement, for all pure atomic formulae $\varphi^n(o_1, p_1, ..., p_{n-1})$ and $\varphi^n(o_2, p_1, ..., p_{n-1})$ the following holds: $\varphi^n(o_1, p_1, ..., p_{n-1}) \in v_{\sigma'}(o_1, s) \cap v_{\sigma'}(p_1, s) \cap ... \cap v_{\sigma'}(p_{n-1}, s) \cap v(\varphi^n, s)$ iff $\varphi^n(o_2, p_1, ..., p_{n-1}) \in v_{\sigma'}(o_2, s) \cap v_{\sigma'}(p_1, s) \cap ... \cap v_{\sigma'}(p_{n-1}, s) \cap v(\varphi^n, s)$; iff, by 2.4.2(2), $\mathcal{M}_{\sigma'} \models_s o_1 \ddot{=} o_2$.

$A = (\Pi y)B$. In case o is substitutable for x in A we have to distinguish the case in which x is free in A and the case in which it is not. Consider the case in which x is free in A. In this case we have the identity claiming that $(\Pi y)B = (\Pi y)B[o/x]$. Using MSPA we obtain the following equivalences: $\mathcal{M}_\sigma \models_s (\Pi y)B[o/x]$ iff, by the identity just mentioned, $\mathcal{M}_\sigma \models_s (\Pi y)B$; iff, by the inductive hypothesis, $\mathcal{M}_{\sigma'} \models_s (\Pi y)B$. Now consider the case in which x is not free in A. In this case y will be not identical with x and so we have $(\Pi y)B[o/x] = (\Pi y)(B[o/x])$. From this we obtain using the induction hypotheses for B and the assignments the following equivalences: $\mathcal{M}_\sigma \models_s (\Pi y)B[o/x]$ iff, by the identity just mentioned, $\mathcal{M}_\sigma \models_s (\Pi y)(B[o/x])$;

iff, by 2.2.4(5), for every (y,s)-variant τ of σ: $\mathcal{M}_\tau \models_s B[o/x]$; iff, by the induction hypothesis, for every (y,s)-variant τ of σ: $\mathcal{M}_{\tau[v_\sigma(o,s)/x]} \models_s B$. Since due to the assumption the nominal term o is substitutable for x in $(\Pi y)B$, the variable y must be distinct from every other variable occurring in o. For this reason $v_\sigma(o,s) = v_\tau(y,s)$. In view of the fact that y and x are distinct, we obtain the following identity: $\tau[v_\sigma(o,s)/x] = \sigma'[v_\tau(o,s)/y]$. With its help we obtain the equivalences: $\mathcal{M}_\sigma \models_s (\Pi y)B[o/x]$ iff, by the identity just mentioned, for every (y,s)-variant τ of σ: $\mathcal{M}_{\sigma'[v_\tau(o,s)/y]} \models_s B$; iff, by clause 2.4.2(5), $\mathcal{M}_{\sigma'} \models_s (\Pi y)B$.

$A = \Box B$. In case o is substitutable for x in $\Box B$ it is substitutable for x in B as well. We get the following equivalences: $\mathcal{M}_\sigma \models_s \Box B[o/x]$ iff, by the substitutability fact just mentioned, $\mathcal{M}_\sigma \models_s \Box(B[o/x])$ iff, by clause 2.4.2(6), for all $t \in S$, such that sRt: $\mathcal{M}_\sigma \models_t B[o/x]$ iff, by the induction hypothesis, for all $t \in S$, such that sRt: $\mathcal{M}_{\sigma'} \models_t B$ iff, by clause 2.4.2(6), $\mathcal{M}_{\sigma'} \models_s \Box B$.

2.5.3 Axiomatization

We now specify for the substitutional language L a series of modal substitutional first-order logics with substitutional identity, SFOL$\doteqdot$+Λ, where Λ is a normal system of propositional modal logic. It is a peculiarity of L that it contains only finitely many predicates (see Subsection 2.2.1). This restriction has been imposed on the set of predicates in view of the second axiom for substitutional identity to be given below.

Axioms

1. *PC*: Every truth-functional tautology.

2. $\Pi 1$: Every formula of L of the form $(\Pi x)A \rightarrow A[o/x]$, where x a nominal variable and o is a nominal term, o is free for x in A and $A[o/x]$ results from replacing each free occurrence of x in A by o.

3. $\doteqdot$1: $o \doteqdot o$, where o is any nominal term.

4. $\doteqdot$2: Let φ^n be a pure n-ary predicate of L. $K^n_{\varphi^n}(o_1, o_2) =_{df} (\Pi z_1)...$ $(\Pi z_{n-1})(\Pi z_n)((\varphi^n o_1 z_2...z_n \leftrightarrow \varphi^n o_2 z_2...z_n) \wedge (\varphi^n z_1 o_1...z_n \leftrightarrow \varphi^n z_1 o_2...z_n) \wedge$

$... \wedge (\varphi^n z_1...z_{n-1}o_1 \leftrightarrow \varphi^n z_1...z_{n-1}o_2))$. Let $\varphi_1^{k_1}, ..., \varphi_m^{k_m}$ be all the pure predicates of L, where φ_i is k_i-ary. $K_L(o_1, o_2) =_{df} K_{\varphi_1}^{k_1}(o_1, o_2) \wedge ... \wedge K_{\varphi_m}^{k_m}(o_1, o_2)$. Axiom: $K_L(o_1, o_2) \leftrightarrow o_1 \ddot{=} o_2$.[7]

5. *S*: Every formula of L of the form S. Where S is some axiom scheme of

   K: $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
   D: $\Box A \rightarrow \Diamond A$
   T: $\Box A \rightarrow A$
   B: $A \rightarrow \Box\Diamond A$
   4: $\Box A \rightarrow \Box\Box A$
   5: $\Diamond A \rightarrow \Box\Diamond A$

   These familiar schemes axiomatize normal propositional modal systems Λ, and thus the modal bases of SFOL$\ddot{=}$ + Λ systems.

Rules

We have the following rules of inference. ('$\vdash A$' indicates that A is a theorem of the system.)

1. *Modus Ponens (MP)* : if $\vdash A$ and $\vdash A \rightarrow B$, then $\vdash B$.
2. $\Pi 2$: if $\vdash A \rightarrow B$ and x is not free in A, then $\vdash A \rightarrow (\Pi x)B$.
3. *Necessitation (Nec)*: if $\vdash A$, then $\vdash \Box A$.

[7] It is because of this axiom that in setting up our substitutional language we required the set of pure predicates P to be finite. When substitutional identity is absent from our language, we are free to assume that P is infinite. I am immensely indebited to my supervisors, in particular to Ede Zimmermann, for spotting a mistake in an earlier axiomatization and for pointing out to me that an axiom to this effect is needed.

If we were prepared to accept infinite conjunctions, the following axiom would be an alternative: $K_L(B_1 \leftrightarrow B_2) \leftrightarrow o_1 \ddot{=} o_2$, where $K_L(B_1 \leftrightarrow B_2)$ is the (possibly infinite) conjunction of all formulae of L of the form $(B_1 \leftrightarrow B_2)$, where B_1 and B_2 are pure atoms and where B_2 is like B_1 except for containing occurrences of the term o_2 at one or all places where B_1 contains o_1.

Here are some derived rules which will be used in the proofs of completeness.

UG1: if $\vdash A$, then $\vdash (\Pi x)A$.
DR1: if $\vdash C \to D$, then $\vdash \Box C \to \Box D$.
DR2: if $\vdash \Diamond C \to D$, then $\vdash C \to \Box D$.

Some Theorems

1. *T1*: $(\Pi x)(A \to B) \to ((\Pi x)A \to (\Pi x)B)$
2. *T2*: $(\Pi x)(A \to B) \to (A \to (\Pi x)B)$ provided x is not free in A
3. *T3*: $\neg(\Pi x)\neg(A[o/x] \to (\Pi x)A)$ provided o is not free in $(\Pi x)A$
4. *T4*: $\neg(\Pi x)\neg A \leftrightarrow (\Sigma x)A$

These theorems will be used in the completeness proofs.

2.5.4 Soundness of Λ+SBF$\ddot{=}$

The familiar soundness results for normal propositional systems together with the following theorem show that whenever a normal propositional system Λ is sound with respect to a certain class of frames, so is the corresponding SFOL system Λ+SBF.

Theorem 2.5.4. Suppose that $\mathcal{F}$ is a frame for a normal propositional modal system Λ. Then $\mathcal{F}$ is a frame for the constant substitution class system Λ+SBF.

Proof of Theorem 2.5.4. The proof is essentialy the same as the proof for the denotational couterpart of that system (cf. [Hughes & Cresswell, 1996] pp. 247-248). However, since the axiomatization of substitutional identity differs from that of its referential counterpart slight deviations are made necessary. We let $\mathcal{C}^{\mathcal{M}}$ be the class of all SBF rigid models based on $\mathcal{F}$ and show that each instance of the axiom schemata of Λ+SBF, viz. tautologies, S, $\Pi 1$, $\ddot{=}1$, $\ddot{=}2$, and SBF is true in every rigid model in $\mathcal{C}^{\mathcal{M}}$ based on $\mathcal{F}$ and that MP, Nec, and $\Pi 2$ are truth-preserving in every such model.

1. We omit the proofs for the tautologies.

2. For each axiom schema S of Λ, we have to verify that if B is a formula of L obtained by substituting formulae $C_1, ..., C_n$ of L for propositional variables $\mathrm{p}_1, ..., \mathrm{p}_n$ in some theorem A of Λ, then B is true in every rigid model in $\mathcal{C}^{\mathcal{M}}$. So suppose that B is not true in every such model, i.e. that for some $\langle \mathcal{F}, C, c, P, v \rangle \in \mathcal{C}^{\mathcal{M}}$ and some $s \in S$ in $\mathcal{F}$, $\mathcal{M} \not\models_s B$. Now let $\langle \mathcal{F}, v' \rangle$ be a model for propositional modal logic in which $\mathcal{F}$ is precisely the same frame as in $\langle \mathcal{F}, C, c, P, v \rangle$ and in which, for every $s \in S$ and every p_i $(1 \leq i \leq n)$, the truth value of p_i at s in $\langle \mathcal{F}, v' \rangle$ is the same as the truth value of C_i at s in $\langle \mathcal{F}, C, c, P, v \rangle$. It can be verified by induction that A is not true at s in $\langle \mathcal{F}, v' \rangle$ and thus not true in that model. But this means that A is not a theorem of Λ, since by hypothesis $\mathcal{F}$ is a frame for Λ. Hence, if A is a theorem of Λ, B is true in every rigid model in $\mathcal{C}^{\mathcal{M}}$.

3. For $\Pi 1$, suppose that for some $s \in S$ in some rigid SBF associative model and for some rigid nominal variable assignment σ based on that model, $\mathcal{M}_\sigma \models_s (\Pi x)A$. Let τ be the rigid (x, s)-variant of σ such that $\tau(x, s) = \sigma(y, s)$. So by clause 2.4.2(5) $\mathcal{M}_\tau \models_s A$. So by MSPR $\mathcal{M}_\sigma \models_s A[o/x]$. Consequently, every instance of $\Pi 1$ is true in every rigid SBF model and thus in every rigid model in $\mathcal{C}^{\mathcal{M}}$.

4. For $\ddot{=}1$ suppose that for some $s \in S$ in some rigid SBF associative model and for some rigid assignment σ based on that model, $\mathcal{M}_\sigma \models_s o \ddot{=} o$. By clause 2.4.2(2) for any model at all indices and all variable assignments, $\mathcal{M}_\sigma \models_s o_1 \ddot{=} o_2$ exactly if for all pure atomic formulae B_1 and B_2 where B_2 is like B_1 except for containing occurrences of the term o_2 at one or all places where B_1 contains the term o_1: $\mathcal{M}_\sigma \models_s B_1$ iff $\mathcal{M}_\sigma \models_s B_2$. But in the case at hand there is no difference between B_2 and B_1 since the former contains occurrences of the term o at one or all places where B_1 contains the term o. So we need not distinguish between B_1 and B_2 and may write B. Hence, $\mathcal{M}_\sigma \models_s B$ iff $\mathcal{M}_\sigma \models_s B$. Which is surely the case and gives us, by clause 2.4.2(2), the required result.

5. For $\ddot{=}2$ suppose that for some $s \in S$ in some rigid SBF associative model

and for some rigid assignment σ based on it we have $\mathcal{M}_\sigma \models_s K_L(o_1, o_2)$. Then, by the definition of that formula, $\mathcal{M}_\sigma \models_s K^{k_1}_{\varphi_1}(o_1, o_2) \wedge ... \wedge K^{k_m}_{\varphi_m}(o_1, o_2)$ for all the pure predicates $\varphi^{k_1}_1, ..., \varphi^{k_m}_m$ of L, where φ_i is of arity k_i. Therefore, we have for each conjunct, where φ^n is a pure n-ary predicate of L: $\mathcal{M}_\sigma \models_s (\Pi z_1) \ ... \ (\Pi z_{n-1}) \ (\Pi z_n)((\varphi^n o_1 z_2 \ ... \ z_n \leftrightarrow \varphi^n o_2 z_2 \ ... \ z_n) \wedge (\varphi^n z_1 o_1 \ ... \ z_n \leftrightarrow \varphi^n z_1 o_2 \ ... \ z_n) \wedge ... \wedge (\varphi^n z_1 \ ... \ z_{n-1} o_1 \leftrightarrow \varphi^n z_1 \ ... \ z_{n-1} o_2))$. Let $(\Pi z_1)(\Pi z_2)...(\Pi z_n)A$ abbreviate that formula. Then, by 2.4.2(5) $\mathcal{M}_\sigma \models_s (\Pi z_1)(\Pi z_2)...(\Pi z_n)A$ just in case for every (z_1, s)-variant τ_1 of σ, $\mathcal{M}_{\tau_1} \models_s (\Pi z_2)...(\Pi z_n)A$. And so on until $\mathcal{M}_{\tau_{n-1}} \models_s (\Pi z_n)A$ exactly if for every (z_n, s)-variant τ_n of τ_{n-1}, $\mathcal{M}_{\tau_n} \models_s A$. Now A is a formula of the form $\bigwedge(B_1 \leftrightarrow B_2)$, that is, a finite conjunction of all formulae of L of the form $(B_1 \leftrightarrow B_2)$, where B_1 and B_2 are atomic formulae built up out of the pure predicate φ^n and where B_2 is like B_1 except for containing occurrences of the term o_2 at one or all places where B_1 contains o_1. So for each such conjunct $(B_1 \leftrightarrow B_2)$ we have, $\mathcal{M}_{\tau_n} \models_s B_1 \leftrightarrow B_2$. In other words, for all such atomic formulae B_1 and B_2: $\mathcal{M}_{\tau_n} \models_s B_1$ iff $\mathcal{M}_{\tau_n} \models_s B_2$. By MSPA, this is so just in case for all such atomic formulae B_1 and B_2: $\mathcal{M}_\sigma \models_s B_1$ iff $\mathcal{M}_\sigma \models_s B_2$. Hence, by clause 2.4.2(2) this means that $\mathcal{M}_\sigma \models_s o_1 \ddot{=} o_2$. The other direction is similar.

6. For SBF. Suppose for some $s \in S$ in some rigid associative SBF model and for some rigid variable assignment σ based on it we have $\mathcal{M}_\sigma \models_s (\Pi x)\Box A$. Letting τ be any (x, s)-variant of σ and sRt we obtain by clause 2.2.4(5), $\mathcal{M}_\tau \models_s \Box A$, and by clause 2.2.4(6) $\mathcal{M}_\tau \models_t A$. In view of the fact that this holds for all (x, s)-variants of σ and all $t \in S$ such that sRt we obtain, by 2.2.4(5), $\mathcal{M}_\sigma \models_t (\Pi x)A$ and then $\mathcal{M}_\sigma \models_s \Box(\Pi x)A$, by clause 2.2.4(6). So $\mathcal{M}_\sigma \models_s (\Pi x)\Box A \rightarrow \Box(\Pi x)A$. So SBF is true in every rigid SBF model, and so in every model in $\mathcal{C}^{\mathcal{M}}$. (In case Λ contains the B scheme as a thesis, SFB will be derivable form Λ and thus will not be included in the list of axioms.)

7. MP and Nec are truth preserving in a model for the same reasons as in propositional modal systems.

8. For $\Pi 2$, assume that $A \rightarrow B$ is true in every model in $\mathcal{C}^{\mathcal{M}}$. We have to show that if this is so then $A \rightarrow (\Pi x)B$ (where x is not free in A) is true in every model in $\mathcal{C}^{\mathcal{M}}$ as well. Suppose that for some model $\mathcal{M} = \langle S, R, C, c, P, v \rangle$ in $\mathcal{C}^{\mathcal{M}}$, some $s \in S$, and some variable assignment σ in $c(s)$, the following holds: $\mathcal{M}_\sigma \not\models_s A \rightarrow (\Pi x)B$. Then $\mathcal{M}_\sigma \models_s A$ and $\mathcal{M}_\sigma \not\models_s (\Pi x)B$. So there is some (x, s)-variant τ of σ such that $\mathcal{M}_\tau \not\models_s B$. Since x is not free in A we obtain, by MSPA, $\mathcal{M}_\tau \models_s A$. So, by the truth conditions for the material conditional, $\mathcal{M}_\sigma \not\models_s A \rightarrow B$. But this contradicts the assumption that $A \rightarrow B$ is true in every rigid model in $\mathcal{C}^{\mathcal{M}}$.

This completes the proof of Theorem 2.5.4. An immediate consequence of this theorem is that every theorem of K+SBF$\ddot{=}$ is valid on every frame.

2.5.5 Soundness of Λ+SBF$\ddot{=}$: Comments

1. On the modal interpretation of the truth conditions for substitutional identity (see clause 2.3.3(2*) in comment 2.3.4(7)) the validity of axiom $\ddot{=}1$ requires that the accessibility relation in the rigidified model be (weakly) reflexive. The simplest countermodel to that axiom will be $\mathcal{M} = \langle S, R, C, c, P, v \rangle$ based on an irreflexive frame where $S = \{s\}$, $R = \emptyset$, $C = c(s) = \{a\}$, $P = \{F\}$ with F monadic, and $v(a, s) = v(F, s) = \{Fa\}$. So axiom T should be among the axioms for the necessity operator in this case.

2. The same observation applies to axiom $\ddot{=}2$. For the modal interpretation of the truth conditions for substitutional identity that axiom would have to be modalized by inserting boxes in front of each biconditional that occurs in $K_L(o_1, o_2)$ and $o_1 \ddot{=} o_2$.

3. Neither NSI, i.e., $o_1 \ddot{=} o_2 \rightarrow \Box(o_1 \ddot{=} o_2)$, nor NSD, $\neg(o_1 \ddot{=} o_2) \rightarrow \Box\neg(o_1 \ddot{=} o_2)$, is valid on rigidified frames. It would take, e.g., static models to make them valid (see 2.3.8(6)).

4. Unlike its denotationally interpreted counterpart, the following axiom is not valid for rigid models:

$\ddot{=}3$ $(o_1 \ddot{=} o_2) \rightarrow (A(o_1) \rightarrow A(o_2))$, where o_1 and o_2 are any nominal terms and the possibly complex formula $A(o_2)$ differs from $A(o_1)$ at most in having an occurrence of o_2 at one or more places where $A(o_1)$ has o_1.

The reason for this is that in the inductive step for $A = \Box B$ the validity of NSI is needed. When validity is analysed in terms of static models the axiom can be incorporated. In the case of rigid models it can be maintained given the obvious restriction.

5. The axiomatization of substitutional identity in substitutional counterpart to an ordinary first-order axiom system with identity will be given by the axioms $\ddot{=}1$, $\ddot{=}2$ and $\ddot{=}3$. The proofs for $\ddot{=}1$ and $\ddot{=}2$ are adaptations of the proofs given in the previous section. To establish the validity of $\ddot{=}3$ as contained in the axiomatization of a substitutional first-order axiom system with $\ddot{=}$ we first prove the following claim as a lemma.

   For any first-order associative model $\mathcal{I} = \langle C, P, v \rangle$ and any variable assignment σ based on that model the following holds: $\mathcal{I}_\sigma \models o_1 \ddot{=} o_2 \rightarrow (B(o_1) \rightarrow B(o_2))$ iff $\mathcal{I}_\sigma \models o_1 \ddot{=} o_2 \rightarrow (C(o_1) \rightarrow C(o_2))$ where: B is of complexity n and C of complexity $n + 1$; $B(o_2)$ is like $B(o_1)$ except for containing occurrences of the term o_2 at one or all places where $B(o_1)$ contains the term o_1; and $C(o_2)$ is like $C(o_1)$ except for containing occurrences of the term o_2 at one or all places where $C(o_1)$ contains the term o_1.

   We prove the lemma by induction on the complexity of B. We first prove the lemma (by reductio) for B of complexity $n = 0$ and so for C of complexity $n + 1$. For complex Bs we then assume as hypothesis that the result holds for B of complexity $< n$ with $n > 0$. Having shown this lemma we turn to the proof for the axiom $\ddot{=}3$. Here we first prove the soundness for the atomic case. For the complex cases we then assume as hypothesis that the result holds for instances of $\ddot{=}3$ with B of complexity $< n$ with $n > 0$ and use the above lemma.

2.5.6 Completeness of Λ+SBF$\ddot{=}$

We now show, following largely familiar lines that for any formula A which is not a theorem of Λ+SBF$\ddot{=}$ it is possible to construct an canonical associative model for Λ+SBF$\ddot{=}$ in which A is invalidated.

The proof differs slightly from the standard proof for a corresponding system with referential identity, =, in that it is free of the complications to which the presence of the referential identity predicate gives rise. When referential identity is present the simple strategy of letting the canonical denotations of individual terms be just themselves is no longer viable. In reaction to this the domain D of an ordinary denotational canonical model M = ⟨S, R, D, d, v⟩ is usually defined in terms of a relation ∼ on the set T of individual terms o, p, ... of the, standardly extended, denotational language with =, such that o ∼ p iff o = p ∈ s, for some s ∈ S. Since ∼ is an equivalence relation, for each o ∈ T the equivalence class [o] for o is then defined to be the set {p ∈ T: o ∼ p} and D is taken to be the set of all these equivalence classes [o]. Moreover, the strategy appeals to the part of canonical model which is based on a cohesive subframe of its frame, since in the cohesive part of the model every canonical index contains the same referential identity formulae, which is shown by appealing to the necessity of referential identity and distinctness.[8] In view of the fact that the systems for which we are going to prove completeness do not contain NSI and NSD an analogue of this strategy is not viable for us. However, these systems do not give rise to such complications anyway, for $\ddot{=}$ is not a referential identity predicate. Moreover, as we shall see shortly, in associative canonical models the interpretation of nominal terms takes an altogether different shape. Let me add that this interpretation remains the same independently of whether $\ddot{=}$ is present or not.

The indices of the associative canonical models to be presented will be, as usual, maximal Λ+SBF$\ddot{=}$-consistent sets of formulae of L which have the Π-property:

A set Θ has the Π-property iff for every formula A and every

[8] See [Hughes & Cresswell, 1996] pp. 315-317. For a survey on completeness proofs in denotational quantified modal logic see [Garson, 2001].

nominal variable x, there is some witnessing nominal term o such that $A[o/x] \rightarrow (\Pi x)A \in \Theta$.

So if Γ is both maximal Λ+SBF$\doteqdot$-consistent and has the Π-property then there must be a witness o such that $A[o/x] \notin \Gamma$, in case $(\Pi x)A \notin \Gamma$. As usual we call a nominal term o which ensures that $(\Pi x)A(x)$ is false at s, a 'term witness'.

Following the standard technique we consider not the language L itself, but an expansion of it, L^+, which contains, in addition to the terms of L, a denumerably infinite set of new terms. So $T^+ = C^+ \cup V^+$ will be the set of L^+-terms.

We assume the standard results about maximal consistent sets of formulae with respect to some system S. We say that a set of formulae Γ is S-consistent iff there is no finite set $\{A_1, ..., A_n\} \subseteq \Gamma$ such that $\vdash_S \neg(A_1 \wedge ... \wedge A_n)$. And we say that a set Γ is maximal iff for every formula A: either $A \in \Gamma$ or $\neg A \in \Gamma$. We assume, omitting the well-known proof, the following to hold:

Theorem 2.5.6(1) (Lindenbaum's Theorem). Any S-consistent set of formulae Γ can be extended to a maximal S-consistent set of formulae Δ.

We may prove the following theorem.

Theorem 2.5.6(2) (Π-property). If Θ is a consistent set of formulae of L then there is a consistent set Δ of formulae of L^+ with the Π-property such that $\Theta \subseteq \Delta$.

The proof is reproduced in Appendix A. By Lindenbaum's Theorem, since Δ is consistent and has the Π-property, it has a maximal consistent extension Γ in L^+ with the Π-property.

We can now prove the following existence theorem concerning maximal consistent sets with the Π-property in modal systems.

Theorem 2.5.6(3) (Existence Theorem). If s is a maximal consistent set of formulae in L^+, and s has the Π-property, and A is a formula such that $\Box A \notin s$ then there is a consistent set t of formulae of L^+ with the Π-property such that $\{B : \Box B \in s\} \cup \{\neg A\} \subseteq t$.

A reproduction of this proof can be found in Appendix A.

We shall now show that for each Λ+SBF$\ddot{=}$-consistent set Γ of formulae of L^+ an associative model $\mathcal{M}_\Gamma^\Lambda$ (for calligraphical reasons letting the superscripted Λ be an abbreviation of Λ+SBF$\ddot{=}$) can be constructed in which all the formulae of Γ are true.

Let Γ be a Λ+SBF$\ddot{=}$-consistent set of L-formulae. We define a rigid associative canonical model for L with the extension L^+ to be a 6-tuple:

$$\mathcal{M}_\Gamma^\Lambda = \langle S^\Lambda, R^\Lambda, C^\Lambda, c^\Lambda, P^\Lambda, v^\Lambda \rangle$$

where:

1. S^Λ is the set of maximal Λ+SBF$\ddot{=}$-consistent set of sets, i.e. canonical indices, $s, t, \ldots$ of L^+-formulae which have the Π-property.

2. $sR^\Lambda t$ iff $\{A : \Box A \in s\} \subseteq t$ iff for every formula $\Box A$ of L^+: if $\Box A \in s$, then $A \in t$.

3. C^Λ is C^+.

4. $c^\Lambda : S^\Lambda \rightarrow \wp(C^\Lambda)$. So $c^\Lambda(s)$ is the canonical substitution class for some $s \in S^\Lambda$.

5. $P^\Lambda = P$.

6. v^Λ is defined as follows:

   $v^\Lambda : C^\Lambda \times S^\Lambda \rightarrow \wp(Atm^\Lambda)$ such that $v^\Lambda(\alpha, s) \subseteq Atm^\Lambda(\alpha)$;

   $v^\Lambda : P^\Lambda \times S^\Lambda \rightarrow \wp(Atm^\Lambda)$ such that $v^\Lambda(\varphi^n, s) \subseteq Atm^\Lambda(\varphi^n)$,

   where Atm^Λ is the set of pure atomic L_0^+-sentences and where $Atm^\Lambda(\alpha)$ and $Atm^\Lambda(\varphi^n)$ are defined as follows:

   $Atm^\Lambda(\alpha) =_{df} \{A \in Atm^\Lambda$: A contains at least one occurrence of the nominal constant $\alpha\}$;
   $Atm^\Lambda(\varphi^n) =_{df} \{A \in Atm^\Lambda$: A contains an occurrence of the predicate $\varphi^n\}$.

   We stipulate that for arbitrary $\alpha \in c^\Lambda(s)$, $\varphi^n \in P^\Lambda$, and $s \in S^\Lambda$ the following condition holds with respect to v^Λ:

$\varphi^n\alpha_1...\alpha_n \in v^\Lambda(\alpha_1, s)\cap...\cap v^\Lambda(\alpha_n, s)\cap v^\Lambda(\varphi^n, s)$ iff $\varphi^n\alpha_1...\alpha_n \in s$.

Alternatively, v^Λ may be defined by putting for arbitrary $\alpha \in c^\Lambda(s)$, $\varphi^n \in P^\Lambda$, and $s \in S^\Lambda$:

$$v^\Lambda(\alpha, s) =_{df} Atm^\Lambda(\alpha) \cap s; \text{ and}$$
$$v^\Lambda(\varphi^n, s) =_{df} Atm^\Lambda(\varphi^n) \cap s.$$

The condition is then met trivially.

Since the model is rigid it satisfies the following rigiditiy condition: for all nominal constants $\alpha \in C^\Lambda$ and for all indices $s, t \in S^\Lambda$: $v^\Lambda(\alpha, s) = v^\Lambda(\alpha, t)$.

Given a rigid associative canonical model $\mathcal{M}_\Gamma^\Lambda = \langle S^\Lambda, R^\Lambda, C^\Lambda, c^\Lambda, P^\Lambda, v^\Lambda\rangle$ a *canonical variable assignment* σ^Λ is defined as follows.

$$\sigma^\Lambda : V^+ \times S^\Lambda \to C^\Lambda$$

such that, where x is some member of V^+, $\sigma^\Lambda(x, s)$ is a nominal constant $\alpha \in c^\Lambda(s)$.

Let σ^Λ and τ^Λ be two canonical variable assignments for $s \in S^\Lambda$. σ^Λ and τ^Λ are *canonical* (x, s)*-variants* exactly if for all variables y except at most x, $\tau^\Lambda(y, s) = \sigma^\Lambda(y, s)$.

Let $\mathcal{M}_\Gamma^\Lambda = \langle S^\Lambda, R^\Lambda, C^\Lambda, c^\Lambda, P^\Lambda, v^\Lambda\rangle$ be a rigid canonical model, s a index in S^Λ, and σ^Λ an assignment in $c^\Lambda(s)$. Then for any term o the *canonical term value* of o with respect to v^Λ and σ^Λ, $v^\Lambda_{\sigma^\Lambda}(o, s)$ is defined as follows.

$$v^\Lambda_{\sigma^\Lambda}(o, s) = \begin{cases} v^\Lambda(o, s) & \text{if } o \text{ is a nominal constant} \\ v^\Lambda(\sigma^\Lambda(o, s), s) & \text{if } o \text{ is a nominal variable.} \end{cases}$$

Since the model is rigid, the term values are rigid, too. So for any term o the canonical rigid term value of o with respect to v^Λ and σ^Λ, $v^\Lambda_{\sigma^\Lambda}(o, s)$, satisfies the condition: for all terms $o \in T^+$ and for all $s, t \in S^\Lambda$: $v^\Lambda_{\sigma^\Lambda}(o, s) = v^\Lambda_{\sigma^\Lambda}(o, t)$.

We can now prove the following theorem for rigid canonical models.

Theorem 2.5.6(4). (Truth Theorem). For any $s \in S^{\Lambda}$, and any formula $A \in L^{+}$: $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \models_s A$ iff $A \in s$.

(Recall that the superscripted Λ abbreviates Λ+SBF$\ddot{=}$.) The proof is by induction on the construction of formulae.

(i) Pure atomic formulae. $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \models_s \varphi^n o_1 ... o_n$ iff, by clause 2.4.2(1), (a) if $o_1, ..., o_n$ are nominal constants, then $o_1, ..., o_n \in c^{\Lambda}(s)$ and if they are nominal variables, then $\sigma^{\Lambda}(o_1, s), ..., \sigma^{\Lambda}(o_n, s) \in c^{\Lambda}(s)$ and (b) $\varphi^n o_1 ... o_n \in v^{\Lambda}_{\sigma^{\Lambda}}(o_1, s) \cap ... \cap v^{\Lambda}_{\sigma^{\Lambda}}(o_n, s) \cap v^{\Lambda}(\varphi^n, s)$; iff, in view of the definition of v^{Λ} and the definition of canonical term values under σ^{Λ}, by 2.4.2(1), $\varphi^n o_1 ... o_n \in s$.

(ii) Substitutional identity formulae. $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \models_s o_1 \ddot{=} o_2$ iff, by clause 2.4.2(2), for all pure atomic formulae B_1 and B_2 where B_2 is like B_1 except for containing occurrences of the term o_2 at one or all places where B_1 contains o_1: $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \models_s B_1$ iff $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \models_s B_2$; iff, by induction hypothesis, for all pure atomic formulae B_1 and B_2 where B_2 is like B_1 except for containing occurrences of the term o_2 at one or all places where B_1 contains o_1: $B_1 \in s$ iff $B_2 \in s$; iff for all B_1 and B_2: $B_1 \leftrightarrow B_2 \in s$; iff, by $\ddot{=}$2, $o_1 \ddot{=} o_2 \in s$.

(iii) Negations. $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \models_s \neg A$ iff, by clause 2.4.2(3), $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \not\models_s A$; iff, by hypothesis, $A \notin s$; iff $\neg A \in s$.

(iv) Conjunctions. $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \models_s A \wedge B$ iff, by clause 2.4.2(4), $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \models_s A$ and $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \models_s B$; iff, by hypothesis, $A \in s$ and $B \in s$; iff, by clause 2.4.2(4), $A \wedge B \in s$.

(v) Universal substitutional quantifications; first part. Suppose $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \not\models_s (\Pi x)A$. Then for some (x, s)-variant τ^{Λ} of σ^{Λ}, $\mathcal{M}^{\Lambda}_{\Gamma,\tau^{\Lambda}} \not\models_s A$. So, by MSPR, for some term o in L^{+}, $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \not\models_s A[o/x]$. Then, by hypothesis, $A[o/x] \notin s$. And so, by Π1, $(\Pi x)A \notin s$.

(vi) Universal substitutional quantifications; second part. Suppose $(\Pi x)A \notin s$. Then $\neg(\Pi x)A \in s$. So, since s has the Π-property in L^{+}, there is some o in L^{+} such that $\neg A[o/x] \in s$. So $A[o/x] \notin s$, and thus, by hypothesis, $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \not\models_s A[o/x]$. From this we conclude, by Π1, $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \not\models_s (\Pi x)A$.

(vii) Necessity formulae; first part. Suppose $\Box A \in s$ and $sR^{\Lambda}t$. Then, by the definition of R^{Λ}, $A \in t$. So, by hypothesis, $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \models_t A$. And since this is so for every t such that $sR^{\Lambda}t$, by clause 2.4.2(6), $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \models_s \Box A$.

(viii) Necessity formulae; second part. Suppose $\Box A \notin s$. Then $\neg\Box A \in s$. But then, by the Existence Theorem 2.5.6(3) and Lindenbaum's Theorem 2.5.6(1), there is some $t \in S^{\Lambda}$, such that $sR^{\Lambda}t$ with the Π-property such that $\neg A \in t$. So $A \notin t$. Thus, by hypothesis, $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \not\models_t A$. Since $sR^{\Lambda}t$, we obtain by clause 2.4.2(6), $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \not\models_s \Box A$.

This completes the proof of the Truth Theorem 2.5.6(4).

Completeness follows immediately.

> *Theorem 2.5.6(5)* (Completeness). If $\models_{\Lambda} A$, then $\vdash_{\Lambda} A$, where Λ is short for Λ+SBF$\ddot{=}$.

Proof of Theorem 2.5.6(5). Suppose $\not\vdash_{\Lambda} A$. So $\neg A$ is Λ-consistent. By the construction of canonical models $\neg A$ will be a member of some index s of the model $\mathcal{M}^{\Lambda}_{\{\neg A\}}$ which is generated by $\{\neg A\}$, that is, $\neg A \in s$. In view of the Truth Theorem 2.5.6(4) we can conclude that $\mathcal{M}^{\Lambda}_{\{\neg A\},\sigma\Lambda} \models_s \neg A$ which means that $\mathcal{M}^{\Lambda}_{\{\neg A\}} \models_{\Lambda}$, and, thus $\not\models_{\Lambda} A$.

2.5.7 The Model-Theoretic *de nomine/de dicto* Distinction

We shall discuss the intuitive semantical interpretation of the *de nomine/de dicto* distinction suggested in Subsection 2.4.7 more formally by means of an adaptation of a result from [Tichý, 1973]. Our discussion will follow the presentation of this result which has been given in [Hughes & Cresswell, 1996] (pp. 251- 254).

We shall first show that it is not the case that in the constant substitution class axiom system S5+SBF (without substitutional identity) all *de nomine* formulae of L (minus substitutional identity) are equivalent to *de dicto* formulae of that language. In order to parallel the denotational result for which it is essential that the semantic values of the individual terms are constant across the indices but the extensions of the predicates may vary, we assume that the models are rigid (in the sense of clause 2.3.1(6)) and assume, accordingly, that the variable assignments, their variants and term values are rigidified.

We now show that, unlike for *de nomine* formulae, for *de dicto* formulae of our language without substitutional identity two rigid associative models $\mathcal{M}^1 = \{S, R, C, c, P, v^1\}$ and $\mathcal{M}^2 = \{S, R, C, c, P, v^2\}$ which differ only in which associate is assigned to a given predicate at a specific index t, are equivalent from the point of view of another index s. So we may say that the truth of *de dicto* formulae does not turn upon the local associates predicates receive across different indices.

Illustration: Our rigid models $\mathcal{M}^1$ and $\mathcal{M}^2$ coincide in the following respects: $S = \{s, t\}$, $R = S \times S$, $c(s) = c(t) = C = \{\alpha, \beta\}$, $P = \{\varphi, \psi\}$, where both predicates are monadic, $v^1(\alpha, s) = v^1(\alpha, t) = \{\varphi\alpha\}$ and $v^1(\beta, s) = v^1(\beta, t) = \{\varphi\beta\}$, and, correspondingly, $v^2(\alpha, s) = v^2(\alpha, t) = \{\varphi\alpha\}$ and $v^2(\beta, s) = v^2(\beta, t) = \{\varphi\beta\}$. However, $\mathcal{M}^1$ and $\mathcal{M}^2$ differ in what associates are assigned to the predicates. $\mathcal{M}^1$: $v^1(\psi, s) = v^1(\psi, t) = \emptyset$ and $v^1(\varphi, s) = v^1(\varphi, t) = \{\varphi\alpha\}$. Whereas for $\mathcal{M}^2$: $v^2(\psi, s) = v^2(\psi, t) = \emptyset$, but $v^2(\varphi, s) = \{\varphi\alpha\}$ and $v^2(\varphi, t) = \{\varphi\beta\}$. Since $\mathcal{M}^1$ and $\mathcal{M}^2$ have the same substitution classes the class of the variable assignments is the same in each of them. The proof shows that in cases like these the difference between the models can be detected only by a *de nomine* formula and not by a *de dicto* formula.

Given a rigidified assignment σ^1 we let, adapting the original strategy, σ^2 be its *anti-assignment* such that for every $x \in V$ and for all $s, t \in S$, $\sigma^1(x, s) \neq \sigma^2(x, s)$. We now prove, sticking to the above example, the following theorem.

> *Theorem 2.5.7.* If A is a *de dicto* formula of L (without substitutional identity) and the models are rigidified then $\mathcal{M}^1_{\sigma^1} \models_s A$ iff $\mathcal{M}^2_{\sigma^1} \models_s A$ and $\mathcal{M}^1_{\sigma^1} \models_t A$ iff $\mathcal{M}^2_{\sigma^2} \models_t A$.

The proof is by induction on the complexity of formulae.

$A = \varphi o$. Since the predicate ψ receives the empty set as associate, the theorem obviously holds for any pure atom A built up out of that predicate. We therefore need to consider only the cases in which A is built up out of φ, i.e., φo.

For the first conjunct of the theorem we obtain the following equivalences: $\mathcal{M}^1_{\sigma^1} \models_s \varphi o$ iff, by clause 2.4.2(1), $\varphi o \in v^1_{\sigma^1}(o, s) \cap v^1(\varphi, s)$; iff, in view of the fact that $\sigma^1(o,s) = \alpha$ and $v^1_{\sigma^1}(o,s) = v^1(\alpha,s)$, $\varphi\alpha \in v^1_{\sigma^1}(o,s) \cap v^1(\varphi,s)$; iff $\varphi o \in v^1_{\sigma^1}(o,s) \cap v^2(\varphi,s)$; iff, by clause 2.4.2(1), $\mathcal{M}^2_{\sigma^1} \models_s \varphi o$.

For the second conjunct we have: $\mathcal{M}^1_{\sigma^1} \models_t \varphi o$ iff, by clause 2.4.2(1), $\varphi o \in v^1_{\sigma^1}(o,t) \cap v^1(\varphi,t)$; iff, in view of the fact that $\sigma^1(o,t) = \alpha$ and $v^1_{\sigma^1}(o,t) = v^1(\alpha,t)$, $\varphi\alpha \in v^1_{\sigma^1}(o,t) \cap v^1(\varphi,t)$; iff, in view of the fact that $\sigma^2(o,t) = \beta$ and $v^2_{\sigma^2}(o,t) = v^2(\beta,t)$, $\varphi o \in v^2_{\sigma^2}(o,t) \cap v^2(\varphi,t)$; iff, by clause 2.4.2(1), $\mathcal{M}^2_{\sigma^2} \models_t \varphi o$.

For the complex cases we assume as an inductive hypothesis that the theorem holds for formulae of complexity $< n$ and consider the cases for formulae of complexity n. Here we omit the cases of negation and conjunction and consider only the quantifier and the box cases.

$A = (\Pi x)B$. By the characterization of *de dicto* the following is the case: if $(\Pi x)B$ is *de dicto* then so is the subformula B. For the first conjunct of the theorem we have $\mathcal{M}^1_{\sigma^1} \models_s (\Pi x)B$ iff, by clause 2.4.2(5), for every (x,s)-variant τ^1 of σ^1: $\mathcal{M}^1_{\tau^1} \models_s B$. Since B is of complexity $n-1$, the induction hypothesis applies. So the latter is the case iff for every (x,s)-variant τ^1 of σ^1: $\mathcal{M}^2_{\tau^1} \models_s B$; iff, by clause 2.4.2(5), $\mathcal{M}^2_{\sigma^1} \models_s (\Pi x)B$.

Second conjunct. $\mathcal{M}^1_{\sigma^1} \models_t (\Pi x)B$ iff, by clause 2.4.2(5), for every (x,t)-variant τ^1 of σ^1: $\mathcal{M}^1_{\tau^1} \models_t B$; iff, by the induction hypothesis, $\mathcal{M}^2_{\tau^2} \models_t B$. Since every (x,t)-variant ρ of σ^1 will be τ^2 for some (x,t)-variant τ^1 of σ^1, this will be the case iff for every (x,t)-variant ρ of $\sigma^2 \mathcal{M}^2_\rho \models_t B$; iff, by clause 2.4.2(5), $\mathcal{M}^2_{\sigma^2} \models_t (\Pi x)B$.

$A = \Box B$. By the characterization of *de dicto*, if $\Box B$ is *de dicto*, then no variable in B is free. So, by MSPA, recalling that the variants are rigidified, the following two biconditionals hold: for every σ, τ, and $u \in S$: $\mathcal{M}^1_\sigma \models_s B$ iff $\mathcal{M}^1_\tau \models_s B$ and $\mathcal{M}^2_\sigma \models_s B$ iff $\mathcal{M}^2_\tau \models_s B$. Now for every $u \in S$, (1) $\mathcal{M}^1_{\sigma^1} \models_u \Box B$ iff (2) $\mathcal{M}^1_{\sigma^1} \models_s B$ and (3) $\mathcal{M}^1_{\sigma^1} \models_t B$. By the induction hypothesis (2) holds iff (4) $\mathcal{M}^2_{\sigma^1} \models_s B$ and (3) holds iff (5) $\mathcal{M}^2_{\sigma^2} \models_t B$. Now, since B does not contain free variables, (5) holds iff (6) $\mathcal{M}^2_{\sigma^1} \models_t B$. Thus, given the equivalences, (6) holds just in case (3) holds as well. So (2) and (3) hold iff (4) and (5) hold. But (4) and (5) hold iff, for any $u \in S$ (7) $\mathcal{M}^2_{\sigma^1} \models_u \Box B$. So (1) holds just in case (7) does; and the theorem follows immediately for $u = s$: (8) $\mathcal{M}^2_{\sigma^1} \models_s \Box B$. For the case $u = t$, (7) will hold, in view of the fact that B does not contain free variables, iff (9) $\mathcal{M}^2_{\sigma^2} \models_t \Box B$.

This concludes the proof of Theorem 2.5.7. We now use that result to show that the *de nomine* modal formula $(\Sigma x)\Box\varphi x$ is not equivalent in the constant substitution class system S5+SBF to any *de dicto* formula.

We again stick to the models from our illustration. $\mathcal{M}^1 = \{S, R, C, c, P, v^1\}$ and $\mathcal{M}^2 = \{S, R, C, c, P, v^2\}$ are both models for S5+SBF. We assume for reductio that $(\Sigma x)\Box\varphi x \leftrightarrow A$ is a theorem of S5+SBF supposing that A is *de dicto*.

But then, in view of the fact that the selected models are rigid S5+SBF models, for every nominal variable assignment σ and for every index u the following equivalences hold: $\mathcal{M}^1_\sigma \models_u (\Sigma x)\Box\varphi x$ iff $\mathcal{M}^1_\sigma \models_u A$ and $\mathcal{M}^2_\sigma \models_u (\Sigma x)\Box\varphi x$ iff $\mathcal{M}^2_\sigma \models_u A$. Now, since A is *de dicto* we obtain, by theorem 2.5.7, putting $u = s$ the following equivalence: $\mathcal{M}^1_\sigma \models_s A$ iff $\mathcal{M}^2_\sigma \models_s A$. But then we have, in view of the above equivalences, $\mathcal{M}^1_\sigma \models_s (\Sigma x)\Box\varphi x$ iff $\mathcal{M}^2_\sigma \models_s (\Sigma x)\Box\varphi x$ and thus a contradiction. For $\mathcal{M}^1_\sigma \models_s (\Sigma x)\Box\varphi x$, but $\mathcal{M}^2_\sigma \not\models_s (\Sigma x)\Box\varphi x$. So the difference between both models cannot be detected by a *de dicto* formula, but only by a *de nomine* formula. This shows that the *de nomine* modal formula $(\Sigma x)\Box\varphi x$ is not equivalent in the constant substitution class system S5+SBF (and so in weaker systems) to any *de dicto* formula.

Can a corresponding result for constant substitution class S5+SBF$\ddot{=}$ be obtained along these lines? It is easy to see that, in the present setting, the inductive step in which A is $o_1 \ddot{=} o_2$ will not go through. In order to prove a corresponding theorem for this system we would have to modify both models so as to obtain the result for that inductive step. In view of the truth conditions for substitutional identity formulae, such a modification would require that at each index in each of the models both $\varphi\alpha$ and $\varphi\beta$ come out true. Clearly, this modification would undermine the original strategy of anti-assignments, which required that the models differ in such a way that in the first model $\varphi\alpha$ be in the local associates for φ at both indices whereas in the second model $\varphi\alpha$ be in the associate for φ at s whereas $\varphi\beta$ be in its associate at t. In effect, such a modification would make both models collapse into a single one and would give us, contrary to what the strategy intended, the truth of $(\Sigma x)\Box\varphi x$ at s with respect to both models.

It might, therefore, seem that the model-theoretic characterization of *de* *dicto* formulae given by the Theorem 2.5.7 does not capture formulae of the form $o_1 \ddot{=} o_2$. Intuitively, though, formulae which are syntactically *de dicto*, and thus formulae of that very form, should have this feature also model-theoretically.

In order to adapt the strategy of anti-assignments also for formulae of the form $o_1 \ddot{=} o_2$ and to retain their *de dicto* character as captured by the theorem, we observe that the interpretation of the predicate φ in both models differs in the following respect. In the first model φ is, loosely speaking, essential to α, since $\varphi\alpha$ comes out true at every index at which α is contained in the substitution class of that index, whereas in the second this is not so.

Let us call, anticipating the terminology of Chapter 3 (though not the exact defintion to be given there) a predicate χ^n characteristic for a nominal constant γ_i with respect to a rigid constant substitution class model $\mathcal{M} = \langle S, R, C, c, P, v \rangle$ just in case for every $\chi^n\gamma_1...\gamma_n \in Atm(\chi^n)$ which is contained in $v(\gamma_i, u)$ at every $u \in S$ $\chi^n\gamma_1...\gamma_n$ is also contained in $v(\chi^n, u)$. The predicate φ is clearly not characteristic for α in $\mathcal{M}^2 = \langle S, R, C, c, P, v^2 \rangle$, whereas it is characteristic for it in $\mathcal{M}^1 = \langle S, R, C, c, P, v^1 \rangle$. Next we modify clause 2.4.2(2) as follows:

> 2.4.2($2^\dagger$): $\mathcal{M}_\sigma \models_s o_1 \ddot{=} o_2$ iff for all pure atomic formulae B_1 and B_2 where B_2 is like B_1 except for containing occurrences of the term o_2 at one or all places where B_1 contains the term o_1 and where the predicate in these formulae is characteristic for o_1 and o_2, respectively: $\mathcal{M}_\sigma \models_s B_1$ iff $\mathcal{M}_\sigma \models_s B_2$.

We now alter the two models used in the proof of the theorem letting $\mathcal{M}'^1$ and $\mathcal{M}'^2$ coincide in the following respects: $S = \{s, t\}$, $R = S \times S$, $c(s) = c(t) = C = \{\alpha, \beta\}$, $P = \{\varphi, \psi, \chi, \omega\}$, where all predicates are monadic, $v'^1(\alpha, s) = v^1(\alpha, t) = \{\varphi\alpha, \chi\alpha, \omega\alpha\}$ and $v'^1(\beta, s) = v^1(\beta, t) = \{\varphi\beta, \chi\beta, \omega\beta\}$, and, correspondingly, $v'^2(\alpha, s) = v'^2(\alpha, t) = \{\varphi\alpha, \chi\alpha, \omega\alpha\}$ and $v'^2(\beta, s) = v'^2(\beta, t) = \{\varphi\beta, \chi\beta, \omega\beta\}$. However, we let $\mathcal{M}'^1$ and $\mathcal{M}'^2$ differ with respect to the interpretation of the predicates. $\mathcal{M}'^1$: $v'^1(\psi, s) = v'^1(\psi, t) = \emptyset$, $v'^1(\varphi, s) = v'^1(\varphi, t) = \{\varphi\alpha\}$; and in addition to the original model we let $v'^1(\chi, s) = v'^1(\chi, t) = \{\chi\alpha, \chi\beta\}$, and $v'^1(\omega, s) = v'^1(\omega, t) = \{\omega\alpha, \omega\beta\}$. Whereas for $\mathcal{M}'^2$, we let as in the original model, $v'^2(\psi, s) = v'^2(\psi, t) = \emptyset$,

$v'^2(\varphi, s) = \{\varphi\alpha\}$ and $v'^2(\varphi, t) = \{\varphi\beta\}$, but require in addition to the original model, $v'^2(\chi, s) = v'^2(\chi, t) = \{\chi\alpha, \chi\beta\}$, and $v'^2(\omega, s) = v'^2(\omega, t) = \{\omega\alpha, \omega\beta\}$. So in effect the new models differ from the old ones in that they involve two additional predicates χ and ω which are characteristic to both α and β.

In order to prove Theorem 2.5.7 we proceed in a way analogous to the proof given above using the truth clauses from Subsection 2.2.4 except for applying clause 2.4.2($2^\dagger$) rather than 2.4.2(2) in the inductive step for $A = o_1 \ddot{=} o_2$.

First conjunct. $\mathcal{M}'^1_{\sigma'1} \models_s o_1 \ddot{=} o_2$ iff, by clause 2.4.2($2^\dagger$), for all pure atomic formulae B_1 and B_2 where B_2 is like B_1 except for containing occurrences of the term o_2 at one or all places where B_1 contains the term o_1 and where the predicate in these formulae is characteristic for o_1 and o_2, respectively: $\mathcal{M}'^1_{\sigma'1} \models_s B_1$ iff $\mathcal{M}'^1_{\sigma'1} \models_s B_2$ (where in this particular case the predicate which occurs in the formulae is either χ or ω). This condition holds iff $v'^1_{\sigma'1}(o_1, s) = v'^2_{\sigma'1}(o_1, s)$ and $v'^1_{\sigma'1}(o_2, s) = v'^2_{\sigma'1}(o_2, s)$ as well as $v'^1(\chi, s) = v'^2(\chi, s)$ and $v'^1(\omega, s) = v'^2(\omega, s)$. And so the above equivalence will hold iff for all pure atomic formulae B_1 and B_2 where B_2 is like B_1 except for containing occurrences of the term o_2 at one or all places where B_1 contains the term o_1 and where the predicate in these formulae is characteristic for o_1 and o_2, respectively: $\mathcal{M}'^2_{\sigma'1} \models_s B_1$ iff $\mathcal{M}'^2_{\sigma'1} \models_s B_2$; iff, by clause 2.4.2($2^\dagger$), $\mathcal{M}'^2_{\sigma'1} \models_s o_1 \ddot{=} o_2$.

Second conjunct. $\mathcal{M}'^1_{\sigma'1} \models_t o_1 \ddot{=} o_2$ iff, by clause 2.4.2($2^\dagger$), for all pure atomic formulae B_1 and B_2 where B_2 is like B_1 except for containing occurrences of the term o_2 at one or all places where B_1 contains the term o_1 and where the predicate in these formulae is characteristic for o_1 and o_2, respectively: $\mathcal{M}'^1_{\sigma'1} \models_t B_1$ iff $\mathcal{M}'^1_{\sigma'1} \models_t B_2$. (As before, the predicate which occurs in the formulae is either χ or ω.) The condition holds iff, by the definition of the notion of an anti-assignment and the fact that the predicates are characteristic, $v'^1_{\sigma'1}(o_1, t) = v'^2_{\sigma'2}(o_1, t)$ and $v'^1_{\sigma'1}(o_2, t) = v'^2_{\sigma'2}(o_2, t)$ as well as $v'^1(\chi, s) = v'^2(\chi, s)$ and $v'^1(\omega, s) = v'^2(\omega, s)$; iff for all pure atomic formulae B_1 and B_2 where B_2 is like B_1 except for containing occurrences of the term o_2 at one or all places where B_1 contains the term o_1 and where the predicate in these formulae is characteristic for o_1 and o_2, respectively: $\mathcal{M}'^2_{\sigma'2} \models_t B_1$ iff $\mathcal{M}'^2_{\sigma'2} \models_t B_2$; iff, by clause 2.4.2($2^\dagger$), $\mathcal{M}'^2_{\sigma'2} \models_t o_1 \ddot{=} o_2$.

2.6 Axiom Systems without SBF

2.6.1 Axiomatization

The systems we shall consider in this section are similar to the axiom systems discussed in the previous section. The only difference is that they do not contain SBF as a thesis (i.e., as an axiom or a theorem). We shall call these new systems SFOL$\ddot{=}$+Λ-systems. Since any normal system Λ which contains the axiom scheme B will have SBF as a theorem, we assume that Λ does not contain B.

2.6.2 Associative Semantics without SBF

To give an account of truth in a model for a system SFOL$\ddot{=}$+Λ which does not always validate SBF we shall employ rigid varying substitution class models. A model of this kind, $\mathcal{M} = \langle S, R, C, c, P, v\rangle$, is just the same kind of thing as a constant substitution class model except for the fact that constancy (i.e., the condition that $c(s) = C$ for every $s \in S$) need not hold. We assume that the models satisfy *monotonicity* (i.e., for all $s, t \in S$, if sRt then $c(s) \subseteq c(t)$). However, since SBF has to be invalidated in every such model, we require that these models violate *antimonotonicity*, that is, the condition that for all $s, t \in S$, if sRt then $c(t) \subseteq c(s)$. For this reason we shall call the models for the semantics which invalidates SBF, *non-antimonotonic models*. When R is symmetric SBF cannot be invalidated, since this feature of R and monotonicity would make the model constant. We therefore require that the frames be not symmetric (and thus frames invalidating the B scheme).

The clauses for truth at an index in a model remain the same as in Subsection 2.4.2, except for the clause for the substitutional universal quantifier. Adapting the usual denotational strategy we replace clause 2.4.2(5) by the following one:

> 2.4.2.(5′). $\mathcal{M}_\sigma \models_s (\Pi x)A$ iff for every (x, s)-variant τ of σ such that $\tau(x, s) \in c(s)$: $\mathcal{M}_\tau \models_s A$.

This clause restricts the evaluation of quantified formulae at an index to the nominal constants which are contained in the substitution class for that index. It will be noted that since the substitution class needs no longer to be C at every index, the first conjunct of the clause for pure atomic formulae, i.e., clause 2.4.2(1), is not satisfied in the trivial way as in constant substitution class models. So in evaluating the truth of pure atomic sentences at some index s we now cannot always take for granted that the nominal constants which occur in them are contained in $c(s)$.

The definition of truth in a model given in Subsection 2.4.5 is now to be replaced by the following definition, where the models, the variable assignments and their variants are taken to be rigid.

> 2.4.5$'$. A formula A of L is *true in an associative substitutional model* (formally, $\mathcal{M} \models A$) iff for every $s \in S$, $\mathcal{M}_\sigma \models_s A$ for every substitutional variable assignment σ such that $\sigma(x, s) \in c(s)$ for every variable x.

Validity in a frame is then defined in terms of this account of truth in a model in a way analogous to the definition given in Subsection 2.4.6.

2.6.3 Soundness without SBF

The proofs for the soundness of varying substitution class systems without SFB are essentially the same as the proofs for the soundness of the SFB systems presented in the previous section. The crucial point of difference is that the proofs now use rigid models which are non-antimonotonic.

1. The proof that every theorem of SFOL$\ddot{=}$+Λ is true in every model in which every instance of a theorem of Λ is true proceeds by induction on the proofs of theorems of SFOL$\ddot{=}$+Λ. The truth in a model of every substitution-instance of a theorem of Λ follows for the same reasons as for systems with SBF.

2. The proof for Π1 is essentially the same as for the constant substitution class system. Suppose that $\mathcal{M}_\sigma \models_s (\Pi x)A$ and let τ be the (x, s)-variant of σ such that $\tau(x, s) = \sigma(y, s)$. So since $\sigma(y, s) \in s$, $\tau(x, s) \in s$. And so, by the clause for the restricted universal quantifier 2.4.2(5$'$) $\mathcal{M}_\tau \models_s A$. And then, by MSPR, $\mathcal{M}_\sigma \models_s A[o/x]$. Hence Π1 is truth preserving in all non-antimonotonic models.

3. $\ddot{=}1$, $\ddot{=}2$, and MP preserve truth in non-antimonotonic models for the same reasons as in models for systems with SBF.

4. For Nec suppose that $\mathcal{M}_\sigma \not\models_s \Box A$ for some $s \in S$, where $\sigma(x,s) \in c(s)$ for every x. Then for some $t \in S$ such that sRt, $\mathcal{M}_\sigma \not\models_t A$. So since sRt, monotonicity ensures that $\alpha \in c(t)$ for every x. Hence A is false in that model as well. Note that if monotonicity fails, although $(\Pi x)Fx \rightarrow Fy$ will be true at an index in some model, $\Box((\Pi x)Fx \rightarrow Fy)$ can be false at that index.

5. Π2 preserves, in the presence of MSPA, truth in every non-antimonotonic model for the same reasons as in the semantics for systems with SBF.

Consequently, every theorem of SFOL$\ddot{=}$+Λ is valid according to the present criterion. However, as the example presented in 2.3.8(2) shows, given the obviuos adjustments to the semantics with variable assignments, SBF is not always a theorem of SFOL$\ddot{=}$+Λ.

2.6.4 Completeness without SBF

The associative canonical models for systems without SBF are construed, following the standard technique, using two languages, that is, L and L^+, where the latter has infinitely many fresh terms not in L.

We allow for variyng substitution classes by letting each index s in the associative canonical model be a maximal consistent set of formulae of L_s which contains all the terms of L and possibly some of the new terms of L^+, granted that there are infinitely many terms of L^+ not in L_s.

Using the terminology of [Hughes & Cresswell, 1996] we say that where A ⊆ B, A is an *infinitely proper* subset of B iff there are infinitely many members of B not in A.

To say that a language $\mathcal{L}$ is an infinitely proper sublanguage of a language $\mathcal{L}'$ is to say that $\mathcal{L}$ and $\mathcal{L}'$ contain the same predicates, and the terms of $\mathcal{L}$ are an infinitely proper subset of the terms of $\mathcal{L}'$.

We can prove the following existence theorem concerning maximal consistent sets of the sort just described.

Theorem 2.6.4(1) (Existence Theorem). If $\Box A \notin s$ then there is a maximal consistent set t with the Π-property in a language L_t containing L_s such that $\{B : \Box B \in s\} \cup \{\neg A\} \subseteq t$.

A reproduction of the standard proof of this theorem can be found in Appendix A.

Let Γ be a SFOL$\ddot{=}$+Λ-consistent set of L-formulae. A rigid associative canonical model for a system SFOL$\ddot{=}$+Λ in L with an extension L^+ is a 6-tuple $\mathcal{M}_\Gamma^\Lambda = \langle S^\Lambda, R^\Lambda, C^\Lambda, c^\Lambda, P^\Lambda, v^\Lambda \rangle$, where

1. S^Λ is the set of all maximal consistent sets $s, t, u, ...$ with the Π-property in some sublanguage L_s of L^+ which contains all the terms of L and perhaps some of the fresh terms from L^+, under the provisio that there are infinitely many terms of L^+ which are not contained in L_s.

2. $sR^\Lambda t$ iff $\{A : \Box A \in s\} \subseteq t$ iff for every formula $\Box A$ of L^+: if $\Box A \in s$, then $A \in t$.

3. C^Λ is C^+, that is to say the set of nominal constants of L_0^+.

4. $c^\Lambda : S^\Lambda \rightarrow \wp(C^\Lambda)$. So $c^\Lambda(s)$ is the set of nominal constants in L_s. Where $\alpha \in c(s)$ then $\Box(\varphi\alpha \rightarrow \varphi\alpha) \in s$ and so where $sR^\Lambda t$, $\varphi\alpha \rightarrow \varphi\alpha \in t$ and thus $\alpha \in c(t)$, i.e. L_s is a sublanguage of L_t when $sR^\Lambda t$ and so monotonicity is satisfied.

5. $P^\Lambda = P$.

6. v^Λ is defined along the lines of Subsection 2.5.6. For arbitrary $\alpha \in c^\Lambda(s)$, $\varphi^n \in P^\Lambda$, and $s \in S^\Lambda$:

   $$v^\Lambda(\alpha, s) =_{df} Atm^\Lambda(\alpha) \cap s\text{; and}$$
   $$v^\Lambda(\varphi^n, s) =_{df} Atm^\Lambda(\varphi^n) \cap s,$$

   where Atm^Λ is the set of pure atomic L_0^+-sentences and where $Atm^\Lambda(\alpha)$ and $Atm^\Lambda(\varphi^n)$ are defined as follows:

   $Atm^\Lambda(\alpha) =_{df} \{A \in Atm^\Lambda$: A contains at least one occurrence of the nominal constant $\alpha\}$

$Atm^{\Lambda}(\varphi^n) \quad =_{df} \quad \{A \in Atm^{\Lambda}$: A contains an occurrence of the predicate $\varphi^n\}$.

In effect, everything except for the fact that the models are non-antimonotonic remains as it was. Given an associative canonical model $\mathcal{M}^{\Lambda}_{\Gamma} = \langle S^{\Lambda}, R^{\Lambda}, C^{\Lambda}, c^{\Lambda}, P^{\Lambda}, v^{\Lambda}\rangle$ canonical variable assignments σ^{Λ}, their variants, and canonical term values are defined in the same way as in Subsection 2.5.6. Again, we assume that they and the canonical models under consideration are rigid.

We can now establish the following theorem concerning non-antimonotonic varying substitution class canonical models.

> *Theorem 2.6.4(2)* (Truth Theorem). For any $s \in S^{\Lambda}$, and any formula $A \in L^+$: $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \models_s A$ iff $A \in s$.

(The superscripted Λ is short for SFOL$\ddot{=} + \Lambda$.)

The proof is similar to the proof of theorem 2.5.6(4). But in making the inductive steps one has to appreciate the fact that the indices in S^{Λ} are all in different sublanguages of L^+.

(i) Pure atomic formulae. $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \models_s \varphi^n o_1 ... o_n$ iff, by clause 2.4.2(1), (a) if the terms $o_1, ..., o_n$ are nominal constant, then $o_1, ..., o_n \in c^{\Lambda}(s)$ and if they are nominal variables, then $\sigma^{\Lambda}(o_1, s), ..., \sigma^{\Lambda}(o_n, s) \in c^{\Lambda}(s)$ and (b) $\varphi^n o_1 ... o_n \in v^{\Lambda}_{\sigma^{\Lambda}}(o_1, s) \cap ... \cap v^{\Lambda}_{\sigma^{\Lambda}}(o_n, s) \cap v^{\Lambda}(\varphi^n, s)$; iff, in view of the definition of v^{Λ} and the definition of canonical term values under σ^{Λ}, by 2.4.2(1), $\varphi^n o_1 ... o_n \in s$.

(ii) Substitutional identity formulae. Again, the proof is essentially the same. Suppose $o_1 \ddot{=} o_2 \in L_s$. Then $B_1 \in L_s$ and $B_2 \in L_s$ for all pure atomic formulae of L_s B_1 and B_2 in Atm^{Λ} where B_2 is like B_1 except for containing occurrences of the term o_2 at one or all places where B_1 contains the term o_1. And then $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \models_s o_1 \ddot{=} o_2$ iff for all pure atomic formulae B_1 and B_2 where B_2 is like B_1 except for containing occurrences of the term o_2 at one or all places where B_1 contains o_1: $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \models_s B_1$ iff $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \models_s B_2$; iff for all B_1 and B_2: $B_1 \in s$ iff $B_2 \in s$; iff for all B_1 and B_2: $B_1 \leftrightarrow B_2 \in s$; iff, by $\ddot{=}$2, $o_1 \ddot{=} o_2 \in s$.

(iii) Negations. Suppose $\neg A \in L_s$. Then $A \in L_s$. And then $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \models_s \neg A$; iff $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \not\models_s A$; iff $A \notin s$; iff $\neg A \in s$. It is vital that $A \in L_s$ since first, the induction hypothesis only applies to such formulae. Further, since s is maximal consistent only in L_s, it is only if $A \in L_s$ that we can be sure that if $A \notin s$ then $\neg A \in s$.

(iv) Conjunctions. Suppose $A \wedge B \in L_s$. Then $A \in L_s$ and $B \in L_s$. Then $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \models_s A \wedge B$; iff $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \models_s A$ and $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \models_s B$; iff $A \in s$ and $B \in s$; iff $A \wedge B \in s$. Again, it is important that A, B, and $A \wedge B$, all be in L_s.

(v) Universal substitutional quantifications; first part. Suppose $(\Pi x)A \in L_s$ and $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \not\models_s (\Pi x)A$. Then for some (x, s)-variant τ^{Λ} of σ^{Λ}, such that $\tau^{\Lambda}(x, s) \in c(s)$, $\mathcal{M}^{\Lambda}_{\Gamma,\tau^{\Lambda}} \not\models_s A$. So by MSPR for some term o in L^+, $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \not\models_s A[o/x]$. So $A[o/x] \notin s$. Now, $A[o/x] \in L_s$ and so, by $\Pi 1$, $(\Pi x)A \notin s$.

(vi) Universal substitutional quantifications; second part. Suppose $(\Pi x)A \in L_s$, but $(\Pi x)A \notin s$ and so $\neg(\Pi x)A \in s$. Thus, since s has the Π-property in L_s, there is some term o in L_s such that $\neg A[o/x] \in s$. Consequently, $A[o/x] \notin s$. And since $A[o/x] \in L_s$, $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \not\models_s A[o/x]$. So, since $\Pi 1$ is valid, $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \not\models_s (\Pi x)A$.

(vii) Necessity formulae; first part. Suppose $\Box A \in s$ and $sR^{\Lambda}t$. Then $A \in t$. So $A \in L_t$. And thus $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \models_t A$. Since this is so for every t such that $sR^{\Lambda}t$, $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \models_s \Box A$.

(viii) Necessity formulae; second part. Suppose $\Box A \in L_s$ but $\Box A \notin s$. Then since s is maximal consistent in L_s, $\neg\Box A \in s$. But then, by the Existence Theorem 2.6.4(1) and Lindenbaum's Theorem 2.5.6(1), there is some $t \in S^{\Lambda}$ with the Π-property such that $\neg A \in t$. So $\neg A \in t$. So $A \notin t$. But L_t is an extension of L_s and $A \in L_s$. So $A \in L_t$. So $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \not\models_t A$. But $sR^{\Lambda}t$, and so $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \not\models_s \Box A$.

This completes the proof of the Truth Theorem 2.6.4(2). Completeness follows just like in Theorem 2.5.6(5).

2.7 Axiom Systems without CSBF

2.7.1 Semantics without CSBF

CSBF is invalidated in models which do not satisfy monotonicity (see, for instance, the countermodel in 2.3.8(3)).

To set out a semantics for varying substitution class axiom systems without CSBF we introduce a new predicate $\ddot{E}$ for *substitutional existence*. This predicate is impure and so does not belong to P. Its semantics is given by the following clause:

2.7.1(1). $\mathcal{M}_\sigma \models_s \ddot{E}o$ iff $\mathcal{M}_\sigma \models_s (\Sigma x)o \ddot{=} x$.

It is somewhat misleading to call $\ddot{E}$ an existence predicate, since its truth clause is not given in terms of the referential existential quantifier, referential identity and individuals. It will be obvious, therefore, that $\ddot{E}$ has no existential import, but we shall continue to speak of substitutional existence in order to parallel the well-established terminology of denotational (modal) semantics.

The notion of truth at an index in a model is defined just like in Subsection 2.4.2 except for being supplemented with clause 2.7.1(1). Truth in a model is defined as in Subsection 2.4.5 and validity in a frame is explained using that definition as in Subsection 2.4.6.

The notion of a variable assignment has beed defined (in Subsection 2.4.1) in such a way as to allow for the case that $\sigma(x, s) = \alpha$ where $\alpha \notin c(s)$. This means that a nominal variable can be assigned a name at an index which is not contained in the substitution class for that very index.

The rest of the semantics for such systems is specified in the same way as for systems without SBF. In particular, the clauses for the substitutional quantifier are 2.4.2(5′) (see Subsection 2.6.2) and

2.4.2(9′). $\mathcal{M}_\sigma \models_s (\Sigma x)A$ iff for some (x, s)-variant τ of σ such that $\tau(x, s) \in c(s)$: $\mathcal{M}_\tau \models_s A$.

It is just this reading of the existential substitutional quantifier which is involved in the clause for $\ddot{E}$ given above.

We observe that in view of the fact that the quantifiers are constrained to the substitution classes at indices, but variable assignments are not, Π1 is no longer valid. Consider a simple instance of that axiom, $(\Pi x)Fx \to Fy$, and the rigid non-monotonic model $\mathcal{M} = \langle S, R, C, c, P, v\rangle$ where $S = \{s, t\}$, $R = \{\langle s, t\rangle\}$, $C = \{a, b\}$, $c(s) = C$, $c(t) = \{a\}$, $v_\sigma(a, s) = v_\sigma(a, t) = \{Fa\}$,

$v_\sigma(b,s) = v_\sigma(b,t) = \{Fb\}$, $v(F,s) = \{Fa, Fb\}$ and $v(F,t) = \{Fa\}$. Let $\sigma(y,t) = b$. Clearly, $\mathcal{M}_\sigma \models_t (\Pi x)Fx$ and $\mathcal{M}_\sigma \not\models_t Fy$. So $\Pi 1$ is invalidated.

Moreover, Nec is no longer truth preserving as this model shows. For $\mathcal{M}_\sigma \models_s (\Pi x)Fx \rightarrow Fy$, but $\mathcal{M}_\sigma \not\models_s \Box((\Pi x)Fx \rightarrow Fy)$.

Since we have the predicate for substitutional existence at our disposal we may replace $\Pi 1$ by

$$\Pi 1\ddot{E} \quad ((\Pi x)A \wedge \ddot{E}o) \rightarrow A[o/x].$$

This axiom is valid as we shall see below and Nec regains the property of being truth preserving.

2.7.2 Semantics without CSBF: Comments

1. Clause 2.7.1(1) is, in effect, an abbreviation of the following, somewhat loosely stated, clause: $\mathcal{M}_\sigma \models_s \ddot{E}o$ iff there is some term p, which is not necessarily distinct from o, such that if p is a nominal constant then $p \in c(s)$ and if p is a nominal variable then $\sigma(p,s) \in c(s)$: $\varphi^n(o, q_1, ..., q_{n-1})$ and $\varphi^n(p, q_1, ..., q_{n-1})$ (where φ^n is an arbitrary pure predicate and $q_1, ..., q_{n-1}$ are arbitrary nominal terms), $\varphi^n(o, q_1, ..., q_{n-1}) \in v_\sigma(o,s) \cap v_\sigma(q_1,s) \cap ... \cap v_\sigma(q_{n-1},s) \cap v(\varphi^n,s)$ iff $\varphi^n(p, q_1, ..., q_{n-1}) \in v_\sigma(p,s) \cap v_\sigma(q_1,s) \cap ... \cap v_\sigma(q_{n-1},s) \cap v(\varphi^n,s)$.

2. The usual clause for the referential existence predicate E says that an individual is an element of the extension of E at some index just in case that individual is an element of the individual domain for that index. In effect, a formula of the form $E\underline{o}$, where $\underline{o}$ is an individual term, is said to be true at an index exactly if that term is assigned an element of the individual domain, its denotatum, for that index. The predicate E and denotational counterparts of such axioms like $\Pi 1\ddot{E}$ are used by free logicians in order to tackle problems concerning the semantics and the logic of sentences containing denotationless terms. Since in the present semantical framework there just are no denotations at all, there is, as it seems to me, no deep philosophical point in introducing $\ddot{E}$ and such systems like the ones presented below. We shall return to the

problems of nondenoting terms and negative existentials in Chapter 4, after the applied semantical interpretation of our framework has been provided. Indeed, the present framework achieves, as I hope, something free logicians seem, among other things, to be aiming at, namely to defend the "purity" of first-order logic.[9]

2.7.3 Axiomatization

The replacement of $\Pi 1$ with $\Pi 1\ddot{E}$ gives rise to some changes in the axiom basis of SFOL. In particular, $\Pi 2$ can no longer be used in its present shape. The following axiomatization recasts the one presented in [Hughes and Cresswell, 1996] on pages 293-295 in substitutional terms.

Axioms

Where $\mathit{\Lambda}$ is any normal system of propositional modal logic, SFOL$\ddot{=}\ddot{E}+\mathit{\Lambda}$ is defined as follows.

1. Λ': Any SFOL substitution-instance of a theorem of $\mathit{\Lambda}$ is an axiom of SFOL$\ddot{E}+\mathit{\Lambda}$.

2. $\Pi 1\ddot{E}$: Where x is any nominal variable and o any nominal term, and A any formula then $((\Pi x)A \wedge \ddot{E}o) \rightarrow A[o/x]$ is an axiom of SFOL$\ddot{E}+\mathit{\Lambda}$.

3. $\Pi^{\rightarrow}$: $(\Pi x)(A \rightarrow B) \rightarrow ((\Pi x)A \rightarrow (\Pi x)B)$ (where A and B are any formulae and x is any substitutional variable).

4. VQ: $A \leftrightarrow (\Pi x)A$ provided x is not free in A.

5. $U\ddot{E}$: $(\Pi x)\ddot{E}x$

6. $\ddot{=}1$ and $\ddot{=}2$ (see Subsection 2.5.3).

[9] For a recent survey on free logics see [Bencivenga, 2002].

Rules

The rules are MP, Nec and the following ones:

1. *UG*: if $\vdash A$, then $\vdash (\Pi x)A$.

2. $UG\Box\Pi^n$: if $\vdash A_1 \to \Box(A_2 \to ... \to \Box(A_n \to \Box B)...)$, then $\vdash A_1 \to \Box(A_2 \to ... \to \Box(A_n \to \Box(\Pi x)B)...)$, where x is not free in $A_1, ..., A_n$.

Theorems and Derived Rules

1. $\Pi 2$: if $\vdash A \to B$ and x is not free in A then $\vdash A \to (\Pi x)B$.

2. $UG^{\to}$: if $\vdash A \to B$ then $\vdash (\Pi x)A \to (\Pi x)B$.

3. *Eq*: if $\vdash A \leftrightarrow B$ and $C[A]$ differs from $C[B]$ only in having A at zero or more places where $C[B]$ has B then $\vdash C[A] \leftrightarrow C[B]$.

4. $\Pi 1'$: Where x and y are any nominal variables, and A is any formula then $\vdash (\Pi y)((\Pi x)A \to A[y/x])$.

5. *RBV*: If A and B differ only in that A has free x where and only where B has free y then $\vdash (\Pi x)A \leftrightarrow (\Pi y)B$.

6. *QR*: $\neg(\Pi y)\neg(A[y/x] \to (\Pi x)A)$.

7. $VQ^{\to}$: $(\Pi x)(A \to B) \to (A \to (\Pi x)B)$, where x is not free in A.

The first three rules are derivable from UG, VQ and $\Pi^{\to}$. $\Pi 1'$ is derivable using these rules. With their help RBV and QR can be proved. The proof of the last theorem in this list uses $\Pi^{\to}$ and VQ. (See [Hughes & Cresswell, 1996] pp. 293-295.)

2.7.4 Soundness without CSBF

We can easily verify that the axiomatization given above is sound with respect to the definition of validity stated in Subsection 2.4.6. We shall confine ourselves to the proofs of the axioms $\Pi 1\ddot{E}$ and $U\ddot{E}$ and of the rule $\text{UG}\Box\Pi^n$. Because of the semantics $\ddot{E}$, the proofs of the axioms differ somewhat from their denotational counterparts. The models considered are rigid.

1. Proof for $\Pi 1\ddot{E}$. The proof is by induction on the complexity of A. In effect, the proof follows, *mutatis mutandis*, the general strategy outlined in 2.5.5(5) which has been discussed in the context of the proof for axiom $\ddot{=}3$. We proceed by reductio and consider only the case where A is a pure atomic formula. Suppose $\mathcal{M}_\sigma \models_s (\Pi x)A$, $\mathcal{M}_\sigma \models_s \ddot{E}o$, but $\mathcal{M}_\sigma \not\models_s A[o/x]$. Let τ be the (x, s)-variant of σ such that $\tau(x, s) = \sigma(y, s)$. Then, by MSPR, $\mathcal{M}_\tau \not\models_s A$. Now $\mathcal{M}_\sigma \models_s \ddot{E}o$ and so, by 2.7.1(1), $\mathcal{M}_\sigma \models_s (\Sigma x)o\ddot{=}x$. And so, by 2.4.2.(9′), for some (x, s)-variant ρ of σ such that $\rho(x, s) \in c(s)$: $\mathcal{M}_\rho \models_s o\ddot{=}x$. From this we obtain, in view of clause 2.4.2(2), for some (x, s)-variant ρ of σ such that $\rho(x, s) \in c(s)$: for all pure atomic formulae $B(o)$ and $B(x)$ where $B(x)$ is like $B(o)$ except for containing occurrences of x at one or all places where $B(o)$ contains the term o: $\mathcal{M}_\rho \models_s B(o)$ iff $\mathcal{M}_\rho \models_s B(x)$. In order to construct the contradiction we let ρ be τ, we let o be x and assume, that $B(o)$ is just the atomic formula A. From this we obtain, using the fact that, $\mathcal{M}_\tau \not\models_s A$ the following: there is an (x, s)-variant of σ, that is, τ, such that $\tau(x, s) \in c(s)$ and $\mathcal{M}_\tau \not\models_s A$. Consequently, it is not the case that for every (x, s)-variant τ of σ such that $\tau(x, s) \in c(s)$ $\mathcal{M}_\tau \models_s A$. Which, by clause 2.4.2.(5′), gives us $\mathcal{M}_\sigma \not\models_s (\Pi x)A$ and thus the required contradiction. The inductive step for the case in which A is a substitutional identity formula is similar. For the complex cases we proceed using the strategy outlined in 2.5.5(5).

2. Proof for $U\ddot{E}$. Suppose $\mathcal{M}_\sigma \not\models_s (\Pi x)\ddot{E}x$. So, by clause 2.4.2.(5′), for every (x, s)-variant τ of σ such that $\tau(x, s) \in c(s)$: $\mathcal{M}_\tau \not\models_s \ddot{E}x$. Thus, by clause 2.7.1(1), $\mathcal{M}_\tau \not\models_s (\Sigma x)x\ddot{=}x$ which, in view of clause 2.4.2(2), is absurd.

3. Proof for UG$\Box\Pi^n$. The proof parallels the denotational proof (cf. [Hughes & Cresswell, 1996] p. 295). Suppose $\mathcal{M}_\sigma \models_s A_1 \rightarrow \Box(A_2 \rightarrow ... \rightarrow \Box(A_n \rightarrow \Box(\Pi x)B) ...)$. Then there is an R-chain $s_1, ..., s_{n+1}$ with $s = s_1$ and $\mathcal{M}_\sigma \models_{s_i} A_i$ for $1 \leq i \leq n$, and $\mathcal{M}_\sigma \not\models_{s_{n+1}} (\Pi x)B$, and so for some (x, s)-variant τ of σ with $\tau(x, s_{n+1}) \in c(s_{n+1})$, $\mathcal{M}_\tau \not\models_{s_{n+1}} B$. But x is not free in A and so, by MSPA, both $\mathcal{M}_\sigma \models_{s_i} A_i$ and $\mathcal{M}_\tau \models_{s_i} A_i$. And so $\mathcal{M}_\tau \not\models_s A_1 \rightarrow \Box(A_2 \rightarrow ... \rightarrow \Box(A_n \rightarrow \Box B)...)$.

2.7.5 Completeness without CSBF

Following the standard procedure (see [Hughes & Cresswell, 1996] pp. 296-301) we shall now prove completeness for some arbitrarily selected system SFOL$\ddot{=}\ddot{E}+\Lambda$. We proceed as in Subsection 2.5.6 and assume that Θ is a consistent set of L-formulae and that L is an infinitely proper sublanguage of L^+. Here, however, the elements of S^Λ have to be maximal and are required to have the $\Box\Pi$-property in L^+. Let Δ be a set of formulae of L^+. Δ will have the $\Box\Pi$-property in L^+ just in case it satisfies the following two conditions:

1. For every formula A of L^+ and variable x there is some nominal term o in L^+ such that $\ddot{E}o \wedge (A[o/x] \rightarrow (\Pi x)A) \in \Delta$.

2. For all formulae of L^+, $B_1, ..., B_n$ $(n \geq 0)$ and A, and every variable x not free in $B_1, ..., B_n$ there is some term p in L^+ such that $\Box(B_1 \rightarrow ... \rightarrow \Box(B_n \rightarrow \Box(\ddot{E}p \rightarrow A[p/x]))...) \rightarrow \Box(B_1 \rightarrow ... \rightarrow \Box(B_n \rightarrow \Box(\Pi x)A)...) \in \Delta$.

We may now prove the following theorem:

> *Theorem 2.7.5(1)* ($\Box\Pi$-property). If Θ is a consistent set of formulae of L then there is a consistent set Δ of formulae of L^+ with the $\Box\Pi$-property, such that $\Theta \subseteq \Delta$.

The proof is standard and is relegated to Appendix A.

We can now also establish the following existence claim:

> *Theorem 2.7.5(2)* (Existence Theorem). If s is a maximal consistent set of formulae in L^+, and s has the $\Box\Pi$-property, and A is a formula such that $\Box A \notin s$, then there is a consistent set t of formulae of L^+ with the $\Box\Pi$-property such that $\{B : \Box B \in s\} \cup \{\neg A\} \subseteq t$.

The proof is reproduced in Appendix A.

Letting Γ be a SFOL$\ddot{=}\ddot{E}+\Lambda$-consistent set of L-formulae. We define a rigid associative canonical model for L with the extension L^+, $\mathcal{M}_\Gamma^\Lambda =$

$\langle S^{\Lambda}, R^{\Lambda}, C^{\Lambda}, c^{\Lambda}, P^{\Lambda}, v^{\Lambda}\rangle$, as before taking the superscripted Λ to be short for SFOL$\ddot{=}\ddot{E}$+Λ, just like in Subsection 2.5.6 except that we require that for arbitrary $\alpha \in c^{\Lambda}(s)$, $\varphi^n \in P$, and $s \in S$ also the following condition to hold with respect to v^{Λ}:

> $\ddot{E}\alpha \in s$ iff there is some $\beta \in c^{\Lambda}(s)$, where α and β need not be distinct, such that for all sentences in Atm^{Λ} $\varphi^n(\alpha, \gamma_1, ..., \gamma_{n-1})$ and $\varphi^n(\beta, \gamma_1, ..., \gamma_{n-1})$ (where $\gamma_1, ..., \gamma_{n-1}$ are arbitrary nominal constants), $\varphi^n(\alpha, \gamma_1, ..., \gamma_{n-1}) \in v^{\Lambda}(\alpha, s) \cap v^{\Lambda}(\gamma_1, s) \cap ... \cap v^{\Lambda}(\gamma_{n-1}, s) \cap v^{\Lambda}(\varphi^n, s)$ iff $\varphi^n(\beta, \gamma_1, ..., \gamma_{n-1}) \in v^{\Lambda}(\beta, s) \cap v^{\Lambda}(\gamma_1, s) \cap ... \cap v^{\Lambda}(\gamma_{n-1}, s) \cap v^{\Lambda}(\varphi^n, s)$.

This condition uses the observation made in 2.7.2(1). In effect, the right-hand side of that condition is designed so as to ensure that the right-hand side of clause 2.7.1.(1) is satisfied.

The canonical assignments to the variables, their variants, and term values are defined as in Subsection 2.5.6 and are taken to be rigid.

We are now in a position to prove the following theorem.

> *Theorem 2.7.5(3)* (Truth Theorem). For any $s \in S^{\Lambda}$ and any formula A in L^+, $\mathcal{M}^{\Lambda}_{\Gamma} \models_s A$ iff $A \in s$.

The proof differs from the proof of Truth Theorem 2.5.6(4) only with respect to additional inductive step for substitutional existence formulae and with respect to the inductive step for quantified formulae.

With respect to substitutional existence the following equivalences hold: $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \models_s \ddot{E}o$ iff, by clause 2.7.1(1), $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \models_s (\Sigma x)o\ddot{=}x$; iff, by clause 2.4.2(9′), for some (x, s)-variant τ^{Λ} of σ^{Λ} such that $\tau^{\Lambda}(x, s) \in c^{\Lambda}(s)$: $\mathcal{M}^{\Lambda}_{\Gamma,\tau^{\Lambda}} \models_s o\ddot{=}x$; iff, by clause 2.4.2(2), for some (x, s)-variant τ^{Λ} of σ^{Λ} such that $\tau^{\Lambda}(x, s) \in c^{\Lambda}(s)$: for all pure atomic formulae $B(o)$ and $B(x)$ where $B(x)$ is like $B(o)$ except for containing occurrences of the variable x at one or all places where $B(o)$ contains the term o: $\mathcal{M}^{\Lambda}_{\Gamma,\tau^{\Lambda}} \models_s B(o)$ iff $\mathcal{M}^{\Lambda}_{\Gamma,\tau^{\Lambda}} \models_s B(x)$; iff, by MSPR, letting σ^{Λ} be any (x, s)-variant of τ^{Λ} such that $\sigma^{\Lambda}(x, s) \in c^{\Lambda}(s)$ which means that $\sigma^{\Lambda}(x, s) = p$ for some term p in L^+, for all pure atomic formulae $B(o)$ and $B[p/x]$ where $B[p/x]$ is like $B(o)$ except for containing occurrences of the term p at one or all places where $B(o)$ contains the term o: $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \models_s B(o)$ iff $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \models_s B[p/x]$; iff, making this abbreviation explicit, there is some term p, which is not necessarily distinct from o, such that if p is a nominal constant then $p \in c^{\Lambda}(s)$ and if p is a nominal variable then $\sigma^{\Lambda}(p, s) \in c^{\Lambda}(s)$: $\varphi^n(o, q_1, ..., q_{n-1})$ and $\varphi^n(p, q_1, ..., q_{n-1})$ (where φ^n is an arbitrary pure predicate and $q_1, ..., q_{n-1}$ are arbitrary nominal terms), $\varphi^n(o, q_1, ..., q_{n-1}) \in v^{\Lambda}_{\sigma^{\Lambda}}(o, s) \cap v^{\Lambda}_{\sigma^{\Lambda}}(q_1, s) \cap ... \cap v^{\Lambda}_{\sigma^{\Lambda}}(q_{n-1}, s) \cap v^{\Lambda}(\varphi^n, s)$ iff $\varphi^n(p, q_1, ..., q_{n-1}) \in v^{\Lambda}_{\sigma^{\Lambda}}(p, s) \cap v^{\Lambda}_{\sigma^{\Lambda}}(q_1, s) \cap ... \cap v^{\Lambda}_{\sigma^{\Lambda}}(q_{n-1}, s) \cap v^{\Lambda}(\varphi^n, s)$; iff, by the condition which has been imposed above on the definition of v^{Λ}, $\ddot{E}o \in s$.

If $(\Pi x)A \notin s$, then by the $\Box\Pi$-property there is some term o such that $\ddot{E}o \in s$ and $A[o/x] \notin s$. Thus, by hypothesis, $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \not\models_s A[o/x]$. From this we obtain, letting τ^{Λ} be the (x, s)-variant of σ^{Λ} with $\tau^{\Lambda}(x, s) = \sigma^{\Lambda}(y, s)$, $\mathcal{M}^{\Lambda}_{\tau^{\Lambda}} \not\models_s A$. Now, since $\ddot{E}o \in s$, by the condition on v^{Λ} there will be some term in $c^{\Lambda}(s)$ which ensures that this is so. Let $\sigma^{\Lambda}(y, s)$ be that term. But then, by clause 2.4.2(5′), $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \not\models_s (\Pi x)A$.

Now, suppose $(\Pi x)A \in s$ and let τ^{Λ} be any (x, s)-variant of σ^{Λ} such that $\tau^{\Lambda}(x, s) = \sigma^{\Lambda}(y, s)$ for some $\sigma^{\Lambda}(y, s) \in c^{\Lambda}(s)$. But then there will be a nominal constant in $c^{\Lambda}(s)$ which will satisfy the condition for $\ddot{E}o \in s$ to hold. By $\Pi 1\ddot{E}$, we then obtain $A[o/x] \in s$. And so, by hypothesis, $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \models_s A[o/x]$. From this we obtain, by MSPR, $\mathcal{M}^{\Lambda}_{\Gamma,\tau^{\Lambda}} \models_s A$ and then, by clause 2.4.2(5′), $\mathcal{M}^{\Lambda}_{\Gamma,\sigma^{\Lambda}} \models_s (\Pi x)A$.

A consequence of this theorem is that the canonical model of SFOL$\ddot{=}\ddot{E}+\Lambda$ validates all and only theorems of SFOL$\ddot{=}\ddot{E} + \Lambda$. This is shown just like in the case of theorem 2.5.6(5).

Chapter 3

Constrained Associative Semantics

3.1 Introduction

For the purposes of an applied semantical interpretation of the formal associative framework we shall impose in this chapter admissibility constraints on associative models, in particular, on the assignments of associates to nominal constants and pure predicates. In Section 3.2 we explain what makes an associative model admissible and natural. And in Section 3.3 we adjust the semantics so as to see how the problem of essential asymmetrical relations as recently considered in [Fine, 1994] could be solved in terms of the associative framework.

3.2 Admissibility Constraints

We first introduce a couple of auxiliary notions and then say in terms of these notions what admissible assignments are and what makes an associative model admissible.

3.2.1 Auxiliary Notions

In order to explain the notion of an admissible assignment we introduce a couple of auxiliary notions: defining associates and predicates for nominal constants, consequential/conforming predicates of a predicate with respect to a nominal constant.

1. For every nominal constant $\alpha \in C$ there is a *defining associate for the nominal constant* α, $v_{def}(\alpha)$, which is the set of all L_0-sentences from $Atm(\alpha)$ which we call "defining of" α. Note that $v_{def}(\alpha)$ is not local. Defining name associates can be empty.

2. For every nominal constant $\alpha \in C$ which has a defining associate, $Def(\alpha) \subseteq P$ is the set of all the pure predicates occurring in the sentences in $v_{def}(\alpha)$. If $Def(\alpha) = \{\varphi, \chi, \psi, ...\}$, we say that $\varphi, \chi, \psi, ...$ are the *defining predicates for the nominal constant* α.

3. Every pure predicate φ in P has a *formal meaning postulate*, $Mp(\varphi)$, associated with it. $Mp(\varphi)$ is a sentence of L. Moreover, we put $Mp =_{def} \{Mp(\varphi) : \varphi \in P\}$. A meaning postulate for φ determines (a) which predicates $\chi \in P$ are consequential upon φ with respect to some nominal constant α and (b) which predicates $\chi \in P$ conform to φ with respect to it.

   (a) A *predicate* χ *is consequential upon* φ *with respect to a nominal constant* α just in case, if $\varphi...\alpha... \in v_{def}(\alpha)$, then $\chi...\alpha...$ is derivable given Mp. We assume that the relavant notion of derivability is relativized to a suitably characterized meaning calculus which involves the postulates in Mp. For example, if $Fa \in v_{def}(a)$, then Ga is derivable given $Mp(F) = (\Pi x)(Fx \leftrightarrow (Gx \wedge \neg Hx))$. Thus the predicate G is consequential upon F with respect to a. According to $Mp(F)$, G is the only predicate in P which is consequential upon F with respect to any nominal constant in C. It is natural to require the calculus to guarantee that F be consequential upon itself.

   (b) A *predicate* χ *conforms to* φ *with respect to a nominal constant* α just in case, if $\varphi...\alpha... \in v_{def}(\alpha)$, then $\chi...\alpha...$ is derivationally consistent with Mp. So, for instance, if $Fa \in v_{def}(a)$ and $Mp(F)$ is the L-sentence $(\Pi x)(Fx \leftrightarrow (Gx \wedge \neg Hx))$, then Ha is not consistent with $Mp(F)$. Consequently, H does not conform to F with respect to a. Thus according to $Mp(F)$, all predicates in P except for H conform to F with respect to any nominal constant in C.

3.2.2 Auxiliary Notions: Comments

Some comments on the auxiliary notions just introduced will now be in place.

1. Informally, we may view—until the applied semantical interpretation will be provided in the next chapter—the defining associates for nominal constants as encoding the meanings of (denoting or denotationless) proper names as specified by their nominal definitions. As for the predicates which are defining of a nominal constant, we may view them as stating what is definitionally individuative of those names. As mentioned above, we take it that every nominal constant has a defining associate and thus a set of predicates which individuates it. In case the defining associate of a constant is empty, so is, of course, the set of its defining predicates. Anticipating the terminology to be introduced in Chapter 4 we shall find it sometimes convenient, especially when intuitive explanations are appropriate, to speak of the "sense-extensions of names/predicates" rather than of associates for nominal constants/predicates.

2. The explanations of the notions of a defining associates and of defining predicates for a name resemble to a certain extent the characterization of essence which has been recently suggested by Kit Fine.[1] According to Fine, "we may identify the being or the essence of x with the collection of propositions that are true in virtue of its identity (or with the corresponding collection of essential properties)".[2] The passage which is enclosed in brackets parallels, as I take it, the definition of the set of defining predicates and the portion of the quotation which precedes that passage parallels the notion of a defining associate for a nominal constant. However, the differences are significant, whereas Fine deals

[1] See [Fine, 1995a] p. 275.

[2] This use of 'essence' corresponds to what we shall later call the 'essence-set' of an object. See Subsection 3.3.3 below.

(intuitively) with objects, properties, and essence (or real definition), we deal (intuitively) with names, predicates, and meaning (or nominal definition). Moreover, whereas the propositions contained in Fine's collection of propositions are taken to be true, the sentences contained in defining associates serve to explain truth. These contrasts will be explained further in Chapter 4.

3. It will be noted that, in view of the fact that defining predicates are atomic, also such trivial logical predicates like $Fx \vee \neg Fx$ are not defining of nominal constants. Furthermore, it will have been noted that the predicates upon which a predicate can be consequential or to which it may conform with respect to a nominal constant are the defining predicates for that constant.

4. Informally, the meaning postulates can be taken to specify the actual meanings of (partially) definable natural language predicates and their analytic interrelations.[3] In the present framework they help to determine in a purely syntactical way how the associates of nominal constants and predicates are structured internally, i.e., which pure atomic sentences they contain. The condition expressed by a meaning postulate for a predicate may be a necessary and sufficient one (as the one given above), but it need not. Intuitively, it will be natural to hold, for instance, that the postulate for '... is a bachelor', say, will be of the former kind, whereas the postulate for '... is red' will state merely some necessary condition.

5. The notion of derivability and consistency involved in the definition of consequential and conforming predicates, respectively, can be specified in various ways. One way would be to use a notion of derivability that is relativized to a classical natural deduction system for the substitutional language L (without substitutional identity) which is enriched by the meaning postulates from Mp. For example, a sentence A from $Atm(\alpha)$ will be in $v_{cns}(\alpha)$, for some $\alpha \in C$, just in case there is a derivation in that system in which the elements from $v_{def}(\alpha, s)$ and the postulates

[3] See [Carnap, 1956 (1947)], Supplement B.

for the defining predicates for α from Mp are the assumptions and A is the conslusion. In case there is a derivation of $\neg A$, A is not contained in $v_{cns}(\alpha)$.[4]

3.2.3 Admissible Assignments

We distinguish admissible assignments to nominal constants and admissible assignments to pure predicates.[5]

1. Let $\mathcal{M} = \langle S, R, C, c, P, v \rangle$ be an associative substitutional model, let α be any nominal constant in C and let s be any index in S. An *associate assignment to a nominal constant* α at s is *admissible* just in case the following conditions are satisfied:

   (a) The resulting local associate $v(\alpha, s)$ for α contains the defining associate for α, that is, $v_{def}(\alpha)$.

   (b) The resulting local associate $v(\alpha, s)$ for α contains all sentences from $Atm(\alpha)$ which are derivable from $v_{def}(\alpha)$ given the meaning postulates Mp.

   We now introduce some terminology. Let $v_{cns}(\alpha)$ be that subset of all the sentences from $v(\alpha, s)$ which contain occurrences of the predicates χ from P which are consequential upon the predicates in $Def(\alpha)$ with respect to α. We call $v_{cns}(\alpha)$ the *consequential associate for the nominal constant* α. $Cns(\alpha)$ is the set of all the predicates which occur in the sentences of $v_{cns}(\alpha)$ which are not already contained in $Def(\alpha)$. We call these predicates the *consequential predicates for* α. So not all predicates which are consequential upon a defining predicate with respect to some nominal constant are consequential predicates for that

[4] An alternative way to explain the notion of derivability which is involved in the definition of consequential/conforming predicates and associates might proceed, perhaps, in terms of a purely syntactical and—for the present setting—nonconstructive adaptation of a meaning calculus of the sort presented in [Kamlah & Lorenzen, 1996 (1967)]. Much more needs to be said on this, though.

[5] I am indebited to Peter Schroeder-Heister for drawing my attention to an unintended consequence of an earlier formulation of definition 3.2.3(1).

constant. Obviously, this is so for the defining predicates themselves. We call the set $v_{chr}(\alpha) = v_{def}(\alpha) \cup v_{cns}(\alpha)$ the *characteristic associate for the nominal constant* α and $Chr(\alpha) = Def(\alpha) \cup Cns(\alpha)$ the set of the *characteristic predicates for* α. In effect, the constraint just stated says that an assignment of a local associate for a nominal constant α at an index $s \in S$, i.e. $v(\alpha, s)$, is admissible just in case $v_{chr}(\alpha) \subseteq v(\alpha, s)$.

2. Let $\mathcal{M} = \langle S, R, C, c, P, v \rangle$ be an associative substitutional model, let α be any nominal constant in C and let s be any index in S. Given the notion of a characteristic associate for any $\alpha \in C$ we say that a *associate assignment to a predicate* $\varphi \in P$ at s is *admissible* just in case the resulting local predicate associate $v(\varphi, s)$ for φ contains all the L_0-sentences from $Atm(\varphi)$ which are contained in the characteristic associates $v_{chr}(\alpha)$ for any $\alpha \in C$.

In Subsection 3.2.5 below these notions will be used in the definition of the notion of an admissible model.

3.2.4 Admissible Assignments: Comments

Before we state the definition of an admissible model, we introduce some further terminology and give a couple of comments on admissible assignments.

1. We call a model $\mathcal{M} = \langle S, R, C, c, P, v \rangle$ a *diversifying model* just in case for some $\alpha \in C$, some $\varphi^n \in P$ and some $s \in S$: (i) $v_{chr}(\alpha) \subseteq v(\alpha, s)$ and $v_{chr}(\alpha) \neq v(\alpha, s)$; and (ii) $v_{chr}(\alpha) \neq \bigcup\{v(\varphi^n, s) : \varphi^n \in Chr(\alpha)\}$. A diversifying model thus involves assignments to nominal constants which do not coincide with their characteristic associates and the associates of their characteristic predicates. In other words, diversifying models allow that atomic sentences which are not members of characteristic associates can be true at their indices.

2. Let $\mathcal{M} = \langle S, R, C, c, P, v \rangle$ be diversifying and let α be a nominal constant in C and let s be an index in S. We define the *definitionally conforming local associate for a nominal constant* α, $v_{cnf_d}(\alpha, s) \subseteq v(\alpha, s) - v_{chr}(\alpha)$, to be the set of sentences form $Atm(\alpha)$ which are

derivationally consistent with $v_{def}(\alpha)$ given Mp. We call the predicates which occur in the sentences from $v_{cnf_d}(\alpha, s)$ and which are not already contained in $Chr(\alpha)$ the *definitionally conforming predicates for* α. These predicates form the set $Cnf_d(\alpha) \subseteq P - Chr(\alpha)$. So in case $v_{chr}(\alpha) \neq v(\alpha, s)$, $v(\alpha, s)$ will be the set $v_{chr}(\alpha) \cup v_{cnf}(\alpha, s)$.

3. Intuitively, it may be controversial whether a predicate is a defining or a consequential predicate of some name. It will be, presumably, less controversial whether a predicate is characteristic or conforming. For this reason it will be more convenient, in many cases, to avoid speaking of the defining/consequential distinction and just to speak of characteristic associates and predicates. This observation also suggests an alternative account of conformity.

4. In order to obtain the notion of conformity which we shall use from now on, we alter the definition just given slightly, by leaving it as it is except for replacing the defining associate of α with its characteristic associate $v_{chr}(\alpha)$. Doing this, we obtain, in an exactly analogous way a set, $v_{cnf_c}(\alpha, s)$, which we call the *characteristically conforming local associate for the nominal constant* α. Similarly, we call the predicates which occur in the sentences from $v_{cnf_c}(\alpha, s)$ and which are not already contained in $Chr(\alpha)$, the *characteristically conforming predicates for* α. These predicates form the set $Cnf_c(\alpha) \subseteq P - Chr(\alpha)$. (We observe that $v_{cnf_c}(\alpha, s)$ and $Cnf_c(\alpha)$ are smaller than their definitonally conforming counterparts.)

5. It should be noted that the (characteristically) conforming predicates for a nominal constant are only required to conform to the characteristic predicates of that constant. They need not conform to each other. So given some diversifying model, for any constant α and any index s, $v(\alpha, s)$, i.e., $v_{chr}(\alpha) \cup v_{cnf_c}(\alpha, s)$ need not be a consistent set in view of Mp.

6. Obviously conforming associates are local, whereas characteristic associates are not. Unless the model is rigid, this locality will be responsible

for the locality of the whole associate of a given nominal constant. We shall return to this in Subsection 3.2.5 below.

7. As has been argued in Chapter 2 (see, in particular, the discussion in comment 2.3.8(6)), the axiom NSI, i.e. $(o_1 \ddot{=} o_2) \rightarrow \Box(o_1 \ddot{=} o_2)$, is invalid on the semantics given in that chapter. But in view of the resources of the constrained framework, there is an intuitively appealing way in which the validity of that axiom can be obtained. We simply alter the clause for substitutional identity as follows:

   2.4.2($2^{\ddagger}$) $\mathcal{M}_\sigma \models_s o_1 \ddot{=} o_2$ iff for all pure atomic formulae B_1 and B_2 where (i) B_2 is like B_1 except for containing occurrences of the term o_2 at one or all places where B_1 contains the term o_1 and where (ii) B_1 and B_2 are contained in the characteristic associates assigned to o_1 and o_2, respectively: $\mathcal{M}_\sigma \models_s B_1$ iff $\mathcal{M}_\sigma \models_s B_2$.

   On the assumptions that the assignments to the nominal constants and predicates involved in the models are admissible and that substitutional identity formulae are to be evaluated according to clause 2.4.2($2^{\ddagger}$), NSI will be obviously valid. Similar remarks apply to NSD.

8. Let $\mathcal{M} = \langle S, R, C, c, P, v \rangle$ be any model and let s be any index in it. We cannot, in view of the definitions given above, take the union of all the associates for the nominal constants for s together with the union of all the associates for the predicates for s to be something like a Carnapian state description, i.e., a consistent set which for any $A \in Atm$ contains either A or $\neg A$,[6] for associates contain only pure atomic sentences and can be, in view of Mp, inconsistent.

9. It will be instructive to consider an example which illustrates the terminology introduced so far. For this purpose we use proper names rather than nominal constants, ordinary elemantary predicates rather than the elements of P and ordinary language counterparts of pure atomic sentences. Let the defining associate for the name 'Nicomachus',

[6] Cf. [Carnap, 1956 (1947)].

v_{def}(Nicomachus), be {Nicomachus is a man, Nicomachus is a son of Aristotle, ...}. (The defining associate for 'Aristotle' will not contain the sentence 'Nicomachus is a son of Aristotle'.) The set of defining predicates for 'Nicomachus', Def(Nicomachus), will be {is a man, is a son of, is a son, ...}. The meaning postulate for '... is a man' will be to the effect that, for example, the predicate '... is an organism' is consequential upon it, the predicate '... is a philosopher' conforms to it but is not consequential upon it, and '... is an artefact' is neither consequential upon nor conforming to '... is a man'. Correspondingly, we have the following kinds of associate for 'Nicomachus'. The consequential associate for that name, v_{cns}(Nicomachus), will be {Nicomachus is an organism, ...}; the characteristic associate for 'Nicomachus', v_{chr}(Nicomachus), will be the union of v_{def}(Nicomachus) and v_{cns}(Nicomachus), that is {Nicomachus is a man, Nicomachus is a son of Aristotle, Nicomachus is a son, Nicomachus is an organism, ...}; the conforming local associate for 'Nicomachus', v_{cnf}(Nicomachus, s), will be {Nicomachus is a philosopher, Nicomachus is married, Nicomachus admires Aristotle, ...}. Since conforming predicates of a name are only required to conform to its defining predicates. They need not conform to each other. Thus, the conforming local associate for 'Nicomachus', v_{cnf}(Nicomachus, s), will also include the sentence 'Nicomachus is a navigator', 'Nicomachus is a bachelor' and all the other conforming atomic sentences of the language. It will be obvious what the consequential, characteristic, and conforming predicates for 'Nicomachus' are. In view of definition 3.2.3(1) an assignment of a local associate to 'Nicomachus' will be admissible iff it contains the sentence 'Nicomachus is a man' and all the other sentences from its defining associate, as well as 'Nicomachus is an organism' and all the other sentences from the consequential associate of that name. Moreover, the local associate for 'Nicomachus' may contain further sentences from that name's conforming local associate, for instance, 'Nicomachus is a philosopher' or 'Nicomachus admires Aristotle'. In view of definition 3.2.3(2) an assignment of an associate to the predicate '... is a man' will be admissible just in case it contains, for instance, the sentences 'Nicomachus is a man' and 'Aristotle is a man' and so on. For if 'Nicomachus' and 'Aristotle' are contained in the substitution class for some index, the constraint requires that the local associate for the predicate '... is a man' contain every pure atomic sentence built up from that predicate which is contained in the characteristic associates for these names. It will be inadmissible, if it contains, for example, 'Athens is a man', for '... is a man' will neither be a characteristic predicate for 'Athens', nor a predicate which conforms to the defining or to the characteristic predicates for that name.

We are now ready to define the notion of an admissible model.

3.2.5 Admissible Models

A model $\mathcal{M} = \langle S, R, C, c, P, v \rangle$ is an *admissible associative model* just in case it satisfies the conditions on admissible assignments 3.2.3(1) and 3.2.3(2).

According to 3.2.3(1) and 3.2.4(2), (characteristically) conforming associates of nominal constants are local, whereas characteristic associates are not. However, nothing prevents us from allowing that a conforming local associate for a nominal constant be rigidified by stipulating that it be the same across all indices. So we can assume that for every $\alpha \in C$ and for all $s, t \in S$, $v_{cnf_c}(\alpha, s) = v_{cnf_c}(\alpha, t) = v_{cnf_c}(\alpha)$. Given this stipulation the whole associate for α will be rigid, that is, for every $\alpha \in C$ and for all $s, t \in S$, $v(\alpha, s) = v(\alpha, t) = v(\alpha)$, where $v(\alpha)$ is the set $v_{chr}(\alpha) \cup v_{cnf_c}(\alpha)$. We call an admissible model which satisfies this rigidity condition a *rigid admissible associative model*.

The associative models for which we shall provide an applied semantical interpretation in Chapter 4 will be admissible diversifying rigid constant substitution class models with a total accessibility relation. We shall call such models *natural models*.

3.3 Asymmetrical Essential Relations

We now turn to the problem of asymmetrical essential relations which arises for constrained and unconstrained associative semantics alike and adjust the language L and the semantics so as to overcome that problem.

3.3.1 The Problem

According to Kit Fine, we may distinguish two kinds of modal account of essential properties, a categorical and a conditional account. On the categorical variant an individual x has a property F essentially just in case it is necessary that x has F (that is, $\Box Fx$); on the conditional account, an individual x has a property F essentially just in case it is necessary that x has F if x exists (or, altenatively, just in case it is necessary that x has F if it is identical to x). Fine has argued, to my mind convincingly, that such accounts of essence in terms of *de re* necessity are inadequate, since they suggest, among other inadequacies, that *de re* necessary statements containing relational predicates like, for example, $\Box Rab$ claim that the individual denoted by a essentially bears relation R to the individual denoted by b and conversely. (As the context suggests we take a and b to be individual constants.) However, as Fine observes this symmetry is counterintuitive. To see this let the values of a and b denote Socrates and the singleton {Socrates} respectively, and let R be the relation of set-membership. Then $\Box Rab$ claims that it is essential to Socrates that he is a member of {Socrates} and that it is essential to {Socrates} that it contain Socrates as a member. But clearly, as Fine insists, the latter claim is intuitively true whereas the former is not.[7]

> "What makes it so easy to overlook this point is the confusion of subject with source. One naturally supposes, given that a subject-predicate proposition is necessary, that the subject of the proposition is the source of the necessity. One naturally supposes, for example, that what makes it necessary that singleton 2 contains (or has the property of containing) the number 2 is something about the singleton. However, the concept of necessity is indifferent to which of the many objects in a proposition is taken to be its subject. The proposition that singleton 2 contains

[7] See [Fine, 1994] pp. 4-5.

2 is necessary whether or not the number or the set is taken to be the subject of the proposition."[8]

The problem with the modal account of essence, thus, is that it explains the concept of essence, a concept which is sensitive to variations in source, in terms of the concept of *de re* necessity which is not. Fine therefore rejects that account and, in response, revives the traditional definitional account of that notion which, in effect, identifies essence with real definition.[9]

As a consequence of this proposal the order of explanation is reversed; the notion of necessity is no longer treated as explanatory prior to the notion of essence, rather necessity is explained in terms of essence. Roughly, a proposition is now said to be metaphysically (or *de re*) necessary if it is true in virtue of the essence or identity of objects.[10]

Crucially, Fine introduces an essentialist operator which unlike the necessity operator (i.e., $\Box$) is sensitive to variations in source. This essentialist operator, $\Box_x$, which is taken to be a primitive notion reads 'it is true in virtue of the essence of object x that'.[11] The subscript of the essentialist operator indicates the source of the truth of essentialist claims. For example, letting R have the meaning of '... is an element of ...' and letting a denote Socrates and b refer to {Socrates} the formula $\Box_b Rab$, unlike $\Box Rab$, makes it clear that the essentialist claim which is symbolized by it owes its truth to the essence of b rather then to the essence of a.

[8][Fine, 1994] p. 9. The notion of proposition Fine has in mind is, as it seems, that of a structured proposition of a Russellian sort.

[9]See, for example, [Fine, 1994] p. 14. According to Fine, real definition does not apply to linguistic items, that is, the expressions of some language, but to the non-linguistic items for which they stand. The distinction is nicely illustrated in the following passage. "Thus the expression 'the number of objects that are not self-identical' may be taken either as a definition of the numeral '0' or as a real definition of the number 0. In the first case, the identity '0 = the number of objects which are not self-identical' is taken to be true because of the meaning of the numeral '0'; and in the second case, it is taken to be true because of the nature or essence of the number 0."([Fine, 2002a] pp. 30-31).

[10]See [Fine, 1994] p. 9, bottom.

[11]Fine uses 'essence', 'nature', 'identity', and 'being' synonymously (cf. [Fine, 1995b] p. 69, note 2.); and he uses 'object' in the wide sense of 'entity', rather than 'individual'. This is also observed in [Hale, 1996] p. 116, note 14.

Due to the insensitivity of necessity (and thus the box-operator $\Box$) to variations in source, Fine not only rejects the modal account of essentialist claims but he also rejects (referential) quantified modal logic as a proper formal tool for the analysis of such claims. Instead, he develops a logic of essence which treats formulae containing the essentialist operator and which is not taken to be a fragment of a modal system, but a logic of its own right.[12] However, we shall not discuss Fine's systems and semantics here, rather we shall be concerned with his diagnosis of the problem and the way in which we can deal with asymmetrical essential relations in terms of the constrained associative framework.

Needless to say, given the distinctions of the previous section, associative semantics must not be taken, intuitively, to treat asymmetrical essential relations which obtain between objects in virtue of their essences. Rather, as I wish to suggest, it should be taken to be concerned with asymmetrical definitional relations which obtain between names in virtue of their sense-extensions. Sense-extensions are, as I have already mentioned, the applied semantical counterparts of associates. (Of course, the notion of sense-extension is a philsophical term of art; but so is the notion of essence.)

3.3.2 A Proposal

Rather than viewing asymmetrical essential relations as arising from the insensitivity of the notion of necessity to variation in source, we should view asymmetrical essential relations as arising from the inability of binary predicates to discern subject and object position in modal contexts.

Moreover, instead of refining the box-operator, we refine binary predicates. To put it differently, rather than proposing an essentialist primitive and developing a logic of essence, we stick to quantified modal logic and enrich its alphabet with binary predicates which are subject-object sensitive in modal contexts. With such "aspectualized" predicates at our disposal the modal account of essentialist claims regains some of its plausibility. In what follows we shall confine our discussion to aspectualizations of binary pure predicates.

[12] See [Fine, 1995c] and [Fine, 2000].

I shall set up this proposal in terms of substitutional quantified modal logic and constrained associative semantics. As a consequence, we discuss the salient issues in terms of *de nomine* necessity and its ilk rather than in *de re* terms. It will become apparent, however, that the strategy I am about to develop is applicable to a suitably adjusted version of denotational modal semantics. But only at the costs already discussed in Chapter 1.

A modal account of essential properties in terms of associative modal semantics, no matter whether constrained or not, has to face the problem of asymmetrical essential relations in essentially the same way as its counterpart working within the framework of denotational modal semantics. Consider the *de nomine* modal claim $\Box Rab$. As before, let the binary predicate R be short for '... is an element of ...' and let a and b be the nominals 'Socrates' and 'the singleton Socrates', respectively. For this claim to be true the name associate for a would have to contain Rab at every index and so would b. But clearly whereas the latter is fine, the former is not. Of course, in both constrained and unconstrained associative semantics the truth of $\Box Rab$ can be blocked, for example by not letting Rab be contained in the name associate for a in the former, or by not letting it be contained in some local name associate for a and/or R on the latter. However, if we impose such restrictions then $\Box Rab$ becomes false, which seems to be counterintuitive with respect to b. What we want, intuitively, is that the claim $\Box Rab$ be true with respect to b but not with respect to a.

As insinuated above, to handle the problem of asymmetric essential relations we refine the apparatus of predication by introducing aspectualized binary predicates. In effect, we will not only evaluate modal statements composed out of a binary predicate φ^2 itself but statements built up out of its aspectualizations as well. These aspectualizations are binary predicates which are sensitive to the subject/object distinction in modal contexts.

The introduction of such predicates requires a further refinement of associative modal semantics.

Aspectualized Binary Predicates

To obtain the language L^* which contains aspectualized binary predicates, we introduce for a selected range of 2-ary pure predicates φ^2 of L *aspectualized* *binary predicates* of the form φ_1^2, φ_2^2, and $\varphi_{1,2}^2$ where

1. $\varphi_1^2 xy$ reads 'x stands in φ^2 to y insofar as x's bearing φ^2 to y is of concern';

2. $\varphi_2^2 xy$ reads 'y stands in φ^2 to x insofar as y's bearing the converse of φ^2 to x is of concern';

3. $\varphi_{1,2}^2 xy$ reads 'x stands in φ^2 to y insofar as x's bearing φ^2 to y is of concern and y stands in φ^2 to x insofar as y's bearing the converse of φ^2 to x is of concern'. (We write φ^{*2} for the converse of φ^2.)

The subscripts indicate the argument place (or places) with respect to which φ^2 is aspectualized. It is easily verified what the aspectualizations for our binary predicate R that abbreviates '... is an element of ...' are. As the reader will note, the utility of this enrichment of L would be less obvious if that language did not contain a box-operator.

Predicate Inclusion

In order to give the truth conditions for atomic sentences which are built up from aspectualized predicates we introduce the notion of predicate inclusion which is rather natural and which has been already implicit in our account of consequential predicates. We explain this notion in terms of the meaning postulates in Mp as introduced in 3.2.1(3). First we give some intuitive examples.

Consider the English predicate '... is the mother of — ' (symbolically: φ^2). In virtue of its meaning, this predicate will include, among other predicates, the following ones: the binary predicates '... is a parent of — ' (symbolically: χ^2) and '— has ... as mother', which is the converse of the predicate we have chosen as our example, (symbolically: φ^{*2}) as well as the monadic predicates '... is a mother' (which we symbolize as ψ_1^{2-1}) and '— has a mother' (symbolically: ψ_2^{2-1}). (A predicate may not only include predicates of equal or lower arity but also of higher arity. For instance, the monadic predicate '... commits suicide' will include the binary (reflexive) predicate '... kills ...'.) It was the purpose of introducing the formal meaning postulates Mp

into the constrained framework in order to mirror such analytic relationships between the pure predicates of English also between the predicates of our formal language L. This is to hold for L^* as well.

In general, the meaning postulate for a binary predicate φ^2, $Mp(\varphi^2) \in Mp$, will take the following shape:

$$(\Pi x)(\Pi y)(\varphi^2 xy \rightarrow (\chi^2 xy \wedge ... \wedge \varphi^{*2} yx \wedge \psi_1^{2-1} x \wedge \psi_2^{2-1} y \wedge \psi_{1,2}^2 xy)).$$

The meaning postulates for its aspectualizations will be as follows:

$$(\Pi x)(\Pi y)(\varphi_1^2 xy \rightarrow \psi_1^{2-1} x).$$

$$(\Pi x)(\Pi y)(\varphi_2^2 xy \rightarrow \psi_2^{2-1} y).$$

$$(\Pi x)(\Pi y)(\varphi_{1,2}^2 xy \rightarrow (\psi_1^{2-1} x \wedge \psi_2^{2-1} y)).$$

We define the notion of *predicate inclusion*, saying that a predicate is included in another predicate just in case it occurs on the right-hand side of the meaning postulate for the latter. It will be noted that every predicate which is consequential upon another with respect to the defining associate for some nominal constant is included in the defining predicate.[13]

Being monadic predicates the aspectualized predicates ψ_1^{2-1} and ψ_2^{2-1} are aspectualized in a trivial sense. We shall take them to be syntactically identical to ordinary pure monadic predicates. So for example the predicate ψ_1^{2-1} and the predicate χ^1 will be syntactically idenitcal symbolizations of '... is a mother'. As a consequence, these predicates will have a single associate.

The definition of the notion of an admissible model for the language L^* is obtained in an exactly analogous way to the definition given in Subsection 3.2.5 by adding the meaning postulates for the aspectualized binary predicates to Mp and adjusting the definition of an admissible assignment in the obvious way.

[13]The account of predicate inclusion presented here is inspired, to a certain extent, by Alan McMichael's characterization of the notion of the inclusion of relations in [McMichael, 1983] p. 83, which takes an altogether different form. I should like to thank Ede Zimmermann for pointing out to me an inadequacy in my earlier account of predicate inclusion.

Aspectualized Binary Predicates: Truth Conditions

We state the truth conditions for pure atomic sentences which are built up out of aspectualizations of 2-ary predicates φ^2, that is φ^2_1, φ^2_2, and $\varphi^2_{1,2}$ in terms of the truth conditions for pure atoms built up out of the predicates ψ_i^{2-1} which are included in φ^2.

Let $\mathcal{M} = \langle S, R, C, c, P, v\rangle$ be a constant model and let s be an index in that model, let φ^2_1, φ^2_2, and $\varphi^2_{1,2}$ be the aspectualizations of some φ^2 of L^*, and let ψ_i^{2-1} be the predicates which are properly included in φ^2, then the truth conditions of such atomic sentences are defined as follows.

1. $\mathcal{M} \models_s \varphi^2_1\alpha_1\alpha_2$ iff $\mathcal{M} \models_s \psi_1^{2-1}\alpha_1$.
2. $\mathcal{M} \models_s \varphi^2_2\alpha_1\alpha_2$ iff $\mathcal{M} \models_s \psi_2^{2-1}\alpha_2$.
3. $\mathcal{M} \models_s \varphi^2_{1,2}\alpha_1\alpha_2$ iff $\mathcal{M} \models_s \psi_1^{2-1}\alpha_1$ and $\mathcal{M} \models_s \psi_2^{2-1}\alpha_2$.

Here, the right-hand sides are abbreviations of the equivalent claims in terms of associates, intersection, and set-membership. So the the truth conditions of such sentences are, just like the conditions for atomic $\doteqdot$-senteces, not inductively defined. We may add these clauses to the conditions for truth at an index in a model without making any further changes. So, in particular, pure atoms of the form $\varphi^n\alpha_1...\alpha_n$ may be evaluated as usual.

To illustrate the effects of these conditions let R^2 abbreviate the 2-ary predicate '... is a set-member of ...' and let F_1^{2-1} and F_2^{2-1} be short for the 1-ary predicates '... is a set-member' and '... has a set-member', respectively, which are included in R^2. Moreover, let a abbreviate 'Socrates' and b abbreviate 'the singleton Socrates'.

Now suppose we are asked to evaluate the claim 'a necessarily bears R to b'. Given the aspectualization of binary predicates this claim may be symbolized in four different ways: (i) $\Box R^2ab$, (ii) $\Box R^2_{1,2}ab$, (iii) $\Box R^2_1ab$, and (iv) $\Box R^2_2ab$.

Each alternative will be evaluated in its own way. (i) will not be true according to constrained semantics, since it will require that R^2ab be contained in the characteristic associates of both a and b. As we have noted this might be correct for b, but it certainly is not for a. Alternative (ii), that is $\Box R^2_{1,2}ab$, will not be true either, for it would require that $F_1^{2-1}a$ and $F_2^{2-1}b$ be true at every index and that would mean that $\Box F_1^{2-1}a$ is true. But this seems to be false, since, intuitively, '... is a set-member' is not a characteristic predicate of 'Socrates'. For exactly this reason (iii) is false as well. However, alternative (iv), that is $\Box R_2^2 ab$, seems to be unobjectionably true since $\Box F_2^{2-1}b$ symbolizes a straightforward *de nomine* necessary truth.

It seems that using $\Box$ and aspectualized binary predicates, we may easily localize the "source" of the necessity by inspecting the subscripts of the predicates. Consequently, given aspectualized predicates and a suitable disambiguation of the statement which is to be analysed there will be no danger of confusing the subject and the "source" of the necessity, since now, loosely speaking, the subscripts point to the subjects which are the bearers of the "source" of the necessity.

The language L^* and the adjustments in the semantics presented in this section have been designed to tackle the specific problem of binary asymmetrical essential relational predication. It should be possible to generalize our strategy to such predications of arity > 2. Certainly, such cases are less natural than the binary ones.

3.3.3 A Note on Fine's Essentialist Operator

The basic primitive of Kit Fine's essentialist systems, the essentialist operator $\Box_x$, which reads 'it is true in virtue of the essence (or the nature or the being or the identity) of the object x that' or, equivalently, 'the truth of proposition P *flows from* the essence of the object x' does not only perform the function of being sensitive to variations of source. Fine's operator achieves, deliberately and somehow by the way, a certain ontological reduction.

> "Although the form of the words 'it is true in virtue of the identity of x' might appear to suggest an analysis of the operator into the notions of the identity of an object and of a proposition being true in virtue of the identity of an object, I do not wish to suggest such an analysis. The notion should be taken to indicate an *unanalysed relation* between an object and a proposition."[14]

[14][Fine, 1995a] p. 273, my emphasis.

In this simple way, a commitment to essences of objects is avoided by not taking the expression 'the essence of x' which occurs in Fine's operator as a notion of its own right which might be taken to refer to essences. Rather, an unanalysed relation is taken as basic.

It seems to me that Fine's operator just does not feel primitive in the way the negation operator 'it is not the case that' does.[15] Nor does this reduction seem to be one of the (broadly modalist) sort which reduces an ontologically problematic notion to a less problematic one, for the relation of *emanation* (or flowing) from an essence which is captured by Fine's operator, seems to be at least as problematic as the essence of some object which is referred to by 'the identity of x'. However, I shall not argue these points here.

My point is that the solution to the problem of asymmetrical essential relations suggested in this section does neither involve perplexities concerning intuitions of primitivity, essence, and emanation. On the present framework all the work is done by the necessity operator $\Box$ and aspectualized binary predicates, that is, by notions which neither raise such perplexities, nor involve essentialist commitments.

Intuitively, on the present approach, the "source" of a necessary statement containing some name will not be the essence of the object denoted by that name, rather it will be the nominal definition which determines the defining (and derivatively, the characteristic) name associate for that name. (I shall return to this in the next chapter.) So we cannot view claims of *de nomine* necessity like $\Box Fa$ or $\Box R^2_2ab$ as essentialist statements which say that certain objects have certain essential properties. Instead, we view claims of *de nomine* necessity as expressing definitional properties of names (or, if one prefers concepts to linguistic expressions, name concepts). (More on this in Chapter 4.)

Accordingly, *de nomine* necessary truths like the above Fa or R^2_2ab do not "flow from" (or emanate from) the nature of objects in some unexplained

[15]Fine seems to have different intuitions. "I do not think of the subexpression 'the identity of Socrates' as a significant grammatical component of the operator 'it is true in virtue of the identity of Socrates that', just as I do not think of 'not' as a significant grammatical component of the operator 'it is not the case that'." ([Fine, 1995b] p. 69, note 2.)

sense. Rather, a statement like Fa is a necessary truth in virtue of its being a member of the characteristic associate for the name a. Similarly, a statement like R_2^2ab is necessary, because $F_2^{2-1}b$ is a member of the characteristic name associate for b. (Recall, F_2^{2-1} is a predicate which is included in R^2.) According to the present proposal then, we need not invoke some notion of emanation, for we can do with the notion of set-membership which presumably is (and does not only sound) less problematic.

Of course, since there is no *de re* necessity in the present framework, there is no metaphysical necessity (as essentialistically understood) either. What we do have is *de nomine* necessity. Instead of saying that a proposition is a metaphysical (or *de re*) necessity just in case it flows from the nature of some object[16], we say that a proposition is *de nomine* necessary in virtue of the fact that, by the semantic rules as captured by the admissibility constraints, it must be an element of the defining (and derivatively, the characteristic) sense-extension for the salient name and of the sense-extension of the salient predicate.[17]

Let me close these remarks with an exegetical observation. It is worth noting that in his writings on essence Fine uses the terms 'essence' and 'identity' in two quite different senses. On the one hand, Fine uses the term 'essence' to indicate essence *qua* something in virtue of which the truths about the entity which has that essence flow ("source-essence", as I shall call it). This is the unanalysed notion of essence which is involved in Fine's essentialist operator $\Box_x$. On the other hand, he uses 'essence' to refer to the very class of truths which are true in virtue of—what we call—the source-essence of some object x, that is $\{A : \Box_x A\}$.[18] Let me dub the referent of this second use the "essence-set" of an object. Of course, Fine cannot make the notion of emanation more palatable by claiming that for a proposition A to emanate from the essence of an object x is for it to be an element of $\{A : \Box_x A\}$. For surely, this membership would be parasitic upon the

[16]See [Fine, 1994] p. 9, bottom.

[17]As noted above the propositions Fine seems to appeal to are Russellian propositions. When the primary truth bearers are taken to be propositions and not linguistic items, the present framework will naturally appeal to (neo-)Fregean propositions.

[18]See, for example, [Fine, 1995a] p. 275: "we may identify the being or the essence of x with the collection of propositions that are true in virtue of its identity".

proposition's flowing from the source-essence and thus upon the very notion of emanation.

Chapter 4

Applied Associative Semantics

4.1 Introduction

The semantics which we have discussed in the previous chapters, both in the unconstrained version of Chapter 2 and in the constrained version of Chapter 3, was pure (or formal) semantics. The formal languages L (see Section 2.2) and L^* (see Subsection 3.3.2) are systems of uninterpreted symbols which are entirely devoid of natural language meanings. Correspondingly, the associative modal models, that is, structures of the form $\langle S, R, C, c, P, v \rangle$, which we have employed to provide these formal languages with interpretations, are purely set-theoretical entities, mere tuples of sets. More exactly, the elements of S and the elements contained in the tuples from R, the "indices", are mathematical entities, pure sets, as I shall assume.[1] By contrast, the elements of C, $c(s)$, and P aren't pure sets; they are sets of expressions of

[1]Some theorists assume that the elements of the sets from which the models of pure or formal semantics are built up need not be taken to be mathematical entities, but may be any kind of entity. So Plantinga, for example, suggests that the elements contained in the index-set of a pure Kripke-model may be chessmen or numbers (see [Plantinga, 1974] p. 127) and Fitting and Mendelsohn allow that these elements can be "numbers, sets, goldfish etc." (see [Fitting & Mendelsohn, 1998] p. 12). To my mind the distinction between formal and applied semantics (cf. [Plantinga, 1974] sect. 7.4) will be sharpened when we take a less neutral attitude towards the metaphysical nature of the elements from which the models of formal semantics are composed and insist that they be mathematical entities– or to be more restrictive, require that they be (pure) sets. This would emphasize both the mathematical nature of model-theory on the one hand, and the need for an applied

L_0, the base language of the language L (and its extension L^*). This feature is distinctive of associative models.

So, since in pure semantics the formulae are devoid of natural language meanings, and since the relation between its models and reality, which they might be intended to represent, is left unspecified, the truth of a formula of L^* does not teach us anything about reality. What is needed, therefore, is a link between the truth of a formula of L^* at an index in an associative model, and the real, absolute truth (i.e., truth *simpliciter*) of suitably regimented sentences of ordinary (modal) language. Obviously, this link will explain how the formal language is related to natural language, and the way in which associative models are related to reality. It is the task of applied semantics, as I understand it, to provide such a link.

So far, we have a good understanding of what truth of a closed formula at an index in a model comes to. When $\mathcal{M} = \langle S, R, s^@, C, c, P, v\rangle$, where $s^@ \in S$ is stipulated to be a distinguished index, is a natural associative model of L^*, σ is a substitutional variable assignment, and A a closed formula of L^*, then the following equivalences hold: (i) A is true in the model $\mathcal{M}$ at index s under the assignment σ (formally, $\mathcal{M}_\sigma \models_s A$) iff (ii) A is true in the model $\mathcal{M}$ at index s under every assignment σ (formally, for every σ, $\mathcal{M}_\sigma \models_s A$); iff (iii) A is true in the model $\mathcal{M}$ at index s (formally, $\mathcal{M} \models_s A$). Our aim now is to forge a link between the notion of truth at an index in a model of a closed formula of L^*, i.e., (iii), and the notion of truth *simpliciter* for sentences of ordinary (modal) language.

As the reader might expect, due to the lack of referential capacity of the formal language, the notion of truth *simpliciter*, which is at stake here, won't be the usual referential notion. Correspondingly, ordinary language whose sentences are true in the intended sense, must not be taken in a referential way, it has to be treated nonreferentially. What the link will have to achieve,

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semantical or intended interpretation of these models on the other hand. Thus rather than allowing that the items in a pure model are numbers or goldfish, we should take these items–respecting the set-theoretic nature of model-theory–to be pure sets (rather than numbers or goldfish), which we may take to represent numbers or goldfish via a suitable embedding of some sort. I admit, though, that the ontologically neutral way of seeing pure models is perfectly intelligible and might be attractive for other reasons.

therefore, is to connect up the formal language with "odinary language taken nonreferentially" and to connect up associative models with the portion of reality which they represent. Some brief introductory remarks concerning this way of understanding language and the portion of reality which is represented by associative models will be helpful before the details are spelled out. My remarks will be concerned with pure denotational semantics and pure associative semantics and the intuitions by which their development is guided.

The portion of reality which constrained associative models are intended to represent is not the part of reality which can be conceived of as a level of reference (or the level of denotata) of the object language. Rather, as I wish to suggest, it is the part of reality which can be conceived of as the level of sense (or, more exactly, the level of sense-extensions) of that object languague. To get at the idea let me compare the relevant aspects of denotational and associative semantics.

An ordinary first-order denotational model $\mathcal{T}$ of the simplest denotational object language $\mathcal{L}^d$ is a pair $\langle D, v \rangle$ where D is a non-empty domain of objects or individuals and v a valuation function which assigns objects from D to the individual constants, sets of objects from D to 1-ary predicates, and sets of n-tuples of objects from D to n-ary predicates with $n \geq 2$. The sets are the extensions of the predicates.

Such a semantics is called "denotational" or "referential", because of the intuition that when the language and the models are suitably informally interpreted the language may be conceived as being about the reality represented by the models. The objects from D may be taken to be the things which actually exist, the individual constants of $\mathcal{L}^d$ are then naturally interpreted in terms of names which refer to (or denote) these things, the predicates are interpreted in terms of natural language predicates which are satisfied by (or true of) the denotata in their extensions, and the closed formulae of $\mathcal{L}^d$ are taken to be sentences which describe the denotata. In sum the intuition underlying denotational semantics is that the object language is about a level of denotata (or that it is intentional with respect to it).

On the denotational view the truth *simpliciter* of natural language sentences depends, roughly, on two factors: first, the meanings of the expressions which compose the sentences and, second, the extralinguistic facts, the level of denotata of the language which that language serves to describe. We shall call this notion of absolute truth truth *simpliciter* with respect to the level of reference (or referential truth).

By contrast, the intuition by which associative semantics is guided is quite different. Let me begin with the pure part. A first-order associative model $\mathcal{I}$ of the simplest substitutional object language which can be obtained from the modal language L^* in the obvious way is a triple $\langle C, P, v\rangle$ where C is a non-empty substitution class of nominal constants from the base language, P a set of pure predicates of the base language, and v a valuation function which assigns to each nominal constant from C a set of pure atomic sentences of the base language which contain an occurrence of that name and which assigns to each predicate from P a set of pure atomic sentences of the base language which contain an occurrence of that very predicate. Since there is no domain of denotata in the model of the object language, that language cannot be taken to be about something. The language is not able to make claims of (referential) existence and the names cannot be taken to refer to items of which the open sentences of that language are true.

But what, then, is the idea by which associative substitutional semantics is underlain? Roughly, it is the idea that C is a set of names of a natural language, P a set of predicates of that language, and that the valuation function v which provides names and predicates with associates should be taken as assigning "sense-extensions", as we shall call them, to the names and predicates. These sense-extensions are determined by the actual meanings of the names and predicates as specified by (partial or total) nominal definitions and meaning postulates of the names and predicates, respectively. Roughly, associates taken as sense-extensions are viewed as encoding a name's or a predicate's combinatorial semantic nature by listing the atomic sentences which can be meaningfully combined from the name or the predicate in view of their meanings. Thus, what v assigns to names and predicates on this interpretation are sense-extensions rather than referents and referential extensions, respectively. Just in the way we might take the actual referents of names and the referential extensions of predicates to constitute the actual "level of reference" (or the "level of denotata"), we may take the actual sense-extensions of the names and predicates of the language to constitute what might be called the (actual) "level of sense".

Now, since the formal object language has no capacity to refer (or to denote), it would be inappropriate to interpret the ordinary object language as describing a level of denotata. It is therefore necessary to take the sentences of ordinary language nonreferentially. Understood in this way, a sentence of a natural language will be true, if true, not because it accurately *describes* a level of denotata of that language, but because it accurately *reflects* its level of sense. Roughly, an atomic sentence of the object language taken nonreferentially reflects the level of sense accurately just in case the sentence is contained in the sense-extension of the predicate and in the sense-extensions of all the names which occur in it. A sentence which truly reflects the level of sense will then be said to be true *simpliciter* with respect to that level.

Given a suitable applied semantical interpretation, associative models are taken to represent the level of sense of natural language as understood nonreferentially. Tarskian models (or, maybe more appropriately, standard first-order denotational models), on the other hand may be taken, given some applied interpretation, to represent the level of reference of that language when it is understood referentially.

On the associative view, then, the truth *simpliciter* of natural language sentences depends, roughly, on two factors: first, the meanings of the expressions which compose the sentences and, second, their sense-extensions comprising the level of sense of the language which is reflected by it. Accordingly, we shall distinguish two notions of *absolute truth*, referential truth, on the one hand, and truth with respect to the level of sense, on the other hand.

The chapter is organized as follows. Section 4.2 gives a detailed explanation of the notions involved in the account of truth with respect to the level of sense, Section 4.3 provides the link between the notion of truth at an index in an associative model in terms of an adaptation of Charles Chihara's Fundamental Theorem to intended* natural models, and a link between truth with respect to the level of sense and truth with respect to the level of reference is given in Section 4.4. The chapter closes with a discussion of a couple of issues in the philosophy of quantified modal logic and the philosophy of language, and a reflection on modal arguments in general (Section 4.5).

4.2 Truth With Respect to the Level of Sense

According to the present semantical framework, truth with respect to the level of reference is not the only notion of absolute truth. There is also another notion of absolute truth: truth with respect to the level of sense. Modal and fictional truth is, as I wish to suggest, truth with respect to that level, rather than truth with respect to the level of reference. In this section I shall explain the notions involved in the account of truth with respect to the level of sense.

4.2.1 The Referential and the Nonreferential Mode of Evaluation

Denotational semantics is guided by the intuition of designation and is thus

> "based squarely on the assumption that the proper business of semantics is to specify how language connects with the world—in other words, to explicate the inherent «aboutness» of language."[2]

Accordingly, denotational semanticists take "the world" to be something at the level of reference of the object language, something which is described by it.

No doubt the intuition of designation is deeply entrenched in our way of viewing the relation between language and world. But it would be a mistake to endow it with the status of a dogma. It would be a mistake, not only because this would obstruct other ways in which assertoric discourse can be understood. It would be a mistake, because an unrestricted compliance with that intuition generates a lot of problems—in particular with respect to fictional and modal discourse—as we have observed in Chapter 1.

That something is wrong with an unrestricted compliance with the intuition of designation is nicely reflected by the usual perplexing formulations which one frequently encounters in the literature on denotation failures and related issues. Consider, for instance, the following arbitrarily selected fragments which expose the influence of this intuition.

[2] [Dowty *et al.*, 1981] p. 5.

> "Noman is not something, and hence, even though 'Noman' refers to him, there is nothing that 'Noman' refers to."[3]

A reaction:

> "Indeed, he [i.e., the author of [Salmon, 1987]] uses 'Noman' as the name for his possible twin brother. But, he concludes, that doesn't mean that Noman exists. *Thus, Noman doesn't exist but we can name 'him'.*"[4]

Another passage is:

> "I will be concerned with the question of why utterances of certain empty names, such as 'Santa Claus' and 'Father Xmass' seem, in at least some loose sense, to be *about the same thing, even though there is nothing in reality that they are about.*"[5]

Such formulations of "speaking about things that don't exist" formidably suggest, I believe, that the referential mode of viewing the relation between language and world is mistaken when it is applied *tout court*.

Clearly, such formulations won't lose their paradoxical flavour, when more or less curious denotata are introduced or when narrowed down notions of reference are proposed to get the semantics right. One cannot, I believe, satisfactorily solve the semantic problems of denotation failures and nonexistence by means of denotational semantical strategies. And, as it seems to me, in view of make-believe theories (see Subsection 1.1.1.e), one cannot solve semantical problems concerning the meaning of fictional and modal sentences, by merely providing considerations about the use of sentences in which nondenoting terms occur.

It seems to me that the intuition of reflection, i.e. the intuition that a sentence may be true without being about something, is not entirely alien to us and that it is in many cases rather natural. Take, for example, such fictional truths like 'Santa Claus has a white beard' or 'Ulysses is a man'—

[3][Salmon, 1987] p. 94. According to Nathan Salmon, "reference preceeds existence".

[4][Linsky & Zalta, 1994] p. 454, note 26; my emphasis.

[5][Everett, 2000] p. 37; my emphasis.

intuitively (or pre-theoretically), we do not take ourselves to be talking about Santa Claus or Ulysses, since there just are no such things to talk about.

Similarly, the intuition of reflection does also apply, I believe, to modal language. For surely the truth of such modal claims like 'Tarski could have been a violinist' does, intuitively, not appeal to, say, Tarski-counterparts or Tarski-haecceities. In a way analogous to our pre-theoretical reluctance to take ourselves to be talking about Santa Claus, we seem normally—not only qua theorists— to be reluctant to take ourselves to be talking about possible objects of some sort when we assent to the truth of modal claims.

In my oppinion, the denotationalist's problems concerning modality, e.g., the inflation of ontology, the problem of modal reference, or the problems of trans-world identity and reidentification are nothing but artefacts of denotational modal semantics, a semantics which rests on a wrong-headed picture of the relation between modal language and reality.

The framework of associative semantics is designed to equip the intuition of reflection, with a formal underpinning. I wish to suggest that fictional and modal sentences (or propositions) are sentences (propositions) which are true without being about something. Rather than being about Santa Claus (or about the name 'Santa Claus' or about the singular concept SANTA CLAUS expressed by that name), the sentence 'Santa Claus has a white beard' does reflect the name 'Santa Claus' (or the concept it expresses) with respect to the level of sense. Similarly, rather than being about Tarski (or, as a denotationalist would suggest, being ultimately about some representative of him), the modal sentence 'Tarski could have been a violinist' does accurately reflect the name 'Tarski' (or the concept that name expresses) with respect to the level of sense. On the present proposal, then, to talk modally about Tarski is, in effect, to reflect the name 'Tarski' with respect to the level of sense rather than to describe a modal reality of some sort. It is the burden of this chapter to provide the associative framework with an applied semantical interpretation.

In view of the denotationalist problems raised in Chapter 1, I wish to suggest that we should distinguish two modes in which a sentence of natural language may be evaluated: the referential and the nonreferential mode.

On the *referential mode of evaluation* a sentence is read referentially. Here the quantifiers and the identity predicate are taken to be referential and then evaluated in the usual way according to the referential extensions of the nonlogical vocabulary. This will be said to be the "descriptive" way of viewing language.

By contrast, on the *nonreferential mode of evaluation* a natural language sentence is read nonreferentially where the quantifiers and the identity predicate are read substitutionally and then evaluated in accordance with what I shall later call the "sense-extensions" of the nonlogical vocabulary. This will be the "reflective" way of viewing language.

I take it that sentences in which denotationless terms occur or sentences which are built up from predicates which lack a referential extension have to be evaluated according to the nonreferential mode. And I take the same to apply to the semantic evaluation of modal sentences (or, as I am inclined to hope, to sentences which occur in intensional contexts in general).

Any natural language sentence can be evaluated, as I wish to suggest, in the nonreferential mode. However, only a certain portion of natural language sentences can be sensibly evaluated referentially. On the present theory, the usual referential mode of evaluation is adequate solely for sentences which (i) involve exclusively denoting singular terms and predicates which do indeed have a referential extension, and (ii) which do not occur in modal (or, more ambitiously, in intensional contexts). Sentences which satisfy these two conditions comprise what we shall call the *referential portion* of the language in question.

Let me now explain the fundamental notions of the applied semantical interpretation of the associative framework. Of course, I do not claim that the associative framework is the only framework which can be proposed to articulate the intuition of reflection and, moreover, I do not claim that the following proposal is the only way to explain the intuition that sentences can be true without being about something within the associative framework.

4.2.2 The Meanings of Names

Not every name has a referent, but every name has, as we take it following Frege, a meaning (or a sense). We assume, in a way similar to Searle, that the meaning of a name is specified by a list of definite descriptions and that such a list provides the definition of the name. The meaning assigned to a name will be the meaning of the definite descriptions which occur in the list. For example, the meaning of the denoting name 'Socrates' is specified by the list 'the son of Sophroniscus, the son of Phaenarete, the husband of Xantippe, the teacher of Plato, the man who saved the life of Alcibiades, ...'. Similarly, the meaning of the nondenoting name 'Ulysses' is specified by the list of definite descriptions 'the son of Laertes, the husband of Penelope, the father of Telemachus, the king of Ithaca, the man who killed Democoon, ...'.

Lists like these are naturally taken to reflect the actual uses of the names in referential or nondedotational discourse. The list for 'Socrates' contains the definite descriptions which we actually use to talk about Socrates; and the definite descriptions for 'Ulysses' are the descriptions which can be extracted from the way in which Homer (or whoever wrote the Homeric epics) employed that name in his 'Ulysses'-statements. (Obviously, the term 'description' is somewhat misleading when it is used to account for the meanings of nondenoting names, since it suggests that nondenoting names refer to objects and that their meanings are given in terms of descriptions which apply to these objects.)

The definite descriptions which occur in a description list of a name each provide a partial definition of the name and the whole list of such partial definitions provides the name's full definition. These definitions are linguistic or nominal in the sense that they define something linguistic, that is a name, in terms of something linguistic, that is a definite description.

Importantly, the meanings of denoting names and the meanings of nondenoting names differ in a certain significant respect. We shall take it, following Kit Fine, that some of the partial definitions which occur in the list for denoting names may not only be regarded as being linguistic partial definitions of names but as simultanously providing real definitions of their bearers.[6]

Let the *defining meaning of a denoting name* be specified by those definite

[6]Cf., for example, [Fine, 2002a] p. 16: "We may also talk of defining a non-linguistic item by means of something linguistic. We may say, for example, that 'the successor of 1' is a definition of the *number* 2 rather than the numeral '2'. But in such cases we define the object by defining, or by providing the means for defining, an expression for the object."

descriptions in the list for that name which may be taken to reflect the essential features of the name's bearer. We shall call these descriptions *defining definite descriptions*. Thus, the defining meaning of 'Socrates' is specified by the defining list (i.e., the list of defining definite descriptions) 'the son of Sophroniscus, the son of Phaenarete, ...'.

An analogous account of defining meaning is not available for the meanings of nondenoting names. For given the fact that nondenoting names do not have bearers, no defining definite description occurring in the list of a nondenoting name can be understood in this way. Thus, the defining definite descriptions 'the son of Sophroniscus' or 'the owner of the genetic code so-and-so' may be treated not only as providing a partial linguistic definition of the name 'Socrates' but also as giving a partial real definition of the object Socrates. By contrast, the defining definite description 'the son of Laertes' cannot, like all the other descriptions in the list for 'Ulysses', be plausibly treated as a partial real definition of the bearer of that name since there is no such thing like the Homeric Ulysses and thus no object to which a real definition could apply. Consequently, the defining meanings of nondenoting names cannot be specified in terms of linguistic definitions which qualify as real definitions.

There seem to be (at least) two natural ways to explain what a *defining meaning of a nondenoting name* is. According to one proposal, the defining meanings just are the meanings of the nondenoting names and the defining definite descriptions are just all the descriptions which occur in the list which specifies these meanings. On this account of the defining meaning of a nondenoting name, 'the son of Laertes' and 'the man who killed Democoon' alike are defining descriptions of 'Ulysses'.

On the second proposal, we may treat some of the definite descriptions in the defining lists not as yielding partial real definitions of the beares of nondenoting names, but *as if* the name had a bearer and *as if* the definite descriptions provided such definitions. The defining meaning of a nondenoting name will then be specified by those definite descriptions in the list for that name which are treated as if they reflected the essential features of a pretended bearer of that name. So on this account, preserving the analogy with the denotational case, 'the son of Laertes' will be a defining definite description of ‘Ulysses’, but the description ‘the man who killed Democoon’ will not. Due to its explanatory simplicity, I am more attracted to the first proposal.

On the present account, a name may be either a proper name or a definite description. Thus only the meanings of proper names have been discussed so far. But what about the meanings of definite descriptions? We shall take it that the meaning of a definite description coincides with its defining meaning. Thus the meaning of a definite description is specified by a defining list which contains just that very description and no further descriptions. For example, the meaning of the definite description ‘the husband of Penelope’ is given by the list ‘the husband of Penelope’ which contains but one defining description.

There is a further distinction which must be appreciated. We have to distinguish the definite description which is to be defined from the defining definite description, which serves to define it. The former is treated syntactically like a proper name and is not parsed further into its components. So, for example, the definite description ‘the husband of Penelope’, *qua definiendum*, is treated in the same way like ‘Ulysses’. In accordance with this assumption we symbolize definite descriptions in L^* with nominal constants and allow that they be included in the substitution classes of associative models. (If we did symbolize definite descriptions in iota-notation as $(\imath x)A$ and included them in the substitution classes then the inductive proof of Subsection 2.3.5 would not go through, since the As might increase complexity.[7])

Clearly, the (applied) associative semantic framework being nonreferential does not give rise to problems of nondenoting definite descriptions which to a considerable extent motivated Russell’s denotationalist theory of descriptions. It is for this reason that we need not stick to the received tradition. (It will be obvious, though, how the Russellian account could be adapted in the associative framework.)

Now what about the defining definite descriptions of names (i.e., proper names and definite descriptions)? They are to be viewed as implicitly prefixed with an ‘is’, the ‘is’ of predication, so as to render a complex predicate which

[7]See [Kripke, 1976] p. 332.

is built up from a pure predicate and proper names depending on the arity of the pure predicate. (So if the predicate has arity n then the complex predicate will be built up from $n-1$ not necessarily distinct proper names.) For example, the defining definite description 'the husband of Penelope' is viewed as part of a complex predicate '... is the husband of Penelope' which is built up from the binary predicate '... is the husband of ...' and the proper name 'Penelope'. In cases where the defining definite description is nested like, for example, 'the father of the husband of Penelope' the corresponding complex predicate will consist of the binary predicate '... is the father of ...' and the definite description 'the husband of Penelope' which is treated like a proper name. (It is easily verified that, in the example at hand, the nested definite description, but not the embedded one is defining of 'Laertes'.)

We assume that the lists of defining definite descriptions for a name are complete. (The dots which occur in description lists of proper names are meant to indicate that they are not regarded as being exhausted by the defining descriptions which occur in them.) Furthermore, we assume that not every description occuring in a defining list must be known to us (or even knowable for us). And although we may disagree on the meanings of proper names, the meaning of a proper name is, as I wish to assume following Frege, an objective matter.

We close this Subsection stating a *criterion for the semantic difference of names*: the meanings of names differ just in case their defining meanings differ.

Thus, for example, the meaning of the proper name 'Aphrodite' does not differ from that of 'Venus', since the defining meanings of both names are specified by a single defining list, i.e., 'the daughter of Zeus, the daughter of Dione, the mother of Aeneas, ...' on the first account of defining meanings for nondenoting names, or, alternatively, 'the daughter of Zeus, the daughter of Dione, ...' on the second account (so that 'the mother of Aeneas' will not be included). In this sense we may say that 'Aphrodite' and 'Venus' are synonymous. Like remarks apply to the meanings of denoting proper names.

By contrast, the defining meanings of definite descriptions, like for example the defining lists for 'the daughter of Zeus' and 'the daughter of Dione' differ in meaning as they do not contain the same defining definite descriptions. Similarly, for the meanings of denoting definite descriptions.

4.2.3 The Meanings of Names: Comments

A couple of comments will elucidate the meanings of names, as they shall be understood here, further.

1. *Defining Meanings and Real Definitions*. In cases in which a partial definition of a denoting name may be regarded as providing a partial real definition of its bearer, the real definition will be said to be *linguistically reducible* to the definition of the name, since the object is partially defined in terms of the partial definition of the name of that object.[8] Thus all the real definitions which can thus be extracted from the present account of the notion of defining meaning of denoting names are linguistically reducible. In case there are nondenumerably many objects but only denumerably many names, as one traditionally assumes, not every object will have a name. And so not every real definition will be linguistically reducible. The notion of a defining meaning of a denoting name should not be confused with the notion of real definition. A confusion of this sort will hardly arise in the case of nondenoting names—at least when the word 'nondenoting' is taken literally. In any event, our defining meanings are explained in terms of linguistic nominal definitions, not in terms of real definitions.

2. *Defining Meanings and Fregean Senses*. Defining meanings differ from Fregean senses in certain respects. First, they are complete lists of definite descriptions rather than single (conjunctive) definite descriptions. Moreover, these lists contain only defining definite descriptions. For example, the list of defining descriptions for the denoting name 'Aristotle' does not contain the definite descriptions 'the pupil of Plato, the teacher of Alexander the Great' by which Frege (as I understand him in view of [Frege, 1994 (1892)] p. 42, note 2) would specify the meaning of that name, since none of them (each being a so called "famous deeds

[8] I take this to be in agreement with [Fine, 2002a] p. 16.

description") is a defining definite description in the sense explicated above. Finally, let me add that Frege's characterization of the sense of a term as the way it presents its denotation would be to coarse to be applied to defining meanings as understood on the present account, for the defining meanings of nondenoting names cannot be sensibely taken to present denotata. (Of course, defining meanings of denoting names can serve to determine the referents of denoting names).

3. *"Frege's Puzzle"*. According to the criterion of the semantic difference of names, the meanings of two names differ just in case they differ in their defining meanings. Now, since the names 'Hesperus' and 'Phosphorus' do not differ in their defining meanings for they are specified by one and the same list of defining descriptions, these names do not have different meanings. Where does this leave us with "Frege's Puzzle"? In a nutshell, the puzzle is that of giving an answer to the question of how two identity statements with coreferential singular terms, for example, the sentence 'Hesperus is Hesperus' (which reports a truism) and the sentence 'Hesperus is Phosphorus' (which reports a discovery) could differ in cognitive significance. Frege suggested that the singular terms which occur in such identity statements do not only have referents but do also have senses or meanings (for example, the sense of 'Hesperus' is given by, say, the definite description 'the brightest object visible in the evening sky' and the sense of 'Phosphorus' by 'the brightest object visible in the morning sky') and that the difference in cognitive significance of such identity statements corresponds to the difference in the meanings of the singular terms. However, on the present nonreferential picture the difference in cognitive significance does not correspond to the meanings of singular terms directly but to a difference in their sense-extensions which are determined by defining meanings. I shall explain the notion of a sense-extension and articulate my proposal for a solution shortly.

4. *Homonyms*. "We must distinguish between homonyms, just as we would distinguish the name of London (England) from the homony-

mous name of London (Ontario)".[9] Homonyms of nondenoting names are to be distinguished along similar lines. For example, the name 'Aeneas' as used in the Homeric *Iliad* is to be distinguished from its homonym as used in Virgil's *Aeneid*, or from its homonym as as used in the union (intersection etc.) of both epics (and other pieces of literature). Clear enough, homonymic names may differ in their defining meanings and will be distinguished accordingly when they do. In this sense the defining meanings of 'Venus (the goddess)' and 'Venus (the planet)' will differ.

5. *Defining Meanings and Quotation-Names*. As the defining lists which specify the defining meanings of denoting names "linguistically reduce" real definitions, the defining descriptions of denoting names will not contain quotation-names. Thus, for example, the definite description 'the bearer of the name 'Socrates' ' won't be defining of 'Socrates'. The case of nondenoting names is somewhat trickier, since their defining meanings cannot be viewed as reducing real definitions. On the second account of their defining meanings, quotation-names may be excluded in essentially the same way by the *as if*-strategy. But this doesn't work on the preferred first account. In this case it will be simplest (though admittedly crude) to stipulate that defining definite descriptions of nondenoting names which involve quotation-names be viewed as implicitly forming monadic predicates. So let me assume, for the sake of argument, that 'the man called 'Ulysses' ' is a defining description of 'Ulysses'. We may then view that defining description as implicitly generating the monadic predicate (whose monadic character does not surface though) '... is the man called 'Ulysses' '.

6. *Names with Empty Meanings*. We shall allow that the meaning of a name may be empty. Obviously, in such cases the list of definite descriptions for the name will be empty. We may call names with empty meanings, somewhat misleadingly, "meaningless names". These names should not be confused, of course, with what the denotationalist usually calls "empty names" which are just denotationless names.

[9] [Lewis, 1983 (1978)] p. 265.

7. *Referentialism (or Millianism)*. It is disputable whether the doctrine of Referenitialism (roughly, the assumption that the sole semantic function of a name is to refer to its bearer) is correct concerning the level of reference. However, that doctrine is entirely irrelevant, when we look to the level of sense. We may, therefore, remain agnostic about it. As is well-known, Referentialism faces certain problems concerning the meaning of nondenoting names (for, if the referent is all there is to the meaning of a name, nondenoting names seem to be meaningless), the semantics of negative existentials such as 'Santa Claus does not exist', or Frege's puzzle. (I shall make suggestions on how these problems could be addressed from the prespective of the present semantical framework in Section 4.5 below.)

4.2.4 The Meanings of Predicates

We take the meanings of predicates to be specified by total and partial definitions, respectively. For instance, the predicate '... is a bachelor' is, presumably, totally defined by the explicit definition 'Everything is such that it is a bachelor just in case it is an unmarried adult man', whereas the predicate '... is red' is, presumably, only partially defined by 'Something is red only if it has a surface which emits light of wavelength such-and-such'. Again, we shall assume that such definitions are objective.

4.2.5 The Sense-Extensions of Names

On the present approach the definitions of names and predicates do not primarily serve for the purpose of fixing the reference of names and of determining the extensions of predicates. Instead, their main purpose is to determine the sense-extensions of names and predicates, respectively. Since the notion of truth with respect to the level of sense is explained in terms of sense-extensions, these items are vital to the present framework. Let me explain the notion of the sense-extension of a name first.

I take it that every name, whether denoting or nondenoting, does not only have a meaning but is also associated with a sense-extension which is determined by its meaning. Roughly, a sense-extension of a name is the collection of all the syntactically simplest, that is, atomic, sentences of the language which can be built up from that name in accordance with both the defining meaning of that name and the meanings of the (pure) predicates of the language. The atomic sentences in a name's sense-extension may thus be taken to encode the name's meaning. More exactly, the sense-extension of a name is a collection of atomic sentences of the language which is composed from three subcollections: the defining sense-extension, the consequential sense-extension, and the conforming sense-extension of the name. Let me explain these components in turn.

The basic portion of a name's sense-extension is its *defining sense-extension*. This portion is determined solely by the name's defining meaning. The defining sense-extension of a name is the collection of all the atomic sentences of the language which we may view as resulting from a sentence forming procedure which first turns each description of the defining list of the name into a (complex) predicate and then saturates the resulting predicate with that very name. The resulting sentences are, as I shall choose to say, the *name's defining conditions*.[10]

Consider, for example, the name 'Socrates'. The defining meaning of that name is specified by its defining list of definite descriptions. Let the defining list of that name be 'the son of Sophroniscus, the son of Phaenarete, ...'. The sentence forming procedure turns the defining description 'the son of Sophroniscus' into the complex predicate (of the sort discussed one section back) '... is the son of Sophroniscus' and then saturates it yielding the sentence 'Socrates is the son of Sophroniscus'. Thus, the portion of the sense-extension of 'Socrates' which is determined in this way will be the collection {Socrates is the son of Sophroniscus, Socrates is the son of Phaenarete, ...}. The pure predicates which occur in the defining sense-extension of a name are the *defining predicates* for that name. Thus, the predicate '... is the son of ...', for instance, will be among the predicates defining of 'Socrates'. In other words, the defining sense-extension of a name contains all the atomic sentences of the language in which that name and its defining predicates occur.

[10]The terminology is borrowed from [Fine, 2002a] p. 17.

These atomic sentences, that is, the name's defining conditions, constitute, so to speak, the name's semantical essence.

Let me now explain the notion of a name's *consequential sense-extension*. This portion of a name's sense-extension is determined by that name's defining sense-extension and the meanings of the name's defining predicates. The consequential sense-extension of a name comprises all the atomic sentences which must result from the atomic sentences contained in the defining-sense extension in view of the meaning postulates of the defining predicates and which are not already contained in the defining sense-extension of that name. The predicates which occur in the sentences which are contained in the consequential sense-extension of a name are that names *consequential predicates*.

Consider, for example, the predicate '... is the son of ...' which is defining of 'Socrates'. The meaning of this predicate will be given by a definiton which will involve, for instance, the predicates '... is a son of ...', '... is a son', '... is male', '... is a human being' and further predicates which are consequential for 'Socrates' in view of the meanings of the defining predicates of that name. The consequential sense-extension of a name is thus the portion of the sense-extension of a name which contains all the atomic sentences of the language in which that name and its consequential predicates occur. For example, the consequential sense-extension of 'Socrates' will be the collection {Socrates is a son, Socrates is a man, Socrates is a human being, ...}.

Taken together the defining portion and the consequential portion of a name's sense-extension yield the name's *characteristic sense-extension*. The pure predicates occuring therein are the *characteristic predicates* of the name. So the characteristic portion of the sense-extension of 'Socrates' will be {Socrates is the son of Sophroniscus, Socrates is the son of Phaenarete, Socrates is a son, Socrates is a man, Socrates is a human being, ...}.

Finally, let me turn to the portion of the sense-extenison of a name which may be called the name's *conforming sense-extension*. This portion of a name's sense-extension is determined by that name's characteristic sense-extension and by the meanings of the name's characteristic predicates.

The conforming sense-extension of a name comprises all the atomic sentences, not already contained in the characteristic sense-extension of that name, which are consistent in view of the meaning postulates of the characteristic predicates of that name with the characteristic portion of that name's sense-extension. The predicates which occur in the sentences which are contained in the conforming sense-extension of a name are that name's *conforming predicates*.

In view of the definition of the notion of a conforming predicate, it is not relevant whether the conforming predicates of the name conform to each other. What matters is whether they conform to the characteristic predicates of the name. Thus, on the present construal of the applied associative framework, the predicates '... is a philosopher', '... is married', '... saves the life of ...', for example, are among the conforming predicates for 'Socrates'. But so are the predicates '... is a navigator', '... is a bachelor', '... kills ...'. This is so, since all these predicates conform, given their definitions, to the characteristic predicates of 'Socrates'. The conforming portion of the sense-extension of 'Socrates' will be {Socrates is a philosopher, Socrates is married, Socrates saves the life of Alcibiades, Socrates is a navigator, Socrates is a bachelor, Socrates kills Alcibiades, ...}.

In sum, the sense-extension of a name consists of its characteristic and its conforming sense-extension. We might view the characteristic sense-extenison of a denoting name as corresponding to the essential properties of the individual it denotes and its conforming sense-extension as corresponding to the contingent properties of that individual. This analogy, however, has only a limited appeal since it does not apply to the sense-extensions of nondenoting names.

The sense-extension of a nondenoting name is determined in a strictly analogous way. What should be kept in mind is that, on the first account of the meaning of nondenoting names, the meaning and the defining meaning of the name coincide. So, for example, the defining sense-extension of 'Ulysses' will contain the sentence 'Ulysses killed Democoon', a sentence which would naturally not be taken as defining of that name if it were denoting; and which would, if it were, occur in the conforming sense-extension. On the second account of the meaning of nondenoting names, the above sentence will be classified in just this "natural" way, but now the defining definite descriptions will be treated only *as if* they could be taken as real definitions.

Given the above distinctions and terminology we may now state a principle by which the sense-extension of any name, whether denoting or not, is governed.

> *The Principle of Nominal Sense-Extension.* For any name the sense-extension of that name contains:
>
> 1. all the atomic sentences of the language which result from a sentence forming procedure which transforms each description of the defining list into a defining condition for that name (= defining sense-extension);
> 2. all the atomic sentences which, taking the meanings of the defining predicates into account, must also be contained in the name's defining sense-extension (= consequential sense-extension);
> 3. all the atomic sentences which, taken seperately, are not ruled out as being contained in the name's characteristic sense-extension (= defining sense-extension + characteristic sense-extension), when the meanings of the characteristic predicates are taken into account (= conforming sense-extension).

The Principle of Nominal Sense-Extension corresponds to the admissibility constraints for the assignment of associates to nominal constants as defined in Section 3.2 and makes explicit the intuition by which these constraints are underlain.

In effect, we may view the sense-extension of a name as encoding the name's meaning by capturing its semantic nature in terms of the collection of all the pure atomic, and thus the most basic, sentences containing that name which result from or are compatible with that name's defining conditions in view of the meanings of the predicates of the language. Thus, on the present account of the notion of the sense-extension of a name, a name's sense-extension cannot vary as to the number of the atomic sentences it contains and to the predicates which occur in these sentences. Moreover, no two names have the same sense-extension, even though they can have the same defining meanings.

The sense-extensions of unrestrictedly all the names, taken together, constitute what we shall call the *sense-spectrum of names*. Given the invariability of the sense-extensions of names the sense-spectrum of names will be invariable as well.

4.2.6 The Sense-Extensions of Names: Comments

The following remarks will help to elucidate the notion of a name's sense-extension further.

1. *Sense-Extensions and Denotata*. On a referential account the object a name denotes could be determined in a natural way by the referential truth of the defining conditions which occur in the name's defining sense-extension.[11] I am, of course, in agreement with this practice. However, since my primary concern is not with referential truth but with the notion of truth with respect to the level of sense, it is not relevant whether these defining conditions are indeed referentially true. In the present setting, these defining conditions serve to delimit the name's sense-extension, and sense-extensions are the items in terms of which truth with respect to the level of sense is to be explained.

2. *"Consequential" and "Conforming"*. The notions of consequential and conforming sense-extensions are explained in modal terms, that is, with the help of such expressions like "must result from" (or "must be contained") and "is not ruled out" as being contained (or "can" be contained) when the meaning of the defining or the characteristic predicates, respectively, is taken into account. We take these modal notions to be primitive and call them *basic necessity* and *basic possibility*, respectively. Due to their involvement in the account of a name's sense-extension, they are explanatorily prior to the notion of definitional modality with which the present theory is primarily concerned. (This notion performs, in effect, a function which the notion of derivability

[11] Cf. [Fine, 2002a] p. 17.

in view of Mp performed in formal associative semantics. It will be elucidated further, in paricular, in Subsections 4.2.13 and 4.5.1.)

3. *Sense-Extensions of Meaningless Names.* Meaningless names are names which have no defining meanings. Their defining sense-extensions are therefore empty. Consequently, in view of the consistency of a defining sense-extension of such a name, the sense-extension of a meaningless name will be a very large conforming sense-extension. It will contain, among other atomic sentences, sentences which are built up from the considered meaningless name and such incompatible predicates like, e.g., '... is a son', '... is a daughter', '... is a palace of ...'.

4. *Defining Sense-Extensions and Finean Essence-Sets.* This is an appropriate place to explain the notion of the defining sense-extension of a name by distinguishing it from Finean essence-sets (see Subsection 3.3.3). The sentences which occur in an essence set of an object are the ones which are referentially true in virtue of the essence of that object. By contrast, the sentences which occur in the defining sense-extension of a name, that is, its defining conditions, encode the defining meaning of the name independently of whether they are referentially true or not. Moreover, meaningless names have empty defining sense-extensions; but since arguably every entity (whether linguistic or not) will have an essence, the essence-set of any entity will never be empty. This is a further reason why meaning and sense-extensions on the one hand and essence and essence-sets on the other hand must not be confused.

5. *Sense-Extensions and "Frege's Puzzle".* The names 'Hesperus' and 'Phosphorus' have the same meaning, but they have different sense-extensions, since they do not have all atomic sentences in common. On the present framework this accounts, as I wish to suggest, for the difference in cognitive significance between the sentences 'Hesperus is Hesperus' and 'Hesperus is Phosphorus'. The sense-extensions of the names which occur in the former sentence are the same, but the sentences which occur in the latter are not. (Indeed, no sentence which

occurs in the sense-extension of the former, does occur in the latter.) In view of the truth conditions for formulae of substitutional identity both sentences are true with respect to the level of sense when the 'is' is interpreted in terms of substitutional identity, since the former sentence will be true with respect to the level of sense just in case the latter is true with respect to that level as well. (For the notion of truth with respect to the level of sense see Subsection 4.2.11 below.) We may, in effect, distinguish the following kinds of semantic sameness of names: sameness of (defining) meaning, sameness of sense-extension, and substitutional identity.

6. *The Sense-Extensions of Definite Descriptions.* On the present construal of the framework the defining sense-extension of a defintite description will contain but one defining condition. So for instance the defining sense-extension of 'the husband of Penelope' will be {the husband of Penelope is the husband of Penelope}. As explained in Subsection 4.2.2, the 'is' which occurs in the defining condition is the 'is' of predication. It will be obvious how loosely specified and multifarious the whole sense-extension of that description will be. Statements of substitutional identity (e.g. 'the husband of Penelope is Ulysses') will be true for the same reasons like in the cases in which the 'is' of substitutional identity is flanked by names which are both proper names. Like remarks apply to denoting definite descriptions.

4.2.7 The Sense-Extensions of Predicates

Every name has a sense-extension associated with it and so does every pure predicate. Roughly, the sense-extension of a predicate is a collection of atomic sentences in which that predicate occurs, where the sentences are built up in accordance with (i) the meaning of that predicate, (ii) the meanings of the other predicates of the language, and (iii) the meanings of the names of the language. In effect, we may view the sense-extension of a predicate as encoding the predicate's meaning in terms of a collection of the atomic sentences which are built up from it.

The sense-extensions of predicates are divided into two groups: the sense-extensions of characteristic predicates on the one hand, and the sense-extensions of the conforming predicates on the other hand.

The principle which governs the sense-extensions of predicates may be stated as follows.

> *The Principle of Predicative Sense-Extension.* For every name and for every pure predicate the following holds.
>
> 1. If the predicate is a *characteristic predicate* of some name, then the sense-extension of that predicate does contain all the atomic sentences in which the name and the predicate occur that are contained in the characteristic sense-extension of that name.
> 2. If, on the other hand, the predicate is a *conforming predicate* of some name, then for every sentence of the language in which the predicate and that name occurs, the sense-extension of that predicate does contain that sentence just in case it is not ruled out by the meanings and the sense-extensions of the other conforming predicates of that name.

The Principle of Predicative Sense-Extension mirror the admissibility constraints for the assignment of associates to predicates as defined in Subsection 3.2.3.

Let me illustrate the condition for conforming predicates first. The sense-extension of the predicate '... is a bachelor', for example, which is a conforming predicate of 'Socrates', will contain the sentence 'Socrates is a bachelor', given the definition of '... is a bachelor' just in case the sentence 'Socrates is married' is not already contained in the sense-extension of '... is married', and *vice versa*. In this way the sense-extension of a conforming predicate is variable in view of the definition of the predicate and the sense-extensions and definitions of the other conforming predicates. A conforming predicate may thus have infinitely many alternative sense-extensions. They may vary as to the number of atomic sentences they contain and to the names which occur in them. Only one of these alternatives is the actual sense-extension.

As things happen to be the sentence ‘Socrates is married’ is contained in the actual sense-extension of the predicate ‘... is married’, the sentence ‘Pegasus is married’ is not.

As for the condition on characteristic predicates, the sense-extension of the predicate ‘... is a man’ which is characteristic of, for instance, ‘Socrates’, ‘Ulysses’, ‘Sophroniscus’, ‘Laertes’ and many more names, will be {Socrates is a man, Ulysses is a man, Sophronsicus is a man, Laertes is a man, ...}. In contrast to conforming predicates, the condition on the sense-extensions of characteristic predicates does not allow for alternatives. The effect of the condition is that the characteristic sense-extensions of all names taken together and the sense-extensions of all characteristic predicates taken together coincide.

Taken together the actual sense-extensions of the conforming predicates and the sense-extensions of the characteristic predicates constitute a collection of atomic sentences which do not rule out each other as contained in view of the meanings of the names and pure predicates of the language. Let me call this collection the *sense-spectrum of predicates*. Unlike the sense-spectrum of names, the sense-spectrum of predicates does allow (on the present construal) for variation; it may vary with respect to the actual sense-extensions of the conforming predicates. Such variations give rise to alternative ways in which the sense-spectrum of predicates could have been arranged.

4.2.8 The Sense-Extensions of Predicates: Comments

Sense-extensions of predicates differ significantly from the sense-extensions of names. The following remarks emphasize and clarify this difference.

1. *Sense-Extensions of Names and Sense-Extensions of Predicates.* Sense-extensions of names and sense-extensions of predicates differ (according to the present construal) in the following respect.

   The sense-extension of a name contains all the atomic sentences of the language which result from the name’s defining meaning, that is, the characteristic portion of its sense-extension, and all the atomic sentences which, taken seperately, conform with that portion. Since the sentences in the conforming portion will not be compatible with each other when the meanings of their predicates are taken into account, not all the sentences in a name's sense-extension will be compatible with each other.

The sense-extension of a predicate, on the other hand, contains only atomic sentences which are compatible with each other. This is so because the sense-extension of a predicate contains those atomic sentences of the language which can be formed from that predicate in accordance with (i) the meaning of that predicate, (ii) the meanings of the other predicates of the language, and (iii) the meanings of the names of the language. Let me illustrate this contrast with an example. Whereas the sense-extension of the name 'Socrates' does contain the sentences 'Socrates is married' and 'Socrates is a bachelor', the sense-extension of the predicate '... is a bachelor' will, in view of the meaning of that predicate, contain the sentence 'Socrates is a bachelor' if the sense-extension of '... is married' does not contain the sentence 'Socrates is married' and *vice versa*.

2. *The Actual Sense-Extensions of Predicates.* Just like every predicate of a natural language taken referentially has its actual referential extension, so every such predicate taken nonreferentially has its actual sense-extension. Intuitively, we might take the sense-spectrum of predicates to be a set of atomic sentences which, if they were referentially true, would provide a sort of Carnapian state-description which would accurately describe the actual world. (I say only "a sort of Carnapian state-description", since it will contain exclusively atomic sentences and never negations thereof.) Alternatives to the sense-spectrum of predicates, then, would yield inaccurate descriptions of the actual world. My appeal to this denotationalist intuitions here is merely a heuristic one. Sense-extensions are, of course, in no way state-descriptions, for there is nothing they describe. This is so for such predicates like '... is a man' which have besides a sense-extension also a referential extension, as well as for such fictional predicates like '... is a fairy'. (That sense-extensions cannot be sensibely viewed as descriptions will

be particularly obvious for the sense-extensions of names, which, in their conforming portion, contain sentences which are built up from incompatible predicates. Consider, for instance, the example concerning the sense-extension of 'Socrates' given in the previous comment. There is nothing which is married and a bachelor at the same time, and so nothing which the sense-extension of 'Socrates' could be taken to describe.)

3. *Hearts and Kidneys*. According to applied denotational semantics, the referential extensions of such predicates like '... has a heart' and '... has kidneys' are the same. However, it is obvious that the sense-extensions of these predicates are distinct. Indeed, no two different predicates have the same sense-extensions (cf. comment 2.3.2(7)). In effect, like denotational extensions of predicates, sense-extensions have clear identity criteria (i.e., they are identical exactly if they contain the same members), but sense-extensions are finer. In this respect they perform a function similar to properties.

Having explained what the sense-extensions of names and predicates are, we are now in a position to explain what the level of sense is, which together with the level of reference constitutes, as I wish to suggest, the reality to which language is related.

4.2.9 The Level of Sense

The level of reference of a natural language is naturally taken to be the totality of the actual denotata of names and the actual referential extensions of predicates we take ourselves to be talking about when we use that language in a referential way. Correspondingly, the *level of sense*, according to the present proposal, is the totality of the actual sense-extensions of the names and the predicates we may take ourselves as reflecting upon when we evaluate natural language sentences using them in a nonreferential way. (The difference between "talking about" or "describing", on the one hand, and "reflecting upon", on the other hand, will be clarified in the next Subsection.) More exactly, the actual level of sense comprises two separate components: *the* *sense-spectrum of names* and *the sense-spectrum of predicates*. As a whole the level of sense is governed by the Principles of Nominal and Predicative Sense-Extension.

Modal discourse does not only concern the (actual) level of sense which comprises the actual sense-extensions of (unrestrictedly) all the names and all the predicates of the language, but also alternative ways in which that level could have been rearranged. The fact that the level of sense could have been arranged in various ways is captured by the following principle of plenitude.

> *The Principle of Rearrangement*. The level of sense could have been arranged in an alternative way just in case it did satisfy the Principle of Nominal Sense-Extension and the Principle of Predicative Sense-Extension when so arranged.

Here the 'could have been' expresses possibility in the basic sense of the word. (See comment 4.2.6(2).) On the present construal these alternative ways will differ only with respect to the sense-spectrum of predicates, more exactly with respect to the sense-extensions of the conforming predicates, since the sense-extensions of names and characteristic predicates are invariant.

4.2.10 Describing and Reflecting

Given its lack of the capacity to refer, a natural object language taken nonreferentially cannot be sensibly said to describe the level of reference of that language. Similarly, it cannot be taken to describe the level of sense. That level can only be described by a metalanguage for that object language that is referentially understood. The object language itself, therefore, is better said, as I wish to suggest, to *reflect the level of sense* and not to describe it. Taken nonreferentially, the object language is "reflective" rather than "descriptive".

Due to this referential impotence, sentences of the object language as nonreferentially understood, cannot be plausibly taken to give rise to any kind of ontological commitment. Only the metalanguage of the semantic theory for that object language may be understood in that way. This is a further respect in which an object language taken nonreferentially differs from its referential interpretation. Understood referentially, the object language will, indeed, be descriptive and ontologically committal.

In view of the discussion of Chapter 1, I wish to suggest that fictional and modal language is naturally viewed as being reflective. Indeed, I wish to suggest that, with respect to fictional and modal discourse, "talking about something" is just a loose way of speaking for "reflecting sense-extensions". So when we purport to be talking about Santa Claus, we reflect upon the level of sense with respect to the name 'Santa Claus' (or the concept SANTA CLAUS). (Of course, since every name has a sense-extension, we may also reflect upon the level of sense with respect to a denoting name like 'Tarski'.) Similarly, our modal talk about Tarski is more adequately understood in terms of reflection upon the various ways the level of sense could have been arranged with respect to the name 'Tarski' (or the concept it expresses).

4.2.11 Truth with Respect to the Level of Sense

According to the standard, i.e., referential way of interpreting the semantics of ordinary language assertoric sentences, an atomic sentence, for example, one which consists of a monadic predicate and a proper name is true just in case the predicate (or the open sentence) is satisfied by (or true of) the object which is denoted by the proper name. For example, the sentence 'Ulysses is a man' will be true just in case the predicate '... is a man' is satisfied by the denotatum of 'Ulysses'.

The only way to accept the truth of 'Ulysses is a man' on the standard referential account will be to admit some sort of fictional object to which 'Ulysses' is taken to refer. But such ontological inflation, which typical of denotational semantics of fiction, inflates the metaphysical agenda and raises, in my opinion, more questions than it purports to solve. There are two venerable alternatives to treat such statements within denotational first-order logic: first, one may declare such sentences to be false, second, one may attach to them some nonclassical truth value. Obviously, both proposals run counter to the intuition that that sentence is true. (Moreover, the Russellian strategy of ruling 'Ulysses is a man' to be false treats that sentence in the same way like such uncontroversially false sentences as 'Ulysses is a woman', without providing an account of the intuitive difference. The second alternative, on the other hand, is runs not only counter to intuitions, but also complicates logic. As I have noted in Subsection 1.1.1.e, there are also other reference involving approaches, for example, of a pretence-theoretic sort. However, such accounts give rise to the problems of reference and truth within the context of pretence or fiction.)

According to the present theory, as will be sufficiently clear by now, there is also a nonreferential way of interpreting natural language sentences. On this theory (absolute) truth does not coincide with referential truth, for there also is truth with respect to the level of sense. An atomic sentence is *true with respect to the level of sense* if and only if it is contained in the sense-extensions of all its constituent terms; or, to put it in other words, it is true with respect to that level just in case all the sense-extensions of the constituent terms coincide on that sentence. Read nonreferentially, the sentence 'Ulysses is a man' will be true with respect to the level of sense just in case the sense-extensions of the name 'Ulysses' and the predicate '... is a man' coincide; since, indeed, they do, the sentence is true with respect to the level of sense. However, the sentence 'Ulysses saved the life of Democoon' is false with respect to the level of sense, because the (actual) sense-extension of the predicate '... saved the life of ...' does not contain that sentence. Accordingly, 'Ulysses is a man' does accurately reflect the (actual) level of sense, but 'Ulysses saved the life of Democoon' does not. In this way, truth with respect to the level of sense is a matter of meaning and of sense-extensions, rather than meaning and referents. The sentence 'Socrates is a man', for instance, is unlike the previous one referentially true when read referentially and true with respect to the level of sense when viewed nonreferentially. (I shall argue later, in Section 4.4, that both kinds of truth coincide, in effect, for nonmodal sentences in which only denoting names occur.)

Modal discourse (and thought) is, as I wish to suggest, to be evaluated nonreferentially. Modal statements are, if true, true only with respect to the level of sense. On the present theory, a possibility statement like for instance 'Socrates could have been a navigator' taken nonreferentially will be said to be true with respect to the level of sense *simpliciter* just in case that level could have been such that the sense-extensions of 'Socrates' and of '... is a navigator' did coincide on 'Socrates is a navigator'. Similarly, 'Agamemnon is necessarily a man' will be true *simpliciter* with respect to the level of sense just in case no matter how the level of sense could have been arranged (in the basic sense of 'could'), the sense-extensions of 'Agamemnon' and '... is a man' would coincide on 'Agamemnon is a man'. On the present proposal, then, sentences of the object language are not taken to describe the vicissitudes of more or less bizarre denotata at more or less bizarre possible worlds; instead, they are taken to reflect how the level of sense could have been arranged in agreement with the Principle of Rearrangement.

4.2.12 From Sentences to Propositions

The notion of truth with respect to the level of sense has been explained in terms of sense-extensions of natural language expressions (or more exactly, in terms of expression types). But it is not mandatory to explain that notion in terms of expressions of some natural language. Instead, we could have explained it, in an analogous way, in terms of a language of thought and mental representations. Moreover, there is also a nonlinguistic alternative. We may explain the notion of truth with respect to the level of sense directly in terms of concepts and structured propositions rather than in terms of their linguistic counterparts.

We shall assume that for every atomic expression of a natural language there is an atomic concept that is expressed by that expression. Following common practice, we take it that the concept captures the content which the expression expresses when it is freed from the accidental features of the natural language to which it belongs. The concepts which we shall employ will be of a Fregean sort; and we shall follow the convention of using English expressions written in capital letters to refer to them. So, for example, the German predicate '... ist ein Mann' and the English predicate '... is a man' both express the predicative Fregean concept IS A MAN. Similarly, the German 'Odysseus' and the English 'Ulysses' express the name concept ULYSSES. Sentences express Fregean propositions. The sentences 'Odysseus ist ein Mann' and 'Ulysses is a man', for instance, both express the atomic Fregean proposition ULYSSES IS A MAN. We shall assume that

Fregean propositions are built up from atomic concepts in a sentence-like way and that they cay be represented as ordered tuples of such concepts, e.g., ⟨ULYSSES, MAN⟩. With respect to their metaphysical nature, we make the common assumption that Fregean propositions and their constituents are abstract, language-independent, and mind-independent entities.[12] Let me be more precise.

We shall assume that for any expression of a natural language there is a counterpart in a system of Fregean propositions or thoughts. This system, Th for short, is a modal first-order "language of concepts" $\langle Lex, Gram\rangle$ whose first element is a lexicon of atomic Fregean concepts containing name concepts, predicative concepts, connective concepts, quantifier concepts, and operator concepts. *Gram* generates from *Lex* well-formed (saturated and unsaturated) Fregean propositions in a way analogous to the way in which a formula of L or L^* is composed.[13] On the conceptual account of truth with respect to the level of sense, these propositions will serve as truthbearers.

For simplicity, we stipulate that for every expression of a language there is exactly one conceptual counterpart in the system of concepts which is expressed (rather than denoted) by it. Examples: The name 'Ulysses' expresses the name (or singular) concept ULYSSES, the predicate '... is a man' expresses the predicative concept MAN, the connective 'and' expresses the connective concept AND, the quantifier 'all' expresses the quantifier concept ALL, and the modal expression 'necessarily' expresses the operator concept NECESSARILY. So, for example, the sentence 'Ulysses is necessarily a man' expresses the proposition NECESSARILY(MAN(ULYSSES)). We adopt the simplifying convention that instead of writing MAN for the concept expressed by '... is a man' and NECESSARILY(MAN(ULYSSES)) for the thought expressed by 'Ulysses is necessarily a man' we just write, simulating the superficial syntactical structure of English predicates and sentences, IS A MAN and ULYSSES IS NECESSARILY A MAN, respectively.

In a way analogous to the linguistic case we assume that name concepts

[12]For a critical discussion of Fregean propositions and their rivals see, in particular, [Schiffer, 2003] ch. 1. See also [Villanueva, 1998] for a representative anthology of recent philosophical work on concepts.

[13]A language system of this sort is offered in [Peacocke, 1999] ch. 4.

and predicative concepts have a definitional structure. And we take it that this structure mirrors the meanings of names and predicates. For example, the definitional structure of the name concept ULYSSES will be mirrored by the list THE SON OF LAERTES, THE HUSBAND OF PENELOPE, THE FATHER OF TELEMACHUS, THE KING OF ITHACA, THE MAN WHO KILLED DEMOCOON, ...; and the definitional structure of the predicative concept IS A BACHELOR may be displayed by the list IS A MAN, IS ADULT, IS NOT MARRIED.

The notions of sense-extensions for name concepts, of sense-extensions of predicative concepts, of the level of sense, and truth with respect to the level of sense may now be explained in conceptual terms in a way strictly analogous to the linguistic account. So the proposition ULYSSES IS A MAN will be true with respect to the level of sense just in case the sense-extensions of ULYSSES and IS A MAN which encode, so to speak, the definitionally determined structure of these concepts coincide on that proposition.

We shall assume that propositions are the primary bearers of truth. Accordingly, we take it that the sentences 'Odysseus ist ein Mann' and 'Ulysses is a man' owe their truth with respect to the level of sense to the truth of the proposition ULYSSES IS A MAN with respect to that level. On this assumption then the conceptual account is more basic than the linguistic version. However, this assumption, to which we shall stick from now on, is not mandatory. And, of course, it should be possible to adapt other views of the nature of concepts and propositions than the Fregean to the conceptual account.

4.2.13 Definitional Necessity

Clearly, on the present framework, a sentence like $(\Sigma x)\Box Fx$ cannot be viewed as claiming that some individual has a certain property necessarily or essentially. For, as will be clear by now, the semantics does not explain the truth of such sentences in terms of individuals (or, more broadly, objects), reference and satisfaction. Consequently, sentences like these do not symbolize *de re* necessities. They symbolize *de nomine* necessities. And since a formula like $\Box Fa$ does not symbolize a *de re* necessity, the box-operator does not symbolize metaphysical necessity either. Associative semantics, therefore, cannot be taken to model the logical behaviour of that notion. According to the present framework, the modal operator symbolizes what we might call *definitional modality*. The meanings of 'definitionally necessary' and 'definitionally possible' may be specified by the following conditions.

> (Necessity) 'It is definitionally necessary that A' is true with respect to the level of sense (i.e., the actual level of sense) just in case no matter how the level of sense could have been arranged (in the basic sense of 'could'), A would be true with respect to it.
>
> (Possibility) 'It is definitionally possible that A' is true with respect to the level of sense just in case the level of sense could have been arranged (in the basic sense of 'could') such that A was true with respect to it.

It is important to note that it would be a mistake to equate definitional necessity with conceptual necessity as usually understood. For example, according to the present nonreferential approach to modality, the propositions expressed by the sentences 'All bachelors are unmarried' and 'Socrates is a man' are both definitional necessities; where the former is *de dicto* and the latter is *de nomine*. These propositions are true with respect to the level of sense in virtue of the sense-extensions of the names and predicates involved in them (and, in case of the first sentence, also in virtue of the meaning of logical vocabulary).

On the usual denotational account only the proposition expressed by the former will qualify for the status of a conceptual necessity (i.e., a proposition which is referentially true in virtue of the meaning of the logical constants and the predicates involved in it). On the other hand, the proposition expressed by the latter sentence, will be a metaphysical necessity *de re*, a proposition which—on the usual essentialist account of such necessities—is referentially true in virtue of the nature or the essence of the object Socrates.[14] Definitional necessity, therefore, must not be confused with metaphysical necessity *de re* nor with conceptual necessity.

[14]See, for example, [Fine, 1994] p. 9. Cf. also [Lowe, 1999] p. 19, bottom.

Moreover, definitional necessity must not be confused with the notion of broadly logical necessity which is standardly equated with metaphysical necessity. On the standard account of this notion, it encompasses, for example, (narrowly or strictly) logical, conceptual, and *de re* metaphysical necessity as special cases.[15] Fine's essentialist account of metaphysical necessity is, as it seems to me, the first proposal on which a definition of this notion is provided that pays attention to questions of source. For him metaphysical necessities (or broadly logical necessities) are those propositions which are true in virtue of the natures of objects.[16] Conceptual and logical necessity, for instance, are regarded as special cases of metaphysical necessity. A proposition is said to be a conceptual necessity if it is true in virtue of the nature of the concepts involved in it. The reason why this is so is obvious. Since the notion of definitional necessity is not explained in terms of objects and their properties, none of its special cases can be explained in that way. According to the present approach, modal discourse is, properly understood, nonreferential and thus has no use for referential predication, referential quantification and an ontology of individuals and their properties. Consequently, there is no room for metaphysical necessity and essence or—more exactly—linguistically irreducible real definitions. (For the latter see comment 4.2.3(1).)

We shall now make a couple of taxonomical remarks on definitional necessity. Definitional necessity can be either *de dicto*, or *de nomine*—where in a way analogous to the formal discussion of Subsection 2.5.7, the former sort of definitional necessity, unlike the latter, does not turn upon other ways the sense-spectrum of predicates could have been arranged.

We may view other notions of necessity, for instance, conceptual, logical necessity, or the kind of necessity which pertains to such propositions like the one expressed by 'Socrates is a man' as special cases of definitional ne-

[15]The standard reference on broadly logical necessity is [Plantinga, 1974] pp. 1-2. Plantinga explains this notion by way of example and so does the tradition following him. See, for example, [Forbes, 1985] pp. 1-2. For a recent discussion of this notion (in terms of possible worlds) see, in particular, [Lowe, 1999], sections 3-6. The notion of broadly logical necessity is standardly equated with that of metaphysical necessity. Some authors seem to deviate from received terminology. In [Hale, 1996] p. 94 and [Chihara, 1998] p. 7, for instance, 'broadly logical necessity' is used interchangeably with 'conceptual necessity'.

[16]See [Fine, 1994] pp. 9-10 and [Fine, 2002b] p. 254.

cessity. We may view them as being defined in terms of a restriction of the notion of definitional necessity. For example, a proposition will be a logical (conceptual) necessity if (i) it is a definitional necessity and if (ii) its truth turns upon the meanings of the the logical terms (and the predicates) which occur in it. (In this way logical necessity may be taken to be a special case of conceptual necessity.) Conceptual (and hence logical) necessities may then be subsumed under the notion of *de dicto* definitional necessity. By contrast, the kind of necessity in which a proposition like the one expressed by 'Socrates is a man' may be said to be necessary will fall under the notion of definitional necessity *de nomine*. It is natural to assume that a proposition is necessary in this sense if (i) it is a definitional necessity and if (ii) its truth does in no substantial way turn upon the meanings of logical terms.[17]

In effect, the work done on denotational account by the notions of essence, individuals and metaphysical necessity (and its special cases) is done, on the present account, by the notions of nominal definition (for names and predicates), sense-extension, and definitional necessity.

4.3 Truth at an Index in an Associative Model and Truth With Respect to the Level of Sense

The strategy I shall apply to connect up the relative notion of truth at an index in an associative model and the absolute notion of truth *simpliciter* with respect to the level of sense is adapted from the denotationalist tradition of intended* Kripke-models as pioneered in [Menzel, 1990] and modified and developed further in [Ray, 1996] and [Chihara, 1998]. The strategy of

[17][Fine, 2002b] p. 255 offers a different kind of definition of new notions of necessity in terms of the restriction of an old one. For example, on his view a proposition may be said to be a mathematical necessity if (i) it is a metaphysical necessity (along the lines suggested in [Fine, 1994]) and if (ii) it is a mathematical truth (where the notion of mathematical truth is hoped to be explicable in non-modal terms or, at least, without appeal to further modal notions). I shall contrast the notion of definitional necessity with Fine's notion of metaphysical necessity further in Subsection 4.5.10.

intended* models has a couple of features which, in my opinion, are desirable for any account of the connection between model relative truth and absolute truth.

One advantage is that it takes the mathematical nature of the models of pure semantics seriously, since it endorses the view that they are purely mathematical items whose constitution involves only mathematical objects. This attitude towards pure models allows to appreciate the chasm between model-theory and robust reality which has to be bridged. Second, this strategy does, arguably, not involve a commitment to possible worlds (standardly figuring in the index-set of the intended model) and possible objects (as contained in the domain of discourse of the intended model) of any sort. Of course, on a substitutional approach a commitment to possible objects is avoided for other and, by now, obvious reasons. Thirdly, the strategy of intended* models provides an explanation of how the models of pure semantics do succeed to represent reality. I doubt that we learn something about the way a pure model represents reality by just being told which entities are contained in (the index set and) the domain of some intended model.

The strategy which we shall use to provide a link between truth at an index in an associative model and the notion of truth with respect to the level of sense is essentially the strategy proposed by Charles Chihara adapted to the needs of the present nonreferential approach.[18] Chihara's account of model-theoretic modal semantics differs from Menzel's approach and Ray's version thereof in a couple of ways which need not concern us here.

The most noteworthy difference is that Menzel's and Ray's account is explicitly representational in the sense of [Etchemendy, 1990].[19] Chihara rejects the correctness of Etchemendy's classification of Tarskian first-order

[18]See [Chihara, 1998] ch. 7. Since a detailed discussion of Chihara's work would lead us to far away from our present concerns, the reader is urged to study that chapter in order to appreciate the differences between Chihara's proposal and my adaptation thereof.

[19]According to Chihara, the difference between representational and interpretational semantics can be generally characterized as follows: "the former keeps the interpretation of the language fixed and considers, by means of the models, different ways the world could have been; whereas the latter keeps the world fixed and considers, by means of the models, different interpretations of the language." ([Chihara, 1998] p. 185).

model theory as interpretational as incorrect and doubts that it is adequately characterized as representational as well.[20]

On Chihara's account of the Tarskian semantics of first-order logic[21] denotational models "provide abstract [i.e., mathematical] representations of different possible domains and extensions of predicates, without settling the question of whether it is the meanings of the parameters [i.e., the nonlogical vocabulary of the denotational first-order language] that are being held fixed, while the different possible situations are up for consideration, or it is the world that is being held fixed, while the different assignments of meanings to the parameters are up for consideration."[22]

First-order associative models involve no other entities than items of the formal language itself and they do not represent some level of denotata. Note, however, that Etchemendy's originial distinction may be restated for first-order associative semantics and the level of sense. As we shall see below, Chihara's account can be adapted to the framework of associative semantics as well. The result of this adaptation will be the account of associative first-order semantics I wish to endorse.

A clear advantage of Chihara's proposal is that where Menzel and Ray provide only a piecemeale inductive evidence for the link between relative and absolute truth, Chihara proves a theorem (his "Fundamental Theorem") to this effect.

Chihara proves his theorem for a relatively complicated varying domain axiom system which is in accordance with his endorsement of Serious Actualism (i.e., the thesis, which put in terms of possible worlds realism says, that if an object has a property in a possible world, then it must exist in that world).[23] We shall adapt Chihara's denotationalist Fundamental Theorem for a constant (and hence very simple and appealing) substitution class axiom system so as to provide a connection between the notions of truth in a natural associative model of the form $\mathcal{M} = \langle S, R, s^{@}, C, c, P, v \rangle$ and truth *simpliciter* with respect to the level of sense. In doing this we shall retain Chihara's terminology and notation as far as we can.

[20]See [Chihara, 1998] sect. 5.1 for the details.

[21]The essentials of Chihara's account are, in effect, given in Subsection 4.4.2 below.

[22][Chihara, 1998] p. 196.

[23]Cf. [Chihara, 1998] sect. 7.2.

4.3.1 Natural Language Proto-Interpretations of L^*

We first provide the symbols of our formal language L^* with meanings so that the resulting meaningful formulae can be treated as expressing propositions which are true or false *simpliciter* with respect to the level of sense. Let $\Im$ be a *natural language proto-interpretation* of L^* be a meaning assignment, which specifies the meanings of the symbols of that language as follows.

1. The truth functional connectives $\neg, \wedge, \vee, \rightarrow$, and $\leftrightarrow$ have the truth functional meanings of the corresponding English connectives 'it is not the case that', 'and', 'or', 'if ... then', and 'if and only if', respectively.

2. The meanings of the modal operators $\Box$ and $\Diamond$ are given by the following conditions:

   (a) A formula $\Box A$ is true iff it is definitionally necessary that A is true.

   (b) A formula $\Diamond A$ is true iff it is definitionally possible that A is true.

   The meanings of 'definitionally necessary' and 'definitionally possible' in turn are captured by the conditions (Necessity) and (Possibility) given in Subsection 4.2.13 above.

3. To give meaning to the substitutional quantifiers we select an English predicate which delimits their substitution class to just those names yielding a true sentence when substituted in an atomic open sentence built up from that predicate. (Unlike Chihara, we assume that this delimiting predicate need not be monadic.) To achieve the widest possible scope for the substitution classes, we let that predicate be, somewhat trivially, the English predicate for self-identity—in the substitutional (not the referential) sense of the word. If the substitution class is delimited by '... is (substitutionally) self-identical with ...', then the meaning of $(\Pi x)A$ is given by the condition: $(\Pi x)A$ is true with respect to the level of sense iff all names which are (substitutionally) self-identical are such that they render A true with respect to the level of sense when substituted for x in A.

4. Each nominal constant is assigned the meaning of an English name (i.e., a proper name or a definite description) which, trivially, is (substitutionally) self-identical.

5. Pure n-ary predicates (incl. aspectualized binary predicates) and the predicate for $\doteqdot$ are assigned the meanings of n-ary English predicates and the meaning of identity in the substitutional sense of '... is identical with ...', respectively.

6. Nominal variables do not obtain English meanings. We take them to perform a function like that of pronouns in open sentences (or sentential functions) of English like, for instance, in 'x is a man'.

4.3.2 Natural Language Proto-Interpretations of L^*: Notation and Explanations

Next we adopt a couple of notational conventions for proto-interpretations.

1. If $\Im$ is a natural language proto-interpretation of L^* and C is a substitution class for the quantifiers, then '$\Im(C)$' designates the English predicate which delimits C according to $\Im$, in our case the predicate '... is (substitutionally) identical with ...'. In accordance with the intuitions expressed in Subsection 4.2.1, the English identity predicate is always substitutional when it is flanked by nondenoting names or when it occurs in a modal context. In this way the "denotational status" of the terms to which the English identity predicate applies or the context in which that predicate occurs indicate how it has to be understood. There is no extra word like 'substitutionally identical' in ordinary English, just like there is no extra word 'referentially identical' in it. (Although there are no two different predicates for identity in ordinary English there will presumably be two different concepts which the single predicate may be taken to express.) We write '(substitutionally) identical' and '(referentially) identical' to indicate how the identity predicate is to be understood, although, it will be always clear when it must be understood substitutionally. (I shall argue later

in Section 4.4 that in the "referential portion" of English the identity predicate can be understood both ways.) Analogous remarks apply to the quantifiers of English.

2. If α is a nominal constant of L^*, then '$[\alpha/\Im]$' designates the nominal constant α taken to have the meaning assigned to it by $\Im$. Consequently, the name and $[\alpha/\Im]$ have the same meaning, namely that of the name. We say that $[\alpha/\Im]$ is the *exact synonym of the name* for the nominal constant α. For example, if the meaning assigned to a is the meaning of 'Pegasus', then the exact synonym of the name for that constant, $[a/\Im]$, will have the same meaning as 'Pegasus'. Both expressions express, as we take it, the concept PEGASUS.

3. If φ^n is a pure predicate of L^*, then '$[\varphi^n/\Im]$' designates the predicate φ^n taken to have the meaning assigned to it by $\Im$. Consequently, the English predicate and $[\varphi^n/\Im]$ have the same meaning; namely that of the English predicate. We say that $[\varphi^n/\Im]$ is the *exact synonym of the predicate* for the formal predicate φ^n. So, for instance, if the meaning assigned to F is the meaning of '... is white' then the exact synonym of the English predicate for F, that is $[F/\Im]$, has the same meaning as '... is white'. Moreover, both express the concept WHITE. The expression $[\ddot{=}/\Im]$ is to be undestood analogously.

4. If A is a formula of L^*, then '$[A/\Im]$' designates the formula A taken to have the meaning assigned to it by $\Im$. For example, if the meaning assigned to a is the meaning of 'Pegasus' and the meaning assigned to F is the meaning of '... is white', then the *exact synonym of the sentence* for the pure atom Fa, $[Fa/\Im]$, has the same meaning like 'Pegasus is white'. Both expressions express the proposition PEGASUS IS WHITE.

Instead of providing the symbols of L^* with meanings of natural language expressions, we could have, equally well, introduced a conceptual analogue to $\Im$ which would assign to each symbol directly a concept together with a meaning specifying condition given in terms of concepts. In order to obtain a broader perspective, i.e., a picture which comprises both the linguistic and the conceptual realm, we shall use the linguistic version of proto-interpretations, on the one hand, and consider conceptual sense-extensions of names and predicates, on the other hand.

4.3.3 Natural Language Proto-Interpretations of L^* and Sense-Extensions

We now introduce *sense-extensions for exact synonyms* of names and predicates.

1. 'senext-$[\alpha/\Im]$' designates the sense-extension of $[\alpha/\Im]$, that is, the sense-extension for the name concept which is expressed by the name $[\alpha/\Im]$ is exactly synonymous with.
2. 'senext-$[\varphi^n/\Im]$' designates the sense-extension of $[\varphi^n/\Im]$, i.e., the sense-extension for the pure predicative concept which is expressed by the English predicate $[\varphi^n/\Im]$ is exactly synonymous with.

Since the meaning of a name and the meaning of its exact synonym are the same and since the meaning of a name determines the sense-extension of that name, the name and its exact synonym have the same sense-extension—to wit, the sense-extension of the name. For example, the sense-extension of 'Agamemnon' and $[a/\Im]$, where $\Im$ assigns to a the meaning of 'Agamemnon' is {AGAMEMNON IS A MAN, AGAMEMNON IS A WORRIER, ...}. Analogously, for English predicates and their exact synonyms.

4.3.4 Representation via Bijections

So far we have explained how the expressions of L^* are related to natural language. What we still need is an account of how natural associative models (as characterized in Subsection 3.2.5) represent reality—more specifically, the portion which is constituted by level of sense— and, speaking loosely, the ways in which the level of sense could have been arranged.

Let $\Im$ be a natural language proto-interpretation, let $\mathcal{M} = \langle S, R, s^{@}, C, c, P, v\rangle$ be a natural (and hence, constant substitution class) associative model of L^*, let C be the set of name concepts (more exactly, the self-identical ones), let P be the set of the predicative concepts, and let Atm be the pure atomic propositions of the system Th presented in Subsection 4.2.12.

Moreover, suppose that, for some $s \in S$, the level of sense could have been (in the basic sense of 'could') such that there was a bijective function for s, f_s, from $c(s)$ (i.e., C which is $\bigcup_{s\in S} c(s)$) to C, from P to P, and from Atm to Atm such that:

1. For for each $\alpha \in c(s)$ and for every $A \in Atm(\alpha)$:

   $\{\varphi^n...\alpha..., \psi^n...\alpha..., \chi^n...\alpha..., ...\} \subseteq v(\alpha, s)$
   iff
   $\{f_s(\varphi^n...\alpha...), f_s(\psi^n...\alpha...), f_s(\chi^n...\alpha...), ...\} \subseteq$ senext-$[\alpha/\Im]$.

2. For each pure n-ary predicate $\varphi^n \in P$ and for every $A \in Atm(\varphi)$:

   $\{\varphi^n...\alpha_1..., \varphi^n...\alpha_2..., \varphi^n...\alpha_3..., ...\} \subseteq v(\varphi^n, s)$
   iff
   $\{f_s(\varphi^n...\alpha_1...), f_s(\varphi^n...\alpha_2...), f_s(\varphi^n...\alpha_3...), ...\} \subseteq$ senext-$[\varphi^n/\Im]$.

The '$[\alpha/\Im]$' in 'senext-$[\alpha/\Im]$' expresses, so to speak, $f_s(\alpha)$ and the '$[\varphi^n/\Im]$' in 'senext-$[\varphi^n/\Im]$' expresses $f_s(\varphi^n)$.

Since f_s is a bijection the sets on which this function operates have the same cardinality. So C (P, Atm) will be finite (denumerable, nondenumerable) if C (P, Atm) is. Since on the present proposal P is finite P will be, presumably, only a proper subset of the predicates of a natural (or mental) language or of a language-like system of concepts.

4.3.5 Representation via Bijections: Conventions

In a way analogous to Chihara we view an *associative bijection-tuple* $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_\Im$ for some s in a model as representing via f_s how the (actual) level of sense was, in so far as what was of concern were the substitution classes of the quantifiers and the sense-extensions of the nominal constants and predicates, respectively, when these are interpreted as specified by $\Im$.

So we say that $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_\Im$ represents the level of sense via f_s. Mimicking Chihara's practice, we take 'the level of sense could have been (in the basic sense of 'could') such that there was a function f_s via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it' to be short for the passage "for some $s \in S$ the level of sense could have been (in the basic sense of 'could') such that there was a bijective function (...) $\subseteq$ senext-$[\varphi^n/\Im]$" from the third paragraph of the preceding section.

4.3.6 Natural Language Interpretations

Natural language interpretations are natural language proto-interpretations $\Im$ that conform to some natural model $\mathcal{M} = \langle S, R, s^@, C, c, P, v\rangle$ of L^*.

For a proto-interpretation to conform to a (natural) model, the model must accurately represent the sense-extensions of the names and predicates. $\mathcal{M} = \langle S, R, s^@, C, c, P, v\rangle$ accurately represents the level of sense if it meets the following three Chiharaian Menzel-style conditions.

The first condition imposed on the model ensures that there are no impossible bijection-tuples.

> (C1) For every $s \in S$, the level of sense could have been (in the basic sense of 'could') such that there was a bijection via which the tuple $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

The second condition guarantees that there are enough bijection-tuples to represent, loosely speaking, every possible way the level of sense could have been arranged.

> (C2) No matter how the level of sense had been arranged (i.e., necessarily in the basic sense), there would have been a bijection via which the tuple $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented how it were arranged for some $s \in S$.

Finally, the third condition aims to ensure that all the bijections which could have represented (the actual way) the level of sense (is arranged), agree with each other. To get at the idea behind this principle, suppose that $\mathcal{M} = \langle S, R, s^@, C, c, P, v\rangle$ accurately represents the level of sense and that, thus, there is a bijection $f_{s^@}$ which maps, say, the nominal constant a from

$c(s^{@})$ to a certain name concept, e.g. AGAMEMNON, from C and which maps, say, the monadic predicate F^1 from P to IS A WORRIER from P. Then the requirement is that a bijection g_s for any other index s in that model does not qualify as a representing function if it maps a to AJAX (even if it still maps F^1 to IS A WORRIER). This condition of agreement, therefore, allows us to study the behaviour of concepts through, loosely speaking, various ways the level of sense could have been arranged. The condition may be stated in more exact terms as follows.

(C3)

(a) For every $s, s_1, ..., s_n \in S$ and for every $\alpha, \alpha_1, ..., \alpha_n \in c(s)$ such that $\alpha_1 \in c(s_1), ..., \alpha_n \in c(s_n)$: if $\alpha_1, ..., \alpha_n \in c(s)$ then it is not the case that the level of sense could have been (in the basic sense of 'could') such that there was a bijection f_1 via which $\langle c(s_1), v(\alpha, s_1), v(\varphi^n, s_1)\rangle_{\mathfrak{I}}$ represented it, ..., the level of sense could have been such that there was a bijection f_n via which $\langle c(s_n), v(\alpha, s_n), v(\varphi^n, s_n)\rangle_{\mathfrak{I}}$ represented it, such that, had the level of sense been such that there was a bijection via which $\langle c(s),\ v(\alpha, s), v(\varphi^n, s)\rangle_{\mathfrak{I}}$ represented it, it is not the case that there would have also been a bijection g via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\mathfrak{I}}$ represented the level of sense, such that $g(\alpha_1) = f_1(\alpha_1)$, ..., $g(\alpha_n) = f_n(\alpha_n)$.

(b) For every representing bijection f_1 for s_1, ..., f_n for s_n, there is a repesenting bijection g for s such that for every $\varphi^n, \psi^n, \chi^n, ... \in P$: $g(\varphi^n) = f_1(\varphi^n), g(\psi^n) = f_2(\psi^n), g(\chi^n) = f_3(\chi^n)$, and so on for every f-subscript $n \geq 1$.

(c) For every representing bijection f_1 for s_1, ..., f_n for s_n, there is a repesenting bijection g for s such that for every $A_1, A_2, A_3, ... \in$ *Atm*: $g(A_1) = f_1(A_1), g(A_2) = f_2(A_2), g(A_3) = f_3(A_3)$, and so on for every f-subscript $n \geq 1$.

Chihara's account involves a further constraint which captures the intuition of Serious Actualism (roughly, if an object has a property in a possible world, then it must exist in that world).[24] As the reader might expect, on the present

[24]See [Chihara, 1998] pp. 235-36 and p. 238.

framework we need not worry about *possibilia* and (serious) actualism, since there are no objectual domains and, therefore, no room for individuals and individual representatives. Moreover, since the substitution classes of natural models are constant, a situation which would motivate an associative analogue of Serious Actualism does not arise. We shall return to actualism later (in Subsection 4.5.5).

4.3.7 Truth With Respect to the Level of Sense Restated

Having explained the notion of natural language interpretation, we may now reformulate the notion of truth *simpliciter* with respect to the level of sense (which has been previously stated for sentences of some suitably regimented ordinary modal language or some language-like system in Subsection 4.2.12) in terms of their exact synonyms, i.e., in terms of formulae of the formal modal language L^* which have received natural language interpretations. This reformulation will allow us to isolate the connecting points between which the link between the relative notion of truth at an index in a (natural) associative model for formulae of L^* and the absolute notion of truth *simpliciter* of ordinary modal claims with respect to the level of sense will be forged.

We shall assume that every expression of the modal language L^* (except for the nominal variables) has received a natural language interpretation and that the interpreted nonlogical vocabulary, more exactly, the exact synonyms of English names (i.e., proper names and definite descriptions) and predicates have their sense-extensions and can be true or false with respect to the level of sense. In order to provide the nominal variables which have hitherto received no natural language interpretations with sense-extensions, we introduce the notion of a *natural language interpreted nominal (or substitutional) variable assignment* $\sigma_{\Im(C)}$ (the subscript indicates that the assignment is relative to the substitution class as restricted by the delimiting English predicate $\Im(C)$). $\sigma_{\Im(C)}$ assigns to every substitutional variable $[x]$ of the natural language interpreted formal language L^* an English name, or more accurately its exact synonym. The sense-extension of that variable, $\sigma_{\Im(C)}[x]$ will then be the sense-extension of that synonym.

We now can redefine the notion of truth *simpliciter* of sentences of a modal natural language with respect to the level of sense in terms of truth *simpliciter* of natural language interpreted formulae of the formal modal language L^* with respect to the level of sense under an interpreted nominal variable assignment $\sigma_{\Im(C)}$ in a way analogous to the formal case in Subsection 2.4.2.

1. If A is an atomic formula of the formal modal language L^* and $o_1, ..., o_n$ are the n nominal terms occuring in A and φ^n is the predicate of that formula, then $[A/\Im]$ is true *simpliciter* with respect to the level of sense under an interpreted nominal variable assignment $\sigma_{\Im(C)}$ just in case the level of sense could have been (in the basic sense of 'could') such that $[A/\Im]$ was contained senext-$[\sigma_{\Im(C)}[o_1]/\Im]$ through senext-$[\sigma_{\Im(C)}[o_n]/\Im]$ and in senext-$[\varphi^n/\Im]$.

2. If $\neg A$ is a formula of the formal modal language L^* then $[\neg A/\Im]$ is true *simpliciter* with respect to the level of sense under an interpreted nominal variable assignment $\sigma_{\Im(C)}$ just in case $[A/\Im]$ is not true *simpliciter* with respect to the level of sense under an interpreted nominal variable assignment $\sigma_{\Im(C)}$.

And so on. In particular, the truth *simpliciter* of modal statements with respect to the level of sense under an interpreted nominal variable assignment $\sigma_{\Im(C)}$ will be characterized as follows:

> (Necessity′). If $\Box A$ is a formula of the formal modal language L^*, then $[\Box A/\Im]$ is true *simpliciter* with respect to the level of sense under an interpreted nominal variable assignment $\sigma_{\Im(C)}$ just in case no matter how the level of sense could have been (i.e., necessarily in the basic sense), $[A/\Im]$ would be true *simpliciter* with respect to it under an interpreted nominal variable assignment $\sigma_{\Im(C)}$.
>
> (Possibility′). If $\Diamond A$ is a formula of the formal modal language L^*, then $[\Diamond A/\Im]$ is true *simpliciter* with respect to the level of

sense under an interpreted nominal variable assignment $\sigma_{\Im(C)}$ just in case the level of sense could have been (in the basic sense of 'could') such that $[A/\Im]$ was true *simpliciter* with respect to it under an interpreted nominal variable assignment $\sigma_{\Im(C)}$.

Truth *simpliciter* with respect to the level of sense is then defined as truth *simpliciter* with respect to every interpreted nominal variable assignment $\sigma_{\Im(C)}$. So we may put the truth conditions for modal sentences of ordinary language as follows:

(Necessity$''$). If $\Box A$ is a closed formula of the formal modal language L^*, then $[\Box A/\Im]$ is true *simpliciter* with respect to the level of sense (under every interpreted nominal variable assignment $\sigma_{\Im(C)}$) just in case no matter how the level of sense could have been (in the basic sense of 'could') $[A/\Im]$ would be true *simpliciter* with respect to it (under every interpreted nominal variable assignment $\sigma_{\Im(C)}$).

(Possibility$''$). If $\Diamond A$ is a closed formula of the formal modal language L^*, then $[\Diamond A/\Im]$ is true *simpliciter* with respect to the level of sense (under an interpreted nominal variable assignment $\sigma_{\Im(C)}$) just in case the level of sense could have been (in the basic sense of 'could') such that $[A/\Im]$ was true *simpliciter* with respect to it (under every interpreted nominal variable assignment $\sigma_{\Im(C)}$).

(From now on we shall no longer indicate that the modal notions involved in the right-hand side of the biconditionals are the basic ones.) This said and done let us next isolate the points of connection between which the link between the relative notion of truth at an index in an associative model and the absolute notion of truth *simpliciter* with respect to the level of sense will be fixed.

Recall, when $\mathcal{M} = \langle S, R, s^@, C, c, P, v \rangle$ is a natural associative model of L^* (where $s^@ \in S$ is stipulated to be a distinguished index), σ is a substitutional variable assignment, and A is a closed formula of L^*, then the following equivalences hold on the "relative side": (i) $\mathcal{M}_\sigma \models_s A$ iff (ii) for every assignment σ, $\mathcal{M}_\sigma \models_s A$; iff (iii) $\mathcal{M} \models_s A$.

Correspondingly, for the "absolute side". When A is a closed formula and $[A/\Im]$ is A taken to have the meaning assigned to it by $\Im$, then we have the following equivalences: (I) $[A/\Im]$ is true *simpliciter* with respect to the level of sense under an interpreted substitutional assignment $\sigma_{\Im(C)}$ iff (II) $[A/\Im]$ is true *simpliciter* with respect to the level of sense under every interpreted substitutional assignment $\sigma_{\Im(C)}$; iff (III) $[A/\Im]$ is true *simpliciter* with respect to the level of sense.

The intended link is the link between (iii) and (III). It will be established by a proof of a biconditional with the effect: (iii) $\mathcal{M} \models_s A$ iff (III) $[A/\Im]$ is true *simpliciter* with respect to the level of sense. The way in which this is will now be explained.

4.3.8 A Chihara-Style Connecting Theorem

We have interpreted the formal language L^*, along Chiharan lines, in terms of two kinds of interpretation. On the one hand, we have interpreted the expressions of L^* in terms of natural language proto-interpretations $\Im$ which provided them with the meanings of natural language expressions. On the other hand, we have interpreted L^* in terms of (natural) associative models which just are set-theoretic entities involving pure sets and expressions of the base language L_0. Given a proto-interpretation $\Im$, we may view these models as representing via bijections how the level of sense could have been arranged in so far as the substitution classes and the sense-extensions of the nominal constants and predicates are of concern. Here we counted $\Im$ as a natural language interpretation, when the models did represent in conformity with conditions (C1) through (C3).

What we still have to do, is to provide a link between the truth *simpliciter* of sentences of ordinary modal language with respect to the level of sense and the truth of a formula of the modal language L^* at an index in a (natural) associative model. As I have already explained, this will be accomplished when we can prove a biconditional to the effect: (iii) $\mathcal{M} \models_s A$ iff (III) $[A/\Im]$ is true *simpliciter* with respect to the level of sense. Now given the equivalences from the previous section, this link will be established when an analogous link between (i) and (I) is provided. We first fix an auxiliary link between (i) and (I) by proving our Chihara-style Connecting Theorem. The desired link will then be captured by a corollary of that theorem—the Chihara-style Principal Corollary.

> *The Chihara-Style Connecting Theorem*. For every (natural) associative model $\mathcal{M} = \langle S, R, s^{@}, C, c, \ P, v\rangle$ of L^* and for every natural language proto-interpretation $\Im$ conforming to $\mathcal{M}$: for every index $s \in S$, for every nominal constant $\alpha_1, ..., \alpha_n \in C$, and for every formula A, if $x_1, ..., x_n$ are the n nominal variables which occur in A, then the following holds: $\mathcal{M}_\sigma \models_s A$ iff for some index $t \in S$, had the level of sense been such that there was a bijective function via which the tuple $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_\Im$ represented it, there would have been a representing bijective function g for t such that some $\Im(C)$-assignment fulfilled the condition: $[A/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_\Im$ represented it.

As an immediate corollary to this theorem we obtain:

> *The Chihara-Style Principal Corollary*. For $s \in S$, for any closed formula A, $\mathcal{M} \models_s A$ iff $[A/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_\Im$ represented it.

We may now simply retrive Chihara's original proof of his Fundamental Theorem[25] in terms of the framework of associative modal semantics. The proof is relegated to Appendix B.

[25] See [Chihara, 1998] pp. 239-259.

4.4 Truth With Respect to the Level of Sense and Truth With Respect to the Level of Reference

Every natural language sentence can be evaluated nonreferentially. However, only a certain portion of natural language sentences can be evaluated referentially. On the present theory, the usual referential mode of evaluation is adequate only for sentences which (i) involve exclusively denoting singular terms and predicates which do have a referential extension, and which (ii) do not occur in modal (or, perhaps, generally in intensional) contexts. In Subsection 4.2.1 we have already dubbed this portion of a (natural) language the *referential portion* of language.

In this section we shall be concerned with the (first-order fragment of the) referential portion of natural language relating it to both the level of sense and the level of reference. To obtain a picture of this interrelation, we shall first state a simple connecting theorem for a substitutional first-order language. After that we shall do the same for a denotational first-order language. With these theorems at hand, we shall link the notion of truth with respect to the level of sense to the notion of truth with respect to the level of reference by proving what we shall call "the Level Connecting Theorem" (LCT). This theorem claims, in effect, that a sentence of the referential portion of natural language is true with respect to the level of sense just in case it is also referentially true. As in Section 4.3 we shall consider the sense-extensions of names and predicates in terms of the sense-extensions of the concepts they express.

4.4.1 The Semantics of a Substitutional First-Order Language

Let $\mathcal{L}^{s\doteq}$ be a substitutional first-order language with substitutional identity. We obtain $\mathcal{L}^{s\doteq}$ from L^* when we erase the box-operator and the corresponding formation rule. Let $\mathcal{I}$ be an nonmodal constrained associative model for

$\mathcal{L}^{s\doteq}$, a tuple $\langle C, P, v \rangle$. As before, let $\Im$ be a nonreferential natural language proto-interpretation, for $\mathcal{L}^{s\doteq}$. So $\Im$ will lack a clause for the box.

Just like $[\alpha/\Im]$ is the nominal constant α of $\mathcal{L}^{s\doteq}$ taken to have the meaning of an English name assigned to it by $\Im$, so $[\varphi^n/\Im]$ is the n-ary predicate φ^n of $\mathcal{L}^{s\doteq}$ taken to have the meaning of an English predicate assigned to it by $\Im$. And as before $[\alpha/\Im]$ and $[\varphi^n/\Im]$ are the exact synonyms of the English name and predicate, respectively. Of course, for the LCT only the synonyms of denoting names and predicates from the referential portion of language will be relevant. As before we let 'senext-$[\alpha/\Im]$' and 'senext-$[\varphi^n/\Im]$' denote the sense-extensions of $[\alpha/\Im]$ and $[\varphi^n/\Im]$, respectively.

We say that a natural language proto-interpretation $\Im$ of $\mathcal{L}^{s\doteq}$ *conforms to* $\mathcal{I} = \langle C, P, v \rangle$ if and only if the following holds: there is a bijective function f, from C to C, from P to P, and from Atm to Atm, such that

1. for each $\alpha \in C$ and for every $A \in Atm(\alpha)$:

   $\{\varphi^n...\alpha..., \psi^n...\alpha..., \chi^n...\alpha..., ...\} \subseteq v(\alpha)$
   iff
   $\{f(\varphi^n...\alpha...), f(\psi^n...\alpha...), f(\chi^n...\alpha...), ...\} \subseteq$ senext-$[\alpha/\Im]$; and

2. for each n-ary pure predicate $\varphi^n \in P$ and for every $A \in Atm(\varphi^n)$:

   $\{\varphi^n...\alpha_1..., \varphi^n...\alpha_2..., \varphi^n...\alpha_3..., ...\} \subseteq v(\varphi^n)$
   iff
   $\{f(\varphi^n...\alpha_1...), f(\varphi^n...\alpha_2...), f(\varphi^n...\alpha_3...), ...\} \subseteq$ senext-$[\varphi^n/\Im]$.

So a natural language proto-interpretation $\Im$ of $\mathcal{L}^{s\doteq}$ is called a conforming natural language proto-interpretation if it determines a first-order associative model.

We can then obtain, in a way which is similar but much simpler than the proof of our Chihara-style Connecting Theorem (see Appendix B), the following theorem.

> *Chiharan Nondenotational First-Order Connecting Theorem.* For every model $\mathcal{I}$ of $\mathcal{L}^{s\doteq}$, every $\Im$ conforming to $\mathcal{I}$, and every formula A of $\mathcal{L}^{s\doteq}$: $\mathcal{I} \models A$ iff $[A/\Im]$ is true with respect to the level of sense.

The resulting applied associative semantics is, as I wish to suggest, not only an adequate semantics for fictional language, but also perfectly appriopriate for its referential portion.

4.4.2 The Semantics of a Denotational First-Order Language

Let $\mathcal{L}^{d=}$ be an ordinary denotational first-order language with referential identity of the kind discussed in Chapter 1. In what follows the metalinguistic variables for the formulae of $\mathcal{L}^{d=}$ will be underlined in order to discern them from the formulae of $\mathcal{L}^{s\ddot{=}}$. So, for example, $(\forall \underline{x})(\exists \underline{y})(\underline{x} = \underline{y})$ is the $\mathcal{L}^{d=}$-counterpart for the $\mathcal{L}^{s\ddot{=}}$-formula $(\Pi x)(\Sigma y)(x \ddot{=} y)$. Similarly $\underline{A}$ will be the $\mathcal{L}^{d=}$-counterpart of an $\mathcal{L}^{s\ddot{=}}$-formula A. Let $\mathcal{T}$ be an ordinary first-order denotational (or "Tarskian model") for that language, i.e., an ordinary couple $\langle D, v\rangle$ consisting of a non-empty domain of denotata (i.e., pure sets) and a denotational valuation function. Importantly, we stipulate that $\mathcal{T}$ be constrained by the following condition: for every element $\mathtt{d}$ of D there is an individual constant $\underline{\alpha}$ of $\mathcal{L}^{d=}$ such that $v(\underline{\alpha}) = \mathtt{d}$. If D is nondenumerable, there will be also nondenumerably many individual constants. In this case the set of nominal constants of $\mathcal{L}^{s\ddot{=}}$ will be stipulated to be nondenumerable as well. We also assume that both languages have finitely many predicates.

Moreover, let $\Im^d$ be a referential natural language interpretation for $\mathcal{L}^{d=}$ such that:

1. The logical connectives receive their usual truth functional meanings.

2. The meanings of the referential quantifiers are given by selecting an English predicate, $\Im^d(D)$, which delimits their domain to just those objects which satisfy that predicate. We take that predicate to have the meaning of '... is (referentially) identical with ...'. The formula $(\forall \underline{x})\varphi\underline{x}$ thus has the meaning of 'Every $\underline{x}$ such that $\underline{x}$ is identical to $\underline{x}$ is such that it is φ'.

3. Each individual constant $\underline{\alpha}$ receives the meaning of a name or a rigidly denoting definite description of an object that satisfies $\Im^d(D)$.

4. Each predicate φ^n receives the meaning of an English predicate of arity n. And $=$ receives the meaning of '... is (referentially) identical to ...'.

In effect, the meaning assignments for the symbols of both languages coincide except for the quantifers and the identity symbols.

In a way analogous to the nonreferential case, we let '$[\underline{\alpha}/\Im^d]$' denote $\underline{\alpha}$ with the meaning given to it by $\Im^d$ and we let '$[\varphi^n/\Im^d]$' denote φ^n with the meaning given to it by $\Im^d$. Again, for the referential portion of language, only the synonyms of denoting names and predicates are of concern. We call $[\underline{\alpha}/\Im^d]$ and $[\varphi^n/\Im^d]$ the *referential exact synonyms* of the English name and predicate, respectively. Moreover, we let 'refext-$[\underline{\alpha}/\Im^d]$' denote the referent of the denoting English proper name or rigidly denoting definite description of which $[\underline{\alpha}/\Im^d]$ is an exact referential synonym; similarly we let 'refext-$[\varphi^n/\Im^d]$' denote the referential extension of the English predicate of which $[\varphi^n/\Im^d]$ is an exact referential synonym.

We say that a referential natural language proto-interpretation $\Im^d$ of the denotational language $\mathcal{L}^{d=}$ *conforms to* $\mathcal{T} = \langle D, v\rangle$ if and only if the following holds: There is a bijective function f, from D to the referential extension of $\Im^d(D)$ such that:

1. for every individual constant $\underline{\alpha}$ of $\mathcal{L}^{d=}$, $f(v(\underline{\alpha}))$ = refext-$[\underline{\alpha}/\Im^d]$; and

2. for every predicate φ^n of $\mathcal{L}^{d=}$, and for every n-tuple of the denumerably many members of D, $\langle \mathtt{d}_1, \dots, \mathtt{d}_n\rangle$: $\langle \mathtt{d}_1, \dots, \mathtt{d}_n\rangle \in v(\varphi^n)$ iff $\langle f(\mathtt{d}_1), \dots, f(\mathtt{d}_n)\rangle \in$ refext-$[\underline{\varphi^n}/\Im^d]$.

A referential natural language proto-interpretation $\Im^d$ of $\mathcal{L}^{d=}$ is called a conforming referential natural language proto-interpretation when it determines a Tarskian model.[26]

We can then prove the following connecting theorem.

> *Chiharan Denotational First-Order Connecting Theorem.* For every model $\mathcal{T}$ of $\mathcal{L}^{d=}$, every $\Im^d$ conforming to $\mathcal{T}$, and every formula $\underline{A}$ of $\mathcal{L}^{d=}$: $\mathcal{T} \models \underline{A}$ iff $[\underline{A}/\Im^d]$ is true with respect to the level of reference.

[26]See [Chihara, 1998] pp. 194-195.

The proof is similar, but much simpler than the proof of Chihara's original Fundamental Theorem.

In this way we obtain an applied interpretation of ordinary denotational semantics for the referential portion of natural language which employs intended*, rather than intended, Tarskian models. As I have already mentioned, I am inclined to think that denotational semantics is adequate, if adequate at all (recall, e.g., Marcus's Venus problem discussed in Subsection 1.1.1.c.), only for the referential portion of language.

4.4.3 The Level Connecting Theorem

In order to connect up the notion of truth with respect to the level of sense and the notion of truth with respect to level of reference for the sentences of the referential portion of language, we shall show the following theorem:

> *The Level Connecting Theorem.* For every nonreferentially interpreted sentence $[A/\Im]$ and for every referentially interpreted sentence $[\underline{A}/\Im^d]$: $[A/\Im]$ is true with respect to the level of sense iff $[\underline{A}/\Im^d]$ is true with respect to the level of reference.

In effect, the Level Connecting Theorem (LCT) links the "absolute" right-hand side of the Chiharan Nondenotational First-Order Connetcing Theorem to the "absolute" right-hand side of the Chiharan Denotational First-Order Connetcing Theorem. With this theorem established, we may conclude that the following theorem holds as well.

> *Formal Level Connecting Theorem.* For every model $\mathcal{I}$ of $\mathcal{L}^{s\doteq}$ and every closed formula A of $\mathcal{L}^{s\doteq}$, and for every denumerable model $\mathcal{T}$ of $\mathcal{L}^{d=}$ and every closed formula $\underline{A}$ of $\mathcal{L}^{d=}$: $\mathcal{I} \models A$ iff $\mathcal{T} \models \underline{A}$.

To prove LCT, we shall appeal to the following principle of harmony which imposes a constraint on the admissibility of assignments of semantic values to the nonlogical vocabulary of the extensional portion of natural language taken referentially and nonreferentially, respectively.

The Principle of Correlation. Any denoting name and any predicate of the referential portion of natural language is assigned its actual sense-extension when it is taken nonreferentially just in case it is also assigned its actual reference-extension when it is taken referentially; where the assignment is such that for any name and any predicate taken nonreferentially and referentially respectively—or equivalently, for their corresponding nonreferential and referential exact synonyms—the following holds:

1. $[\alpha/\Im]$ is assigned senext-$[\alpha/\Im]$ iff $[\underline{\alpha}/\Im^d]$ is assigned refext-$[\underline{\alpha}/\Im^d]$; and
2. senext-$[\varphi^n/\Im] = \{\varphi^n\alpha_1...\alpha_n, ...\}$ iff refext-$[\varphi^n/\Im^d] = \{\langle$refext-$[\underline{\alpha}_1/\Im^d], ...,$ refext-$[\underline{\alpha}_n/\Im^d]\rangle, ...\}$.

For example, according to the first part of the Correlation Principle, the name 'Socrates' receives its actual referent (i.e., Socrates) just in case it also receives its actual sense-extension (i.e. {SOCRATES IS THE SON OF SOPHRONISCUS, SOCRATES IS A MAN, SOCRATES SAVES THE LIFE OF ALCIBIADES, SOCRATES IS A BACHELOR, SOCRATES IS MARRIED, ... }). And according to the second part of the principle, the predicate '... saves the life of ...' will receive its actual reference-extension (i.e., {⟨Socrates, Alcibiades⟩, ...}) just in case it also receives its actual sense-extension (i.e., {SOCRATES SAVES THE LIVE OF ALCIBIADES, ...}).

To show LCT, we first evaluate any assertoric sentence of the (first-order fragment of the) referential portion of natural language according to the referential and the nonreferential mode and then apply the Principle of Correlation in a simple inductive proof of that theorem.

Let us first consider the evaluations. We begin with the nonreferential mode of evaluation. We symbolize natural language expressions in the nonreferential formal language $\mathcal{L}^{s\doteq}$ so as to obtain formulae A. Next we provide each of the constituent symbols of the resulting formula A with its nonreferential natural language interpretation $\Im$ so as to obtain the nonreferential exact synonym $[A/\Im]$ of the natural language sentence which it symbolizes.

Every nonlogical term which occurs in $[A/\Im]$ will receive its actual sense-extension. An $\Im$-interpreted nominal constant α, that is, the nonreferential exact synonym $[\alpha/\Im]$, will receive the actual sense-extension of the name which it is an exact synonym of, i.e., senext-$[\alpha/\Im]$.

Similarly, an $\Im$-interpreted predicate φ^n, that is, the nonreferential exact synonym $[\varphi^n/\Im]$, will receive the actual sense-extension of the natural language predicate of which it is an exact synonym. Let 'senext-$[\varphi^n/\Im]$' denote this sense-extension.

The truth conditions of nonreferentially natural language interpreted formulae of $\mathcal{L}^{s\ddot{=}}$ (where the pure atomic formulae of that language belong to the base language of $\mathcal{L}_0^{s\ddot{=}}$) are given (along the lines of Subsection 4.3.7) as follows:

1. $[\varphi^n o_1 ... o_n/\Im]$ will be true with respect to the level of sense under an interpreted nominal variable assignment $\sigma_{\Im(C)}$ just in case $[\varphi^n o_1 ... o_n/\Im]$ is contained in senext-$[\sigma_{\Im(C)}[o_1]/\Im]$ through senext-$[\sigma_{\Im(C)}[o_n]/\Im]$ and in senext-$[\varphi^n/\Im]$.

2. $[o_1 \ddot{=} o_2/\Im]$ will be true with respect to the level of sense under an interpreted nominal variable assignment $\sigma_{\Im(C)}$ just in case for every natural language interpreted atomic formula $[A_2/\Im]$ which is just like $[A_1/\Im]$ except for containing occurrences of $[\sigma_{\Im(C)}[o_2]/\Im]$ where $[A_1/\Im]$ contains occurrences of $[\sigma_{\Im(C)}[o_1]/\Im]$ the following holds: $[A_1/\Im]$ is true with respect to the level of sense under $\sigma_{\Im(C)}$ just in case $[A_2/\Im]$ is true with respect to the level of sense under $\sigma_{\Im(C)}$.

3. $[\neg A/\Im]$ will be true with respect to the level of sense under an interpreted nominal variable assignment $\sigma_{\Im(C)}$ just in case $[A/\Im]$ is not true with respect to the level of sense under $\sigma_{\Im(C)}$.

4. $[A \wedge B/\Im]$ will be true with respect to the level of sense under an interpreted nominal variable assignment $\sigma_{\Im(C)}$ just in case both $[A/\Im]$ and $[B/\Im]$ are true with respect to the level of sense under $\sigma_{\Im(C)}$.

5. $[(\Pi x)A/\Im]$ will be true with respect to the level of sense under an interpreted nominal variable assignment $\sigma_{\Im(C)}$ just in case for every interpreted substitutional variant $\tau_{\Im(C)}$ of $\sigma_{\Im(C)}$ $[A'/\Im]$ is true with respect to the level of sense under $\tau_{\Im(C)}$.

Let us now turn to the referential mode of evaluation. We symbolize the natural language sentence in the referential formal language $\mathcal{L}^{d=}$ so as to obtain a formula $\underline{A}$. Next we provide each of the constituent symbols of the resulting formula $\underline{A}$ with its referential natural language interpretation $\Im^d$ so as to obtain the referential exact synonym $[\underline{A}/\Im^d]$ of the natural language sentence which it symbolizes.

Now every nonlogical term which occurs in $[\underline{A}/\Im^d]$ will receive its actual referential extension. An $\Im^d$-interpreted individual constant $\underline{\alpha}$, that is the referential exact synonym $[\underline{\alpha}/\Im^d]$, will receive the actual referent of the name of which it is an exact synonym. Let 'refext-$[\underline{\alpha}/\Im^d]$' denote this referent.

Similarly, an $\Im^d$-interpreted predicate φ^n, that is the referential exact synonym $[\varphi^n/\Im^d]$, will receive the actual referential extension of the natural language predicate of which it is an exact synonym. Let 'refext-$[\varphi^n/\Im^d]$' denote this extension.

We let a referential natural language interpreted variable assignment, $\sigma_{\Im^d(D)}$, be the referential analogue of $\sigma_{\Im(C)}$. Such an assignment is just a natural language interpreted version of the usual denotational assignments.

The truth conditions of referentially natural language interpreted formulae of $\mathcal{L}^{d=}$ are given as follows.

1. $[\varphi^n \underline{\varrho}_1 ... \underline{\varrho}_n/\Im^d]$ will be true with respect to the level of reference under a denotational interpreted variable assignment $\sigma_{\Im^d(D)}$ just in case the n-tuple $\langle$refext-$[\sigma_{\Im^d(D)}[\underline{\varrho}_1]/\Im^d]$, ..., refext-$[\sigma_{\Im^d(D)}[\underline{\varrho}_n]/\Im^d]]\rangle$ is in refext-$[\varphi^n/\Im^d]$.

2. $[\underline{\varrho}_1 = \underline{\varrho}_2/\Im^d]$ will be true with respect to the level of reference under a denotational interpreted variable assignment $\sigma_{\Im^d(D)}$ just in case refext-$[\sigma_{\Im^d(D)}[\underline{\varrho}_1]/\Im^d]]$ = refext-$[\sigma_{\Im^d(D)}[\underline{\varrho}_2]/\Im^d]]$.

3. $[\neg\underline{A}/\Im^d]$ will be true with respect to the level of reference under a denotational interpreted variable assignment $\sigma_{\Im^d(D)}$ just in case $[\underline{A}/\Im^d]$ is not true with respect to the level of reference under $\sigma_{\Im^d(D)}$.

4. $[\underline{A} \wedge \underline{B}/\Im^d]$ will be true with respect to the level of reference under a denotational interpreted variable assignment $\sigma_{\Im^d(D)}$ just in case both

$[\underline{A}/\Im^d]$ and $[\underline{B}/\Im^d]$ are true with respect to the level of reference under $\sigma_{\Im^d(D)}$.

5. $[(\forall \underline{x})\underline{A}/\Im^d]$ will be true with respect to the level of reference under a denotational interpreted variable assignment $\sigma_{\Im^d(D)}$ just in case for every interpreted denotational variant $\tau_{\Im^d(D)}$ of $\sigma_{\Im^d(D)}$ $[\underline{A}/\Im^d]$ is true with respect to the level of reference under $\tau_{\Im^d(D)}$.

We can now prove the LCT using the Principle of Correlation via a straightforward induction on the complexity of formulae of $\mathcal{L}^{s\ddot{=}}$ and $\mathcal{L}^{d=}$, respectively.

Case 1: The natural language expression is an interpreted atomic formula. Then $[\varphi^n o_1 ... o_n/\Im]$ will be true with respect to the level of sense under an interpreted nominal variable assignment $\sigma_{\Im(C)}$ iff $[\varphi^n o_1 ... o_n/\Im]$ is contained in senext-$[\sigma_{\Im(C)}[o_1]/\Im]$ through senext-$[\sigma_{\Im(C)}[o_n]/\Im]$ and in senext-$[\varphi^n/\Im]$ (where o_1 through o_n are nominal constants). But, in view of the Principle of Correlation, the right-hand side will hold iff n-tuple $\langle$refext-$[\sigma_{\Im^d(D)}[\underline{o}_1]/\Im^d]$, ..., refext-$[\sigma_{\Im^d(D)}$ $[\underline{o}_n]/\Im^d]]\rangle$ is in refext-$[\varphi^n/\ \Im^d]$ (where $\underline{o}_1$ through $\underline{o}_n$ are individual constants). And, given the relevant truth conditions, this in turn will hold iff $[\varphi^n \underline{o}_1 ... \underline{o}_n/\Im^d]$ will be true with respect to the level of reference under a denotational interpreted variable assignment $\sigma_{\Im^d(D)}$.

Case 2: The natural language sentence is an identity sentence. $[\underline{o}_1 = \underline{o}_2/\Im^d]$ will be true with respect to the level of reference under a denotational interpreted variable assignment $\sigma_{\Im^d(D)}$ iff refext-$[\sigma_{\Im^d(D)}[\underline{o}_1]/\Im^d]$ = refext-$[\sigma_{\Im^d(D)}[\underline{o}_2]/\Im^d]$, or equivalently, iff $\langle$refext-$[\sigma_{\Im^d(D)}[\underline{o}_1]/\Im^d]$, refext-$[\sigma_{\Im^d(D)}$ $[\underline{o}_2]/\Im^d]\rangle \in$ refext-$[=\ /\Im^d]$. But in view of the Principle of Correlation, the right-hand side will hold iff senext-$[\sigma_{\Im(C)}[o_1]/\Im]$ = senext-$[\sigma_{\Im(C)}[o_2]/\Im]$ and every predicate $[\varphi^n/\Im]$ receives its actual sense-extension. By the Principle of Correlation, the latter will hold exactly if every $[\varphi^n/\Im^d]$ receives its actual referential extension. With respect to the actual level of sense this condition is trivially satisfied. So senext-$[\sigma_{\Im(C)}[o_1]/\Im]$ = senext-$[\sigma_{\Im(C)}[o_2]/\Im]]$ and every $[\varphi^n/\Im]$ has its actual sense-extension But this is the case just in case for every natural language interpreted atomic formula $[A_2/\Im]$ which is just like $[A_1/\Im]$ except for containing occurrences of $[\sigma_{\Im(C)}[o_2]/\Im]$ where $[A_1/\Im]$ contains occurrences of $[\sigma_{\Im(C)}[o_1]/\Im]$ the following holds: $[A_1/\Im]$ is true with respect to the level of sense under $\sigma_{\Im(C)}$ just in case $[A_2/\Im]$ is true with respect to the level of sense under $\sigma_{\Im(C)}$. And this will hold iff $[o_1 \ddot{=} o_2/\Im]$ is true with respect to the level of sense under an interpreted nominal variable assignment $\sigma_{\Im(C)}$.

Inductive hypothesis. Let LTC hold for formulae of $\mathcal{L}^{s\ddot{=}}$ and $\mathcal{L}^{d=}$ respectively of complexity $< k$, where $k > 0$. Then any formula complexity k will be either a negation, a conjunction or a quantified (substitutionally or, alternatively, referentially) formula.

Case 3: Negations. $[\neg A/\Im]$ will be true with respect to the level of sense under an interpreted nominal variable assignment $\sigma_{\Im(C)}$ iff $[A/\Im]$ is not true with respect to the level of sense under $\sigma_{\Im(C)}$. By the inductive hypothesis, this will be the case iff $[\underline{A}/\Im^d]$ is not true with respect to the level of reference under $\sigma_{\Im^d(D)}$. Which will hold iff $[\neg\underline{A}/\Im^d]$ is true with respect to the level of reference under a denotational interpreted variable assignment $\sigma_{\Im^d(D)}$.

Case 4: Conjunctions. $[A \wedge B/\Im]$ is true with respect to the level of sense under an interpreted nominal variable assignment $\sigma_{\Im(C)}$ iff both $[A/\Im]$ and $[B/\Im]$ are true with respect to the level of sense under $\sigma_{\Im(C)}$. By the hypothesis, this will be the case exactly when both $[\underline{A}/\Im^d]$ and $[\underline{B}/\Im^d]$ are true with respect to the level of reference under $\sigma_{\Im^d(D)}$. And this is so iff $[\underline{A} \wedge \underline{B}/\Im^d]$ will be true with respect to the level of reference under a denotational interpreted variable assignment $\sigma_{\Im^d(D)}$.

Case 5: Quantifications. $[(\Pi x)A/\Im]$ is true with respect to the level of sense under an interpreted nominal variable assignment $\sigma_{\Im(C)}$ iff for every interpreted substitutional variant $\tau_{\Im(C)}$ of $\sigma_{\Im(C)}$ $[A'/\Im]$ is true with respect to the level of sense under $\tau_{\Im(C)}$; iff, by hypothesis, for every interpreted denotational variant $\tau_{\Im^d(D)}$ of $\sigma_{\Im^d(D)}$ $[\underline{A}/\Im^d]$ is true with respect to the level of reference under $\tau_{\Im^d(D)}$; iff $[(\forall \underline{x})\underline{A}/\Im^d]$ will be true with respect to the level of reference under a denotational interpreted variable assignment $\sigma_{\Im^d(D)}$.

The notion of truth *simpliciter* with respect to the level of sense for a sentence is defined, in the way already mentioned, as truth *simpliciter* under all interpreted substitutional variable assignments. The notion of truth *simpliciter* with respect to the level of reference is defined in an analogous way. With these definitions we can complete the proof of LCT. The Formal LCT follows in the way already described.

Let us conclude this section with an illustration of the claim that we may evaluate natural language sentences in two ways, that is the referential and the nonreferential mode. Consider, for example, the following sentences:

1. 'Santa Claus is male';
2. 'Santa Claus is a goddess';
3. 'Lou Reed is male';
4. 'Lou Reed is a goddess'.

Naturally, interpreted nonreferentially (1) will be true with respect to the level of sense and interpreted referentially, it will be false with respect to the level of reference; (2) and (4) will be false with respect to both levels on the appropriate interpretations; and (3) will be true with respect to both levels on the appropriate interpretations.

In our discussion of the LCT we have been dealing exclusively with named objects. Sentences *about* unnamed objects (such sentences will not be atomic, nor will they involve atomic sentences of this sort) will be evaluated only according to the referential mode. (But maybe it is not wholly absurd to assume that every object has a name (in some non-Lagadonian sense[27]) or that there is a name concept for every object. After all, we have endorsed the common metaphysical assumption that Fregean concepts are mind- and language independent abstract objects. If we did also declare that every object has a concept associated with it (and that every concept is associated with an, not necessarily tokened, expression-type that expresses it), all Fregean propositions (or the sentences which express them) would be evaluable in the nonreferential mode.)

4.4.4 Modal Contexts and Modal Environments

According to the semantic theory presented in this text, we may distinguish two independent modes of semantic evaluation of natural language sentences: the referential and the nonreferential mode. The present theory treats modal discourse as a species of nonreferential discourse and suggests that sentences

[27]According to David Lewis's Lagadonian method, we may treat every object as naming itself (see [Lewis, 1986] p. 145).

which occur in the context of modal discourse should be evaluated according to the nonreferential mode (see Subsection 4.2.1).

It is important to appreciate the difference between sentences which occur in the context of modal discourse and modal sentences. A sentence which occurs in the context of modal discourse, for example, at some place in a modal argument, does not need to be a modal sentence (one which contains a modal operator) itself.

Let me make this distinction somewhat more precise. We say that a sentence of natural language is a modal sentence if it is a sentence of the modal portion of natural language *and* contains a modal operator; and we say that a sentence occurs in the context of modal discourse if it is a sentence of the modal portion of natural language. The present theory of modality applies to the latter kind of sentences, not only to sentences of the former sort.

We say that a sentence occurs in a *modal context* just in case it is a subsentence of a sentence which contains a modal operator. Since a sentence which contains a modal operator may be taken to be its own subsentence, a modal sentence may be said to occur in a modal context itself. In order not to confuse sentences which occur in modal contexts with sentences which occur in the context of modal discourse, it might be helpful to replace talk of sentences which occur in the context of modal discourse with talk of sentences which occur in a *modal environment*.

On the present theory, then, sentences which occur in modal environments are nonreferential and are to be evaluated according to the nonreferential mode. This proposal, as I shall argue later in Subsection 4.5.12, has consequences for the prospects of certain modal arguments.

4.5 Some Philosophical Consequences

We conclude this chapter with considerations of a couple of widely discussed issues in the philosophy of quantified modal logic and the philosophy of language from the perspective of the applied associative framework.

4.5.1 On Analysing Modality

Do the absolute truth conditions given by (Necessity) and (Possibility), or their primed versions from Subsection 4.3.7, provide analyses of the modal operators (i.e., 'definitionally necessarily' and 'definitionally possibly') in more basic terms? Recall that (Possibility′), for instance, makes the following biconditional claim:

> If $\Diamond A$ is a closed formula of the formal modal language L^* then $[\Diamond A/\Im]$ is true *simpliciter* with respect to the level of sense iff the level of sense could have been (in the basic sense of 'could') such that $[A/\Im]$ was true *simpliciter* with respect to it.

Letting A be some sentence of ordinary language this claim is tantamount to (Possibility):

> 'It is definitionally possible that A' is true *simpliciter* with respect to the level of sense iff the level of sense could have been (in the basic sense of 'could') such that 'A' was true *simpliciter* with respect to it.

To be sure these truth conditions do not provide extensional explanations of these expressions in terms of quantification over possible worlds or, alternatively, in terms of quantification over rearrangements of the level of sense. The present proposal is, therefore, in opposition to theories which seek to explain modal operators in terms of quantification.

Moreover, our truth conditions obviously do not provide an eliminative analysis of definitional modality, an analysis, that is, whose right-hand side does in no way involve modal notions.[28]

[28]Of course, this is not a distinctive feature of our approach. As is well-known, on most quantificational accounts of modality, except perhaps for Lewis's proposal, modal notions are involved in these explanations. A common feature of ersatzist approaches is that they involve modal notions in their explanations of what a possible world is. Thus, for instance, Plantinga's worlds are *maximal* states of affairs, Adams's worlds are *maximal consistent* sets, Stalnaker's worlds are *instantiable* complex properties and Armstrong's worlds are built up from *possible* atomic states of affairs. (See Subsection 1.1.2.g.) For an informative discussion of the possibility of giving a non-modal analysis of modal notions within ersatzist frameworks see [Divers, 2002] ch. 11.

Furthermore, (Possibility) and (Necessity) as well as their primed versions do not provide a modalist treatment of definitional possibility (necessity). (On the "classical" account of *modalism*, modalism is the view that modal operators are primitive notions, that is, notions which cannot be explained in more basic terms.[29]) This is so, since, on the present proposal, the primitive notion of basic possibility serves to explain the notion of definitional possibility. (The classical modalist explanation might be correct, as I am inclined to think, for the notion of basic possibility.[30])

[29]See [Fine, 1977] for the classical formulation. Our characterization of modalism is taken from page 116.

[30]It is, maybe, worth noting that our account of definitional modality in terms of (Possibility) and (Necessity) (and their primed versions) can be neither subsumed under what Christopher Peacocke has recently called "constraint modalism" nor under what he calls "ontological modalism":

> "*Constraint modalism* is the doctrine that there are constraints involving the notion of possibility which are explanatorily prior to whether a world is possible or not. A supporter of constraint modalism can consistently quantify over possible worlds of an ersatz kind, and use such quantification in the explanation of modal discourse, so long as the possible worlds he so uses are conceived as derivative from the satisfaction of various constraints involving the notion of possibility. *Ontological modalism*, by contrast, states that possible worlds have no part to play in the elucidation and understanding of modal discourse. The ontological modalist will insist that 'necessarily' is never in any way to be understood as a quantifier." ([Peacocke, 1999] p. 156, my italics).

Obviously, ontological modalism is a special case of modalism as "classically" understood. (For ontological modalism see [Forbes, 1989] ch. 4. and the discussion thereof in [Melia, 1992] and [Forbes, 1992]. Forbes's earlier discussion of modalism, in which he did not make use of quantification over modalistically introduced "possibilities" may be found in [Forbes, 1985] ch. 4; for a discussion of this earlier account see, in particular, [Chihara, 1998] ch. 4.)

Moreover, it will be noted that the ersatzist accounts mentioned in a footnote back are, in principle, compatible with Peacocke's constraint modalism. (In my opinion Peacocke's terminology is rather misleading. For Peacocke's term 'constraint modalism' is, as it seems to me, essentially another word for 'principle-based theory of modality' (which refers to the theory he has developed in [Peacocke, 1999] and defended further in [Peacocke, 2002b]) and that theory is primarily concerned with providing a tie between the metaphysics and the epistemology of modality rather than with the issue of whether modal operators are

It will be noted that the absolute truth conditions for definitional necessity as given by the versions of (Necessity) and (Possibility) do not have the nonextensional form

> 'Definitionally necessarily, A' is true iff definitionally necessarily, A.
>
> 'Definitionally possibly, A' is true iff definitionally possibly, A.

which is characteristic of homophonic accounts of (broadly logical) modality, or alternatively, of model-theoretic accounts of (broadly logical) modality which pursue the strategy of intended* Kripke-models.[31] The truth conditions of the present proposal do have a form which differs from the truth conditions offered hitherto; they are neither quantificational nor modalist.

Moreover, we might add that our truth conditions are neither circular nor infinitely regressive. For, on the one hand, definitional necessity is not explained in terms of definitional necessity and, on the other hand, the analysis of that notion comes to an end at the primitive notion of basic possibility. In this way the present account of definitional necessity is (as I hope) illuminating, even though it is not eliminative. We allow for a distinction between a modal notion which is to be analysed and a (primitive) modal notion which helps to analyse it. In this way we elucidate a nonbasic notion of modality in terms of a basic one.[32]

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primitives. I shall present the main ideas of Peacocke's principle based account in Chapter 5.)

[31]For homophonic truth theories of necessity along these lines see, in particular, [Davies, 1978], [Gupta, 1978], and [Peacocke, 1978]. The following claim of Menzel's may be taken as *pars pro toto* for the supporters of intended* Kripke-models: "if we are going to take modality in the broadly logical sense at face value, then there is no reason to ask for any more than a homophonic theory of modal truth conditions: for a modal statement to be true—just as in the nonmodal case—is for things to be as the statement says." ([Menzel, 1990] p. 385; see also *ibid.* p. 383.) Both of these approaches are in agreement with classical modalism.

[32]As it seems to me, a suggestion like this allows, to view the merits of non-Lewisian explanations of modality (i.e., accounts which do not involve genuine realist worlds) more optimistically than David Lewis suggests (cf. his well-known remarks at page 85 of [Lewis, 1973]). (For a critical discussion of Lewis's analysis of modality in terms of

Of course, if for some reason or other we decided to pursue an approach along quantificational lines, we could have admitted quantification over alternative ways the level of sense could have been arranged. After all, these rearrangements would be just sets of atomic sentences of a natural (or mental) language or, alternatively, atomic propositions of a language-like system of concepts. Such an ontology, would be even more agreeable than the ontology of what David Lewis calls "linguistic ersatzism". For whereas linguistic possible worlds *represent* various ways the world might have been, rearrangements of the level of sense would not represent anything at all—they would be sets of atomic sentences (or propositions) which just *constitute* the alternative levels. Unlike linguistic ersatz worlds, they would therefore not appeal to something dubious they represent.[33] As the reader will have noted, on the construction offered in this chapter, the only kind of item that represents are associative bijection-tuples. They represent how the (actual) level of sense—being a set of atoms of some language-like system—could have been arranged; and they do so in a precise, mathematically specified way.

4.5.2 On Modal Truth With Respect to the Level of Sense

When the conditions on the conformity of natural language proto-interpretations themselves (see Subsection 4.3.6) are taken into account, the truth of closed formulae which are prefixed with modal operators may be linked to the modal truth of the exact synonyms of the formulae in the unprefixed form with respect to the level of sense. Thus, from the Principal Corollary of our Chihara-style Connecting Theorem, we may obtain a couple of corollar-

quantification over Lewisian worlds see, in particular, [Chihara, 1998] ch. 8, sect. 6 and the references therein.)

Maybe Robert Stalnaker is right when he writes: "But it seems to me that modal notions are basic notions, like truth and existence, which can be eliminated only at the cost of distorting them. One clarifies such notions, not by reducing them to something else, but by developing one's theories in terms of them." ([Stalnaker, 2003] p. 7.) From the perspective of the present theory this dictum will be correct for basic modality.

[33] The differences between linguistic possible worlds and rearrangements of the level of sense will be discussed further in Subsection 4.5.4 below.

ies concerning the notion of modal truth *simpliciter* with respect to the level of sense.[34] Here we have to be aware of the distinction between definitional modality (which will be below symbolized by boxes and diamonds) and basic modality.

> *Chihara-style Connecting Theorem.* For every natural associative model $\mathcal{M} = \langle S, R, s^{@}, C, c, P, v\rangle$ of L^*, for every natural language proto-interpretation $\Im$ which conforms to $\mathcal{M}$ the following holds: for $s \in S$, any closed formula A of L^* is such that $\mathcal{M} \models_s A$ iff $[A/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

As a corollary to this we obtain:

> (i) For every associative model $\mathcal{M} = \langle S, R, s^{@}, C, c, P, v\rangle$ of L^*, for every natural language proto-interpretation $\Im$ which conforms to $\mathcal{M}$ the following holds: for any closed formula A of L^* there is a $s \in S$, such that $\mathcal{M} \models_s A$ iff there is a $t \in S$ such that, had the level of sense been such that there was a bijective function via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, then $[A/\Im]$ would have been true with respect to the level of sense.

The left-hand side of this corollary holds, in particular, just in case:

> (A) $\mathcal{M} \models_{s^{@}} \Diamond A$.

In view of (C1) the right-hand side of the corollary holds just in case:

> (B) The level of sense could have been (in the basic sense of 'could') such that $[A/\Im]$ was true with respect to the level of sense.

But (B) will hold just in case:

[34] These corollaries and the reasoning to follow essentially parallel the considerations in [Chihara, 1998] pp. 258-259.

(C) It is basically possible that $[A/\Im]$ is true with respect to the level of sense.

So from (A) and (C) we obtain the following biconditional as a corollary to corollary (a):

(ii) $\mathcal{M} \models_{s^@} \Diamond A$ iff it is basically possible that $[A/\Im]$ is true with respect to the level of sense.

In a similar way we also obtain:

(iii) $\mathcal{M} \models_{s^@} \Box A$ iff it is basically necessary that $[A/\Im]$ is true with respect to the level of sense.

(Here 'basically necessary' just means 'no matter how the level of sense could have been arranged (in the basic sense of 'could') '.) In this way the model-relative truth of closed formulae of L^* which are prefixed by modal operators is mirrored by absolute, basically modal truth of exact synonyms of the originally unprefixed sentences of ordinary modal language with respect to the level of sense.

4.5.3 On the Ground of Modal Truth

A theory of modality should not only give an account of the form of the truth conditions of modal statements, it should also address the question of what, if anything, is "the metaphysical ground of modal truth"?[35] Proponents of theories which appeal to intended Kripke-models are likely to locate the metaphysical ground of modal truth in the reality which is modeled by them. It is not clear, however, how the proponents of intended* Kripke-models are to approach this issue. Indeed, neither of them addresses this metaphysical question.[36]

[35] [Menzel, 1990] p. 385.

[36] Christopher Menzel, for example, confines himself to questions of the form of modal truth conditions without considering questions of ground: "Freed from their [i.e., the intended models'] grip in modal semantics, nothing deeper (and nothing less deep) than a homophonic account of modal truth conditions—silent as it is on questions of the ground of modal truth—is to be expected." ([Menzel, 1990] *ibid.*

Before we attempt to give an answer to the question of what grounds definitional modality according to the present theory (which makes use of intended* modal associative models), it will be helpful to distinguish a couple of things concerning the "ground of modal truth".

When we discuss the question of the ground of necessary truth in the context of denotational theories, we should be aware of the distinction between that what a necessary proposition is intuitively taken to be about, i.e., the *subject* of the proposition, on the one hand, and the *source* of that proposition's broadly logical (or, equivalently, metaphysical) necessity, on the other hand.[37] For example, whereas on the denotational view the statement 'Necessarily, all bachelors are unmarried men' which expresses a conceptual necessity is about bachelors, the source of that truth will be located in the meaning of the terms which figure in that statement. Similarly, any logical necessity will be true solely in virtue of the meaning of the logical terms involved in it, but the source of this necessity will not be taken to coincide with the things it is about.[38] Finally, a *de re* modal claim like 'Socrates is necessarily a man' is—by the denotationalist's lights—about Socrates, whereas the source of that necessity is not Socrates or any old feature of him, but his essence.

Moreover, with respect to denotational theories of modality, we have to be aware of a further distinction, the distinction between the *subject* of a modal truth and its ultimate *truthmaker* (or its constitutents). According to denotational approaches along the lines of intended (or, altenatively, intended*) Kripke-models, the truthmakers of modal truths (or their constituents) are the items which make up the models. So, for example, the subject of the modal sentence 'Socrates is necessarily a man' will be—intuitively—Socrates,

[37]The difference between the subject and the source of metaphysically necessary statements has been emphasized in [Fine, 1994].

[38]As I have already noted in Subsection 4.2.13, conceptual necessities and (narrow) logical necessites are usually taken to be special cases of broadly logical (or metaphysical) necessity. See, for example, [Plantinga, 1974] pp. 1-2, [Chihara, 1998] p. 7 (where, deviating somewhat from the usual terminology, broadly logical necessity is equated with conceptual necessity) or [Fine, 1994] p. 9 (who gives an essentialist account of metaphysical necessity). For a recent discussion of metaphysical necessity and its special cases see also [Lowe, 1999] ch. 1. sections 4 and 5.

but the ultimate truthmaker will be a representative of him (e.g., his haecceity, a counterpart of him, or some pure set representing him via a bijection). (I write 'intuitively', since, as I have argued in Subsection 1.1.2.h, on the denotationalist's account the intuitive subject of the modal sentence 'Socrates is necessarily a man' (which in this case just is the intuitive referent of 'Socrates') and the denotationalist's referent (which is his representative) come apart.)

These distinctions made, we may now turn to the question of the metaphysical ground of modal truth. How is that question to be approached on the present nondenotational account? Since modal language is viewed as being reflective, it is not treated as being about anything at all. And since the nonreferential mode of evaluation is adequate for modal language, the question of subject does, on our approach, not arise for modal truths. In effect, the question of ground reduces to the question of source. But what, then, is the source of definitional necessity?

To get at an answer consider, for example, the absolute truth conditions for the following *de nomine* definitional possibility.

> 'Tarski could have been (in the definitional sense) a violinist' is true *simpliciter* with respect to the level of sense iff the level of sense could have been (in the basic sense of 'could') such that the sense-extensions of 'Tarski' and '... is a violinist' did coincide on 'Tarski is a violinist'.

(*Mutatis mutandis*, for the conceptual version.) That the right-hand side of this truth condition holds is guaranteed by the Principle of Rearrangement by which the level of sense is governed. In the case at hand, this principle will require that the right-hand side holds just in case the level of sense could have been (in the basic sense of 'could') arranged that way in accordance with the Principle of Nominal Sense-Extension and the Principle of Predicative Sense-Extension. These principles, however, require that the sense-extensions of names and predicates be determined by their meanings. It is, therefore, the meanings of the constituting nonlogical terms in which the truth of 'Tarski could have been (in the definitional sense) a violinist' with respect to the level of sense has its source.

Similarly, the definitional necessity of the propositions expressed by 'All bachelors are male' or 'All rectangles are rectangles' have their source in the meanings of the terms involved. In the latter case the source of the necessity will be the meaning of the logical vocabulary, in the former it will be also the meaning of the nonlogical terms.

One might be tempted to ask what the source of the primitive notion of basic modality is, which serves to explain definitional necessity. As I wish to understand this notion, there are no sentences, whether natural or mental, and there are no propositions that are necessary or possible in this sense. This notion merely serves as a structuring device of the level of sense by determining which atomic sentences of the language must or can be contained in the sense-extensions of the nonlogical vocabulary (see, in particular, Subsection 4.2.5). In effect, this notion merely operates on sense-extensions. According to this way of understanding basic possibility (necessity), there just are no statements which are possibly (necessarily) true in this sense. There is, therefore, no good reason to ask what the source of basic modal truth is.

4.5.4 On Ontological Commitments

Since on the nonreferential mode of evaluation of existential claims of the object language no referents are involved, this sort of evaluation cannot be taken to give rise to any kind of ontological commitment. Only the quantifications of the metalanguage of the semantic theory for that object language can be understood in this way. Like remarks, apply to the formal semantics of the theory: the formal language L^* is unable to express claims of referential existence, but the metalanguage used to study that language can be taken to express such claims, at least, when its quantifiers are assumed (as seems natural) to have existential import. A crucial difference between our nonreferential theory and the referential theories is that we construe only the language of the theorist in an ontologically committal way but not the language (or the language-like system) of the speaker.

In Section 1.1. we have assumed that a semantic theory is committed to the existence of those entities which are involved in its account of truth (see, in particular, Subsection 1.1.2.c). What kinds of ontological commitment, then, does our theory involve? Well, on the formal side it appeals to the entities of which constitute modal associative models, that is, pure sets, the expressions of the formal language, and sets thereof. On the applied side, as developed in terms of the strategy of intended* modal associative models, the theory involves, first, the model-theoretic apparatus of associative models and, second, depending upon ones choice of the object language (or language-like system), the expressions of that language (which may be natural or mental), and the sense-extensions of the nonlogical expressions of that language which amount to sets of pure atomic sentences. On the conceptual construal of sense-extensions given in Section 4.3, the ontology of the theory amounts to Fregean concepts, Fregean propositions and sets. On a nominalist account it will involve expression-types and sentence-types instead of concepts and propositions. In effect, expressions of some preferred language-like system and sets is all the ontology needed in applied associative semantics.

What about possible worlds? As the present theory proceeds squarely in terms of intended* models, it is not committed to the existence of such things. However, the theory is committed to the existence of the level of sense. This level is, in effect, a set of sentences (or structured propositions) of some language system. In this respect it involves an ontology very much like that of linguistic ersatzism (in a broad sense of 'linguistic').[39]

(In view of our discussion of the level of sense in Section 4.2, it would be a mistake to view the level of sense as a (broadly) linguistic ersatz world. This would be so for several reasons. First, linguistic worlds are simply maximally consistent sets, whereas the level of sense is a set of a rather different sort. In particular, the level of sense is not consistent, since the sense-extension of any name (and thus the sense-spectrum of names) is not consistent. More exactly, for each name, given the meaning postulates of all the predicates (not only of the ones characteristic for it), an atomic sentence and a negation will be derivable from its sense-extension. Moreover, the present theory, because of the intuition of reflection it is intended to capture,

[39]For an influential criticism of linguistic ersatzism (in a sense of 'linguistic' which is less broad than ours), see [Lewis, 1986] sect. 3.2.

is immune to the problems which are characteristic of linguistic ersatzism—in particular, to problems concerning the descriptive power of the "worldmaking language" as raised by David Lewis (e.g., the problem that there can be many indiscernible possible individuals, but no two indiscernible descriptions, or the problem that possibilities can outrun the means of describing them[40]). In addition, the present theory is not prone to the more general problems of intended Kripke-models discussed in Chapter 1, which the linguistic ersatzer has to face (e.g., the problem of modal reference raised in Subsection 1.1.2.h). However, unlike the denotational linguistic ersatzer our approach has the disadvantage that, in view of our axiomatization of substitutional identity, our language will contain only finitely many predicates, in case it includes the predicate for substitutional identity.)

In my opinion the relatively modest ontological inventory of our applied semantical framework (which will be more parsimpnious on the nominalist than on the conceptualist account of sense-extensions) is an advantage of the present theory. In contrast to the denotationalist approaches which use intended Kripke-models, we need not care about such things like haecceities, contingently nonconcrete objects, Lewisian (or other sorts of) counterparts, about other sorts of *possibilia* or about possible worlds. And we do not have to face all the metaphysical questions to which these entities give rise. This does also apply to theories which proceed in terms of intended* Kripke-models. However, unlike our approach, these theories have to face, besides other difficulties (see Subsection 1.1.2.i), metaphysical problems, e.g., problems concerning the metaphysical grounds of modal truth or problems about truth in fiction.

The advantages of ontological parsimony and a significantly narrowed down metaphysical agenda are, of course, advantages which the present theory also shares with theories that appeal to applied standard substitutional semantics (or truth-value semantics). However, on the present framework questions concerning the intended or the intended* interpretations of 1 and 0 do not arise. We simply need not be concerned with metaphysical questions about the nature of the True and the False or with truthmaker monism (see Subsection 1.2.3).

[40]See [Lewis, 1986] pp. 157-165 for the relevant discussion.

4.5.5 On Actualism

The ontological commitments of the present semantic theory are intended to be in agreement with actualism, i.e., the view captured by the thesis that unrestrictedly everything that exists (or, equivalently, everything there is) is actual (see Subsection 1.1.2.g).

We may claim that unrestrictedly all possible names and predicates (or that unrestrictedly all possible name concepts and predicative concepts) do actually exist even if not all of them are (or, even, can be) deployed. On that assumption there will be no "alien" names and predicates. An idealizing assumption of this sort, however, does not need to be endorsed. We may be more modest and confine ourselves to the names (or name concepts) and predicates (or predicative concepts) of some language (natural or mental) as developed at a certain stage or, alternatively, to some suitable language-like system of concepts.

The applied semantical framework presented in this chapter is a framework for constant substitution class semantics. However, applied varying substitution class semantics can be reconciled with actualism as well. In general, we could stipulate that there be a substitution class which contains unrestrictedly all names and that there be also substitution classes of names which are subsets thereof which are restricted to some suitable parameter. But whatever the motivation for an account of this sort might be, it will involve, as it seems to me, complications (e.g., complications of representation, since an additional condition on conformity might have to be imposed which did in some sense correspond to the denotationalist's intuition of Serious Actualism) without philosophical gains.

4.5.6 On BF, CBF, and NE

We have mentioned in Chapter 1 that the technically simple—and, therefore, nice—constant domain systems run counter to certain modal intuitions which actualists typically share, since such systems allow for the derivation of theorems like BF, CBF, and NE.

Recall that, on the paradigmatic account, to endorse the truth of the formula $\Diamond(\exists \underline{x})\varphi\underline{x} \rightarrow (\exists \underline{x})\Diamond\varphi\underline{x}$, which is equivalent to an instance of BF, and its antecedent is to accept the existence of at least one antiessentialist *actuale*. So, for example, when φ is taken to have the meaning of '... is a harpy', then, accepting that formula and its antecedent, one will be committed to the view that among the actually existing objects there is an object which is possibly a harpy. But this would run counter to the intuition that the fundamental kind properties of an object are essential to it. Moreover, the acceptance of BF, again in the form of $\Diamond(\exists \underline{x})A \rightarrow (\exists \underline{x})\Diamond A$, seems to violate the intuition of distinctness which says that there might have been some possible object which is distinct from every actual object. CBF, on the other hand, allows the derivation of NE, that is the formula $(\forall \underline{x})\Box(\exists \underline{y})(\underline{y} = \underline{x})$, which on the paradigmatic reading violates the intuition that there are things which might not have existed and thus leads to necessitarianism.

The only (to the best of my knowledge) presently available strategy of reconciling constant domains with actualism, does invoke the actual existence of contingently abstract (or nonconcrete) objects.[41]

With regard to the problem of antiessentialist *actualia*, BF and the actualist's essentialist intuitions may be reconciled on this sort of account, by claiming that the *actuale* is a contingently abstract object which is essentially a harpy in every possible world where it is concrete.

Similarly, the intuition that there might have been some possible object which is distinct from every actual object is recaptured by claiming that there might have been a concrete object which is distinct from every object which is actually concrete; where this object is a contingently abstract *actuale*.

Finally, the intuition that there are things which might not have existed is rejected in favour of the Leibnizian view that everything there is exists of necessity. However this antinecessitarian intuition is recaptured by claiming that there are things which might not have been concrete. Thus, for example, Linsky and Zalta are concrete at this world but at others they are contingently abstract objects which cannot be found in spacetime there.[42]

[41]This strategy is due to [Linsky & Zalta, 1994]. For an exchange on this proposal see [Tomberlin, 1996] and [Linsky & Zalta, 1996]. Essentially the same approach is proposed in [Williamson, 1998].

[42]See, for example, [Linsky & Zalta, 1994] p. 448.

I do not wish to discuss the plausibility of the assumption that there are such things like contingently nonconcrete objects.[43] My present point is that either way the denotational modal semanticist is confronted with the dilemma of either complicating quantified modal logic (for instance, by admitting varying domains, introducing elements from free logic and so on), or inflating ontolology (for example, by introducing a new category of being like Linsky and Zalta do). Needless to say, theorists who combine the former strategy with intended Kripke-models, do not only complicate logic, but also inflate their ontology and metaphysical agenda.[44] Actualists who endorse the strategy of intended* models are better off, since they complicate logic but do not inflate ontology.[45] Neither option is appealing.

On the present approach to modal semantics, however, we may find a

[43]Let me only report some doubts. Since the solution to the problem of antiessentialist *actualia* appeals to contingently abstract objects, it might, arguably be taken merely to shift the problem. For more radical essentialists will surely insist that things are either essentially concrete or essentially abstract and that, therefore, Linsky's and Zalta's restriction of the range of properties which may count as essential is somewhat *ad hoc* and not supported by independent reasons. Moreover, even granting the intelligibility of their assumption, it will be legitimate to ask how the concrete Linsky of the actual world differs, exactly, from the Linsky who exists *qua* abstract object at another world w, or how the latter differs at w from Zalta who exists *qua* abstract object at that world (irrespectively of the difference which might obtain between them at other worlds). Finally, let me add that it is not clear—as Linsky and Zalta do not discuss that question—what a possible world is supposed to be on their and how their possible objects (e.g. the contingently abstract ones) are metaphysically related to them. This incompleteness of their metaphysical views surely makes their proposal less credible. For further criticism see [Divers, 2002] pp. 214-215.

[44]For the technical complications see [Hughes & Cresswell, 1996] and [Garson, 2001]; for philosophical discussion cf., in particular, [Linsky & Zalta, 1994] sect. 3 and [Menzel, 2000].

[45]Menzel and Chihara both develop systems of S5 modal first-order free logics see [Menzel, 1991] and [Chihara, 1998]. Interestingly, Menzel's system involves a restriction on the rule of necessitation: If $\vdash A$, then $\vdash \Box A$, so long as A is provable without any instance of $t = t$, for any individual term t. This blocks the derivation of necessitarianism. In Chihara's system neither BF and CBF is valid, and so is NE. In particular, $\Box(\exists \underline{x})E!\underline{x}$ fails to be provable, since the domain at some index may be empty. Moreover, Chihara has to add further axioms to his Serious Actualist system; see [Chihara, 1998] ch. 7, sect. 2.

way out of this dilemma. We leave the denotationalist picture behind and adopt a picture on which modal language is nonreferential. We take the correct modal logic to be substitutional quantified modal logic. This logic—given our affection for technical simplicity—will be a constant substitution class system (which, to make matters even simpler, will preferably be S5, for the accessibility relation may then be "dropped"). Consequently, we do not discuss the troublesome denotationalist BF, CBF, and NE but their substitutional counterparts.

So, to endorse SBF is not to accept the existence of some sort of antiessentialist *actualia*, since on this modal semantics of the modal object language there are no objects and thus no *actualia* of this sort at all. On the present account the truth of the ordinary language counterpart of $\Diamond(\Sigma x)\varphi x \rightarrow (\Sigma x)\Diamond\varphi x$ with respect to the level of sense amounts to saying that (taking φ to have the meaning of '... is a harpy') if the level of sense could have been (in the basic sense of 'could') such that the sense-extension of some name did coincide with the sense-extension of '... is a harpy', then for some name the level of sense could have been (in the basic sense of 'could') such that its sense-extension did coincide with that of '... is a harpy'.

In defense of the intuition of distinctness we do not need to face the choice between accepting necessitarianism like the inflators of ontology do, or reject it like the complicators of logic. We may be entirely agnostic on this issue. We may claim that it suffices that the names and predicates and the concepts they express exist without caring about the modal status of this existence claim.

How, then, is the intuition of distinctness to be dealt with on the present approach? We do not need to claim that to say that there could have been more (or fewer) things than there actually are just is to say that more (or fewer) things, all of which exist of necessity, could have been concrete. My preferred suggestion is, instead, that to say that there could have been more things than there actually are is to claim that the sense-extensions of more names (or name concepts, unrestrictedly all of which do actually exist and unrestrictedly all of which are contained in the substitution class of the substitutional model) could have (in the basic sense of 'could') coincided with the sense-extension of some relevant predicate (or predicative concept). Analogously, to say that there could have been fewer things than there actually are is to claim that some name could have been such that its sense-extension did not coincide with the sense-extension of some relevant predicate.

Now, what about NE? As already mentioned, we have no use for that formula on the reflective picture of the relation of modal language and reality. Instead we consider NSE. Clearly, given the fact that there are no objectual domains according to the present semantics NSE, i.e., $(\Pi x)\Box(\Sigma y)(y \doteq x)$, does not claim that every object does necessarily exist. Rather it says that every nominal constant is necessarily substitutionally identical to a nominal constant (e.g., to itself). Here we have to bear in mind that substitutional identity is a relation which (ultimately) obtains between nominal constants in virtue of the logical equivalence of pure atomic sentences in which these names occur. NSE will be true with respect to the level of sense just in case unrestrictedly all names are such that no matter how the level of sense could have been arranged (in the basic sense of 'could'), there would be some name to which they would be substitutionally identical. In effect, this claim reflects the relations which obtain between names (or name concepts) in virtue of their sense-extensions; it does not make a necessitarian claim of any sort. Now, since NSE is not objectionable, so is CSBF.

Of course, we may justify the intelligibility of BF, CBF, and NE by considering their substitutional counterparts in terms of the present framework along the lines suggested by Linsky and Zalta. We may stipulate, adapting their recipe, that the predicate '... is concrete' be not characteristic for some suitably selected class of names. This will be another way of justifying constant systems of quantified modal logic without inflating ontology.

4.5.7 On *Possibilia*

In substitutional modal semantics we consider SBF rather than BF. The rejection of SBF and thus to acceptance of the truth of the formula $\Diamond(\Sigma x)\varphi x \wedge \neg(\Sigma x)\Diamond\varphi x$, which is equivalent to the negation of SBF, does in no way give rise to perplexities concerning *possibilia*. If φ is taken to have the meaning of '... is a harpy' and the rest of the symbols receives its usual natural language interpretations, then the natural language counterpart of that formula says, in effect, that it is definitionally possible that something is a harpy and that at the same time it is not the case that something is definitionally possibly a harpy. This claim will be true with respect to the level of sense just in case if the level of sense could have been (in the basic sense of 'could') such that the sense-extension of some name did coincide with the sense-extension of '... is a harpy', then it is not the case that for some name the level of sense could have been such that its sense-extension did coincide with the sense-extension of '... is a harpy'.

Although the acceptance of the negation of the substitutional counterpart of the Barcan Formula does not give rise to worries about *possibilia*, there is no use in opting for varying substitution classes and the systems discussed in Sections 2.6 and 2.7. The reason is, of course, that these systems and their semantics are more complicated and that their metaphysical benefits can be enjoyed already in the simpler constant substitution class framework. So in contrast to the denotationalists who opt for varying domain frameworks, we neither complicate logic,[46] nor do we inflate ontology with actualistically acceptable substitutes for *possibilia*.

It is obvious, as it seems to me, that most of the currently discussed philosophical problems concerning quantification in modal contexts originate, in the end, from the same source: the transplantation of denotational semantics from first-order logic to first-order modal logic which has been carried out, most influentially, in [Kripke, 1963]. The present semantical framework is a model-theoretic alternative to denotational semantics of quantified modal logic which provides us with all the benefits of constant systems and semantics, I hope, without foisting metaphysical perplexities upon us. It seems to me that the framework might also be taken to provide an alternative to free logics, since considerations of reference get irrelevant and the axiomatization need not involve the related complications.

4.5.8 On Truth in Fiction

Truth in fiction is not truth with respect to some level of denotata, it is truth with respect to the level of sense. Fictional discourse, being reflective,

[46] See [Garson, 2001] for a survey.

is to be evaluated in the nonreferential mode. So, on the present proposal, fictional statements do neither describe a reality of fictional or other objects, nor do they describe such a reality make-believedly. Accordingly, fictional names do not refer outright, nor do they refer in some context of pretence.

On the present account, fictional truths accurately reflect the sense-extensions of the names and predicates (or name concepts and predicative concepts) involved in them. It will be obvious by now how the truth of, say, 'Holmes lived at 221B Baker Street' or 'Holmes was a person of flesh and blood' is to be explained.[47] Moreover, we may also see how the truth of a mixed claim like 'Sherlock Holmes isn't more intelligent than Saul Kripke' can be accounted for. We may proceed in the usual way purely nonreferentially. Alternatively, we may adapt the associative framework to a mixed semantics (see Section 1.3).

On the present proposal, then, the perplexing formulation "talking about something that does not exist" is a loose expression for "reflecting the sense-extensions of nondenoting names".

4.5.9 On the Puzzle of Non-Existence

From the point of view of the present theory, we should insist that negative existentials containing denotationless names be always evaluated according to the nonreferential mode. A statement like 'Santa Claus does not exist' taken substitutionally (formally: $\neg(\Sigma x)\ (x \doteq\!\!\doteq a)$) will, then, be *false* with respect to the level of sense. In this way, viewing fictional language as reflective rather than referential, we may remain ignorant about the puzzle of how the *truth* of that senence taken referentially is to be explained. Clearly, this is one of the problems which are particularly pressing for advocates of Referentialism (or Millianism), roughly, the doctrine that the sole semantic function of a name is to refer to its bearer.[48] The reasons why a statement like 'Round squares do not exist' is false with respect to the level of sense will be apparent.

[47]These examples are taken from [Lewis, 1983 (1978)].

[48]For recent proposals see, for example, [Everett, 2000] and [Taylor, 2000].

4.5.10 On Essentialism and Metaphysical Necessity

To endorse the truth of a natural language sentence like, say, 'Socrates is necessarily a man' (formally: $\Box Fa$) with respect to the level of sense does in no way involve a commitment to essentialism. Rather than claiming that a particular individual, i.e. Socrates, has the property of being a man of metaphysical necessity or essentially, the statement says that no matter how the level of sense could have been (in the basic sense of 'could') the sense-extensions of the name 'Socrates' (or the concept SOCRATES) and the predicate '... is a man' (or IS A MAN) would coincide on 'Socrates is a man' (or the proposition SOCRATES IS A MAN). In this way 'Socrates is a man', taken nonreferentially, expresses a definitionaly necessary proposition which is true in virtue of the sense-extensions of the name and the predicate, rather than a metaphysical necessity which is true in virtue of the essence of Socrates.

Let me close this section with a critical remark on Kit Fine's essentialist account of the notion of metaphysical necessity which he characterizes as follows:

> "Indeed, it seems that far from viewing essence as a special case of metaphysical necessity, we should view metaphysical necessity as a special case of essence. For each class of objects, be they concepts or individuals or entities of some other kind, will give rise to its own domain of necessary truths, the truths which flow from the nature of the objects in question. The *metaphysically necessary* truths can then be identified with the propositions which are true in virtue of the nature of all objects whatever. Other familiar concepts of necessity (though not all of them) can be understood in a similar manner."[49]

For Fine conceptual and logical necessity are special cases of metaphysical necessity so understood.

> "The *conceptual necessities* can be taken to be the propositions which are true in virtue of the nature of all concepts; the *logical*

[49] [Fine, 1994] p. 9, my emphasis.

> *necessities* can be taken to be the propositions which are true in virtue of the nature of all logical concepts."[50]

Fine's identification of the source of conceptual necessity with the essence of concepts is open to the following objection.[51] Suppose that concepts are abstract entities of some kind or other. And suppose that they are essentially abstract. Then something cannot be a concept unless it is abstract. If all concepts are abstract entities so is, for instance, the concept BACHELOR. Now, according to Fine, a proposition is a conceptual necessity just in case it is true in virtue of the nature of concepts. But then the proposition that the concept BACHELOR is an abstract entity will be a conceptual necessity, for it will be true in virtue of the essence of the concept BACHELOR that the concept BACHELOR is an abstract entity. The proposition that all bachelors are unmarried or that if something is red then it is coloured, seem to be good examples of conceptual necessities, but the proposition that the concept BACHELOR is abstract does not seem to be much of a conceptual necessity. It is clearly propositions of the former, analyticity related, kind that Fine originally aimed at.

To make the point more perspicious, let me rephrase that argument for the case of logical necessity. Suppose that concepts are language-independent entities of some sort. And suppose that they are essentially language-independent. Then something cannot be a concept unless it is language-independent. If all concepts are language-independent entities so is, for instance, the concept of logical conjunction, AND. Now, according to Fine's proposal, a proposition is a logical necessity just in case it is true in virtue of the nature of logical concepts. But then the proposition that the concept AND is language-independent will be a logical necessity, for it will be true in virtue of the essence of the concept AND that the concept AND is language-independent. Clearly, the proposition that all concepts are concepts, say, is a logical necessity. But the proposition that the concept AND is language-independent

[50] *Ibid.*, my emphasis.

[51] In stating my objection I shall not pay attention to the the complexities which arise from Fine's claim that conceptual necessities are true in virtue of the nature of *all* concepts, since these subtleties are irrelevant for the point I wish to make. See section 6 of [Hale, 1996] for a discussion of these issues.

is surely not. Again, it is the former kind of logical necessity (usually called "narrow logical" necessity) which the essentialist definition aims at.

Of course, this kind of objection can be restated for any concept (e.g. the concepts expressed by 'red', 'number', or 'Socrates') and any essential property one might wish to apply to concepts (e.g. being mind-independent, being productive, or existing of necessity).

The problem with Fine's account is, as it seems to me, that it takes conceptual necessity to be a special case of metaphysical necessity (as understood on the essentialist account). On the view expressed in the above passage, conceptual necessities are those metaphysical necessities which flow from the essence of objects of a certain kind—to wit, from the essence of concepts.[52]

Interestingly, Fine suggests to regard essence as "a kind of definition"[53], where the appropiate kind of definition for objects is real definition.[54] This sort of definition takes the ontological nature of the thing it defines into account. But the ontological nature of concepts is clearly irrelevant for conceptual necessity. What is relevant to conceptual necessity is meaning, not essence. And meaning is not a special case of essence (just as linguistic definition is not a special case of real definition). Accordingly, conceptual necessity should not be viewed as a special case of metaphysical necessity as understood on the essentialist construal.

In [Fine, 2002b] (at page 255) Fine offers a somewhat different account of the special cases of metaphysical necessity.[55] On this proposal the notion of conceptual necessity can be defined via a restriction of the definition of metaphysical necessity (as suggested in [Fine, 1994]) along the following lines. A proposition may be said to be a conceptual (or, alternatively, logical) necessity if (i) it is a metaphysical necessity and if (ii) it is a conceptual (logical) truth—where the notion of conceptual (logical) truth is hoped to be explicable in non-modal terms or, at least, without appeal to further modal notions.

[52]Clearly, concepts are treated as objects, where 'object' is obviously to be understood in the sense of 'entity'. This inclusive reading of 'object' is supported by the remarks in [Hale, 1996] p. 116, note 14.

[53]Cf., for example, [Fine, 1994] p. 14 or [Fine, 1995a] p. 273.

[54]Cf. [Fine, 1995a] p. 275.

[55]We have encountered that proposal already in a footnote to Subsection 4.2.13.

Clearly, on this restriction the proposition that the concept BACHELOR is abstract (or the proposition that the concept AND is mind-independent) will not qualify for the status of a conceptual (logical) necessity, since condition (ii) is not met. But—even granting that the restricting condition does not involve modality in an illicit way—it is doubtful, as it seems to me, that this way of defining conceptual (logical) necessity is satisfactory? For even if the propositions expressed by 'All bachelors are male' or 'All rectangles are rectangles' meet the restricting conditions on conceptual and logical necessity, respectively, they do not meet condition (i). As I have argued above, neither is true in virtue of the essence (or real definition) of objects—in the case under consideration—in virtue of the essence of concepts.[56]

Let me add a further worry. If correct, the conclusion of the previous argumentation will have repercussions for Fine's taxonomical views on necessity as presented in [Fine, 2002b]. There he suggests that we should distinguish three fundamental, mutually irreducible, kinds of necessity: metaphysical, natural, and normative necessity. But if the above argumentation is sound, there should be a further fundamental kind of necessity: conceptual necessity. On our account, conceptual necessity may be viewed as a special case

[56]It is worth noting in this context that Fine frequently illustrates his essentialist views in terms of an analogy between two pairs of notions: the couple essence and necessity on the one hand, and the couple meaning and analyticity on the other hand (see, for example, [Fine, 1994] pp. 10-11 or [Fine, 1995b] p. 56). On this analogy, essence applies to objects, whereas meaning applies to terms and, furthermore, objects are defined in terms of real definition, whereas terms are defined in terms of nominal definition (see [Fine, 1995a] p. 275). It seems to me that Fine could avoid the present criticism, if he pursued the more traditional way of explaining conceptual necessity in terms of meaning. He could use nominal definition as applied to terms under his "thick" conception of terms on which terms amount to concepts (cf. [Fine, 1994] pp. 13-14), instead of explaining that notion in terms of essence. However, this strategy will work only if both pairs of the analogy do not collapse. But it is not clear from Fine's writings that this is so: "(...) there exists a certain *analogy* between defining a term and giving the essence of an object; for the one results in a sentence which is true in virtue of the meaning of the term, while the other results in a proposition which is true in virtue of the identity of the object. However, I am inclined to think that the two cases are *not merely parallel but are, at bottom, the same*." ([Fine, 1994] p. 13, my emphasis). No doubt, to give in to this inclination would be objectionable, for it would result in a conflation of nominal and real definiton. It is presumably an inclination of this sort which is responsible for the problem exposed above.

of definitional necessity along the lines suggested in Subsection 4.2.13; and there are, unlike for Fine, no problems of this source related kind with it.

In the present setting I wish to remain agnostic on taxonomical issues that go beyond the remarks made in this section. (I am inclined to think, though, that there is no such thing like a substantial notion of natural or of normative necessity. At best such notions should be viewed, as I tentatively believe, as being relative forms of necessity which can be explained in terms of definitional necessity. Where L is a set of propositions having the status of natural (or, alternatively, normative or any other) laws, a proposition is a natural (or normative or other) necessity, if it is entailed by L—where the entailment is to be explained in terms of logical necessity which is a special case of definitional necessity. As a consequence, the natural (normative or other) necessity of the members of L becomes a trivial matter. Many writers, though, think that an account of this "insubstantial" sort is in conflict with intuitions.[57] But, of course, these intuitions are the denotationalist's ones.)

Let me close this section with a note on basic necessity. In view of the remarks at the end of Subsection 4.5.3, it will be noted, that basic necessity does not serve to define definitional necessity by way of restriction (e.g., in the way definitional necessity defines logical necessity) or by way of relativization (in the way logical necessity may be taken to define, say, natural necessity).

4.5.11 On Transworld Identity and Reidentification

The issue of transworld identity (i.e., on the canonical account of intended Kripke-models, the issue of explaining how one and the same individual can be contained in the domain of different possible worlds and thereby exist, in some sense or other, at more than one world) does not arise on the present approach. We do not need to care whether one representative represents the same individual as another across worlds.

The evaluation of *de nomine* modal claims (i.e., claims in which modal operators occur in the scope of a quantifier or atomic sentences which are prefixed with these operators) does in no way involve objects. On the present account a sentence like 'Socrates is necessarily a man' is true with respect to

[57] See, for instance, [Fine, 2002b].

the level of sense, because no matter how it would be arranged, the sense-extensions of 'Socrates' and '... is a man' would coincide on 'Socrates is a man'. This account does in no way appeal to the fact that Socrates is a man in more than one possible world (in which he exists). Moreover, since there are no individuals and worlds in our framework, we need not be worried about how individuals are to be represented across possible worlds. The discussions of transworld identity (and counterpart-relations) simply need not concern us.

Similarly, we need not worry how to reidentify individuals or surrogates in more than one possible world. Instead, to put it in a somewhat simplifying way, we reflect (perhaps along the lines discussed in Chapter 5) the sense-extensions of names and predicates by exercising our knowledge of their meanings and the rules by which their sense-extensions and the level of sense are governed.[58]

4.5.12 On Modal Arguments

According to the present theory of modality, sentences which occur in modal environments (see Subsection 4.4.4) are nonreferential and are to be evaluated according to the nonreferential mode. This proposal has certain consequences for the success of metaphysically motivated modal arguments.

For example, the conclusion of any modal version of the Ontological Argument being itself a nonmodal existence claim (i.e., 'God exists'), will occur in a modal environment. If such a conclusion is true, it will be true only with respect to the level of sense. A claim of truth with respect to the level of sense, though, will not be enough for the advocate of such an argument, since the conclusion will not amount to a claim of referential existence.

Whatever ones views about modal existence arguments are, the assumption that modal discourse is nonreferential, need not result in the view that

[58]See [Divers, 2002] ch. 16 (and the references therein) for a survey of ersatzist strategies of dealing with the problems of transworld identity and reidentification. For a discussion of the problem of transworld identity which does also deal with genuine realist and modalist approaches see [Chihara, 1998] ch. 2. For Chihara's own proposal see [Chihara, 1998] sect. 8.4.

the conclusions of modal arguments have no ontological impact, for the Level Connecting Theorem can allow to transform the conclusion of the modal argument, which we assume for the sake of argument to be true with respect to the level of sense, into the desired claim of referential existence. However, as has been argued in Subsection 4.4.3, this transition will be licensed just in case the conclusion of the modal argument, which is a nonmodal existence claim, turns out to be true with respect to the level of reference when it is evaluated according to the referential mode. But for that existence claim to be referentially true, there must be a suitable denotatum in the domain of discourse which is responsible for the truth of that claim. And it is exactly this fact which, to my mind, is at stake, and which is presupposed by the denotationalist proponents of such arguments.

Similar remarks apply to modal arguments for the distinctness of certain phenomena. Arguments for anti-materialism in the philosophy of mind, for instance, purport to establish the distinctness of mental and physical phenomena (e.g., pain and C-fibre stimulation, respectively). Such arguments usually appeal to the thesis of the necessity of referential identity and involve a step from the possibility of referential distinctness to referential distinctness.[59] According to the present account of modality, claims which occur in modal arguments are embedded in a modal environment and are to be evaluated according to the nonreferential mode. Consequently, these arguments arrive, if sound, merely at a conclusion which amounts to a claim of substitutional distinctness. Such arguments, therefore, do not give us the referential distinctness the proponent of such an argument intends to establish. In order to obtain referential distinctness, he would have to argue, in a nonmodal way, that the conclusion is also true according to the referential mode of evaluation. If this were so he would be entitled to proceed, using something to the effect of the Level Connecting Theorem, from the claim of substitutional distinctness to the desired conclusion.

[59]For a survey on anti-materialist arguments and the role the notions of conceivability and possibility play in them see, in particular, [Gendler & Hawthorne, 2002] subsection 3.3, section 4 and the references therein.

Chapter 5

Integrated Associative Semantics

5.1 Introduction

As a short epistemological coda to the previous chapters, I wish to discuss how the main ideas, or at least the spirit, of Christopher Peacocke's principle-based account of modality can be adapted to our semantical framework.[1] The only purpose of the discussion to follow, is to demonstrate how that framework could be elucidated further. We shall, therefore, remain neutral on Peacocke's views and neither accept, nor reject them. Moreover, we shall not discuss the merits of his account.[2]

Peacocke's original motivation for his principle-based account of modality is to meet what he calls the "Integration Challenge" for metaphysical modality, i.e., the task of connecting up the truth conditions for modal statements with the conditions for modal knowledge. According to Peacocke, the pivotal role in meeting this challenge is played by the theory of understanding for metaphysical necessity (or, correspondingly, by the theory of the possession

[1] Peacocke's principle-based account is presented, in its most actual form, in [Peacocke, 1999] ch. 4 and developed further in [Peacocke, 2002c].

[2] For a critical discussion of Peacocke's principle-based approach see, in particular, [Rosen, 2002], [Williamson, 2002], and [Wright, 2002]. [Peacocke, 2002b] replies to these contributions.

of the concept of metaphysical necessity). He argues, in effect, that the very possession of the concept of metaphysical necessity ensures that a thinker can have knowledge of propositions which contain that concept. For Peacocke the concept of metaphysical necessity belongs to the special category of epistemically-individuated concepts, i.e., concepts which are individuated in terms of conditions under which a thinker comes to know propositions (or, equivalently, contents) which contain such concepts.[3]

The model which Peacocke favours for the account of the possession of the concept of metaphysical necessity is the "model of implicitly known principles":

> "Under this treatment, grasp of some concept consists in having tacit knowledge of a set of principles involving that concept, where this set of principles plays a dual role, both metaphysical and epistemological. First, the principles themselves fix the extension of the concept. For a content containing the concept to be true is just for it to be counted as true by these principles. That is a statement about the metaphysics of the domain. Second, the thinker draws on his tacit knowledge of these principles, and possibly other information, in evaluating contents containing the concept as true or as false. If, in the process of evaluation, the thinker uses the very principles that determine, at the level of metaphysics, whether the content really is true or false, this will be a way of coming to know those contents."[4]

In the case of the concept of metaphysical necessity, the theory of possession for that concept involves certain constraints, the "Principles of Possibility", a world-description must satisfy if it is to represent a genuine possibility. Taken together, these principles fix the concept of metaphysical necessity. On the metaphysical side a world-description represents a genuine possibility just in case it satisfies all the Principles of Possibility. On the epistemological side, to possess the concept of metaphysical necessity is to have tacit knowledge of these principles, and to employ them correctly in evaluating modal claims.

[3]See [Peacocke, 1999] pp. 13-14.

[4][Peacocke, 2002a] p. 637.

In what follows I will first summarize Peacocke's approach in more detail and then sketch how the main ideas can be adapted to definitional necessity.

5.2 Exposition of Peacocke's Principle-Based Theory of Necessity

According to Peacocke's account, a proposition is a metaphysical necessity just in case it is true under all admissible assignments. In effect, the account adapts the usual account of the notion of truth-functional truth (when the assignments are viewed as models). Now, what is the language Peacocke for which he formulates his semantical framework and what does he take to be an admissible assignment?

Peacocke does not consider uninterpreted schematic expressions as the alphabet of the language of his semantic apparatus, but an alphabet of atomic concepts. This alphabet contains singular concepts like SOCRATES, n-ary predicative concepts like BACHELOR, quantifier concepts like ALL, propositional operator concepts like NOT (for negation) or AND (for conjunction). Following Peacocke, we shall use as metavariables for singular concepts m, $m_1, ..., m_n$ and as metavariables for concepts in general C, $C_1, ..., C_n$. We shall refer to the language with this alphabet with '**L**'. The grammar of **L** can then be given in the usual first-order way.[5]

A Peacocke-style assignment s is, as we shall take it,[6] a quintuple $<D_s, U_s, val_s, propval_s, ext_s>$, where: D_s is a non-empty set of individuals a, b, ... for which we shall use $a, b, ...$ as metavariables; U_s is a non-empty set of n-ary universals P, Q, ... (i.e., of properties and relations) for which we shall use $P, Q, ...$ as metavariables; val_s is a function from singular concepts into elements of D_s; from n-ary predicative concepts into subsets of D_s^n, and so forth; $propval_s$ is a function from n-ary concepts to elements of U_s; and ext_s is a function from universals to extensions of the right sort.

[5]We have offered a language system of this sort back in Subsection 4.2.12.

[6]Unfortunately, Peacocke does not give a formal presentation of his own. Our presentation is patterned after the one presented in [Rosen, 2002].

The elements assigned by val_s are the semantic values of a given concept C according to s, referred to as $val_s(C)$, i.e., individuals or sets of them. The elements assigned by $propval_s$ are the property values of a given concept C according to s, referred to as $propval_s(C)$, i.e., universals.

The extension of a property P according to s is abbreviated as $ext_s(P)$. Let f stand for the extension of the appropriate kind. Peacocke stipulates that $val_s(C) = f$ iff $ext_s(propval_s(C)) = f$.[7] The assignments are taken to be total and comprehensive (i.e., they determine the extensions of all properties and relations on which the actual semantic value of a given atomic concept can depend).[8]

The truth of a Fregean proposition (or Thought) or, alternatively, a Russellian proposition A on an assignment s (formally: $s \models A$) is then defined in the usual way. The truth condition for the simplest case of an atomic Fregean proposition will be:

$$s \models Cm \quad \text{iff} \quad val_s(m) \in val_s(C).$$

Similarly, the truth conditions of a singular Russellian proposition, i.e., a proposition which is built up from n individuals and an n-ary universal (e.g., the proposition expressed by the sentence 'Russell is a philosopher', symbolized as Pa, and represented by the proposition tuple ⟨Russell, the property of being a philosopher⟩) will be defined as follows:

$$s \models Pa \quad \text{iff} \quad a \in ext_s(propval_s(C)), \text{ where } propval_s(C) = P.$$

On Peacocke's account, there is for any given assignment s a corresponding total world-description (or specification) w. A specification w is just the set of propositions the assignment s counts as true. So where A stands for a proposition of either the Fregean or the Russellian kind, a specification w for any assignment s and proposition A is just the set $\{A : s \models A\}$. Such specifications are not yet what Peacocke calls his ersatz possible worlds, for these specifications are not by themselves genuine possibilities. This is so, since for a specification to be a genuine possiblity, its elements must be counted as true by an *admissible assignment*.[9]

[7]Cf. [Peacocke, 1999]: 128.

[8]Cf. [Peacocke, 1999]: 135.

[9]Let me just note that it is somewhat unclear whether Peacocke's specifications represent worlds (cf., for instance, [Peacocke, 1999]: 125) or whether they are the worlds (cf., e.g., [Peacocke, 1999]: 126).

According to Peacocke, there is nothing to this semantic framework as developed so far which prevents inadmissible assignments (e.g., an assignment of the truth function for conjunction to the concept of disjunction). What is needed, therefore, is an explanation of why such assignments are inadmissible and why the corresponding specifications are not genuine. Peacocke's strategy is to characterize the admissibility of an assignment s in such a way that for each genuine specification w^* there is some admissible assignment s^* which counts all the propositions A as true. Thus, a genuine specification w^*, i.e., a Peacocke-style ersatz world, is a set $\{A : s^* \models A\}$. (What is distinctive of these ersatz worlds is that they are hybrid in that they involve Fregean as well as Russellian propositions.)

An assignment is said to be admissible if it satisfies a number of constraints. These constraints are given in terms of the Principles of Possibility:

1. *Unified Modal Extension Principle.*
   "An assignment s is admissible only if: for any concept C, the semantic value of C according to s is the result of applying the same [semantic] rule as is applied in the determination of the actual semantic value of C."[10]

2. *Constitutive Principles.*

   (a) Kind Essence. If P is a property (e.g., the property of being human) which is an object a's (e.g., Socrates) fundamental kind, then an assignment is inadmissible if it counts the proposition Pa as false.[11]

   (b) Individual Essence. In any case in which it is constitutive of the object a (e.g., Socrates) that it bear R (e.g., being a son of) to the object b (e.g., Sophroniscus), an assignment is inadmissible if

[10][Peacocke, 1999] p. 136. The UMEP thus, first, constrains the extension a concept may receive from an assignment and, second, extends the way in which the extension of a concept is fixed in the actual world to genuine specifications (it does not extend the actual extensions of concepts or properties to them).

[11]See [Peacocke, 1999]: 145.

it both counts as true the proposition that a exists and counts the proposition that Rab as false.[12]

3. *Principle of Constrained Recombination.* An assignment is admissible if it respects the Unified Modal Extension Principle and the Constitutive Principles.[13]

In effect, the Unified Modal Extension Principle is a constraint on admissibility which ensures that admissible assignments respect the semantic nature of concepts. Similarly, the Constitutive Principles guarantee that admissible assignments respect the essences of entities at the level individuals, properties and relations.[14] The Principle of Constrained Recombination is to the effect that whichever assignment is not ruled out by these principles (being separately necessary and jointly sufficient for admissibility) qualifies as admissible.

Having explained what is involved in an assignment's being admissible and a specification's being genuinely possible, Peacocke states the contribution made to truth conditions by the modal operators in terms of the following characterizations (Chzns):

> Chzn□: A proposition A is metaphysically necessary iff it is true according to all admissible assignments.
>
> Chzn◇: A proposition A is metaphysically possible iff it is true according to some admissible assignment.[15]

---

[12]See [Peacocke, 1999]: 146. Peacocke allows that the list of the constitutive principles may be open-ended.

[13]See [Peacocke, 1999]: 149. Our Principle of Rearrangement resembles this principle to some extent.

[14]See [Peacocke, 1999]: 148-149.

[15]Cf. [Peacocke, 1999]: 150. Given the correspondence between assignments and specifications, these characterizations can be stated, alternatively, in terms of genuinely possible specifications. Let S^* be the set of all admissible assignments s^*, let W^* be the set of all genuinely possible specifications w^* and let A be a proposition of either kind, then the clauses may be recaptured as follows: Chzn′□: $\Box A$ iff $A \in \bigcup_{s^* \in S^*}\{A : s^* \models A\}$ iff $A \in w^*$ for all $w^* \in W^*$. Chzn′◇: $\Diamond A$ iff $A \in \{A : s^* \models A\}$ for some $s^* \in S^*$ iff $A \in w^*$ for some $w^* \in W^*$.

So much for the metaphysical aspect of the Integration Challenge.

Let us now turn to the epistemological aspect. On Peacocke's account it is readily met as follows.

> "Provided that any non-modal principles upon which she relies are known, a thinker's modal judgements reached by the proper use of the implicit knowledge of the Principles of Possibility will, in the nature of the case, be knowledge."[16]

When implicit knowledge of the principles is guaranteed, the solution to the Integration Challenge for the case of metaphysical necessity amounts to putting both aspects together.

> "The materials above permit formulation of a theory of possession of the concept of metaphysical necessity in which the very Principles of Possibility which fix modal truth are also mentioned in an account of the conditions under which modal contents are known, by way of those principles being contents of the understander's tacit knowledge. It is because of this connection between truth, understanding, and knowledge that the ordinary means we take to establish modal truth do not fall short of reaching modal truth and genuine modal knowledge."[17]

In this way Peacocke's principle-based account of modality meets the Integration Challenge, i.e., the challenge of linking up the truth conditions of modal statements with the conditions for modal knowledge.

5.3 The Integration Challenge Met

How can the Integration Challenge for definitional necessity be met in the spirit of Peacocke's proposal? Well, this can be accomplished without much ado. Recall that the truth conditions for statements of definitional necessity have been given by (Necessity) which says:

[16][Peacocke, 1999] p. 162.
[17][Peacocke, 1999] p. 163.

> 'It is definitionally necessary that A' is true *simpliciter* with respect to the level of sense just in case no matter how the level of sense could have been (in the basic sense of 'could') arranged 'A' would be true *simpliciter* with respect to the level of sense.

Recall, moreover, that the level of sense is governed by a couple of principles, i.e., the Principle of Nominal Sense-Extension, the Principle of Predicative Sense-Extension, and the Principle of Rearrangement. Since these principles also determine how the level of sense could have been arranged, a statement of definitional necessity cannot be true with respect to that level unless its rearrangements satisfy these principles. So much for the metaphysical portion of the challenge.

As regards the epistemological part we simply insist, adapting Peacocke's suggestion, that a thinker has implicit knowledge of these principles and, moreover, take it for granted that if he employs these governing principles in reaching modal judgements in an appropriate way he thereby gains knowledge of definitional necessity. In this way the Integration Challenge for definitional necessity can be met along the lines suggested by Peacocke.

There are numerous differences between Peacocke's theory and ours. Among the most important ways in which they differ are the following. First, our account is concerned with definitional necessity, not with broadly logical (or metaphysical) necessity. Second, our principles are openly modal in that they involve the notion of basic possibility, whereas Peacocke is at pains to avoid the use of modal idiom in the formulation of his Principles of Possibility. Thirdly, unlike Peacocke's account which (in the version he favours) appeals to ersatz possible worlds, ours does not appeal to possible worlds at all. Also, our account is, immune to the denotationalist's problems concerning actualism.[18] Finally, our account of modal truth does not appeal to individuals and universals.

The ordinary methods by which knowledge of definitional necessities is achieved involve the means of argument and proof. Let me briefly illustrate an application of that method with a sketch of a Kripke-style argument for a simple instance of an a posteriori definitional necessity.

[18]For Peacocke's accommodation of actualist intuitions into his principle-based approach see [Peacocke, 2002c].

5.4 Definitional Necessity A Posteriori

According to the present theory of modality, modal truth is truth with respect to the level of sense. True a posteriori necessities are, therefore, truths with respect to that level. In contrast to the demonstration of a priori definitional necessities, the demonstration of a posteriori definitional necessities will appeal at some point to something like the Level Connecting Theorem (see Subsection 4.4.3).

Suppose that we discover that Hesperus and Phosphorus are referentially identical. Our knowledge of the truth expressed by 'Hesperus and Phosphorus are (referentially) identical' will thus be gained by a posteriori (or empirical) means. By appeal to the Principle of Correlation and the Level Connecting Theorem, which are both a priori, we then arrive at the truth of 'Hesperus and Phosphorus are (substitutionally) identical' with respect to the level of sense. From this we can reason by the relevant instance of the necessity of substitutional identity (NSI), entering a modal environment, to 'Necessarily, Hesperus is (substitutionally) identical to Phosphorus'.[19] The truth of this claim of definitional necessity with respect to the level of sense will be guaranteed by the a priori principles which govern that level. However, it should be noted that the last step of this argumentation will be licensed only if we deviate from our official semantics for statements of substitutional identity, for example, by evaluating them exclusively with respect to the characteristic portions of the sense-extensions of 'Hesperus' and 'Phosphorus'—for otherwise NSI will not hold (see, in particular, comment 3.2.4(7)).

Moreover, if we assume, in a way analogous to Peacocke's principle-based approach to modality, that we have implicit knowledge of the principles which govern the level of sense, we may also conclude that the judgement reached in this way does indeed amount to knowledge.

By contrast, knowledge of a priori definitional necessities *de nomine* like 'Necessarily, Santa Claus is (substitutionally) identical to Father Christmas' which contain nondenoting names will be reached without appeal to the level

[19] The classical denotationalist argumentation of this sort is given in [Kripke, 1980 (1972)] pp. 97-105; cf. also [Peacocke, 1999] p. 168.

connecting theorem. This also applies to such a priori definitional necessities *de dicto* like, say, ‘Necessarily, every fairy has magic powers’ in which predicates occur which do not have a referential extension.

Appendix A

This appendix reproduces, in substitutional terms, the standard proofs which have been omitted in Chapter 2. The exposition is taken from [Hughes & Cresswell, 1996].

Proof of Theorem 2.5.6(2) (Π*-property).*

Once a set Δ has the Π-property then any set (in the same language) of which Δ is a subset still has the Π-property. By Lindenbaum's Lemma then since Δ is consistent there is a maximal consistent set Γ such that $\Delta \subseteq \Gamma$, and so since Δ has the Π-property so does Γ.

As usual we assume that all formulae of the form $(\Pi x)A$ for any formula A of L^+ and any nominal variable x are enumerated. We then define a sequence of sets $\Delta_0, \Delta_1, ...$ etc. as follows:

Δ_0 is Θ

Δ_{n+1} is $\Delta_n \cup \{A[y/x] \rightarrow (\Pi x)A\}$

where $(\Pi x)A$ is the $n+1$th formula in the enumeration of formulae of that form and y is the first nominal variable not in Δ_n or in A. Since Δ_0 is in L and Δ_n has been formed from it by the addition of only n formulae there will be infinitely many variables from L^+ left over to provide such a witnessing y. We assume Δ_n to be consistent. We now show that Δ_{n+1} is consistent if Δ_n is. Suppose Δ_{n+1} is inconsistent. So there will be formulae $B_1, ..., B_n$ in Δ_n such that both

(i) $\vdash (B_1 \wedge ... \wedge B_n) \rightarrow A[y/x]$

and

(ii) $\vdash (B_1 \wedge ... \wedge B_n) \rightarrow \neg(\Pi x)A$.

Since y does not occur in Δ_n is not free in $(B_1 \wedge ... \wedge B_n)$. So from (i) by $\Pi 2$

(iii) $\vdash (B_1 \wedge ... \wedge B_n) \rightarrow (\Pi y)A[y/x]$.

Since y does not occur in A, $(\Pi y)A[y/x]$ is a bound alphabetic variant of $(\Pi x)A$, and so by the equivalence of both formulae,

(iv) $\vdash (B_1 \wedge ... \wedge B_n) \rightarrow (\Pi x)A$.

But by transposition and syllogism, (ii) and (iv) give

(v) $\vdash \neg(B_1 \wedge ... \wedge B_n)$

and (v) makes Δ_n inconsistent which contradicts the assumption. So let Δ be $\bigcup_{n\geq 0} \Delta_n$. Clearly, Δ is consistent and has the Π-property. This proves theorem 2.5.6(2) for variables. The reasoning for constants is similar.

Proof of Theorem 2.5.6(3) (Existence Theorem).

We define a sequence of formulae $C_0, C_1, C_2, ...$ etc. C_0 is $\neg A$. Given C_n, formula C_{n+1} is defined as follows. Let $(\Pi x)D$ be the $n+1$th formula of that form and let o be the first nominal term such that

$$(\star) \quad \{B : \Box B \in s\} \cup \{C_n \wedge (D[o/x] \to (\Pi x)D)\}$$

is consistent.

So let C_{n+1} be $C_n \wedge (D[o/x] \to (\Pi x)D)$. We have to ensure that there always will be a witness o satisfying $(\star)$. We use the following lemma which we state without giving a proof (cf. [Hughes & Cresswell, 1996] p. 117),

> (PC) Let Λ be any normal system of propositional modal logic, and let Θ be an Λ-consistent set of formulae containing $\neg\Box A$. Then $\{B : \Box B \in \Theta\} \cup \{\neg A\}$ is Λ-consistent.

By this lemma, $\{B : \Box B \in s\} \cup \{C_0\}$ is consistent, since C_0 is $\neg A$. We now show that provided $\{B : \Box B \in s\} \cup \{C_n\}$ is consistent there will always be an o which satisfies $(\star)$. Note that we cannot assume as we did in proving theorem 2.5.6(2) concerning the Π-property that o is a new nominal term, since, as we have remarked above, all the nominal terms of L^+ will already occur in $\{B : \Box B \in s\}$. However, it can be shown that there always will be an appropriate o.

So suppose for reductio there were no such o. Then for every nominal term o of L^+ there will exist some $\{\Box B_1, ..., \Box B_k\} \subseteq \{B : \Box B \in s\}$ such that

$$\vdash (B_1 \wedge ... \wedge B_k) \to (C_n \to \neg(D[o/x] \to (\Pi x)D))$$

so, by DR1 and $\Box$-distribution

$$\text{(i)} \quad \vdash (\Box B_1 \wedge ... \wedge \Box B_k) \to \Box(C_n \to \neg(D[o/x] \to (\Pi x)D))$$

But s is maximal consistent and

$$\Box B_1, ..., \Box B_k \in s,$$

and so

$$\Box(C_n \to \neg(D[o/x] \to (\Pi x)D)) \in s.$$

Now this is so for every nominal term o and s has the Π-property.

So let z be some nominal variable not occurring in D or in C_n, and consider

$$(\Pi z)\Box(C_n \to \neg(D[z/x] \to (\Pi x)D)).$$

Since s has the Π-property there will be a term o such that

(ii) $\Box(C_n \to \neg(D[o/x] \to (\Pi x)D)) \to (\Pi z)\Box(C_n \to \neg(D[z/x] \to (\Pi x)D)) \in s.$

But we already noted that for every term o $\Box(C_n \to \neg(D[o/x] \to (\Pi x)D)) \in s$. And so

(iii) $(\Pi z)\Box(C_n \to \neg(D[z/x] \to (\Pi x)D)) \in s.$

But s is maximal consistent in Λ+SBF$\doteqdot$, a system which contains SBF as a thesis, and so by SBF

(iv) $\Box(\Pi z)(C_n \to \neg(D[z/x] \to (\Pi x)D)) \in s.$

Since z does not occur in C_n or D and thus is not free in C_n then by T2 we have

(v) $\Box(C_n \to (\Pi z)\neg(D[z/x] \to (\Pi x)D)) \in s$

But by T3,

$$\vdash \neg(\Pi z)\neg(D[z/x] \to (\Pi x)D).$$

Thus $\Box\neg C_n \in s$ and so $\neg C_n \in \{B : \Box B \in s\}$ which, in the light of the fact that $\{B : \Box B \in s\} \subseteq t$, would make t inconsistent.

Now let t be the union of $\{B : \Box B \in s\}$ and all the C_ns. Since each set $\{B : \Box B \in s\} \cup \{C_n\}$ is consistent, and since

$$\vdash C_m \to C_n \text{ for } m \geq n,$$

so is their union t. So t has all the properties we wanted it to have. This completes the proof of theorem 2.5.6(3).

Proof of Theorem 2.6.4(1) (Existence Theorem).

We let L_t be an infinitely proper sublanguage of L^+ such that L_s is an infinitely proper sublanguage of L_t containing infinitely many of the terms of L^+ which are not contained in L_s. As $s \in S^{\Lambda}$, L_s lacks infinitely many terms of L^+. By theorem 2.5.6(3), then, $\{B : \Box B \in s\} \cup \{\neg A\}$ is consistent. Moreover, the formulae in $\{B : \Box B \in s\} \cup \{\neg A\}$ are from L_s. And since L_t contains infinitely many terms not in L_s, theorem 2.5.6(2) ensures that $\{B : \Box B \in s\} \cup \{\neg A\}$ has a consistent extension t with the Π-property in L_t. By Lindenbaum's Theorem 2.5.6(1), s has an extension t which is maximal consistent.

Proof of Theorem 2.7.5(1) ($\Box\Pi$-property).

We form Δ in a way similar to that used in the proof of theorem 2.5.6(2) as the union of a sequence $\Delta_0, \Delta_1, ...$ etc.

$$\Delta_0 = \Theta.$$

Let T and Υ be two infinite disjoint sets of terms of L^+ not in L, and assume the terms of T and Υ are enumerated.

We now assume a double ordering of formulae of L^+, an ordering of all formulae of L^+ which begin with a substitutional universal quantifier, and a further ordering of the set Ω of all formulae of the form $\Box(C_1 \to ... \to \Box(C_j \to \Box(\Pi x)D)...)$ for $j \geq 0$, with x not free in $C_1, ..., C_j$.

When $(\Pi r)A$ is the $n+1$th formula of L^+ beginning with a substitutional universal quantifier and $\Box(C_1 \to ... \to \Box(C_j \to \Box(\Pi x)D)...)$ is the $n+1$th member of Ω and y is the first variable in T and z is the first variable in Υ alien to Δ_n or in A or in $C_1, ..., C_j$ or in D, then Δ_{n+1} is

$$\Delta_n \cup \{\ddot{E}y, A[y/r] \to (\Pi r)A, \Box(C_1 \to ... \to \Box(C_j \to \Box(\ddot{E}z \to D[z/x])) \to \Box(C_1 \to ... \to \Box(C_j \to \Box(\Pi x)D))\}.$$

We show that Δ_{n+1} is consistent if Δ_n is. So suppose Δ_{n+1} were not consistent. Then for some $B_1, ..., B_k \in \Delta_n$

$$\text{(i)} \vdash (B_1 \wedge ... \wedge B_k \wedge (\ddot{E}y \wedge (A[y/r] \to (\Pi r)A)) \to \Box(C_1 \to ... \to \Box(C_j \to \Box(\ddot{E}z \to D[z/x])...)))$$

and

(ii) $\vdash (B_1 \wedge ... \wedge B_k \wedge (\ddot{E}y \wedge (A[y/r] \rightarrow (\Pi r)A))) \rightarrow \neg\Box(C_1 \rightarrow ... \rightarrow \Box(C_j \rightarrow \Box(\Pi x)D)...)$.

Now z does not occur free in Δ_n or in $\ddot{E}y \wedge (A[y/r] \rightarrow (\Pi r)A)$, and so from (i) by UG$\Box\Pi^{j+1}$

(iii) $\vdash (B_1 \wedge ... \wedge B_k \wedge (\ddot{E}y \wedge (A[y/r] \rightarrow (\Pi r)A))) \rightarrow \Box(C_1 \rightarrow ... \rightarrow \Box(C_j \rightarrow \Box(\Pi z)(\ddot{E}z \rightarrow D[z/x])))$

and so by $\Pi^{\rightarrow}$,

(iv) $\vdash (B_1 \wedge ... \wedge B_k \wedge (\ddot{E}y \wedge (A[y/r] \rightarrow (\Pi r)A))) \rightarrow \Box(C_1 \rightarrow ... \rightarrow \Box(C_j \rightarrow \Box((\Pi z)\ddot{E}z \rightarrow (\Pi z)D[z/x])))$.

So by K,

(v) $\vdash (B_1 \wedge ... \wedge B_k \wedge (\ddot{E}y \wedge (A[y/r] \rightarrow (\Pi r)A))) \rightarrow \Box(C_1 \rightarrow ... \rightarrow \Box(C_j \rightarrow (\Box(\Pi z)\ddot{E}z \rightarrow \Box(\Pi z)D[z/x])))$.

Then by PC (i.e., $(A \rightarrow (B \rightarrow C)) \leftrightarrow (B \rightarrow (A \rightarrow C))$,

(vi) $\vdash \Box^j(\Pi z)\ddot{E}z \rightarrow ((B_1 \wedge ... \wedge B_k \wedge (\ddot{E}y \wedge (A[y/r] \rightarrow (\Pi r)A)) \rightarrow \Box(C_1 \rightarrow ... \rightarrow \Box(C_j \rightarrow \Box(\Pi x)D[z/x])))$.

So by RBV,

(vii) $\vdash \Box^j(\Pi z)\ddot{E}z \rightarrow ((B_1 \wedge ... \wedge B_k \wedge (\ddot{E}y \wedge (A[y/r] \rightarrow (\Pi r)A)) \rightarrow \Box(C_1 \rightarrow ... \rightarrow \Box(C_j \rightarrow \Box(\Pi x)D)))$.

Now by U$\ddot{E}$ and Nec,

(viii) $\vdash (B_1 \wedge ... \wedge B_k \wedge (\ddot{E}y \wedge (A[y/r] \rightarrow (\Pi r)A))) \rightarrow \Box(C_i \rightarrow ... \rightarrow \Box(C_j \rightarrow \Box(\Pi x)D))$.

From (ii) and (viii) we get

$$\vdash (B_1 \wedge ... \wedge B_k) \rightarrow (\ddot{E}y \rightarrow \neg(A[y/r] \rightarrow (\Pi r)A))$$

by syllogism and obvious steps. Then by $\Pi 2$

$$\vdash (B_1 \wedge ... \wedge B_k) \rightarrow (\Pi y)(\ddot{E}y \rightarrow \neg(A[y/r] \rightarrow (\Pi r)A)).$$

So by $\Pi^{\rightarrow}$

$$\vdash (B_1 \wedge ... \wedge B_k) \rightarrow ((\Pi y)\ddot{E}y \rightarrow (\Pi y)\neg(A[y/r] \rightarrow (\Pi r)A)).$$

So by U$\ddot{E}$

$$\text{(ix)} \vdash (B_1 \wedge ... \wedge B_k) \rightarrow (\Pi y)\neg(A[y/r] \rightarrow (\Pi r)A).$$

Since y does not occur in A, by QR,

$$\vdash \neg(\Pi y)\neg(A[y/r] \rightarrow (\Pi r)A).$$

But then from (ix)

$$\vdash \neg(B_1 \wedge ... \wedge B_k).$$

But this contradicts the hypothesis that Δ_n is consistent. Since each Δ_n is consistent Δ is also consistent and has the $\Box\Pi$-property. The reasoning for constants is similar.

Proof of Theorem 2.7.5(2) (Existence Theorem).

We assume, as in the proof of theorem 2.7.5(1) that Ω is the set of all formulae of the form $\Box(F_1 \rightarrow ... \rightarrow \Box(F_j \rightarrow \Box(\Pi x)G))...)$, where x is not free in $F_1, ..., F_n$.

We then define a sequence of formulae $C_1, C_2, ...$ etc. as follows. C_0 is $\neg A$. Given C_n we define C_{n+1} in the following way. We first define a formula C_n^+, and then show how to extend C_n^+ to C_{n+1}.

Let $(\Pi x)D$ be the $n+1$th formula of that form and let o be the first term such that

$$(\star) \quad \{B : \Box B \in s\} \cup \{C_n \wedge (\ddot{E}o \wedge (D[o/x] \rightarrow (\Pi x)D))\}$$

is consistent.

Let C_n^+ be $C_n \wedge \ddot{E}o \wedge (D[o/x] \rightarrow (\Pi x)D)$. We have to ensure that there always will be a term o satisfying $(\star)$. Since C_0 is $\neg A$, $\{B : \Box B \in s\} \cup \{C_0\}$ is consistent according to lemma (PC) which has been already used in the proof of theorem 2.5.6(3) above. Given that $\{B : \Box B \in s\} \cup \{C_n\}$ is consistent there always will be a y which satisfies $(\star)$ and thus guarantees that $\{B : \Box B \in s\} \cup \{C_n^+\}$ is consistent.

To show this we cannot assume that o is a new term. Nevertheless we can show that there always will be a suitable witness o. So suppose there were no such o. Then for every term o of L^+ there will exist some set $\{\Box B_1, ..., \Box B_k\} \subseteq s$ such that

$$\vdash (B_1 \wedge ... \wedge B_k) \rightarrow (C_n \rightarrow (\ddot{E}o \rightarrow \neg(D[o/x] \rightarrow (\Pi x)D))).$$

Then by DR1 and $\Box$-distribution,

(i) $\vdash (\Box B_1 \wedge ... \wedge \Box B_k) \rightarrow \Box(C_n \rightarrow (\ddot{E}o \rightarrow \neg(D[o/x] \rightarrow (\Pi x)D))).$

But s is maximal consistent and $\Box B_1, ..., \Box B_k \in s$, and so $\Box(C_n \rightarrow \neg(D[o/x] \rightarrow (\Pi x)D)) \in s$. This is the case for every term o.

Now s has the $\Box\Pi$-property, and so there will be a term o such that

(ii) $\Box(C_n \rightarrow ((\ddot{E}o \rightarrow \neg(D[o/x] \rightarrow (\Pi x)D))) \rightarrow \Box(\Pi z)(C_n \rightarrow (\ddot{E}o \rightarrow \neg(D[z/x] \rightarrow (\Pi x)D))) \in s,$

where z is chosen so that it does not occur in C_n or in D. So, since, $\Box(C_n \rightarrow \neg(D[o/x] \rightarrow (\Pi x)D)) \in s$ for every o,

(iii) $\Box(\Pi z)(C_n \rightarrow (\ddot{E}z \rightarrow \neg(D[z/x] \rightarrow (\Pi x)D))) \in s.$

But s is maximal in SFOL$\ddot{=}\ddot{E} + \Lambda$ and so,

(iv) $\Box(\Pi z)\ddot{E}z \rightarrow \Box(\Pi z)(C_n \rightarrow \neg(D[z/x] \rightarrow (\Pi x)D)) \in s.$

So by U$\ddot{E}$ and Nec,

(v) $\Box(\Pi z)(C_n \rightarrow \neg(D[z/x] \rightarrow (\Pi x)D)) \in s.$

But z does not occur in C_n or D and so by VQ$^{\rightarrow}$

(vi) $\Box(C_n \rightarrow (\Pi z)\neg(D[z/x] \rightarrow (\Pi x)D)) \in s.$

But by QR

$$\vdash \neg(\Pi z)\neg(D[z/x] \rightarrow (\Pi x)D)).$$

So $\vdash \Box\neg C_n$. But then $\Box\neg C_n \in s$ and so $\neg C_n \in \{B : \Box B \in s\}$ which would render $\{B : \Box B \in s\} \cup \{C_n\}$ inconsistent. Consequently, $\{B : \Box B \in s\} \cup \{C_n^+\}$ is consistent if $\{B : \Box B \in s\} \cup \{C_n\}$ is.

The next step is to show how to extend C_n^+ to C_{n+1}. Let $\Box(F_1 \rightarrow ... \rightarrow \Box(F_j \rightarrow \Box(\Pi x)G)...)$ be the nth formula in Ω and let p be the first nominal term such that

(*) $\{B : \Box B \in s\} \cup \{C_n^+ \wedge (\Box(F_1 \rightarrow ... \rightarrow \Box(F_j \rightarrow \Box(\ddot{E}p \rightarrow G[p/x]))...) \rightarrow \Box(F_1 \rightarrow ... \rightarrow \Box(F_j \rightarrow \Box(\Pi x)G)...))\}$

is consistent.

Let C_{n+1} be

$$C_n^+ \wedge (\Box(F_1 \rightarrow ... \rightarrow \Box(F_j \rightarrow \Box(\ddot{E}p \rightarrow G[p/x]))...) \rightarrow \Box(F_1 \rightarrow ... \rightarrow \Box(F_j \rightarrow \Box(\Pi x)G)...)).$$

We may assume that x is not free in C_n^+ or in $F_1, ..., F_j$ since if it is we may choose a bound alphabetic variant of $(\Pi x)G$ in which the variable that replaces x is not free in these formulae. So suppose there were no p satisfying $(*)$. Then for some $B_1, ..., B_k \in \{B : \Box B \in s\}$

(i) $\vdash (B_1 \wedge ... \wedge B_k) \rightarrow (C_n^+ \rightarrow \neg(\Box(F_1 \rightarrow ... \rightarrow \Box(F_j \rightarrow (\ddot{E}p \rightarrow G[p/x]))...) \rightarrow \Box(F_1 \rightarrow ... \rightarrow \Box(F_j \rightarrow (\Pi x)G)...))).$

So

(ii) $\vdash (B_1 \wedge ... \wedge B_k) \rightarrow (C_n^+ \rightarrow \Box(F_1 \rightarrow ... \rightarrow \Box(F_j \rightarrow \Box(\ddot{E}p \rightarrow G[p/x]))...))$

and

(iii) $\vdash (B_1 \wedge ... \wedge B_k) \rightarrow (C_n^+ \rightarrow \neg\Box(F_1 \rightarrow ... \rightarrow \Box(F_j \rightarrow (\Pi x)G)...)).$

With DR1 and $\Box$-distribution we get from (ii),

(iv) $\vdash (\Box B_1 \wedge ... \wedge \Box B_k) \rightarrow \Box(C_n^+ \rightarrow \Box(F_1 \rightarrow ... \rightarrow \Box(F_j \rightarrow \Box(\ddot{E}p \rightarrow G[p/x]))...))$

Now every formula $\Box B_1, ..., \Box B_k$ is in s so we get

(v) $\Box(C_n^+ \rightarrow \Box(F_1 \rightarrow ... \rightarrow \Box(F_j \rightarrow \Box(\ddot{E}p \rightarrow G[p/x]))...)) \in s.$

from (iv). This is so for every term p.

Since s has the $\Box\Pi$-property we also have

(vi) $\Box(C_n^+ \rightarrow \Box(F_1 \rightarrow ... \rightarrow \Box(F_j \rightarrow \Box(\Pi x)G)...)) \in s.$

So

(vii) $C_n^+ \rightarrow \Box(F_1 \rightarrow ... \rightarrow \Box(F_j \rightarrow \Box(\Pi x)G)...) \in \{B : \Box B \in s\}.$

But (vii) and (iii) would make $\{B : \Box B \in s\} \cup \{C_n^+\}$ inconsistent. Thus if $\{B : \Box B \in s\} \cup \{C_n^+\}$ is consistent, so is $\{B : \Box B \in s\} \cup \{C_{n+1}\}$. Since $\{B : \Box B \in s\} \cup \{C_n^+\}$ is consistent if $\{B : \Box B \in s\} \cup \{C_n\}$ is then, given this result, $\{B : \Box B \in s\} \cup \{C_{n+1}\}$ is consistent if $\{B : \Box B \in s\} \cup \{C_n\}$ is. So $\{B : \Box B \in s\} \cup \{C_n\}$ is consistent for every n.

As before let t be the union of $\{B : \Box B \in s\}$ and all the C_ns. So since each $\{B : \Box B \in s\} \cup \{C_n\}$ is consistent, and since

$$\vdash C_m \to C_n \text{ (for } m \geq n),$$

their union t is consistent as well.

By construction t has the $\Box\Pi$-property and so theorem 2.7.5(2) is proved.

Appendix B

We now prove our Chihara-style Connecting Theorem restating Chihara's proof of his Fundamental Theorem in terms of associative semantics.[20] The models we shall consider are natural associative models, and thus models which involve constant substitution classes (see Subsection 3.2.5). In order to obtain the desired result we need a couple of lemmas.

Lemma 1

> For every natural associative model $\mathcal{M} = \langle S, R, s^{@}, C, c, P, v\rangle$ of L^* and for every natural language proto-interpretation $\Im$ conforming to $\mathcal{M}$: for every $s \in S$, for every n-ary predicate φ^n, and for every $\alpha_1, ..., \alpha_n \in c(s)$, had the (actual) level of sense been such that there was a bijection via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it, then it would have been the case that for any representing function f of s, (a) either $\{f_s(\varphi^n...\alpha...), f_s(\psi^n...\alpha...), f_s(\chi^n...\alpha...), ...\} \subseteq$ senext- $[\alpha/\Im]$ or $\{f_s(\varphi^n...\alpha...), f_s(\psi^n...\alpha...), f_s(\chi^n...\alpha...), ...\} \nsubseteq$ senext-$[\alpha/\Im]$; and (b) either $\{f_s(\varphi^n...\alpha_1...), f_s(\varphi^n...\alpha_2...), f_s(\varphi^n...\alpha_3...), ...\} \subseteq$ senext-$[\varphi^n/\Im]$ or $\{f_s(\varphi^n...\alpha_1...), f_s(\varphi^n...\alpha_2...), f_s(\varphi^n...\alpha_3...), ...\} \nsubseteq$ senext-$[\varphi^n/\Im]$.

Proof of Lemma 1. The proof is trivial.

Lemma 2

> For every natural associative model $\mathcal{M} = \langle S, R, s^{@}, C, c, P, v\rangle$ of L^* and for every natural language proto-interpretation $\Im$ conforming to $\mathcal{M}$: for every $s \in S$, for every n-ary predicate φ^n, and for every $\alpha_1, ..., \alpha_n \in c(s)$, had the level of sense been such that there was a bijection via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it, then for any f and g which were representing functions of s, (a) $\{f_s(\varphi^n...\alpha...), f_s(\psi^n...\alpha...), f_s(\chi^n...\alpha...), ...\} \subseteq$ senext- $[\alpha/\Im]$ iff $\{g_s(\varphi^n...\alpha...), g_s(\psi^n...\alpha...), g_s(\chi^n...\alpha...), ...\} \subseteq$ senext- $[\alpha/\Im]$; and (b) $\{f_s(\varphi^n...\alpha_1...), f_s(\varphi^n...\alpha_2...), f_s(\varphi^n...\alpha_3...), ...\} \subseteq$ senext-$[\varphi^n/\Im]$ iff $\{g_s(\varphi^n...\alpha_1...), g_s(\varphi^n...\alpha_2...), g_s(\varphi^n...\alpha_3...), ...\} \subseteq$ senext-$[\varphi^n/\Im]$.

[20]See [Chihara, 1998] pp. 239-259.

Proof of Lemma 2. Part (a). Had the level of sense been such that there was a bijection via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it, then since this tuple represented the level of sense via both f and g, it would have been the case (by the definiton of representation via bijections) that:

$$\{\varphi^n...\alpha..., \psi^n...\alpha..., \chi^n...\alpha..., ...\} \subseteq v(\alpha, s) \text{ iff}$$
$$\{f_s(\varphi^n...\alpha...), f_s(\psi^n...\alpha...), f_s(\chi^n...\alpha...), ...\} \subseteq \text{senext-}[\alpha/\Im];$$

and:

$$\{\varphi^n...\alpha..., \psi^n...\alpha..., \chi^n...\alpha..., ...\} \subseteq v(\alpha, s) \text{ iff}$$
$$\{g_s(\varphi^n...\alpha...), g_s(\psi^n...\alpha...), g_s(\chi^n...\alpha...), ...\} \subseteq \text{senext-}[\alpha/\Im].$$

Hence, had the level of sense been such that there was a bijection via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it, it would have been the case that:

$$\{f_s(\varphi^n...\alpha...), f_s(\psi^n...\alpha...), f_s(\chi^n...\alpha...), ...\} \subseteq \text{senext-}[\alpha/\Im] \text{ iff}$$
$$\{g_s(\varphi^n...\alpha...), g_s(\psi^n...\alpha...), g_s(\chi^n...\alpha...), ...\} \subseteq \text{senext-}[\alpha/\Im].$$

Similarly for part (b). Had the level of sense been such that there was a bijection via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it, then since this tuple represented the level of sense via both f and g, it would have been the case that:

$$\{\varphi^n...\alpha_1..., \varphi^n...\alpha_2..., \varphi^n...\alpha_3..., ...\} \subseteq v(\varphi^n, s) \text{ iff}$$
$$\{f_s(\varphi^n...\alpha_1...), f_s(\varphi^n...\alpha_2...), f_s(\varphi^n...\alpha_3...), ...\} \subseteq \text{senext-}[\varphi^n/\Im];$$

and

$$\{\varphi^n...\alpha_1..., \varphi^n...\alpha_2..., \varphi^n...\alpha_3..., ...\} \subseteq v(\varphi^n, s) \text{ iff}$$
$$\{g_s(\varphi^n...\alpha_1...), g_s(\varphi^n...\alpha_2...), g_s(\varphi^n...\alpha_3...), ...\} \subseteq \text{senext-}[\varphi^n/\Im].$$

Hence it would have been the case that

$$\{f_s(\varphi^n...\alpha_1...), f_s(\varphi^n...\alpha_2...), f_s(\varphi^n...\alpha_3...), ...\} \subseteq \text{senext-}[\varphi^n/\Im] \text{ iff}$$
$$\{g_s(\varphi^n...\alpha_1...), g_s(\varphi^n...\alpha_2...), g_s(\varphi^n...\alpha_3...), ...\} \subseteq \text{senext-}[\varphi^n/\Im].$$

In effect, Lemma 2 says that had the level of sense been such that there was a bijection via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it, it would not make any difference which bijection was the representing bijection of s.

Lemma 3

For every natural associative model $\mathcal{M} = \langle S, R, s^@, C, c, P, v\rangle$ of L^* and for every natural language proto-interpretation $\Im$ conforming to $\mathcal{M}$: for every $s \in S$, for every $\alpha_1, ..., \alpha_n \in c(s)$, and for every *atomic formula* A, if $x_1, ..., x_n$ are the n nominal variables that occur in A, then the following holds:

If $\mathcal{M}_\sigma \models_s A$, *then*, for some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_\Im$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[A/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_\Im$ represented it.

If $\mathcal{M}_\sigma \not\models_s A$, *then*, for some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_\Im$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[\neg A/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_\Im$ represented it.

Proof of Lemma 3. By hypothesis, A is an atomic formula, for instance $\varphi^m o_1...o_m$, where each nominal variable x_k must occur among the $o_1, ..., o_m$.

Now, had the level of sense been such that there was a bijection via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_\Im$ represented it, then, for every nominal constant $\alpha_1, ..., \alpha_n \in c(s)$ and for every $A \in Atm(\alpha)$, for each $\alpha \in c(s)$:

$$\{\varphi^n...\alpha..., \psi^n...\alpha..., \chi^n...\alpha..., ...\} \subseteq v(\alpha, s) \text{ iff}$$
$$\{f_s(\varphi^n...\alpha...), f_s(\psi^n...\alpha...), f_s(\chi^n...\alpha...), ...\} \subseteq \text{senext-}[\alpha/\Im];$$

and for every n-ary pure predicate $\varphi^n \in P$ and for every $A \in Atm(\varphi^n)$, for each $\varphi^n \in P$:

$$\{\varphi^n...\alpha_1..., \varphi^n...\alpha_2..., \varphi^n...\alpha_3..., ...\} \subseteq v(\varphi^n, s) \text{ iff}$$
$$\{f_s(\varphi^n...\alpha_1...), f_s(\varphi^n...\alpha_2...), f_s(\varphi^n...\alpha_3...), ...\} \subseteq \text{senext-}[\varphi^n/\Im].$$

Moreover, from condition (C1), we know that, had the level of sense been such that there was a bijection via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it, there would have been a representing function f for s.

We suppose that

> (A) $\mathcal{M}_\sigma \models_s A$.

So:

> (B) Had the level of sense been such that there was a bijection via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it, there would have been a representing bijection g for s such that $[A/\Im]$ was true *simpliciter* with respect to the level of sense under some interpreted substitutional assignment $\sigma_{\Im(C)}$.

We may then infer (by existential generalization) that

> (C) For some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing bijection g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[A/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

Now, we assume that

> (D) $\mathcal{M}_\sigma \not\models_s A$.

So $\mathcal{M}_\sigma \models_s \neg A$. So we can infer (using Lemma 1) that

> (E) Had the level of sense been such that there was a bijection via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it, there would have been a representing bijection g for s such that $[\neg A/\Im]$ was true *simpliciter* with respect to the level of sense under some interpreted substitutional assignment $\sigma_{\Im(C)}$.

So

(F) For some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing bijection g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[\neg A/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

Lemma 4

For every natural associative model $\mathcal{M} = \langle S, R, s^{@}, C, c, P, v\rangle$ of L^* and for every natural language proto-interpretation $\Im$ conforming to $\mathcal{M}$: for every $s_1, s_2 \in S$, if there could be representing functions f_1 and f_2 of s_1 and s_2 respectively, then if $\alpha_1 \in c(s_1) \cap c(s_2)$, $f_1(\alpha_1) = f_2(\alpha_1)$; and if $\alpha_1, \alpha_2 \in c(s_1)$, where $\alpha_1 \neq \alpha_2$, then $f_1(\alpha_1) \neq f_2(\alpha_2)$.

Proof of Lemma 4.

We assume that the antecedent is the case and that $\alpha_1 \in c(s_1) \cap c(s_2)$. (In the present case $c(s_1)$ and $c(s_2)$ intersect since the substitution classes are constant.) Then, had the level of sense been such that for s_2 there was a bijection via which $\langle c(s_2), v(\alpha, s_2), v(\varphi^n, s_2)\rangle_{\Im}$ represented it, there would have been a representing bijection g for s_2 such that $g(\alpha_1) = f_1(\alpha_1)$ and $g(\alpha_1) = f_2(\alpha_1)$. From this it follows that $f_1(\alpha_1) = f_2(\alpha_1)$.

This time we assume that the antecedent is the case and that $\alpha_1, \alpha_2 \in c(s_1)$, where $\alpha_1 \neq \alpha_2$. So, had the level of sense been such that for s_1 there was a bijection via which $\langle c(s_1), v(\alpha, s_1), v(\varphi^n, s_1)\rangle_{\Im}$ represented it, there would have been a representing bijection g for s_1 such that $g(\alpha_1) = f_1(\alpha_1)$ and $g(\alpha_2) = f_2(\alpha_2)$. Now, since $\alpha_1 \neq \alpha_2$, it follows that $g(\alpha_1) \neq g(\alpha_2)$ and hence that $f_1(\alpha_1) \neq f_2(\alpha_2)$.

Similarly, for the predicates and pure atoms. Lemma 4 points out that, for any $s \in S$ there could not have been more than one representing function for s.

Moreover, it also says that the representing functions of s_1 and s_2 respectively must agree on what they assign to any element the two substitution classes have in common. Similarly, for P and *Atm*. From this the first two corollaries of the lemma can easily be established.

Corollary 4.1.

For every natural associative model $\mathcal{M} = \langle S, R, s^{@}, C, c, P, v \rangle$ of L^* and for every natural language proto-interpretation $\Im$ conforming to $\mathcal{M}$: for every $s \in S$, for every $\alpha_1, ..., \alpha_n \in C$, and for every formula A, if $x_1, ..., x_n$ are the n nominal variables that occur in A, then the following holds:

If, for some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t) \rangle_\Im$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[A/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s)$, $v(\alpha, s)$, $v(\varphi^n, s) \rangle_\Im$ represented it;

then it is not the case that, for some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t) \rangle_\Im$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ did not fulfill the condition: $[A/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s) \rangle_\Im$ represented it.

Corollary 4.2.

For every natural associative model $\mathcal{M} = \langle S, R, s^{@}, C, c, P, v \rangle$ of L^* and for every natural language proto-interpretation $\Im$ conforming to $\mathcal{M}$: for every $s \in S$, for every $\alpha_1, ..., \alpha_n \in C$, and for every formula A, if $x_1, ..., x_n$ are the n nominal variables that occur in A, then the following holds:

If, for some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t) \rangle_\Im$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[A/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it;

then it is not the case that, for some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[\neg A/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

Lemma 5

For every natural associative model $\mathcal{M} = \langle S, R, s^@, C, c, P, v\rangle$ of L^* and for every natural language proto-interpretation $\Im$ conforming to $\mathcal{M}$: for every $s \in S$, for every $\alpha_1, ..., \alpha_n \in c(s)$, and for every *atomic formula* A, if $x_1, ..., x_n$ are the n nominal variables that occur in A, then the following holds:

$\mathcal{M}_\sigma \models_s A$ *if and only if*, for some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[A/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

Proof of Lemma 5.

It follows directly from Lemma 3

For every natural associative model $\mathcal{M} = \langle S, R, s^@, C, c, P, v\rangle$ of L^* and for every natural language proto-interpretation $\Im$ conforming to $\mathcal{M}$: for every $s \in S$, for every $\alpha_1, ..., \alpha_n \in c(s)$, and for every *atomic formula* A, if $x_1, ..., x_n$ are the n nominal variables that occur in A, then the following holds:

If $\mathcal{M}_\sigma \models_s A$, *then*, for some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t

such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[A/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

If $\mathcal{M}_\sigma \not\models_s A$, *then*, for some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[\neg A/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

and Corollary 4.2

For every natural associative model $\mathcal{M} = \langle S, R, s^@, C, c, P, v\rangle$ of L^* and for every natural language proto-interpretation $\Im$ conforming to $\mathcal{M}$: for every $s \in S$, for every $\alpha_1, ..., \alpha_n \in C$, and for every formula A, if $x_1, ..., x_n$ are the n nominal variables that occur in A, then the following holds:

If, for some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[A/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ... , taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it;

then it is not the case that, for some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[\neg A/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

Lemma 6

For every natural associative model $\mathcal{M} = \langle S, R, s^@, C, c, P, v\rangle$ of L^* and for every natural language proto-interpretation $\Im$ conforming to $\mathcal{M}$: for every $s \in S$, for every $\alpha_1, ..., \alpha_n \in C$, and for every *formula* A, if $x_1, ..., x_n$ are the n nominal variables that occur in A, then the following holds:

Either, for some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_\Im$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[A/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_\Im$ represented it;

or, for some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_\Im$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[\neg A/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ... , taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_\Im$ represented it.

Proof of Lemma 6.

The proof is by induction on the complexity of A. We begin as usual with the case in which A has complexity 0. So A will be an atomic formula, say $\varphi^m o_1...o_m$, where each nominal variable x_k must occur among the $o_1, ..., o_m$.

Either (i) $\{\alpha_1, ..., \alpha_n\} \subseteq c(s)$ or (ii) $\{\alpha_1, ..., \alpha_n\} \nsubseteq c(s)$.

Suppose first that (i). Now excluded middle applies to A in so far as either $\mathcal{M}_\sigma \models_s A$ or $\mathcal{M}_\sigma \not\models_s A$.

Suppose $\mathcal{M}_\sigma \models_s A$. Then, by Lemma 3, we have:

For some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_\Im$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[A/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

Suppose $\mathcal{M}_\sigma \not\models_s A$. Then, again by Lemma 3, we have:

> For some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[\neg A/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

Suppose, this time, that (ii), i.e., $\{\alpha_1, ..., \alpha_n\} \not\subseteq c(s)$. So we first assume that:

> (A) *It is not the case that*, for some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[A/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ... , taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

Thus,

> (B) For every $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that there was *no* interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[A/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

But the following holds:

(C) For some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ existed.

So from (B) and (C), we get

(D) For some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ did *not* fulfil the condition: $[A/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ... , taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

Since A is atomic, it is obvius that such an interpreted substitutional assignment $\sigma_{\Im(C)}$ did *not* fulfil the condition:

$[A/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ... , taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

only if it fulfilled the condition:

$[\neg A/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ... , taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

Hence, if (A) is the case, the lemma holds. On the other hand, if (A) is not the case, the lemma holds as well. So in either case, the lemma holds; so it holds for $k = 0$.

We now take it as an inductive hypothesis that the lemma holds for formulae of complexity less than k with $k > 0$. So for any formula A of complexity k, A must be a formula of one of the following forms: 1. $\neg B$; 2. $(C \wedge D)$; 3. $(C \vee D)$; 4. $(C \rightarrow D)$; 5. $(C \leftrightarrow D)$; 6. $(\Pi x)C$; 7. $(\Sigma x)C$; 8. $\Box C$; 9. $\Diamond C$. We shall consider each of these possibilities in the above order.

Case (1). $A = \neg B$. The subformula B is of complexity $k - 1$. So, by the inductive hypothesis, we have:

> (A) *Either*, for some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[B/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it;
>
> *or*, for some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[\neg B/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

Now suppose the first disjunct of (A) holds. Then:

> (B) For some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[\neg\neg B/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

Clearly, if the second disjunct of (A) holds, the result follows as well.

Case (2). $A = (C \wedge D)$.

Both C and D have complexity $< k$. So it follows that:

(A) *Either*, for some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[C/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it;

or, for some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[\neg C/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ... , taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

and

(B) *Either*, for some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[D/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it;

or, for some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[\neg D/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

First we suppose that the first disjuncts of (A) and (B) respectively hold. So

(C) For some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[C/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it;

and

for some $u \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function h for u such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[D/\Im]$ is such that taking x_1 to stand in for $h(\alpha_1)$, ... , taking x_n to stand in for $h(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it

as well. Because of Lemma 4, the following obtains:

(D) For some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition:

$[C/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it;

and also:

$[D/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it;

and hence the condition:

$[(C \wedge D)/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_\Im$ represented it.

Second, we assume that the first disjunct of (A) and the second disjunct of (B) hold. Then, we can infer in the previous manner that:

(E) For some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_\Im$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition:

$[C/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_\Im$ represented it;

and, furthermore, the condition:

$[\neg D/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_\Im$ represented it;

and thus the condition:

$[\neg(C \wedge D)/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_\Im$ represented it.

In the remaining two cases, are handled analogusly. So we are done with case (2).

Cases (3)-(5). The proofs for the rest of logical connectives are similar.

Case (6). $A = (\Pi x)C$.

So let y be the first nominal variable not occurring in the subformula C. Then,

For some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_\Im$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[A/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it,

iff

for all $\alpha \in c(s)$,for some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that every interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[C'/\Im]$ is such that taking y to stand in for $g(\alpha)$, taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

Now C' is a formula of complexity $k - 1$. Thus, for every $\alpha \in c(s)$, we have:

(A) *Either*, for some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[C'/\Im]$ is such that taking y to stand in for $g(\alpha)$, taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it;

or, for some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[\neg C'/\Im]$ is such that taking y to stand in for $g(\alpha)$, taking x_1 to stand in for $g(\alpha_1)$, ... , taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

Suppose that, for all $\alpha \in c(s)$, the first disjunct holds. That is,

(B) For some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[C'/\Im]$ is such that taking y to stand in for $g(\alpha)$, taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

Consequently,

(C) For some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[A/\Im]$ is such that taking y to stand in for $g(\alpha)$, taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

Assume this time that, for some $\alpha \in c(s)$, it is not the case that (B). Then, for some $\alpha \in c(s)$ the lower disjunct of (A) must hold. So it follows that:

(D) For some $\alpha \in c(s)$, for some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[\neg C'/\Im]$ is such that taking y to stand in for $g(\alpha)$, taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

Thus,

(E) For some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[(\Sigma x)\neg C/\Im]$ is such that taking y to stand in for $g(\alpha)$, taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_\Im$ represented it.

Hence:

(F) For some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_\Im$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[\neg A/\Im]$ is such that taking y to stand in for $g(\alpha)$, taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_\Im$ represented it.

Case (7). This case is similar to case (6).

Case (8). $A = \Box C$. Since the subformula C is of complexity $k - 1$ then, by inductive hypothesis, for every $u \in S$:

(A) *Either*, for some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_\Im$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[C/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(u), v(\alpha, u), v(\varphi^n, u)\rangle_\Im$ represented it.

or, for some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_\Im$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[\neg C/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(u), v(\alpha, u), v(\varphi^n, u)\rangle_\Im$ represented it.

First, we assume that, for every $u \in S$,

> (B) For some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[C/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(u), v(\alpha, u), v(\varphi^n, u)\rangle_{\Im}$ represented it.

From this it follows

> (C) For some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[\Box C/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(u), v(\alpha, u), v(\varphi^n, u)\rangle_{\Im}$ represented it.

However,

> $[\Box C/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ... , taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(u), v(\alpha, u), v(\varphi^n, u)\rangle_{\Im}$ represented it.

only if

> $[\Box C/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense no matter how it may have been, and so, in particular, had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it;

and we are done.

This time we take it that it is not the case that for every $u \in S$, do we have

(D) For some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[C/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(u), v(\alpha, u), v(\varphi^n, u)\rangle_{\Im}$ represented it.

By the inductive hypothesis it follows that, for some $u \in W$:

(E) For some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[\neg C/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(u), v(\alpha, u), v(\varphi^n, u)\rangle_{\Im}$ represented it.

Hence we can conclude, because of condition (C2), that:

(F) For some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[\Diamond\neg C/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense no matter how it may have been, and hence had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

So the proof of case (8) is done.

Case (9). $A = \Diamond C$. This case is similar to (8). This concludes the inductive proof of Lemma 6.

Chihara-Style Connecting Theorem.

For every natural associative model $\mathcal{M} = \langle S, R, s^{@}, C, c, P, v \rangle$ of L^* and for every natural language proto-interpretation $\Im$ conforming to $\mathcal{M}$: for every index $s \in S$, for every nominal constant $\alpha_1, ..., \alpha_n \in C$, and for every formula A, if $x_1, ..., x_n$ are the n nominal variables which occur in A, then the following holds: $\mathcal{M}_\sigma \models_s A$ iff for some index $t \in S$, had the level of sense been such that there was a bijective function via which the tuple $\langle c(t), v(\alpha, t), v(\varphi^n, t) \rangle_\Im$ represented it, there would have been a representing bijective function g for t such that some $\Im(C)$-assignment fulfilled the condition: $[A/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s) \rangle_\Im$ represented it.

Proof of the Chihara-style Connecting Theorem.

We prove this theorem, again following the steps of Chihara, by induction on the complexity k of the formulae of L^*. Again we start with $k = 0$.

So A is atomic, say $\varphi^m o_1 ... o_m$, where each nominal variable x_k must occur among the $o_1, ..., o_m$.

In principle, there are two possibilities. Either all the members of the set $\{\alpha_1, ..., \alpha_n\}$ are members of $c(s)$ or at least one member of the set $\{\alpha_1, ..., \alpha_n\}$ is not a member of $c(s)$.

Suppose the former. In that case the theorem holds by Lemma 5. The latter case is excluded by the fact that the (natural) models under consideration are constant substitution class models. So the atomic case is done.

We take it as an inductive hypothesis that the theorem holds for all formulae of complexity $< k$ with $k > 0$. Then, for any formula A of complexity k, A must be a formula of one of the forms given in the proof of Lemma 6.

Case (1). $A = \neg B$.

So B is of complexity $k - 1$. Hence, by the inductive hypothesis, we get:

(A) $\mathcal{M}_\sigma \models_s B$ iff for some index $t \in S$, had the level of sense been such that there was a bijective function via which the tuple

$\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing bijective function g for t such that some $\Im(C)$-assignment fulfilled the condition: $[B/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

Consequently,

(B) $\mathcal{M}_\sigma \not\models_s B$ iff *it is not the case that* for some index $t \in S$, had the level of sense been such that there was a bijective function via which the tuple $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing bijective function g for t such that some $\Im(C)$-assignment fulfilled the condition: $[B/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

We then obtain from (B) and Lemma (6):

(C) *If* $\mathcal{M}_\sigma \models_s \neg B$, *then* for some index $t \in S$, had the level of sense been such that there was a bijective function via which the tuple $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing bijective function g for t such that some $\Im(C)$-assignment fulfilled the condition: $[\neg B/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

Now for the other direction. Suppose that:

(D) For some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[\neg B/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

Then, in because of Lemma 4, it follows that:

(E) *It is not the case that* for some $t \in S$, had the level of sense been such that there was a bijection via which $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing function g for t such that some interpreted substitutional assignment $\sigma_{\Im(C)}$ fulfilled the condition: $[B/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

But then from (B) and (E) we get:

(F) $\mathcal{M}_\sigma \not\models B$.

So:

(G) $\mathcal{M}_\sigma \models \neg B$.

So we are done with Case (1).

Case (2). $A = (C \wedge D)$.
Both subformulae C and D are of complexity $< k$. So we get:

(A) $\mathcal{M}_\sigma \models_s C$ iff for some index $t \in S$, had the level of sense been such that there was a bijective function via which the tuple $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing bijective function g for t such that some $\Im(C)$-assignment fulfilled the condition: $[C/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it;

and

(B) $\mathcal{M}_\sigma \models_s D$ iff for some index $t \in S$, had the level of sense been such that there was a bijective function via which the tuple $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing bijective function g for t such that some $\Im(C)$-assignment fulfilled the condition: $[D/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_\Im$ represented it.

But

(C) $\mathcal{M}_\sigma \models_s (C \wedge D)$ iff $\mathcal{M}_\sigma \models_s C$ and $\mathcal{M}_\sigma \models_s D$.

So:

(D) $\mathcal{M}_\sigma \models_s (C \wedge D)$ iff for some index $t \in S$, had the level of sense been such that there was a bijective function via which the tuple $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_\Im$ represented it, there would have been a representing bijective function g for t such that some $\Im(C)$-assignment fulfilled the condition: $[C/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_\Im$ represented it,

and

for some index $t \in S$, had the level of sense been such that there was a bijective function via which the tuple $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_\Im$ represented it, there would have been a representing bijective function g for t such that some $\Im(C)$-assignment fulfilled the condition: $[D/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_\Im$ represented it.

Because of Lemma 4, the right-hand side of the biconditional (D) holds iff:

(E) For some index $t \in S$, had the level of sense been such that there was a bijective function via which the tuple $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_\Im$ represented it, there would have been a representing bijective function g for t such that some $\Im(C)$-assignment fulfilled the condition: $[C \wedge D/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_\Im$ represented it.

This concludes the step for Case (2).

Cases (3)-(5). Are similar to Case (2).

Case (6). $A = (\Pi x)C$.

Because of the definition of 'true at s in $\mathcal{M}$', we may conclude, letting y be the first nominal variable not occuring in C, that:

> (A) $\mathcal{M}_\sigma \models_s (\Pi x)C$ iff, for every nominal constant α of $c(s)$, $\mathcal{M}_\sigma \models_s C'$.

Since C' is a formula of complexity $< k$, it follows by the inductive hypothesis that:

> (B) For every nominal constant α of $c(s)$, $\mathcal{M}_\sigma \models_s C'$ iff, for some index $t \in S$, had the level of sense been such that there was a bijective function via which the tuple $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_\Im$ represented it, there would have been a representing bijective function g for t such that some $\Im(C)$-assignment fulfilled the condition: $[C'/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_\Im$ represented it.

From this we get by an obvious inference:

> (C) For every nominal constant α of $c(s)$, $\mathcal{M}_\sigma \models_s C'$ iff, for every nominal constant α of $c(s)$, and for some index $t \in S$, had the level of sense been such that there was a bijective function via which the tuple $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_\Im$ represented it, there would have been a representing bijective function g for t such that some $\Im(C)$-assignment fulfilled the condition: $[C'/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_\Im$ represented it.

Moreover, the following holds as well:

(D) For every nominal constant α of $c(s)$, and for some index $t \in S$, had the level of sense been such that there was a bijective function via which the tuple $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing bijective function g for t such that some $\Im(C)$-assignment fulfilled the condition: $[C'/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it

iff

for some index $t \in S$, had the level of sense been such that there was a bijective function via which the tuple $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing bijective function g for t such that some $\Im(C)$-assignment fulfilled the condition: $[(\Pi x)C/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it

With (A), (C), and (D), we obtain the desired conclusion.

Case (7) is similar to (6).

Case (8). $A = \Box C$.

Analogously, from the definition of 'true at s in $\mathcal{M}$', we infer:

(A) $\mathcal{M}_\sigma \models_s \Box C$ iff, for every index $u \in C$ such that sRu, $\mathcal{M}_\sigma \models_u C$.

Since C is a formula of complexity $< k$, it follows by hypothesis that:

(B) For every $u \in S$, $\mathcal{M}_\sigma \models_u C$ iff, for every index $t \in S$, had the level of sense been such that there was a bijective function via which the tuple $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing bijective function g for t such that some $\Im(C)$-assignment fulfilled the condition: $[C/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(u), v(\alpha, u), v(\varphi^n, u)\rangle_{\Im}$ represented it.

By an obvious inference we get from (B):

(C) For every index $u \in S$, $\mathcal{M}_\sigma \models_u C$ iff, for every index $u \in S$, and for every index $t \in S$, had the level of sense been such that there was a bijective function via which the tuple $\langle c(t), v(\alpha, t),\ v(\varphi^n,\ t)\rangle_{\Im}$ represented it, there would have been a representing bijective function g for t such that some $\Im(C)$-assignment fulfilled the condition: $[C/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(u), v(\alpha, u), v(\varphi^n, u)\rangle_{\Im}$ represented it.

Moreover, the following holds:

(D) For every index $u \in S$, and for every index $t \in S$, had the level of sense been such that there was a bijective function via which the tuple $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing bijective function g for t such that some $\Im(C)$-assignment fulfilled the condition: $[C/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(u), v(\alpha, u), v(\varphi^n, u)\rangle_{\Im}$ represented it

iff

for every index $t \in S$, had the level of sense been such that there was a bijective function via which the tuple $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing bijective function g for t such that some $\Im(C)$-assignment fulfilled the condition: $[\Box C/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense no matter how it may have been; and hence had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

Since an $\Im(C)$-assignment would fulfil the condition:

$[\Box C/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense no matter how it may have been; and hence had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

only if it did fulfil the condition:

> $[\Box C/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it,

we obtain from (D):

> (E) For every index $u \in S$, and for every index $t \in S$, had the level of sense been such that there was a bijective function via which the tuple $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing bijective function g for t such that some $\Im(C)$-assignment fulfilled the condition: $[C/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(u), v(\alpha, u), v(\varphi^n, u)\rangle_{\Im}$ represented it
>
> iff
>
> for every index $t \in S$, had the level of sense been such that there was a bijective function via which the tuple $\langle c(t), v(\alpha, t), v(\varphi^n, t)\rangle_{\Im}$ represented it, there would have been a representing bijective function g for t such that some $\Im(C)$-assignment fulfilled the condition: $[\Box C/\Im]$ is such that taking x_1 to stand in for $g(\alpha_1)$, ..., taking x_n to stand in for $g(\alpha_n)$ results in a formula that accurately reflects the level of sense had it been such that there was a bijective function via which $\langle c(s), v(\alpha, s), v(\varphi^n, s)\rangle_{\Im}$ represented it.

With (A), (C), and (E), we obtain the desired result for Case (8).

Case (9). $A = \Diamond C$. This is similar to the previous case. This concludes the proof of our Chihara-style Connecting Theorem. As a corollary to this we obtain our Chihara-style Principal Corollary.

Bibliography

[Adams, 1979 (1974)] R. M. Adams. Theories of actuality. In M. J. Loux (ed.) *The Actual and the Possible*, pp. 190-209, Cornell University Press, Ithaca and London, 1979. (First published in *Noûs*, 8 (1974), 211-31.)

[Armstrong, 1989] D. M. Armstrong. *A Combinatorial Theory of Possibility*, Cambridge University Press, 1989.

[Bencivenga, 2002] E. Bencivenga. Free logics. In D.M. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic*, 2nd Edition, Volume 5, pp. 147-196, Kluwer Academic Publishers, Dordrecht, 2002.

[Carnap, 1956 (1947)] R. Carnap. *Meaning and Necessity. A Study in Semantics and Modal Logic*, second edition, The University of Chicago Press, Chicago and London, 1956.

[Chalmers, 2002] D. J. Chalmers. Does conceivability entail possibility? In T. Szabó Gendler and J. Hawthorne (eds.). *Conceivability and Possibility*, pp. 145-200, Clarendon Press, Oxford, 2002.

[Chihara, 1998] C. S. Chihara. *The Worlds of Possibility. Modal Realism and the Semantics of Modal Logic*, Clarendon Press, Oxford, 1998.

[Copeland, 1982] B. J. Copeland. A note on the Barcan Formula and substitutional quantification, *Logique et Analyse*, 25 (1982), 83-86.

[Copeland, 1985] B. J. Copeland. Substitutional quantification and existence, *Analysis*, 45 (1985), 1-4.

[Currie, 1990] G. Currie. *The Nature of Fiction*, Cambridge University Press, Cambridge, 1990.

[Davies, 1978] M. Davies. Weak necessity and truth theories, *Journal of Philosophical Logic*, 7 (1978), 415-439.

[Davies, 1980] M. Davies. A note on substitutional quantification, *Noûs*, 14 (1980), 619-622.

[Divers, 2002] J. Divers. *Possible Worlds*, Routledge, London and New York, 2002.

[Dowty *et al.*, 1981] D. R. Dowty, R. E. Wall, & S. Peters. *Introduction to Montague Semantics*, D. Reidel, Dordrecht, Boston, and London, 1981.

[Dunn & Belnap, 1968] J. M. Dunn & N. D. Belnap, Jr. The substitution interpretation of the quantifiers, *Noûs*, 2 (1968), 177-185.

[Dunn, 1973] J. M. Dunn. A truth-value semantics for modal logic. In [Leblanc, 1973], pp. 87-100.

[Etchemendy, 1990] J. Etchemendy. *The Concept of Logical Consequence*, Harvard University Press, Cambridge Mass. and London, 1990.

[Everett, 2000] A. Everett, Referentialism and empty names. In [Everett & Hofweber, 2000], pp. 37-60.

[Everett & Hofweber, 2000] A. Everett & T. Hofweber (eds.) *Empty Names, Fiction, and the Puzzles of Non-Existence*, CSLI Publications, Stanford, Calif., 2000.

[Everett & Hofweber, 2000] A. Everett and T. Hofweber, Much ado about nothing. In [Everett & Hofweber, 2000], pp. ix-xxi.

[Fine, 1977] K. Fine. Postscript to *Worlds, Times, and Selves*. In A.N. Prior *Worlds, Times, and Selves*, pp. 116-169, University of Massachusetts Press, Amherst, 1977.

[Fine, 1978] K. Fine. Model theory for modal logic: Part I, the *de re/de dicto* distinction, *Journal of Philosophical Logic*, 7 (1978), 125-156.

[Fine, 1994] K. Fine. Essence and modality. In J. Tomberlin (ed.) *Philosophical Perspectives 8, Logic and Language*, pp. 1-16, Ridgeview, Atascadero, Calif., 1994.

[Fine, 1995a] K. Fine. Ontological dependence. *Proceedings of the Aristotelian Society*, 95 (1994-95), 269-290.

[Fine, 1995b] K. Fine. Senses of essence. In W. Sinnott-Armstrong (ed.) *Modality, Morality, and Belief*, pp. 53-73, Cambridge University Press, 1995.

[Fine, 1995c] K. Fine. The logic of essence, *Journal of Philosophical Logic*, 24 (1995), 241-273.

[Fine, 2000] K. Fine. Semantics for the logic of essence, *Journal of Philosophical Logic*, 29 (2000), 543-584.

[Fine, 2002a] K. Fine. *The Limits of Abstraction*, Clarendon Press, Oxford, 2002.

[Fine, 2002b] K. Fine. The varieties of necessity. In T. Szabó Gendler and J. Hawthorne (eds.). *Conceivability and Possibility*, pp. 253-281, Clarendon Press, Oxford, 2002.

[Fitting, 2002] M. Fitting. *Types, Tableaus and Gödel's God*, Kluwer, Dordrecht/Boston/London, 2002.

[Fitting & Mendelsohn, 1998] M. Fitting & R. L. Mendelsohn. *First-order Modal Logic*, Kluwer, Dordrecht/Boston/London, 1998.

[Forbes, 1985] G. Forbes. *The Metaphysics of Modality*, Clarendon Press, Oxford, 1985.

[Forbes, 1989] G. Forbes. *Languages of Possibility*, Blackwell, Oxford, 1989.

[Forbes, 1992] G. Forbes. Melia on modalism, *Philosophical Studies* 68, 1992, 57-63.

[Frege, 1994 (1892)] G. Frege. Sinn und Bedeutung. In G. Frege. *Funktion, Begriff, Bedeutung*, G. Patzig (ed.), pp. 40-65, Vandenhoek und Ruprecht, Göttingen, 1994.

[Friedrichsdorf, 1992] U. Friedrichsdorf. *Einführung in die klassische und intensionale Logik*, Vieweg, Braunschweig/Wiesbaden, 1992.

[Garson, 2001] J.W. Garson. Quantification in modal logic. In D.M. Gabbay & F. Guenthner (eds.), *Handbook of Philosophical Logic*, 2nd Edition, Volume 3, pp. 267-323, Kluwer Academic Publishers, Dordrecht, 2001.

[Gendler & Hawthorne, 2002] T. Szabó Gendler & J. Hawthorne. Introduction: conceivability and possibility. In T. Szabó Gendler & J. Hawthorne (eds.) *Conceivability and Possibility*, pp. 1-70, Clarendon Press, Oxford, 2002.

[Grover, 1973] D. L. Grover. Propositional quantification and quotation contexts. In [Leblanc, 1973], pp. 101-110.

[Gupta, 1978] A. Gupta. Modal logic and truth, *Journal of Philosophical Logic*, 7 (1978), 441-472.

[Hale, 1996] B. Hale. Absolute necessities. In J. Tomberlin (ed.). *Philosophical Perspectives 10. Metaphysics*, pp. 93-117, Ridgeview, Atascadero, Calif., 1996.

[Hughes & Cresswell, 1996] G. E. Hughes & M. J. Cresswell. *A New Introduction to Modal Logic*, Routledge, London and New York, 1996.

[Jager, 1982] T. Jager. An actualistic semantics for quantified modal logic, *Notre Dame Journal of Formal Logic*, 23 (1982), 335-349.

[Kamlah & Lorenzen, 1996 (1967)] W. Kamlah & P. Lorenzen. *Logische Propädeutik oder Vorschule des vernünftigen Redens*, 3. Auflage, Metzler, Stuttgart, 1996.

[Kripke, 1963] S. Kripke. Semantical considerations on modal logic, *Acta Philosophica Fennica*, 16 (1963), 83-94.

[Kripke, 1975] S. Kripke. Outline of a theory of truth, *Journal of Philosophy*, 72 (1975), 690-716.

[Kripke, 1976] S. Kripke. Is there a problem about substitutional quantification? in: G. Evans & J. McDowell (eds.) *Truth and Meaning: Essays in Semantics*, pp. 325-419, Oxford University Press, New York and Oxford, 1976.

[Kripke, 1980 (1972)] S. Kripke. *Naming and Necessity*, Blackwell, Oxford and Cambridge Mass., 1980. (First published in D. Davidson (ed.) *Semantics of Natural Language*, pp. 253-355, addenda: pp. 763-769, D. Reidel, Dordrecht, 1972.)

[Kroon, 1992] F. W. Kroon. Was Meinong only pretending?, *Philosophy and Phenomenological Research*, 52 (1992), 499-527.

[Larson & Segal, 1995] R. Larson & G. Segal. *Knowledge of Meaning. An Introduction to Semantic Theory*, The MIT Press, Cambridge Mass. and London, 1995.

[Leblanc, 1971] H. Leblanc. Truth-value semantics for a logic of existence, *Notre Dame Journal of Formal Logic*, 12 (1971), 153-168.

[Leblanc, 1973] H. Leblanc (ed.). *Truth, Syntax and Modality. Proceedings of the Temple University Conference on Alternative Semantics*, North-Holland, Amsterdam, 1973.

[Leblanc, 1973a] H. Leblanc. Semantic deviations. In [Leblanc, 1973], pp. 1-16.

[Leblanc, 1973b] H. Leblanc. On dispensing with things and worlds. In M. Munitz (ed.) *Logic and Ontology*, pp. 241-259, New York University Press, New York, 1973.

[Leblanc, 1976] H. Leblanc. *Truth-Value Semantics*, North-Holland, Amsterdam, 1976.

[Leblanc, 2001] H. Leblanc. Alternatives to standard first-order semantics. In D.M. Gabbay & F. Guenthner (eds.), *Handbook of Philosophical Logic*, 2nd Edition, Volume 2, pp. 53-131, Kluwer Academic Publishers, Dordrecht, 2001.

[Lewis, 1983 (1968)] D. Lewis. Counterpart theory and quantified modal logic. In [Lewis, 1983], pp. 26-39. (Postscripts, pp. 39-46.) (First published in *Journal of Philosophy*, 65 (1968), 113-126.)

[Lewis, 1983 (1970)] D. Lewis. General semantics. In [Lewis, 1983], pp. 189-229. (Postscripts, pp. 230-232.) (First published in *Synthese*, 22 (1970), 18-67.)

[Lewis, 1973] D. Lewis. *Counterfactuals*, Blackwell, Cambridge Mass. and Oxford, 1973, reissued 2001.

[Lewis, 1983 (1978)] D. Lewis. Truth in fiction. In [Lewis, 1983], pp. 261-275. (First published in *American Philosophical Quarterly*, 15 (1978), 37-46.)

[Lewis, 1983] D. Lewis. *Philosophical Papers, Vol. I*, Oxford University Press, New York and Oxford, 1983.

[Lewis, 1986] D. Lewis. *On the Plurality of Worlds*, Blackwell, Cambridge Mass. and Oxford, 1986.

[Linsky & Zalta, 1994] B. Linsky & E. N. Zalta. In defense of the simplest quantified modal logic. In J. Tomberlin (ed.) *Philosophical Perspectives 8, Logic and Language*, pp. 431-458, Ridgeview, Atascadero, 1994.

[Linsky & Zalta, 1996] B. Linsky & E. N. Zalta. In defense of the contingently nonconcrete, *Philosophical Studies*, 84 (1996), 283-294.

[Löffler, 2000] W. Löffler. *Notwendigkeit, S5 und Gott*, Lit, Münster, 2000.

[Lowe, 1999] E. J. Lowe. *The Possibility of Metaphysics. Substance, Identity, and Time*, Clarendon Press, Oxford, 1999.

[Marcus, 1993 (1961)] R. Barcan Marcus. Modalities and intensional languages. In [Marcus, 1993], pp. 3-23. (First published version in *Synthese*, 12 (1961), 303-322.)

[Marcus, 1962] R. Barcan Marcus. Interpreting quantification, *Inquiry*, 5 (1962), 252-259.

[Marcus, 1993 (1972)] R. Barcan Marcus. Quantification and ontology. In [Marcus, 1993], pp. 75-87. (First published version in *Noûs*, 6 (1972), 240-250.)

[Marcus, 1976] R. Barcan Marcus. Dispensing with possibilia. In *Proceedings of the American Philosophical Association*, 49 (1976), 39-51.

[Marcus, 1993 (1978)] R. Barcan Marcus. Nominalism and the substitutional quantifier. In [Marcus, 1993], pp. 111-124. (First published in *Monist*, 61 (1978), 351-362.)

[Marcus, 1993 (1985/86)] R. Barcan Marcus. Possibilia and possible worlds. In [Marcus, 1993], pp. 189-213. (First published version in *Grazer Philosophische Studien*, 25/26 (1985/86), 107-133.)

[Marcus, 1993] R. Barcan Marcus. *Modalities: Philosophical Essays*, Oxford University Press, New York and Oxford, 1993.

[McMichael, 1983] A. McMichael. A new actualist modal semantics, *Journal of Philosophical Logic*, 12 (1983), 73-99.

[Melia, 1992] J. Melia. Against modalism, *Philosophical Studies* 68, 1992, 35-56.

[Menzel, 1990] C. Menzel. Actualism, ontological commitment, and possible worlds semantics, *Synthese*, 85 (1990), 355-389.

[Menzel, 1991] C. Menzel. The true modal logic, *Journal of Philosophical Logic*, 20 (1991), 331-374.

[Menzel, 2000] C. Menzel. Actualism. In: E. Zalta (ed.) *Stanford Encyclopedia of Philosophy*, http:// plato.stanford.edu/ archives/ sum2000/ entries/ actualism, 2000.

[Parsons, 1980] T. Parsons. *Nonexistent Objects*, Yale University Press, New Haven, 1980.

[Peacocke, 1978] C. Peacocke. Necessity and truth theories, *Journal of Philosophical Logic*, 7 (1978), 473-500.

[Peacocke, 1999] C. Peacocke. *Being Known*, Clarendon Press, Oxford, 1999.

[Peacocke, 2002a] C. Peacocke. Précis of *Being Known*, *Philosophy and Phenomenological Research*, 63 (2002), 636-640.

[Peacocke, 2002b] C. Peacocke. The principle-based account of modality: elucidations and resources, *Philosophy and Phenomenological Research*, 63 (2002), 663-679.

[Peacocke, 2002c] C. Peacocke. Principles for possibilia, *Noûs*, 36 (2002), 486-508.

[Plantinga, 1974] A. Plantinga. *The Nature of Necessity*, Clarendon Press, Oxford, 1974.

[Plantinga, 1976] A. Plantinga. Actualism and possible worlds, *Theoria*, 42 (1976), 139-60.

[Quine, 1969 (1965)] W. V. Quine. Ontological relativity. In [Quine, 1969], pp. 26-68. (First published in *Journal of Philosophy*, 65 (1965), 185-212.)

[Quine, 1969 (1968)] W. V. Quine. Existence and quantification. In [Quine, 1969], pp. 91-113. (First published in *L'Age de la Science*, 1 (1968), 151-164.)

[Quine, 1969] W. V. Quine. *Ontological Relativity and Other Essays*, Columbia University Press, New York, 1969.

[Ray, 1996] G. Ray. Ontology-free modal semantics, *Journal of Philosophical Logic*, 25 (1996), 333-361.

[Rosen, 2002] G. Rosen. Peacocke on modality, *Philosophy and Phenomenological Research*, 63 (2002), 641-648.

[Salmon, 1987] N. Salmon. Existence. In J. Tomberlin (ed.) *Philosophical Perspectives 1, Metaphysics*, pp. 49-108, Ridgeview, Atascadero, 1987.

[Schiffer, 2003] S. Schiffer. *The Things We Mean*, Clarendon Press, Oxford, 2003.

[Soames, 1999] S. Soames. *Understanding Truth*, Oxford University Press, New York and Oxford, 1999.

[Stalnaker, 2003 (1976)] R. C. Stalnaker. Possible worlds. In [Stalnaker, 2003], pp. 25-39. (First published version in *Noûs*, 10 (1976), 65-75. Also reprinted in [Stalnaker, 1984], pp. 43-58.)

[Stalnaker, 1984] R. C. Stalnaker. *Inquiry*, MIT Press, Cambridge Mass., 1984.

[Stalnaker, 2003 (1997)] R. C. Stalnaker. Reference and necessity. In [Stalnaker, 2003], pp. 165-187. (First published in B. Hale and C. Wright (eds.) *A Compainion to the Philosophy of Language*, pp. 534-554, Blackwell, Oxford 1997.)

[Stalnaker, 2003] R. C. Stalnaker. *Ways a World Might Be*, Clarendon Press, Oxford 2003.

[Tarski, 1995 (1933)] A. Tarski. Pojęcie prawdy w językach nauk dedukcyjnych. In [Tarski, 1995], pp. 13-172. (First published in: Prace Towarzystwa Naukowego Warszawskiego, Wydział III Nauk Matematyczno-Fizycznych, nr 34, Warszawa 1933. German translation: Der Wahrheitsbegriff in den formalisierten Sprachen, *Studia Philosophica 1*, 1935, 261-405. English translation of the German translation: The concept of truth in formalized languages. In [Tarski, 1983], pp. 152-278.)

[Tarski, 1995 (1936)] A. Tarski. O ugruntowaniu naukowej semantyki. In [Tarski, 1995], pp. 173-202. (First published in *Przegląd Filozoficzny*, 39 (1936), 50-57. German translation: Grundlegung der wissenschaftlichen Semantik. In *Actes du Congrès International de Philosophie Scientifique*, vol. 3, Actualités Scientifiques et Industrielles, vol. 390), pp. 1-8, Paris, 1936. English translation of the German translation: The establishment of scientific semantics. In [Tarski, 1983], pp. 401-408.)

[Tarski, 1944] A. Tarski. The semantic conception of truth and the foundations of semantics, *Philosophy and Phenomenological Research*, 4 (1944), 341-375.

[Tarski, 1983] A. Tarski. *Logic, Semantics, Metamathematics*, second edition, J. Corcoran (ed.), Hackett, Indianapolis, Ind., 1983.

[Tarski, 1995] A. Tarski. *Pisma logiczno-filozoficzne*, Tom 1, Prawda, J. Zygmunt (ed.), Wydawnictwo Naukowe PWN, Warszawa 1995.

[Taylor, 2000] K. A. Taylor, Emptyness without compromise: a Referentialist semantics for empty names. In [Everett & Hofweber, 2000], pp. 17-36.

[Thomason, 1970] R. H. Thomason. Some completeness results for modal predicate calculi. In K. Lambert. *Philosophical Problems in Logic*, pp. 56-76, Reidel, Dordrecht, 1970.

[Tichý, 1973] P. Tichý. On *de dicto* modalities in quantified S5, *Journal of Philosophical Logic*, 2 (1973), 687-692.

[Tomberlin, 1996] J. Tomberlin. Actualism or possibilism?, *Philosophical Studies*, 84 (1996), 263-281.

[Villanueva, 1998] E. Villanueva (ed.). *Concepts*, Philosophical Issues, 9, Ridgeview Publishing Company, Atascadero, Calif., 1998.

[Walton, 1990] K. Walton. *Mimesis and Make-Believe*, Harvard University Press, Cambridge Mass., 1990.

[Williamson, 1998] T. Williamson. Bare possibilia, *Erkenntnis*, 48 (1998) 257-273.

[Williamson, 2002] T. Williamson. Peacocke's theory of modality, *Philosophy and Phenomenological Research*, 63 (2002), 649-654.

[Wright, 2002] C. Wright. On knowing what is necessary: three limitations of Peacocke's account, *Philosophy and Phenomenological Research*, 63 (2002), 655-662.

[Zalta, 1988] E. N. Zalta. *Intensional Objects and the Metaphysics of Intentionality*, MIT Press, Cambridge Mass., 1988.

[Zimmermann, 1999] T. E. Zimmermann. Meaning postulates and the model-theoretic approach to natural language semantics, *Linguistics and Philosophy*, 22 (1999), 529-561.

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