Rational Polytopes in Combinatorics and Algebraic Geometry

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1. Deutsche Zusammenfassung

In dieser Dissertation untersuchen wir rationale Polytope, also die konvexen Hüllen endlich vieler Punkte mit rationalen Koordinaten im n-dimensionalen affinen Raum. Wichtige rationale Polytope sind die sogenannten Gitterpolytope, das sind diejenigen rationalen Polytope, deren Eckpunkte im Gitter $\mathbb{Z}^n \subseteq \mathbb{R}^n$ liegen. Da Gitterpolytope und die algebraische Geometrie über die Sprache der torischen Varietäten eng miteinander verbunden sind (siehe zum Beispiel [CLS11]), betrachten wir rationale Polytope nicht nur um ihrer selbst willen, sondern auch im Zusammenhang mit der torischen Geometrie. Wir fokussieren uns daher auf zwei Situationen, welche in einem gewissen Sinn die einfachsten nichttrivialen Beispiele rationaler Polytope liefern. Auf der einen Seite betrachten wir den zweidimensionalen Fall rationaler Polygone und dabei insbesondere diejenigen mit Nenner 2, deren Zweifaches ein Gitterpolygon ist, obwohl sie selbst kein Gitterpolygon sind, und auf der anderen Seite betrachten wir rationale Polytope, die sich auf natürliche Weise aus einem Gitterpolytop ergeben, wobei wir aus der torischen Geometrie motivierte Operationen verwenden.

Für ein gutes Verständnis der zweidimensionalen Situation beginnen wir mit allgemeineren Objekten, nämlich mit ebenen konvexen Mengen in $\mathbb{R}^2$ und ihrer Wechselwirkung mit dem Gitter $\mathbb{Z}^2 \subseteq \mathbb{R}^2$. Motiviert durch klassische Resultate zu Volumenschranken wie die von Minkowski für konvexe zentralsymmetrische Mengen mit einem einzigen inneren Gitterpunkt, die von Lagarias und Ziegler für Gitterpolytope mit einer bestimmten Anzahl an inneren Punkten in einem Untergitter [LZ91] und die von Scott für Gitterpolygone mit einer bestimmten Anzahl innerer Gitterpunkte [Sco76], beschäftigen wir uns im ersten Manuskript [Boh23] mit Flächenschranken für ebene konvexe Mengen mit einer festen Anzahl an inneren Gitterpunkten. Die genannten klassischen Resultate haben unterschiedliche Einschränkungen in Bezug auf rationale Polygone. So kann Minkowskis Schranke zwar für eine beliebige ungerade Anzahl innerer Punkte für zentralsymmetrische Mengen verallgemeinert werden, aber ein rationales Polygon muss im Allgemeinen nicht zentralsymmetrisch sein. Die Volumenschranke von Lagarias und Ziegler kann zwar für rationale Polygone verwendet werden, allerdings ist sie weit entfernt von tatsächlich beobachteten großen Volumina wie etwa in den Beispielen in [LZ91]. Und Scotts Ungleichung liefert zwar eine scharfe Schranke für Gitterpolygone, aber es ist nicht klar, wie diese Schranke direkt auf rationale Polygone verallgemeinert werden kann.

Wir überwinden diese Einschränkungen in Dimension 2 in folgender Weise. Wir versuchen nicht Minkowskis Theorem mit Hilfe eines Asymmetriekoeffizienten zu verwenden, wie dies etwa von Lagarias und Ziegler getan wurde, sondern wir verwenden geeignete Koordinaten und elementare Integrationsmethoden wie Abschätzungen durch geeignete Dreiecke und Parallelogramme um die Fläche zu beschränken. Als geeignet erweisen sich dabei Koordinaten, die so gewählt werden, dass die erste Koordinate die Gitterbreite der konvexen Menge misst, das heißt, wir haben für die

Gitterbreite $\operatorname{lw}(P)$ den Ausdruck

$$\operatorname{lw}(P) := \min_{y\in\mathbb{Z}^2\setminus\{0\}} (\max_{x\in P}\langle x,y\rangle - \min_{x\in P}\langle x,y\rangle) = \max_{(x_1,x_2)\in P} x_1 - \min_{(x_1,x_2)\in P} x_1.$$

Dies liefert uns in [Boh23, Theorem 1.1.] für nicht notwendigerweise zentralsymmetrische (!) ebene konvexe Mengen mit einer großen Gitterbreite sogar eine bessere Schranke als die von Minkowski, genauer erhalten wir einen zusätzlichen quadratischen Term in $\operatorname{lw}(P)$. Für $\operatorname{lw}(P) \in (2,5]$ können wir die Fläche mit Scotts Schranke $2(i+1)+\frac{1}{2}$ beschränken und für kleinere Breite erhalten wir ein gutes Verständnis für Flächenmaximierer in Abhängigkeit von den genauen Daten zur Gitterbreite (siehe [Boh23, Theorem 1.2.]).

Als eine Folgerung erhalten wir für rationale Polygone $P \subseteq \mathbb{R}^2$ mit Nenner d und $i \in \mathbb{Z}_{\geq 2}$ inneren Gitterpunkten in [Boh23, Corollary 3.8.] die scharfe Schranke

$$\operatorname{vol}(P) \leq \frac{(d+1)^2}{2d}(i+1),$$

welche für die Beispiele in [LZ91] angenommen wird.

Für Gitterpolygone können wir mit unserer elementaren Integrationsmethode mit geeigneten Koordinaten nicht nur Scotts Flächenschranke auf alternativem Weg beweisen, sondern erhalten auch einen alternativen Beweis für Picks Theorem und die Klassifikation von Gitterpolygonen ohne innere Gitterpukte. Mit Hilfe von etwas algebraischer Geometrie, genauer gesagt durch die Anwendung von Noethers Formel für den Fall von glatten vollständigen torischen Flächen bzw. entsprechenden kombinatorischen Resultaten, erhalten wir eine genaue Beschreibung der Differenz zwischen der Fläche eines Gitterpolygons und der Schranke von Scott in [Boh23, Theorem 1.3.].

Im zweiten Manuskript zu halbzahligen Polygonen, die ein Ehrhartpolynom besitzen, ([Boh24a]) untersuchen wir halbzahlige Polygone, die sich wie ein Gitterpolygon bezüglich des Zählens von Gitterpunkten in ganzzahligen Vielfachen verhalten. Wir betrachten also halbzahlige Polygone $P \subseteq \mathbb{R}^2$, für welche wir die Werte der Gitterpunktzählfunktion

$$\operatorname{ehr}_P\colon \mathbb{Z}_{\geq 1} \to \mathbb{Z}, t \mapsto |tP \cap \mathbb{Z}^2|$$

durch ein Polynom ausdrücken können, das zugehörige Ehrhartpolynom. Im Allgemeinen ist ehr für ein halbzahliges Polygon nur ein Quasipolynom, das heißt, dass wir unterschiedliche Polynome verwenden müssen in Abhängigkeit davon, ob t gerade oder ungerade ist. Als Hauptresultat erhalten wir eine Klassifikation der Ehrhartpolynome aller halbzahligen Polygone in [Boh24a, Theorem 1.6.]. Wir können diese Klassifikation dadurch geben, dass wir die möglichen Anzahlen an inneren Gitterpunkten $i(P)$ und Randgitterpunkten $b(P)$ bestimmen. Für $i(P) > 1$ erhalten wir als mögliche Paare $(i(P), b(P))$ genau diejenigen Paare mit

$$2 \leq b(P) \leq 2i(P) + 7.$$

Für Gitterpolygone gilt jedoch

$$3 \leq b(P) \leq 2i(P) + 6 \quad \text{oder} \quad i(P) = 1, b(P) = 9$$

aufgrund von Scotts Ungleichung und dem Satz von Pick, d.h. wir bekommen zusätzliche Ehrhartpolynome für den halbzahligen Fall. Daher klassifizieren wir in [Boh24a, Theorem 7.2.] alle halbzahligen Polygone mit Ehrhartpolynom und $b(P) = 2i(P) + 7 > 9$. Außerdem untersuchen wir den Fall $i(P) = 1$ in [Boh24a, 4. and 5.] genauer, da wir für diesen Fall eine Verbindung zur torischen Geometrie haben. Wir erhalten genau 30 halbzahlige Polygone mit Ehrhartpolynom für $i(P) = 1$ ([Boh24a, Classification 4.1.]) und es handelt sich dabei genau um die dualen Polygone der 30 LDP-Polygone von Gorensteinindex 2, welche zuerst in [KKN10] klassifiziert wurden (siehe [Boh24a, Theorem 5.10.]).

Einige Modifikationen an Beispielen aus [Boh24a] ermöglichten in [MAW24, 7.] die Konstruktionen weiterer neuer Ehrhartpolynom für Polygone mit höheren Nennern.

Halbzahlige Polygone treten als *Mittelpolygone* von dreidimensionalen Gitterpolytopen mit Gitterbreite 2 auf. Innerhalb des Mittelpolygons gibt es in diesem Fall ein weiteres halbzahliges Polygon, das Fine interior des Gitterpolygons. Dieses ist relevant für die Theorie von Hyperflächen in einem Torus und diese spezielle Situation wird in [Boh24c] betrachtet.

Wenn wir Gitterpolytope als Newtonpolytop einer Hyperfläche in einem Torus betrachten und uns für die birationale Geometrie der Hyperfläche interessieren, bemerken wir die Relevanz des Fine interiors $F(P)$ des Newtonpolytops $P \subseteq \mathbb{R}^n$. Hierbei ist das Fine interior als die Menge aller Punkt definiert, die mindestens Gitterabstand 1 zu allen Stützhyperebenen mit ganzzahligem Normalenvektor des Polytopes haben. Es zeigt sich, dass das Fine interior selbst ein rationales Polytop ist, und einige wichtige birationale Invarianten der Hyperfläche können mit Hilfe des Fine interiors bestimmt werden. So wurde zum Beispiel von Victor Batyrev in [Bat23a] gezeigt, dass die Kodairadimension κ der Hyperfläche mit der Dimension des Fine interiors des Newtonpolytops zusammenhängt und bestimmt werden kann durch

$$\kappa = \min(\dim(F(P)), \dim(P) - 1).$$

Für dreidimensionale Gitterpolytope mit Gitterbreite 2 zeigen wir in [Boh24c, Theorem 3.2.], dass das Fine interior ein halbzahliges Teilpolygon des Mittelpolygons ist und wir diesen Fall mit Hilfe von zweidimensionaler Kombinatorik betrachten können. Wir geben genügend notwendige Bedingungen für ein solches Fine interior an, um in [Boh24c, 4.] einen effizienten Algorithmus zu erhalten, um diese Fine interior zu klassifizieren. Die Klassifikation wurde für Fine interior mit bis zu 40 Gitterpunkten durchgeführt und wir erhalten auf diesem Weg viele Beispiele für Flächen von allgemeinem Typ und von Geschlecht höchstens 40, die als Hyperfläche in einem Torus beschrieben werden können.

Hat die Hyperfläche negative Kodairadimension, das Newtonpolytop also leeres Fine interior, dann spielt das Fine interior von Vielfachen des Newtonpolytops eine wichtige Rolle. Insbesondere liefert in diesem Fall das kleinste Vielfache mit nichtleerem Fine interior zusätzliche Information (vgl. [Bat23b]). Wir haben diese Situation im Manuskript [Boh24b] weiter untersucht. Dort geben wir zum Beispiel eine rein kombinatorische, simultane Beschreibung für das Fine interior aller Vielfachen eines rationalen Polytops, wodurch wir die Möglichkeit haben spezielle Vielfache zu bestimmen, für welche sich die Kombinatorik des Fine interiors verändert. Außerdem erhalten wir eine vollständige Klassifikation aller schwach sporadischen F-hohlen dreidimensionalen Gitterpolytope, d.h. von allen dreidimensionalen Gitterpolytopen mit leerem Fine interior, die ein Vielfaches mit einem nulldimensionalen Fine interior haben.

In den letzten beiden Manuskripten, die gemeinsam mit Justus Springer verfasst wurden, verwenden wir die Resultate zu Flächenschranken rationaler Polygone aus [Boh23] und zu den zweidimensionalen Fine interior von dreidimensionalen Gitterpolytopen aus [Boh24c], um in [BS24a] einen effizienten Algorithmus zur Klassifikation rationaler Polygone mit kleinem Nenner und wenigen inneren Gitterpunkten zu erhalten. Ein anderer Versuch der Klassifikation und Untersuchung der Ehrhart-Theorie halbzahliger Polygone wurde in [HHK24] unternommen.

Im Klassifikationsalgorithmus werden zunächst für gegebenen Nenner und gegebene Anzahl innerer Gitterpunkte alle maximalen rationalen Polygone zu klassifiziert und dann alle Teilpolygone mit denselben Eigenschaften bestimmt. Die Klassifikation der maximalen Polygone ist mit Hilfe einiger Einschränkungen an die Daten zur Gitterbreite aus [Boh23] und der Kenntnis der Fine interior aus [Boh24c] möglich. Besonders für halbzahlige Polygone erhalten wir eine sehr umfassende Klassifikation, die es uns ermöglicht, Vermutungen über die Anzahl der halbzahligen Polygone mit kollinearen inneren Gitterpunkten und die Anzahl der Ehrhartquasipolynome für halbzahlige Polygone mit einer gegebenen Anzahl innerer Gitterpunkte und mindestens zwei Randgitterpunkten aufzustellen ([BS24a, Conjecture 6.8.]). Die Klassifikation umfasst insbesondere die Polygone, welche in [HHK24] klassifiziert wurden.

Mit Hilfe der Klassifikation erhalten wir in [BS24b] Verallgemeinerungen von Scotts Ungleichung und Picks Formel für rationale Polygone (siehe [BS24b, Theorem 1.1.-1.4.]). Außerdem erhalten wir Klassifikationen für die extremalen Polygone in diesen Fällen. Dies kann als ein Schritt zum Verständnis der Ehrhart-Theorie rationaler Polygone angesehen werden.

2. Introduction

In this dissertation, we study rational polytopes, i.e. convex hulls of finitely many points with rational coordinates in the n-dimensional affine space $\mathbb{R}^n$. An important subset of rational polytopes are lattice polytopes, i.e. polytopes whose vertices are elements of the lattice $\mathbb{Z}^n \subseteq \mathbb{R}^n$. Since there is a very fruitful connection between lattice polytopes and algebraic geometry via the language of toric varieties (see for example [CLS11]), we also look at rational polytopes not only for their own sake, but also in connection with the toric language. Therefore, we focus on two situations which are, in a sense, the simplest non-trivial examples of rational polytopes. On the one hand, we consider the 2-dimensional case of rational polygons, in particular those of denominator 2, which means that although they are not lattice polygons themselves, their double is a lattice polygon. On the other hand, we consider rational polytopes that arise naturally from a lattice polytope through operations inspired by toric geometry.

To better understand the 2-dimensional situation of rational polygons $P \subseteq \mathbb{R}^2$, we start from a more general framework and consider planar convex bodies in $\mathbb{R}^2$ and their interaction with the lattice $\mathbb{Z}^2 \subseteq \mathbb{R}^2$. Motivated by classical theorems on volume bounds by Minkowski for convex centrally symmetric bodies with a single interior integral point, by Largarias and Ziegler for *lattice polytopes containing a fixed number of interior points in a sublattice* [LZ91] and by Scott for lattice polygons with a fixed number of interior integral points [Sco76], we look for *area bounds for planar convex bodies containing a fixed number of interior integral points* in the first manuscript [Boh23]. The classical theorems have some limitations with respect to rational polygons. Although it is possible to generalize Minkowski's bound to an arbitrary odd number i of interior lattice points for centrally symmetric bodies $P \subseteq \mathbb{R}^n$, it is not necessary for a rational polygon to be centrally symmetric. The volume bound of Lagarias and Ziegler can be used for rational polytopes $P \subseteq \mathbb{R}^n$ with an arbitrary number $i \in \mathbb{Z}_{\geq 1}$ of interior lattice points. However, this bound is far away from known examples as given in [LZ91]. For lattice polygons with i interior lattice points, Scott's inequality gives a sharp area bound, but it is not obvious how to generalize this to all rational polygons.

How do we solve these constraints in dimension 2 and what results do we get? Instead of trying to use Minkowski's theorem with the modification of some *coefficient of asymmetry* as it was done by Lagarias and Ziegler, we use suitable coordinates and elementary integration methods, i.e. estimates by suitable triangles and parallelograms, to get bounds for the area. Suitable coordinates here means that we choose coordinates so that the first coordinate is measured along a lattice width direction of the convex body, i.e. for the lattice width $\mathrm{lw}(P)$ we have

$$\mathrm{lw}(P) := \min_{y \in \mathbb{Z}^2 \setminus \{0\}} (\max_{x \in P} \langle x, y \rangle - \min_{x \in P} \langle x, y \rangle) = \max_{(x_1,x_2) \in P} x_1 - \min_{(x_1,x_2) \in P} x_1.$$

This gives us in [Boh23, Theorem 1.1.] for not necessarily symmetric (!) plane convex bodies with a large lattice width even a better bound than Minkowski's theorem with a quadratic correction term in $\mathrm{lw}(P)$. For $\mathrm{lw}(P) \in (2, 5]$ we can control the area with Scott's bound $2(i+1)+\frac{1}{2}$ and for smaller lattice widths we have a good understanding for the area maximizers depending on the lattice width data (see [Boh23, Theorem 1.2.]).

As a corollary for rational polygons $P \subseteq \mathbb{R}^2$ with $i \in \mathbb{Z}_{\geq 2}$ interior lattice points, for which dP is the smallest integer dilation of P, which is a lattice polygon, we obtain in [Boh23, Corollary 3.8.] the sharp bound

$$\operatorname{vol}(P) \leq \frac{(d+1)^2}{2d}(i+1),$$

which is sharp for the examples from [LZ91].

For lattice polygons, we can use our elementary integration method with suitable coordinates to prove not only Scott's area bound, but also Pick's theorem and the known classification of lattice polygons without interior lattice points. With the help of some algebraic geometry, more precisely Noether's formula for smooth complete toric surfaces, and its combinatorial 'number 12'-analogues, we even obtain an exact description of the difference between the area of a lattice polygon and Scott's bound in [Boh23, Theorem 1.3.] .

In the second manuscript on *quasi-period collapse in half-integral polygons* [Boh24a], we study some rational polygons which are not lattice polygons but behave like a lattice polygon when we want to count lattice points in integer dilations. More precisely, we consider half-integral polygons $P \subseteq \mathbb{R}^2$ for which we can get the values of the lattice point counting function

$$\operatorname{ehr}_P \colon \mathbb{Z}_{\geq 1} \to \mathbb{Z}, t \mapsto |tP \cap \mathbb{Z}^2|$$

from a polynomial, its Ehrhart polynomial. In general, for a half-integral polygon, ehr is only a quasi-polynomial of period 2, i.e. we can have different polynomials depending on whether t is even or odd. Therefore, the situation with a polynomial, i.e. a quasi-polynomial of period 1, is called *quasi-period collapse* or alternatively we call the polygon *pseudo-integral*. The Ehrhart polynomial then, as for lattice polygons, always has the form

$$\operatorname{ehr}_P(t) = \left(i(P) + \frac{b(P)}{2} - 1\right)t^2 + \frac{b(P)}{2}t + 1,$$

where $i(P) = |\operatorname{int}(P) \cap \mathbb{Z}^2|$ denotes the number of interior and $b(P) = |\partial P \cap \mathbb{Z}^2|$ the number of boundary lattice points of P. As a main result, we classify all Ehrhart polynomials for half-integral polygons in [Boh24a, Theorem 1.6.] by specifying all possible pairs $(i(P), b(P))$ for the polygons with period collapse. The possible pairs for $i(P) > 1$ are exactly those pairs with

$$2 \leq b(P) \leq 2i(P) + 7.$$

If we compare this with the situation of lattice polygons, where

$$3 \leq b(P) \leq 2i(P) + 6 \quad \text{or} \quad i(P) = 1, b(P) = 9$$

by Scott's inequality and Pick's theorem, we see that we obtain new Ehrhart polynomials in the half-integral case. We therefore classify in [Boh24a, Theorem 7.2.] all half-integral polygons with period collapse and $b(P) = 2i(P) + 7 > 9$, since they yield new Ehrhart polynomials. Furthermore, we consider the case $i(P) = 1$ in [Boh24a, 4. and 5.] in particular, since there we have a connection to toric geometry. We see that there are exactly 30 half-integral pseudo-integral polygons with $i(P) = 1$ ([Boh24a, Classification 4.1.]), and these are exactly the duals of the 30 LDP polygons of Gorenstein index 2, first classified in [KKN10] (see [Boh24a, Theorem 5.10.]). Thus, we have also given an alternative way to classify the toric log del Pezzo surfaces of Gorenstein index 2 and linked the lattice point count of the pseudo-integral polygons to the Hilbert series of these surfaces.

Some modifications of the examples from [Boh24a] also led to further new Ehrhart polynomials, i.e. pseudo-integral polygons with denominator greater than 2 and even more boundary lattice points, in [MAW24, 7.].

In addition to considering the simplest non-lattice polygons, there is another reason for considering half-integer polygons. Half-integral polygons appear as 'middlepolygons' of 3-dimensional lattice polytopes of lattice width 2. There is also an important subpolygon of the middlepolygons, the *Fine interior* of the 3-dimensional lattice polytopes of lattice width 2, which is half-integral. Looking at the Fine interior is motivated by the theory of nondegenerate toric hypersurfaces. This is done in the manuscript *lattice 3-polytopes of lattice width 2 and corresponding toric hypersurfaces* [Boh24c].

If one considers lattice polytopes as Newton polytopes of nondegenerate toric hypersurfaces and is interested in the birational geometry of the hypersurface, one finds that the Fine interior $F(P)$ of the Newton polytope $P \subseteq \mathbb{R}^n$ is an important combinatorial object. The Fine interior of a rational polytope is the set of all points with a lattice distance at least one to all supporting integral hyperplanes of the polytope. It turns out that the Fine interior itself is a rational polytope and, in general, is only a rational and not a lattice polytope, even if P itself is a lattice polytope. Some important birational invariants of the toric hypersurface can be computed from the Fine interior and some additional combinatorial data. For example, Victor Batyrev showed in [Bat23a] that the Kodaira dimension κ of the hypersurface is related to the dimension of the Fine interior of the Newton polytope by

$$\kappa = \min(\dim(F(P)), \dim(P) - 1)$$

and if the Fine interior is non empty, it can be used with some additional combinatorial data to construct a minimal model of the toric hypersurface.

For lattice 3-polytopes of lattice width 2 we show in [Boh24c, Theorem 3.2.] that the Fine interior is a half-integral subpolygon of the middlepolygon and so it is enough to use 2-dimensional combinatorics to understand this case. We find enough conditions for a half-integral polygon to be a Fine interior of a lattice 3-polytope to describe an efficient algorithm ([Boh24c, 4.]) to classify all those Fine interiors if they have a small number of lattice points. The classification is done for up to 40 lattice points and so we get many examples of surfaces of general type and genus at most 40, which can be described as a toric hypersurface.

For toric hypersurfaces of negative Kodaira dimension, i.e., toric hypersurfaces whose Newton polytope has empty Fine interior, the understanding of the Fine interior of dilations of the Newton polytope is important. In particular, the smallest dilation with non-empty Fine interior provides additional information in this case (see [Bat23b]). We investigate this situation in manuscript [Boh24b]. In particular, we give a purely combinatorial description of the Fine interiors of all dilations of a rational polytope, which allows us to determine special dilations of the polytope for which we have have a change in the combinatorics of the Fine interior. Moreover, we give there a complete classification of all *weakly sporadic F-hollow* lattice 3-polytope there, i.e., of all lattice 3-polytopes with empty Fine interior that have a dilation with Fine interior of dimension 0.

In the last two manuscripts, written jointly with Justus Springer, we use the results on the area bounds of rational polygons from [Boh23] and on 2-dimensional Fine interiors of lattice 3-polytopes from [Boh24c] to develop efficient algorithms for *classifying rational polygons with small denominator and few interior lattice points* in [BS24a] and to gain a better understanding of the Ehrhart theory of rational polygons in [BS24b]. Another recent approach to the classification and understanding of the Ehrhart theory of denominator 2 polygons can be found in [HHK24].

The idea behind the classification algorithms is to first classify all maximal rational polygons for a given denominator and a given number of interior lattice points and then to compute all subpolygons with these properties. The classification of the maximal polygons is possible, taking into account some restrictions on the lattice width data from [Boh23] and the Fine interiors from [Boh24c]. The calculation of the subpolygons up to affine unimodular equivalence is done with a suitable normal form of the polygons given in [BS24a, 2.1]. The classification is particularly rich for half-integral polygons, for which we obtain conjectures on the number of half-integral polygons with a given number of collinear interior lattice points ([BS24a, Conjecture 3.7.]) and on the number of Ehrhart quasipolynomials for half-integral polygons with a given number of interior lattice points and at least two boundary lattice points ([BS24a, Conjecture 6.8.]). The classification results from [HHK24] can also be obtained from the dataset.

Inspired by the classification, we obtain in [BS24b] generalizations of Scott's inequality and Pick's formula to rational polygons (see [BS24b, Theorem 1.1.-1.4.]). Furthermore, we give there classifications for the extreme polygons in both cases. Both can be seen as a step towards understanding the Ehrhart theory of rational polygons.

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3. Area bounds for planar convex bodies containing a fixed number of interior integral points

The following manuscript is consistent with the preprint published on arXiv at arXiv:2305.11485v2.

AREA BOUNDS FOR PLANAR CONVEX BODIES CONTAINING A FIXED NUMBER OF INTERIOR INTEGRAL POINTS

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ABSTRACT. We prove area bounds for planar convex bodies in terms of their number of interior integral points and their lattice width data. As an application, we obtain sharp area bounds for rational polygons with a fixed number of interior integral points depending on their denominator. For lattice polygons, we also present an equation for the area based on Noether's formula.

1. INTRODUCTION

For any $d \in \mathbb{Z}_{\geq 1}$, we obtain from Minkowski's theorem [Min10, p. 76] the sharp volume bound 2^d for centrally symmetric convex bodies in $\mathbb{R}^d$ with a single interior integral point. Blichfeld [Bli21, 10.] and van der Corput [vdC36] noted that we can generalize Minkowski's bound to $2^{d-1}(k+1)$ if we have an arbitrary odd number k of interior integral points.

However, if the convex body is not centrally symmetric, then there are already examples for arbitrarily large volumes for $d = 2$. For example, for any $(k, l) \in \mathbb{Z}^2_{\geq 1}$ we already have in [LZ91, 2.6.] the rational triangle

$$T_{k,l} := \operatorname{convhull}\left((0,0), \left(1 + \frac{1}{l}, 0\right), (0, (l+1)(k+1))\right) \subseteq \mathbb{R}^2$$

with $k = |\operatorname{int}(T_{k,l}) \cap \mathbb{Z}^2|$ interior integral points and

$$\operatorname{area}(T_{k,l}) = \frac{(l+1)^2}{2l}(k+1) > l.$$

However, it is possible to replace the central symmetry with other assumptions and still obtain a bound on the area. Examples of such modifications of Minkowski's theorem can be found in [Sco88, 3.].

Our goal is to use the concept of *lattice width data* introduced in the next section to prove area bounds depending on this data.

Theorem 1.1. *Let $K \subseteq \mathbb{R}^2$ be a convex body with $k := |\operatorname{int}(K) \cap \mathbb{Z}^2| \in \mathbb{Z}_{\geq 1}$. If K has symmetric lattice width data or a lattice width $\operatorname{lw}(K) > 5$, then we have*

$$\operatorname{area}(K) \leq 2(k+1) - \left\lfloor \max\left(\frac{\operatorname{lw}(K)}{2} - 3, 0\right)\right\rfloor \cdot \left\lfloor \frac{\operatorname{lw}(K)}{2} - 2 \right\rfloor.$$

Moreover, if there is a function $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ with $\operatorname{area}(K) \leq 2(k+1) - f(\operatorname{lw}(K))$ for every convex body K, then

$$\limsup_{x\to\infty} \frac{f(x)}{x^2} \leq \frac{3}{8}.$$

So in the case of large lattice width we can even sharpen the classical bound $2(k+1)$. But what can we say for smaller lattice widths, where the classical bound cannot be correct because of the existence of triangles $T_{k,l}$? The following theorem gives an answer.

Theorem 1.2. *Let $K \subseteq \mathbb{R}^2$ be a convex body with $k := |\mathrm{int}(K) \cap \mathbb{Z}^2| \in \mathbb{Z}_{\geq 1}$. Then we can explicitly describe area maximizers in the case of $\mathrm{lw}(K) \leq 1$ for given lattice width data, and we have*

$$\mathrm{area}(K) \leq \begin{cases} \frac{\mathrm{lw}(K)^2}{2(\mathrm{lw}(K)-1)}(k+1) & \text{if } \mathrm{lw}(K) \in (1,2] \\ 2(k+1) + \frac{1}{2} & \text{if } \mathrm{lw}(K) \in (2,5], \end{cases}$$

where the bounds for $\mathrm{lw}(K) \in (1,2]$ are sharp.

In particular, this theorem implies that the rational triangles $T_{k,l}$ are area maximizers for $\mathrm{lw}(K) = 1 + \frac{1}{l}$.

Probably polytopes with integral points as vertices, henceforth also called *lattice polytopes* (for the lattice $\mathbb{Z}^d \subseteq \mathbb{R}^d$), are among the best-studied non-symmetric convex bodies that have finite volume bounds if they have a fixed number of interior integral points. This was first proved in [Hen83] and improved volume bounds can be found in [LZ91, Theorem 1], [Pik01, p. 17], and [AKN20, Theorem 1.4.].

However, these bounds do not seem to be sharp, since the largest known volumes for lattice polytopes with a fixed number of interior points from [WZP82] are much smaller. Nevertheless, we do have sharp bounds for some special types of lattice polygons, e.g., for reflexive polytopes (see [BKN22, Corollary 1.2.]) and for lattice simplices with a single interior integral point (see [AKN15]).

We also know from [LZ91, p. 1024] that, up to affine unimodular equivalence, there are only finitely many lattice polytopes in $\mathbb{R}^d$ whose volume is less than a fixed bound. Thus, we can also obtain sharp bounds through the classification of all lattice polytopes in $\mathbb{R}^d$ with a fixed number of interior points. This was done for lattice 3-polytopes with exactly 1 or 2 interior integral points in [Kas10] and [BK16].

In [LZ91, Theorem 1] and [Pik01, p. 17] we also have volume bounds for lattice polytopes with exactly k interior points in a sublattice $(l\mathbb{Z})^d, l \in \mathbb{Z}_{\geq 2}$, which corresponds to the case of rational polytopes with denominator l. But also these bounds are far from the largest known examples in [LZ91, 2.6.], and it was even shown in [Pik01, §6] that the methods used are not good enough to show such small bounds.

There are also some classification results for these polytopes. In [AKW17, Theorem 4], maximal half-integral lattice-free polygons - which is the case $d = 2, k = 0, l = 2$ - were classified on the way of classifying all $\mathbb{Z}^3$-maximal lattice-free lattice 3-polytopes. In [HHS25, 4.11.] we have the classification of almost 2- and 3-hollow LDP-polygons, a subcase of $d = 2, k = 1, l \in \{2, 3\}$, which was necessary for the work on ε-log canonical del Pezzo $\mathbb{K}^*$-surfaces.

It turns out that our methods are good enough to handle the case of rational polygons. As a corollary to the theorems above, we obtain in 3.8 that the triangle $T_{k,l}$ is indeed also an area maximizing rational polygon with denominator $l \in \mathbb{Z}_{\geq 2}$.

The case of lattice polygons is different. For lattice polygons $P_{k,1} \subseteq \mathbb{R}^2$ with k interior integral points, there is an additional $+\frac{1}{2}$ and so

$$\operatorname{area}(P_{k,1}) \leq 2(k+1) + \frac{1}{2}$$

and the convex hull $\operatorname{convhull}((0,0),(3,0),(0,3))$ is the single area maximizer up to affine unimodular equivalence in this case, a result first proven combinatorially by Scott in [Sco76]. However, there is also an area bound

$$\operatorname{area}(P_{k,1}) \leq 2(k+1) + 2 - \frac{n}{2}$$

additionally depending on the number of vertices n, which was conjectured by Coleman in [Col78, p. 54] and then proved combinatorially in [KO07]. In these cases, we write *combinatorially proved* because there are older results from algebraic geometry that imply these results via the language of toric geometry. For example, Castryck showed in [Cas12, p. 506] that Coleman's conjecture was essentially already proven by Koelman in [Koe91, 4.5.2.2], and in [HS09, 2.3] we see that Scott's volume bound corresponds to algebro-geometric results of del Pezzo and Jung.

In the final step, we combine two *number 12-formula*, which originated from Noether's formula via toric geometry, to derive an equation for the area of a lattice polygon. This equation is presented in the following theorem, which not only implies Coleman's conjecture, but also helps us to handle the asymptotics in Theorem 1.1.

Theorem 1.3. *Let $K \subseteq \mathbb{R}^2$ be a lattice polygon with $k := |\operatorname{int}(K) \cap \mathbb{Z}^2| \in \mathbb{Z}_{\geq 1}$, $n(\Delta_{P,smooth})$ the number of rays in the smooth refinement of the normal fan Δ_P of P, which we get from the rays generated by the boundary lattice points on the bounded edges of* $\operatorname{convhull}(\sigma \cap (\mathbb{Z}^2 \setminus \{0\}))$ *for each cone σ of Δ_P. Then*

$$\operatorname{area}(P) = 2(k+1) + 2 - \frac{n(\Delta_{P,smooth})}{2} - \operatorname{area}(\operatorname{convhull}(\operatorname{int}(P) \cap \mathbb{Z}^2))).$$

The article is organized as follows: In the second section, we introduce the concept of lattice width data for planar convex bodies. A central result is that we prove bounds for the vertical slicing lengths when we have $(1,0)$ as a lattice width direction. This also gives bounds for the number of interior integral points on integral vertical lines.

In the third section, we compute area bounds for planar convex bodies based on their their lattice width data and the number of interior integral points. As a corollary, we obtain sharp area bounds for rational polygons, depending on the number of interior integral points and the denominator of the polygon. The proof of Theorem 1.2 is also provided at the end of the third section.

In the final section, we focus on lattice polygons. First, we provide a straightforward proof of Scott's area bound using the lattice width. Then, we use two variants of 'number 12'-theorems to prove Theorem 1.3, which allows us to estimate the error term of Scott's area bound in terms of the number of vertices and the lattice width. This also gives the asymptotic behavior of the bound in Theorem 1.1, which we can prove at the end of the last section.

2. Lattice width data for planar convex bodies

In this section, we define the *lattice width data*, which is a generalization of the concept of lattice width. By choosing appropriate coordinates that respect these lattice width data in 2.10, we can control the vertical slicing lengths and the number of integral points on vertical lines of a planar convex body. These are the main results of this section in Theorem 2.12 and Corollary 2.13, and we can use these results to obtain area bounds by integration later in the next section.

Definition 2.1. Let $K \subseteq \mathbb{R}^2$ be compact and convex with a nonempty interior. Then we call K a planar convex body.

Definition 2.2. Let $K \subseteq \mathbb{R}^2$ be a convex body. The *width function of* K is defined by

$$\operatorname{width}_K \colon (\mathbb{R}^2)^* \setminus \{0\} \to \mathbb{R}_{>0}, v \mapsto \max_{x \in K} v(x) - \min_{x \in K} v(x)$$

and we call $width_K(v)$ the *width of* K *in direction* v.

Remark 2.3. The width function is a continuous function and therefore attains its infimum on any compact set, in particular on $S^1 := \{v \in (\mathbb{R}^2)^* \mid \|v\| = 1\}$. We can use this minimum to define a *geometrical width* of K.

Since $\operatorname{width}_K(\lambda v) = |\lambda| \operatorname{width}_K(v)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$, the width function also attains its infimum on $(\mathbb{Z}^2)^* \setminus \{0\}$ and so we can define a *lattice width* depending on our lattice $\mathbb{Z}^2$ as follows.

Definition 2.4. The *lattice width of* K - denoted as $\operatorname{lw}(K)$ - is defined by

$$\operatorname{lw}(K) := \min_{v \in (\mathbb{Z}^2)^* \setminus \{0\}} \operatorname{width}_K(v).$$

A non-zero dual lattice vector $w \in (\mathbb{Z}^2)^* \setminus \{0\}$ is called a *lattice width direction* of K, if $\operatorname{width}_K(w) = \operatorname{lw}(K)$.

Remark 2.5. If $w = (w_1, w_2) \in (\mathbb{Z}^2)^*$ is a lattice width direction of K, then w is a primitive dual vector, i.e. $\gcd(w_1, w_2) = 1$, and $-w$ is also a lattice width direction of K. By [DMN12, Theorem 1.1.] there are at most 8 different lattice width directions for a planar convex body.

Lemma 2.6. *Let $K \subseteq \mathbb{R}^2$ be a convex body, $\pi_1 : \mathbb{R}^2 \to \mathbb{R}, (x_1, x_2) \mapsto x_1$. Then we have a concave function*

$$l_{K,\pi_1} : \pi_1(K) \to \mathbb{R}, t \mapsto |K \cap \{x_1 = t\}|$$

by measuring the length $|\cdot|$ of the vertical line segments. In particular, l_{K,π_1} attains its maximum on a compact interval and is increasing before and decreasing afterwards.

Proof. If we intersect K with a line through two points, one with a minimal and the other with a maximal first coordinate, we can decompose the area of K into a convex top between the graph of a concave and a linear function, and a convex bottom between the graph of a linear function and a convex function. So we get the vertical line segment length function by adding two concave functions, and so l_{K,π_1} is also concave. □

Remark 2.7. Using the Brunn-Minkowski inequality, we can get a similar result for the slice volumes of convex bodies of any dimension (e.g. [Mat02, 12.2.1]).

Definition 2.8. Let $K \subseteq \mathbb{R}^2$ be a convex body. We call the midpoint of the compact interval from Lemma 2.6, in which K has the longest vertical slicing length, the *position of the longest vertical slicing length* and write it as $\mathrm{plvsl}(K)$.

Remark 2.9. A vertical line segment of K is of maximum length if and only if it is an affine diameter of K, i.e., there are parallel supporting lines of K at the endpoints of the segment. For an overview of affine diameters, see [Sol05], where we can also find this characterization in [Sol05, 3.1.].

Lemma 2.10. *Let $K \subseteq \mathbb{R}^2$ be a convex body, $w \in (\mathbb{Z}^2)^* \setminus \{0\}$ a lattice width direction of K and $\pi_1 : \mathbb{R}^2 \to \mathbb{R}, (x_1, x_2) \mapsto x_1$. Then there is an affine unimodular transformation $U_{A,b}$, i.e.*

$$U_{A,b} : \mathbb{R}^2 \to \mathbb{R}^2, x \mapsto Ax + b, \text{ with } A \in \mathbf{GL}(2, \mathbb{Z}), b \in \mathbb{Z}^2,$$

so that the image $U_{A,b}(K)$ has a lattice width direction $wA^{-1} \in \{(-1, 0), (1, 0)\}$, we have $x_l, x_r \in \mathbb{R}$, $0 \le x_l < 1$ with $[x_l, x_r] = \pi_1(U_{A,b}(K))$ and

$$2 \cdot \mathrm{plvsl}(U_{A,b}(K)) \le \lceil x_l \rceil + \lfloor x_r \rfloor .$$

Moreover, the interval $[x_l, x_r]$ and $\mathrm{plvsl}(U_{A,b}(K))$ are well-defined by K and w, if we additionally ask for $\lceil x_l \rceil - x_l \le x_r - \lfloor x_r \rfloor$ in the case with

$$2 \cdot \mathrm{plvsl}(U_{A,b}(K)) = \lceil x_l \rceil + \lfloor x_r \rfloor .$$

Proof. Since w is a primitive dual vector, we can find another dual vector that complements w to a lattice basis of $(\mathbb{Z}^2)^*$. By choosing w as the first row and the other dual vector as the second row, we get a suitable unimodular matrix A. Now we have a unique first coordinate for b to additionally get $0 \le x_l < 1$.
If we were in the situation

$$2 \cdot \mathrm{plvsl}(U_{A,b}(K)) > \lceil x_l \rceil + \lfloor x_r \rfloor$$

or

$$\lceil x_l \rceil - x_l > x_r - \lfloor x_r \rfloor \text{ and } 2 \cdot \mathrm{plvsl}(U_{A,b}(K)) = \lceil x_l \rceil + \lfloor x_r \rfloor ,$$

then we would choose $-A$ instead of A.

Note that the first row of A and the first coordinate of b are determined by the conditions, so both $[x_l, x_r]$ and $\mathrm{plvsl}(U(K))$ are well-defined. □

Definition 2.11. Let $K \subseteq \mathbb{R}^2$ be a convex body.
If w is a lattice width direction for K and $U_{A,b}$, $[x_l, x_r]$ are as in 2.10, then we define the *lattice width data of K for the lattice width direction w* as the pair

$$\mathrm{lwd}_w(K) := ([x_l, x_r], \mathrm{plvsl}(U_{A,b}(K)))$$

and we call $\lceil x_r \rceil - \lfloor x_l \rfloor - 1$ the *number of interior integral vertical lines for the lattice width direction w*.

Lemma 2.12. *Let $K \subseteq \mathbb{R}^2$ be a planar convex body, $(1,0)$ a lattice width direction of K, and $\pi_1(K) = [x_l, x_r]$.*
Then we have for the length $|\cdot|$ of the vertical line segments

$$|K \cap \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 = h\}| \geq \begin{cases} \min\left(\frac{x_r - x_l}{2}, h - x_l\right) & \textit{if } x_l \leq h \leq \operatorname{plvsl}(K) \\ \min\left(\frac{x_r - x_l}{2}, x_r - h\right) & \textit{if } x_r \geq h \geq \operatorname{plvsl}(K). \end{cases}$$

Proof. It is enough to prove that

$$|K \cap \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 = h\}| \geq h - x_l$$

for

$$x_l < h < \min\left(\frac{x_l + x_r}{2}, \operatorname{plvsl}(K)\right)$$

because of the monotony of the function measuring the vertical slicing length on $[x_l, \operatorname{plvsl}(K)]$ by 2.6 and the symmetric situation in the second case.

We assume that there is a h_0 with

$$x_l < h_0 < \min\left(\frac{x_l + x_r}{2}, \operatorname{plvsl}(K)\right)$$

and

$$|K \cap \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 = h_0\}| < h_0 - x_l.$$

Now, we look at two supporting lines t_a, t_b - thought as graphs of two affine linear functions $x_1 \mapsto t_a(x_1), x_1 \mapsto t_b(x_1)$ - supporting K in $x_1 = h_0$ from above in the point $(h_0, y_a) := (h_0, t_a(h_0))$ and from below in $(h_0, y_b) := (h_0, t_b(h_0))$.

Since an integral shearing $U_s, s \in \mathbb{Z}$, in the direction of the x_2 axis, i.e., a unimodular transformation

$$U_s : \mathbb{R}^2 \to \mathbb{R}^2, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

does not change the length of the vertical line segments and $(1,0)$ as a lattice width direction, we can assume without loss of generality that

$$0 \leq y_a - t_a(h_0 - 1) < 1.$$

In particular, we can assume that t_a is increasing.

Now we can look at the two cases

$$y_b - t_b(h_0 - 1) \leq 0$$

and

$$0 < y_b - t_b(h_0 - 1) < 1,$$

because the case $y_b - t_b(h_0 - 1) \geq 1$ is impossible due to $y_a - t_a(h_0 - 1) < 1$ and the monotonically increasing vertical length on $[x_l, \operatorname{plvsl}(K)]$ by 2.6.

In the first case, we have $y_b - t_b(h_0-1) \leq 0 \leq y_a - t_a(h_0-1)$ and therefore t_b is decreasing and t_a is increasing, so we get for $\pi_2 \colon \mathbb{R}^2 \to \mathbb{R}, (x_1,x_2) \mapsto x_2$ that

$$\begin{aligned}\mathrm{width}_K((0,1)) =& \max \pi_2(K) - \min \pi_2(K)\\ \leq& \frac{x_r - x_l}{h_0 - x_l} \cdot |K \cap \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 = h_0\}|\\ <& x_r - x_l\\ =& \mathrm{width}_K((1,0)),\end{aligned}$$

so $(1,0)$ is not a width direction and we have a contradiction.

The second case, $0 < y_b - t_b(h_0-1) < 1$, is more complicated, because now t_b increases as well. However, in the second case we can also assume that

$$y_a - t_a(h_0-1) \leq 1 - (y_b - t_b(h_0-1)),$$

because if this is not the case, we can use the integral shearing U_{-1} defined above and a reflection on the x_1-axis to achieve this.

Thus, we obtain

$$\begin{aligned}&\mathrm{width}_K((0,1))\\ =& \max \pi_2(K) - \min \pi_2(K)\\ \leq& (x_r - h_0)(y_a - t_a(h_0-1)) + |K \cap \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 = h_0\}| +\\ &\qquad (h_0 - x_l)(y_b - t_b(h_0-1))\\ <& (x_r - h_0)(1 - (y_b - t_b(h_0-1))) + h_0 - x_l + (h_0 - x_l)(y_b - t_b(h_0-1))\\ =& x_r - x_l - (x_l + x_r)y_b - 2h_0 t_b(h_0-1) + (x_l + x_r)t_b(h_0-1) + 2h_0 y_b\\ =& x_r - x_l + (2h_0 - (x_l + x_r))(y_b - t_b(h_0-1))\\ \leq& x_r - x_l\\ =& \mathrm{width}_K((1,0)),\end{aligned}$$

where the last inequality follows from the fact that $2h_0 \leq x_l + x_r$ by definition of h_0.

Thus, $(1,0)$ is not a width direction in this case either, and we have a contradiction.

□

Corollary 2.13. *Let $K \subseteq \mathbb{R}^2$ be a planar convex body, $(1,0)$ a width direction of K, and $\pi_1(K) = [x_l, x_r]$.*
Then we have for the number of interior integral points on vertical line segments

$$\begin{aligned}&|\mathrm{int}(K) \cap \{(x_1,x_2) \in \mathbb{Z}^2 \mid x_1 = h\}|\\ \geq& \begin{cases} \left\lceil \min\left(\frac{x_r - x_l}{2}, h - x_l\right)\right\rceil - 1 & \text{if } x_l \leq h \leq \mathrm{plvsl}(K), h \in \mathbb{Z}\\ \left\lceil \min\left(\frac{x_r - x_l}{2}, x_r - h\right)\right\rceil - 1 & \text{if } x_r \geq h \geq \mathrm{plvsl}(K), h \in \mathbb{Z}.\end{cases}\end{aligned}$$

Proof. This follows directly from Theorem 2.12, since a line segment

$$K \cap \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 = h\}, h \in \mathbb{Z}$$

with length greater than $k \in \mathbb{Z}_{\geq 1}$ has at least k interior integral points. □

3. Area bound for planar convex bodies with respect to their lattice width data

In this section, we compute area bounds for convex bodies $K \subseteq \mathbb{R}^2$ with $k \geq 1$ interior integral points with respect to their lattice width data.

We use only elementary integration tools as methods. After choosing the coordinates appropriately, we approximate the area of the convex body using triangles and trapezoids to obtain our results.

The choice of coordinates was already done in 2.10, where we essentially noticed that we can assume that $(1,0)$ is a lattice width direction of K. Moreover, we can assume the situations in the propositions of this section without loss of generality by 2.10. For a suitable subdivision into triangles and trapezoids we use the vertical integral lines $x_1 = h \in \mathbb{Z}$, supporting lines and chords of K.

3.1. **Planar convex bodies with one integral vertical line.**

Proposition 3.1. *Let $(1,0)$ be a lattice width direction of K, $0 < a, b \leq 1$ with $\pi_1(K) = [1-a, 1+b]$, $\mathrm{plvsl}(K) \leq 1$ and $k := |\mathrm{int}(K) \cap \mathbb{Z}^2|$ the number of interior integral points. Then*

$$area(K) \leq \begin{cases} \frac{(a+b)^2}{2b}(k+1) & \text{if } a > b \\ (a+b)(k+1) \leq 2(k+1) & \text{if } a \leq b. \end{cases}$$

Furthermore, the bounds in the first inequality are sharp.

Proof. We take two supporting lines of K in $x_1 = 1$, one from above and one from below, and intersect the area between them with the strip $1-a \leq x_1 \leq 1+b$ to get a triangle or trapezoid T_K containing K. The dual vector $(1,0)$ is also a lattice width direction of T_K since $\mathrm{width}_K((1,0)) = \mathrm{width}_{T_K}((1,0))$ and $\mathrm{width}_K(v) \leq \mathrm{width}_{T_K}(v)$ for all $v \in (\mathbb{Z}^2)^* \setminus \{0\}$. We also have $\pi_1(T_K) = [1-a, 1+b]$, $|\mathrm{int}(T_K) \cap \mathbb{Z}^2| = k$ and $\mathrm{plvsl}(T_K) \leq 1$ by construction and so it is enough to look for sharp area bounds for triangles or trapezoids T_K.

The length of the line segment

$$T_K \cap \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 1\}$$

is less than or equal to $k+1$ due to the k interior integral points, so the area maximizers for T_K are due to $\mathrm{plvsl}(T_K) \leq 1$ triangles with base length $\frac{a+b}{b}(k+1)$ and height $a+b$ in the case $a > b$ or parallelograms with base length $k+1$ and height $a+b$ for $a \leq b$. Calculating their area yields the result. $\square$

Remark 3.2. In the Proposition, we not only gave sharp bounds, but we also explicitly described the area maximizers in the proof. In particular, we saw that for $\mathrm{lw}(K) \leq 1$, convex bodies with k interior integral points can be found with arbitrarily large area. However, we also found, that if the planar convex body has one integral vertical line and $\mathrm{lw}(K) > 1$, then the area is bounded by

$$\frac{\mathrm{lw}(K)^2}{2(\mathrm{lw}(K)-1)}(k+1).$$

3.2. Planar convex bodies with two integral vertical lines.

Proposition 3.3. *Let $(1,0)$ be a lattice width direction of K, $0 < a, b \leq 1$ with $\pi_1(K) = [1-a, 2+b]$ and $k_i := |\text{int}(K) \cap \{x_1 = i\} \cap \mathbb{Z}^2|$ for $i \in \{1,2\}$, $k := k_1 + k_2 > 0$. If $\frac{3}{2} \geq \text{plvsl}(K) > 1$, then*

$$area(K) \leq \begin{cases} \frac{(a+b+1)^2(a(k_1+1)+b(k_2+1))}{2(a+b)^2} & \text{if } a+b<1 \\ (a+1)(k_1+1)+b(1-k_1+2k_2) & \text{if } a+b \geq 1 \end{cases}$$
$$\leq \begin{cases} \frac{(a+b+1)^2}{2(a+b)}(k+1) & \text{if } a+b<1 \\ 2(k+1) & \text{if } a+b \geq 1. \end{cases}.$$

If $\text{plvsl}(K) \leq 1$, *then we have $l_1 \geq l_2$ for $l_i := |K \cap \{x_1 = i\}|, i \in \{1,2\}$ and*

$$area(K) \leq \left(1+a-\frac{b^2}{2}\right) l_1 + \left(\frac{b^2}{2}+b\right) l_2$$
$$\leq \begin{cases} \left(\frac{1}{2}+a\right) k_1 + \frac{3}{2}k_2 + 2 + a & \text{if } l_1 \leq 2l_2 \\ (1+a)k_1 + \frac{k_2}{2} + \frac{3}{2} + a & \text{if } l_1 > 2l_2 \text{ and } k_2 > 0 \\ 2(k_1+1) + \frac{1}{2k_1} - (1-a)(k_1-1) & \text{if } l_1 > 2l_2 \text{ and } k_2 = 0 \end{cases}$$
$$\leq \begin{cases} 2(k+1) & \text{if } k_2 > 0 \\ 2(k+1) + \frac{1}{2k} & \text{if } k_2 = 0. \end{cases}$$

Proof. In the case $\frac{3}{2} \geq \text{plvsl}(K) > 1, a+b < 1$, we can use the results for one integral width line from Proposition 3.1 twice, once for the part of K with $1-a \leq x_1 \leq 1+\frac{a}{a+b}$ and a second time for the part with $2 - \frac{b}{a+b} \leq x_1 \leq 2+b$. Since $\frac{a}{a+b} > a$, $\frac{b}{a+b} > b$ and $\frac{3}{2} \geq \text{plvsl}(K) > 1$ we get

$$\begin{aligned} \text{area}(K) &\overset{3.1}{\leq} \frac{\left(a+\frac{a}{a+b}\right)^2}{2a}(k_1+1) + \frac{\left(b+\frac{b}{a+b}\right)^2}{2b}(k_2+1) \\ &= \frac{(a+b+1)^2(a(k_1+1)+b(k_2+1))}{2(a+b)^2} \\ &\leq \frac{(a+b+1)^2}{2(a+b)}(k+1). \end{aligned}$$

In the case $\frac{3}{2} \geq \text{plvsl}(K) > 1, a+b \geq 1$ we can assume without loss of generality that $k_1 \geq 1$ and use Proposition 3.1 for the parts $1-a \leq x_1 \leq 2-b$ and $2-b \leq x_1 \leq 2+b$. Since $a \geq 1-b$ and $\text{plvsl}(K) > 1$ we get

$$\begin{aligned} \text{area}(K) &\overset{3.1}{\leq} (a-b+1)(k_1+1) + 2b(k_2+1) \\ &\leq (a+1)(k_1+1) + b(1-k_1+2k_2) \\ &\overset{(*)}{\leq} (a+1)(k_1+1) + 2bk_2 \\ &\leq 2(k+1), \end{aligned}$$

where we have used $k_1 \geq 1$ in $(*)$.

In the case $\operatorname{plvsl}(K) < 1$ we have $l_1 \geq l_2$ due to the monotonically decreasing vertical length on $[\operatorname{plvsl}(K), 2]$ by 2.6. We bound K by the union of two trapezoids, namely a first trapezoid bounded by $1 - a \leq x_1 \leq 2$, one supporting line in $x_1 = 1$ from above and one from below, and a second trapezoid bounded by $2 \leq x_1 \leq 2 + b$, a line through the points of K on $x_1 = 1$ and $x_1 = 2$ with maximal x_2 coordinate and a line through the points of K on $x_1 = 1$ and $x_1 = 2$ with minimal x_2 coordinate.

The first trapezoid has an area less than $(1 + a)l_1$ because l_1 is longer than the midsegment of the trapezoid. The second trapezoid has parallel sides of length l_2 and $l_2 - b(l_1 - l_2)$ and height b, so we get

$$\operatorname{area}(K) \leq (1 + a)l_1 + \frac{b}{2}\left(l_2 + (l_2 - b(l_1 - l_2))\right) = \left(1 + a - \frac{b^2}{2}\right) l_1 + \left(\frac{b^2}{2} + b\right) l_2.$$

If $l_1 \leq 2l_2$ we can enlarge the second trapezoid by taking $x_1 \leq 3$ instead of $x_1 \leq 2 + b$, i.e. use $b = 1$, and with $a \leq 1$ and $l_i \leq k_i + 1$ we get

$$\operatorname{area}(K) \leq \left(\frac{1}{2} + a\right) l_1 + \frac{3}{2} l_2 \leq \left(\frac{1}{2} + a\right) k_1 + \frac{3}{2} k_2 + 2 + a.$$

If $l_1 > 2l_2$ we can enlarge the second trapezoid to the triangle T bounded by $x_1 = 2$ and the lines through the extrema of K at $x_1 = 1$ and $x_1 = 2$. This triangle has base length l_2 and height $\frac{l_2}{l_1 - l_2}$, so with $l_1 > 2l_2$ we get

$$\operatorname{area}(T) = \frac{l_2^2}{2(l_1 - l_2)} < \frac{l_2}{2}$$

We can now use $l_i \leq k_i + 1$ to get

$$\operatorname{area}(K) \leq (1 + a)l_1 + \operatorname{area}(T) \leq (1 + a)l_1 + \frac{l_2}{2} \leq (1 + a)k_1 + \frac{k_2}{2} + \frac{3}{2} + a.$$

In the case $l_1 > 2l_2, k_2 = 0$ we can go even further. We can assume $l_1 > \frac{16}{9}$, because for $l_1 \leq \frac{16}{9}$ the given bound is correct since with $k \geq 1$ we have $k_1 \geq 1$ and

$$2l_1 + \frac{l_2}{2} \leq \frac{9}{4} l_1 \leq 4 \leq 2(k_1 + 1).$$

Thus, assume $l_1 > \frac{16}{9}$. Since $k_2 = 0$ we also have $l_2 \leq 1$ and so the triangle T has an area of

$$\operatorname{area}(T) = \frac{l_2^2}{2(l_1 - l_2)} \leq \frac{1}{2(l_1 - 1)}.$$

The function $x \mapsto x + \frac{1}{x}$ is increasing on $[1, \infty)$. Since $l_1 > \frac{16}{9}$ we have $2(l_1 - 1) > 1$ and so $2(l_1 - 1) + \frac{1}{2(l_1 - 1)} \leq 2k_1 + \frac{1}{2k_1}$. Thus, we get with $k_1 - 1 \leq l_1 \leq k_1 + 1$ the bound

$$\operatorname{area}(K) \leq (1 + a)l_1 + \frac{1}{2(l_1 - 1)} \leq 2(k_1 + 1) + \frac{1}{2k_1} - (1 - a)(k_1 - 1).$$

The last inequality in the proposition follows directly in all cases except in the case $k_1 = 0, k_2 = 1$. But then $1 \geq l_1 \geq l_2$ and we get

$$\operatorname{area}(K) \leq 2l_1 + \frac{3}{2} l_2 \leq 4 = 2(k + 1).$$

□

Remark 3.4. If we look in the case $\operatorname{plvsl}(K) \leq 1$ at those convex bodies with $a+b<1$ or equivalently $b<1-a$, we get

$$\begin{aligned}\operatorname{area}(K) &\leq (1+a)\, l_1 - \frac{b^2}{2}(l_1 - l_2) + bl_2 \\ &\leq (1+a)\, l_1 + (1-a)l_2 \\ &\leq 2l_1 \\ &\leq 2(k+1),\end{aligned}$$

Together with the case $\operatorname{plvsl}(K) > 1, a+b < 1$ and 3.2 we get that for $\operatorname{lw}(K) \in (1,2]$ we have a sharp area bound and can even describe area maximizers.

3.3. **Planar convex bodies with at least three integral vertical lines.**

Proposition 3.5. *Let be $m \in \mathbb{Z}_{\geq 3}$, $(1,0)$ a lattice width direction of K, $0 < a, b \leq 1$ with $\pi_1(K) = [1-a, m+b]$, $\operatorname{plvsl}(K) \leq \frac{m+1}{2}$, $l_i := |K \cap \{x_1 = i\}|$ and $k_i := |\operatorname{int}(K) \cap \{x_1 = i\} \cap \mathbb{Z}^2|$. Then*

$$\operatorname{area}(K) \leq \begin{cases} 2k+2 - \left\lfloor \frac{m-5}{2} \right\rfloor \cdot \left\lfloor \frac{m-3}{2} \right\rfloor & \textit{if } k_m \neq 0 \textit{ or } m \geq 5 \\ 2k+2+\frac{1}{2} & \textit{else.} \end{cases}$$

Proof. Let be $i_{\max} := \lceil \operatorname{plvsl}(K) \rceil$. Then $1 \leq i_{\max} \leq \lceil \frac{m+1}{2} \rceil < m$ and by definition of $\operatorname{plvsl}(K)$ for $i \in \mathbb{Z}$ we have

$$\operatorname{area}(K \cap \{(x_1,x_2) \in \mathbb{R}^2 \mid i \leq x_1 \leq i+1\}) \leq \begin{cases} l_{i+1} & \textit{if } i+1 < i_{\max} \\ l_i & \textit{if } i \geq i_{\max} \end{cases}$$

and with the trapezoid defined by $i_{\max} - 1 \leq x_1 \leq i_{\max} + 1$ and supporting lines in $x_1 = i_{\max}$ we get

$$\operatorname{area}(K \cap \{(x_1,x_2) \in \mathbb{R}^2 \mid i_{\max} - 1 \leq x_1 \leq i_{\max} + 1\}) \leq 2l_{i_{\max}}.$$

If $k_1 k_m \neq 0$, then we get from these bounds with $l_i \leq k_i + 1$

$$\begin{aligned}\operatorname{area}(K) &\leq 2l_{i_{\max}} + \sum_{i=1, i \neq i_{\max}}^{m} l_i \\ &\leq 2k_{i_{\max}} + 2 + \sum_{i=1, i\neq i_{\max}}^{m} (k_i + 1) \\ &\leq k + k_{i_{\max}} + m + 1 \\ &\overset{(*)}{\leq} 2k + 2 - 2 \cdot \sum_{i=1}^{\lfloor (m-5)/2 \rfloor} i \\ &\leq 2k + 2 - \left\lfloor \frac{m-5}{2} \right\rfloor \cdot \left\lfloor \frac{m-3}{2} \right\rfloor,\end{aligned}$$

where we used in $(*)$ that we can count the interior integral points of K by counting the points of $\operatorname{int}(K)$ on $x_1 = i_{\max}$, count at least one point on the other $m-1$

integral vertical lines (remember $k_1k_m \neq 0$!) and also some more points on these lines according to 2.13 (at least one point more for each line till $\operatorname{plvsl}(K)$). So we get

$$k_{i_{\max}} + (m-1) + 2 \cdot \sum_{i=1}^{\lfloor (m-5)/2 \rfloor} i \leq k.$$

For the above calculation, we need $k_1k_m \neq 0$ to have at least one point on each integral width line. But we will now see that everything still works fine if we only have $k_m \neq 0, k_1 = 0$, because we can avoid the $k_1 + 1$ term of

$$\operatorname{area}(K \cap \{(x_1, x_2) \in \mathbb{R}^2 \mid 1 - a \leq x_1 \leq 1\})$$

with the following argument divided in cases with respect to $\operatorname{plvsl}(K)$.

If $\operatorname{plvsl}(K) \in (1, 2]$, then we have $i_{\max} = 2$ and with $k_2 \geq 1$ by 2.13 we get

$$\begin{aligned}
&\operatorname{area}(K \cap \{(x_1, x_2) \in \mathbb{R}^2 \mid 1 - a \leq x_1 \leq 3\}) \\
\leq &\operatorname{area}(K \cap \{(x_1, x_2) \in \mathbb{R}^2 \mid 1 - a \leq x_1 \leq 1 + a\}) + \\
&\operatorname{area}(K \cap \{(x_1, x_2) \in \mathbb{R}^2 \mid 1 + a \leq x_1 \leq 3\}) \\
\leq &2a + (2 - a)l_2 \leq 2a + (2 - a)(k_2 + 1) \leq 2(k_{i_{\max}} + 1)
\end{aligned}$$

and if $\operatorname{plvsl}(K) > 2$ we get with $k_2 \geq 1$

$$\begin{aligned}
&\operatorname{area}(K \cap \{(x_1, x_2) \in \mathbb{R}^2 \mid 1 - a \leq x_1 \leq 2\}) \\
\leq &\operatorname{area}(K \cap \{(x_1, x_2) \in \mathbb{R}^2 \mid 1 - a \leq x_1 \leq 1 + a\}) + \\
&\operatorname{area}(K \cap \{(x_1, x_2) \in \mathbb{R}^2 \mid 1 + a \leq x_1 \leq 2\}) \\
\leq &2a + (1 - a)l_2 \leq 2a + (1 - a)(k_2 + 1) \leq k_2 + 1
\end{aligned}$$

and the case $\operatorname{plvsl}(K) \leq 1$ is not possible, because of $k_1 = 0$ and 2.13.

We can use similar arguments as in the case $\operatorname{plvsl}(K) > 2$ above to avoid the $k_m + 1$ term in the case $k_m = 0$ when $m \geq 5$, because then we get

$$\operatorname{area}(K \cap \{(x_1, x_2) \in \mathbb{R}^2 \mid m - 1 \leq x_1 \leq m + b\}) \leq k_{m-1} + 1.$$

But if $m < 5$, we cannot use this argument, because then $i_{\max} - 1 \leq x_1 \leq i_{\max} + 1$ and $m - 1 \leq x_1$ can have non-empty intersection. So if $k_m = 0$ and $m < 5$, then in the cases $m = 3, i_{\max} = 2$ or $m = 4, i_{\max} = 3$ we have to use the last inequality of 3.3 to get

$$\operatorname{area}(K \cap \{(x_1, x_2) \in \mathbb{R}^2 \mid m - 2 \leq x_1 \leq m + b\}) \leq 2k_{i_{\max}} + 2 + \frac{1}{2},$$

i.e. we get an additional $+\frac{1}{2}$ compared to the calculations above. □

Proof of Theorem 1.2. Combining the Propositions 3.1, 3.3, 3.5 and the Remarks 3.2, 3.4 we get the result. □

3.4. Consequences for area bounds of some specific planar convex bodies.

Corollary 3.6. *Let $K \subseteq \mathbb{R}^2$ be a convex body with* $\operatorname{lw}(K) > 5$ *or with symmetric lattice width data, i.e., there is an odd number n, $0 < a \leq 1$ and a lattice width direction w with* $\operatorname{lwd}_w(K) = ([1-a, n+a], \frac{n+1}{2})$. *Then* $\operatorname{area}(K) \leq 2(k+1)$.

Proof. If $\operatorname{lw}(K) > 5$, then we have at least 5 integral width lines and the result follows with 3.5.

For symmetric lattice width data, we also have an odd number of integral width lines. Therefore, the claim follows with 3.1 for one integral vertical line and with 3.5 for 5 and more integral vertical lines. For three integral vertical lines and symmetric lattice width data we get $\operatorname{plvsl}(K) = 2$, so we can bound $K \cap \{1 \leq x_1 \leq 3\}$ not only by a trapezoid but even by a parallelogram due to 2.9. Thus, we can avoid an additional $+\frac{1}{2}$ similar to the proof of 3.5. □

Definition 3.7. Let $P \subseteq \mathbb{R}^2$ be a polygon, whose vertices have rational coordinates. Then we call P a *rational polygon* and the *denominator of* P is the smallest $l \in \mathbb{Z}_{\geq 1}$, so that lP has integral vertices.

Corollary 3.8. *Let $P \subseteq \mathbb{R}^2$ be a rational polygon with denominator $l \in \mathbb{Z}_{\geq 2}$ and $k > 1$ interior integral points. Then we have the sharp bound*

$$\operatorname{area}(P) \leq \operatorname{area}(T_{k,l}) = \frac{(l+1)^2}{2l}(k+1).$$

Proof. This is the bound from 3.1, and all the other bounds that we have proven are smaller. □

Remark 3.9. In particular, Corollary 3.8 shows that the triangles $T_{k,l}$ from the introduction and already described in [LZ91, 2.6.] are examples of area-maximizing rational polygons with k interior integral points and denominator l. The other area maximizers for $l > 2$ can be constructed from $T_{k,l}$ by a shearing in the direction of the x_2 axis. For $l = 2, k = 1$ there could be even more maximizers, since the bound in 3.8 is in this case $\frac{9}{2} = 2(k+1) + \frac{1}{2}$.

Note also that 3.8 is incorrect for $l = 1, k = 1$, because for the standard triangle $\Delta_2 := \operatorname{convhull}((0,0),(1,0),(0,1))$ we have $\operatorname{area}(3 \cdot \Delta_2) = \frac{9}{2} > 4$.

Remark 3.10. Rational polygons with one interior integral point are also important in toric geometry. For IP-polygons, i.e., lattice polygons with 0 as an interior point, and LDP-polygons, i.e., IP-polygons with primitive lattice points as vertices, we have the *order o_p* of such a polygon P as

$$o_P := \min\{k \in \mathbb{Z}_{\geq 1} \mid \operatorname{int}(P/k) \cap \mathbb{Z}^2 = \{(0,0)\}\}.$$

In [KKN10, 4.6.] there is the open question whether there is a bound on the area of IP-polygons that is cubic in o_P. We get the positive answer from 3.8, that the sharp bound is given by

$$\operatorname{area}(P) \leq o_P(o_P+1)^2.$$

This also implies better bounds relative to the maximal local index of the polygon than the bounds given in [KKN10, 4.5.].

4. Area bounds for lattice polygons

The methods, we used for planar convex bodies, work even better for lattice polygons. In particular the following lemma leads directly to the classification of hollow lattice polygons (e.g. done in [Koe91, 4.1.2]) or to a direct simple proof of Scott's area bound, which also gives an interesting proof of Pick's theorem en passant.

On the other hand we can use results from algebraic geometry via the language of toric geometry to obtain area bounds for lattice polygons. This approach provides an estimate for the error term in Scott's area bound on this way, implying the correct asymptotic behavior in Theorem 1.1.

Definition 4.1. The *standard lattice triangle* is given as

$$\Delta_2 := \operatorname{convhull}((0,0),(1,0),(0,1)).$$

Remark 4.2. Lattice polygons that are affine unimodular equivalent to integer multiples of Δ_2 are often special cases in the results of this section. This starts already with the following lemma.

Lemma 4.3. *Let $P \subseteq \mathbb{R}^2$ be a lattice polygon with lattice width direction $(1,0)$ and lattice width $\operatorname{lw}(P) \geq 2$ which is not affine unimodular equivalent to an integer multiple of the standard lattice triangle Δ_2. Then every vertical integral line $x_1 = i \in \mathbb{Z}$, which has non-empty intersection with $\operatorname{int}(P)$, contains at least one interior lattice point of P.*

Proof. By 2.13 we know, that the intersection of a vertical integral line with the interior of the lattice polygon P is either empty or has a length greater than or equal to 1, where the length 1 is only possible for those vertical integral lines with minimal or maximal x_1 coordinate among them with non-empty intersection. But if, without loss of generality, the integral vertical line with minimal x_1 coordinate among them with non-empty intersection has length 1 and contains no interior lattice point of P, then we have after a translation and a shearing in the direction of the x_2-axis (both of them do not change the lattice width direction $(1,0)$) the points $(0,0),(1,0)$ as boundary points at the bottom and $(0,1)$ as boundary point at the top of the lattice polygon. Therefore, we get an integer multiple of Δ_2 because $(1,0)$ is a lattice width direction, which contradicts the assumptions. □

Corollary 4.4. *Let $P \subseteq \mathbb{R}^2$ be a lattice polygon.*
Then $|\operatorname{int}(P) \cap \mathbb{Z}^2| = 0$ if and only if P has $\operatorname{lw}(P) = 1$ or $P \cong 2 \cdot \Delta_2$.

Proof. From Lemma 4.3 we have that P has $\operatorname{lw}(P) = 1$ or P is unimodular equivalent to an integer multiple of Δ_2. All $m \cdot \Delta_2$ with $2 \neq m \in \mathbb{Z}_{\geq 1}$ have interior lattice points or lattice width 1, so we get $m = 2$, if $\operatorname{lw}(P) \neq 1$. □

Theorem 4.5 ([Sco76])**.** *Let $P \subseteq \mathbb{R}^2$ be a lattice polygon with $k \geq 1$ interior integral points. Then*

$$\operatorname{area}(P) \leq 2(k+1) + \frac{1}{2}$$

and equality holds if and only if $P \cong 3 \cdot \Delta_2$.

Proof. We chose *lattice width coordinates* introduced in 2.10, i.e. we can assume without loss of generality, that $P \subseteq \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq \mathrm{lw}(P)\}$. We also define $l_i := |P \cap \{x_1 = i\}|, k_i := |\mathrm{int}(P) \cap \{x_1 = i\} \cap \mathbb{Z}^2|$ and as length of line segments

$$\begin{aligned} r_{i,a} :=& |\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = i, \\ & \lfloor \max\{j \in \mathbb{R} : (i, j) \in P\} \rfloor \leq x_2 \leq \max\{j \in \mathbb{R} : (i, j) \in P\}\}|, \\ r_{i,b} :=& |\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = i, \\ & \lceil \min\{j \in \mathbb{R} : (i, j) \in P\} \rceil \geq x_2 \geq \min\{j \in \mathbb{R} : (i, j) \in P\}\}|. \end{aligned}$$

for all $i \in [0, \mathrm{lw}(P)] \cap \mathbb{Z}$.

We start with a triangulation of P in special (non-lattice!) triangles with one of the segments $P \cap \{x_1 = i\}$ as base, height 1 and another edge at the boundary of P. Using the triangulation in the first step, the definition of the $r_{i,a}, r_{i,b}, k_i$ in the second and $0 \leq r_{i,a}, r_{i,b} \leq 1$ in the third, we get

$$\begin{aligned} \mathrm{area}(P) =& \frac{l_0}{2} + \frac{l_{\mathrm{lw}(P)}}{2} + 2 \cdot \sum_{i=1}^{\mathrm{lw}(P)-1} \frac{l_i}{2} \\ =& \frac{l_0}{2} + \frac{l_{\mathrm{lw}(P)}}{2} + \sum_{i=1}^{\mathrm{lw}(P)-1} (k_i - 1 + r_{i,a} + r_{i,b}) \\ =& \frac{l_0}{2} + \frac{l_{\mathrm{lw}(P)}}{2} + k - (\mathrm{lw}(P) - 1) + \sum_{i=1}^{\mathrm{lw}(P)-1} (r_{i,a} + r_{i,b}) \\ \leq& \frac{l_0}{2} + \frac{l_{\mathrm{lw}(P)}}{2} + k - (\mathrm{lw}(P) - 1) + 2(\mathrm{lw}(P) - 1) \\ =& \frac{l_0}{2} + \frac{l_{\mathrm{lw}(P)}}{2} + k + \mathrm{lw}(P) - 1. \end{aligned}$$

Since P is a lattice polygon, its longest possible slicing length is equal to one of the l_i, which we call $l_{i_{\max}}$, and this is longer than the midsegment of the trapezoid

$$\mathrm{convhull}(P \cap \{x_1 = 0\}, P \cap \{x_1 = \mathrm{lw}(P)\}) \subseteq P,$$

which has the length $\frac{l_0 + l_{\mathrm{lw}(P)}}{2}$. If P is not affine unimodular equivalent to an integer multiple of Δ_2, then $k_i \geq 1$ for all $1 \leq i \leq \mathrm{lw}(P) - 1$ by 4.3 and in this case we get

$$area(P) \leq \frac{l_0}{2} + \frac{l_{\mathrm{lw}(P)}}{2} + k + \mathrm{lw}(P) - 1 \leq k_{i_{\max}} + 1 + k + 1 + \sum_{i=1, i \neq i_{\max}}^{\mathrm{lw}(P)-1} k_i = 2k + 2.$$

If P is unimodular equivalent to $\mathrm{lw}(P) \cdot \Delta_2$, then we have

$$\mathrm{area}(P) = \frac{\mathrm{lw}(P)^2}{2}, \quad |\mathrm{int}(P) \cap \mathbb{Z}^2| = \sum_{i=1}^{\mathrm{lw}(P)-2} i = \frac{(\mathrm{lw}(P) - 1)(\mathrm{lw}(P) - 2)}{2}.$$

So we have $\mathrm{area}(3 \cdot \Delta_2) = 2(k+1) + \frac{1}{2}$ and $\mathrm{area}(P) = 2k - \frac{\mathrm{lw}(P)^2}{2} + 3\mathrm{lw}(P) - 2 \leq 2(k+1)$ for $\mathrm{lw}(P) \geq 4$. $\square$

Remark 4.6. We can also get the result in 4.5 as a corollary of the area bounds in Proposition 3.1, 3.3 and 3.5.

Remark 4.7. If we look at the details of our proof of 4.5, then we see that we have also almost given a proof of *Pick's formula*

$$\operatorname{area}(P) = k + \frac{b}{2} - 1$$

from [Pic99, §. I.] for a lattice polygon P with k interior lattice points and b boundary lattice points. This is because $r_{i,a} = 1$ or $r_{i,b} = 1$ if and only if we have a boundary lattice point of P on $x_1 = i$ and the other $r_{i,a}, r_{i,b}$ add up to 1 in pairs, because $\mathbb{Z}^2$ is centrally symmetric to the center of any line segment between two lattice points. If we use b_i for the number of boundary lattice points of P at $\{x_1 = i\}$, b_a for their number at the upper boundary, b_b for their number at the lower boundary below and b for the number of all boundary lattice points of P, then we get Pick's formula via

$$\begin{aligned}
area(P) =& \frac{l_0}{2} + \frac{l_{\mathrm{lw}(P)}}{2} + k - (\mathrm{lw}(P) - 1) + \sum_{i=1}^{\mathrm{lw}(P)-1} (r_{i,a} + r_{i,b}) \\
=& k + \frac{b_0 - 1 + b_{l+1} - 1}{2} - (\mathrm{lw}(P) - 1) + \sum_{i=1}^{\mathrm{lw}(P)-1} (r_{i,a} + r_{i,b}) \\
=& k + \frac{b_0 + b_{l+1}}{2} - \mathrm{lw}(P) + b_a + b_b + \sum_{i=1, r_{i,a}<1}^{\mathrm{lw}(P)-1} r_{i,a} + \sum_{i=1, r_{i,b}<1}^{\mathrm{lw}(P)-1} r_{i,b} \\
=& k + \frac{b_0 + b_{l+1}}{2} - \mathrm{lw}(P) + b_a + b_b + \frac{\mathrm{lw}(P) - 1 - b_a}{2} + \frac{\mathrm{lw}(P) - 1 - b_b}{2} \\
=& k + \frac{b}{2} - 1.
\end{aligned}$$

We can even do this without choosing width coordinates first, so we just need our *integration method*. Note again that we have triangulated in non-lattice (!) triangles, as opposed to the triangulations in the classical proofs of Pick's theorem (see e.g. [AZ18, 13.]).

Remark 4.8. For lattice polygons, we can also use results from algebraic geometry via the language of toric varieties. Max Noether's formula

$$\chi(\mathcal{O}_X) = \frac{K_X \cdot K_X + e(X)}{12},$$

which is the topological part in the surface case of the Hirzebruch-Riemann-Roch theorem, implies two combinatorial 'number 12'-theorems in the toric setting, which were essentially already part of the classical textbooks of toric geometry. These combinatorial 'number 12'-theorems lead to a formula for the area of a lattice polygon, which not only implies Coelman's conjecture but also gives a description of the error term. Parts of this story were already mentioned in [Cas12], but the explicit area formula was not written down there.

Lemma 4.9. *Let $P \subseteq \mathbb{R}^2$ be a lattice polygon, $F(P) := \operatorname{convhull}((int(P) \cap \mathbb{Z}^2)$ the convex hull of its interior lattice points.*
If $\dim(F(P)) = 0$, then we have for the numbers of boundary lattice points $b(P)$ and $b(P^*)$ of P and the dual polygon $P^* = \{v \in (\mathbb{R}^2)^* \mid v(x) \geq -1 \ \forall x \in P\}$

$$b(P) + b(P^*) = 12.$$

If $\dim(F(P)) = 2$, then we have for the lattice points on the boundary

$$b(P) - b(F(P)) = 12 - n(\Delta_{P,smooth}),$$

where $n(\Delta_{P,smooth})$ is the number of rays in the smooth refinement of the normal fan Δ_P of P that we get from the rays generated by the boundary lattice points on the bounded edges of

$$\operatorname{convhull}(\sigma \cap (\mathbb{Z}^2 \setminus \{0\}))$$

for each cone σ of Δ_P.

Proof. Both formulas are consequences of Noether's formula in the case of smooth complete toric surfaces. For the first formula see [PRV00], for the second formula see [Koe91, 4.5.2]. For the background of the formulas in the classical textbooks see [Oda78, p. 57 f.], [Oda88, p. 45 f.], [Ful93, p.44] and [CLS11, Theorem 10.5.10.]. □

Proof of Theorem 1.3. If $\dim(F(P)) = 0$, then we have $\operatorname{area}(F(P)) = 0$ and also $n(\Delta_{P,smooth}) = b(P^*)$. So we get with Pick's formula, $k = 1$ and the result from the first formula in 4.9

$$\begin{aligned}\operatorname{area}(P) =& k + \frac{b}{2} - 1 = 1 + \frac{12 - n(\Delta_{P,smooth})}{2} - 1\\ =& \underbrace{2(k+1) + 2}_{=6} - \frac{n(\Delta_{P,smooth})}{2} - \underbrace{\operatorname{area}(F(P))}_{=0}.\end{aligned}$$

If $\dim(F(P)) = 1$, then we also have $\operatorname{area}(F(P)) = 0$. The formula is correct for a rectangle $\operatorname{convhull}((0,-1),(0,1),(n,-1),(n,1))$ for any $n \in \mathbb{Z}_{\geq 3}$. By [Koe91, 4.3] we can obtain a lattice polygon affine unimodular equivalent to P by intersecting such a rectangle with up to four half-planes, whose normal vectors have ± 1 or ± 2 as first coordinate. We see that the equation remains correct for each intersection, because the loss of area is compensated by the loss of interior integral points and the change in $n(\Delta_{P,smooth})$.

If $\dim(F(P)) = 2$ we have from Pick's theorem for the number of boundary and interior lattice points $b(P), k(P)$ and $b(F(P)), k(F(P))$ of P and $F(P)$ due to $k(P) = k(F(P)) + b(F(P))$

$$\begin{aligned}\operatorname{area}(P) + \operatorname{area}(F(P)) =& k(P) + \frac{b(P)}{2} - 1 + k(F(P)) + \frac{b(F(P))}{2} - 1\\ =& 2k(P) + \frac{b(P) - b(F(P))}{2} - 2.\end{aligned}$$

Using the second formula from 4.9, we get the result. □

Let n be the number of vertices of the lattice polygon. We can use Theorem 1.3 to give a proof of Coleman's conjecture $\text{area}(P) \leq 2(k+1) + 2 - \frac{n}{2}$ from [Col78], because $n(\Delta_{P,smooth})$ is the number of rays in a refinement of the normal fan that has n vertices, and so we get $n_{\Delta_{P,smooth}} \geq n$. We also get the additional correction term $\text{area}(F(P))$. To have the correction term depending on the lattice width, we can use the classical result $\text{area}(P) \geq \frac{3}{8}\text{lw}(P)^2$ from [FTM75] for the polygon $F(P)$ together with $\text{lw}(F(P)) = \text{lw}(P) - 2$ from [LS11, Theorem 13] for polygons not affine unimodular equivalent to a multiple of Δ_2. So we have the following corollary.

Corollary 4.10. *Let $P \subseteq \mathbb{R}^2$ be a lattice polygon with $\text{lw}(P) \geq 3$ and not affine unimodular equivalent to an integer multiple of Δ_2.*
Then we have

$$\text{area}(P) \leq 2k + 4 - \frac{n}{2} - \frac{3}{8}(\text{lw}(P) - 2)^2.$$

Remark 4.11. Because for integer multiples of Δ_2 we have

$$\text{area}(P) = 2k - \frac{\text{lw}(P)^2}{2} + 3\text{lw}(P) - 2,$$

the corollary holds for all lattice polygons with $\text{lw}(P) \geq 10$.

For large $\text{lw}(P)$ the inequality is also sharp up to a constant less than or equal to 3, since the inequality $\text{area}(P) \geq \frac{3}{8}\text{lw}(P)^2$ is sharp for integer multiples of $Q := \text{convhull}((1,0),(0,1),(-1,1))$, we have $F(mQ) = (m-1)Q$ for $m \in \mathbb{Z}_{\geq 1}$ and $n(\Delta_{Q,smooth}) = 9$.

Proof of Theorem 1.1. If $\text{lw}(K) > 5$ we have at least five integral vertical lines and thus the area bound follows from 3.5 and 3.6.

For the asymptotic behavior of the correction term f in $\text{lw}(K)$ to the classical bound $2(k+1)$, we must have

$$\limsup_{x\to\infty} \frac{f(x)}{x^2} \leq \frac{3}{8},$$

because of the example mQ in 4.11. □

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4. Quasi-period collapse in half-integral polygons

The following manuscript is consistent with the preprint published on arXiv at arXiv:2405.13404v2.

QUASI-PERIOD COLLAPSE IN HALF-INTEGRAL POLYGONS

MARTIN BOHNERT

ABSTRACT. A half-integral polygon with quasi-period collapse behaves similarly to a lattice polygon in the sense that the number of lattice points in its integer dilates can be calculated as values of a polynomial, its Ehrhart polynomial. As a main result, we classify the Ehrhart polynomials of all half-integral non-lattice polygons with quasi-period collapse. In particular, we obtain that for any positive integer i, the polynomial $\frac{4i+5}{2}t^2 + \frac{2i+7}{2}t + 1 \in \mathbb{Q}[t]$ is an Ehrhart polynomial of a rational polygon, which was an open question for $i > 1$.

We also study some extreme cases in detail. In particular, we show that up to affine unimodular equivalence there exist exactly 30 half-integral non-lattice polygons with quasi-periodic collapse with exactly one interior lattice point, which are the dual polygons of the 30 LDP polygons of Gorenstein index 2. Furthermore, we classify all half-integral polygons with quasi-period collapse with at most 6 interior lattice points or with $i \geq 1$ interior lattice points and the maximum possible number $2i + 7$ of boundary lattice points.

1. INTRODUCTION

Rational non-lattice polytopes, whose number of lattice points in their integer dilates can be calculated as the values of a polynomial as for lattice polytopes, first appeared as random single examples, e.g., the half-integral polytope with the vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,0)$ and $(\frac{1}{2},0,\frac{1}{2})$ in Stanley's *Enumerative Combinatorics* [Sta12, 4.6.10]. In [DLMA04] some Gelfand-Tsetlin polytopes, a class of polytopes relevant to representation theory, are seen as the first infinite family of such *pseudo-integral* non-lattice polytopes. Systematic constructions in [MAW05] led to examples for any given denominator of the rational polytope in any dimension greater than 1 and to a better understanding of the 2 dimensional case. An explanation of the more general phenomena of quasi-period collapse by certain piecewise affine unimodular transformations between the rational polytope and a lattice polytope was proposed in [HMA08]. For the special case of duals of LDP polygons, [KW18] explained quasi-period collapse with methods of algebraic geometry motivated by mirror symmetry. The classification of Ehrhart quasi-polynomials of half-integral polygons was started in [Her10], and in [MAM17] the classification of Ehrhart polynomials of pseudo-integral polygons was started, leaving open whether there are Ehrhart polynomials of the form

$$\frac{4i+k-2}{2}t^2 + \frac{2i+k}{2}t + 1 \in \mathbb{Q}[t]$$

for $i \geq 1, k \geq 7$ and $(i,k) \neq (1,7)$. Our main theorem will show in particular that there are half-integral polygons with quasi-period collapse and such an Ehrhart polynomial with $i > 1, k = 7$.

Why are we focusing on half-integral polygons here? In a sense, they lead to the simplest polytopes with quasi-period collapse, since according to [MAW05, 2.1.] there is no quasi-period collapse in dimension 1. Their denominator is only two, so half-integral polygons with quasi-period collapse are automatically pseudo-integral. Additionally, we will have an easy criterion to detect quasi-periodic collapse for them. We should emphasize that half-integral polygons are relevant in several areas of combinatorics. For example, they naturally arise when studying lattice 3-polytopes of lattice width 2, because then we have a half-integral polygon as a natural 'mid-polygon'. This happens, for instance, when classifying maximal lattice 3-polytopes without interior lattice points in [AKW17] or when studying the Fine interior of lattice 3-polytopes of lattice width 2 in [Boh24b].

We will now begin to introduce our notation and provide the necessary background of Ehrhart theory for our topic. For more information on Ehrhart theory, see [Sta12], [BR15], or [HNP12].

Let be $n \in \mathbb{Z}_{\geq 1}$ and $P \subseteq \mathbb{R}^n$ be a full-dimensional *rational polytope*, i.e., the convex hull of finitely many points of $\mathbb{Q}^n \subseteq \mathbb{R}^n$, whose affine hull is $\mathbb{R}^n$. We denote convex hulls by $\operatorname{convhull}(\cdot, \ldots, \cdot)$. If P can even be described as the convex hull of finitely many points of the lattice $\mathbb{Z}^n \subseteq \mathbb{R}^n$, then we call P a *lattice polytope*. For each rational polygon $P \subseteq \mathbb{R}^n$, the smallest $d \in \mathbb{Z}_{\geq 1}$ such that $dP := \{d \cdot x \in \mathbb{R}^n \mid x \in P\}$ is a lattice polytope is called the *denomiator of* P and is denoted by $\operatorname{denom}(P)$. We are particularly interested in the case of polygons, i.e. $n = 2$, and especially in the case $n = 2, \operatorname{denom}(P) = 2$, where we call P a *half-integral polygon*.

To study the interaction between the rational polytope $P \subseteq \mathbb{R}^n$ and the lattice $\mathbb{Z}^n$, we are interested in the *number of lattice points* $l(P) := |P \cap \mathbb{Z}^n|$, the *number of interior lattice points* $i(P) := |\operatorname{int}(P) \cap \mathbb{Z}^n|$ and the *number of boundary lattice points* $b(P) := |\partial P \cap \mathbb{Z}^n|$ of P, where $\operatorname{int}(P)$ is the topological interior and ∂P is the topological boundary of $P \subseteq \mathbb{R}^n$. If we denote a polygon by $P_{d,(i,b)}$, then it is a rational polygon with denominator d, i interior and b boundary lattice points. We study rational polygons only up to *affine unimodular equivalence*, i.e. up to automorphisms of our lattice. Thus, two rational polygons $P \subseteq \mathbb{R}^n$ and $P' \subseteq \mathbb{R}^n$ are considered equivalent if and only if there exists an *affine unimodular transformation*

$$T_{A,b} \colon \mathbb{R}^n \to \mathbb{R}^n, x \mapsto Ax + b, \qquad A \in \mathbf{GL}(n, \mathbb{Z}), b \in \mathbb{Z}^n$$

with $T_{A,b}(P) = P'$. The numbers $l(P), i(P)$ and $b(P)$ are invariant under such affine unimodular transformations. Since $|\det(A)| = 1$, the volume of P, denoted by $\operatorname{vol}(P)$ or $\operatorname{area}(P)$ for $n = 2$, is also invariant.

Ehrhart theory gives a connection between these invariants via a *quasi-polynomial* of degree n. A quasi-polynomial of degree n is a generalized polynomial

$$q(t) = \sum_{k=0}^{n} q_k(t) t^k,$$

where the coefficient functions $q_k(t)$ are periodic functions with integral periods, rational values on integers, and $q_n \neq 0$. The least common multiple of the periods of $q_0, \ldots, q_n$ is called the *quasi-period* of q. A quasi-polynomial is just a polynomial if its quasi-period is 1. The main result of Ehrhart theory is as follows:

Theorem 1.1 ([Ehr62], [BR15, 3.23]). *Let $P \subseteq \mathbb{R}^n$ be a rational polytope. Then there exists a quasi-polynomial of degree n called the Ehrhart quasi-polynomial of P*

$$\mathrm{ehr}_P(t) = \sum_{k=0}^{n} e_k(t) t^k,$$

with quasi-period dividing $\mathrm{denom}(P)$*,* $e_n(t) = \mathrm{vol}(P)$ *and* $\mathrm{ehr}_P(k) = l(kP)$ *for all positive integers k.*

We call the vector with the periods of the coefficient functions $e_0, \ldots, e_n$ of $\mathrm{ehr}_P(t)$ as coefficients the *period sequence of P* and the quasi-period of $\mathrm{ehr}_P(t)$ the *quasi-period of P*. Furthermore, we say that P has *quasi-period collapse* if its quasi-period is strictly less than $\mathrm{denom}(P)$. If the quasi-period of P is 1, i.e., if the Ehrhart quasi-polynomial is actually a polynomial, then we call P a *pseudo-integral polytope*. In particular, the half-integral polygons with quasi-period collapse are exactly the pseudo-integral ones.

An important property of the Ehrhart quasi-polynomial is the following *Ehrhart-Macdonald reciprocity*, which connects $\mathrm{ehr}_P(k)$ and $i(kP)$.

Theorem 1.2 ([Mac71, (4.6).], [BR15, 4.1]). *Let $P \subseteq \mathbb{R}^n$ be a rational polygon with the Ehrhart quasi-polynomial* $\mathrm{ehr}_P(t)$*. Then we have for $k \in \mathbb{Z}_{\geq 1}$ that*

$$i(kP) = (-1)^n \mathrm{ehr}_P(-k).$$

For lattice polygons, the relation between $\mathrm{area}(P)$, $i(P)$ and $b(P)$ is particularly simple and can be described by the following formula of Pick.

Theorem 1.3 ([Pic99], [BR15, 2.8]). *Let $P \subseteq \mathbb{R}^2$ be a lattice polygon. Then P obeys Pick's formula*

$$\mathrm{area}(P) = i(P) + \frac{b(P)}{2} - 1.$$

Ehrhart-Macdonald reciprocity and Pick's theorem 1.3 allow us to calculate the Ehrhart polynomial of a lattice polygon P from $\mathrm{area}(P)$ and $b(P)$. By 1.1 we have $\mathrm{ehr}_P(t) = \mathrm{area}(P)\, t^2 + e_1 t + e_0$ and by Ehrhart-Macdonald reciprocity we have

$$e_1 = \frac{1}{2}\left(\mathrm{ehr}_P(1) - \mathrm{ehr}_P(-1)\right) = \frac{1}{2}(l(P) - i(P)) = \frac{b(P)}{2}.$$

From Pick's theorem we obtain

$$e_0 = \mathrm{ehr}_P(1) - \mathrm{area}(P) - \frac{b(P)}{2} = 1$$

and thus

$$\mathrm{ehr}_P(t) = \mathrm{area}(P)\, t^2 + \frac{b(P)}{2} t + 1.$$

To fully classify the Ehrhart polynomials for lattice polygons, another important connection between $b(P)$ and $i(P)$ is required. The following inequality of Scott was originally proved in [Sco76] and since then several times, e.g., in [HS09] or recently in [Boh23].

Theorem 1.4 ([Sco76]). *Let $P \subseteq \mathbb{R}^2$ be a lattice polygon with $i(P) > 0$. Then either $b(P) \leq 2i(P) + 6$ or P is affine unimodular equivalent to the lattice triangle* $\operatorname{convhull}((0,0),(3,0),(0,3))$ *and we have $i(P) = 1, b(P) = 9$.*

For all pairs $(i,b) \in \mathbb{Z}^2_{\geq 1}$ with $3 \leq b \leq 2i+6$ there exists a lattice polygon $P_{1,(i,b)}$ with $i(P_{1,(i,b)}) = i$ and $b(P_{1,(i,b)}) = b$.

For example, for $b = 3$, we can use

$$P_{1,(i,3)} := \operatorname{convhull}((0,1),(1,-1),(i+1,0))$$

and for $i > 0, 3 < b \leq 2i+6$, we can use

$$P_{1,(i,b)} := \operatorname{convhull}((0,-1),(b-4,-1),(i+1,0),(0,1)).$$

We also have for $b \geq 3$ the triangle

$$P_{1,(0,b)} := \operatorname{convhull}((0,0),(b-2,0),(0,1))$$

with $i(P_{1,(0,b)}) = 0, b(P_{1,(0,b)}) = b$.

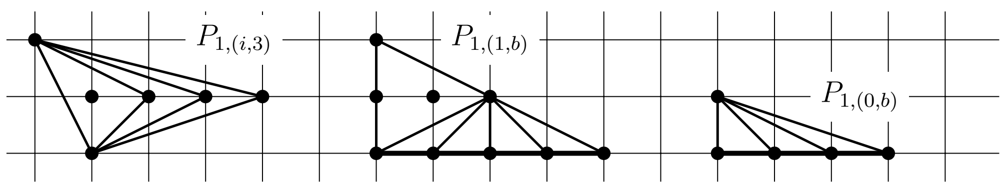

FIGURE 1. Lattice polygons, which show how to produce families, which are affine unimodular equivalent to $P_{1,(i,3)}$, $P_{1,(1,b)}$ and $P_{1,(0,b)}$.

Thus, Scott's inequality gives a complete classification of Ehrhart polynomials of lattice polygons, as formulated in the following corollary.

Corollary 1.5 ([HNP12, 3.3.34]). *The polynomial $e_2t^2 + e_1t + e_0 \in \mathbb{Q}[t]$ is an Ehrhart polynomial of a lattice polygon if and only if $e_0 = 1$ and there exists a pair*

$$(i,b) \in \{0\} \times \mathbb{Z}_{\geq 3} \;\dot\cup\; \{(x,y) \in \mathbb{Z}^2_{\geq 0} \mid x > 0,\ 3 \leq y \leq 2x+6\} \;\dot\cup\; \{(1,9)\}$$

with $e_2 = i + \frac{b}{2} - 1$ and $e_1 = \frac{b}{2}$.

The main result of this paper is the following complete classification of Ehrhart polynomials of pseudo-integral polygons with denominator 2.

Theorem 1.6. *The polynomial $e_2t^2 + e_1t + e_0 \in \mathbb{Q}[t]$ is an Ehrhart polynomial of a pseudo-integral polygon with denominator 2 if and only if $e_0 = 1$ and there exists a pair*

$$(i,b) \in \{(0,3)\} \;\dot\cup\; \{(x,y) \in \mathbb{Z}^2_{\geq 0} \mid x > 0,\ 2 \leq y \leq 2x+7\}$$

with $e_2 = i + \frac{b}{2} - 1$ and $e_1 = \frac{b}{2}$.

To see the difference between the Ehrhart polynomials of lattice polygons and of the pseudo-integral polygons of denominator 2, we can visualize possible pairs $(i(P), b(P)) \in \mathbb{Z}^2$, as done in figure 2.

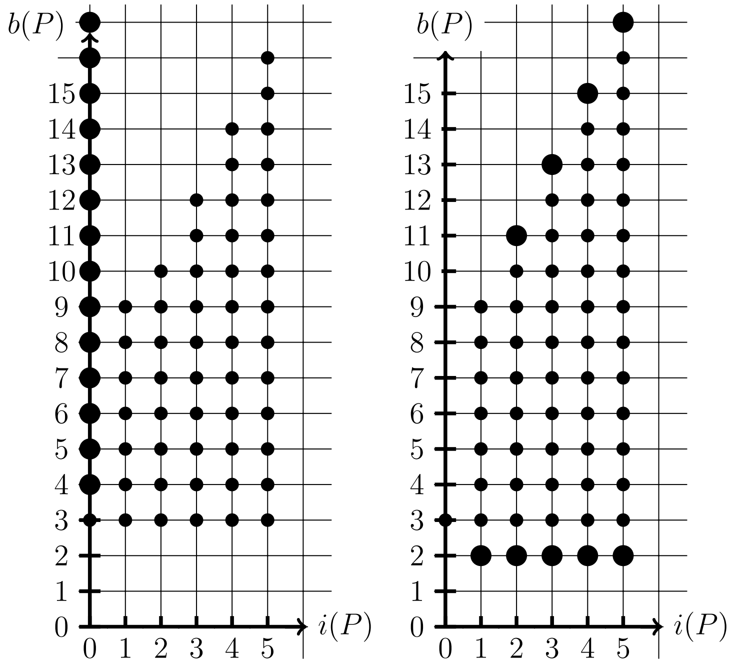

FIGURE 2. All possible pairs $(i(P), b(P))$ with $i(P) \leq 5, b(P) \leq 17$ for lattice polygons on the left and for half-integral pseudo-integral polygons, which are not lattice polygons, on the right. Points with differences are marked with big nodes.

In this paper, mainly experimental methods are used. Since we can generate many examples of pseudo-integral polygons from data of suitable lattice polygons in [KKN10], [Bae25] and [BS24b], we find suitable families of pseudo-integral polygons for our theorems just by looking at these examples. Moreover, the knowledge of area bounds and suitable coordinates for half-integral polygons from [Boh23] proves to be very helpful for the proof of our main theorem.

The paper is organized as follows. In section 2, we describe the Ehrhart quasi-polynomial of a half-integral polygon and characterize the period-collapse for these polygons. We also show how to classify pseudo-integral polygons of denominator 2 using huge lists of lattice polygons. In section 3, we give a family of pseudo-integral polygons of denominator 2 with exactly 2 boundary lattice points, and we will see that there is only one pseudo-integral polygon of denominator 2 without interior lattice points. In section 4, we classify the pseudo-integral polygons of denominator 2 with exactly one interior lattice point, which are duals of the 30 LDP polygons of Gorenstein index 2, as we see in section 5. We also get a formula for the sum of the numbers of boundary lattice points in the pseudo-integral polygon of denominator 2 and its dual polygon there. In section 6, we obtain a two-parameter family of pseudo-integral polygons of denominator 2 with $i \geq 1$ interior and $3 \leq b \leq 2i + 7$ boundary lattice points, and for the extremal polygons with $2i + 7$ boundary lattice points we give a complete classification in the last section.

2. Ehrhart quasi-polynomials and quasi-period collapse for half-integral polygons

Since the quasi-period divides the denominator of a rational polygon by 1.1, the quasi-period of a half-integral polygon $P \subseteq \mathbb{R}^2$ is either 1 or 2. So we can describe the Ehrhart quasi-polynomial of a half-integral polygon by two polynomials $\mathrm{ehr}_{P,even}(t)$ and $\mathrm{ehr}_{P,odd}(t)$ of degree 2, which are defined by the values on integers $k \in \mathbb{Z}$ as

$$\mathrm{ehr}_P(k) = \begin{cases} \mathrm{ehr}_{P,even}(k) & \text{for } k \equiv 0 \mod 2 \\ \mathrm{ehr}_{P,odd}(k) & \text{for } k \equiv 1 \mod 2. \end{cases}$$

The even part is easy to understand via the lattice polygon $2P$ and its Ehrhart polynomial $\mathrm{ehr}_{2P}(t)$, and the linear coefficient of $\mathrm{ehr}_{P,odd}(t)$ is $\frac{b(P)}{2}$ as in the case of a lattice polygon, which was already seen in [Her10, Lemma 3.3]. We give the complete description and argumentation for all coefficients in the following proposition.

Proposition 2.1. *Let $P \subseteq \mathbb{R}^2$ be a half-integral polygon. Then $\mathrm{ehr}_P(t)$ is given by*

$$\begin{aligned} \mathrm{ehr}_{P,odd}(t) =& \mathrm{area}(P)\, t^2 + \frac{b(P)}{2} t + l(P) - \mathrm{area}(P) - \frac{b(P)}{2} \\ \mathrm{ehr}_{P,even}(t) =& \mathrm{area}(P)\, t^2 + \frac{b(2P)}{4} t + 1. \end{aligned}$$

Proof. From 1.1 we know, that there exist rational numbers e_0 and e_1 with

$$\mathrm{ehr}_{P,odd}(t) = \mathrm{area}(P)\, t^2 + e_1 t + e_0.$$

Using the Ehrhart-Macdonald reciprocity 1.2, we have $\mathrm{ehr}_P(-1) = i(P)$ and so we get

$$e_1 = \frac{\mathrm{ehr}_{P,odd}(1) - \mathrm{ehr}_{P,odd}(-1)}{2} = \frac{\mathrm{ehr}_P(1) - \mathrm{ehr}_P(-1)}{2} = \frac{l(P) - i(P)}{2} = \frac{b(P)}{2}.$$

Now it follows from $\mathrm{ehr}_P(1) = \mathrm{ehr}_{P,odd}(1) = l(P)$ that

$$e_0 = \mathrm{ehr}_{P,odd}(1) - \mathrm{area}(P) - e_1 = l(P) - \mathrm{area}(P) - \frac{b(P)}{2}.$$

We have for all $k \in \mathbb{Z}_{\geq 0}$ that $\mathrm{ehr}_{P,even}(2k) = l(2kP) = \mathrm{ehr}_{2P}(k)$ and so we get $\mathrm{ehr}_{P,even}(2t) = \mathrm{ehr}_{2P}(t)$. We know the Ehrhart polynomial of the lattice polygon $2P$ from the introduction, and with the quadratic scaling of the area we have

$$\mathrm{ehr}_{P,even}(t) = \mathrm{ehr}_{2P}\left(\frac{t}{2}\right) = \mathrm{area}(2P)\, \frac{t^2}{4} + \frac{b(2P)}{2}\frac{t}{2} + 1 = \mathrm{area}(P)\, t^2 + \frac{b(2P)}{4} t + 1.$$

$\square$

Knowing the Ehrhart quasi-polynomial of a half-integral polygon, we can now directly characterize the period collapse of its coefficient functions.

Corollary 2.2. *Let $P \subseteq \mathbb{R}^2$ be a half-integral polygon. Then we have the period sequence $(1, *, 1)$ for P if and only if $l(P) - \mathrm{area}(P) - \frac{b(P)}{2} = 1$, i.e., P obeys Pick's formula, and $(*, 1, 1)$ if and only if for every edge e of P we have $e \cap \mathbb{Z}^2 \neq \emptyset$. In particular, P is pseudo-integral if and only if it obeys Pick's formula and has a lattice point on every edge.*

Proof. The only thing to show is that $\frac{b(P)}{2} = \frac{b(2P)}{4}$ if and only if every edge of P contains a lattice point. But we have a lattice point on every edge if and only if there is always exactly one half-integral boundary point between two neighboring boundary lattice points, and this is equivalent to $2b(P) = b(2P)$, since $b(2P)$ counts the half-integral boundary points of P. $\square$

Pseudo-integral polygons are also generally characterized using Pick's formula and boundary lattice points of dilates, as the following theorem shows.

Theorem 2.3 ([MAW05, 3.1])**.** *Let $P \subseteq \mathbb{R}^2$ be a rational polygon. Then the following conditions are equivalent.*

- *P is pseudo-integral.*
- $\mathrm{ehr}_P(t) = \mathrm{area}(P)\, t^2 + \frac{b(P)}{2} t + 1$.
- *kP obeys Pick's theorem and $b(kP) = kb(P)$ for all $k \in \mathbb{Z}_{\geq 1}, k \leq \mathrm{denom}(P)$.*

Therefore, we can compute the number of interior and boundary lattice points of $\mathrm{denom}(P) \cdot P$ for any pseudo-integral polygon using the following corollary.

Corollary 2.4. *Let $P \subseteq \mathbb{R}^2$ be a pseudo-integral polygon with $d := \mathrm{denom}(P)$. Then dP is a lattice polygon with $i(dP) = d^2 \cdot i(P) + \frac{d^2-d}{2} \cdot b(P) - d^2 + 1$ and $b(dP) = d \cdot b(P)$.*

Proof. From theorem 2.3 we have $b(dP) = d \cdot b(P)$ and can calculate $i(dP)$ with the Ehrhart-Macdonald reciprocity 1.2 as

$$\begin{aligned} i(dP) =& \mathrm{ehr}_P(-d) = \mathrm{area}(P) \cdot d^2 - \frac{b(P)}{2} \cdot d + 1 \\ =& d^2 \left(i(P) + \frac{b(P)}{2} - 1 \right) - \frac{b(P)}{2} \cdot d + 1 \\ =& d^2 \cdot i(P) + \frac{d^2 - d}{2} \cdot b(P) - d^2 + 1. \end{aligned}$$

$\square$

Specializing to $\mathrm{denom}(P) = 2$ we get the following way to classify half-integral pseudo-integral polygons.

Corollary 2.5. *Let $P \subseteq \mathbb{R}^2$ be a half-integral pseudo-integral polygon with n vertices. Then P is affine unimodular equivalent to a polygon with vertices*

$$\left\{\frac{1}{2}v_i\right\}_{1\leq i\leq n}, \left\{\frac{1}{2}v_i - \left(0, \frac{1}{2}\right)\right\}_{1\leq i\leq n}, \left\{\frac{1}{2}v_i - \left(\frac{1}{2}, 0\right)\right\}_{1\leq i\leq n} \quad \textit{or}$$
$$\left\{\frac{1}{2}v_i - \left(\frac{1}{2}, \frac{1}{2}\right)\right\}_{1\leq i\leq n},$$

where $\{v_i\}_{1\leq i\leq n}$ are the vertices of a lattice polygon Q with $i(Q) = 4i(P) + b(P) - 3$ and $b(Q) = 2b(P)$.

There are several exhaustive classifications of lattice polygons, according to the number of lattice points (up to $l(P) = 42$ in [Koe91]), the number of interior points (up to $i(P) = 30$ in [Cas12]), and the area (up to $\mathrm{area}(a(P)) = 25$ in [Bal21]). Koelman's list was recently extended in [BS24b].

From all these lists we can create sublists of lattice polygons with a fixed number of interior and boundary lattice points. To classify up to affine unimodular equivalence all half-integral pseudo-integral polygons $P \subseteq \mathbb{R}^2$ with a given number of interior and boundary lattice points, we can test Pick's formula and $2b(P) = b(2P)$ for all polygons from the sublists that are suitable according to 2.5. We did this using lattice polygons with up to 78 lattice points and got the following result (for the complete list of the polygons in the result see [Boh24a]).

Classification 2.6. *Up to affine unimodular equivalence there are exactly* 16688 *pseudo-integral polygons* P *with denominator* 2 *and at most* 6 *interior lattice points. Depending on* $i(P)$ *and* $b(P)$ *the polygons are distributed as follows. We give not only the number of pseudo-integral polygons with denominator* 2 *at* $\#(d=2)$*, but also the number of lattice polygons at* $\#(d=1)$ *for comparison.*

(i,b)	$(0,3)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(1,6)$	$(1,7)$	$(1,8)$	$(1,9)$
$\#(d=1)$	1	0	1	3	2	4	2	3	1
$\#(d=2)$	1	6	6	4	7	3	2	1	1

(i,b)	$(2,2)$	$(2,3)$	$(2,4)$	$(2,5)$	$(2,6)$	$(2,7)$	$(2,8)$	$(2,9)$	$(2,10)$	$(2,11)$
$\#(d=1)$	0	1	5	5	11	7	9	3	4	0
$\#(d=2)$	8	35	59	39	27	27	11	7	5	2

(i,b)	$(3,2)$	$(3,3)$	$(3,4)$	$(3,5)$	$(3,6)$	$(3,7)$
$\#(d=1)$	0	2	8	12	19	17
$\#(d=2)$	29	103	138	124	122	72
(i,b)	$(3,8)$	$(3,9)$	$(3,10)$	$(3,11)$	$(3,12)$	$(3,13)$
$\#(d=1)$	23	14	14	5	6	0
$\#(d=2)$	44	39	15	13	8	4

(i,b)	$(4,2)$	$(4,3)$	$(4,4)$	$(4,5)$	$(4,6)$	$(4,7)$	$(4,8)$
$\#(d=1)$	0	1	10	15	33	29	31
$\#(d=2)$	29	224	400	366	270	164	148
(i,b)	$(4,9)$	$(4,10)$	$(4,11)$	$(4,12)$	$(4,13)$	$(4,14)$	$(4,15)$
$\#(d=1)$	22	27	15	17	5	6	0
$\#(d=2)$	86	48	42	14	10	7	3

(i,b)	$(5,2)$	$(5,3)$	$(5,4)$	$(5,5)$	$(5,6)$	$(5,7)$	$(5,8)$	$(5,9)$
$\#(d=1)$	0	2	16	28	52	61	61	46
$\#(d=2)$	44	420	900	1035	784	482	271	160
(i,b)	$(5,10)$	$(5,11)$	$(5,12)$	$(5,13)$	$(5,14)$	$(5,15)$	$(5,16)$	$(5,17)$
$\#(d=1)$	36	25	28	17	18	6	7	0
$\#(d=2)$	159	90	55	49	16	12	8	4

(i,b)	$(6,2)$	$(6,3)$	$(6,4)$	$(6,5)$	$(6,6)$	$(6,7)$	$(6,8)$	$(6,9)$	$(6,10)$
$\#(d=1)$	0	3	17	39	84	92	111	87	76
$\#(d=2)$	80	718	1868	2148	1664	1111	663	367	217
(i,b)	$(6,11)$	$(6,12)$	$(6,13)$	$(6,14)$	$(6,15)$	$(6,16)$	$(6,17)$	$(6,18)$	$(6,19)$
$\#(d=1)$	49	40	26	34	20	21	7	8	0
$\#(d=2)$	150	174	103	63	56	18	13	9	4

Proof. The classification algorithm has already been justified above. It only remains to show that every pseudo-integral polygon P with denominator 2 has $b(P) = 3$ if $i(P) = 0$ and it has $b(P) \leq 2i(P) + 7$ if $i(P) > 0$. This is done in 3.3 and in 6.3. $\square$

We see that there are examples with $b(P) = 2i(P) + 7$ and $i(P) > 0$, which are not possible for lattice polygons by Scott's inequality 1.4. There are also examples with $b(P) = 2$, which we will discuss in the next section. We end this section by defining a special lattice polytope for each rational polytope P, which will be very helpful in the following.

Definition 2.7. Let $P \subseteq \mathbb{R}^n$ be a rational polytope.
Then the *integer hull of* P is the lattice polytope $\operatorname{convhull}(P \cap \mathbb{Z}^n)$, and we denote it by $\operatorname{inthull}(P)$.

Proposition 2.8. *Let $P \subseteq \mathbb{R}^2$ be a pseudo-integral polygon that is not a lattice polygon. If $\dim(\operatorname{inthull}(P)) = 2$, then the following statements hold:*

- *We have $i(P) > i(\operatorname{inthull}(P))$.*
- *We have $b(P) = b(\operatorname{inthull}(P)) - (i(P) - i(\operatorname{inthull}(P)))$.*
- *We have $\operatorname{area}(P) = \operatorname{area}(\operatorname{inthull}(P)) + \frac{1}{2}(i(P) - i(\operatorname{inthull}(P)))$.*

Proof. Since $\operatorname{denom}(P) > 1$, we have $\operatorname{inthull}(P) \subsetneq P$, and therefore we also get $\operatorname{area}(\operatorname{inthull}(P)) < \operatorname{area}(P)$. By definition, $\operatorname{inthull}(P)$ and P have the same number of lattice points. Since P as a pseudo-integral polygon obeys Pick's formula by 2.3, we obtain the claimed identities. $\square$

3. Half-integral pseudo-integral polygons with at most two boundary lattice points or without interior lattice points

There are no pseudo-integral polygons without boundary lattice points since by 2.3 a pseudo-integral polygon with denominator d has at least $b(P) = \frac{b(dP)}{d} \geq \left\lceil \frac{3}{d} \right\rceil \geq 1$ boundary lattice point, since the lattice polygon dP has at least 3 boundary lattice points. This was already seen in [MAM17, Theorem 1.2]. However, for any given number of interior lattice points $i > 1$ there are examples of pseudo-integral polygons with one boundary lattice point and denominator $2i+1$ in [MAM17, Theorem 1.2]. We can also construct a family of pseudo-integral triangles of denominator 3 with exactly one boundary and i interior lattice points by

$$T_{3,(i,1)} = \operatorname{convhull}\left(\left(\frac{2}{3}, 0\right), (1,0), \left(\frac{1}{3} + i, 6i - 3\right)\right).$$

In the half-integral case, this cannot happen because we have a lattice point on every edge by 2.2.

Nevertheless, we have half-integral pseudo-integral polygons with two boundary lattice points for any given number of interior lattice points greater than 0. For example, we can get such polygons from the family of polygons in the following proposition.

Proposition 3.1. *Let be $i \in \mathbb{Z}_{\geq 1}$. Then*

$$T_{2,(i,2)} := \operatorname{convhull}\left((0,1),(1,-1),\left(i+\frac{1}{2},0\right)\right) \subseteq \mathbb{R}^2$$

is a half-integral pseudo-integral triangle with $i(T_{2,(i,2)}) = i$ and $b(T_{2,(i,2)}) = 2$.

Proof. $T_{2,(i,2)}$ is half-integral, $|\operatorname{int}(T_{2,(i,2)}) \cap \mathbb{Z}^2| = |\{(k,0) \mid k \in \mathbb{Z}, 1 \leq k \leq i\}| = i$, and

$$b(T_{2,(i,2)}) = |\{(0,1),(1,-1)\}| = 2.$$

Since $\operatorname{area}\left(T_{2,(i,2)}\right) = i$ we get $\operatorname{area}\left(T_{2,(i,2)}\right) = i(T_{2,(i,2)}) + \frac{b(T_{2,(i,2)})}{2} - 1$ and we have lattice points on every edge of $T_{2,(i,2)}$, so with 2.2 we also get that $T_{2,(i,2)}$ is pseudo-integral. ☐

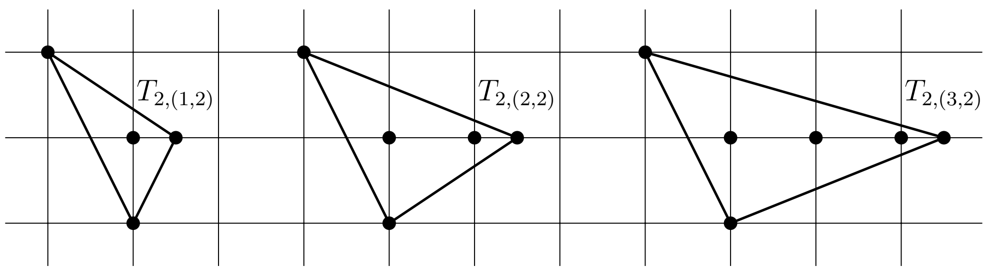

FIGURE 3. Triangles, which are affine unimodular equivalent to the triangles $T_{2,(1,2)}$, $T_{2,(2,2)}$ and $T_{2,(3,2)}$.

We now turn to polygons without interior lattice points, which is a rich class for lattice polygons, but consists of only one polygon for half-integral pseudo-integral polygons, as we will see.

There is no pseudo-integral polygon P without interior lattice points with less than 3 boundary lattice points since P must obey Pick's formula by 2.2 and so we have

$$b(P) = 2 \cdot \operatorname{area}(P) - 2i(P) + 2 = 2 \cdot \operatorname{area}(P) + 2 > 2.$$

This has already been seen in [MAM17, 1.2]. There are also no non-integral pseudo-integral polygons without interior lattice points and non-collinear boundary lattice points, as the following lemma shows.

Lemma 3.2. *Let $P \subseteq \mathbb{R}^2$ be pseudo-integral polygon without interior lattice points and with non-collinear boundary lattice points. Then P is a lattice polygon.*

Proof. Since the boundary lattice points are not collinear, $\dim(\operatorname{inthull}(P)) = 2$. So the lemma follows from 2.8. ☐

It remains to examine polygons with collinear boundary lattice points. For half-integral polygons we get only one such polygon, which is a non-lattice pseudo-integral polygon.

Proposition 3.3. *Let $P \subseteq \mathbb{R}^2$ be half-integral pseudo-integral polygon with $i(P) = 0$. Then P is either a lattice polygon or P is affine unimodular equivalent to the triangle*

$$T_{2,(0,3)} := \operatorname{convhull}\left((0,0),(2,0),\left(0,\frac{1}{2}\right)\right)$$

Proof. If the boundary lattice points are not collinear, we have a lattice polygon by 3.2.

If the boundary lattice points are collinear, P is a triangle and $e := \operatorname{inthull}(P)$ is an edge of P, since we must have a lattice point on every edge of P by 2.2. If d is the lattice distance of the non-integral vertex from the edge e and $\operatorname{length}(e) := l(e) - 1$ the lattice length of e, then we have $2d \in \mathbb{Z}_{\geq 1}$ and with Pick's formula

$$\frac{b(P)}{2} - 1 = \operatorname{area}(P) = \operatorname{length}(e) \cdot \frac{d}{2} = (b(P) - 1) \cdot \frac{d}{2}.$$

Therefore $2d = \frac{2b(P)-4}{b(P)-1} < 2$. We get $d = \frac{1}{2}$ and $b(P) = 3$, so P is affine unimodular equivalent to $\operatorname{convhull}\left((0,0),(2,0),(0,\frac{1}{2})\right)$. □

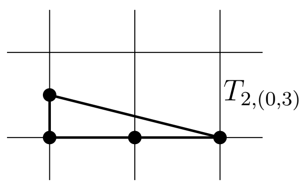

FIGURE 4. Up to affine unimodular equivalence the only pseudo-integral polygon with $\operatorname{denom}(P) = 2$ and $i(P) = 0$.

Remark 3.4. The lattice polygons without interior lattice points are classified (e.g. [Koe91, 4.1.2]). They are either affine unimodular equivalent to the lattice triangle $\operatorname{convhull}((0,0),(2,0),(0,2))$ or to exactly one of the lattice polygons

$$P_{1,(0,(u,d))} = \operatorname{convhull}((1,0),(d,0),(u,1),(1,1)) \text{ with } u,d \in \mathbb{Z}_{\geq 1}, d > 1, d \geq u.$$

In particular, since $b(P_{1,(0,(u,d))}) = u + d$, there is a lattice polygon without interior lattice points for any given number of boundary lattice points $b \geq 3$.

Remark 3.5. The situation is quite different for denominators greater than 2. We cannot generally expect that there is only a finite number of pseudo-integral polygons without interior lattice points for a fixed denominator $d > 2$. For example, for $d = 3$ and any $b \in \mathbb{Z}_{\geq 4}$, one can show that there is a pseudo-integral polygon

$$Q_{3,(0,b)} := \operatorname{convhull}\left((0,0),\left(\frac{1}{3},0\right),\left(b-\frac{7}{3},2b-5\right),(b-1,2b-2)\right)$$

without interior lattice points and with

$$b(Q_{3,(0,b)}) = |\{(l,2l) \mid l \in \mathbb{Z}, 0 \leq l \leq b-1\}| = b.$$

In particular, for every $b \in \mathbb{Z}_{\geq 3}$ we have a pseudo-integral polygon with denominator 2 or 3, without interior lattice points and with b boundary lattice points.

Remark 3.6. The triangle in 3.3 is affine unimodular equivalent to the triangle convhull $\big((0,0),(\frac{1}{2},0),(2,2)\big)$. It seems to be possible to generalize this example and get for each denominator $d \in \mathbb{Z}_{\geq 2}$ a triangle

$$T_{d,(0,d+1)} := \text{convhull}\left((0,0),\left(\frac{1}{d},0\right),(d,(d-1)d)\right),$$

which is pseudo-integral without interior lattice points and with the $d+1$ boundary lattice points of the set $\{(l,(d-1)l) \mid l \in \mathbb{Z}, 0 \leq l \leq d\}$.

4. Half-integral pseudo-integral polygons with exactly one interior lattice point

For lattice polygons, up to affine unimodular equivalence, we have exactly 16 polygons P with $i(P) = 1$, the 16 *reflexive polygons*, independently classified in [Bat85], [Rab89] and [Koe91]. Pseudo-integral polygons with denominator 2 and exactly one interior lattice point have some nice properties in common with reflexive polygons, which we will use for a deeper investigation in the next section. In this section, we give the complete classification of pseudo-integral polygons with denominator 2 and exactly one interior lattice point without using these properties.

Classification 4.1. *There are exactly* 30 *pseudo-integral polygons* $P \subseteq \mathbb{R}^2$ *with denominator* 2 *and* $i(P) = 1$ *up to affine unimodular equivalence. Each of them is affine unimodular equivalent to exactly one of the polygons in figure 5.*

Proof. Since $\text{denom}(P) = 2$, the polygon P has at least 2 lattice points due to 2.2, and so $\text{inthull}(P)$ is at least of dimension 1.

We first consider the case $\dim(\text{inthull}(P)) = 1$. Since $\text{denom}(P) = 2$, by 2.2 we have a lattice point on every edge of P, and so with $i(P) = 1$ we get that P is the convex hull of $\text{inthull}(P)$, which must be a segment with 3 lattice points, and of two half-integral points, one on each side of the affine hull of $\text{inthull}(P)$. So we have $b(P) = 2$ and with Pick's formula we get $\text{area}(P) = 1$, so that the two half-integral vertices have lattice distance $\frac{1}{2}$ to $\text{inthull}(P)$. After a suitable affine unimodular transformation we can assume that $\text{inthull}(P) = \text{convhull}((0,1),(0,-1))$ and the half-integral point $(\frac{1}{2},-1)$ is a vertex of P. Because of the convexity of P, the other half-integral vertex of P must be a point of $\{(-\frac{1}{2},\frac{l}{2}-1) \mid l \in \mathbb{Z}, 0 \leq l \leq 8\}$. With suitable affine unimodular transformations, we can limit ourselves to $l \leq 4$, so we get up to affine unimodular equivalence one of the first five polygons in the first row of figure 5.

Now we turn to the case where $\text{inthull}(P)$ is of dimension 2. Then we have by 2.8 $i(\text{inthull}(P)) = 0$ and $b(\text{inthull}(P)) = b(P) + 1$, since $i(P) = 1$. Moreover, by 2.8 we get the relation $\text{area}(P) = \text{area}(\text{inthull}(P)) + \frac{1}{2}$. Therefore, we obtain P as the convex hull of $\text{inthull}(P)$ and a single half-integral point not in $\text{inthull}(P)$, and the closure in $\mathbb{R}^2$ of $P \setminus \text{inthull}(P)$ is either a triangle affine unimodular equivalent to the triangle convhull $\big((0,0),(2,0),(0,\frac{1}{2})\big)$, or the union of two triangles sharing an edge, while each of them is affine unimodular equivalent to convhull $\big((0,0),(1,0),(0,\frac{1}{2})\big)$. In particular, $\text{inthull}(P)$ has one edge with exactly 3 lattice points or two adjacent edges with exactly 2 lattice points.

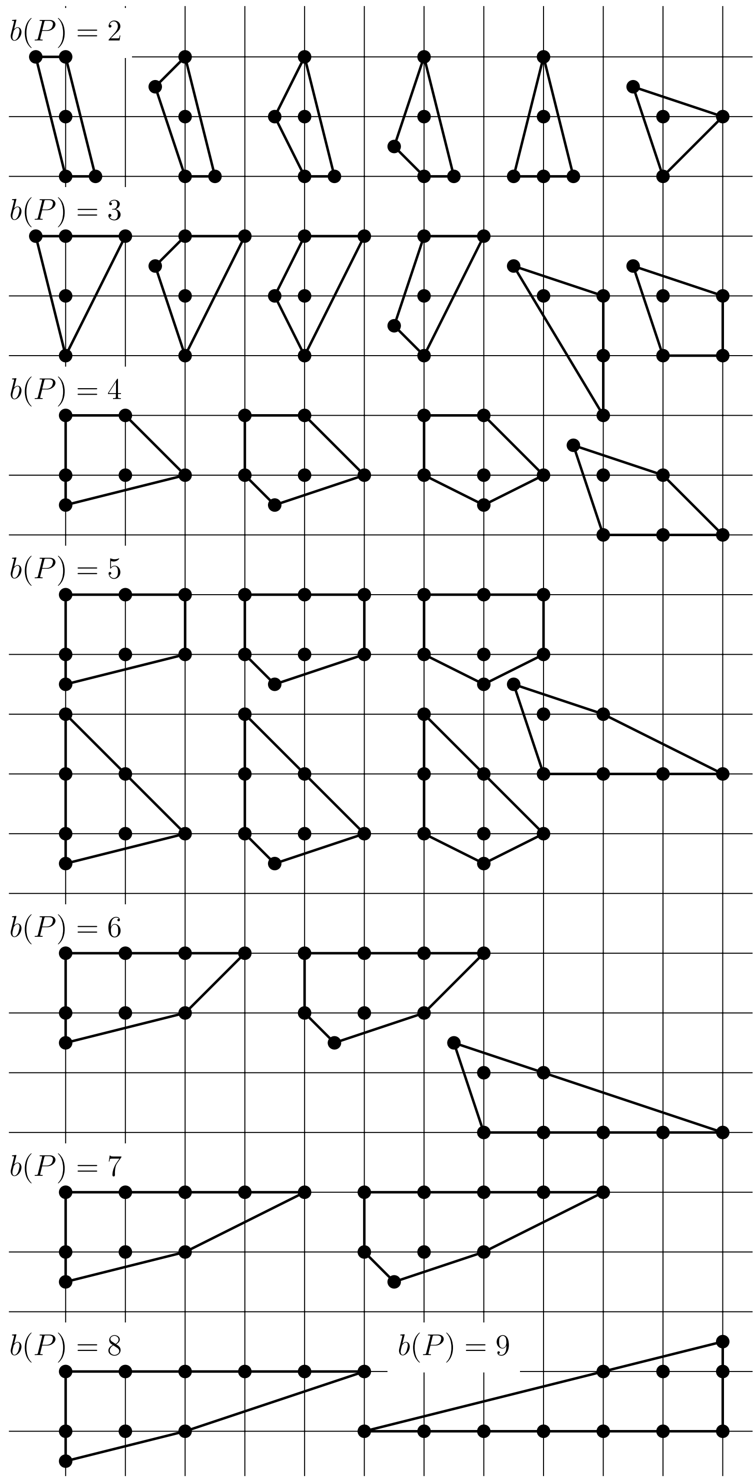

FIGURE 5. All 30 pseudo-integral polygons with denominator 2 and exactly 1 interior lattice points up to affine unimodular equivalence.

We get all lattice polygons without interior lattice points, $b(P)+1$ boundary lattice points and one edge with exactly 3 lattice points or two adjacent edges with exactly 2 lattice points by 3.4. So we obtain that $\operatorname{inthull}(P)$ is either a triangle affine unimodular equivalent to

$$\operatorname{convhull}((1,0),(b(P),0),(1,1)) \text{ or } \operatorname{convhull}((0,0),(2,0),(0,2))$$

or it is a quadrilateral affine unimodular equivalent to

$$\begin{aligned}&\operatorname{convhull}((1,0),(b(P)-1,0),(2,1),(1,1)) \text{ or}\\ &\operatorname{convhull}((1,0),(b(P)-2,0),(3,1),(1,1)),\end{aligned}$$

where we should restrict to $b(P) \geq 3$ in the first case and $b(P) \geq 5$ in the second case to really get a quadrilateral and have each polygon only once.

If the closure of $P \setminus \operatorname{inthull}(P)$ in $\mathbb{R}^2$ is a triangle affine unimodular equivalent to $\operatorname{convhull}\big((0,0),(2,0),(0,\frac{1}{2})\big)$, then P is either affine unimodular equivalent to a subpolygon of $\operatorname{convhull}\big((0,-1),(6,-1),(0,\frac{1}{2})\big)$ with half-integral vertex in $(0,\frac{1}{2})$ and $(0,0),(2,0) \in \partial P$, or has $\operatorname{inthull}(P) \cong \operatorname{convhull}((0,0),(2,0),(0,2))$ and is therefore affine unimodular equivalent to the convex hull of $(0,0),(2,0),(0,2)$ and a half-integral vertex which can be chosen from the set $\{(\frac{l}{2},-\frac{1}{2}) \mid l \in \mathbb{Z}_{\geq 0}, l \leq 5\}$. This is due to the convexity of P and the possibilities for $\operatorname{inthull}(P)$ listed above. Thus, we have only a few possibilities for P and can check all of them for affine unimodular equivalence.

If the closure of $P \setminus \operatorname{inthull}(P)$ in $\mathbb{R}^2$ is the union of two triangles sharing an edge, while each of them is affine unimodular equivalent to $\operatorname{convhull}\big((0,0),(1,0),(0,\frac{1}{2})\big)$, we must have two edges with exactly 2 lattice points for $\operatorname{inthull}(P)$, and so we can assume by the above possibilities for $\operatorname{inthull}(P)$ and by a suitable affine unimodular transformation that we have either

$$\begin{aligned}&\operatorname{inthull}(P) = \operatorname{convhull}((1,0),(b(P),0),(1,1)) \text{ or}\\ &\operatorname{inthull}(P) = \operatorname{convhull}((1,0),(b(P)-1,0),(2,1),(1,1)),\ b(P) \geq 3.\end{aligned}$$

In the first case $\operatorname{inthull}(P) = \operatorname{convhull}((1,0),(b(P),0),(1,1))$, we can assume that the second coordinate of the half integral vertex is at least $\frac{3}{2}$ and get

$$\frac{3}{2} \cdot \frac{b(P)-1}{2} \leq \operatorname{area}(P) = \frac{b(P)}{2}$$

and therefore $b(P) \leq 3$. So we have either $P = \operatorname{convhull}\big((1,0),(2,0),(\frac{1}{2},2)\big)$ or $P = \operatorname{convhull}\big((1,0),(3,0),(\frac{1}{2},\frac{3}{2})\big)$.

In the second case $\operatorname{inthull}(P) = \operatorname{convhull}((1,0),(b(P)-1,0),(2,1),(1,1))$ and $b(P) \geq 3$, we can assume with the help of the symmetry of P that the half-integral vertex of P is $(\frac{1}{2},\frac{3}{2})$, since it must be a point with lattice distance $\frac{1}{2}$ to two edges with 2 lattice points. Due to the convexity of P, we get P as a subpolygon of $\operatorname{convhull}\big((\frac{1}{2},\frac{3}{2}),(1,0),(5,0)\big)$ with half-integral vertex $(\frac{1}{2},\frac{3}{2})$. Thus P is the polygon $\operatorname{convhull}\big((\frac{1}{2},\frac{3}{2}),(1,0),(b(P)-1,0),(2,1)\big)$.

All in all, we get up to affine unimodular equivalence the polygons shown in figure 5, and these polygons are all pseudo-integral with denominator 2, since they obey Pick's formula and have a lattice point on every edge. □

5. Pseudo-integral polygons with exactly one interior lattice point and duals of LDP polygons

To understand the case of half-integral pseudo-integral polygons $P \subseteq \mathbb{R}^2$ with exactly one interior lattice point on a deeper level, it is helpful to note that their duals belong to a special class of lattice polygons, namely *LDP polygons*, which we want to introduce at the beginning of this section.

Period-collapse for duals of LDP polygons is well understood via so-called *mutations*, i.e. some combinatorial transformations introduced in [ACGK12] to study mirror symmetry for Fano manifolds ([CCG13], [ACC16]). Mutations are discussed in the context of period collapse especially in [KW18]. We do not use these tools here and work purely combinatorially.

Definition 5.1. Let $P \subseteq \mathbb{R}^n$ be a rational polytope, $0 \in \operatorname{int}(P)$, and we denote by $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ the standard inner product of $\mathbb{R}^n$. If we identify $(\mathbb{R}^n)^*$ with $\mathbb{R}^n$ using the inner product, then we define the *dual polytope* $P^* \subseteq (\mathbb{R}^n)^* \cong \mathbb{R}^n$ of P by

$$P^* := \{y \in \mathbb{R}^n \mid \langle y, x \rangle \geq -1 \ \forall x \in P\}.$$

Definition 5.2. Let $H_{n,b} \subseteq \mathbb{R}^d$ be a rational hyperplane defined by a dual lattice vector $\langle \cdot, n \rangle \in (\mathbb{Z}^d)^*$ and $b \in \mathbb{Z}$ as

$$H_{n,b} = \{x \in \mathbb{R}^n \mid \langle x, n \rangle = b\}.$$

Then the *lattice distance* $\operatorname{ldist}_{H_{n,b}}(l)$ of a rational point $l \in \mathbb{Q}^d$ to the rational hyperplane $H_{n,b}$ is defined as

$$\operatorname{ldist}_{H_{n,b}}(l) := |\langle l, n \rangle - b|.$$

The lattice distance to a rational polytope of codimension 1 is defined as the lattice distance to the affine hull of the polytope.

In general, P^* is just a rational polytope, and we cannot even expect it to be a lattice polytope if P is a lattice polytope. However, if P^* is a lattice polytope, we have a theorem of Hibi which describes the behavior of the lattice points of P.

Theorem 5.3 ([Hib92])**.** *Let $P \subseteq \mathbb{R}^n$ be a rational polytope with $0 \in \operatorname{int}(P)$. Then the following conditions are equivalent*

- *P^* is a lattice polytope.*
- *For every facet $F \preceq P$ there is $a \in \mathbb{Z}_{\geq 1}$, so that $\operatorname{ldist}_F(0) = \frac{1}{a}$.*
- *$l(kP) = i((k+1)P)$ for all $k \in \mathbb{Z}_{\geq 0}$.*

In particular, we see from the theorem that P has only 0 as an interior lattice point, if P^* is a lattice polytope. We can force the rational polytope P to behave more like a lattice polytope by requiring that P^* has primitive vertices. This results in the following definitions.

Definition 5.4. Let $P \subseteq \mathbb{R}^n$ be a lattice polytope with $0 \in \operatorname{int}(P)$. If the coordinates of every vertex of P are relatively prime, then P is called a *Fano polytope*. The number $\operatorname{denom}(P^*)$ is called the *Gorenstein index i_g of P*. If P^* is also a lattice polytope, then P is called a *reflexive polytope*, which is the same as a Fano polytope of Gorenstein index 1. Fano polytopes of dimension 2 are also called *LDP polygons*.

Fano polytopes correspond to toric Fano varieties via toric geometry. For a survey of Fano polytopes see [KN12]. Reflexive polytopes are especially known for their role in combinatorial mirror symmetry, since they provide a way to construct mirror pairs of Calabi-Yau varieties ([Bat94]).

If P^* is a Fano polytope, we get the following corollary of Hibi's theorem.

Corollary 5.5. *Let $P \subseteq \mathbb{R}^n$ be a rational polygon with $0 \in \operatorname{int}(P)$. Then the following conditions are equivalent*

- *P^* is a Fano polytope.*
- *For each facet $F \preceq P$ we have $\operatorname{ldist}_F(0) = 1$.*
- *$l(kP) = i((k+1)P)$ for all $k \in \mathbb{Z}_{\geq 0}$ and the affine hull of every facet of P contains lattice points.*

We now focus on the 2 dimensional pseudo-integral case and in particular get all pseudo-integral polygons with exactly one interior lattice point as duals of LDP polygons.

Corollary 5.6. *Let $P \subseteq \mathbb{R}^2$ be a rational pseudo-integral polygon with $0 \in \operatorname{int}(P)$. Then the following conditions are equivalent*

- *P^* is an LDP polygon.*
- *For each edge $E \preceq P$ we have $\operatorname{ldist}_E(0) = 1$.*
- *$l(kP) = i((k+1)P)$ for all $k \in \mathbb{Z}_{\geq 0}$ and the affine hull of every edge of P contains lattice points.*
- $\operatorname{ehr}_P(t) = \operatorname{area}(P)\, t^2 + \operatorname{area}(P)\, t + 1$.
- $\operatorname{int}(P) \cap \mathbb{Z}^2 = \{(0,0)\}$.

Proof. Since P is pseudo-integral, we have $\operatorname{ehr}_P(t) = \operatorname{area}(P)\, t^2 + \frac{b(P)}{2} t + 1$ and we have Pick's formula for P by 2.3.

We already know that the first three conditions are equivalent by 5.5 and they imply $\operatorname{int}(P) \cap \mathbb{Z}^2 = \{(0,0)\}$, since we cannot have more interior lattice points, if every edge $E \preceq P$ has $\operatorname{ldist}_E(0) = 1$. With Pick's formula we get from $i(P) = 1$ that $b(P) = 2\operatorname{area}(P) - 2i(P) + 2 = 2\operatorname{area}(P)$ and therefore we have the Ehrhart polynomial $\operatorname{ehr}_P(t) = \operatorname{area}(P)\, t^2 + \operatorname{area}(P)\, t + 1$.

Conversely, if $\operatorname{ehr}_P(t) = \operatorname{area}(P)\, t^2 + \operatorname{area}(P)\, t + 1$ and therefore $2\operatorname{area}(P) = b(P)$, we get for all $k \in \mathbb{Z}_{\geq 0}$ by Ehrhart-Macdonald reciprocity 1.2

$$\begin{aligned} l(kP) =& \operatorname{area}(P)\, k^2 + \operatorname{area}(P)\, k + 1 = \operatorname{area}(P)\,(k+1)^2 - \operatorname{area}(P)\,(k+1) + 1 \\ =& i((k+1)P). \end{aligned}$$

Thus we also have $\operatorname{ldist}_E(0) \leq 1$ for all edges $E \preceq P$ by 5.3 and so we get with $\operatorname{denom}(P) \cdot b(P) = b(\operatorname{denom}(P) \cdot P)$ from 2.3 and using the lattice length of an edge $\operatorname{length}(E) = \frac{l(\operatorname{denom}(P) \cdot E) - 1}{2\operatorname{denom}(P)}$ that

$$\operatorname{area}(P) = \sum_{E \preceq P} \operatorname{ldist}_E(0) \cdot \frac{\operatorname{length}(E)}{2} = \sum_{E \preceq P} \operatorname{ldist}_E(0) \cdot \frac{l(\operatorname{denom}(P) \cdot E) - 1}{2\operatorname{denom}(P)} \leq \frac{b(P)}{2}.$$

Since $2\operatorname{area}(P) = b(P)$ we must have equality and so every edge must have lattice distance 1 from 0. $\square$

We know all LDP polygons up to Gorenstein index $i_g = 17$ by [KKN10]. Thus, by 5.6, the pseudo-integral polygons with $i(P) = 1$ and denominator at most 17 are up to affine unimodular equivalence among the duals of these polygons. We can test whether the duals are really pseudo-integral and get the following result, which in particular gives a second classification of pseudo-integral polygons with denominator 2 and exactly one interior lattice point (for the complete list of the polygons in the result see [Boh24a]).

Classification 5.7. *Up to affine unimodular equivalence there are exactly* 320 *pseudo-integral polygons with* $i(P) = 1$ *and denominator* d *at most* 17*. They have at most* 9 *boundary lattice points and there are no examples for* $d \in \{7, 13, 17\}$*.*

The number of polygons depending on the denominator and the number of boundary lattice points is given in the following table, where the b*-th entry in the vector in the third column gives the number of pseudo-integral polygons with* $i(P) = 1$ *and* b *boundary lattice points up to affine unimodular equivalence.*

d	*# LDP polygons with* $i_g = d$	*# pseudo-integral polygons*
1	16	$(0, 0, 1, 3, 2, 4, 2, 3, 1)$
2	30	$(0, 6, 6, 4, 7, 3, 2, 1, 1)$
3	99	$(1, 6, 1, 3, 2, 1, 1, 1, 0)$
4	91	$(1, 1, 3, 2, 1, 1, 1, 1, 0)$
5	250	$(1, 1, 2, 2, 1, 2, 1, 1, 1)$
6	379	$(11, 10, 8, 12, 10, 4, 1, 0)$
7	429	$(0, 0, 0, 0, 0, 0, 0, 0, 0)$
8	307	$(0, 0, 0, 0, 1, 0, 0, 0, 0)$
9	690	$(0, 0, 0, 1, 0, 0, 1, 0, 0)$
10	916	$(12, 2, 6, 4, 3, 1, 1, 1, 1)$
11	939	$(0, 0, 0, 1, 0, 0, 0, 1, 0)$
12	1279	$(12, 10, 14, 15, 6, 5, 3, 1, 0)$
13	1142	$(0, 0, 0, 0, 0, 0, 0, 0, 0)$
14	1545	$(0, 1, 1, 1, 2, 0, 0, 0, 0)$
15	4312	$(2, 13, 11, 21, 9, 6, 5, 4, 0)$
16	1030	$(0, 0, 0, 0, 0, 0, 1, 0, 0)$
17	1892	$(0, 0, 0, 0, 0, 0, 0, 0, 0)$

Even more is known about LDP triangles. The manual classifications for $i_g = 2$ in [Dai06] and $i_g = 3$ in [Dai09] have recently been extended to $i_g = 200$ in the toric part of [HHHS25] and to $i_g = 1000$ in [Bae25]. We have done the computations for the LDP triangles up to $i_g = 1000$ and get the following result (for the complete list of the polygons in the result see [Boh24a]).

Classification 5.8. *There are up to affine unimodular equivalence exactly* 135 *pseudo-integral triangles with* $i(P) = 1$ *and denominator at most* 1000*. All triangles have between* 1 *and* 9 *boundary lattice points, and there is no example for* $b(P) = 7$*. Depending on the number of boundary lattice points, these triangles are distributed as shown in the following table.*

$b(P)$	1	2	3	4	5	6	7	8	9
# pseudo-integral triangles	39	26	19	8	15	14	0	8	6

Remark 5.9. Experimental data lead to several questions about pseudo-integral polygons with exactly one interior lattice point. Are there always at most 9 boundary lattice points? A positive answer to this question for triangles was announced in a talk by Tyrrell B. McAllister [MAW22]. Are there infinitely many examples that are triangles? Are there infinitely many examples for a given number of boundary lattice points? Is there an example which is a triangle and has exactly 7 boundary lattice points?

The article [MAW24], that was written after the first preprint of this article, provides answers to the questions about triangles. There are pseudo-integral triangles with one interior lattice point and b boundary lattice points if and only if $1 \leq b \leq 9, b \neq 7$. Moreover, there are infinitely many of them in each allowed case. These triangles can be considered a generalization of the *Markov triangles* mentioned in, for example, [KW18, 3.8]. For more connections to algebraic geometry, see also [HK24].

We leave the other questions open and concentrate now on the half-integral case. In the half-integral case we see that every dual of an LDP polygon is pseudo-integral. This gives a new possibility to classify LDP polygons with $i_g = 2$ as duals of pseudo-integral polygons with denominator 2 and exactly one interior lattice point. Therefore, we give a proof of this observation without using the above data in the following theorem.

Theorem 5.10. *Let $P \subseteq \mathbb{R}^2$ be a half-integral polygon with $0 \in \operatorname{int}(P)$. Then the following conditions are equivalent*

- *P^* is an LDP polygon.*
- *Each edge of P has lattice distance 1 from 0.*
- *$|kP \cap \mathbb{Z}^2| = |\operatorname{int}((k+1)P) \cap \mathbb{Z}^2|$ for all $k \geq 0$ and the affine hull of every edge of P contains lattice points.*
- *$\operatorname{ehr}_P(t) = \operatorname{area}(P)\, t^2 + \operatorname{area}(P)\, t + 1$.*
- *$\operatorname{int}(P) \cap \mathbb{Z}^2 = \{(0,0)\}$ and P is pseudo-integral.*

Proof. By 5.5 and 5.6 it is sufficient to show that the first three conditions imply that P is pseudo-integral. If the affine hull of every edge of P contains lattice points, then we have lattice points on every edge since $\operatorname{denom}(P) = 2$ and we get $b(2P) = 2b(P)$. Thus, with 2.2 it suffices to show that the first three conditions imply that P obeys Pick's formula.

Since every edge of P has lattice distance 1 from 0, every edge of $2P$ has lattice distance 2 from 0. Thus, $b(2P) = \operatorname{area}(2P) = 4 \cdot \operatorname{area}(P)$ and we get

$$i(P) + \frac{b(P)}{2} - 1 = 1 + \frac{b(2P)}{4} - 1 = \operatorname{area}(P).$$

□

Remark 5.11. We can also see with [KW18, 3.4] that all duals of LDP polygons with Gorensteinindex 2 are pseudo-integral. This is because all cyclic quotient singularities of local index 2 are T-singularities. We see this combinatorially because every lattice polygon with lattice height 2 has an even number of boundary lattice points due to Pick's formula.

Do not forget, that the theorem is not correct for polygons with higher denominators, as we saw in 5.7. For example, we have $\Delta := \operatorname{convhull}\big((1,0),(0,-\frac{1}{3}),(-1,2)\big)$ with dual $\Delta^* = \operatorname{convhull}((7,3),(-1,-1),(-1,3))$, which is an LDP polygon of Gorenstein index 3, but the period sequence of Δ is $(3,1,1)$.

We end this section with a formula for the number of boundary lattice points of P in the half-integral case, depending on the number of interior and boundary lattice points of the dual LDP polygon. This generalizes the equation $b(P) + b(P^*) = 12$, which we have for reflexive polygons ([PRV00]). We therefore use the following even more general theorem for LDP polygons, which is itself a combinatorial interpretation of the *stringy Libgober-Wood identity* from [LW90]. A purely combinatorial proof of this general theorem was given in [BHSdW23].

Theorem 5.12 ([BS17, Corollary 4.5])**.** *Let $P \subseteq \mathbb{R}^2$ be dual to an LDP polygon. Then*

$$2 \cdot (\operatorname{area}(P) + \operatorname{area}(P^*)) = 12 \sum_{l \in P^* \cap \mathbb{Z}^2} (\kappa_{P^*}(l) + 1)^2$$

with

$$\kappa_{P^*}(l) = -\min\{\lambda \in \mathbb{R}_{\geq 0} \mid l \in \lambda P^*\}.$$

In particular, $2(\operatorname{area}(P) + \operatorname{area}(P^*)) \geq 12$ *with equality if and only if P is reflexive.*

Corollary 5.13. *Let $P \subseteq \mathbb{R}^2$ be a pseudo-integral polygon with* $\operatorname{int}(P) \cap \mathbb{Z}^2 = \{(0,0)\}$ *and* $\operatorname{denom}(P) \leq 2$*. Then*

$$b(P) + b(P^*) = 12 + (i(P^*) - 1).$$

Proof. Since P is pseudo-integral with $i(P) = 1$, we have $2 \cdot \operatorname{area}(P) = b(P)$ by Pick's formula. Since P^* is a lattice polygon by 5.6, we have by Pick's theorem $2 \cdot \operatorname{area}(P^*) = 2 \cdot i(P^*) + b(P^*) - 2$. So we get with 5.12

$$\begin{aligned} b(P) + b(P^*) =& 2 \cdot (\operatorname{area}(P) + \operatorname{area}(P^*)) - 2 \cdot i(P^*) + 2 \\ =& -2 \cdot i(P^*) + 2 + 12 \sum_{l \in P^* \cap \mathbb{Z}^2} (\kappa_{P^*}(l) + 1)^2. \end{aligned}$$

Note that $\kappa_{P^*}(l) = -1$ if $l \in \partial P^*$ and $\kappa_{P^*}((0,0)) = 0$. Moreover, since P is half-integral, P^* has Gorenstein index $i_g \leq 2$ and so every interior lattice point l of P^* different from $(0,0)$ has $\kappa_{P^*}(l) = -\frac{1}{2}$. So we get

$$\begin{aligned} b(P) + b(P^*) =& -2 \cdot i(P^*) + 2 + 12 \sum_{l \in P^* \cap \mathbb{Z}^2} (\kappa_{P^*}(l) + 1)^2 \\ =& 12 + \sum_{l \in \operatorname{int}(P^*) \cap \mathbb{Z}^2, l \neq (0,0)} \left(12 \cdot (-\frac{1}{2} + 1)^2 - 2 \right) \\ =& 12 + (i(P^*) - 1). \end{aligned}$$

□

6. A two parameter family of half-integral pseudo-integral polygons with $i \geq 1$ interior and $3 \leq b \leq 2i+7$ boundary lattice points

In the last sections we classified all pseudo-integral polygons of denominator 2 with at most 1 interior lattice point, and we gave an example of such a polygon for any given positive number of interior lattice points if $b(P) = 2$.

In the following proposition we give an example of a pseudo-integral polygon of denominator 2 with $i(P) \in \mathbb{Z}_{\geq 1}, 3 \leq b(P) \leq 2i(P) + 7$.

Proposition 6.1. *Let be $i, b \in \mathbb{Z}_{\geq 1}$ with $3 \leq b \leq 2i + 7$. Then*

$$P_{2,(i,b)} := \operatorname{convhull}\left(\left(0, \frac{3}{2}\right), (0,1), (2i+7-b, 0), (2i+4, 0), \left(2i+2, \frac{1}{2}\right)\right) \subseteq \mathbb{R}^2$$

is a pseudo-integral polygon with $\operatorname{denom}(P) = 2$*,* $i(P_{2,(i,b)}) = i$ *and* $b(P_{2,(i,b)}) = b$*.*

Proof. We have $\operatorname{denom}(P_{2,(i,b)}) = 2$,

$$|\operatorname{int}(P_{2,(i,b)}) \cap \mathbb{Z}^2| = |\{(k,1) \mid k \in \mathbb{Z}, 1 \leq k \leq i\}| = i$$

and

$$\begin{aligned} |\partial P_{2,(i,b)} \cap \mathbb{Z}^2| =& |\{(0,1), (i+1,1)\} \cup \{(k,0) \mid k \in \mathbb{Z}, 2i+7-b \leq k \leq 2i+4\}| \\ =& 2 + b - 2 = b. \end{aligned}$$

Since

$$\begin{aligned} \operatorname{area}\left(P_{2,(i,b)}\right) =& \operatorname{area}\left(\operatorname{convhull}\left(\left(0, \frac{3}{2}\right), (0,0), (3i+3, 0)\right)\right) \\ & - \operatorname{area}\left(\operatorname{convhull}\left((0,1), (0,0), (2i+7-b, 0)\right)\right) \\ & - \operatorname{area}\left(\operatorname{convhull}\left(\left(2i+2, \frac{1}{2}\right), (2i+4, 0), (3i+3, 0)\right)\right) \\ =& \frac{3(3i+3)}{4} - \frac{2i+7-b}{2} - \frac{i-1}{4} \\ =& i + \frac{b}{2} - 1, \end{aligned}$$

the polygon $P_{2,(i,b)}$ satisfies Pick's formula and we have lattice points on every edge of $P_{2,(i,b)}$, so with 2.2 we get also that $P_{2,(i,b)}$ is pseudo-integral. $\square$

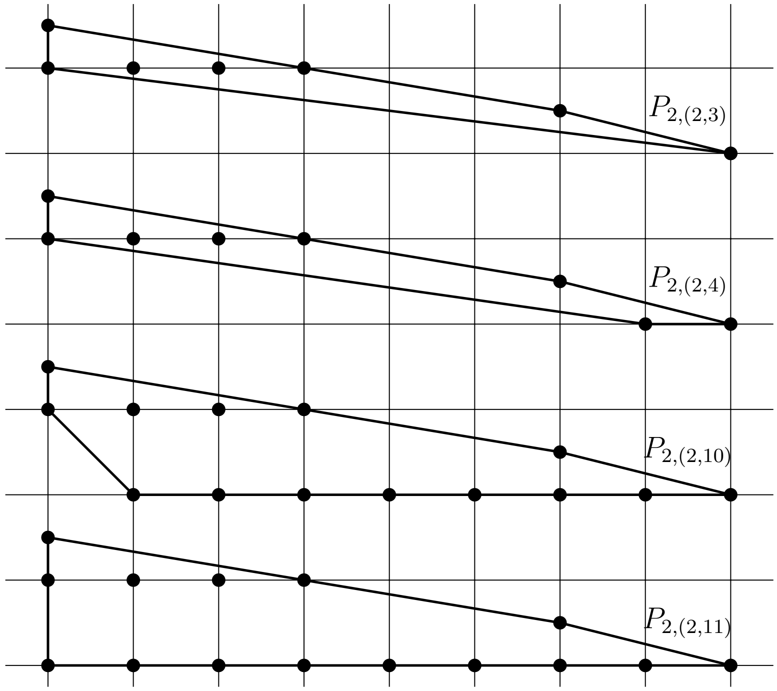

FIGURE 6. Some pseudo-integral half-integral polygons with 2 interior lattice points. We have drawn polygons affine unimodular equivalent to $P_{2,(2,3)}$, $P_{2,(2,4)}$, $P_{2,(2,10)}$ and $P_{2,(2,11)}$.

We have seen explicit examples with $b(P) = 2i(P) + 7$ and $i(P) > 1$ in the last proposition, i.e. we have examples that do not obey Scott's inequality 1.4. It was an open question in [MAM17] whether such pseudo-integral polygons exist. Can we classify all such polygons, and can we have even more boundary lattice points for pseudo-integral polygons of denominator 2? The answer to the first question is 'yes', we will do the classification in the next section. For the second question, we will see below that this is not the case, starting with a result that gives us suitable coordinates for half-integral polygons of large area.

Lemma 6.2. *Let $P \subseteq \mathbb{R}^2$ be a half-integral polygon with* $\operatorname{area}(P) \geq 2i(P) + 3$. *Then P is affine unimodular equivalent to a polygon in $[0, \frac{3}{2}] \times \mathbb{R}$. If $P \subseteq [0, 2] \times \mathbb{R}$ with $P \cap \{0\} \times \mathbb{R} \neq \emptyset$ and $P \cap \{2\} \times \mathbb{R} \neq \emptyset$, then* $\operatorname{area}(P) \leq 2i(P) + 2$.

Proof. This is a consequence of [Boh23, 3.1, 3.3, 3.5]. □

Theorem 6.3. *Let $P \subseteq \mathbb{R}^2$ be a half-integral pseudo-integral polygon with $i(P) > 0$ and $b(P) \geq 2i(P) + 7$. Then $b(P) = 2i(P) + 7$.*

Proof. Suppose P is a half-integral pseudo-integral polygon with $b(P) > 2i(P) + 7$. Then $\operatorname{area}(P) \geq 2i(P) + 3$ by Pick's formula and so P is by 6.2 affine unimodular equivalent to a half-integral polytope Q in $[0, \frac{3}{2}] \times \mathbb{R}$. Since P has $i(P)$ interior lattice points, we can also assume, with the help of a suitable translation and shearing, that

$$Q \cap \{1\} \times \mathbb{R} \subseteq \{1\} \times [0, i(P) + 1]$$

and $(\frac{3}{2}, 0)$ is a vertex of Q. Since Q is pseudo-integral, we have

$$i(2Q) = 4i(P) + b(P) - 3$$

by 2.4 and therefore $i(2Q) > 6i(P)+4$. But since $Q \cap \{1\} \times \mathbb{R} \subseteq \{1\} \times [0, i(P)+1]$ and $(\frac{3}{2}, 0)$ is a vertex of Q, we have

$$\mathrm{int}(2Q) \subseteq \mathrm{convhull}\left((1,1),(2,1),(2,2i(P)+1),(1,4i(P)+3)\right)$$

and so $i(2Q) \leq 2i(P)+1+4i(P)+3 = 6i(P)+4$, a contradiction. $\square$

We summarize our results from the last sections on the possible values of the pair $(i(P), b(P))$ in the following theorem. With 2.3 this theorem gives the proof of our main theorem 1.6.

Theorem 6.4. *Let $P \subseteq \mathbb{R}^2$ be a half-integral pseudo-integral polygon that is not a lattice polygon. Then*

$$(i(P), b(P)) \in \{(0,3)\} \cup \{(i,b) \mid i, b \in \mathbb{Z}_{\geq 1}, 2 \leq b \leq 2i+7\}.$$

Conversely, there is a half-integral pseudo-integral polygon $P \subseteq \mathbb{R}^2$, which is not a lattice polygon, with $(i(P), b(P)) = (i,b)$ for every

$$(i,b) \in \{(0,3)\} \cup \{(i,b) \mid i, b \in \mathbb{Z}_{\geq 1}, 2 \leq b \leq 2i+7\}.$$

7. Classification of half-integral pseudo-integral polygons with $i \geq 1$ interior and $b = 2i+7$ boundary lattice points

For the classification, we need the following lemma about the integer hull of pseudo-integral polygons with many boundary lattice points.

Lemma 7.1. *Let $P \subseteq \mathbb{R}^2$ be a rational polygon with* $\mathrm{denom}(P) > 1$, *$i(P) \geq 1$ and $b(P) \geq 2i(P)+7$. Then* $\mathrm{inthull}(P)$ *has no interior lattice points.*

Proof. The case $i(P) > 1, b(P) \geq 2i(P)+7$ follows from [MAM17, 3.1]. Suppose $i(P) = 1$, $b(P) \geq 9$ and $i(\mathrm{inthull}(P)) > 0$, then we have $b(\mathrm{inthull}(P)) = b(P) = 9$ and $\mathrm{inthull}(P) \cong \mathrm{convhull}((0,3),(0,0),(3,0))$ and so we get $P = \mathrm{inthull}(P)$, since any larger P would have more interior lattice points. So P has denominator 1, a contradiction. $\square$

Theorem 7.2. *$P \subseteq \mathbb{R}^2$ is a pseudo-integral polygon with* $\mathrm{denom}(P) = 2, i(P) = i$ *and $b(P) = 2i+7$ if and only if either*

$$P \cong P_{2,(i,2i+7),a} := \mathrm{convhull}\left(\left(0, \frac{3}{2}\right), \left(0, \frac{1}{2}\right), (a,0), (a+2i+4, 0), \left(2i+2, \frac{1}{2}\right)\right)$$

for an $a \in \mathbb{Z}, 0 \leq a \leq \frac{i-1}{2}$ or

$$i > 1, P \cong P_{2,(i,2i+7),s} := \mathrm{convhull}\left(\left(0, \frac{3}{2}\right), (0,1), \left(\frac{1}{2}, 0\right), (2i+5, 0), \left(2i+2, \frac{1}{2}\right)\right)$$

or

$$i = 3, P \cong P_{2,(3,13),s_3} := \mathrm{convhull}\left(\left(0, \frac{3}{2}\right), (0,1), \left(\frac{1}{2}, 0\right), \left(\frac{23}{2}, 0\right), (4,1)\right).$$

Proof. Let $P \subseteq \mathbb{R}^2$ be a pseudo-integral polygon with $\operatorname{denom}(P) = 2, i(P) = i$ and $b(P) = 2i(P) + 7$. Due to explicit computations in 2.6 we can assume $i(P) > 6$.

As a first step, we show that we can choose suitable coordinates so that we can realize P as a subpolygon of $\operatorname{convhull}\big((0, \frac{3}{2}), (0, 0), (3i + 3)\big)$. By 7.1 we know that $\operatorname{inthull}(P)$ has no interior lattice points. Since there are at least 7 boundary lattice points, we can assume by 3.4 after a suitable affine unimodular transformation that we have $\operatorname{inthull}(P) \subseteq \mathbb{R} \times [0, 1]$. Moreover, since $b(P) > 4$, we cannot have points of P in both $\mathbb{R} \times (-\infty, 0)$ and $\mathbb{R} \times (1, \infty)$, and so, after a suitable affine unimodular transformation, we can also assume that $P \subseteq \mathbb{R} \times [0, \infty)$, and so we get in particular $\operatorname{int}(P) \cap \mathbb{Z}^2 \in \mathbb{Z} \times \{1\}$. Since $i(P) > 6$ we have $\operatorname{length}(P \cap \mathbb{R} \times \{1\}) > 5$ and so a point of P in $(\frac{5}{2}, \infty)$ would imply interior lattice points of P on $\{2\} \times \mathbb{R}$. So we have $P \subseteq \mathbb{R} \times [0, 2]$. We can even go further to $P \subseteq \mathbb{R} \times [0, \frac{3}{2}]$, since a vertex on $\mathbb{R} \times \{0\}$ and on $\mathbb{R} \times \{2\}$ would imply that $\operatorname{area}(P) \leq 2i + 2$ by 6.2, which contradicts $\operatorname{area}(P) = i(P) + \frac{b(P)}{2} - 1 = 2i + \frac{5}{2}$. Since P is half-integral pseudo-integral, we have lattice points on every edge of P by 2.2 and so we have only one vertex of P on $\mathbb{R} \times \{\frac{3}{2}\}$. After a suitable affine unimodular transformation we have that this vertex is $(0, \frac{3}{2})$ and $P \cap \mathbb{R} \times \{1\} \subseteq [0, i + 1] \times \{1\}$, so we can assume all in all $P \subseteq \operatorname{convhull}\big((0, \frac{3}{2}), (0, 0), (3i + 3, 0)\big)$.

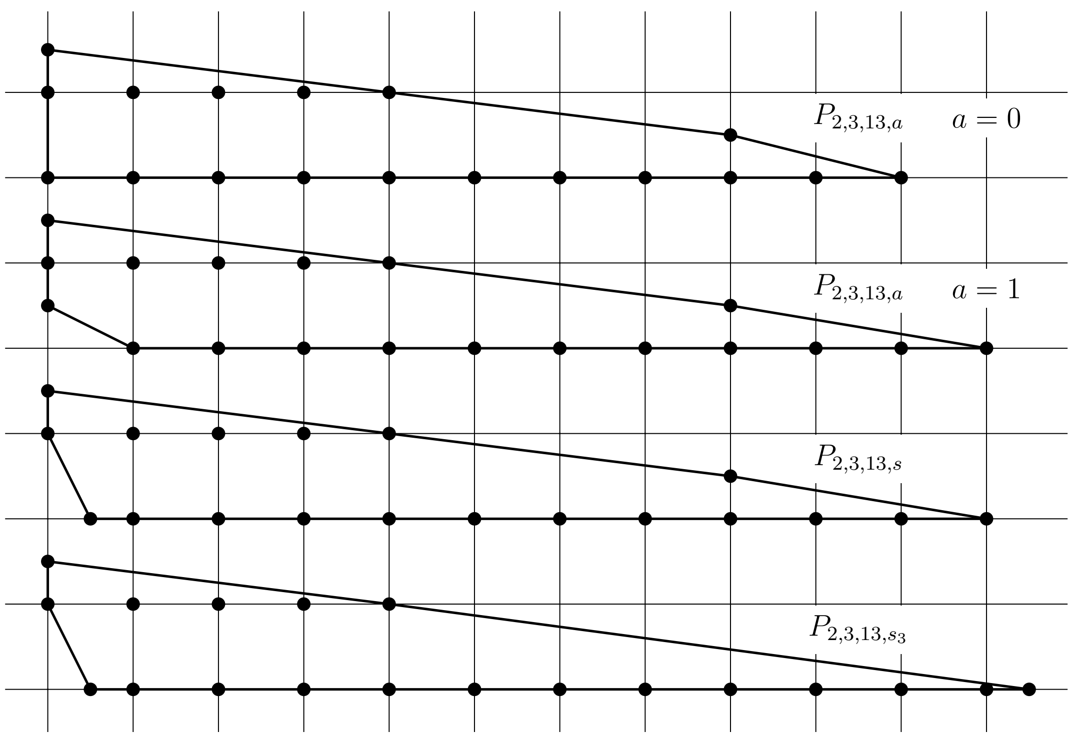

FIGURE 7. Polygons, which are affine unimodular equivalent to the four pseudo-integral polygons with denominator 2, 3 interior lattice points and 13 boundary lattice points, i.e. equivalent to $P_{2,3,13,0}$, $P_{2,3,13,1}$, $P_{2,3,13,s}$ and $P_{2,3,13,s_3}$.

In the second step, we show that if P is realized as subpolygon of the triangle convhull $\left((0,\frac{3}{2}),(0,0),(3i+3,0)\right)$, then $(0,1),(i+1,1)\in P$. Since P is pseudo-integral, we have by 2.5 for the number of half-integral interior points $i(2P)$ of P that $i(2P)=6i+4$. This is the same number of half-integral interior points as in convhull $\left((0,\frac{3}{2}),(0,0),(3i+3,0)\right)$ because

$$\begin{aligned}&|\mathrm{int}(\mathrm{convhull}\,((0,3),(0,0),(6i+6,0)))\cap\mathbb{Z}^2|\\ =&|\{(l,1)\mid 1\le l<4i+4\}\cup\{(l,2)\mid 1\le l<2i+2\}|\\ =&6i+4.\end{aligned}$$

We also know from the pseudo-integrality of P that there must be a lattice point between every two half-integral points on the boundary of P, which are not lattice points. Therefore, and to get $i(2P)=6i+4$, we must have $(0,1),(i+1,1)\in P$.

To get the complete description of P in the last step, we distinguish whether $(0,\frac{1}{2})$ and $(2i+2,\frac{1}{2})$ are both points of P or not.

We start with the case where $(0,\frac{1}{2})$ and $(2i+2,\frac{1}{2})$ are both points of P. Since $(0,\frac{1}{2})\in P$, the next half-integral point counterclockwise on the boundary of P after $(0,\frac{1}{2})$ must be a lattice point $(a,0), a\in\mathbb{Z}_{\ge 0}$ because of the pseudointegrality.

To get $b(2P)=2b(P)=4i+14$ the next vertex of P counterclockwise after $(a,0)$ must be $(a+2i+4,0)$. So

$$P\cong P_{2,i,2i+7,a}:=\mathrm{convhull}\left(\left(0,\frac{3}{2}\right),\left(0,\frac{1}{2}\right),(a,0),(a+2i+4,0),\left(2i+2,\frac{1}{2}\right)\right).$$

To have $(2i+2,\frac{1}{2})$ really as a boundary point, we must additionally have the inequality $a+2i+4\le 3i+3$, i.e. $a\le i-1$. Moreover, we get affine unimodular equivalent polygons if and only if we take $a'=i-1-a$ instead of a. So we must restrict to $0\le a\le\frac{i-1}{2}$ to have each polygon only once.

In the case that $(0,\frac{1}{2})$ and $(2i+2,\frac{1}{2})$ are not both points of P, we can assume after a suitable affine unimodular transformation that $(0,\frac{1}{2})$ is not a point of P. To get $i(2P)=6i+4$, we must have $(\frac{1}{2},\frac{1}{2})$ as an interior point of P, and so the next half-integral point on the boundary of P counterclockwise after $(0,1)$ must be $(\frac{1}{2},0)$. To get $b(2P)=2b(P)=4i+14$, the next vertex counterclockwise from P after $(\frac{1}{2},0)$ must be $(2i+5,0)$ if $(2i+2,\frac{1}{2})\in P$ and $(2i+\frac{11}{2},0)$ if $(2i+2,\frac{1}{2})\notin P$. So we have

$$P\cong P_{2,(i,2i+7),s}=\mathrm{convhull}\left(\left(0,\frac{3}{2}\right),(0,1),\left(\frac{1}{2},0\right),(2i+5,0),\left(2i+2,\frac{1}{2}\right)\right)$$

or

$$P\cong\mathrm{convhull}\left(\left(0,\frac{3}{2}\right),(0,1),\left(\frac{1}{2},0\right),\left(2i+\frac{11}{2},0\right),(i+1,1)\right).$$

In the first case we see that $(2i+2,\frac{1}{2})$ is a boundary point only if $2i+5\le 3i+3$, i.e. $i\ge 2$, so we have this additional constraint. In the second case, we have $(\frac{3}{2}i+\frac{13}{4},\frac{1}{2})$ as a boundary point, so we get $(2i+\frac{3}{2},\frac{1}{2})\in P$, $(2i+2,\frac{1}{2})\notin P$ if and only if $\frac{3}{2}i+\frac{13}{4}=2i+\frac{7}{4}$, i.e. $i=3$.

Conversely, we see that all given polygons have the correct number of interior and boundary lattice points by construction and the correct area to satisfy Pick's formula. Moreover, they all have denominator 2 and lattice points on all edges, so we conclude that they are all pseudo-integral with denominator 2 and have exactly $2i(P) + 7$ boundary lattice points. $\square$

Corollary 7.3. *For $i \in \mathbb{Z}_{\geq 1}$ there are exactly $\lfloor \frac{i-1}{2} \rfloor + 2 + \delta(i-3) - \delta(i-1)$ pseudo-integral polygons $P \subseteq \mathbb{R}^2$ with $\operatorname{denom}(P) = 2$, $i(P) = i$ and $b(P) = 2i(P) + 7$, where*

$$\delta \colon \mathbb{Z} \to \mathbb{Z}, k \mapsto \begin{cases} 1 & \textit{for } k = 0 \\ 0 & \textit{else} \end{cases}.$$

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5. Lattice 3-polytopes of lattice width 2 and corresponding toric hypersurfaces

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LATTICE 3-POLYTOPES OF LATTICE WIDTH 2 AND CORRESPONDING TORIC HYPERSURFACES

MARTIN BOHNERT

Abstract. The Kodaira dimension of a nondegenerate toric hypersurface can be computed from the dimension of the Fine interior of its Newton polytope according to recent work of Victor Batyrev, where the Fine interior of the Newton polytope is the subpolytope consisting of all points which have an integral distance of at least 1 to all integral supporting hyperplanes. In particular, if we have a Fine interior of codimension 1, then the hypersurface is of general type and the Newton polytope has lattice width 2. In this article we study this situation for lattice 3-polytopes and the corresponding surfaces of general type. In particular, we classify all 2-dimensional Fine interiors of those lattice 3-polytopes which have at most 40 interior lattice points, thus obtaining many examples of surfaces of general type and genus at most 40.

1. Introduction

Let $M \cong \mathbb{Z}^d$ be a lattice of rank $d \in \mathbb{Z}_{\geq 1}$ and $N = \mathrm{Hom}(M\mathbb{Z}) \cong \mathbb{Z}^d$ the dual lattice. We have the natural pairing $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$, which extends to the real vector spaces $M_{\mathbb{R}} = M \otimes \mathbb{R}$ and $N_{\mathbb{R}} = N \otimes \mathbb{R}$. For a rational d-polytope P in $M_{\mathbb{R}}$, i.e. $P \subseteq M_{\mathbb{R}}$ is the convex hull of a finite number of points of $M_{\mathbb{Q}} = M \otimes \mathbb{Q}$ and P is of full dimension, we have the *Fine interior* of P defined as

$$F(P) := \{x \in M_{\mathbb{R}} \mid \langle x, n\rangle \geq \mathrm{Min}_P(n) + 1 \ \forall n \in N \setminus \{0\}\} \subseteq P$$

with

$$\mathrm{Min}_P(n) := \min\{\langle x, n\rangle \mid x \in P\}.$$

The Fine interior is itself a rational polytope, and if P is the Newton polytope of a nondegenerate toric hypersurface Z and has a non-empty Fine interior, then recent work by Victor Batyrev [Bat23] shows that the Fine interior and additional combinatorial data can be used to construct a minimal model $\hat{Z}$ of the hypersurface. Moreover, we can compute some important invariants of the minimal model directly from the Fine interior. The Kodaira dimension is given by [Bat23, 9.2] as

$$\kappa(\hat{Z}) = \min\{\dim F(P), d - 1\}$$

and, for example, for the top intersection number in the case $\dim F(P) = d - 1$ we have the formula ([Gie22, 5.1], [Bat23, 9.4])

$$(K_{\hat{Z}})^{d-1} = 2\mathrm{Vol}_{d-1}(F(P)),$$

where Vol_{d-1} means that we are looking at the normalized volume in the affine subspace generated by $F(P)$, where normalized means that we have volume 1 for a simplex whose vertices form an affine lattice basis of the sublattice in the subspace.

We focus in this article on the case $d = 3$ with $\dim F(P) = 2$, so our nondegenerate toric hypersurface becomes a surface of general type. This implies that our Newton polytope $P \subseteq M_{\mathbb{R}}$ is a lattice 3-polytope with lattice width 2, where we should remember that the *lattice width* $\mathrm{lw}(P)$ is given by

$$\mathrm{lw}(P) := \min\{-\mathrm{Min}_P(-n) - \mathrm{Min}_P(n) \mid n \in N \setminus \{0\}.$$

and we call $n_{lw} \in N$ a *lattice width direction* if $\mathrm{lw}(P) = -\mathrm{Min}_P(-n_{lw}) - \mathrm{Min}_P(n_{lw})$.

Why is the combinatorics in this situation easy enough to handle? As a first step, we will see in Theorem 2.5 in section 2, that the Fine interior of a lattice d-polytope with lattice width 2 and lattice width direction n_{lw} is completely determined by its half-integral *middle polytope*, i.e. the rational polytope of dimension $d-1$, which we get by the intersection of P with the hyperplane

$$\{x \in M_{\mathbb{R}} \mid \langle x, n_{lw}\rangle = \mathrm{Min}_P(n_{lw}) + 1\}.$$

In particular, we can work for lattice 3-polytopes with a *middlepolygon* and only have to do combinatorics from there on only in dimension 2. As a second step, we will see in section 3 that in this case not only the middle polygon but also the Fine interior is a half-integral polygon (Theorem 3.2). Moreover, we will see there that the Fine interiors can always be described by the union of a lattice polygon and some very special triangles that we can have on edges of the lattice polygons having two lattice points (see Theorem 3.9). It turns out that this description of the Fine interior is restrictive enough that we end up with an efficient classification algorithm for these Fine interiors in section 4. We will use this algorithm there to obtain our main classification result in 4.1: There are up to affine unimodular equivalence exactly 24324158 different 2-dimensional Fine interiors those lattice 3-polytopes which have at most 40 interior lattice points. In the last section we use this classification to classify as a corollary the Chern numbers for nondegenerated toric hypersurfaces of genus at most 40 which have as Newton polytope a lattice 3-polytope with 2-dimensional Fine interior. Having the Chern numbers of the surfaces, we can arrange them in the geography of the surfaces of general type.

2. The Fine interior of lattice polytopes of lattice width 2

In this section we will see that the Fine interior of a lattice d-polytope of lattice width 2 for arbitrary dimension d is determined by its half-integral middle polytope, which allows us to do our Fine interior computations in codimension 1, or in other words, without loss of generality, we can restrict ourselves for Fine interior computations to the case of pyramids of heigth 2 which have the same middle polytope.

We begin with the following lemma, which shows why the case of lattice width 2 is important for Fine interior computations.

Lemma 2.1. *Let $P \subseteq M_{\mathbb{R}}$ be a rational d-polytope. If $\mathrm{lw}(P) < 2$, then $F(P) = \emptyset$. If $\mathrm{lw}(P) = 2$, then $\dim F(P) < \dim P$, and if $\dim F(P) = \dim(P) - 1$, then $\mathrm{lw}(P) = 2$.*

Proof. If $n_{lw} \in N$ is a lattice width direction, then we have

$$F(P) \subseteq \{x \in M_{\mathbb{R}} \mid \langle x, n_{lw}\rangle \geq \mathrm{Min}_P(n_{lw}) + 1, \langle x, -n_{lw}\rangle \geq \mathrm{Min}_P(-n_{lw}) + 1\}$$

and the set on the right side is empty for $\mathrm{lw}(P) < 2$ and at of dimension $d-1$ for $\mathrm{lw}(P) = 2$. If we have $\dim F(P) = \dim(P) - 1$, then we must have a normal vector $n \in N$ for $F(P)$ such that $-n$ is also a normal vector and thus $-\mathrm{Min}_P(-n) - \mathrm{Min}_P(n) = 2$. This implies $\mathrm{lw}(P) \leq 2$ and we have $\mathrm{lw}(P) \geq 2$ since this is always the case for non-empty Fine interior by the first part of the lemma. □

Remark 2.2. There are many examples of lattice polytopes $P \subseteq M_{\mathbb{R}}$ with lattice width greater than 2 and $\dim F(P) < \dim P - 1$. Since $\dim F((d+1)\Delta_d) = 0$ for the standard simplex Δ_d, i.e. the convex hull of an affine lattice basis, and $\mathrm{lw}((d+1)\Delta_d) = d+1$, we have for $d, k, w \in \mathbb{Z}, d \geq 2, 0 \leq k < d-1, 3 \leq w \leq d-k+1$ the lattice d-polytope $(d-k+1)\Delta_{d-k} \times [0, w]^k$ with lattice width $w > 2$ and $\dim(F(P)) = k$.

Remark 2.3. We can see that $\dim(F(P)) = d-1$ implies $\mathrm{lw}(P) \leq 2$ also from the general theory about lattice projections along the Fine interior. By [Bat23, 9.1] we have a lattice projection along the Fine interior on a lattice polytope with a Fine interior of dimension 0. So from a Fine interior of dimension $d-1$ we get a projection onto a lattice polytope of dimension 1 with a Fine interior of dimension 0. But up to lattice translation there is only $[-1, 1]$ with this property, and a lattice projection onto $[-1, 1]$ is obviously equivalent to $\mathrm{lw}(P) \leq 2$.

We now rearrange our coordinates for a good working setup with the middle polytope and split the dual lattice N with respect to the middle polytope.

Definition 2.4. Let $P \subseteq \mathbb{R} \times M_{\mathbb{R}}$ be a lattice polytope with $\mathrm{lw}(P) = 2$ and $P \subseteq [-1, 1] \times M_{\mathbb{R}}$. Then we have the half-integral middle polytope $P_0 \subseteq M_{\mathbb{R}}$ of P defined by

$$\{0\} \times P_0 = P \cap (\{0\} \times M_{\mathbb{R}}).$$

Using P_0, we define a partition $N \setminus \{0\} = N_1(P_0) \cup N_2(P_0)$ of the non-zero dual lattice vectors by

$$N_1(P_0) := \{n \in N \setminus \{0\} \mid \mathrm{Min}_{P_0}(n) \in \mathbb{Z}\}, N_2(P_0) := \{n \in N \setminus \{0\} \mid \mathrm{Min}_{P_0}(n) \notin \mathbb{Z}\}.$$

Now we are ready to show how the middle polytope determines the Fine interior of a lattice polytope of lattice width 2.

Theorem 2.5. *Let $P \subseteq \mathbb{R} \times M_{\mathbb{R}}$ be a lattice polytope of lattice width 2, $P \subseteq [-1, 1] \times M_{\mathbb{R}}$ and $P_0 \subseteq M_{\mathbb{R}}$ the middle polytope of P. Then the Fine interior $F(P)$ of P is*

$$F(P) = F_1(P) \cap F_2(P)$$

with

$$\begin{aligned} F_1(P) :=& \{0\} \times \{x \in M_{\mathbb{R}} \mid \langle x, n \rangle \geq \mathrm{Min}_{P_0}(n) + 1 \ \forall n \in N_1(P_0)\} \\ F_2(P) :=& \{0\} \times \{x \in M_{\mathbb{R}} \mid \langle x, 2n \rangle \geq \mathrm{Min}_{P_0}(2n) + 1 \ \forall n \in N_2(P_0)\}. \end{aligned}$$

In particular, $F(P)$ is completely determined by P_0.

Proof. Since $P \subseteq [-1,1] \times M_{\mathbb{R}}$ and $\mathrm{lw}(P) = 2$, we have $F(P) = F(P) \cap (\{0\} \times M_{\mathbb{R}})$ and

$$F(P) = \{0\} \times \{x \in M_{\mathbb{R}} \mid \langle (0,x), (n_0,n)\rangle \geq \mathrm{Min}_P((n_0,n)) + 1 \ \forall n_0 \in \mathbb{Z}, n \in N \setminus \{0\}\}.$$

So we have $F(P) \subseteq \{0\} \times P_0$, and since P is a lattice polytope and P_0 is half-integral, it is sufficient enough to look at those pairs (n_0, n) with

$$\mathrm{Min}_P((n_0,n)) + \frac{1}{2} \geq \mathrm{Min}_{\{0\}\times P_0}((n_0,n)).$$

With

$$\begin{aligned}
\tilde{N}_1 :=& \{(n_0,n) \in \mathbb{Z} \times (N \setminus \{0\}) \mid \mathrm{Min}_P((n_0,n)) = \mathrm{Min}_{\{0\}\times P_0}((n_0,n))\},\\
\tilde{N}_2 :=& \{(n_0,n) \in \mathbb{Z} \times (N \setminus \{0\}) \mid \mathrm{Min}_P((n_0,n)) + \frac{1}{2} = \mathrm{Min}_{\{0\}\times P_0}((n_0,n))\}
\end{aligned}$$

we get

$$\begin{aligned}
F(P) =& (\{0\} \times \{x \in M_{\mathbb{R}} \mid \langle (0,x),(n_0,n)\rangle \geq \mathrm{Min}_{\{0\}\times P_0}((n_0,n)) + 1 \ \forall (n_0,n) \in \tilde{N}_1\}) \cap\\
& (\{0\} \times \{x \in M_{\mathbb{R}} \mid \langle (0,x),(n_0,n)\rangle \geq \mathrm{Min}_{\{0\}\times P_0}((n_0,n)) + \frac{1}{2} \ \forall (n,\nu) \in \tilde{N}_2\}\\
=& (\{0\} \times \{x \in M_{\mathbb{R}} \mid \langle x,n\rangle \geq \mathrm{Min}_{P_0}(n) + 1 \ \forall (n_0,n) \in \tilde{N}_1\}) \cap\\
& (\{0\} \times \{x \in M_{\mathbb{R}} \mid \langle x,2n\rangle \geq \mathrm{Min}_{P_0}(2n) + 1 \ \forall (n_0,n) \in \tilde{N}_2\})
\end{aligned}$$

We have for $(n_0,n) \in \tilde{N}_1$ that $\mathrm{Min}_{P_0}(n) = \mathrm{Min}_P((n_0,n)) \in \mathbb{Z}$ and so $n \in N_1(P_0)$. With the same argument we get $n \in N_2(P_0)$ for all $(n_0,n) \in \tilde{N}_2$. It remains to show that for all $n \in N_1(P_0)$ we have a $n_0 \in \mathbb{Z}$ with $(n_0,n) \in \tilde{N}_1$ and analogously for $N_2(P_0)$ and $\tilde{N}_1$.

Let be $n \in N_1(P_0)$, i. e. $\mathrm{Min}_{P_0}(n) \in \mathbb{Z}$. Then there is a vertex e of P_0 with $\mathrm{Min}_{P_0}(n) = \langle e,n\rangle$. We choose such a vertex $f = (1,f')$ of P such that $n_0 := \langle e - f', n\rangle \in \mathbb{Z}$ is maximal. We have

$$\langle (1,f'),(n_0,n)\rangle = \langle f',n\rangle + n_0 = \langle e,n\rangle = \mathrm{Min}_{P_0}(n)$$

and since $\langle (0,e),(n_0,n)\rangle = \langle e,n\rangle = \mathrm{Min}_{P_0}(n)$ we get that the dual lattice vector (n_0,n) defines a hyperplane containing $(0,e)$ and $(1,f')$ that is also a supporting hyperplane for P since we have chose n_0 maximal. So we get

$$\mathrm{Min}_P((n_0,n)) = \langle e,n\rangle = \mathrm{Min}_{\{0\}\times P_0}((n_0,n))$$

and so we have $(n_0,n) \in \tilde{N}_1$.

Let be $n \in N_2(P_0)$, i.e. $2\mathrm{Min}_{P_0}(n) \in \mathbb{Z} \setminus 2\mathbb{Z}$. Then there is a vertex e of P_0 with $\mathrm{Min}_{P_0}(n) = \langle e,n\rangle$. We now choose a vertex $f = (1,f')$ of P, so that $n_0 := \langle e - f', n\rangle - \frac{1}{2} \in \mathbb{Z}$ is maximal. Then we have

$$\langle (1,f'),(n_0,n)\rangle = \langle f',n\rangle + n_0 = \langle e,n\rangle - \frac{1}{2} = \mathrm{Min}_{P_0}(n) - \frac{1}{2}$$

and so the dual lattice vector (n_0, n) defines a hyperplane containing $(1, f')$ and this is also a supporting hyperplane for P since we chose n_0 maximal. So we get

$$\mathrm{Min}_P((n_0, n)) + \frac{1}{2} = \langle e, n\rangle = \mathrm{Min}_{\{0\}\times P_0}((n_0, n))$$

and so we have $(n_0, n) \in \tilde{N}_2$. □

If the middle polytope is a lattice polytope, then the situation becomes easier. This was already seen in [Boh24b, 4.3]. We now understand this situation as a corollary.

Corollary 2.6. *Let $P \subseteq \mathbb{R} \times M_{\mathbb{R}}$ be a lattice polytope of lattice width 2 with a lattice polytope $P_0 \subseteq M_{\mathbb{R}}$ as middle polytope of P. Then $F(P) = \{0\} \times F(P_0)$.*

Proof. Since P_0 is a lattice polytope, we have $N_2(P_0) = \emptyset, N_1(P_0) = N \setminus \{0\}$ and therefore $F(P) = F_1(P) = \{0\} \times F(P_0)$. □

We now introduce some new notation to work directly in $M_{\mathbb{R}}$.

Definition 2.7. Let $P_0 \subseteq M_{\mathbb{R}}$ be a half-integral polytope. Then we set

$$\bar{F}(P_0) := \bar{F}_1(P_0) \cap \bar{F}_2(P_0)$$

with

$$\bar{F}_1(P_0) := \{x \in M_{\mathbb{R}} \mid \langle x, n\rangle \geq \mathrm{Min}_{P_0}(n) + 1 \ \forall n \in N_1(P_0)\}$$
$$\bar{F}_2(P_0) := \{x \in M_{\mathbb{R}} \mid \langle x, 2n\rangle \geq \mathrm{Min}_{P_0}(2n) + 1 \ \forall n \in N_2(P_0)\}.$$

With this notation we now have the following short version of 2.5.

Corollary 2.8. *Let $P \subseteq \mathbb{R} \times M_{\mathbb{R}}$ be a lattice polytope of lattice width 2, $P \subseteq [-1, 1] \times M_{\mathbb{R}}$ and $P_0 \subseteq M_{\mathbb{R}}$ the middle polytope of P. Then we have*

$$F(P) = \{0\} \times \bar{F}(P_0).$$

Remark 2.9. Note that $\bar{F}(P_0)$ is not in general the Fine interior of the half-integral polytope $P_0 \subseteq M_{\mathbb{R}}$. We have that $\bar{F}(P_0)$ is the Fine interior of P_0 if and only if the primitive normal vectors to $F(P_0)$ are all in $N_1(P_0)$. In particular, $\bar{F}(P_0) = F(P_0)$ if P_0 is a lattice polytope. We also have the following inclusions

$$F(F(2P_0)) \subseteq 2F(P_0) \subseteq 2\bar{F}(P_0) \subseteq F(2P_0).$$

If the middle polytope completely determines the Fine interior, we can focus on some special polytopes with this middle polytope. For a rational polytope $P_0 \subseteq M_{\mathbb{R}}$ we have the pyramid over P defined by $\mathrm{Pyr}(P_0) := \mathrm{conv}\,((1, 0), \{0\} \times P_0) \subseteq \mathbb{R} \times M_{\mathbb{R}}$ and if we translate $2\mathrm{Pyr}(P_0)$ by the lattice vector $(-1, 0)$ to a subpolytope of $[-1, 1] \times M_{\mathbb{R}}$, then we get P_0 as the middle polytope. So we have the following corollary.

Corollary 2.10. *In the situation from 2.5 we have*

$$F(P) = \{0\} \times \bar{F}(P_0) \cong F(2 \cdot \mathrm{Pyr}(P_0))$$

Moreover, if P_0 is even a lattice polytope, then we also have

$$F(P) = F([-1, 1] \times P_0).$$

3. The Fine interior of a lattice 3-polytope of lattice width 2

Remark 3.1. Let P be as in the theorem, with a lattice polygon $P_0 \subseteq M_{\mathbb{R}}$ as middle polytope of P.
Then $F(P) = F(P_0) \times \{0\} = \operatorname{conv}(\operatorname{int}(P) \cap M)$ is a lattice polygon.

3.1. **The Fine interior of a lattice 3-polytope of lattice width 2 is a half-integral polygon.**

Theorem 3.2. *Let $M \cong \mathbb{Z}^2$ be a lattice of rank 2, $P \subseteq \mathbb{R} \times M_{\mathbb{R}}$ a lattice 3-polytope of lattice width 2, $P \subseteq [-1,1] \times M_{\mathbb{R}}$, and $\dim(F(P)) = 2$. Then $F(P) \subseteq \{0\} \times M_{\mathbb{R}}$ is a half-integral polygon.*

Proof. Let be $P_0 \subseteq M_{\mathbb{R}}$ the middle polygon of P. By 2.5 it is sufficient to show that $\bar{F}(P_0)$ is a half-integral polygon. To see this, we will prove that

$$\bar{F}(P_0) \subseteq \operatorname{conv}\left(x \in \bar{F}(P_0) \mid 2x \in M\right).$$

We can assume that we have at least two lattice points in $\bar{F}(P_0)$, since lattice 3-polytopes with at most 1 interior lattice point cannot have a Fine interior of dimension 2 by [BKS22]. So $\operatorname{conv}\left(x \in \bar{F}(P_0) \mid 2x \in M\right)$ has at least dimension 1 and we can look at any edge $e \preceq \operatorname{conv}\left(x \in \bar{F}(P_0) \mid 2x \in M\right)$. Now it is enough to see that e is also an edge of $\bar{F}(P_0)$. We have two cases, either the affine hull of e contains lattice points or it does not.

If the affine hull of e contains lattice points, then after a suitable affine unimodular transformation we can assume that the line segment $\operatorname{conv}((0,0),(1/2,0))$ is a subset of e and that we have no points of $\operatorname{conv}\left(x \in \bar{F}(P_0) \mid 2x \in M\right)$ with positive second coordinate. If we can now show that there are no points of P_0 with second coordinate greater than 1, we get that there are no points of $\bar{F}(P_0)$ with positive second coordinate and so we are done. Let $x_m = (x_{m1}, x_{m2}) \in P_0$ be a half-integral point with maximal second coordinate. Then the triangle $\operatorname{conv}((0,0),(1/2,0),x_m)$ contains a half-integral point with second coordinate $1/2$ by Pick's theorem. With a suitable shearing we can assume that this point is $(1/2,1/2)$ and so we get that $1/2 \leq x_{m1} \leq x_{m2}$ by convexity of the triangle. If $x_{m2} > 1$, then we have for $n \in N \setminus \{0\}$ at least one of the following inequalities

$$\begin{aligned}
\langle (0,0), n\rangle &\leq \langle (1/2,1/2), n\rangle \text{ foror}\\
\langle (1/2,0), n\rangle &\leq \langle (1/2,1/2), n\rangle \text{ or}\\
\langle (x_m, n\rangle < \langle (1,1), n\rangle &< \langle (1/2,1/2), n\rangle \text{ or}\\
\langle (x_m, n\rangle < \langle (1/2,1), n\rangle &< \langle (1/2,1/2), n\rangle.
\end{aligned}$$

These inequalities imply $\langle (1/2,1/2), n\rangle \geq \operatorname{Min}_{\bar{F}(P_0)}(n)$ and so we get $(1/2,1/2) \in \operatorname{conv}\left(x \in \bar{F}(P_0) \mid 2x \in M\right)$ which contradicts $x_{m2} > 1$. So we must have $x_{m2} \leq 1$ and so we are done.

If the affine hull of e contains no lattice points then we start our argument in a similar way. After a suitable affine unimodular transformation we can assume that the line segment $\operatorname{conv}((0,-1/2),(1/2,-1/2))$ is a subset of e and we have no points of $\operatorname{conv}\left(x \in \bar{F}(P_0) \mid 2x \in M\right)$ with second coordinate greater than

$-1/2$. If we can now proof that there are no points of P_0 with second coordinate greater than 0, we are done. Let $x_m = (x_{m1}, x_{m2}) \in P_0$ again be a half-integral point with maximal second coordinate. Then, by Pick's theorem, the triangle $\operatorname{conv}((0,-1/2),(1/2,-1/2),x_m)$ contains a half-integral point with second coordinate 0. With another affine unimodular transformation we can assume that this point is $(1/2,0)$ and get so that $1/2 \leq x_{m1} \leq x_{m2} + 1/2$ from the convexity of the triangle. With analogous inequalities as in the first case, we now get that $x_{m2} \leq 1/2$. So if $x_{m2} > 0$, then we have $x_m = (1/2,1/2)$ or $x_m = (1,1/2)$. In the first case we get $\operatorname{Min}_{P_0}((1,-1)) = 0$, because otherwise we get $(0,0) \in \operatorname{int}(P_0)$ and therefore the contradiction $(0,0) \in \bar{F}(P_0)$. But $\operatorname{Min}_{P_0}((1,-1)) = 0$ also leads to a contradiction, since $(0,-1/2) \in \bar{F}(P_0)$. Similary we get contradictions if $x_m = (1,1/2)$. All in all we see that $x_{m2} \leq 0$ and so we are done. □

The theorem also gives us also possibilities to compute $\bar{F}(P_0)$, since we only have to check which half-integral points are contained. If we know the support of the Fine interior of the lattice polygon $2P_0$, i.e. all the dual lattice vectors $n \in N$ with $\operatorname{Min}_{2P_0}(n) + 1 = \operatorname{Min}_{F(2P_0)}(n)$, we can use the following corollary. Note that the Fine interior here is just the convex hull of the interior lattice points, since $2P_0$ is a lattice polygon ([Bat17, 2.9]).

Corollary 3.3. *Let P be as in the theorem, P_0 its half-integral middle polygon. Then we get $2\bar{F}(P_0)$ as the convex hull of the following finite sets of lattice points*

$$\{x \in \partial F(2P_0) \cap M \mid \langle x, n\rangle \neq \operatorname{Min}_{2P_0}(n) + 1 \forall n \in S_F(2P_0) \cap N_1(P_0)\},$$
$$F(F(2P_0)) \cap M.$$

From the proof of the theorem we get another corollary. We have shown there that an edge of $\bar{F}(P_0)$ without lattice point has only lattice distance $1/2$ to a supporting line which contains therefore lattice points. But this means that such an edge of $\bar{F}(P_0)$ cannot exist and we get the following corollary.

Corollary 3.4. *Let P be as in the theorem with $\dim F(P) = 2$. Then we have for all edges $E \preceq F(P)$ that $E \cap M \neq \emptyset$.*

3.2. **The Fine interior vs. the convex hull of the interior lattice points.** For a lattice polygon, we know from [Bat17, 2.9] that the Fine interior is just the convex hull of the interior lattice points. For $\bar{F}$ of a half-integral polygon the situation is more complicated, but there are also some connections between $\bar{F}$ and the convex hull of the interior lattice points that we want to study in this subsection.

Proposition 3.5. *Let $P_0 \subseteq M_{\mathbb{R}}$ be a half-integral polygon with at least two interior lattice points, $e \preceq \operatorname{conv}(\operatorname{int}(P_0) \cap M)$ an edge and $v \preceq \operatorname{conv}(\operatorname{int}(P_0) \cap M)$ a vertex. Then e is an edge of $\bar{F}(P_0)$ or $|e \cap M| = 2$ and v is a boundary lattice point of $\bar{F}(P_0)$.*

Proof. First, we show that an edge $e \preceq \operatorname{conv}(\operatorname{int}(P_0) \cap M)$ with $|e \cap M| > 2$ is an edge of $\bar{F}(P_0)$. After a suitable affine unimodular transformation we can assume that $\operatorname{conv}((0,0),(2,0)) \subseteq e$ and the second coordinate of all interior lattice points of P_0 is at most 0. Suppose now that e is not an edge of $\bar{F}(P_0)$, i.e. we have a half-integral point with positive second coordinate in $\bar{F}(P_0)$ and we cah choose $S =$

$(s_1, s_2) \in \bar{F}(P_0)$ as a half-integral point with maximal possible second coordinate s_2. By a suitable shearing we can assume that $s_2 \geq s_1 > 0$. We conclude directly that $s_2 < \frac{3}{2}$, since otherwise $\operatorname{conv}((0,0),(2,0),(s_1,s_2)) \subseteq \operatorname{int}(P_0)$ contains the lattice point $(1,1) \in M$, which contradicts our choice of coordinates. It remains to consider the cases $S = (1/2, 1)$ and $S = (1/2, 1/2)$. Since s_2 was chosen maximal and every edge of $\bar{F}(P_0)$ contains a lattice point by 3.4, S must be a vertex of $\bar{F}(P_0)$. Let $e_{S,l}, e_{S,r} \preceq \bar{F}(P_0)$ be the edges of $\bar{F}(P_0)$ containing S, where the indices l and r indicate that the first coordinates of the points of e_{S_l} are not greater than those of e_{S_r} and so e_{S_l} is on the left. Take the parallel lines to the affine hull of e_{S_l} through $(0,1)$ and to the affine hull of e_{S_2} through $(1,1)$. Then P_0 must be below both of these parallel. This implies that all points of P_0 have second coordinate of at most 1, which contradicts the fact that $S \in \bar{F}(P_0)$.

We now show that every vertex of $\operatorname{conv}(\operatorname{int}(P_0) \cap M)$ is a boundary lattice point of $\bar{F}(P_0)$. From the first part of the proof we already know that this is true for vertices on edges of $\bar{F}(P_0)$ with at least 3 lattice points. So we only need to consider edges of $\operatorname{conv}(\operatorname{int}(P_0) \cap M)$ with exactly two lattice points. After a suitable affine unimodular transformation we can assume that this edge is $\operatorname{conv}((0,0),(1,0))$ and that the second coordinate of all interior lattice points of P_0 to be at most 0. Furthermore, we can still assume that $S = (s_1, s_2) \in \bar{F}(P_0)$ is a half-integral point with the maximum possible second coordinate s_2 and $s_2 \geq s_1 > 0$. We now get $s_1 = \frac{1}{2}$, since otherwise $\operatorname{conv}((0,0),(1,0),(s_1,s_2)) \subseteq \operatorname{int}(P_0)$ contains the lattice point $(1,1) \in M$, which contradicts our choice of coordinates. By symmetry it is now sufficient to show that the line segment $\operatorname{conv}((0,0), S)$ is a subset of an edge of $\bar{F}(P_0)$. As in the first part of the proof, we consider parallel lines to the affine hull of e_{S_l} through $(0,1)$ and to the affine hull of e_{S_2} through $(1,1)$. If $\operatorname{conv}((0,0), S)$ is not a subset of an edge of $\bar{F}(P_0)$, then we get that all points of P_0 have a second coordinate of at most $s_2 + \frac{1}{2}$. If $s_2 \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, then this contradicts the fact that $S \in \bar{F}(P_0)$.If $s_2 \in \mathbb{Z}$, then the affine hull of $(0,1)$ and $(1/2, s_2 + 1/2)$ must be a supporting line of P_0. But this also contradicts $S \in \bar{F}(P_0)$ and so we are done. $\square$

Since we have seen, that edges of $\operatorname{conv}(\operatorname{int}(P_0) \cap M)$ with 3 or more lattice points are always edges of $\bar{F}(P_0)$, we get the following corollary

Corollary 3.6. *Let $P_0 \subseteq M_{\mathbb{R}}$ be a half-integral polygon with at least three interior lattice points, which are collinear. Then $\dim \bar{F}(P_0) = 1$ and we have in particular that P_0 is affine unimodular equivalent to a polygon in $\mathbb{R} \times [-1, 1]$.*

We now focus on the edges of $\operatorname{conv}(\operatorname{int}(P_0) \cap M)$ which are edges of $\bar{F}(P_0)$. For them we have the following proposition.

Proposition 3.7. *Let $P_0 \subseteq M_{\mathbb{R}}$ be a half-integral polygon with at least two interior lattice points, $e \preceq \operatorname{conv}(\operatorname{int}(P_0) \cap M)$ an edge which is not an edge of $\bar{F}(P_0)$. Then there is only one vertex v of $\bar{F}(P)$, which is on the other side of e than $\operatorname{int}(P_0) \cap M$. Moreover, if the lattice distance of v from e is h, then P_0 has at least $2(h-1)$ interior lattice points, and the triangle $\operatorname{conv}(e, v)$ is unimodular affine equivalent to the half-integral triangle $\operatorname{conv}((0,0),(1,0),(1/2,h))$.*

Proof. We have already seen in the proof of the second part of 3.5, that there is only one vertex v of $\bar{F}(P)$, which is on the other side of e than $\operatorname{int}(P_0) \cap M$ and that the triangle $\operatorname{conv}(e, v)$ is unimodular affine equivalent to the half-integral triangle $\operatorname{conv}((0,0),(1,0),(1/2,h))$. So we can assume that the triangle is $\operatorname{conv}((0,0),(1,0),(1/2,h))$ and we can shift the affine hulls of $(0,0),(1/2,h)$ and $(1,0),(1/2,h)$ by one integral step outside and bound with them an area which contains P_0. In particular, since $(0,0),(1,0)$ are interior lattice points of P_0 there must a points $(x_1,x_2) \in P_0$ with $x_1 \leq -1/2, x_2 \leq -h+1$ as well as a point with $x_1 \geq 3/2, x_2 \leq -h+1$. Therefore all the lattice points in

$$\operatorname{conv}((0,0),(1,0),(0,-h-2),(1,h-2))$$

are interior lattice point of P_0 and we therefore have at least $(2(h-1))$ interior lattice points of P_0. $\square$

Corollary 3.8. *The lattice height h of the triangles on edges of* $\operatorname{conv}(\operatorname{int}(P_0) \cap M)$ *with two lattice points from the last proposition is bounded by* $\frac{1}{2}(|P_0 \cap M| + 2)$.

Let us end this subsection with summerizing some propeties of $\bar{F}(P_0)$ we have proven so far.

Theorem 3.9. *Let $P_0 \subseteq M_{\mathbb{R}}$ be a half-integral polygon with* $\dim(\bar{F}(P_0)) = 2$. *Then $\bar{F}(P_0)$ is a half-integral polygon, every edge of $\bar{F}(P_0)$ contains lattice points and we get $\bar{F}(P_0)$ as union of* $\operatorname{conv}(\operatorname{int}(P_0) \cap M)$ *with some triangles, whereby the base of each of these triangle is an edge of* $\operatorname{conv}(\operatorname{int}(P_0) \cap M)$ *with two lattice point and the third vertex of the triangle has a lattice height of at most* $\frac{1}{2}(|P_0 \cap M| + 2)$ *over this edge. Moreover, every vertex of* $\operatorname{conv}(\operatorname{int}(P_0) \cap M)$ *is a boundary lattice point of $\bar{F}(P_0)$.*

3.3. **The maximal half-integral polygon to a given Fine interior.** In this subsection we want to describe the unique inclusion maximal lattice polytope to a given Fine interior and the unique inclusion maximal half-integral polygon for given $\bar{F}$. This leads also to a test, if a given polytope can occure as a Fine interior.

We start by moving out the facets of a polytope by lattice distance 1.

Definition 3.10. Let $P \subseteq M_{\mathbb{R}}$ be a rational polytope. Then we define

$$P^{(-1)} := \{x \in M_{\mathbb{R}} \mid \langle x, n \rangle \geq \operatorname{Min}_P(n) - 1 \ \forall n \in \Sigma_P[1]\}.$$

Remark 3.11. Moving out the facets is partially an inverse operation of computing the Fine interior. We get directly from the definition the following relations

$$F(\operatorname{conv}(P^{(-1)} \cap M)) \subseteq F(P^{(-1)}) \subseteq P.$$

The opposite inclusions are not correct in general but they hold for Fine interiors as we see in the next lemma.

Lemma 3.12. *A rational d-polytope $Q \subseteq M_{\mathbb{R}}$ is a Fine interior of a rational polytope if and only if $Q \subseteq F(Q^{(-1)})$. Moreover, Q is a Fine interior of a lattice polytope if and only if $Q \subseteq F(\operatorname{conv}(Q^{(-1)} \cap M))$.*

Proof. If the inclusions hold then Q is a Fine interior since the opposite inclusions hold in general. If Q is a Fine interior, then there is a d-polytope $Q' \subseteq M_{\mathbb{R}}$ with $F(Q') = Q$. We get $Q' \subseteq Q^{(-1)}$ and so by the monotony of the Fine interior $Q = F(Q') \subseteq F(Q^{(-1)})$. □

The situation becomes especially easy for lattice polygons. We have by [HS09, Lemma 9] the following proposition.

Proposition 3.13. *A lattice polygon Q is a Fine interior of a lattice polygon if and only if $Q^{(-1)}$ is a lattice polygon.*

Remark 3.14. All lattice polygons with up to 112 lattice points which are a Fine interior a lattice polygon were classified for calculations in [BS24a]. The data are available on zenodo [BS24b].

We now want similar results as for the Fine interior also for $\bar{F}$.

Definition 3.15. Let $P_0 \subseteq M$ be a half-integral polygon with $\dim \bar{F}(P_0) = 2$. Then we call P_0 a maximal half-integral polygon for given $\bar{F}$ if and only if

$$P_0 = \operatorname{conv}\left((\bar{F}(P_0))^{(-1)} \cap M/2\right).$$

Remark 3.16. The maximal half-integral polygon for given $\bar{F}$ is unique by definition and contains all other half-integral polygons with the same $\bar{F}$. If a half-integral polygon is maximal with respect to inclusion among all half-integral polygons with the same number of interior lattice points, it is in particular maximal for given $\bar{F}$. Thus we get all inclusion maximal half-integral polygons as a subset of the polygons maximal for given $\bar{F}$. This was used to classify half-integral polygons in [BS24a].

We get now analogue to above a test for Fine interiors of dimension 2 of lattice 3-polytopes.

Corollary 3.17. *A half-integral polygon $Q \subseteq M_{\mathbb{R}}$ is unimodular equivalent to the Fine interior of a lattice 3-polytope if and only if*

$$Q = \bar{F}(\operatorname{conv}\left(Q^{(-1)} \cap M/2\right)).$$

4. Classification of Fine interiors

The previous section gives us the following algorithm to classify the two dimensional Fine interiors of lattice 3 polytopes with given number of lattice points.

1. Take all lattice polygons with the given number of lattice points, e.g. from the dataset [BS24b].
2. For each lattice polygon with the given number of lattice points consider all possibilities to put triangles on the edges with 2 lattice points, which are permitted by 3.9. We get a finite number of lattice polygons with hats, since the height of the hats is bounded and the lattice polygon with hats needs to be convex.
3. Check for each lattice polygon with hats from the last step with 3.17, if it is indeed a Fine interior or not and discard the polygons which are not a Fine interior.

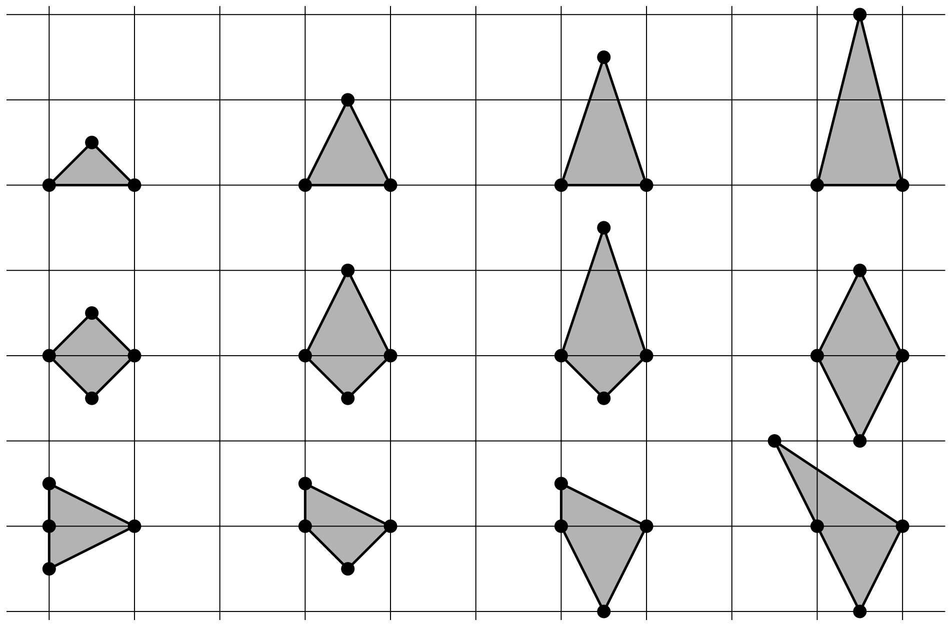

FIGURE 1. All Fine interiors of dimension 2 for lattice 3-polytopes with 2 interior lattice points

(4) Eliminate affine unimodular equivalent Fine interior, e. g. by use of a normal form as described in [BS24a, section 2.1.].

This algorithm was used for calculations with up to 40 lattice points and we got the following result.

Theorem 4.1. *There are exactly 24 324 158 half-integral polygons with at most 40 lattice points, which are Fine interiors of a lattice 3-polytope. The data for the vertices of these polygons are available on* [Boh24a]*. The examples with* 2 *or* 3 *lattice points can be seen in figure 1 and 2.*

5. Geography of minimal surfaces of general type arising from lattice 3-polytopes of lattice width 2

We can also visualize invariants of the minimal surfaces which correspond the lattice 3-polytopes with Fine interior of dimension 2.

Theorem 5.1. *(*[Bat23, 9.4], [Gie22, 5.1]*) Let $\hat{Z}$ be a minimal surface coming from a nondegenerated toric hypersurface with Newton polytope P of dimension d with* $\dim(F(P)) = d - 1$*. Then*

$$(K_{\hat{Z}})^{d-1} = 2\mathrm{Vol}_{d-1}(F(P))$$

Corollary 5.2. *If* $\dim(F(P)) \geq d - 1$*, then* $2\mathrm{Vol}_{\dim F(P)}(F(P)) \in \mathbb{Z}$*.*

Proof. If $\dim(F(P)) = d - 1$, then $2\mathrm{Vol}_{d-1}(F(P)) = (K_{\hat{Z}})^{d-1} \in \mathbb{Z}$. If $\dim(F(P)) = d$, then we have for $Q := P \times [-1, 1]$, that $F(Q) = F(P)$ and $2\mathrm{Vol}_d(F(P)) = 2\mathrm{Vol}_d(F(Q)) = (K_{\hat{Z}})^d \in \mathbb{Z}$. $\square$

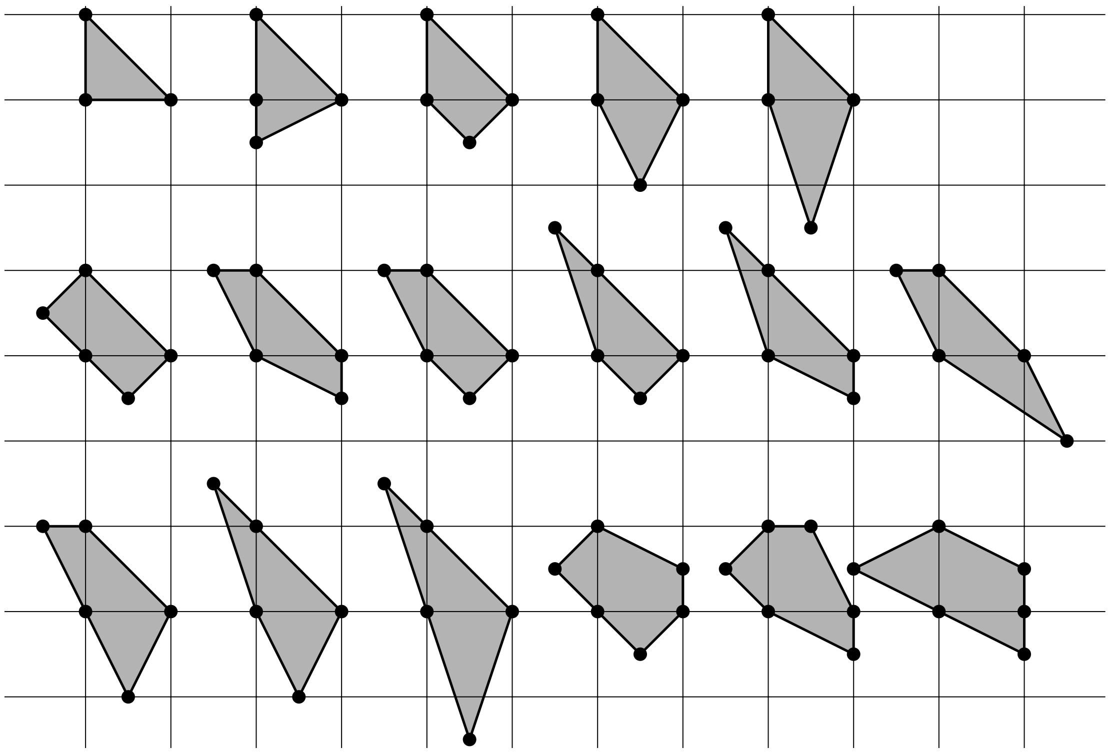

FIGURE 2. All Fine interiors of dimension 2 for lattice 3-polytopes with 3 interior lattice points

If $d = 3$, then we get for the Chern numbers the following corollary.

Corollary 5.3. [Bat23, 10.1]

$$\begin{aligned} c_1^2(\hat{Z}) =& 2\mathrm{Vol}_2(F(P)) \\ c_2(\hat{Z}) =& 12(1+\chi) - c_1^2 = 12(1 + |F(P) \cap M|) - c_1^2 \end{aligned}$$

Remember that the Chern numbers of surfaces of general type are restricted by general theory. We have the following theorem

Theorem 5.4. [BHPV04, VII. (1.1)] *$c_1^2, c_2 \in \mathbb{Z}_{\geq 0}$, $c_1^2 + c_2 = 12\chi \equiv 0 \mod 12$ and*

$$\begin{aligned} c_1^2 \leq& 3c_2 && (\textit{Bogomolov–Miyaoka–Yau inequality}) \\ c_1^2 \geq 2\chi - 4 =& \frac{c_2 - 36}{5} && (\textit{Noether inequality}) \end{aligned}$$

For surfaces on the Noether line we have the following corollary.

Corollary 5.5. *P defines a surface with Chern numbers on the Noether line, i.e. with $c_1^2 = 2\chi - 4$, if and only if $F(P)$ is a hollow polygon, i.e. it has no interior lattice points. Moreover, for every point on the Noether line exists a suitable hollow $F(P)$.*

Proof. $c_1^2 \geq 2\chi - 4$ gives for the Fine interior $2\mathrm{Vol}_2(F(P)) \geq 2|F(P) \cap M| - 4$. By Pick's formula $2\mathrm{Vol}_2(F(P)) \geq 4|F(P) \cap M| - 2|\partial F(P) \cap M| - 4 \geq 2|F(P) \cap M| - 4$ with equality if and only if $F(P)$ is a hollow polygon. □

If the Fine interior is a lattice polygon, we get $c_1^2, c_2 \in 2\mathbb{Z}_{\geq 0}$ and the following additional restrictions from the theory of lattice polygons:

Theorem 5.6 ([Sco76]). *$|\partial P \cap M| \leq 2|\mathrm{int}(P) \cap M| + 6$ or $P \cong 3\Delta_2$*

Corollary 5.7.

$$c_1^2 = 2\mathrm{Vol}_2(F(P)) \geq \frac{8}{3}|F(P) \cap M| - 8 = \frac{2}{7}(c_2 - 48)$$

Proof. We have $3|\partial P \cap M| \leq 2|P \cap M| + 6$ or $P \cong 3\Delta_2$ from Scott's inequality. So we get by Pick's theorem

$$\begin{aligned} c_1^2 =& 2\mathrm{Vol}_2(F(P)) = 4|F(P) \cap M| - 2|\partial F(P) \cap M| - 4 \geq \frac{8}{3}|F(P) \cap M| - 8 \\ =& \frac{2}{7}(c_2 - 48). \end{aligned}$$

□

Proposition 5.8. *$vol(P) \leq l - \frac{5}{2}$ and $c_1^2 \leq \frac{1}{2}(c_2 - 42)$.*

Proof. We have at least 3 boundary points, so by Pick's theorem $vol(P) \leq l - \frac{5}{2}$. Now we have

$$c_1^2 = 2\mathrm{Vol}_2(F(P)) \leq 4|F(P) \cap M| - 10 = \frac{c_2 + c_1^2}{3} - 14 \leq \frac{1}{2}(c_2 - 42).$$

□

Our classification of Fine interiors 4.1 can be translated into the following theorem, result of which is visualized in figure 3. Notably, we observe many half-integral Fine interiors that realize Chern numbers not achievable by examining only Fine interiors that are lattice polygons. For instance, there are Chern numbers in areas not permitted by Scott's inequality. These Fine interiors arise in the algorithm from putting hats on hollow lattice polygons. Corresponding surfaces can be thought of as kind of a generalization of the surfaces with Chern numbers on the Noether line and Fine interiors, which are hollow polygons. Surfaces with Chern numbers on the Noether line are often referred to as *Horikawa* surfaces, named after Eiji Horikawa, who studied them, for example, in [Hor76].

Theorem 5.9. *A pair (c_1^2, c_2) is a pair of Chern numbers of a non-degenate toric hypersurface corresponding to a lattice polytope of width 2 with up to 40 interior lattice points and 2 dimensional Fine interior if and only if it correspondents to a pair (χ, c_1^2) of the following table*

χ	*number of Fine interiors*	c_1^2 *is element of*
2	12	$[1,4] \cap \mathbb{Z}$
3	17	$[2,6] \cap \mathbb{Z}$
4	48	$[4,10] \cap \mathbb{Z}$
5	86	$[6,12] \cap \mathbb{Z}$
6	177	$[8,17] \cap \mathbb{Z}$
7	279	$[10,19] \cap \mathbb{Z}$
8	504	$[12,24] \cap \mathbb{Z}$
9	768	$[14,26] \cap \mathbb{Z}$
10	1222	$[16,31] \cap \mathbb{Z}$
11	1850	$[18,34] \cap \mathbb{Z}$
12	2881	$[20,39] \cap \mathbb{Z}$
13	4160	$[22,41] \cap \mathbb{Z}$
14	6150	$[24,47] \cap \mathbb{Z}$
15	8480	$[26,49] \cap \mathbb{Z}$
16	12066	$[28,54] \cap \mathbb{Z}$
17	16746	$[30,56] \cap \mathbb{Z}$
18	23462	$[32,62] \cap \mathbb{Z}$
19	31601	$[34,64] \cap \mathbb{Z}$
20	42914	$[36,69] \cap \mathbb{Z}$
21	56675	$[38,72] \cap \mathbb{Z}$
22	75457	$[40,77] \cap \mathbb{Z}$
23	98713	$[42,79] \cap \mathbb{Z}$
24	129468	$[44,85] \cap \mathbb{Z}$
25	167366	$[46,87] \cap \mathbb{Z}$
26	216764	$[48,92] \cap \mathbb{Z}$
27	276569	$[50,95] \cap \mathbb{Z}$
28	352907	$[52,100] \cap \mathbb{Z}$
29	446184	$[54,103] \cap \mathbb{Z}$
30	564041	$[56,108] \cap \mathbb{Z}$
31	706531	$[58,110] \cap \mathbb{Z}$
32	884749	$[60,116] \cap \mathbb{Z}$
33	1097809	$[62,118] \cap \mathbb{Z}$
34	1362551	$[64,124] \cap \mathbb{Z}$
35	1681298	$[66,127] \cap \mathbb{Z}$
36	2071958	$[68,131] \cap \mathbb{Z}$
37	2535238	$[70,134] \cap \mathbb{Z}$
38	3099580	$[72,139] \cap \mathbb{Z}$
39	3768629	$[74,142] \cap \mathbb{Z}$
40	4578248	$[76,147] \cap \mathbb{Z}$

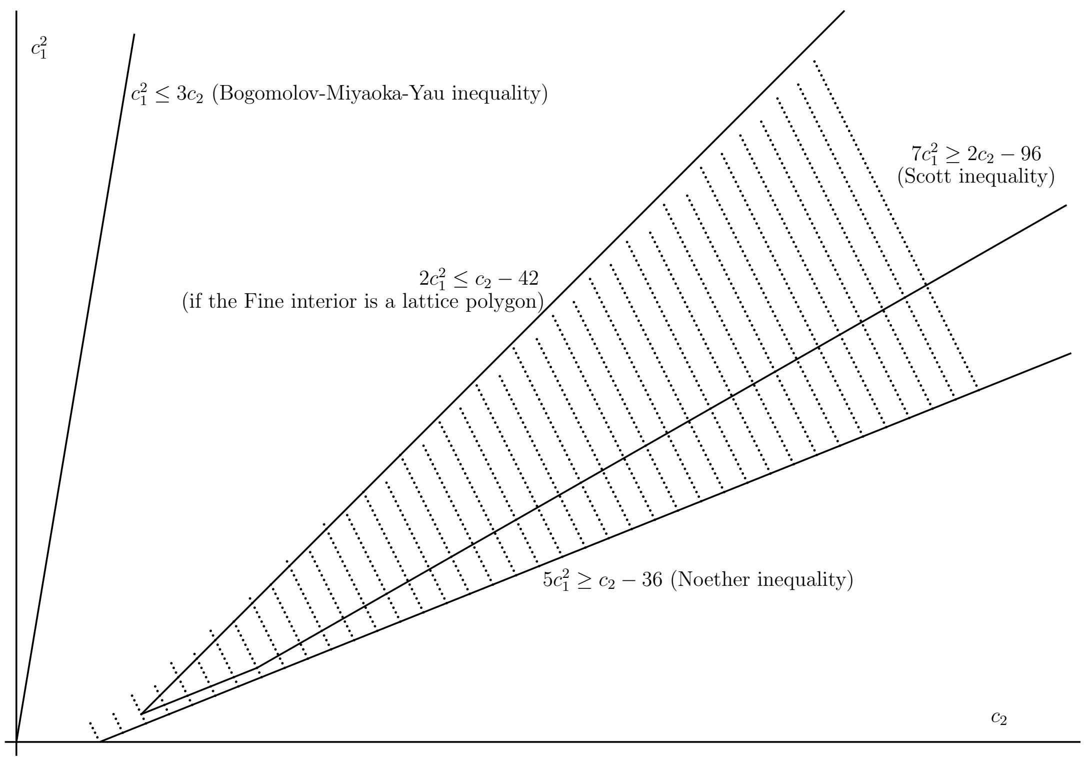

FIGURE 3. The Chern numbers of the classified surfaces of general type

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6. The Fine interior of dilations of a rational polytope

The following manuscript is consistent with the preprint published on arXiv at arXiv:2412.11163v2.

THE FINE INTERIOR OF DILATIONS OF A RATIONAL POLYTOPE

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Abstract. A nondegenerate toric hypersurface of negative Kodaira dimension can be characterized by the empty Fine interior of its Newton polytope according to recent work by Victor Batyrev, where the Fine interior is the rational subpolytope consisting of all points which have an integral distance of at least 1 to all integral supporting hyperplanes of the Newton polytope. Moreover, we get more information in this situation if we can describe how the Fine interior behaves for dilations of the Newton polytope, e.g. if we can determine the smallest dilation with a non-empty Fine interior. Therefore, in this article we give a purely combinatorial description of the Fine interiors of all dilations of a rational polytope, which allows us in particular to compute this smallest dilation and to classify all lattice 3-polytopes with empty Fine interior, for which we have only one point as Fine interior of the smallest dilation with non-empty Fine interior.

1. Introduction

The Fine interior $F(P)$ of a lattice d-polytope $P \subseteq \mathbb{R}^d$ was introduced by Jonathan Fine in [Fin83, §4.2.] to study top degree differentials of hypersurfaces using their Newton polytope. Originally called *heart* of the Newton polytope, we now use the term *Fine interior* as introduced by Miles Reid when working with plurigenera and canonical models of nondegenerate toric hypersurfaces in [Rei87, II, 4 Appendix].

If the Fine interior of the Newton polytope of a nondegenerate hypersurface in a torus is not empty, then we can construct both a unique projective model with at worst canonical singularities and minimal models of the hypersurface using the Fine interior and related combinatorial data, as recently shown by Victor Batyrev in [Bat23]. However, if the Fine interior is empty, the hypersurface has negative Kodaira dimension, and we either get the Zariski closure of the hypersurface in a suitable canonical toric $\mathbb{Q}-$Fano variety as a canonical $\mathbb{Q}$-Fano hypersurface, or we get a $\mathbb{Q}$-Fano fibration for the compactification of the hypersurface by [Bat24].

We focus here on the second case with an empty Fine interior and call a lattice polytope with an empty Fine interior *F-hollow*. To study this case in detail, it is necessary to understand how the Fine interior behaves for dilations of the lattice polytope. The most important affine unimodular invariants of the lattice polytope P in this context are the *minimal multiplier* $\mu = \mu(P)$, which is the smallest real number λ with $F(\lambda P) \neq \emptyset$, and the associated Fine interior $F(\mu P)$. We call an F-hollow lattice polytope P *weakly sporadic*, if $\dim(F(\mu P)) = 0$, which corresponds to the first situation with a canonical $\mathbb{Q}$-Fano hypersurface. If $\dim(F(\mu P)) > 0$, we get the $\mathbb{Q}$-Fano fibration with a combinatorial description from the lattice projection of P along $F(\mu P)$.

There are two main goals of this article: first to give a simple purely combinatorial description of the Fine interiors of all dilations of a rational polytope (Theorem 3.3), which allows in particular to compute the minimal multiplier and to see it as a rational number (Corollary 5.3), and, as a second goal, to use these tools to classify all weakly sporadic F-hollow 3-polytopes (Theorem 6.8, Theorem 6.11).

We begin by fixing the notation and give precise definitions of the Fine interior and related combinatorial objects in the next section. The third section gives the description of the Fine interior of dilations of a lattice polytope as the intersection of a hyperplane with a polyhedron of dimension $d+1$, before we take a closer look at the commutativity of intersections with hyperplanes and the computation of the Fine interior in section 4. In section 5 we introduce the minimal and some other special multipliers that mark combinatorial changes for the Fine interior during the dilation of the polytope. And in the last section we give the classification of the weakly sporadic F-hollow polytopes in dimension 3.

2. The Fine interior and associated combinatorial objects

Let be $d \in \mathbb{Z}_{\geq 1}$, $M \cong \mathbb{Z}^d$ be a lattice of rank d, $N = \operatorname{Hom}(M, \mathbb{Z})$ be the lattice dual to M and $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$ be the natural pairing. We have real vector spaces $M_{\mathbb{R}} = M \otimes \mathbb{R}$ and $N_{\mathbb{R}} = N \otimes \mathbb{R}$ and extend the natural pairing to them.

We will look now at convex geometric objects in $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ and give a brief description of the objects we will need later. For more details see e.g. [Zie07]. The convex hull of a finite number of points in $M_{\mathbb{Q}} \subseteq M_{\mathbb{R}}$ is called a *rational d-polytope* if there are $d+1$ affine independent points among these points. A *lattice d-polytope* is a rational d-polytope with all vertices in M. For $\lambda \in \mathbb{R}$ and a rational d-polytope $P \subseteq M_{\mathbb{R}}$ we define the real dilation

$$\lambda P := \{\lambda \cdot x \in M_{\mathbb{R}} \mid x \in P\} \subseteq M_{\mathbb{R}}.$$

By a *d-polytop* in $M_{\mathbb{R}}$ we mean in the following a real dilation of some rational d-polytope, and these are the most general polytopes we work with.

If a dual vector $y \in N_{\mathbb{R}} \setminus \{0\}$ attains its minimum on a set $P \subseteq M_{\mathbb{R}}$, we denote this minimum by

$$\operatorname{Min}_P(y) := \min\{\langle x, y\rangle \mid x \in P\}$$

and we write $\operatorname{Min}_P(y) = -\infty$ if for all $n \in \mathbb{Z}_{\geq 0}$ there is a point $p_n \in P$ with $\langle p_n, y\rangle < -n$. We can use this notation, for example, to describe a convex, closed set $P \subseteq M_{\mathbb{R}}$ with help of the supporting hyperplane theorem as an intersection of closed half-spaces by

$$P = \{x \in M_{\mathbb{R}} \mid \langle x, y\rangle \geq \operatorname{Min}_P(y)\ \forall y \in N_{\mathbb{R}} \setminus \{0\}\}.$$

For a d-polytope $P \subseteq M_{\mathbb{R}}$ we define additional data to simplify this dual description. For a face $F \preceq P$, i.e. the intersection of P with a supporting hyperplane, we have a cone

$$\sigma_F := \{y \in N_{\mathbb{R}} \mid \operatorname{Min}_P(y) = \operatorname{Min}_F(y)\} \subseteq N_{\mathbb{R}},$$

the *normal cone of the face F*. All normal cones together form a fan, which we call the *normal fan* Σ_P of the polytope P. We write $\Sigma_P[1] \subseteq N$ for the set of primitive ray generators of the normal fan, i.e. the primitive dual lattice vectors that generate the cones of dimension 1. We can now use this to uniquely describe our d-polytope as the intersection of the finitely many closed half-spaces associated to facets, i.e. faces of dimension $d-1$, by

$$P = \{x \in M_{\mathbb{R}} \mid \langle x, n\rangle \geq \mathrm{Min}_P(n)\ \forall n \in \Sigma_P[1]\}.$$

More generally, we call the possibly unbounded intersection of finitely many closed half-spaces in $M_{\mathbb{R}}$ a *polyhedron*. A *rational polyhedron* is a polyhedron, where we can chose for each facet of the polyhedron a normal vector from N. If we have 0 in each facet, we have a *polyhederal cone*. We can associate to each polytope in $M_{\mathbb{R}}$ the *cone over* P, i.e. the polyhedron cone defined by

$$\mathrm{cone}(P) := \mathbb{R}_{\geq 0}(\{1\} \times P) \subseteq \mathbb{R} \times M_R.$$

Moreover, every polyhedron has a decomposition as a Minkowski sum of a polytope and a polyhedral cone where the polyhedral cone is uniquely determined and is called the *recession cone of the polyhedron*.

We will now introduce the Fine interior of a d-polytope as the central combinatorial object of the article.

Definition 2.1. Let $P \subseteq M_{\mathbb{R}}$ be a d-polytope. Then the *Fine interior* $F(P)$ of P is defined as

$$F(P) := \{x \in M_{\mathbb{R}} \mid \langle x, n\rangle \geq \mathrm{Min}_P(n) + 1\ \forall n \in N \setminus \{0\}\} \subseteq P.$$

In fact, we do not need to use all the dual vectors of $N \setminus \{0\}$ in the definition 2.1. It is enough to choose the finitely many dual vectors that are part of a Hilbert basis of some cone of the normal fan Σ_P ([Bat23, 3.10]). In particular, since $F(P)$ is bounded as a subset of P, the Fine interior of a d-polytope is itself a d-polytope. By [Bat23, 5.4] it is even possible to restrict to primitive ray generators in the canonical refinement of Σ_P, i.e. elements of

$$\Sigma_P^{\mathrm{can}}[1] = \{n \in N \mid n \text{ is vertex of } \mathrm{conv}\,(\sigma \cap (N \setminus \{0\})) \text{ for a maximal cone } \sigma \in \Sigma_P\}$$

and so we get

$$F(P) = \{x \in M_{\mathbb{R}} \mid \langle x, n\rangle \geq \mathrm{Min}_P(n) + 1\ \forall n \in \Sigma_P^{\mathrm{can}}[1]\}.$$

But this is the strongest possible a priori restriction on dual lattice vectors, since by [Bat23, 5.3] we need all $n \in \Sigma_P^{\mathrm{can}}[1]$ if we look at λP for λ large enough instead of P.

A posteriori, we could use in the definition 2.1 only those dual vectors that define half-spaces which really touch $F(P)$ after the shift by 1. We have the following formal definition for these dual vectors:

Definition 2.2. Let $P \subseteq M_{\mathbb{R}}$ be a d-polytope with $F(P) \neq \emptyset$. Then the *support of* $F(P)$ is the set of dual lattice vectors

$$S_F(P) := \{n \in N \mid \mathrm{Min}_{F(P)}(n) = \mathrm{Min}_P(n) + 1\} \subseteq N.$$

In the construction of canonical models of nondegenerate hypersurfaces [Bat23] there is another rational polytope defined by the support of the Fine interior which turns out to be very helpful, which we introduce now.

Definition 2.3. Let $P \subseteq M_{\mathbb{R}}$ be a d-polytope with $F(P) \neq \emptyset$. Then the *canonical hull of* P is the d-polytope

$$C(P) := \{x \in M_{\mathbb{R}} \mid \langle x, \nu \rangle \geq \mathrm{Min}_P(\nu)\ \forall \nu \in S_F(P)\} \subseteq P.$$

Moreover, P is called *canonically closed* if $C(P) = P$, i.e. if and only if we have $\Sigma_P[1] \subseteq S_F(P)$.

We end this section with two important examples of canonically closed lattice polytopes that we will need later.

Example 2.4. All lattice polygons, i.e. lattice polytopes of dimension $d = 2$, with non-empty Fine interior are canonically closed by [Bat23, 4.4].

Example 2.5. In arbitrary dimension the *reflexive polytopes*, i.e. the lattice polygons $P \subseteq M_{\mathbb{R}}$, which have $0 \in \mathrm{int}(P)$ and for which the dual polytope

$$P^* := \{y \in N_{\mathbb{R}} \mid \langle x, y \rangle \geq -1\ \forall x \in P\} \subseteq N_{\mathbb{R}}$$

is a lattice polytope, are canonically closed. More precisely, they are exactly those canonically closed lattice polytopes which have $\{0\}$ as their Fine interior ([Bat23, 4.9]). Reflexive polytopes are well known because of their important role in combinatorial mirror symmetry ([Bat94]). Note that we also get interesting objects for combinatorial mirror symmetry in some cases when the lattice polytope has only $\{0\}$ as its Fine interior, but it is not canonically closed ([Bat17]).

3. Simultaneous description of the Fine interiors of all dilations

In this section we want to describe the Fine interiors of all real dilations of a d-polytope simultaneously. As a first step, we want to understand the behavior of $\mathrm{Min}_P(y)$, Σ_P and $S_F(P)$ under real dilations. We do this in the following two lemmata.

Lemma 3.1. *Let $P \subseteq M_{\mathbb{R}}$ be a d-polytope and $\lambda \in \mathbb{R}_{>0}$. Then we have for all $y \in N_{\mathbb{R}}$ that $\lambda \mathrm{Min}_P(y) = \mathrm{Min}_{\lambda P}(y)$ and $\Sigma_P = \Sigma_{\lambda P}$.*

Proof. Since $\lambda > 0$ the multiplication with λ respects the minimum and due to the linearity of $\langle \cdot, y \rangle$ we have

$$\begin{aligned}\lambda \mathrm{Min}_P(y) =& \lambda \cdot \min\{\langle x, y \rangle \mid x \in P\} = \min\{\langle \lambda x, y \rangle \mid x \in P\} = \min\{\langle x, y \rangle \mid x \in \lambda P\} \\ =& \mathrm{Min}_{\lambda P}(y).\end{aligned}$$

If we use this not only for P, but also for a face $F \preceq P$, we get for the normal cones

$$\sigma_F = \{y \in N_{\mathbb{R}} \mid \mathrm{Min}_P(y) = \mathrm{Min}_F(y)\} = \{y \in N_{\mathbb{R}} \mid \mathrm{Min}_{\lambda P}(y) = \mathrm{Min}_{\lambda F}(y)\} = \sigma_{\lambda F}$$

and so we have $\Sigma_P = \Sigma_{\lambda P}$. □

Lemma 3.2. *Let $P \subseteq M_{\mathbb{R}}$ be a d-polytope with $F(P) \neq \emptyset$. If $\nu \in S_F(P)$, then $\nu \in S_F(\lambda P)$ for all $\lambda \in \mathbb{R}_{\geq 1}$. If $F(P)$ is full-dimensional and $\nu \in \Sigma_{F(P)}[1]$, then we also have $\nu \in \Sigma_{F(\lambda P)}[1]$ for all $\lambda \in \mathbb{R}_{\geq 1}$.*

Proof. Since $\nu \in S_F(P)$, there is a point $x \in F(P)$ with $\langle x, \nu \rangle = \mathrm{Min}_P(\nu) + 1$ and $\langle x, n \rangle \geq \mathrm{Min}_P(n) + 1$ for all $n \in N \setminus \{0\}$. For each $n \in N \setminus \{0\}$ we take some $x_n \in P$ with $\langle x_n, n \rangle = \mathrm{Min}_P(n)$ and set $w_n := x - x_n \in M_{\mathbb{R}}$. Then we have $x = x_\nu + w_\nu = x_n + w_n$ for all $n \in N \setminus \{0\}$ and

$$\langle w_\nu, \nu \rangle = \langle x - x_\nu, \nu \rangle = 1, \langle w_n, n \rangle = \langle x - x_n, n \rangle \geq 1.$$

Thus we get for $\lambda \in \mathbb{R}_{\geq 1}$ and $n \in N \setminus \{0\}$ with 3.1

$$\begin{aligned}
\langle \lambda x_\nu + w_\nu, n \rangle =& \langle \lambda(x_n + w_n - w_\nu) + w_\nu, n \rangle \\
=& \lambda \langle x_n, n \rangle + \lambda \langle w_n, n \rangle - (\lambda - 1) \langle w_\nu, n \rangle \\
=& \lambda \mathrm{Min}_P(n) + \lambda \langle w_n, n \rangle - (\lambda - 1) \langle x_n + w_n - x_\nu, n \rangle \\
=& \mathrm{Min}_{\lambda P}(n) + \langle w_n, n \rangle - (\lambda - 1)\mathrm{Min}_P(n) + (\lambda - 1) \langle x_\nu, n \rangle \\
\geq& \mathrm{Min}_{\lambda P}(n) + 1,
\end{aligned}$$

because $x_\nu \in P$ and so $\langle x_\nu, n \rangle \geq \mathrm{Min}_P(n)$. So $\lambda x_\nu + w_\nu \in F(\lambda P)$ and we get $\nu \in S_F(\lambda P)$ by

$$\langle \lambda x_\nu + w_\nu, \nu \rangle = \lambda \mathrm{Min}_P(\nu) + 1 = \mathrm{Min}_{\lambda P}(\nu) + 1.$$

If $\nu \in \Sigma_{F(P)}[1]$, then we can do the same calculations not only for x but also for d affine independent points on the corresponding facet to ν of $F(P)$ and get d affine independent points on a facet of $F(\lambda P)$ defined by ν and thus $\nu \in \Sigma_{F(\lambda P)}[1]$ for all $\lambda \in \mathbb{R}_{\geq 1}$. $\square$

We are now ready to describe the Fine interior of all real dilations of a d-polytope in $M_{\mathbb{R}}$ simultaneously with the help of a suitable polyhedron in $\mathbb{R} \times M_{\mathbb{R}}$, which lies in the cone over the polytope.

Theorem 3.3. *Let $P \subseteq M_{\mathbb{R}}$ be a d-polytope and $\lambda \in \mathbb{R}_{\geq 0}$. Then*

$$\mathcal{F}(P) := \{(x_0, x) \in \mathbb{R} \times M_{\mathbb{R}} \mid \langle (x_0, x), (-\mathrm{Min}_P(\nu), \nu) \rangle \geq 1 \ \forall \nu \in \Sigma_P^{\mathrm{can}}[1]\} \subseteq \mathbb{R} \times M_{\mathbb{R}}$$

is a polyhedron without compact facets, the number of facets is $|\Sigma_P^{\mathrm{can}}[1]|$, the recession cone of $\mathcal{F}(P)$ is $\mathrm{cone}(P) = \mathbb{R}_{\geq 0}(\{1\} \times P)$*, and for all $\lambda \in \mathbb{R}_{\geq 0}$ we get the Fine interior of λP from $\mathcal{F}(P)$ by the identity*

$$\{\lambda\} \times F(\lambda P) = \mathcal{F}(P) \cap \{(x_0, x) \in \mathbb{R} \times M_{\mathbb{R}} \mid x_0 = \lambda\}.$$

Proof. The Fine interior of λP is given by

$$F(\lambda P) = \{x \in M_{\mathbb{R}} \mid \langle x, \nu \rangle \geq \mathrm{Min}_{\lambda P}(\nu) + 1 \ \forall \nu \in \Sigma_{\lambda P}^{\mathrm{can}}[1]\}.$$

With 3.1 we get

$$\begin{aligned}
F(\lambda P) =& \{x \in M_{\mathbb{R}} \mid \langle x, \nu \rangle \geq \lambda \mathrm{Min}_P(\nu) + 1 \ \forall \nu \in \Sigma_P^{\mathrm{can}}[1]\} \\
=& \{x \in M_{\mathbb{R}} \mid \langle (\lambda, x), (-\mathrm{Min}_P(\nu), \nu) \rangle \geq 1 \ \forall \nu \in \Sigma_P^{\mathrm{can}}[1]\}
\end{aligned}$$

and so we have

$$\begin{aligned}
&\{\lambda\} \times F(\lambda P) \\
=& \{(x_0, x) \in \mathbb{R} \times M_{\mathbb{R}} \mid \langle (x_0, x), (-\mathrm{Min}_P(\nu), \nu) \rangle \geq 1 \ \forall \nu \in \Sigma_P^{\mathrm{can}}[1]\} \cap \{x_0 = \lambda\}.
\end{aligned}$$

Since $\Sigma_P[1] \subseteq \Sigma_P^{\text{can}}[1]$ we get the recession cone of $\mathcal{F}(P)$ by [Zie07, Prop. 1.12.] as

$$\begin{aligned} &\{(x_0, x) \in \mathbb{R} \times M_{\mathbb{R}} \mid \langle (x_0, x), (-\mathrm{Min}_P(\nu), \nu) \rangle \geq 0 \ \forall \nu \in \Sigma_P^{\text{can}}[1]\} \\ =&\{(x_0, x) \in \mathbb{R} \times M_{\mathbb{R}} \mid \langle x, \nu \rangle \geq x_0 \cdot \mathrm{Min}_P(\nu) \ \forall \nu \in \Sigma_P^{\text{can}}[1]\} \\ =&\{(x_0, x) \in \mathbb{R} \times M_{\mathbb{R}} \mid x \in x_0 \cdot P\} \\ =&\mathrm{cone}(P). \end{aligned}$$

The facets of $\mathcal{F}(P)$ are unbounded because from $\nu \in \Sigma_{F(\lambda_0 P)}[1]$ we get $\nu \in \Sigma_{F(\lambda P)}[1]$ for all $\lambda \geq \lambda_0$ by 3.2 and the number of facets is $|\Sigma_P^{\text{can}}[1]|$ since we need all vertices of the canonical refinement to describe $F(\lambda P)$ for λ large enough. □

We now focus on the situation where P is at least a rational polytope, so that $\mathcal{F}(P)$ also becomes a rational polyhedron. For lattice polytopes we want to give an alternative description of $\mathcal{F}(P)$ using the Fine interior of cone(P). Although we have not yet given a formal definition of the Fine interior of cones, it seems reasonable to have this definition completely analogous. Since $n \in N \setminus \{0\}$ attains a finite minimum on σ if and only if n is an element of the dual cone

$$\check{\sigma} = \{y \in N_{\mathbb{R}} \mid \langle x, y \rangle \geq 0 \ \forall x \in \sigma\} \subseteq N_{\mathbb{R}}$$

and then the minimum is 0, we have

$$\begin{aligned} F(\sigma) :=&\{x \in M_{\mathbb{R}} \mid \langle x, n \rangle \geq \mathrm{Min}_\sigma(n) + 1 \ \forall n \in N \setminus \{0\}\} \\ =&\{x \in M_{\mathbb{R}} \mid \langle x, n \rangle \geq 1 \ \forall n \in \check{\sigma} \cap N \setminus \{0\}\}. \end{aligned}$$

This allows us in some cases to get $\mathcal{F}(P)$ directly as the Fine interior of cone(P), as the following corollary shows.

Corollary 3.4. *If P is a lattice polytope, then we have for all $\lambda \in \mathbb{R}_{\geq 1}$ that*

$$\begin{aligned} \mathcal{F}(P) \cap \{x_0 = \lambda\} =&F(\lambda P) \times \{\lambda\} = F(\mathrm{cone}(P) \cap \{x_0 = \lambda\}) \times \{\lambda\} \\ =&F(\mathrm{cone}(P)) \cap \{x_0 = \lambda\}. \end{aligned}$$

In particular, we have $\mathcal{F}(P) = F(\mathrm{cone}(P))$ if and only if $F(\lambda P) = \emptyset$ for all $\lambda < 1$.

Proof. By 3.3 and the definition of cone(P) we get

$$\mathcal{F}(P) \cap \{x_0 = \lambda\} = F(\lambda P) \times \{\lambda\} = F(\mathrm{cone}(P) \cap \{x_0 = \lambda\}) \times \{\lambda\}.$$

Since we have by definition

$F(\mathrm{cone}(P)) =$

$\{(x_0, x) \in \mathbb{R} \times M_{\mathbb{R}} \mid \langle (x_0, x), (n_0, n) \rangle \geq 1 \ \forall (n_0, n) \in \mathrm{cone}\check{}(P) \cap (\mathbb{Z} \times N) \setminus \{(0, 0)\}\}$,

it is by 3.3 and for $\lambda \geq 1$ enough to see that the vertices of

$$\mathrm{conv}\left(\mathrm{cone}\check{}(P) \cap (\mathbb{Z} \times N \setminus \{0\})\right)$$

are exactly given by $(-\mathrm{Min}_P(\nu), \nu)$ with $\nu \in \Sigma_P^{\text{can}}$. First notice that $-\mathrm{Min}_P(\nu) \in \mathbb{Z}$ since P is a lattice polytope. Also, for all $x \in P$ we have

$$\langle (-\mathrm{Min}_P(\nu), \nu), (\lambda, \lambda x) \rangle = -\lambda \mathrm{Min}_P(\nu) + \lambda \langle \nu, x \rangle) \geq 0$$

and therefore $(-\mathrm{Min}_P(\nu), \nu) \in \check{\mathrm{cone}}(P)$. With the same calculation we see that the facet $F_p \preceq \check{\mathrm{cone}}(P)$ corresponding to a vertex $p \preceq P$ is

$$F_p = \{(y_0, y) \in (\mathbb{Z} \times N)_{\mathbb{R}} \mid y \in \sigma_p, y_0 = -\mathrm{Min}_P(y)\}.$$

Thus every lattice point $(n_0, n) \in \check{\mathrm{cone}}(P) \cap (\mathbb{Z} \times N \setminus \{0\})$ shares its N-projection with a lattice point in a facet, namely $(-\mathrm{Min}_P(n), n)$. Thus the convex hull of $\check{\mathrm{cone}}(P) \cap (\mathbb{Z} \times N \setminus \{0\})$ is the convex hull of the lattice points different from 0 in the facets, and the vertices of this convex hull are the points $(-\mathrm{Min}_P(\nu), \nu)$ with $\nu \in \Sigma_P^{\mathrm{can}}$ by definition of the canonical refinement. □

We end this section with a characterization of the cases where $\mathcal{F}(P)$ is as simple as possible, i.e. it is $\mathcal{F}(P)$ is a rational cone shifted by a lattice vector. For this, recall the following notions for Fano polytopes (for more background on Fano polytopes see e.g. the survey [KN12]).

Definition 3.5. Let $P \subseteq M_{\mathbb{R}}$ be a lattice d-polytope. If $\mathrm{int}(P) \cap M = \{0\}$, then we call P a *canonical Fano polytope*. If there is some $k \in \mathbb{Z}_{\geq 1}$ such that kP is affine unimodular equivalent to a reflexive polytope, then we call P a *Gorenstein polytope of index k*.

We now get the following characterizations for the cases where $\mathcal{F}(P)$ is a rational cone shifted by a lattice vector.

Corollary 3.6. *Let P be a rational polytope. Then $\mathcal{F}(P)$ is a rational cone shifted by $(1, 0)$ if and only if P^* is a canonical Fano polytope.*

Proof. Since $\mathrm{cone}(P)$ is the recession cone of $\mathcal{F}(P)$ by 3.3, $\mathcal{F}(P)$ is a rational cone shifted by $(1, 0)$ if and only if $\mathcal{F}(P) = (1, 0) + \mathrm{cone}(P)$. But by 3.3 this is equivalent to $\Sigma_P[1] = \Sigma_P^{\mathrm{can}}[1]$, $F(P) = \{0\}$, and $\mathrm{Min}_P(\nu) = -1$ for all $\nu \in \Sigma_P[1]$, which we have if and only if P^* is a lattice polytope having only 0 as an interior lattice point. □

Corollary 3.7. *Let P be a lattice polytope. Then $\mathcal{F}(P)$ is a lattice cone with vertex $(k, x), k \in \mathbb{Z}_{\geq 1}, x \in M$ if and only if $kP - x$ is a reflexive polytope. In particular, P is then a Gorenstein polytope of index k.*

Proof. $\mathcal{F}(P)$ is a lattice cone with vertex $(k, x) \in \mathbb{Z} \times M$ if and only if $\mathcal{F}(kP - x)$ is a lattice cone with vertex $(0, 1)$. This is equivalent by 3.6 to the situation where $(kP - x)^*$ is a canonical Fano polytope, and since $kP - x$ is a lattice polytope, this is the case if and only if $kP - x$ is reflexive. □

4. The Fine interior and intersections with hyperplanes

The result in 3.4 motivates us to look at cases where we can do Fine interior computations in codimension 1 using intersections with hyperplanes. We will see in this section that we have such results for dilations of lattice polytopes with small lattice width. Recall that the *lattice width* $\mathrm{lw}(P)$ of a lattice polytope $P \subseteq M_{\mathbb{R}}$ is defined by

$$\mathrm{lw}(P) := \min\{-\mathrm{Min}_P(-n) - \mathrm{Min}_P(n) \mid n \in N \setminus \{0\}\}$$

and a dual vector $n_{lw} \in N$ with

$$\operatorname{lw}(P) = -\operatorname{Min}_P(-n_{lw}) - \operatorname{Min}_P(n_{lw})$$

is called a *lattice width direction*.

We start with the lattice pyramid $\operatorname{Pyr}(P)$ of a lattice d-polytope $P \subseteq M_{\mathbb{R}}$, which is defined by

$$\operatorname{Pyr}(P) := \operatorname{conv}\left(\{1\} \times P, (0,0)\right).$$

and clearly has $\operatorname{lw}(\operatorname{Pyr}(P)) = 1$. Since we can also consider $\operatorname{Pyr}(P)$ as the intersection of $\operatorname{cone}(P)$ with the half-space $x_0 \leq 1$, we get the following corollary from 3.4.

Corollary 4.1. *Let $P \subseteq M_{\mathbb{R}}$ be a lattice d-polytope, $\mu, \lambda \in \mathbb{Q}_{\geq 0}$. Then we have for dilations of the lattice pyramid $\operatorname{Pyr}(P) \subseteq \mathbb{R} \times M_{\mathbb{R}}$*

$$F(\mu\operatorname{Pyr}(P)) \cap \{x_0 = \lambda\} \cong \begin{cases} F(\lambda P) & \text{if } 1 \leq \lambda \leq \mu - 1 \\ \emptyset & \text{else.} \end{cases}$$

and if $F(\mu\operatorname{Pyr}(P)) \neq \emptyset$, then $\mu \geq 2$ and

$$\begin{aligned} &S_F(\mu\operatorname{Pyr}(P)) \\ &= \begin{cases} \{(-\operatorname{Min}_P(\nu), \nu) \in \mathbb{Z} \times N \mid \nu \in S_F(\mu-1)P\} \cup \{(0, \pm 1)\} & \text{if } F(P) \neq \emptyset \\ \{(-\operatorname{Min}_P(\nu), \nu) \in \mathbb{Z} \times N \mid \nu \in S_F(\mu-1)P\} \cup \{(0, -1)\} & \text{if } F(P) = \emptyset. \end{cases} \end{aligned}$$

In particular, $F(2\operatorname{Pyr}(P)) = \{1\} \times F(P)$ and for $F(P) \neq \emptyset$ we have that $2\operatorname{Pyr}(P)$ is canonically closed if and only if P is canonically closed.

Proof. For $(x_0, x) \in F(\mu\operatorname{Pyr}(P)) \subseteq \mathbb{R} \times M_{\mathbb{R}}$ we have

$$\langle (x_0, x), (1, 0) \rangle = x_0 \geq \operatorname{Min}_{\mu\operatorname{Pyr}(P)}((1,0)) + 1 = 1$$

and

$$\langle (x_0, x), (-1, 0) \rangle = -x_0 \geq \operatorname{Min}_{\mu\operatorname{Pyr}(P)}((-1,0)) + 1 = -\mu + 1.$$

So we get $F(\mu\operatorname{Pyr}(P)) \cap \{x_0 = \lambda\} = \emptyset$ for $\lambda < 1$ and $\lambda \geq \mu - 1$ and support vectors $(0, \pm 1)$ in the appropriate cases. If we can show that all the other support vectors of $F(\mu\operatorname{Pyr}(P))$ attain their minimum on $\mu\operatorname{Pyr}(P)$ on a face of dimension 1 or greater, we get the result as a corollary of 3.4. If $\mu \in \mathbb{Z}$, this is clear, because $\mu\operatorname{Pyr}(P) \cap \{x_0 = 1\} \cong P$ and $\mu\operatorname{Pyr}(P) \cap \{y = \mu - 1\} \cong (\mu - 1)P$ are lattice polytopes in this case, and so there is no addition support vector which attains its minimum only on a vertex of $\mu\operatorname{Pyr}(P)$. For $\mu \in \mathbb{Q}$ we get this, since the situation near vertices after a translation with a suitable vector from $\mathbb{Q}^{d+1}$ is locally the same as for $\lfloor \mu \rfloor \operatorname{Pyr}(P)$ and the calculation of the Fine interior commutes with translations. □

We can note expect, that we can always commute Fine interior calculations with intersections with hyperplanes. Nevertheless, we have at least the partial result of the following lemma.

Lemma 4.2. *Let $P \subseteq \mathbb{R} \times M_{\mathbb{R}}$ be a rational d-polytope with $P \cap (\{1\} \times M_{\mathbb{R}}) \neq \emptyset$ and $P \cap (\{-1\} \times M_{\mathbb{R}}) \neq \emptyset$, and $P_0 \subseteq M_{\mathbb{R}}$ defined by $\{0\} \times P_0 = P \cap (\{0\} \times M_{\mathbb{R}})$. Then $\{0\} \times F(P_0) \subseteq F(P) \cap (\{0\} \times M_{\mathbb{R}})$.*

Proof. If $(0,x) \notin F(P)\cap(\{0\}\times M_{\mathbb{R}})$, then we have some $(\nu_0,\nu) \in (\mathbb{Z}\times N)\setminus\{(0,0)\}$ with

$$\langle(0,x),(\nu_0,\nu)\rangle < \mathrm{Min}_P((\nu_0,\nu)) + 1.$$

Note that $\nu \neq 0$, because otherwise we had

$$\mathrm{Min}_P((\nu_0,0)) + 1 \leq -|\nu_0| + 1 \leq 0 = \langle(0,x),(\nu_0,0)\rangle$$

for all $\nu_0 \neq 0$, since $P\cap P\cap(\{1\}\times M_{\mathbb{R}}) \neq \emptyset \neq P\cap P\cap(\{-1\}\times M_{\mathbb{R}})$. So we get $\nu \neq 0$ with

$$\langle x,\nu\rangle = \langle(0,x),(\nu_0,\nu)\rangle < \mathrm{Min}_P((\nu_0,\nu)) + 1 \leq \mathrm{Min}_{\{0\}\times P_0}((\nu_0,\nu)) + 1 = \mathrm{Min}_{P_0}(\nu) + 1$$

and so $x \notin F(P_0)$. $\square$

As the main result of the section we show now, that everything is fine if we have a lattice polytopes of lattice width 2 whose middle-polytope is also a lattice polytope.

Proposition 4.3. *Let $P \subseteq \mathbb{R}\times M_{\mathbb{R}}$ be lattice polytope of lattice width 2 with $P \subseteq [-1,1]\times M_{\mathbb{R}}$, such that $\{0\}\times P_0 := P\cap(\{0\}\times M_{\mathbb{R}})$ is also a lattice polytope. Then*

$$F(P) = \{0\}\times F(P_0)$$

and if $F(P) \neq \emptyset$, then

$$S_F(P_0) = \{\nu \in N\setminus\{0\} \mid \exists \nu_0 \in \mathbb{Z}, (\nu_0,\nu) \in S_F(P)\}$$

and $S_F(P)$ is given as union of $\{(-1,0),(1,0)\}$ with

$$\{(\nu_0,\nu) \in \mathbb{Z}\times N \mid \nu \in S_F(P_0), \mathrm{Min}_P((\nu_0,\nu)) = \mathrm{Min}_{\{0\}\times P_0}((\nu_0,\nu))\}.$$

Proof. Since $P \subseteq [1,1]\times M_{\mathbb{R}}$ we have $F(P) = F(P)\cap(\{0\}\times M_{\mathbb{R}})$ and so we have $F(P) \supseteq \{0\}\times F(P_0)$ by 4.2.

Now we show $F(P) \subseteq F(P_0)\times\{0\}$. Since $P \subseteq [1,1]\times M_{\mathbb{R}}$ we have $F(P) \subseteq \{0\}\times P_0$ and it is enough to show that $(0,x) \notin F(P)$ for all $x \notin F(P_0)$. For $x \in M_{\mathbb{R}}\setminus F(P_0)$ we have some $0 \neq \nu \in N$ with

$$\langle x,\nu\rangle < \mathrm{Min}_{P_0}(\nu) + 1.$$

Since P_0 is a lattice polytope, we have a vertex $e \in M$ of P_0 with $\mathrm{Min}_{P_0}(\nu) = \langle e,\nu\rangle$. Let $f = (1,f') \in M$ be a vertex of P such that $\nu_0 := \langle e - f',\nu\rangle \in \mathbb{Z}$ is maximal. Then we have

$$\langle(1,f'),(\nu_0,\nu)\rangle = \nu_0 + \langle f',\nu\rangle = \langle e,\nu\rangle = \mathrm{Min}_{P_0}(\nu)$$

and since $\langle(0,e),(\nu_0,\nu)\rangle = \langle e,\nu\rangle = \mathrm{Min}_{P_0}(\nu)$ and ν_0 was chosen maximal, we get that (ν_0,ν) defines a hyperplane supporting P on on a face containing e and f and we have

$$\mathrm{Min}_P((\nu_0,\nu)) = \mathrm{Min}_{P_0}(\nu)$$

and

$$\langle(0,x),(\nu_0,\nu)\rangle = \langle x,\nu\rangle < \mathrm{Min}_{P_0}(\nu) + 1 = \mathrm{Min}_P((\nu_0,\nu)) + 1,$$

i.e. $(0,x) \notin F(P)$.

By the calculations above we get also that for every $\nu \in S_F(P_0)$ there exists a $\nu_0 \in \mathbb{Z}$, for example with ν_0 defined as above, with $(\nu_0, \nu) \in S_F(P)$. Since P_0 is a lattice polytope and $F(P) \subsetneq P_0$, we have $\mathrm{Min}_P((\nu_0, \nu)) = \mathrm{Min}_{\{0\}\times P_0}((\nu_0, \nu))$ for all $(\nu_0, \nu) \in S_F(P) \setminus \{(\pm 1, 0)\}$, and from $F(P) = \{0\} \times F(P_0)$ we get $\nu \in S_F(P_0)$. $\square$

As a corollary we can use two-dimensional results from [BS24] to explain some experimental results on the Fine interior of lattice 3-polytopes in [BKS22].

Corollary 4.4. *Let $P \subseteq \mathbb{R}^3$ be a lattice 3-polytope with $|int(P) \cap \mathbb{Z}^3| \leq 1$. Then $\dim F(P) \neq 2$.*

Proof. Suppose $\dim F(P) = 2$. Then P must have lattice width 2 and we can assume $P \subseteq [-1, 1] \times \mathbb{R}^2$. We have that the half-integral polygon $P_0 \subseteq \mathbb{R}^2$ definied by $\{0\} \times P_0 := P \cap (\{0\} \times \mathbb{R}^2)$ has at least 1 interior integral point. By classification results of maximal half-integral polygons in [BS24, section 5 and 6] we know, that then we can assume $P_0 \subseteq \mathbb{R} \times [-1, 1]$ or $P_0 \subseteq \mathrm{conv}\left((0, 0), (3, 0), (0, 3)\right)$. In particular, we can assume that P_0 is part of a lattice polygon $\bar{P}_0$ which has Fine interior of dimension smaller than 1. Thus by 4.3 we get $\dim F(\mathrm{conv}\left(P, \bar{P}_0\right)) < 2$ for the lattice polytope $\mathrm{conv}\left(P, \bar{P}_0\right)$ and from $F(P) \subseteq F(\mathrm{conv}\left(P, \bar{P}_0\right))$ we get the contradiction $\dim F(P) < 2$. $\square$

The following example shows that we cannot omit central premises in 4.3.

Example 4.5. For $P := \mathrm{conv}\left((-1, -1, -1), (1, 0, -1), (0, 1, -1), (0, 0, 1)\right)$, we have

$$F(P \cap \{x_3 = 0\}) = \emptyset \neq \{(0, 0, 0)\} = F(P) \cap \{x_3 = 0\} = F(P).$$

Note, that we have $\mathrm{lw}(P) = 2$ but

$$P \cap \{x_3 = 0\} = \mathrm{conv}\left((-1/2, -1/2), (1/2, 0, 0), (0, 1/2, 0)\right)$$

is not a lattice polytope. If the intersection gives a lattice polytope, we also cannot commute the calculation of the Fine interior and the intersection in general. See for example

$$\begin{aligned} F(2P \cap \{x_3 = 0\}) &= F((-1, -1), (1, 0, 0), (0, 1, 0)) = \{(0, 0, 0)\} \\ \neq &\mathrm{conv}\left((-1/2, -1/2, 0), (1/2, 0, 0), (0, 1/2, 0)\right) = F(2P) \cap \{x_3 = 0\}. \end{aligned}$$

We will later use the following corollary which helps us to understand the Fine interior for dilations of lattice polytopes of lattice width 1.

Corollary 4.6. *Let P be a lattice polytope of lattice width 1, $P \subseteq [0, 1] \times M_{\mathbb{R}}$ and $\{\frac{1}{2}\} \times P_{1/2} := P \cap (\{\frac{1}{2}\} \times M_{\mathbb{R}})$. Then $2P_{1/2}$ is a lattice polytope with*

$$F(2P) = \{1\} \times F(2P_{1/2}).$$

Moreover, if $F(2P) \neq \emptyset$, then we get that $2P$ is canonically closed if and only if $2P_{1/2}$ is canonically closed.

Proof. From 4.3 we get $F(2P) = \{1\} \times F(2P_{1/2})$ and since $2P$ has no vertices with first coordinate 1, we have for all $\nu \in S_F(2P_{1/2})$ exactly one $\nu_0 \in \mathbb{Z}$ with $(\nu_0, \nu) \in S_F(2P)$ and (ν_0, ν) attains its minimum on P on an edge. So $\Sigma_P[1] \in S_F(2P)$ if and only if $\Sigma_{P_{1/2}}[1] \in S_F(2P_{1/2})$. $\square$

We can even go further for $d = 2$ since the Fine interior of a lattice polygon is the convex hull of the interior lattice points by [Bat17, 2.9] and every lattice polygon with non-empty Fine interior is canonically closed by 2.4.

Corollary 4.7. *Let $P \subseteq \mathbb{R} \times M_{\mathbb{R}}$ be a lattice 3-polytope of lattice width 1. Then $F(2P) = \operatorname{conv}(\operatorname{int}(2P) \cap M)$ and if $F(2P) \neq \emptyset$, then $2P$ is canonically closed.*

We finish this section with an important example for the situation in the corollary.

Example 4.8. Let $\Delta \subseteq \mathbb{R}^3$ be an empty lattice 3-simplex of normalized volume q, i.e. $|\Delta \cap \mathbb{Z}^3| = 4$. Then we have by [Whi64] some $p \in \mathbb{Z}$ with $gcd(p,q) = 1$ and

$$\Delta \cong \Delta(p,q) := \operatorname{conv}\left((0,0,0),(1,0,0),(0,0,1),(p,q,1)\right),$$

in particular, the lattice width of Δ is 1. By 4.7 we get

$$\begin{aligned} F(2\Delta(p,q)) =& \operatorname{conv}\left(\operatorname{int}(2\Delta(p,q)) \cap \mathbb{Z}^3\right) \\ =& \operatorname{conv}\left(\operatorname{int}(\operatorname{conv}((0,0),(1,0),(p,q),(p+1,q))) \cap \mathbb{Z}^2\right) \times \{1\} \\ =& \operatorname{conv}\left(\left\{\left(\left\lceil \frac{jp}{q} \right\rceil, j, 1\right) \mid 0 < j < q\right\}\right). \end{aligned}$$

Thus $F(2\Delta(p,q)) = \emptyset$ if and only if $q = 1$. If the interior lattice points are collinear, we get after a shearing $p = \pm 1$. So we get for $q > 1$

$$\dim(F(2\Delta(p,q))) = \begin{cases} 0 & \text{if } q = 2 \\ 1 & \text{if } q \geq 3 \text{ and } p \equiv \pm 1 \mod q \\ 2 & \text{else.} \end{cases}$$

5. Special multipliers of a lattice polytope

In this section we focus on special dilations of a rational d-polytope P corresponding to vertices of $\mathcal{F}(P)$.

Definition 5.1. Let $P \subseteq M_{\mathbb{R}}$ be a rational d-polytope, $\{(\mu_1, p_1), \ldots, (\mu_n, p_n)\}$ the set of vertices of $\mathcal{F}(P) \subseteq \mathbb{R} \times M_{\mathbb{R}}$.
We call every μ_i a *special multipier of P*, $\mu(P) := \min_i \mu_i$ the *minimal multiplier*, and $\mu_{max}(P) := \max_i \mu_i$ the *maximal multiplier of P*.

Since P is rational, we now directly get the rationality of all special multipliers.

Corollary 5.2. *Let $P \subseteq M_{\mathbb{R}}$ be a rational d-polytope and $(-\operatorname{Min}_P(\nu_k), \nu_k) \in \mathbb{Q}^{d+1}$ for $1 \leq k \leq d+1$ linear independent vectors defining a vertex (μ_i, p_i) of the polyhedron $\mathcal{F}(P) \subseteq \mathbb{R} \times M_{\mathbb{R}}$.
Then we have*

$$\mu_i = \frac{\sum_{j=1}^{d+1} (-1)^{j+d+1} \left|\nu_1 \quad \ldots \quad \hat{\nu_j} \quad \ldots \quad \nu_{d+1}\right|}{\sum_{j=1}^{d+1} (-1)^{j+d} \operatorname{Min}_P(\nu_j) \left|\nu_1 \quad \ldots \quad \hat{\nu_j} \quad \ldots \quad \nu_{d+1}\right|} \in \mathbb{Q},$$

were the vectors $\hat{\nu_j}$ are canceled.

Proof. This follows directly from Cramer's rule and the Laplace expansion of the occurring determinants. □

In particular, we get now an easy combinatorial proof for the characterization of the minimal multiplier in [Bat23, 3.8.] and for its rationality proven in [Bat24, 3.2.] by methods from toric geometry.

Corollary 5.3. *Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional rational polytope. Then $\mu(P) \in \mathbb{Q}$ and $0 \leq \dim F(\lambda P) \leq d-1$ if and only if $\lambda = \mu(P)$.*

Proof. We have $\mu(P) \in \mathbb{Q}$ by 5.2. Moreover, the intersection of $\mathcal{F}(P) \subseteq M_{\mathbb{R}} \times \mathbb{R}$ with $M_{\mathbb{R}} \times \lambda$ has dimension between 0 and $d-1$ if and only if $\lambda = \mu(P)$ because $\mathcal{F}(P)$ has no facet parallel to $M_{\mathbb{R}} \times \{0\}$ since $\nu \neq 0$ for all $\nu \in \Sigma_P^{\mathrm{can}}[1]$. □

For the lattice pyramid $\mathrm{Pyr}(P)$ we can compute the minimal multiplier from the minimal multiplier of P using the results from the last section.

Corollary 5.4. *Let $P \subseteq M_{\mathbb{R}}$ be a lattice polytope with $\mu := \mu(P) \geq 1$. Then we get $\mu(\mathrm{Pyr}(P)) = \mu + 1$ for the minimal multiplier of the lattice pyramid and*

$$F((\mu+1)\mathrm{Pyr}(P)) \cong F(\mu P).$$

Proof. Since $\mu \geq 1$ we have by 4.1

$$F((\mu+1)\mathrm{Pyr}(P)) \cap (\{\lambda\} \times M_{\mathbb{R}}) = \begin{cases} F(\mu P) & \text{if } \lambda = \mu \\ \emptyset & \text{else.} \end{cases}$$

Thus we get

$$F((\mu+1)\mathrm{Pyr}(P)) = F((\mu+1)\mathrm{Pyr}(P)) \cap (\{\mu\} \times M_{\mathbb{R}}) \cong F(\mu P).$$

□

We now introduce some invariants of Ehrhart theory related to the minimal multiplier. Recall that for a lattice d-polytope $P \subseteq M_{\mathbb{R}}$, by results of Eugène Ehrhart and Richard P. Stanley, we have a unique polynomial $h_P^* \in \mathbb{Z}[x]$ with non-negative coefficients and $\deg h_P^* \leq d$ such that

$$\sum_{k \in \mathbb{Z}_{\geq 0}} |kP \cap M| x^k = \frac{h_P^*(x)}{(1-x)^{d+1}}.$$

We write $\deg(P) := \deg(h_P^*)$ and have for the *codegree of* P, i.e. $\mathrm{codeg}(P) := d + 1 - \deg(P)$ that $\mathrm{codegree}(P) = \min\{k \in \mathbb{Z}_{>0} \mid |\mathrm{int}(kP) \cap M| > 0\}$ and the highest non-zero coefficient of the $h*$ polynomial gives the number of interior lattice points of $\mathrm{codeg}(P) \cdot P$. For this and more results on Ehrhart theory see for example [BS15]. We get the following connection with the minimal multiplier.

Corollary 5.5. *Let $P \subseteq M_{\mathbb{R}}$ be a lattice d-polytope. Then we have the as a upper bound for the minimal multiplier $\mu(P) \leq \mathrm{codegree}(P) \leq d+1$.*

Since a rational d-polytope with lattice width smaller than 2 has empty Fine interior, we get also a lower bound for $\mu(P)$.

Corollary 5.6. *Let P be a rational d-polytope. Then $\frac{2}{lw(P)} \leq \mu(P)$ and if we have $\dim(F(\mu P)) = d-1$ then $\frac{2}{lw(P)} = \mu(P)$.*

We look now at the special situation with $\mu(P) > 1$ and $\dim(F(\mu(P)P)) = 0$ since we want to understand this situation for $d = 3$ in details later. First recall some definitions from [Bat24].

Definition 5.7. Let $P \subseteq M_{\mathbb{R}}$ be a lattice d-polytope. We call P *F-hollow* if $\mu(P) > 1$. If $\mu(P) > 1$ and $\dim(F(\mu(P)P)) = 0$, then we call P *weakly sporadic F-hollow*. A weakly sporadic F-hollow lattice d-polytope, which is not affine unimodular equivalent to a subset of $P' \times \mathbb{R}^{d-k}$ for some lattice k-polytope P' with $k < d$, is called a *sporadic F-hollow polytope.*

We will no give to classes of examples to illustrate these definitions.

Example 5.8 ([AWW09])**.** For any unit fraction partition of 1 of length d, i.e. a d-tuple $(k_1, \ldots, k_d) \in \mathbb{Z}^d_{\geq 2}$ with $\sum_{i=1}^{d} \frac{1}{k_i} = 1$, the simplex

$$\Delta_{(k_1,\ldots,k_d)} := \operatorname{conv}(0, k_1e_1, \ldots, k_de_d) \subseteq \mathbb{R}^d,$$

where e_i is the standard basis in $\mathbb{R}^d$, is a sporadic F-hollow simplex with

$$\mu(\Delta_{(k_1,\ldots,k_d)}) = \frac{k+1}{k}$$

Moreover, it is a maximal hollow lattice polytope, i.e. there are no hollow lattice polytopes, that contain it as a subpolytope. How do we see this? With $k := lcm(k_i)$ we can describe the points $x = (x_1, \ldots, x_d) \in \Delta_{(k_1,\ldots,k_d)}$ as the points in the intersection of the half-spaces $x_i \geq 0$ and $\sum_{i=1}^{d} \frac{k}{k_i} x_i \leq k$. We have no interior lattice points, since every interior point has $x_i > 0$ and $\frac{k}{k_i} x_i < k$. Moreover, every facet has at least one lattice point in its relative interior, we can pick the points $\sum_{i\in\{1,\ldots,d\}, i\neq j} e_i$ and $\sum_{i\in\{1,\ldots,d\}} e_i$ to see this. So we see that $\Delta_{(k_1,\ldots,k_d)}$ is a maximal hollow lattice simplex. So it is enough to show that

$$F\left(\frac{k+1}{k}\Delta_{(k_1,\ldots,k_d)}\right) = \{(1, 1, \ldots, 1)\}.$$

We get $F\left(\frac{k+1}{k}\Delta_{(k_1,\ldots,k_d)}\right) \subseteq \{(1, 1, \ldots, 1)\}$ since every facet has lattice distance 1 from $(1, 1, \ldots, 1)$. Every other supporting integral hyperplane has also at least lattice distance 1 to $(1, 1, \ldots, 1)$, since there is a vertex of $\Delta_{(k_1,\ldots,k_d)}$ which has a smaller lattice distance to the hyperplane than $(1, 1, \ldots, 1)$ and the vertex is a lattice point.

It is worth noting, that unit fraction partitions of 1 and generalized forms of them are also used in other classifications of lattice simplices, e.g. in recent classification results of Fano simplices for various Gorenstein indices in [Bae25].

Example 5.9. We can construct some special weakly sporadic lattice polytopes from canonical Fano polytopes. For some canonical Fano polytopes P there is a $\lambda < 1$ and some $x \in \mathbb{Q}^d$ such that $Q := \lambda P^* - x$ is an F-hollow lattice polytope. Then Q is weakly sporadic with $\mu(P) = \mu_{max}(P) = \frac{1}{\lambda}$ by 3.6. If P is not only canonical Fano but also reflexive, we get all Gorenstein polytopes and all their integral dilations without interior lattice points this way. In particular, since all canonical Fano polygons are reflexive, we get in dimension 2 only the three Gorenstein polygons of index greater 1 and the twofold standard lattice triangle, and we will see later that these are already all weakly sporadic lattice polygons. In dimension 3 we get 53 weakly sporadic lattice polytopes this way, among them the 34 Gorenstein polytopes with index greater than 1 and the 3 maximal sporadic simplizes $\Delta_{(3,3,3)}, \Delta_{(2,4,4)}, \Delta_{(2,3,6)}$. If we allow Q to have a non-empty Fine interior but still want it to be hollow, then we also get all maximal hollow lattice polytopes with Fine interior of dimension 0 and (perhaps surprisingly) also the only maximal hollow lattice polytope with Fine interior of dimension 3, which are described in [BKS22].

We end this section with some words on the other special multipliers.

Lemma 5.10. *Let P be a rational d-polytope, $n \in S_F(\lambda P)$ for some $\lambda \in \mathbb{R}_{\geq 0}$. Then there is a special multiplier μ_i of P, with $n \in S_F(\lambda P)$ if and only if $\lambda \geq \mu_i$.*

Proof. Since $n \in S_F(\lambda P)$ for some $\lambda \in \mathbb{R}_{\geq 0}$, there is a supporting hyperplane of $\mathcal{F}(P)$ with normal vector $(-\mathrm{Min}_P(n), n)$. The intersection of this hyperplane with $\mathcal{F}(P)$ is a unbounded face of $\mathcal{F}(P)$ and the lowest last coordinate of the vertices of this facet is a special multiplier μ_i with $\nu \in S_F(\lambda P)$ if and only if $\lambda \geq \mu_i$. $\square$

Definition 5.11. Let P be a rational d-polytope, $n \in S_F(\lambda P)$ for some $\lambda \in \mathbb{R}_{\geq 0}$. Then we define μ_n as the special multiplier from 5.10 and call it the *special multiplier of the support vector n.*

There is a special multiplier connected to the property of being canonically closed introduced in [Bat23, 4.17.]. We also get the characterization and rationality of this multiplier now as a corollary.

Corollary 5.12. *Let P be a rational polytope.*
There is a multiplier $\mu_{cc}(P) \in \mathbb{Q}$ of P, so that λP is canonically closed if and only if $\lambda \geq \mu_{cc}$.

Proof. By [Bat23, 4.3] λP is canonically closed if and only if $\Sigma_P[1] \subseteq S_F(\lambda P)$. So by 5.10 the multiplier μ_{cc} can be defined by $\mu_{cc} := \max\{\mu_n \mid n \in \Sigma_P[1]\}$. $\square$

There are examples with $\mu_{cc}(P) \neq \mu_{max}(P)$ e.g. one can see that

$$\mathrm{conv}\,((-4,-7,-9,-5),(0,1,0,0),(1,0,0,0),(2,5,9,5),(0,1,0,3))$$

is a maximal hollow lattice 4-simplex with minimal multiplier 1, in particular it is canonically closed. But one can calculate that $\mu_{max} = \frac{4}{3}$.

Thus, contrary to [Bat23, 5.8], it is possible that for a canonically closed lattice polytope P the combinatorial type of some $F(\lambda P)$ with $\lambda > 1$ is different from the combinatorial type of $F(P)$.

Nevertheless, for $\lambda \geq \mu_{max}(P)$ we still have a Minkowski sum decomposition of $F(\lambda P)$ as in [Bat23, 5.5]. But here this result is now just a corollary of 3.3 and the usual decomposition of a polyhedron into the Minkowski sum of a polytope and the recession cone.

Corollary 5.13. *Let P be a rational polytope and μ_{max} the maximal multiplier of P. Then we have*

$$F(\lambda\mu_{max}P) = F(\mu_{max}P) + (\lambda - 1)\mu_{max}P$$

for all $\lambda \geq 1$ and μ_{max} is the smallest rational number with this property.

Proof. We look at $\mathcal{F}(\mu_{max}P)$ for $s_0 \geq 1$. If we decompose this polyhedron as Minkowski sum of a polytope and the recession cone, we get since there are only vertices with first coordinate 1

$$\mathcal{F}(\mu_{max}P) \cap \{x_0 \geq 1\} = \{1\} \times F(\mu_{max}P) + \mathbb{R}_{\geq 0}(\{1\} \times \mu_{max}P).$$

So we have for all $\lambda \geq 1$

$$\begin{aligned}\{\lambda\} \times F(\lambda\mu_{max}P) =&\mathcal{F}(\mu_{max}P) \cap \{x_0 = \lambda\}\\ =&\{1\} \times F(\mu_{max}P) + (\lambda - 1)(\{1\} \times \mu_{max}P)\\ =&\{\lambda\} \times (F(\mu_{max}P) + (\lambda - 1)\mu_{max}P).\end{aligned}$$

For smaller rational numbers $\lambda < \mu_{max}$ the polyhedron $\mathcal{F}(\lambda P) \cap \{y \geq 1\}$ has vertices with first coordinate greater than 1 and so λ can not fulfil the property. $\square$

6. Weakly sporadic F-hollow lattice 3-polytopes

Our aim in this section is to classify all weakly sporadic F-hollow lattice 3-polytopes.

In a first step we look in arbitrary dimension $d \geq 2$ on the weakly sporadic F-hollow lattice d-polytopes with degree at most 1, where the degree is defined as the degree of the h^* polynomial as before. Since lattice d-polytopes with degree at most 1 are classified in [BN07], we can use this classification to describe all weakly sporadic F-hollow lattice d-polytopes among them.

Proposition 6.1. *Let P be a lattice d-polytope with $\deg(P) \leq 1$. Then P is a weakly sporadic F-hollow lattice polytope if and only if P is either the standard simplex Δ_d, a Gorenstein polytope of index d, or affine unimodular equivalent to the $(d-2)$-fold pyramid over $2\Delta_2$.*

Proof. If $\deg(P) = 0$, then P is by [BN07, Prop. 1.4.] affine unimodular equvivalent to the standard simplex Δ_d which is weakly sporadic by 5.4.

So only the case $\deg(P) = 1$ remains. By [BN07, Theorem 2.5] every 3-polytope of degree 1 is affine unimodular equivalent to the the $(d-2)$-fold pyramid over $2\Delta_2$, which is weakly sporadic F-hollow by 5.4, or to the a Lawrence prism $P_{h_1,\ldots,h_d}$. Since every Lawrence prism $P_{h_1,\ldots,h_d}$ projects to the standard simplex Δ_{d-1}, we get $\mu(P_{h_1,\ldots,h_d}) \geq \mu(\Delta_{d-1}) = d$. So $\mu(P_{h_1,\ldots,h_d}) = d$, because $dP_{h_1,\ldots,h_d}$ has interior lattice points since $\operatorname{codeg}(P_{h_1,\ldots,h_d}) = d + 1 - \deg(P_{h_1,\ldots,h_d}) = d$. To be weakly sporadic, we must therefore have exaktly one interior lattice point in the lattice polytope $dP_{h_1,\ldots,h_d}$, so we get $h^*(P_{h_1,\ldots,h_d})(t) = t + 1$ and so $P_{h_1,\ldots,h_d}$ is a Gorenstein polytope of index d. $\square$

Remark 6.2. We can explicitly describe the Gorenstein d-polytopes of index d for $d \geq 2$. They all have the h^*-polynomial $t + 1$ and so they are affine unimodular equivalent to a Lawrence prism $P_{h_1,\ldots,h_d}$ with

$$h^*_{P_{h_1,\ldots,h_3}}(t) = (h_1 + \ldots + h_d - 1)t + 1 = t + 1$$

by [BN07, 2.4. f.]. Because of the automorphisms of Δ_d we get without restriction $(h_1, h_2, \ldots, h_d) \in \{(1, 1, 0, \ldots, 0), (2, 0, 0, \ldots, 0)\}$ and so there are exactly two Gorenstein polytopes of index d, namely $P_{1,1,0,\ldots,0}$ and $P_{2,0,\ldots,0}$.

In dimension 2 every F-hollow lattice polygon has degree at most 1 and so we get already the complete classification of weakly sporadic lattice polygons from this general result, i.e. the weakly sporadic lattice polygons are $\Delta_2, P_{1,1}, P_{2,0}$ and $2\Delta_2$.

We will see now, that in dimension 3 the only addional examples of lattice width 1 are Gorenstein polytopes of index 2.

Proposition 6.3. *Let P be a lattice 3-polytope of lattice width 1 with $\deg(P) > 1$. Then P is a weakly sporadic F-hollow lattice polytope if and only if P is a Gorenstein polytope of index 2.*

Proof. Since P has lattice width 1, we have $\mu(P) \geq 2$. We can not have $\mu(P) > 2$, since then $\deg P = d + 1 - \operatorname{codeg}(P) \leq 4 - 3 = 1$.

It remains to look at the case $\mu(P) = 2$, and so we have $\dim F(2P) = 0$. From 4.6 we also get $\dim F(2P') = 0$ for the middle lattice polygon $2P'$, and so $F(2P')$ is a lattice point and $2P'$ is canonically closed. So $F(2P)$ is a lattice point and $2P$ is canonically closed by 4.6, so $2P$ is a reflexive polytope and therefore P is a Gorenstein polytope of index 2. □

Remark 6.4. The Gorenstein d-polytopes of index $d-1$, or equivalently degree 2, are completely classified in [BJ10].

In dimension 3 there are exactly 31 Gorenstein polytopes of degree 2, among them the 16 lattice pyramids over the 16 reflexive polygons. The polytope $2\Delta_3$ is the only one of these polytopes, which has lattice width greater than 1.

We describe now the weakly sporadic F-hollow lattice 3-polytopes projecting to $2\Delta_2$ but not to Δ_1 as lattice polytopes in $2\Delta_2 \times \mathbb{R}$.

Lemma 6.5. *Let P be a weakly sporadic F-hollow lattice 3-tope.*
If the lattice width of P is greater than 1 and P is not a sporadic F-hollow polytope, then P is affine unimodular equivalent to a lattice polytope in $2\Delta_2 \times [0, 4]$.

Proof. Since P is not sporadic and has a lattice width greater than 1, it must have a lattice projection on $2\Delta_2$ and is therefore affine unimodular equivalent to a lattice polytope Q in $2\Delta_2 \times \mathbb{R}$.

Let $c = (\frac{2}{3}, \frac{2}{3})$ be the barycenter of $2\Delta_2$ and c_l, c_u the lower and upper intersection point of $\{c\} \times \mathbb{R}$ and the boundary of Q. The intersection points are points of a lower and an upper facet F_l and F_u of Q. The lower facet F_l projects along $(0, 0, 1)$ onto a lattice subpolygon of $2\Delta_2 \times \{0\}$. There are up to automorphisms of $2\Delta_2 \times \{0\}$ two lattice subpolygons without three consecutive vertices which form an affine lattice basis of $\mathbb{Z}^2 \times \{0\}$, namely

$$\Delta_2 \text{ and } \operatorname{conv}((0, 0, 0), (2, 0, 0), (0, 1, 0)).$$

So, after a appropriate affine unimodular transformation we can assume that either

$$\begin{aligned} F_l \subseteq& \Delta_2 \times \{0\}, \\ F_l =& \operatorname{conv}((0, 0, 0), (2, 0, 0), (0, 2, 1)) \text{ or} \\ F_l =& \operatorname{conv}((0, 0, 1), (2, 0, 0), (0, 1, 0)) \end{aligned}$$

and in particular, we have $Q \subseteq 2\Delta_2 \times [0, \infty)$.

Next, we will show that the third coordinate of c_u is at most $\frac{4}{3}$. If $F_l \subseteq \Delta_2 \times \{0\}$ or $F_l = \operatorname{conv}((0,0,1),(2,0,0),(0,1,0))$, then we have $\frac{3}{2}c_l = (1,1,0)$, and if $F_l = \operatorname{conv}((0,0,0),(2,0,0),(0,2,1))$, then we have $\frac{3}{2}c_l = (1,1,\frac{1}{2})$. But since c_l is in the second case an interior point of a facet with integral points of Q, we get in both cases that every hyperplane supporting $\frac{3}{2}Q$ from below has at least lattice distance 1 to $(1,1,1)$. Similarly every hyperplane supporting $\frac{3}{2}Q$ from above has at least lattice distance 1 to $\frac{3}{2}c_u - (0,0,1)$. If we assume that the third coordinate of c_u is larger than $\frac{4}{3}$, we get $F(\frac{3}{2}Q) \supseteq \operatorname{conv}\big((1,1,1), \frac{3}{2}c_u - (0,0,1)\big)$ and $\dim(F(\frac{3}{2}Q)) = 1$, which contradicts the assumption that Q is weakly sporadic.

Let h be a vertex of Q with a maximum third coordinate. It remains to show, that this third coordinate is at most 4. The line through h and c_u has two intersection points h and s with the boundary of $\Delta_2 \times \mathbb{R}$ and the third coordinate of s is at least 0, because between c_u and s the intersection of the line with the interior of Q is empty and the segment between c_u and s lies above the polytope Q. Since the length of the segment between h and c_u is at most twice the length of the segment between c_u and s, we get that the third coordinate of h is at most $3 \cdot \frac{4}{3} = 4$.

Therefore, P is affine unimodular equivalent to a subpolytope Q of $2\Delta_2 \times [0,4]$. □

Remark 6.6. The polytope $2P_{2,0,0} = \operatorname{conv}((0,0,0),(2,0,0),(0,2,0),(0,0,4))$ is weakly sporadic F-hollow because $P_{2,0,0}$ is a Gorenstein polytope of index 3 by 6.2 and has only the lattice width directions $\pm(1,0,0), \pm(0,1,0), \pm(1,1,0)$. This shows that the prism $2\Delta_2 \times [0,4]$ in Lemma 6.5 is the best possible choice.

We can explicitly classify the lattice subpolytopes of $2\Delta_2 \times [0,4]$ by recursively deleting vertices from the lattice polytope and looking at the convex hull of the remaining lattice points. To do this efficiently, we should skip all the lattice polytopes we have already seen interms of affine unimodular equivalence. This can be done, for example, by using the PALP normal form of a lattice polytope ([KS04], [GK13]). We get the following result.

Proposition 6.7. *There are exactly* 80 *weakly sporadic non sporadic F-hollow lattice 3-polytopes of lattice width* 2*. All of them are lattice subpolytopes of* $2P_{2,0,0}$ *or* $2P_{1,1,0}$*.* $2\Delta_3$ *is the only one of them with minimal multiplier $\mu = 2$, the others have $\mu = \frac{3}{2}$.*

Theorem 6.8. *There are exactly* 114 *weakly sporadic but non sporadic F-hollow lattice* 3*-polytopes. Among them are exactly* 31 *Gorenstein polytopes of index* 2*,* 2 *Gorenstein polytopes of index* 3*,* 1 *Gorenstein polytope of index* 4*,* 1 *exceptional simplex with $\mu = \frac{5}{2}$ and* 79 *polytopes of width* 2 *with $\mu = \frac{3}{2}$. Coordinates for the vertices of all these polytopes are available on* [Boh24]*.*

By [Bat24, Theorem 1.11.] up to affine unimodular equivalence there are only finitely many sporadic $F-$hollow lattice d-polytopes, i.e. F-hollow lattice polytopes without F-hollow projection. A similar result holds for hollow lattice polytopes [NZ11]. In dimension 3 all sporadic F-hollow polytopes are subpolytopes of the 12 maximal hollow lattice polytopes classified in [AWW11], [AKW17].

Deciding whether a lattice polytope projects to $[0,1]$ is easy, since we have such a projection if and only if the polytope has lattice width 1. Since the only F-hollow polygon with lattice with greater than 1 is $2\Delta_2 = \operatorname{conv}((0,0),(2,0),(0,2))$, we need the following lemma to understand projections on this polygon.

Lemma 6.9. *Let P be a lattice d-polytope, $d \geq 3$, with lattice width greater than 1. Then P projects to $2 \cdot \Delta_2$ if and only if P has lattice width 2 and there are linearly dependent lattice width directions w_1, w_2, w_3 of P such that 0 is a interior point of $Q := \operatorname{conv}(w_1, w_2, w_3)$, the normalized area of Q is 3 and the supporting hyperplanes of P corresponding to the lattice width directions $\pm w_1, \pm w_2, \pm w_3$ define a polyhedron with exactly 3 facets.*

Proof. If P projects to $2\Delta_2$, then P is affine unimodular equivalent to a subpolytope of $2\Delta^2 \times \mathbb{R}^{d-1}$. Since $2\Delta_2$ has lattice width 2 and width directions $(1,0),(0,1),(-1,-1)$, we have lattice width directions $w_1 = (1,0,0,\dots,0), w_2 = (0,1,0,\dots,0), w_3 = (-1,-1,0,\dots,0)$ of P and w_1, w_2, w_3 satisfy the required conditions.

Conversely, if we have lattice width directions w_1, w_2, w_3 that satisfy the conditions, then $\operatorname{conv}(0, w_1, w_2)$ must be an empty triangle, since 0 is an interior point of the polygon $\operatorname{conv}(w_1, w_2, w_3)$ with normalized area 3. Thus $0, w_1, w_2$ is an affine lattice basis of a 2-dimensional sublattice, which we can extend to a lattice basis of the whole lattice $\mathbb{Z}^d$. Mapping this lattice basis to the standard basis we get that P is affine unimodular equivalent to a subpolytope of $[0,2]^2 \times \mathbb{R}^{d-2}$. Under this map, the width direction $w_3 = -w_1 - w_2$ maps to $(-1,-1,0,\dots,0)$ and so P is even affine unimodular equivalent to a subpolytope of $2\Delta_2 \times \mathbb{R}^{d-2}$ or $\operatorname{conv}((1,0),(0,1),(-1,1),(-1,0),(0,-1),(1,-1)) \times \mathbb{R}^{d-2}$. But the latter is not possible, since the supporting hyperplanes of P, corresponding to the lattice width directions, define a polyhedron with exactly 3 facets, and so P is affine unimodular equivalent to a subpolytope of $2\Delta_2 \times \mathbb{R}^{d-2}$ and thus projects to $2\Delta_2$. □

Remark 6.10. Working with the lattice width has been helpful for various classifications of lattice polytopes e.g. for the classification of empty 4-simplices in [IVS21]. There are also classifications looking at a multi-width, which is even a generalization of our situation with lattice width 2 for three different width directions, see [Ham24] for this approach.

With the help of the lemma, we can now determine all sporadic F-hollow lattice 3-polytopes as lattice polygons of the maximal hollow ones. We get the following result.

Theorem 6.11. *There are exactly 1368 sporadic F-hollow lattice 3-polytopes up to affine unimodular equivalence. All of them are subpolygons of $\Delta_{(3,3,3)}$, $\Delta_{(2,4,4)}$ or $\Delta_{(2,3,6)}$. 300 have $\mu = \frac{4}{3}$, 632 have $\mu = \frac{5}{4}$ and 436 have $\mu = \frac{7}{6}$. Coordinates for the vertices of all these polytopes are available on [Boh24].*

Let us end with two remarks on other classifications.

Remark 6.12. Since by [BKS22, Appendix B] there are 9 hollow lattice 3-polytopes, which are not F-hollow, we have all in all 1377 sporadic hollow lattice 3-polytopes, not projecting to a hollow lattice polytope of smaller dimension.

Remark 6.13. Among the sporadic F-hollow polytopes are 52 bipyramids, which were also classified in [IVS21, Lemma 4.2.] to determine all empty 4-simplices, which have a hollow projection to a hollow 3-polytope. 29 primitive bipyramids out of the 52 correspond to the 29 *stable quintuples* in the classification of 4-dimensional terminal quotient singularities in [MMM88].

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7. Classifying rational polygons with small denominator and few interior lattice points

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CLASSIFYING RATIONAL POLYGONS WITH SMALL DENOMINATOR AND FEW INTERIOR LATTICE POINTS

MARTIN BOHNERT AND JUSTUS SPRINGER

Abstract. We present algorithms for classifying rational polygons with fixed denominator and number of interior lattice points. Our approach is to first describe maximal polygons and then compute all subpolygons, where we eliminate redundancy by a suitable normal form. Executing our classification, we obtained a dataset of several billions of polygons covering a wide variety of cases.

1. Introduction

A polytope is called k-*rational*, if its k-fold dilation is a lattice polytope, i.e. has integral vertices. Lagarias and Ziegler [26] showed that for fixed number i of interior lattice points, there exist up to lattice automorphism only finitely many k-rational polytopes in each dimension, provided $i \geq 1$. Even though this does not hold for $i = 0$, Averkov, Wagner and Weismantel [2] showed that in this case there are still only finitely *maximal* k-rational polytopes, i.e. which are not strictly included in another k-rational polytope with equal number of interior lattice points.

In this article, we develop algorithms to classify polytopes in two dimensions. We used Julia [9] to carry out the classification and made the implementation available as part of a software package on rational polygons [33]. Moreover, large parts of our data are available for download [34–39].

Theorem 1.1. *The numbers of k-rational polygons with i interior lattice points is given by the following table. Each cell contains the number of maximal polygons on top, the number of distinct Ehrhart quasipolynomials in the middle and the total number of polygons below. For $i = 0$, only the polygons which cannot be realized in $\mathbb{R} \times [0, 1]$ are counted.*

$k \downarrow i \rightarrow$	0	1	2	3	4	5	6	...
1	1 1 1	3 7 16	4 8 45	6 10 120	9 12 211	11 14 403	13 16 714	...
2	4 34 79	10 270 5 145	25 586 48 639	33 1 060 249 540	63 1 701 893 402	106 2 525 2 798 638	178 3 577 7 299 589	...
3	14 803 6 723	39 8 124 924 042	146 25 892 17 656 886	303 62 347 156 777 687	687 124 021 909 056 858	1 452 218 169 4 211 988 753		
4	39 18 916 399 294	145 320 256 101 267 212	670 1 593 475 3 452 922 966					
5	134 253 631 18 935 385	698 5 034 566 8 544 548 186						
6	299 8 930 335 820 697 679							

The first row in the table above is the case of *lattice polygons*, which has already been studied by several authors. Starting from the topleft, the unique maximal lattice polygon without interior lattice points is the twofold standard lattice triangle, see e.g. [23, sec. 4.1], [22, §5] and [31, Theorem 2]. The classification of the 16 lattice polygons with exactly one interior lattice point has been done e.g. in [7, 29] and [23, sec. 4.2]. Lattice polygons with collinear interior lattice points, which includes the case $i = 2$, were classified by Koelman [23, sec. 4.3] and a general recursive algorithm was given by Castryck [13], which he ran up to $i = 30$. There are also algorithms classifying lattice polygons by their number of lattice points [23, sec. 4.4] and by their area [3]. Pick's formula [28] relates the area to the numbers of lattice and interior lattice points, hence one can also obtain the classification by interior lattice points from those algorithms.

To the authors knowledge, besides the first row in our table, only the four maximal 2-rational polygons without interior lattice points have directly appeared in the literature before [1]. The classification of almost k-hollow LDP polygons, which was done in [19] for $k \in \{2, 3\}$ is also worth mentioning, as these can be viewed as a subset of all k-rational polygons with exactly one interior lattice point. Hence they appear in our classification among the 5 145 resp. 924 042 polygons in the second column of the table above.

Let us also mention that in three dimensions, the lattice polytopes with one and two interior lattice points have been classified [4, 20] and the case of no interior lattice points has been done in [1, 2]. Moreover, there is the classification of four-dimensional reflexive polytopes [24], which is an important subset of those having exactly one interior lattice point.

This article is organised as follows. In section 2, we go over some topics that are neccessary for our classifications, namely normal forms, computing subpolygons and the notion of maximality. Sections 3, 4 and 5 describe our classification, first for the case of polygons contained in $\mathbb{R} \times [-1, 1]$, then for polygons with no and exactly one interior lattice point, filling out the first two columns of the table in Theorem 1.1. The methods we use are mainly constraints on the lattice width data developed in [10]. Section 6 is about the case $k = 2$. There, we were able to generalize the idea of moving out the edges from Castryck's classification [13] using the classification of two-dimensional Fine interiors of three-dimensional lattice polytopes [11]. Finally, in section 7, we discuss a generic (yet inefficient) method of classifying rational polygons for any pair $(k, i) \in \mathbb{Z}^2_{\geq 1}$ that we used to fill in the rest of the table in Theorem 1.1.

2. Prerequesites for classifying rational polygons

By a *polygon*, we mean a convex bounded polyhedral set $P \subseteq \mathbb{R}^2$. We call a polygon *rational* (*integral*), if its vertices are contained in $\mathbb{Q}^2$ ($\mathbb{Z}^2$). Integral polygons will also be called *lattice polygons*. For $k \in \mathbb{Z}_{\geq 1}$, we say P is *k-rational*, if kP is integral. The *denominator* $\operatorname{denom}(P)$ of a rational polygon is the smallest positive integer k such that P is k-rational.

We call two polygons P and P' *unimodular equivalent* ($P \sim^{\mathrm{u}} P'$) if $P' = UP$ for some $U \in \mathrm{GL}(2, \mathbb{Z})$. We call them *affine unimodular equivalent* ($P \sim^{\mathrm{a}} P'$), if $P' = UP + b$ for some $U \in \mathrm{GL}(2, \mathbb{Z})$ and $b \in \mathbb{Z}^2$. We say that P can be *realized* in some subset $A \subseteq \mathbb{R}^2$, if $P \sim^{\mathrm{a}} Q$ for some $Q \subseteq A$. We call P a *subpolygon* of P', if P can be realized in P'.

We write $\mathrm{Vol}(P)$ for the *normalized volume* of a polygon, which is defined to be twice its euclidian volume. Moreover, we define the *k-normalized* volume as $\mathrm{Vol}_k(P) := k^2 \,\mathrm{Vol}(P)$. Note that if P is k-rational, $\mathrm{Vol}_k(P)$ is an integer. We define the number of interior, boundary and total number of lattice points as:

$$i(P) \ := \ |P^\circ \cap \mathbb{Z}^2\,|, \qquad b(P) \ := \ |\partial P \cap \mathbb{Z}^2\,|, \qquad l(P) \ := \ |P \cap \mathbb{Z}^2\,|.$$

All these are invariants under (affine) unimodular equivalence and there exist several relations between them, e.g. Pick's formula states that $\mathrm{Vol}(P) = 2i(P)+b(P)-2$ holds for lattice polygons [28]. Another important invariant is the *Ehrhart quasipolynomial*, which counts the numbers of lattice points in integral dilations of a rational polygon, i.e.

$$\mathrm{ehr}_P(t) \ := \ l(tP) \ = \ At^2 + c_1(t)t + c_2(t), \qquad t \in \mathbb{Z}_{\geq 1},$$

where A is the euclidian volume of P and $c_1, c_2 : \mathbb{Z} \to \mathbb{Q}$ are uniquely determined periodic functions whose period divides $\mathrm{denom}(P)$. For lattice polygons, $c_1(t) = {}^{b(P)}\!/2$ and $c_2(t) = 1$ holds for all t. For the existence of the Ehrhart quasipolynomial and more insights on Ehrhart theory see for example [8].

2.1. **Normal forms.** Since we want to classify polygons up to lattice automorphism, we need an way to decide whether two polygons are equivalent by an affine unimodular transformation, which can be done by introducing a normal form. PALP [25], see also [17], is such a normal form that works for lattice polytopes of any dimension. However, we argue that the two-dimensional case deserves to be treated seperately. This is because the vertices of a polygon are ordered naturally by walking along its boundary, up the choice of the starting point and the direction. In this subsection, we present our approach to the normal form of rational polygons, which is based on this natural ordering of the vertices. An implementation of our normal form is available in `RationalPolygons.jl` [33].

We begin by describing how we encode k-rational polygons by integral matrices.

Definition 2.1. Fix $n \in \mathbb{Z}_{\geq 1}$. By a *vertex matrix*, we mean an integral matrix $V = [v_1 \ \ldots \ v_n] \in \mathbb{Z}^{2\times n}$ such that $v_1, \ldots, v_n$ are the vertices of $\mathrm{convhull}(v_1, \ldots, v_n)$ and v_i is connected to v_{i+1} by an edge for all $i = 1, \ldots, n-1$. For $k \in \mathbb{Z}_{\geq 1}$, we obtain a k-rational polygon

$$P_{k,V} \ := \ \mathrm{convhull}(v_1/k, \ldots, v_n/k).$$

Remark 2.2. Using a planar convex hull algorithm, we can reduce any finite set of lattice points in the plane to a set of vertices ordered by angle. For example, a Graham scan [16] achieves this with time complexity $O(n \log n)$.

Definition 2.3. For a vertex matrix $V = [v_1 \ \ldots \ v_n] \in \mathbb{Z}^{2\times n}$, we obtain new vertex matrices by reversing or shifting the order of its columns:

$$V^{(-1)} \ := \ [v_n \ \ldots \ v_1], \qquad V_i := [v_i \ \ldots v_n \ v_1 \ \ldots v_{i-1}].$$

Moreover, we write $V + w := [v_1 + w \ \ldots \ v_n + w]$. Two vertex matrices $V, V' \in \mathbb{Z}^{2\times n}$ are called

(i) *strictly equivalent* ($V \sim V'$), if $V' = V_i^{(s)}$ for some $s \in \{\pm 1\}$ and $i = 1, \ldots, n$,
(ii) *unimodular equivalent* ($V \sim^{\mathrm{u}} V'$), if $V' \sim UV$ for some $U \in \mathrm{GL}(2, \mathbb{Z})$,
(iii) *k-affine unimodular equivalent* ($V \sim_k^{\mathrm{a}} V'$), if $V' \sim UV + b$ for some $U \in \mathrm{GL}(2, \mathbb{Z})$ and $b \in k\,\mathbb{Z}^2$.

Remark 2.4. Let $V, V' \in \mathbb{Z}^{2\times n}$ be vertex matrices and $k \in \mathbb{Z}_{\geq 1}$. Then we have

(i) $P_{k,V} = P_{k,V'}$ if and only if $V \sim V'$,
(ii) $P_{k,V} \sim^{\mathrm{u}} P_{k,V'}$ if and only if $V \sim^{\mathrm{u}} V'$,
(iii) $P_{k,V} \sim^{\mathrm{a}} P_{k,V'}$ if and only if $V \sim_k^{\mathrm{a}} V'$.

Definition 2.5. Let $V = [v_1 \ \ldots \ v_n]$ be a vertex matrix.

(i) The *unimodular normal form* of V is

$$\mathrm{unf}(V) \ := \ \max\{\mathrm{hnf}(V_i^{(s)}) \ \mid \ s \in \{\pm 1\}, \ i = 1, \dots, n \ \},$$

where the maximum is taken by lexicographical ordering and we write $\mathrm{hnf}(M)$ for the hermite normal form of an integral matrix M.

(ii) The *1-affine normal form* of V is

$$\mathrm{anf}_1(V) \ := \ \max\{\mathrm{hnf}(V_i^{(s)} - v_i) \ \mid \ s \in \{\pm 1\}, \ i = 1, \dots, n\}.$$

(iii) Let $k \in \mathbb{Z}_{>1}$ and consider the lexicographically smallest vector $(x, y) \in \{0, \dots, k-1\}^2$ such that $\mathrm{anf}_1(V) + (x, y) \sim_k^{\mathrm{a}} V$. Then the *k-affine normal form* of V is

$$\mathrm{anf}_k(V) \ := \ \mathrm{anf}_1(V) + (x, y).$$

Proposition 2.6. *Let $k \in \mathbb{Z}_{\geq 1}$ and consider vertex matrices $V, V' \in \mathbb{Z}^{2\times n}$. Then*

(i) $V \sim^{\mathrm{u}} V'$ if and only if $\mathrm{unf}(V) = \mathrm{unf}(V')$,
(ii) $V \sim_1^{\mathrm{a}} V'$ if and only if $\mathrm{anf}_1(V) = \mathrm{anf}_1(V')$,
(iii) $V \sim_k^{\mathrm{a}} V'$ if and only if $\mathrm{anf}_k(V) = \mathrm{anf}_k(V')$.

Proof. Note that we have $V \sim^{\mathrm{u}} \mathrm{unf}(V)$ as well as $\mathrm{unf}(UV) = \mathrm{unf}(V)$ and $\mathrm{unf}(V_i^{(s)}) = \mathrm{unf}(V)$ for all $U \in \mathrm{GL}(2, \mathbb{Z})$ and $i \in \{1, \dots, n\}, s \in \{\pm 1\}$. Thus (i) follows and (ii) is handled analogously. For (iii), we again have $V \sim_k^{\mathrm{a}} \mathrm{anf}_k(V)$, hence $\mathrm{anf}_k(V) = \mathrm{anf}_k(V')$ implies $V \sim_k^{\mathrm{a}} V'$. For the converse, note that $V \sim_k^{\mathrm{a}} V'$ implies $V \sim_1^{\mathrm{a}} V'$, hence by (ii), we have $\mathrm{anf}_1(V) = \mathrm{anf}_1(V')$. But this means $\mathrm{anf}_1(V) + (x, y) \sim_k^{\mathrm{a}} V$ if and only if $\mathrm{anf}_1(V') + (x, y) \sim_k^{\mathrm{a}} V'$ for all $(x, y) \in \mathbb{Z}^2$, hence $\mathrm{anf}_k(V) = \mathrm{anf}_k(V')$. ☐

Remark 2.7. Using Definition 2.5, computing the unimodular or affine normal form requires computing $2n$ hermite normal forms, where n is the number of vertices. This can be improved by first distinguishing a subset of vertices that maximize a suitable invariant. For example, we can call a vertex v_i *special*, if it maximizes the area $\det(v_{i+1} - v_i, v_i - v_{i-1})$. Then we obtain an alternative unimodular normal form by

$$\mathrm{unf}'(V) \ := \ \max\{\mathrm{hnf}(V_i^{(s)}) \ \mid \ s \in \{\pm 1\}, \ v_i \text{ special}\}.$$

For polygons with few symmetries, there tend to be few special vertices, hence for those polygons $\mathrm{unf}'(V)$ is faster to compute than $\mathrm{unf}(V)$. More sophisticated invariants lead to fewer special vertices on average, however then the computation of the invariant itself becomes more expensive.

Remark 2.8. When computing the unimodular or affine normal form of a rational polygon, we get its respective automorphism group for free: Consider the set I of index pairs (i, s) with $i = 1, \dots, n$ and $s \in \{\pm 1\}$ such that $\mathrm{hnf}(V_i^{(s)}) = \mathrm{unf}(V)$. Then the number of rotational symmetries of the polygon is equal to the cardinality of I. Moreover, there exists a reflectional symmetry if and only if I contains two index pairs with opposite second coordinate. In this case, the unimodular automorphism group is the dihedral group $D_{|I|/2}$. Otherwise, it is the cyclic group of order $|I|$.

2.2. **Subpolygons.** We address the problem of finding all subpolygons of a given rational polygon up to equivalence. We use the same approach of *succesive shaving* that the authors of [12] used to classify lattice polygons contained in a square of given size. In our implementation, we noticed a significant speed-up by using hilbert bases in the shaving process, hence we describe this in detail. As an application, we reproduce and extend the classification of [12], see 2.13.

Fix $k \in \mathbb{Z}_{\geq 1}$ and let P be a k-rational polygon. Recall that a *subpolygon* of P is a k-rational polygon Q such that $Q \sim^{\mathrm{a}} Q'$ for some $Q' \subseteq P$. Up to affine unimodular equivalence, there exist only finitely many subpolygons to any given k-rational polygon. For a vertex v, we obtain a subpolygon by taking the convex hull of all k-rational points of P except v:

$$P \backslash v \ := \ \mathrm{convhull}\left(\left(P \cap \frac{1}{k}\,\mathbb{Z}^2\right) \backslash \{v\}\right).$$

Proposition 2.9. *Fix $k \in \mathbb{Z}_{\geq 1}$ and let P and Q be k-rational polygons with Q a subpolygon of P. Then there exists a chain of k-rational polygons*

$$P \ = \ P_0 \ \supsetneq \ P_1 \ \supsetneq \ \ldots \ \supsetneq \ P_n \ \sim^{\mathrm{a}} \ Q$$

such that for all $i = 0, \ldots, n-1$, we have $P_{i+1} = P_i \backslash v_i$ for some vertex v_i of P_i.

Proof. We may assume that $Q \subsetneq P$ holds. Note that the number of k-rational points of P must be strictly greater than the number of k-rational points of Q. Let's consider the case that they differ by one, i.e. there is a unique k-rational point $v \in \frac{1}{k}\,\mathbb{Z}$ with $v \in P \backslash Q$. By convexity, v must be a vertex, hence we have $Q = P \backslash v$. The general case now follows by induction. □

Algorithm 1 Computation of subpolygons

Input: A set of k-rational polygons M.

Output: A set of k-rational polygons S such that for all $P \in M$ and all subpolygons Q of P, there exists precisely one $Q' \in S$ such that $Q \sim^{\mathrm{a}} Q'$.

1: **procedure** SUBPOLYGONS(M)
2:     $A \ := \ \max\{\mathrm{Vol}_k(P) \ \mid \ P \in M\}$
3:     **for** a from 1 to A **do**
4:         $S_a := \{P \in M \ \mid \ \mathrm{Vol}_k(P) = a\}$
5:     **end for**
6:     **for** a from A to 1 **do**
7:         **for** $P \in S_a$ **do**
8:             **for** v vertex of P **do**
9:                 $Q \ := \ \mathrm{anf}_k(P \backslash v)$
10:                 $a' \ := \ \mathrm{Vol}_k(Q)$
11:                 **if** $\dim(Q) = 2$ and $Q \notin S_{a'}$ **then**
12:                     add Q to $S_{a'}$
13:                 **end if**
14:             **end for**
15:         **end for**
16:     **end for**
17:     **return** $S_3 \cup \cdots \cup S_A$
18: **end procedure**

Proof. Proposition 2.9 ensures that the algorithm encounters every subpolygon at least once. By Propisition 2.6, the resulting set contains each polygon precisely once up to affine unimodular equivalence. □

In line 9 of Algorithm 1, we need to compute the polygon $P \backslash v$. This can be done by first determining all k-rational points of P, removing v and computing the convex hull of the remaining points (for example, by performing a Graham scan). In the rest of this section, we show how to significantly speed this up using an algorithm involving hilbert bases. First, let us quickly summarize the neccessary facts around hilbert bases and their computation in two dimensions using continued fractions. We refer to [14, Chapter 10] for details.

Let $\sigma \subseteq \mathbb{R}^2$ be a rational pointed cone. After applying a unimodular transformation, we may assume $\sigma = \operatorname{cone}(e_2, de_1 - ke_2)$ where $0 \leq k < d$ are coprime integers. The *hilbert basis* $\mathcal{H}_\sigma$ of σ is the unique minimal generating set of the monoid $\sigma \cap \mathbb{Z}^2$. It can be computed as follows: First, using the modified euclidian algorithm, we express d/k as a Hirzebruch-Jung continued fraction $[\![b_1, \ldots, b_r]\!]$, which can be defined succinctly as

$$[\![b]\!] := b, \qquad [\![b_1, \ldots, b_r]\!] := b_1 - \frac{1}{[\![b_2, \ldots, b_r]\!]}.$$

Given $d/k = [\![b_1, \ldots, b_r]\!]$, the Hilbert basis is given by $\mathcal{H}_\sigma = \{u_0, \ldots, u_{r+1}\}$, where

$$u_0 := e_2, \qquad u_1 := e_1, \qquad u_{i+1} := b_i u_i - u_{i-1}.$$

Proposition 2.10. *Let $\sigma \subseteq \mathbb{R}^2$ be a rational pointed cone with hilbert basis $\mathcal{H}_\sigma = \{u_0, \ldots, u_{r+1}\}$ as constructed above. Consider the unbounded polyhedron*

$$\Theta_\sigma := \operatorname{convhull}((\sigma \cap \mathbb{Z}^2) \backslash \{0\}).$$

Then the following holds.

(i) $\mathcal{H}_\sigma$ is the set of lattice points on the bounded edges of Θ_σ.
(ii) $\mathcal{H}'_\sigma := \{u_i \in \mathcal{H}_\sigma \mid b_i \neq 2\}$ is the set of vertices of Θ_σ.

Proof. For (i), we refer to [14, Theorem 10.2.8 (b)]. For part (ii), note that we have $u_{i+1} - u_i = (b_i - 1)u_i - u_{i-1}$ for all $i = 1, \ldots, r$. Hence u_i lies on the line segment between u_{i-1} and u_{i+1} if and only if $b_i = 2$. $\square$

Corollary 2.11. *Let P be a lattice polygon with vertices $v_1, \ldots, v_n \in \mathbb{Z}^2$ and assume $v_j = 0$. With $\sigma = \operatorname{cone}(v_{j-1}, v_{j+1})$ and $\mathcal{H}'_\sigma$ as in the proposition, the vertices of $P \backslash v_j$ are given by*

$$(\{v_1, \ldots, v_n\} \backslash \{v_j\}) \cup \mathcal{H}'_\sigma .$$

Remark 2.12. The vertices $\mathcal{H}'_\sigma = \{u'_1, \ldots, u'_s\}$ are already ordered naturally, i.e. u'_i and u'_{i+1} are connected by an edge. This means that if we start with ordered vertices $v_1, \ldots, v_n$ of P, Corollary 2.11 allows us to easily write down the ordered vertices of $P \backslash v_j$, which ties in nicely with our normal form.

Note that the condition $v_j = 0$ in Corollary 2.11 is not a restriction, since we can translate any lattice polygon to move v_j to the origin. Moreover, for any k-rational polygon P, we can apply the Corollary to kP to compute the vertices of $P \backslash v$ for any vertex v of P.

Using this improved algorithm for determining subpolygons, we were able to reproduce and extend a classification by Brown and Kasprzyk, see [12, Table 1].

Classification 2.13 (Data available at [37])**.** *Below are the numbers $\#(m)$ of lattice subpolygons of $[0, m] \times [0, m]$ that are not subpolygons of $[0, m-1] \times [0, m-1]$, up to affine unimodular equivalence. We also include the maximally attained number of vertices $N(m)$ and the number of vertex maximizers $M(m)$.*

m	*1*	*2*	*3*	*4*	*5*	*6*
$\#(m)$	*2*	*15*	*131*	*1 369*	*13 842*	*129 185*
$N(m)$	*4*	*6*	*8*	*9*	*10*	*12*
$M(m)$	*1*	*1*	*1*	*1*	*15*	*2*

m	*7*	*8*	*9*	*10*	*11*
$\#(m)$	*1 104 895*	*8 750 964*	*64 714 465*	*450 686 225*	*2 976 189 422*
$N(m)$	*13*	*14*	*16*	*16*	*17*
$M(m)$	*3*	*13*	*1*	*102*	*153*

Remark 2.14. In our application, we are rarely interested in all subpolygons of a k-rational polygon P, but only in those sharing the same number of interior lattice points with P. This can be integrated rather easily into our Algorithm 1: Whenever we compute the vertices of $P\backslash v$ using Corollary 2.11, we check whether $\mathcal{H}'_\sigma \cap k\,\mathbb{Z}^2 \neq \emptyset$. If so, that means an interior lattice point of P has moved to the boundary of $P\backslash v$. Hence $P\backslash v$ has stricly fewer interior lattice points than P and we can discard it immediately.

2.3. **Maximal polygons.** By a *k-maximal* polygon, we mean a k-rational polygon P such that for any k-rational polygon $Q \supseteq P$ with $i(P) = i(Q)$, we have $P = Q$. Clearly, every k-rational polygon is a subpolygon of a k-maximal one. This means if we classify all k-maximal polygons with $i \in \mathbb{Z}_{\geq 0}$ interior lattice points, we can use Algorithm 1 to obtain all k-rational polygons with i interior lattice points. In Proposition 2.16, we give a criterion for k-maximality that can be checked efficiently. Let us first fix some notation.

A k-rational polygon P can be written as the intersection of the affine halfplanes H_e associated with its edges, where

$$H_e = \{x \in \mathbb{R}^2 \;\mid\; \langle n_e, x\rangle \geq c_e\}, \qquad n_e \in \mathbb{Z}^2 \text{ primitive}, \qquad c_e \in \mathbb{Q}\,.$$

For any $x \in \mathbb{R}^2$, we define the *lattice distance* of the edge e with x as $\mathrm{ldist}_e(x) := \langle n_e, x\rangle - c_e$. Hence H_e consists of those points having nonnegative lattice distance with e. For any rational number $q \in \mathbb{Q}$, we define the *translated halfplane* by

$$H_e + q \;:=\; \{x \in \mathbb{R}^2 \;\mid\; \mathrm{ldist}_e(x) \geq q\}.$$

By moving out the affine halfplanes, we obtain a new rational polygon $P^{(-1)} := \bigcap_e \left(H_e - \frac{1}{k}\right)$, where the intersection runs through all edges of P.

Lemma 2.15. *Let $\Delta = \mathrm{convhull}(x, y, z) \subseteq \mathbb{R}^2$ be a lattice triangle containing no lattice points other than its vertices. Then $\mathrm{ldist}_e(z) = 1$, where e is the edge connecting x and y.*

Proof. By Pick's formula, Δ has area $i(\Delta) + \frac{b(\Delta)}{2} - 1 = \frac{1}{2}$. This forces $\mathrm{ldist}_e(z) = 1$. $\square$

Proposition 2.16. *For a k-rational polygon P, the following are equivalent:*

(i) P is k-maximal,

(ii) for all $v \in \partial P^{(-1)} \cap \frac{1}{k}\,\mathbb{Z}^2$, we have $i(\mathrm{convhull}(P \cup \{v\})) > i(P)$,

(iii) for all $v \in \partial P^{(-1)} \cap \frac{1}{k}\,\mathbb{Z}^2$, one of the following holds:

- *(1) There exists an edge e of P and a point $w \in e^\circ \cap \mathbb{Z}^2$ such that $v \notin H_e$,*
- *(2) There exists a vertex $w \in \mathbb{Z}^2$ of P such that $v \notin H_e$ and $v \notin H_{e'}$, where e and e' are the edges adjacent to w.*

Proof. "(i) $\Rightarrow$ (ii)" is clear. To show "(ii) $\Rightarrow$ (i)", assume P is not k-maximal. Then we have $P \subsetneq Q$ for some k-rational Q with $i(P) = i(Q)$. Let $z \in Q$ be a vertex and e an edge of P such that $z \notin H_e$. Pick two k-rational points $x, y \in e$ containing no k-rational points on the line segment between them and consider the triangle $\Delta := \mathrm{convhull}(x, y, z)$. Let $z' \in \Delta$ be a k-rational point with minimal lattice distance to e. Then $\Delta' := \mathrm{convhull}(x, y, z')$ contains no k-rational points other than its vertices, hence Lemma 2.15 implies $\operatorname{ldist}_e(z') = -\frac{1}{k}$. We obtain

$$i(P) \;\leq\; i(\operatorname{convhull}(P \cup \{z'\})) \;\leq\; i(Q).$$

Since $i(P) = i(Q)$, we get equality, which contradicts (ii).

To see that (ii) implies (iii), note that $Q := \operatorname{convhull}(P \cup \{v\})$ must contain a lattice point $w \in \partial P \cap \mathbb{Z}^2$ in its interior. If w lies in the relative interior of an edge of P, we are in case (1). If w is a vertex of P, we are in case (2). The converse (iii) $\Rightarrow$ (ii) is immediate. $\square$

Remark 2.17. A k-maximal polygon is not necessarily maximal as a convex set, i.e. there could be a convex set with the same number of interior lattice points which strictly contains the k-maximal polygon. A k-rational polygon is maximal as a convex set if and only if every edge contains an interior lattice point. For example, the quadrilateral

$$\operatorname{convhull}\left(\left(0, 1 + \frac{1}{k}\right), (0,0), (k(i+1)+i, 0), \left(\frac{1}{k}, 1 + \frac{1}{k}\right)\right), \quad i \in \mathbb{Z}_{\geq 0}, k \in \mathbb{Z}_{\geq 1}$$

is k-maximal for $(i,k) \notin \{(0,1),(0,2),(1,1)\}$ by Proposition 2.16, but it is not a maximal convex set since there is no interior lattice point on the edge between $\left(0, 1 + \frac{1}{k}\right)$ and $\left(\frac{1}{k}, 1 + \frac{1}{k}\right)$.

We end this section by showing that for lattice polygons, our notion of maximality coincides with the one used by [23] and [13]. Recall that the *interior hull* of a rational polygon is $P^{(1)} := \operatorname{convhull}(P^\circ \cap \mathbb{Z}^2)$.

Proposition 2.18. *Let P be a lattice polygon with two-dimensional interior hull $P^{(1)}$. Then P is 1-maximal if and only if $P = P^{(1)(-1)}$.*

Proof. Assume P is 1-maximal. Then by [13, Theorem 5], $P^{(1)(-1)}$ is a lattice polygon containing P. Clearly, we have $i(P) = i(P^{(1)(-1)})$, hence we obtain $P = P^{(1)(-1)}$. Conversely, if $P = P^{(1)(-1)}$, pick a 1-maximal polygon Q containing P with $i(P) = i(Q)$. Then we must have $Q^{(1)} = P^{(1)}$, hence $Q = Q^{(1)(-1)} = P^{(1)(-1)} = P$. $\square$

3. Rational polygons in $\mathbb{R} \times [-1,1]$ or with collinear interior points

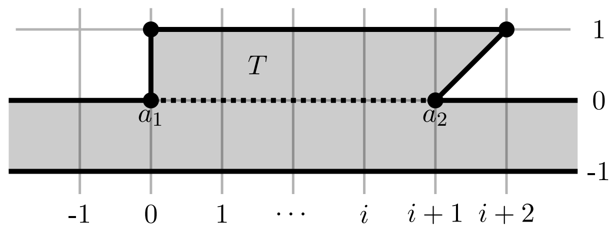
In this short section, we will classify k-maximal polygons that can be realized in $\mathbb{R} \times [-1,1]$ with given number of interior lattice points. This case occurs in our classifications of k-maximal polygons with no interior lattice points in section 4 and exactly one interior lattice point in section 5, hence we treat it here seperately. As an application, we classify k-rational polygons with collinear interior lattice points.

Proposition 3.1. *Every rational polygon $P \subseteq \mathbb{R} \times [-1,1]$ with $i \in \mathbb{Z}_{\geq 0}$ interior lattice points can be realized in $\mathbb{R} \times [-1,0] \,\cup\, T$, with the half-open trapezoid*

$$T := \operatorname{convhull}((0,0),(i+1,0),(i+2,1),(0,1)) \setminus (\mathbb{R} \times \{0\}).$$

Proof. After a suitable translation, we can assume the intersection of the interior of P with $\mathbb{R} \times \{0\}$ is contained in $(0, i+1) \times \{0\}$. Furthermore, we can assume this intersection to be non-empty, otherwise we can realize P in $\mathbb{R} \times [-1,0]$. Consider the points $a_1 := (0,0)$ and $a_2 := (i+1,0)$. Then we find unique supporting lines L_1 and L_2 of P with $a_j \in L_j$ such that L_1 and L_2 both contain a point of P with positive second coordinate. Since we can reflect on $\mathbb{R} \times \{0\}$, we can assume L_1 and L_2 to be either parallel or intersect in a point of $\mathbb{R} \times (0,\infty)$. Therefore, after a suitable horizontal shearing, we can assume that L_1 intersects $\mathbb{R} \times \{1\}$ in a point of $[0,1] \times \{1\}$ and L_2 intersects $\mathbb{R} \times \{1\}$ in a point of $(-\infty, i+2] \times \{1\}$. But this implies $P \subseteq \mathbb{R} \times [-1,0] \cup T$. $\square$

FIGURE 1. All rational polygons in $\mathbb{R}\times[-1,1]$ with i interior lattice points can be realized in $\mathbb{R}\times[-1,0]\cup T$



Construction 3.2. Consider a k-rational polygon $P \subseteq \mathbb{R}\times[-1,0]\cup T$ with $P\cap T \neq \emptyset$. Then there are supporting lines L_1 and L_2 as in the proof of the Proposition above. Consider the associated affine halfplanes H_1 and H_2 with $P \subseteq H_j$. We define the *surrounding polygon* of P to be the k-rational polygon

$$P_{\mathrm{sur}} \ := \ \mathrm{convhull}\left(Q \cap \frac{1}{k}\mathbb{Z}^2\right), \quad \text{where } Q \ := \ \mathbb{R}\times[-1,1] \ \cap \ H_1 \cap H_2.$$

Note that we have $P \subseteq P_{\mathrm{sur}} \subseteq \mathbb{R}\times[-1,0]\cup T$ and $i(P) = i(P_{\mathrm{sur}}) = i$. In particular, if P is k-maximal, we have $P = P_{\mathrm{sur}}$.

Definition 3.3. For two distinct points $v, w \in \mathbb{R}^2$, we write $H(v,w)$ for the closed affine halfplane given by all points to the right of the line through v and w with respect to the direction vector $w - v$

Algorithm 3.4. Given integers $k \in \mathbb{Z}_{\geq 1}$ and $i \in \mathbb{Z}_{\geq 0}$, we can classify all k-maximal polygons with i interior lattice points in $\mathbb{R}\times[-1,1]$ as follows: Consider $a_1 := (0,0)$ and $a_2 := (i+1,0)$. To any pair of points $x_1, x_2 \in T$, we associate the polygon

$$P := \mathrm{convhull}\left(Q \cap \frac{1}{k}\mathbb{Z}^2\right), \quad \text{where } Q := \mathbb{R}\times[-1,1] \cap H(a_1,x_1) \cap H(x_2,a_2)$$

This defines a possible surrounding polygon. Going through all polygons that arise this way and filtering out those that are k-maximal (for instance, using Proposition 2.16) gives us all k-maximal in $\mathbb{R}\times[-1,1]$ having i interior lattice points, excluding those with $P \cap T = \emptyset$, i.e. those that can be realized in $\mathbb{R}\times[0,1]$. However, polygons contained in $\mathbb{R}\times[0,1]$ are not k-maximal, hence we already have all k-maximal polygons.

We turn to the case of collinear interior lattice points, which is tightly related to being realizable in $\mathbb{R}\times[-1,1]$. Clearly, a polygon in $\mathbb{R}\times[-1,1]$ has collinear interior lattice points. The converse holds for lattice polygons with at least two collinear interior lattice points, which Koelman [23, Section 4.3] used to completely describe them. The following proposition generalizes this fact to k-rational polygons.

Proposition 3.5. *Let $P \subseteq \mathbb{R}^2$ be k-rational polygon with collinear interior lattice points and $i(P) > k$. Then P can be realized in $\mathbb{R}\times[-1,1]$.*

Proof. After an affine unimodular transformation, we can assume the interior lattice points of P to be of the form $(j,0)$ for $0 \leq j \leq i(P)-1$. Suppose P is not contained in $\mathbb{R}\times[-1,1]$. After a reflexion, we can assume that there exists a vertex $v = (x,y)$ of P with $y > 1$. We intend to show that there exists a k-rational point $q \in P \cap (\mathbb{R}\times\{1+\frac{1}{k}\})$. If $y = 1 + \frac{1}{k}$, we can take $q = v$. Otherwise, we have $y \geq 1 + \frac{2}{k}$ and we consider the line segment $L := P \cap (\mathbb{R}\times\{1+\frac{1}{k}\}$. We can estimate its length using the intercept theorem to obtain

$\text{length}(L) \geq \frac{k}{k+2}$. If $k \geq 2$, this implies $\text{length}(L) \geq \frac{1}{k}$ and therefore there exists a k-rational point $q = (\frac{a}{k}, 1 + \frac{1}{k}) \in L$. After a horizontal shераring, we can assume $0 \leq a \leq k$. Consider the triangle $Q := \text{convhull}(q, (0,0), (i(P)-1, 0))$ and note that $Q \setminus \{q\} \subseteq P^\circ$. If $a = 0$, we have $(0,1) \in Q$. If $a > 0$, then $i(P) > k$ implies $(1,1) \in Q$. Thus in both cases, we have an additional interior lattice point of P, a contradiction.

It remains to consider the case $k = 1$. Take any vertex $v = (x,y) \in P$ with $y \geq 1$. Then the triangle $\text{convhull}(v, (0,0), (1,0))$ does not contain any lattice points other than its vertices. Lemma 2.15 then implies $y = 1$. □

Using Algorithm 1, we can determine all subpolygons of the k-maximal polygons in $\mathbb{R} \times [-1,1]$. For $i > k$, the preceding Proposition implies that this gives us all k-rational polygons with i collinear interior lattice points. We arrive at the following classification.

Classification 3.6 (Data available at [35])**.** *Below are the numbers of* 2*-rational polygons with i collinear interior lattice points.*

i	3	4	5	6	7	8
#Polygons	168 640	504 530	1 279 695	2 881 106	5 924 808	11 343 912
i	9	10	11	12	13	14
#Polygons	20 496 555	35 295 876	58 364 056	93 212 470	144 449 999	218 021 550
i	15	16	17	18	19	20
#Polygons	321 478 832	464 285 436	658 158 267	917 447 376	1 259 556 240	1 705 404 538

Below are the numbers of 3*-rational polygons with i collinear interior lattice points.*

i	4	5	6
#Polygons	546 989 533	2 150 324 427	7 124 538 405

Recall that lattice polygons with collinear interior lattice points were classified by Koelman [23, Section 4.3]. In particular, he showed that there are exactly $\frac{1}{6}(i+3)(2i^2+15i+16)$ lattice polygons with $i \geq 2$ collinear interior lattice points. Using our values for $k = 2$, we found that they too can be described by a polynomial. Hence we raise the following conjecture, which we know to be true up to $i = 20$.

Conjecture 3.7. *There are exactly*

$$\frac{1}{1260}(i+1)(512i^6 + 12\,928i^5 + 137\,740i^4 + 685\,145i^3 + 1\,582\,743i^2 + 1\,665\,222i + 710\,640)$$

half-integral polygons with $i \geq 3$ collinear interior lattice points.

4. Rational polygons without interior lattice points

We give a classification algorithm for k-maximal polygons with no interior lattice points. Note that for $k = 2$, the four k-maximal polygons with no interior lattice points are known [1, Section 2]. With our approach, we obtain results up to $k = 20$, see Classification 4.5.

The main tool that allows us to get good bounds for maximal polygons with no interior lattice points is the lattice width. Recall that the *lattice width* of a rational polygon P in direction $w \in \mathbb{Z}^2 \setminus \{0\}$ is defined as

$$\text{width}_w(P) := \max_{v \in P} \langle v, w \rangle - \min_{v \in P} \langle v, w \rangle \in \mathbb{Z}^2.$$

The *lattice width* $\text{lw}(P)$ of P is defined to be the minimum over all $\text{width}_w(P)$ with $w \in \mathbb{Z}^2 \setminus \{0\}$. We call $w \in \mathbb{Z}^2 \setminus \{0\}$ a *lattice width direction* of P, if $\text{lw}(P) = \text{width}_w(P)$.

Lemma 4.1. *Every rational polygon P without interior lattice points can be realized in $\mathbb{R} \times [-1,2]$.*

Proof. After an affine unimodular transformation, we may assume that $(1,0)$ is a lattice width direction vector of P and $P \subseteq [x_l, x_r] \times \mathbb{R}$ for $x_l, x_r \in \mathbb{R}$. Then [10, Corollary 2.13] gives lower bounds for the numbers $|P^\circ \cap (\{h\} \times \mathbb{Z})|$, where $x_l \leq h \leq x_r$ and $h \in \mathbb{Z}$. Since P does not have interior lattice points, these imply that there are at most two interior integral vertical lines of P, hence $x_r - x_l \leq 3$ and the claim follows. □

Proposition 4.2. *Every rational polygon P without interior lattice points can be realized in $A \cup (\mathbb{R} \times [0,1]) \cup B$, with the two half-open trapezoids*

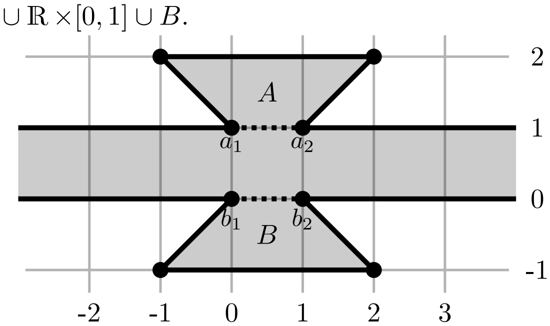
$$\begin{aligned} A &:= \operatorname{convhull}((0,1),(1,1),(2,2),(-1,2)) \setminus (\mathbb{R} \times \{1\}), \\ B &:= \operatorname{convhull}((0,0),(-1,-1),(2,-1),(1,0)) \setminus (\mathbb{R} \times \{0\}). \end{aligned}$$

Proof. By 4.1, we can assume $P \subseteq \mathbb{R} \times [-1,2]$. Consider the line segments

$$P_0 := P^\circ \cap (\mathbb{R} \times \{0\}), \qquad P_1 := P^\circ \cap (\mathbb{R} \times \{1\}).$$

We may assume P_0 and P_1 to be non-empty, otherwise we are in the situation of Proposition 3.1. Since P_0 and P_1 do not contain lattice points, we can apply a translation and shearing to achieve $P_0 \subseteq [0,1] \times \{0\}$ and $P_1 \subseteq [0,1] \times \{1\}$. By symmetry, it is enough to show that P does not contain a point $v = (x,y)$ with $x < 0$ and $y > x$. Suppose there is such a point. Then $\operatorname{convhull}(P_0 \cup P_1 \cup \{v\}) \subseteq P$ has $(0,0)$ as an interior lattice point, a contradiction. □

FIGURE 2. All rational polygons without interior lattice points can be realized in $A \cup \mathbb{R} \times [0,1] \cup B$.



Construction 4.3. Consider a k-rational polygon $P \subseteq A \cup \mathbb{R} \times [0,1] \cup B$ with $P \cap A \neq \emptyset$ and $P \cap B \neq \emptyset$ where A and B are as in the last Proposition. Consider the points

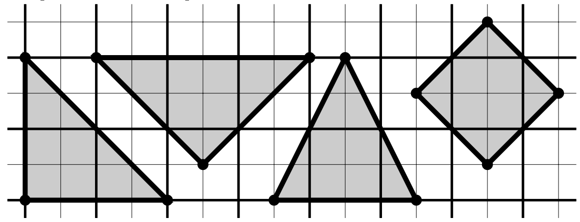
$$a_1 := (0,1), \quad a_2 := (1,1), \quad b_1 := (0,0), \quad b_2 := (1,0).$$

Then we have unique supporting lines $L_{a_1}, L_{a_2}, L_{b_1}, L_{b_2}$ of P with $a_j \in L_{a_j}, b_j \in L_{b_j}$ such that L_{a_j} contains a point of $P \cap A$ and L_{b_j} contains a point of $P \cap B$. These define affine halfplanes $H_{a_1}, H_{a_2}, H_{b_1}, H_{b_2}$ with $P \subseteq H_{a_j}$ and $P \subseteq H_{b_j}$. We define the *surrounding polygon* of P to be the k-rational polygon

$$P_{\mathrm{sur}} := \operatorname{convhull}\left(Q \cap \frac{1}{k}\mathbb{Z}^2\right), \qquad Q := \mathbb{R} \times [-1,2] \cap H_{a_1} \cap H_{a_2} \cap H_{b_1} \cap H_{b_2}.$$

Note that we have $P \subseteq P_{\mathrm{sur}} \subseteq A \cup \mathbb{R} \times [0,1] \cup B$ and $i(P) = i(P_{\mathrm{sur}}) = 0$. In particular, if P is k-maximal, we have $P = P_{\mathrm{sur}}$.

FIGURE 3. All four 2-maximal rational polygons without interior integral points, see also [1, Section 2]. All of them can be realized in $\mathbb{R} \times [-1, 1]$.



Algorithm 4.4. Given $k \in \mathbb{Z}_{\geq 1}$, we can classify all k-maximal polygons without interior lattice points as follows: To any choice of points $v_1, v_2 \in A$ and $w_1, w_2 \in B$, we associate the polygon $P := \mathrm{convhull}\left(Q \cap \frac{1}{k}\mathbb{Z}^2\right)$, where

$$Q := \mathbb{R} \times [-1, 2] \cap H(a_1, v_1) \cap H(v_2, a_2) \cap H(b_1, w_1) \cap H(w_2, b_2).$$

This defines a possible surrounding polygon. Going through all polygons that arise this way and filtering out those that are k-maximal (for instance, using Proposition 2.16) gives us all k-maximal polygons without interior lattice points, excluding those with $P \cap A = \emptyset$ or $P \cap B = \emptyset$. The latter ones can be realized in $\mathbb{R} \times [-1, 1]$ and hence can be obtained with Algorithm 3.4 applied to $i = 0$.

An implementation of Algorithm 4.4 is available in `RationalPolygons.jl` [33]. We ran the algorithm up to $k = 20$ and obtained the following classification, see also figures 3, 4 and 5.

Classification 4.5 (Data available at [39])**.** *Below are the numbers of k-maximal polygons without interior lattice points, split into the two cases of Algorithm 4.4.*

k	1	2	3	4	5	6	7	8	9	10
$\mathbb{R} \times [-1, 1]$	1	4	12	24	54	85	164	244	380	517
$\mathbb{R} \times [-1, 2]$	0	0	2	15	80	214	791	1 652	4 101	8 368
Total	1	4	14	39	134	299	955	1 896	4 481	8 885

k	11	12	13	14	15	16	17	18	19	20
$\mathbb{R} \times [-1, 1]$	809	1 021	1 506	1 878	2 398	2 987	4 039	4 743	6 239	7 263
$\mathbb{R} \times [-1, 2]$	17 757	29 637	57 937	91 009	161 418	258 728	409 990	595 051	930 544	1 304 553
Total	18 566	30 658	59 443	92 887	163 816	261 715	414 029	599 794	936 783	1 311 816

Using Algorithm 1, we computed all subpolygons of the k-maximal polygons from Classification 4.5 up to $k = 6$. This gives us all k-rational polygons without interior lattice points that cannot be realized in $\mathbb{R} \times [0, 1]$ (there are infinitely many in $\mathbb{R} \times [0, 1]$), see Theorem 1.1. The data is available at [39].

Figure 4. All fourteen 3-maximal rational polygons without interior integral points. The two rightmost polygons in the lower row are the only ones that cannot be realized in $\mathbb{R}\times[-1,1]$.

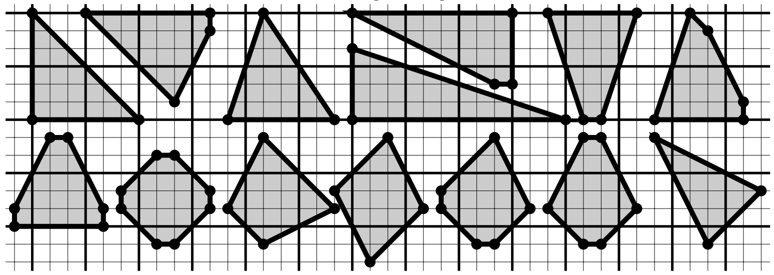
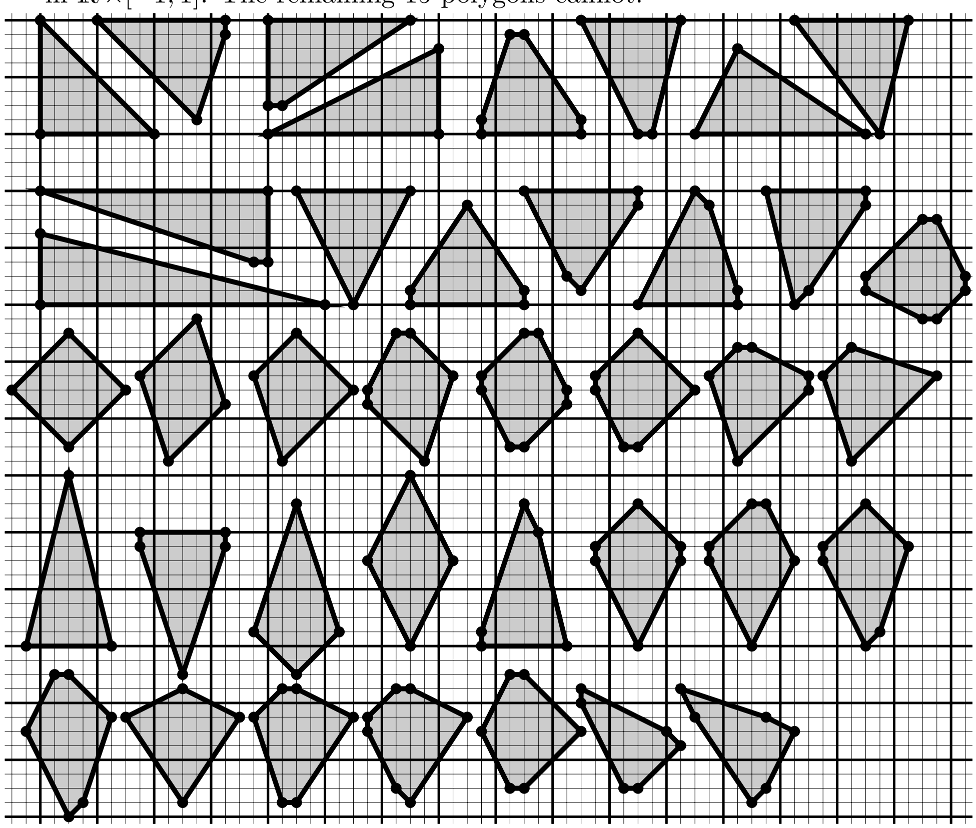


Figure 5. All thirtynine 4-maximal rational polygons without interior lattice points. The 24 polygons in the upper three rows can be realized in $\mathbb{R}\times[-1,1]$. The remaining 15 polygons cannot.



5. Rational polygons with exactly one interior lattice point and LDP polygons

We give a classification algorithm for k-maximal polygons with no interior lattice points and obtain explicit results up to $k = 10$, see Classification 5.5. Moreover, we apply this to

LDP polygons and extend the classification of k-hollow LDP polygons from [19] to $k = 6$, see Classification 5.6.

Lemma 5.1. *Every rational polygon P with exactly one interior lattice point can be realized in $\mathbb{R} \times [-2, 2]$.*

Proof. After an affine unimodular transformation, we may assume that $(1, 0)$ is a lattice width direction vector of P and $P \subseteq [x_l, x_r] \times \mathbb{R}$ for $x_l, x_r \in \mathbb{R}$. Then [10, Corollary 2.13] gives lower bounds for the numbers $|P^\circ \cap (\{h\} \times \mathbb{Z})|$, where $x_l \leq h \leq x_r$ and $h \in \mathbb{Z}$. Since P only has one interior lattice point, these imply that there are at most three interior integral vertical lines of P, hence $x_r - x_l \leq 4$ and the claim follows. $\square$

Proposition 5.2. *Every rational polygon with exactly one interior lattice point can be realized in $\mathbb{R} \times [0, 1] \cup A \cup B \cup C_q$ for some $q \in \{1, 2, 3, 4, 5\}$, where*

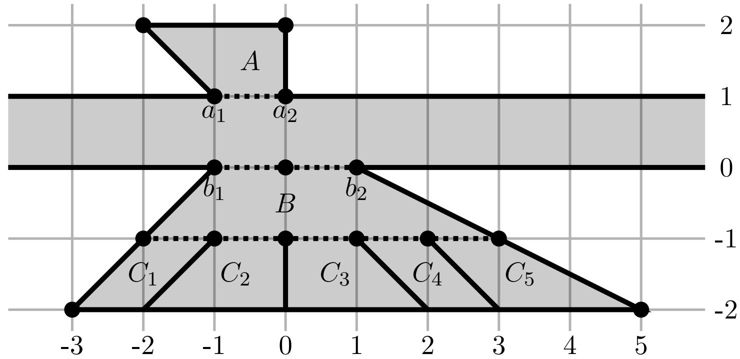
$$\begin{aligned}
A &:= \operatorname{convhull}((-1,1),(0,1),(0,2),(-2,2)) \ \setminus \ (\mathbb{R} \times \{1\}), \\
B &:= \operatorname{convhull}((-1,0),(-2,-1),(3,-1),(1,0)) \ \setminus \ (\mathbb{R} \times \{0\}, \\
C_1 &:= \operatorname{convhull}((-3,-2),(-2,-2),(-1,-1),(-2,-1)) \ \setminus \ (\mathbb{R} \times \{-1\}), \\
C_2 &:= \operatorname{convhull}((-2,-2),(0,-2),(0,-1),(-1,-1)) \ \setminus \ (\mathbb{R} \times \{-1\}), \\
C_3 &:= \operatorname{convhull}((0,-2),(2,-2),(1,-1),(0,-1)) \ \setminus \ (\mathbb{R} \times \{-1\}), \\
C_4 &:= \operatorname{convhull}((2,-2),(4,-2),(5,-2),(1,0)) \ \setminus \ (\mathbb{R} \times \{-1\}), \\
C_5 &:= \operatorname{convhull}((-1,0),(-3,-2),(5,-2),(1,0)) \ \setminus \ (\mathbb{R} \times \{-1\}).
\end{aligned}$$

Proof. By 5.1, we can assume $P \subseteq \mathbb{R} \times [-2, 2]$ and after translation, we can assume the interior lattice point of P to be the origin. Consider the line segments

$$P_1 := P^\circ \cap (\mathbb{R} \times \{1\}), \quad P_0 := P^\circ \cap (\mathbb{R} \times \{0\}), \quad P_{-1} := P^\circ \cap (\mathbb{R} \times \{-1\}).$$

Clearly, we have $P_0 \neq \emptyset$ and convexity forces $P_0 \subseteq [-1, 1] \times \{0\}$. If $P_1 = P_{-1} = \emptyset$, we are in the situation of Proposition 3.1, hence we are done after suitable translation and reflexion. Thus, after a reflexion, we can assume $P_1 \neq \emptyset$. If P were to contain any point outside of $\mathbb{R} \times [0, 1] \cup A \cup B \cup \bigcup_q C_q$, convexity would force P to contain an additional lattice point, which is impossible. Moreover, if there were points $v \in P^\circ \cap C_q$ and $w \in P^\circ \cap C_r$ for $q \neq r$, we would have an interior lattice point in the triangle $\operatorname{convhull}(v, w, (0, 0)) \subseteq P$, which is impossible. Hence $P^\circ \cap C_q \neq \emptyset$ for at most one $q \in \{1, 2, 3, 4, 5\}$ and we arrive at the claim. $\square$

FIGURE 6. All rational polygons with exactly one interior lattice point can be realized in $\mathbb{R} \times [0, 1] \cup A \cup B \cup C_q$ for some $q \in \{1, 2, 3, 4, 5\}$.



Construction 5.3. Consider a k-rational polygon $P \subseteq \mathbb{R}\times[0,1] \cup A \cup B \cup C_q$ with $P \cap A \neq \emptyset$, where $q \in \{1,2,3,4,5\}$. Consider the points

$$a_1 := (-1,1), \quad a_2 := (0,1), \quad b_1 := (-1,0), \quad b_2 := (1,0),$$
$$c_1 := (q-3,-1), \quad c_2 := (q-2,-1).$$

Then we have unique supporting lines $L_{a_1}, L_{a_2}, L_{b_1}, L_{b_2}$ of P with $a_j \in L_{a_j}, b_j \in L_{b_j}$ such that L_{a_j} contains a point of $P \cap A$ and L_{b_j} contains a point of $P \cap B$. Moreover, if $P \cap C_q \neq \emptyset$, we have unique supporting lines L_{c_1}, L_{c_2} defined analogously. All these supporting lines define affine halfplanes H_{a_j}, H_{b_j} and H_{c_j} that contain P. We define the *surrounding polygon* of P to be the k-rational polygon $P_{\mathrm{sur}} := \operatorname{convhull}\left(Q \cap \frac{1}{k}\mathbb{Z}^2\right)$ where

$$Q := \begin{cases} \mathbb{R}\times[-1,2] \cap H_{a_1} \cap H_{a_2} \cap H_{b_1} \cap H_{b_2}, & P \cap C_q = \emptyset, \\ \mathbb{R}\times[-2,2] \cap H_{a_1} \cap H_{a_2} \cap H_{b_1} \cap H_{b_2} \cap H_{c_1} \cap H_{c_2}, & P \cap C_q \neq \emptyset. \end{cases}$$

Note that we have $P \subseteq P_{\mathrm{sur}} \subseteq \mathbb{R}\times[0,1] \cup A \cup B \cup C_q$ and $i(P) = i(P_{\mathrm{sur}}) = 1$. In paticular, if P is k-maximal, we have $P = P_{\mathrm{sur}}$.

Algorithm 5.4. Given $k \in \mathbb{Z}_{\geq 1}$, we can classify all k-maximal polygons with exactly one interior lattice point as follows: Pick $q \in \{1,\dots,5\}$. To any choice of points $v_1, v_2 \in A, w_1, w_2 \in B$ and $u_1, u_2 \in C_q$, we associate the polygon $P := \operatorname{convhull}\left(Q \cap \frac{1}{k}\mathbb{Z}^2\right)$, where

$$Q := \mathbb{R}\times[-2,2] \cap H(a_1,v_1) \cap H(v_2,a_2) \cap H(b_1,w_1) \cap H(w_2,b_2) \cap H(c_1,u_1) \cap H(u_2,c_2).$$

This defines a possible surrounding polygon. Going through all polygons that arise this way and filtering out those that are k-maximal (for instance, using Proposition 2.16) gives us all k-maximal polygons with one interior lattice point that can be realized in $\mathbb{R}\times[-2,2]$, excluding those that can be realized in $\mathbb{R}\times[-1,2]$. The latter ones are obtained in an analogous way but without the halfplanes given by c_1 and c_2. Finally, the polygons that can be realized in $\mathbb{R}\times[-1,1]$ are obtained by Algorithm 3.4 applied to $i = 1$.

An implementation of Algorithm 5.4 is available in `RationalPolygons.jl` [33]. We ran the algorithm up to $k = 10$ and obtained the following classification, see also figures 7 and 8.

Classification 5.5 (Data available at [38])**.** *Below are the numbers of k-rational polygons with exactly one interior lattice point, split into the three cases according to Algorithm 5.4.*

k	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
$\mathbb{R}\times[-1,1]$	*2*	*9*	*26*	*57*	*132*	*199*	*396*	*605*	*937*	*1 260*
$\mathbb{R}\times[-1,2]$	*1*	*1*	*12*	*83*	*470*	*1 390*	*4 964*	*11 426*	*27 801*	*56 303*
$\mathbb{R}\times[-2,2]$	*0*	*0*	*1*	*5*	*96*	*329*	*2 874*	*8 241*	*32 176*	*81 950*
Total	*3*	*10*	*39*	*145*	*698*	*1 918*	*8 234*	*20 272*	*60 914*	*139 513*

Using Algorithm 1, we computed all subpolygons of the k-maximal polygons from Classification 5.5 up to $k = 5$. This gives us all k-rational polygons with exactly one interior lattice point, see Theorem 1.1.

We turn to LDP polygons, which are of special interest to toric geometry. Recall that a lattice polygon P is called *LDP*, if $0 \in P^\circ$ and its vertices are primitive vectors of $\mathbb{Z}^2$. By taking the fan generated by its faces, an LDP polygon P defines a toric log del Pezzo surface X_P and this assignment is a bijection on isomorphy classes. We refer to [21] for more information on LDP polygons as well as their classification up to index 17.

In [19], the authors describe a classification algorithm for *almost k-hollow* LDP polygons, i.e. LDP polygons with $P \cap k\mathbb{Z}^2 = \{0\}$. These correspond to toric $\frac{1}{k}$-log canonical

Figure 7. All ten 2-maximal rational polygons with exactly one interior integral point up to affine unimodular equivalence. The threefold standard triangle is the only one that cannot be realized in $\mathbb{R} \times [-1, 1]$.

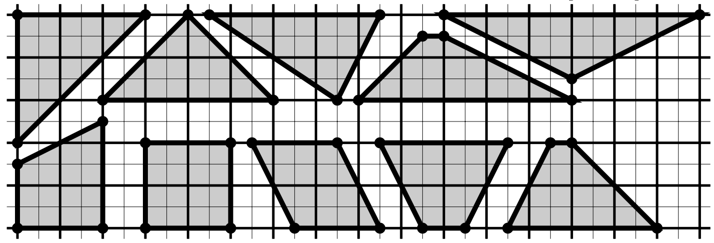
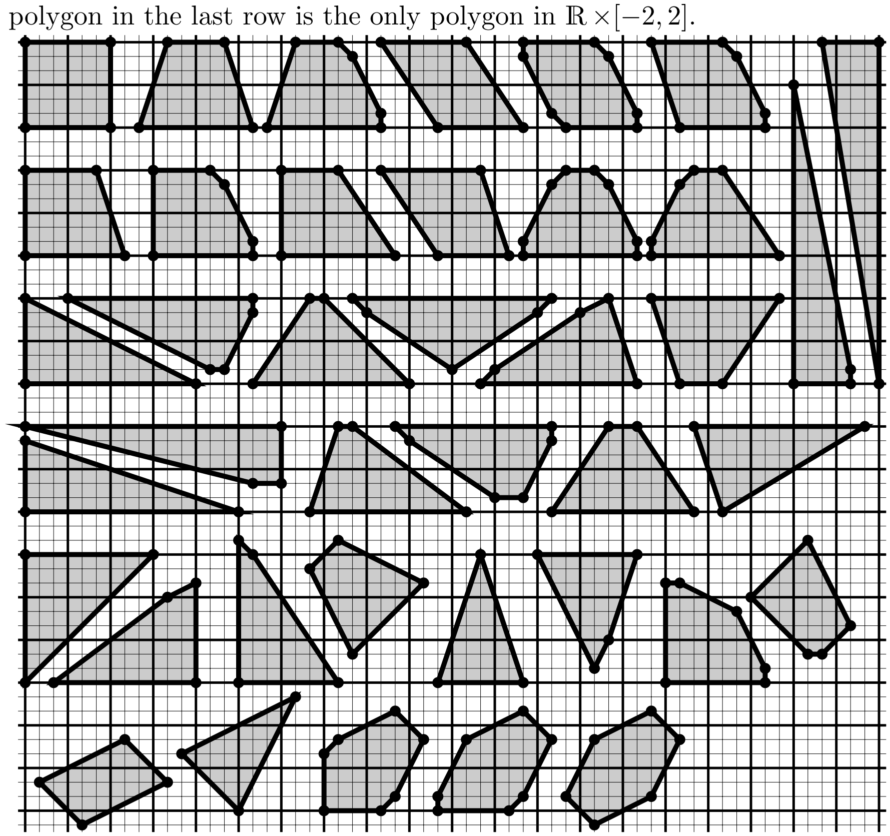


Figure 8. All thirtynine 3-maximal polygons with exactly one interior lattice points. The upper four rows are the 26 polygons in $\mathbb{R} \times [-1, 1]$. The lower two rows contain the 12 polygons in $\mathbb{R} \times [-1, 2]$ and the rightmost polygon in the last row is the only polygon in $\mathbb{R} \times [-2, 2]$.



del Pezzo surfaces, which means the discrepancies of the exceptional divisors in the canonical resolution are all greater or equal to $\frac{1}{k} - 1$. Dividing an almost k-hollow LDP polygon by k gives a k-rational polygon having only the origin as an interior lattice point. Hence we can also classify k-hollow LDP polygons by taking our k-maximal polygons with one interior lattice point and computing all subpolygons having primitive vertices. We have

FIGURE 9. The k-rational polygons with exactly one interior lattice point and maximal number of vertices for $k \in \{1, 2, 3, 5\}$.

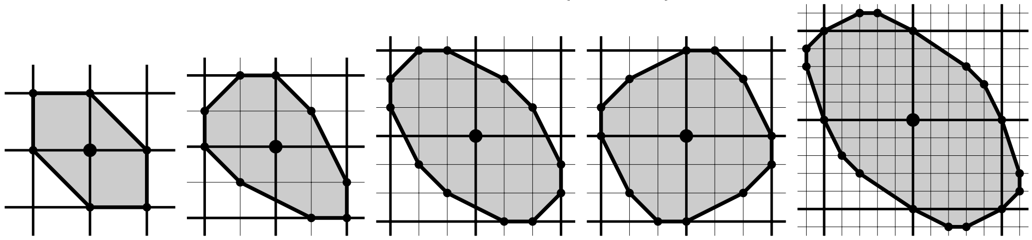
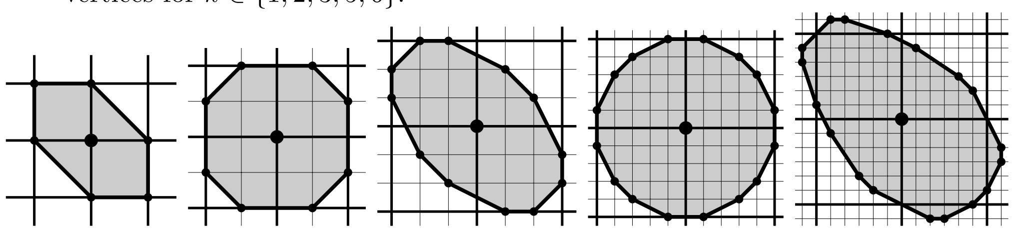


FIGURE 10. The almost k-hollow LDP polygons with maximal number of vertices for $k \in \{1, 2, 3, 5, 6\}$.



done this up to $k = 6$, in particular reproducing the classification of the almost 3-hollow LDP polygons from [19]:

Classification 5.6. *Below are the numbers $\#_{\mathrm{ldp}}(k)$ of almost k-hollow LDP polygons, their maximimally attained number of vertices $N_{\mathrm{ldp}}(k)$ and the number of vertex maximizers $M_{\mathrm{ldp}}(k)$. Equivalently, $\#_{\mathrm{ldp}}(k)$ is the number of toric $\frac{1}{k}$-log canonical del Pezzo surfaces and $N_{\mathrm{ldp}}(k) - 2$ is their maximally attained Picard number. For comparison, we include the analogous numbers for all k-rational polygons with exactly one interior lattice points.*

k	1	2	3	4	5	6
$\#_{\mathrm{ldp}}$	16	505	48 032	1 741 603	154 233 886	2 444 400 116
N_{ldp}	6	8	12	12	16	18
M_{ldp}	1	1	1	20	1	1
$\#_{\mathrm{all}}$	16	5 145	924 042	101 267 212	8 544 548 186	
N_{all}	6	9	12	14	18	
M_{all}	1	1	2	23	1	

6. Half-integral polygons

In this section, we classify half-integral polygons with few interior lattice points. Our method is inspired by Castryck's algorithm for classifying of lattice polygons [13], where the notion of an *internal polygon* (i.e. a polygon of the form $P^{(1)}$) is replaced by the *Fine interior* as introduced in [15, §4.2.] (see also [30, II, 4 Appendix]). For more information on the Fine interior and its recent use in combinatorial mirror symmetry and combinatorial birational geometry, see [5, 6].

Recall that for a lattice polytope $P \subseteq \mathbb{Z}^d$, we set $P^{(1)} := \mathrm{convhull}(P^\circ \cap \mathbb{Z}^d)$. On the other hand $P^{(-1)}$ is the rational polytope obtained by "moving out" all facets of P by one, see section 2.3.

Definition 6.1. The *Fine interior* of a rational polytope $P \subseteq \mathbb{R}^d$ is

$$F(P) \;:=\; \{x \in \mathbb{R}^d \mid \langle x, n\rangle \geq \min\{\langle y, n\rangle \mid y \in P\} + 1 \;\forall n \in \mathbb{Z}^d \setminus \{0\}\}.$$

Remark 6.2. $F(P)$ is a rational polytope. Moreover, if P is a lattice polytope, we have $P^{(1)} \subseteq F(P)$ by [5, 2.6 f.]. For lattice polygons, we even have equality by [5, p. 2.9]. In this sense, a rational polytope that is a Fine interior of another one can be seen as a generalization of an internal polygon.

We have seen a criterion for the maximality of lattice polygons using internal polygons in 2.18. Here we focus first on the polytopes that are maximal with respect to inclusion among those sharing same Fine interior. The following is a direct consequence of the definition of the Fine interior.

Proposition 6.3. *A full-dimensional rational polytope P is the Fine interior of a rational polytope if and only if $F(P^{(-1)}) = P$. In this case $P^{(-1)}$ contains every rational polytope, which has P as its Fine interior.*

This result implies that we can classify all lattice polytopes with given number of interior lattice points by first classifying their Fine interiors, then getting all maximal ones by moving out the facets and finally computing all subpolytopes. Recently [11] gave a classification of two-dimensional Fine interiors of lattice 3-polytopes, which allows us to follow this path with a small detour in dimension 3. We give the essential results from [11] for our classification in three steps. The first one gives us a connection between dimension 2 and 3.

Proposition 6.4. *The Fine interior of a lattice 3-polytope $P \subseteq \mathbb{R}^2 \times [-1, 1]$ with at least two interior lattice points and $\mathrm{lw}(P) = 2$ has vertices in $\frac{1}{2}\mathbb{Z}^2 \times \{0\}$ and depends only on the half-integral polygon $P \cap (\mathbb{R}^2 \times \{0\})$. Moreover, the Fine interior of such a lattice 3-polytope is of dimension 1 if and only if $P \cap (\mathbb{R}^2 \times \{0\})$ is affine unimodular equivalent to a polygon in $\mathbb{R} \times [-1, 1] \times \{0\}$.*

The second result gives a suitable notion of maximality in this context.

Proposition 6.5. *A half-integral polygon Q is the Fine interior of a lattice 3-tope if and only if there is a half-integral polygon $P_0 \subseteq \mathbb{R}^2$ such that Q is the Fine interior of the lattice 3-tope $P := \mathrm{convhull}\,(2P_0 \times \{-1\}, (0, 0, 1))$. Furthermore, there is a unique half-integral polygon Q^- with $F(\mathrm{convhull}\,(2Q^- \times \{-1\}, (0, 0, 1))) = Q$, which contains every half-integral polygon P_0' for which*

$$F\big(\mathrm{convhull}\,\big(2P_0' \times \{-1\}, (0, 0, 1)\big)\big) = Q.$$

And the third gives us the classification which was done there.

Classification 6.6. *There are exactly 24 324 158 half-integral polygons with at most* 40 *lattice points, which are Fine interiors of a lattice 3-polytope.*

For each polygon from Classification 6.6, we can compute the polygon Q^- from Proposition 6.5 and check for 2-maximality. This gives us all 2-maximal polygons with up to 40 interior lattice points that cannot be realized in $\mathbb{R} \times [-1, 1]$. The latter ones can be obtained with Algorithm 3.4, hence we arrive at the following classification.

Classification 6.7 (Data available at [35])**.** *Below are the numbers $\#_{\max}(i)$ of* 2*-maximal polygons with i interior lattice points.*

i	1	2	3	4	5	6	7	8	9	10
$\#_{\max}(i)$	10	25	33	63	106	178	274	476	693	1 058
i	11	12	13	14	15	16	17	18	19	20
$\#_{\max}(i)$	1 561	2 345	3 431	4 970	6 712	9 345	12 883	17 667	23 809	31 806
i	21	22	23	24	25	26	27	28	29	30
$\#_{\max}(i)$	41 394	54 071	70 088	90 805	116 613	148 877	187 818	236 793	296 708	370 444
i	31	32	33	34	35	36	37	38	39	40
$\#_{\max}(i)$	459 667	569 098	698 986	857 540	1 047 830	1 277 349	1 549 049	1 875 058	2 257 565	2 714 004

Moreover, we obtain the numbers $\#_{\mathrm{all}}(i)$ of all 2-rational polygons as well as the numbers $\#_{\mathrm{ehr}}(i)$ of distinct Ehrhart quasipolynomials:

i	1	2	3	4	5	6
$\#_{\mathrm{all}}(i)$	5 145	48 639	249 540	893 402	2 798 638	7 299 589
$\#_{\mathrm{ehr}}(i)$	270	586	1 060	1 701	2 525	3 577
i	7	8	9	10	11	12
$\#_{\mathrm{all}}(i)$	17 556 059	41 005 529	83 848 960	170 081 198	339 219 561	617 457 338
$\#_{\mathrm{ehr}}(i)$	4 875	6 459	8 343	10 565	13 138	16 111
i	13	14	15	16		
$\#_{\mathrm{all}}(i)$	1 140 740 735	2 040 032 893	3 454 142 981	5 957 874 274		
$\#_{\mathrm{ehr}}(i)$	19 488	23 307	27 596	32 381		

We end this section with a conjecture on the number of Ehrhart quasipolynomials of half-integral polygons. Recall that for a lattice polygon P, the Ehrhart polynomial is of the form $At^2 + \frac{b}{2}t + 1$, where A is the euclidian area and b the number of boundary lattice points. By Pick's formula [28], fixing i and b uniquely determines A. Moreover, Scott's inequality [32] states that for lattice polygons, $b \leq 2i + 6$ holds for $i \geq 2$. Lattice polygons clearly satisfy $b \geq 3$ and one can show that every intermediate value of b is also attained, see e.g. [18, Section 1]. This implies that the number of distinct Ehrhart polynomials of lattice polygons is $2i + 4$ for $i \geq 2$. Looking at our data for $k = 2$, we found that there is also is a polynomial expression for the number of distinct Ehrhart quasipolynomials, provided $b \geq 2$. Hence we raise the following conjecture, which we have verified up to $i = 16$.

Conjecture 6.8. *There are exactly*

$$\frac{9}{2}i^3 + 36i^2 + \frac{175}{2}i + 53$$

Ehrhart quasipolynomials of half-integral polygons with $i \in \mathbb{Z}_{\geq 2}$ interior lattice points and at least 2 boundary lattice points.

7. The general case

In this section, we show how a classification of k-maximal polygons with $i \in \mathbb{Z}_{\geq 0}$ interior lattice points can be derived from a sufficiently large classification of lattice polygons. First, we need a suitable upper volume bound of rational polygons in terms of their number of interior lattice points.

Lemma 7.1. *Let P be a rational polygon with $k = \mathrm{denom}(P)$ and $i := i(P)$ which cannot be realized in $\mathbb{R} \times [-1, 1]$. Then we have*

$$\mathrm{Vol}_k(P) \leq \max\left(k^2(4i + 5),\ \frac{1}{2}k(k + 2)^2(i + 1)\right).$$

Proof. After an affine unimodular tranformation, we can assume that $(0, 1)$ is a lattice width direction of P and $P \subseteq \mathbb{R} \times [0, \infty)$. Let $m \in \mathbb{Z}$ be minimal such that $P \subseteq \mathbb{R} \times [0, m+$

1]. By assumption, we have $m \geq 2$. If $m \geq 3$, [10, Proposition 3.5] implies

$$\mathrm{Vol}_k(P) \leq 2k^2\left(2i+2+\frac{1}{2}\right) = k^2(4i+5)$$

and we are done. Consider $m = 2$ and let $0 < a, b \leq 1$ be minimal such that $P \subseteq \mathbb{R}\times[1-a, 2+b]$. If $a+b \geq 1$, [10, Proposition 3.3] implies again $\mathrm{Vol}_k(P) \leq 4k^2(i+1)$. Otherwise, it gives

$$\mathrm{Vol}_k(P) \leq \frac{(a+b+1)^2}{a+b}(i+1)k^2,$$

which attains its maximum for $a = b = \frac{1}{k}$, hence we get $\mathrm{Vol}_k(P) \leq \frac{1}{2}k(k+2)^2(i+1)$. □

Proposition 7.2. *Let P be a k-maximal polygon with* $\mathrm{denom}(P) = k \geq 2$ *and* $i := i(P)$ *that cannot be realized in* $\mathbb{R}\times[-1,1]$. *Then there exists a lattice polygon Q such that $P = \frac{1}{k}Q^{(-1)}$, where*

$$l(Q) \ \leq\ \frac{1}{2}\,\mathrm{Vol}_k(P) - \frac{1}{2} \ \leq\ \max\left(k^2(2i+\frac{5}{2}),\ \frac{1}{4}k(k+2)^2(i+1)\right) - \frac{1}{2}.$$

Proof. Since P is k-maximal, the lattice polygon kP is 1-maximal by Proposition 2.16. Set $Q := (kP)^{(1)}$. Since P cannot be realized in $\mathbb{R}\times[-1,1]$ and $k \geq 2$, it follows that Q is two-dimensional. Thus Proposition 2.18 implies $P = \frac{1}{k}Q^{(-1)}$. Since $l(Q) = i(kP)$, the bound for $l(Q)$ follows with Pick's formula and then Lemma 7.1. □

This proposition tells us that if we know all lattice polygons with sufficiently many lattice points, we can use them to classify k-maximal polygons with $i \in \mathbb{Z}_{\geq 0}$ interior lattice points (those in $\mathbb{R}\times[-1,1]$ can be classified directly using Algorithm 3.4). A classification algorithm for lattice polygons by number of lattice points was given by Koelman [23, Section 4.4]. How far we would have to know this classification for is given by the bound in Proposition 7.2, i.e.

$$l_{\max}(k,i) \ :=\ \left\lfloor \max\left(k^2(2i+\frac{5}{2}),\ \frac{1}{4}k(k+2)^2(i+1)\right) - \frac{1}{2}\right\rfloor.$$

Below are some values for $l_{\max}(k,i)$, where k runs vertically and i horizontally. Note however that this bound is not sharp and can likely be improved.

$l_{\max}(k,i)$	0	1	2	3	4	5	6	7	8
2	9	17	25	33	41	49	57	65	73
3	22	40	58	76	94	112	130	149	168
4	39	71	107	143	179	215	251	287	323
5	62	122	183	244	305	367	428	489	550
6	95	191	287	383	479	575	671	767	863
7	141	283	424	566	708	850	991	1133	1275

We have implemented Koelman's classification algorithm in `RationalPolygons.jl` [33], in particular reproducing his results up to $l = 42$ and extending it up to $l = 112$. In total, we have found 115 449 011 813 lattice polygons with at most 112 lattice points, 79 140 459 of which are internal, i.e. of the form $P^{(1)}$. We made the internal ones as well as all polygons up to 70 lattice points available at [36], see also [27, A371917] for their numbers. Using Proposition 7.2, we then obtained the classification of 3-maximal polygons with up to five and 4-maximal polygons with up to two interior lattice points, see Theorem 1.1. While this approach works in principle for all pairs (k,i), it is very inefficient compared to our direct classifications for $i \in \{0,1\}$ in Sections 4 and 5 as well as $k = 2$ in Section 6.

However, it did prove useful as an independent verification of our results in these sections for low values of k and i respectively.

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8. Generalizations of Scott's inequality and Pick's formula to rational polygons

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GENERALIZATIONS OF SCOTT'S INEQUALITY AND PICK'S FORMULA TO RATIONAL POLYGONS

MARTIN BOHNERT AND JUSTUS SPRINGER

Abstract. We prove a sharp upper bound on the number of boundary lattice points of a rational polygon in terms of its denominator and the number of interior lattice points, generalizing Scott's inequality. We then give sharp lower and upper bounds on the area in terms of the denominator, the number of interior lattice points, and the number of boundary lattice points, which can be seen as a generalization of Pick's formula. Minimizers and maximizers are described in detail. As an application, we derive bounds for the coefficients of Ehrhart quasipolymials of half-integral polygons.

1. Introduction

By a *rational polygon* $P \subseteq \mathbb{R}^2$, we mean the convex hull of finitely many points in $\mathbb{Q}^2$. We call the smallest positive integer k such that kP has integral vertices the *denominator* of P. Scott [9] showed that for rational polygons of denominator one (henceforth called *lattice polygons*), the numbers i and b of interior and boundary lattice points of P satisfy $b \leq 9$ for $i = 1$ and $b \leq 2i+6$ for $i \geq 2$ and this bound is sharp. In particular, $(i, b) = (1, 9)$ holds if and only if P is affine unimodular equivalent to the threefold standard lattice triangle. We prove the following generalization of Scott's inequality to rational polygons:

Theorem 1.1. *Let P be a rational polygon of denominator $k \geq 2$ with $i \geq 1$ interior and b boundary lattice points. Then $b \leq (k+1)(i+1)+3$ and equality holds if and only if P is affine unimodular equivalent to the triangle* $\operatorname{conv}\big((0, \frac{1}{k}), (0, -1), ((k+1)(i+1), -1)\big)$.

Next, we provide area bounds. Recall that Pick's formula [8] states that the area of a lattice polygon with i interior and b boundary lattice points equals $i + \frac{b}{2} - 1$. For rational polygons of denominator at least two, fixing i and b no longer uniquely determines the area. Instead, we provide sharp lower and upper bounds, where we also completely describe minimizers and maximizers.

Theorem 1.2 (See also figures 1 and 2)**.** *Let P be a rational polygon of denominator $k \geq 2$ with $i \geq 1$ interior and b boundary lattice points. Set $b_{\max} = (k+1)(i+1)+3$. If $Q := \operatorname{conv}(P \cap \mathbb{Z}^2)$ is two-dimensional and $b \geq 2$, we have*

$$\operatorname{area}(P) \;\geq\; \frac{i(k+1)+1}{2k} + \frac{b}{2} - 1.$$

Here, equality holds if and only if P is equivalent to one of the following polygons:

(0a)	$\operatorname{conv}((0,-1),(i(k+1)-k+1,-1),(-\frac{1}{k},\frac{1}{k}))$	*for* $b = b_{\max} - 2(k+1)$
(1a)	$\operatorname{conv}((0,-1),(b-2,-1),(i,0),(-\frac{1}{k},\frac{1}{k}))$	*for* $2 \leq b \leq b_{\max} - (k+1)$
(1b)	$\operatorname{conv}((0,-2),(2,0),(-\frac{1}{k},\frac{1}{k}))$,	*for* $(i,b) = (3,3)$
(2a)	$\operatorname{conv}((0,0),(0,-1),(b-3,-1),(i+1,0),(\frac{x}{k},\frac{1}{k}))$	*for* $3 \leq b \leq b_{\max}$
(2b)	$\operatorname{conv}((0,0),(0,-2),(2,0),(\frac{x'}{k},\frac{1}{k}))$	*for* $(i,b) = (1,5)$,

where $x = 0, \ldots, \lfloor \frac{b_{\max}-b}{2} \rfloor$ *and* $x' = 0, \ldots, k$. *Moreover, if* $\dim(Q) < 2$, *we have* $b \leq 2$ *and*

$$\operatorname{area}(P) \geq \begin{cases} \frac{i}{k} - \frac{1}{k} + \frac{3}{2k^2}, & b = 0 \\ \frac{i}{k} + \frac{1}{2k^2}, & b = 1 \\ \frac{i}{k} + \frac{1}{k}, & b = 2. \end{cases}$$

Here, equality holds if and only if P *is equivalent to one of the following polygons:*

$$\begin{array}{lll} \text{(0c)} & \operatorname{conv}((1, \frac{1}{k}), (1 - \frac{1}{k}, -\frac{1}{k}), (i + \frac{1}{k}, 0)), & \textit{for } b = 0, \\ \text{(1c)} & \operatorname{conv}((1, \frac{1}{k}), (1 - \frac{1}{k}, -\frac{1}{k}), (i + 1, 0)), & \textit{for } b = 1, \\ \text{(2c)} & \operatorname{conv}((0, 0), (0, \frac{1}{k}), (\frac{x}{k}, -\frac{1}{k}), (i + 1, 0)), & \textit{for } b = 2, \end{array}$$

where $x = 0, \ldots, k(i + 1)$. *All polygons listed above are pairwise non-equivalent.*

Theorem 1.3 (See also figure 3). *Let* P *be a rational polygon of denominator* $k \geq 4$ *with* $i \geq 1$ *interior and* b *boundary lattice points. Set* $b_{\max} := (k+1)(i+1)+3$ *and* $\tilde{b} := b_{\max} - b$. *Then*

$$2k^2 \operatorname{area}(P) \;\leq\; k(k+1)^2(i+1) - \begin{cases} \tilde{b}, & b \in \{b_{\max}, b_{\max} - 1\} \\ (2 + k(\tilde{b} - 2)), & 1 \leq b \leq b_{\max} - 2 \\ (3 + k(\tilde{b} - 2)), & b = 0. \end{cases}$$

Here, equality holds if and only if P *is equivalent to one the following polygons:*

$$\begin{array}{lll} \text{(0a)} & \operatorname{conv}\left((0, \frac{1}{k}), (\frac{1}{k}, -1), ((k+1)(i+1) - \frac{1}{k}, -1)\right) & \textit{for } b = b_{\max} - 4 \\ \text{(0b)} & \operatorname{conv}\left((0, \frac{1}{k}), (\frac{1}{k}, -1), (1 - \frac{1}{k}, -1), (k(i+1) - \frac{1}{k}, \frac{1}{k} - 1)\right) & \textit{for } b = 0 \\ \text{(1a)} & \operatorname{conv}\left((0, \frac{1}{k}), (\frac{1}{k}, -1), (b - \frac{1}{k}, -1), (k(i+1), \frac{1}{k} - 1)\right) & \textit{for } 1 \leq b \leq b_{\max} - 3, \\ \text{(2a)} & \begin{array}{l}\operatorname{conv}\left((0, \frac{1}{k}), (0, \frac{1}{k} - 1), (x + \frac{1}{k}, -1),\right. \\ \qquad \left.(x + b - 1 - \frac{1}{k}, -1), (k(i+1), \frac{1}{k} - 1)\right)\end{array} & \textit{for } 2 \leq b \leq b_{\max} - 2, \\ \text{(2b)} & \operatorname{conv}\left((0, \frac{1}{k}), (0, \frac{1}{k} - 1), (\frac{1}{k}, -1), ((k+1)(i+1), -1)\right) & \textit{for } b = b_{\max} - 1, \\ \text{(2c)} & \operatorname{conv}\left((0, \frac{1}{k}), (0, -1), ((k+1)(i+1), -1)\right) & \textit{for } b = b_{\max}, \end{array}$$

where $x = 0, \ldots, \lfloor \frac{\tilde{b}}{2} \rfloor - 1$. *All polygons listed above are pairwise non-equivalent.*

After we prove Theorems 1.1-1.3 in section 2, we turn to the case $k = 2$ in section 3. We found that in this case, the area bound from Theorem 1.3 can be violated precisely if $i = 1$ or $b = 0$. We obtain the following sharp upper bound:

Theorem 1.4. *Let* P *be rational polygon of denominator* 2 *having* $i \geq 1$ *interior and* b *boundary lattice points. If* $i \geq 2$, *we have the sharp bound*

$$\operatorname{area}(P) \;\leq\; \frac{3}{2}i + \frac{1}{4}b + \frac{1}{8} \cdot \begin{cases} 8, & 0 \leq b \leq 3i + 4 \\ 7, & b = 3i + 5 \\ 6, & b = 3i + 6. \end{cases}$$

Moreover, if $i = 1$, *we have the sharp bound*

$$\operatorname{area}(P) \;\leq\; \frac{1}{4}b + \frac{1}{8} \cdot \begin{cases} 21, & 0 \leq b \leq 6 \\ 20, & b = 7 \\ 19, & b = 8 \\ 18, & b = 9. \end{cases}$$

2. Proofs of Theorems 1.1, 1.2 and 1.3

For any polygon P, we write $i(P)$ for the number of interior lattice points and $b(P)$ for the number of boundary lattice points. We call the lattice polygon $\operatorname{conv}(P \cap \mathbb{Z}^2)$ the *integer hull* of P. We call two polygons P and Q *equivalent*, if $P = UQ + z$ holds for a unimodular matrix $U \in \mathbb{Z}^{2\times 2}$ and some $z \in \mathbb{Z}^2$.

Let us recall the classification of lattice polygons without interior lattice points.

Proposition 2.1 (See [7, Theorem 4.1.2])**.** *Let P be a lattice polygon with $i(P) = 0$. Then P is either equivalent to the twofold standard triangle* $\operatorname{conv}((0,0),(2,0),(0,-2))$ *or to a polygon of the form* $\operatorname{conv}((0,0),(0,-1),(n,-1),(m,0))$ *with integers $n \geq m \geq 0$.*

Proof of Theorem 1.1. Consider the integer hull $Q := \operatorname{conv}(P \cap \mathbb{Z}^2)$. We may assume Q to be two-dimensional, otherwise we have $b \leq 2$ and the claim holds trivially. If $i(Q) > 0$, then we get with Scott's inequality

$$b \ \leq \ b(Q) \ \leq \ 2i(Q) + 7 \ \leq \ 2i + 7 \ \leq \ (k+1)(i+1) + 3.$$

This can only be an equality for $(k,i) = (2,1)$ and Q being the threefold standard lattice triangle. But since P is not a lattice polygon, it must contain a non-integral vertex outside of Q. Since the threefold standard lattice triangle has a lattice point in the relative interior of each edge, this implies $b < b(Q)$, hence the inequality is strict after all.

If $i(Q) = 0$, Proposition 2.1 says that Q is either the twofold standard lattice triangle or of the form $\operatorname{conv}((0,0),(0,-1),(n,-1),(m,0))$ for integers $n \geq m \geq 0$. In the former case, we have $b \leq b(Q) = 6$ and the claim holds trivially. Let us treat the latter case. Since P has at least one interior lattice point, there must be a vertex $v = (x,y)$ of P which lies outside of the strip $\mathbb{R} \times [-1,0]$. If $y < -1$, we have $i \geq n-1$, hence

$$b \ \leq \ m+3 \ \leq \ n+3 \ \leq \ i+4 \ < \ (k+1)(i+1)+3.$$

Now assume $y > 0$. Then we maximize the number of boundary lattice points if $(-1,l)$ are all boundary points of P for $0 \leq l \leq n$ and n is as large as possible. This happens when y is as small as possible i.e. $y = \frac{1}{k}$. By convexity, the largest possible n is $(k+1)(i+1)$. Then we obtain the triangle from the assertion and $b = n+3$ as claimed. □

Remark 2.2. Plugging $k = 1$ into Theorem 1.1 gives $i \leq 2i+5$, which is less than Scott's inequality. The difference is explained as follows: First, the threefold standard lattice triangle is an exception which only exists for $k = 1$. Second, if we take $k = 1$ in the triangle from Theorem 1.1, the vertex $(0, \frac{1}{k})$ is itself a lattice point, hence it has one more boundary lattice point than expected.

In the following proof, it will be convenient for us to normalize our area measure with respect to the $\frac{1}{k}$-fold standard triangle instead of the unit square. To that end, we define the *k-normalized area* of P as

$$\operatorname{Area}_k(P) \ := \ 2k^2 \operatorname{area}(P),$$

where $\operatorname{area}(P)$ denotes the standard euclidian area. Note that $\operatorname{Area}_k(P)$ is always an integer.

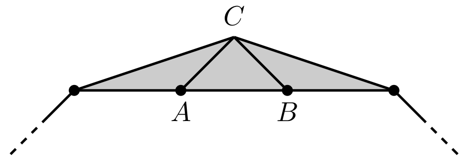
Proof of Theorem 1.2. We treat first the case $\dim(Q) = 2$. In terms of the k-normalized area, we have to show the following:

$$\operatorname{Area}_k(P) \ \geq \ k^2(i+b-2) + k(i+1).$$

Pick's formula gives us a first bound

$$\operatorname{Area}_k(P) \ \geq \ \operatorname{Area}_k(Q) \ = \ k^2\left(2i(Q) + b(Q) - 2\right).$$

Consider an interior lattice point A of P which lies on the boundary of Q and let B be a boundary lattice point of Q adjacent to A. Pick a rational point $C \in (P \setminus Q) \cap \frac{1}{k}\mathbb{Z}^2$ lying over the edge of Q which contains A and B and consider the triangle with vertices A, B and C. Its base has length one and its height is at least $\frac{1}{k}$. Hence it has k-normalized area of at least k. Choosing the same C for as many pairs (A, B) as possible gives us at least $i - i(Q) + 1$ triangles of this form with disjoint interiors. We obtain $\mathrm{Area}_k(P) - \mathrm{Area}_k(Q) \geq k(i - i(Q) + 1)$.



Using $b(Q) = b + i - i(Q)$, we compute

$$\begin{aligned}\mathrm{Area}_k(P) &\geq k^2(2i(Q) + b(Q) - 2) + k(i - i(Q) + 1)\\ &= k^2(i + b - 2) + k(i + 1) + i(Q)(k^2 - k)\\ &\geq k^2(i + b - 2) + k(i + 1).\end{aligned}$$

To get equality, we need $i(Q) = 0$ and $\mathrm{Area}_k(P) - \mathrm{Area}_k(Q) = k(i+1)$. The latter happens if and only if there is a unique point $C \in (P \setminus Q) \cap \frac{1}{k}\mathbb{Z}^2$ having lattice distance $\frac{1}{k}$ to all relevant edges of Q. By Proposition 2.1, Q is either the twofold standard triangle or of the form $\mathrm{conv}((0, 0), (0, -1), (n, -1), (m, 0))$ for $n \geq m \geq 0$. In the former case, we get the polygons (1b) and (2b) from the assertion. In the latter case, we get the polygons (0a), (1a) and (2a). Here, the number before the letter denotes the number of boundary lattice points of P on $\mathbb{R} \times \{0\}$, which is at most two. Note that by symmetry, for the polygons (2a) it is enough to consider $x = 0, \dots, \lfloor \frac{b_{\max} - b}{2} \rfloor$. Similarly, for (2b), $x' = 0, \dots, k$ is enough. See also figure 2 for an illustration.

Now we consider $\dim(Q) < 2$, i.e. the lattice points of P are collinear. In particular, we have $b \leq 2$. After an affine unimodular transformation, we may assume $Q \subseteq \mathbb{R} \times \{0\}$. In terms of the k-normalized area, we have to show

$$\mathrm{Area}_k(P) \geq \begin{cases} 2k(i + 1), & b = 2,\\ 2ki + 1, & b = 1,\\ 2k(i - 1) + 3, & b = 0.\end{cases}$$

Note that by Pick's formula, we have

$$\mathrm{Area}_k(P) \;=\; \mathrm{Area}_1(kP) \;=\; 2i(kP) + b(kP) - 2. \tag{1}$$

We start with the case $b = 2$. Then kQ has exactly $k(i + 1) - 1$ lattice points in its relative interior, hence $i(kP) \geq k(i + 1) - 1$. Moreover, kP must have at least one lattice point (a vertex) above and below of Q. Together with the two boundary lattice points on Q itself, this gives $b(kP) \geq 4$. Plugging this into equation (1), we arrive at the desired bound and equality holds if and only if $i(kP) = k(i + 1) - 1$ and $b(kP) = 4$. Up to affine unimodular equivalence, there are exactly $k(i + 1) + 1$ ways to choose boundary points of kP without getting additional interior lattice points, which are exactly the polygons (2c) from the assertion.

For $b = 1$, we obtain with analogous reasoning $i(kP) \geq ki$ and $b(kP) \geq 3$. The bound again follows from Equation (1). To get equality, there is up to affine unimodular transformation only one possible choice of boundary points of kP, which gives us the polygon (1c) from the assertion.

For $b = 0$, we have $i(kP) \geq k(i-1)+1$ and $b(kP) \geq 3$, which yields the desired bound. Again, to get equality, there is only one possibility, namely the polygon (0c) from the assertion. See also figure 1 for an illustration. $\square$

Remark 2.3. In Theorem 1.2, the polygon (1a) for $b = 2$ plays a special role: While it is always an area minimizer with respect to polygons with two boundary lattice points having a two-dimensional integer hull, it is not in general an area minimizer with respect to all polygons with two boundary lattice points. This is because the bound for $\dim(Q) < 2$ and $b = 2$ (namely $\operatorname{area}(P) \geq \frac{i+1}{k}$) is in general smaller. The only exception is the case $(k, i) = (2, 1)$, where both bounds agree. This is the only case where (1a) for $b = 2$ truly is an area minimizer.

We turn to upper bounds on the area, i.e. Theorem 1.3. Let $k \geq 2$ and $i \geq 1$. Set $b_{\max} := (k+1)(i+1)+3$ and let $0 \leq b \leq b_{\max}$. Then we intend to prove that the k-normalized area of a rational polygon with denominator k having i interior and b boundary lattice points is not greater than

$$\operatorname{Maxarea}(k,i,b) \;:=\; k(k+1)^2(i+1) - \begin{cases} \tilde{b}, & b \in \{b_{\max}, b_{\max}-1\} \\ (2+k(\tilde{b}-2)), & 1 \leq b \leq b_{\max}-2 \\ (3+k(\tilde{b}-2)), & b = 0 \end{cases}$$

where $\tilde{b} := b_{\max} - b$.

Lemma 2.4. *Let P be a rational polygon with denominator k having i interior and b boundary lattice points. Assume P is not equivalent to a polygon in $\mathbb{R}\times[-1,1]$. Then we have*

$$\operatorname{Area}_k(P) \leq \max\left(k^2(4i+5),\ \frac{1}{2}k(k+2)^2(i+1)\right).$$

In particular, for $k \geq 4$ we have $\operatorname{Area}_k(P) < \operatorname{Maxarea}(k,i,b)$.

Proof. See [4, Proposition 7.2] for the first bound on $\operatorname{Area}_k(P)$. For the supplement, note that $k \geq 4$ implies

$$\max\left(k^2(4i+5),\ \frac{1}{2}k(k+2)^2(i+1)\right) \;<\; k^2(k+1)(i+1)-k-3.$$

and $k^2(k+1)(i+1)-k-3 = \operatorname{Maxarea}(k,i,0) \leq \operatorname{Maxarea}(k,i,b)$. $\square$

Lemma 2.5. *Let P be a rational polygon of denominator $k \geq 4$ having i interior and b boundary lattice points. Assume P is not equivalent to a polygon in $\mathbb{R}\times[-1,\frac{1}{k}]$. Then $\operatorname{Area}_k(P) < \operatorname{Maxarea}(k,i,b)$.*

Proof. By Lemma 2.4, we can assume that $P \subseteq \mathbb{R}\times[-1,1]$. Let $h \in \{2,\dots,k\}$ be minimal such that $P \subseteq \mathbb{R}\times[-1,\frac{h}{k}]$. Then by [2, Proposition 3.1], we have $\operatorname{Area}_k(P) \leq k\left(\frac{k^2}{h}+2k+h\right)(i+1)$. For $k \geq 4$, this attains its maximum at $h = 2$. We obtain

$$\operatorname{Area}_k(P) \leq k\left(\frac{k^2}{2}+2k+2\right)(i+1) < k^2(k+1)(i+1)-k-3 \leq \operatorname{Maxarea}(k,i,b),$$

where we again used $k \geq 4$ to get the strict inequality in the middle. $\square$

Proof of Theorem 1.3. By Lemma 2.5, we may assume $P \subseteq \mathbb{R} \times [-1, \frac{1}{k}]$. Throughout our proof, we use the following shorthand notation: For any $-1 \leq y \leq \frac{1}{k}$ as well as $-1 \leq y_1 < y_2 \leq \frac{1}{k}$, we write

$$P_y \;:=\; P \cap (\mathbb{R} \times \{y\}), \qquad P_{[y_1,y_2]} \;:=\; P \cap (\mathbb{R} \times [y_1, y_2]).$$

After a translation, we can assume $P_0 \subseteq [0, i+1] \times \{0\}$. Next, we apply a shearing to achieve that the leftmost vertex in $P_{1/k}$ is $(0, \frac{1}{k})$. Convexity then implies that $P_{[-1,0]}$ is contained in the convex hull of $(0,0), (0,-1), ((k+1)(i+1), -1)$ and $(i+1, 0)$. In particular, the line segment P_{-1} is contained in $[0, (k+1)(i+1)] \times \{-1\}$. Write b_0 and b_{-1} for the number of boundary point of P in P_0 and P_1 respectively. Then $b = b_0 + b_{-1}$ and $b_0 \in \{0, 1, 2\}$. The length of P_{-1} is restricted by the number of lattice points b_{-1} on it, i.e.

$$\text{length}(P_{-1}) \;\leq\; \begin{cases} (k+1)(i+1), & b_{-1} = b_{\max} - 2 \\ (k+1)(i+1) - \frac{1}{k}, & b_{-1} = b_{\max} - 3 \\ b_{-1} + 1 - \frac{2}{k}, & 0 \leq b_{-1} \leq b_{\max} - 4. \end{cases} \tag{2}$$

Consider now the line segment $P_{-1+1/k}$, which is contained in $[0, k(i+1)] \times \{-1 + 1/k\}$. We obtain the following bound:

$$\text{length}(P_{-1+1/k}) \;\leq\; k(i+1) - \frac{2 - b_0}{k+1}. \tag{3}$$

We consider the decomposition $P = P_{[-1,-1+1/k]} \cup P_{[-1+1/k,1/k]}$. Note that the area of $P_{[-1+1/k,1/k]}$ is maximized if and only if (3) is an equality and $(0, \frac{1}{k})$ is the only vertex in $P_{1/k}$. This is because $P_{[-1+1/k,0]}$ cannot be larger anyway and using an additional vertex in $P_{1/k}$ would lose more area in $P_{[-1+1/k,0]}$ than we could gain in $P_{[0,1/k]}$. On the other hand, $P_{[-1,-1+1/k]}$ is just a trapezoid, hence its area is maximized if and only if (2) and (3) are both equalities. We now proceed by case distinction on $b_0 \in \{0, 1, 2\}$ and describe all polygons satisfying these conditions, which gives us all area maximizers.

We first treat $b_0 = 2$, which implies $b \geq 2$ and $b_{-1} = b - 2$. Here, we achieve equality in (3) if we have edges connecting the vertex $(0, \frac{1}{k})$ to $(0, -1 + \frac{1}{k})$ and $(k(i+1), -1 + \frac{1}{k})$. For $b = b_{\max}$, there is only one possibility to achieve equality in (2), namely the polygon (2c). For $b = b_{\max} - 1$, there are a priori two possiblieties of maximizing the length of P_{-1}. However, they are equivalent by symmetry, hence there is only one polygon (2b). For $2 \leq b \leq b_{\max} - 2$, we obtain the family of polygons (2a), which arises from different translations of the line segment P_{-1}. A priori, we have $x = 0, \ldots, \tilde{b} - 2$. However, by symmetry, it is enough to consider $x = 0, \ldots, \lfloor \frac{\tilde{b}}{2} \rfloor - 1$. Computing the area of (2a), (2b) and (2c), we arrive at the bound from the assertion.

Next, we treat $b_0 = 1$. By symmetry, we can assume that the unique boundary lattice point on P_0 is $(i+1, 0)$. Then we achieve equality in (3) if we have edges connecting the vertex $(0, \frac{1}{k})$ to $(\frac{1}{k}, -1)$ and $(k(i+1), -1 + \frac{1}{k})$. By convexity, we can maximize length(P_{-1}) (i.e. achieve equality in (2)) only for $b \leq b_{\max} - 3$, which gives us the polygons (1a) from the assertion. Note that these have exactly the same area as the polygons (2a) for fixed $b \leq b_{\max} - 3$ and we cannot reach this bound for b larger and length(P_{-1}) not maximal. Moreover, since $(0, \frac{1}{k} - 1)$ is no longer a vertex for (1a), there is no flexibility in moving P_{-1} around. Hence for each $1 \leq b \leq b_{\max} - 3$, we only get one polygon of type (1a).

Lastly, for $b_0 = 0$, simultaneous equality in (2) and (3) can only be reached for $b = b_{\max} - 4$, in which case connecting $(0, \frac{1}{k})$ to $(\frac{1}{k}, -1)$ and $(b + 1 - \frac{1}{k}, -1)$ gives the triangle (0a). It has the same area as the polygons corresponding polygons (1a) and (2a). This leaves open the case $b = 0$. Here, we cannot reach equality in (3), since this would fix P_{-1} and lead to $b > 0$. The best we can do in this case is to connect $(0, \frac{1}{k})$ to $(\frac{1}{k}, -1)$ and $(k(i+1) - \frac{1}{k}, \frac{1}{k} - 1)$. The line segment P_{-1} is then already uniquely determined, hence we arrive at the polygon (0b). It has the claimed area from the assertion, which is notably smaller than in the other cases by one unit of k-normalized area. □

Remark 2.6. We can compare our area bounds to Pick's formula by plugging $k = 1$ into Theorems 1.2 and 1.3. Ignoring the cases $b \leq 2$ (which cannot happen for lattice polygons), we arrive at $i + \frac{b}{2} - \frac{1}{2}$ for both the lower and upper bound. This is notably $\frac{1}{2}$ more than what Pick's formula says. The difference can be explained as follows: All of our area maximizers and minimizers described in Theorems 1.2 and 1.3 have a rational vertex at $y = \frac{1}{k}$. For $k = 1$, this would become a boundary lattice point, hence the polygon has one more boundary lattice point than it would have for $k > 1$. Substituting $b \mapsto b + 1$ into Pick's formula, we recover the same expression $i + \frac{b}{2} - \frac{1}{2}$.

Remark 2.7. In the proof of Theorem 1.3, the condition $k \geq 4$ was only neccessary to ensure $P \subseteq \mathbb{R} \times [-1, \frac{1}{k}]$ by applying the bounds from lemmas 2.4 and 2.5. In particular, the area bound of Theorem 1.3 holds for all polygons in $\mathbb{R} \times [-1, \frac{1}{k}]$, even if $k \in \{2, 3\}$. However, as we will see in Section 3, not all area maximizers for $k = 2$ are contained in $\mathbb{R} \times [-1, \frac{1}{2}]$. For $k = 3$, we used our classification of rational polygons [4] to verify that Theorem 1.3 holds completely for $1 \leq i \leq 5$. In particular, there are no maximizers outside of $\mathbb{R} \times [-1, \frac{1}{3}]$. This leads us to conjecture that the bounds from lemmas 2.4 and 2.5 can be improved to work also for $k = 3$.

Remark 2.8. Let P be an area maximizer of type (2a) as described in Theorem 1.3 with at least three boundary lattice points. Taking $Q := \operatorname{conv}(P \cap \mathbb{Z}^2)$, the polygon $P' := \operatorname{conv}(Q \cup \{(0, 1/k)\})$ is then an area minimizer with the same number of interior and boundary lattice points as P. Moreover, we can find a polygon that attains any area between $\operatorname{Area}_k(P)$ and $\operatorname{Area}_k(P')$ by means of the following three operations:

(1) Shorten or lengthen the line segment $P \cap \mathbb{R} \times \{-1\}$ by l/k on either side for $1 \leq l \leq k - 1$. This lowers or increases $\operatorname{Area}_k(P)$ by l.
(2) Add or remove the vertex $(l/k, 1/k)$ for $1 \leq l \leq k - 1$. This increases or lowers $\operatorname{Area}_k(P)$ by l.
(3) Move in the vertices on $P \cap \mathbb{R} \times \{-1 + 1/k\}$ by $1/k$. This lowers $\operatorname{Area}_k(P)$ by k.

See figure 6 for an example of how these operations can be used to attain any area between the lower and upper bounds.

3. Half-integral polygons

In this section, we prove the sharp upper bound for the area of half-integral polygons given in Theorem 1.4. Note that it differs from the bound given in Theorem 1.3 exactly in the cases $i = 1$ and $b = 0$, where it exceeds the general bound by $\frac{1}{8}$ (= one unit of 2-normalized area). For $i = 1$, the half-integral polygons that violate the bound from Theorem 1.3 are all subpolygons of the threefold standard lattice triangle, see figure 4. For $b = 0$, the polygons violating Theorem 1.3 are all realized in $\mathbb{R} \times [-1, 1]$. This class of half-integral area maximizers also occurs for $b > 0$, where their area agrees with the generic bound. The following lemma completely describes them.

Lemma 3.1 (See also figure 5)**.** *Let $P \subseteq \mathbb{R} \times [-1, 1]$ be a rational polygon of denominator two with i interior and b boundary lattice points which cannot be realized in $\mathbb{R} \times [-1, \frac{1}{2}]$.*

Then we have

$$b \le 2i+6, \qquad \mathrm{Area}_2(P) \le 12i+2b+8.$$

Moreover, for $-1 \le y \le 1$, *write* ℓ_y *for the length of the line segment* $P \cap (\mathbb{R}\times\{y\})$ *and* b_y *for the number of boundary lattice points on them. Then* $\mathrm{Area}_2(P) = 12i+2b+8$ *if and only if the following hold:*

- $(\ell_1, \ell_{-1}) = (b_1, b_{-1})$ *and* $\ell_0 = i + \frac{2}{3} + \frac{1}{6}b_0$,
- $P \cap (\mathbb{R}\times[-\frac{1}{2},\frac{1}{2}])$ *is a trapezoid, i.e.* P *has no vertex on* $\mathbb{R}\times\{0\}$.

Proof. We clearly have $b_1 \le \ell_1+1$ and $b_{-1} \le \ell_{-1}+1$ as well as $b_0 \le 2$. Moreover, $\ell_0 \le i+1$ holds and by convexity, we have $\ell_{-1}+\ell_1 \le 2\ell_0$. We obtain

$$b \;=\; b_{-1}+b_0+b_1 \;\le\; \ell_{-1}+\ell_1+4 \;\le\; 2\ell_0+4 \;\le\; 2i+6.$$

To estimate the area, consider the trapezoids $P_{-1} := P \cap (\mathbb{R}\times[-1,-\frac{1}{2}])$ and $P_1 := P \cap (\mathbb{R}\times[\frac{1}{2},1])$. Note that these are non-trivial, since we assume P cannot be realized in $\mathbb{R}\times[-1,\frac{1}{2}]$. Their area is given by

$$\mathrm{Area}_2(P_{-1}) \;=\; 2(\ell_{-1}+\ell_{-\frac{1}{2}}), \qquad \mathrm{Area}_2(P_1) \;=\; 2(\ell_{\frac{1}{2}}+\ell_1).$$

Moreover, the area of $P_0 := P \cap (\mathbb{R}\times[-\frac{1}{2},\frac{1}{2}])$ is

$$\mathrm{Area}_2(P_0) \;=\; 2(\ell_{-\frac{1}{2}}+\ell_0)+2(\ell_0+\ell_{\frac{1}{2}}) \;=\; 4\ell_0+2(\ell_{-\frac{1}{2}}+\ell_{\frac{1}{2}}).$$

Since P has denominator two, we have $\ell_{-1} \le b_{-1}$ and $\ell_1 \le b_1$. Moreover, convexity forces $\ell_{-\frac{1}{2}}+\ell_{\frac{1}{2}} \le 2\ell_0$. In total, we obtain

$$\mathrm{Area}_2(P) = \mathrm{Area}_2(P_{-1}) + \mathrm{Area}_2(P_0) + \mathrm{Area}_2(P_1) \le 12\ell_0 + 2(b_{-1}+b_1).$$

Consider now the line segment $[s, s+\ell_0]\times\{0\} := P \cap (\mathbb{R}\times\{0\})$. Since $P \subseteq \mathbb{R}\times[-1,1]$ and the denominator of P is two, the possible values for the rational part of s are $\{0, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}\}$. This implies the bound $\ell_0 \le i + \frac{2}{3} + \frac{1}{6}b_0$. Hence we obtain

$$\mathrm{Area}_2(P) \le 12i+8+2(b_0+b_{-1}+b_1) = 12i+2b+8.$$

Moreover, equality holds if and only if the conditions from the assertion hold. Note that P_0 being a trapezoid is equivalent to $\ell_{-\frac{1}{2}}+\ell_{\frac{1}{2}} = 2\ell_0$. See also figure 5 for an illustration of some area maximizers of the described form. $\square$

In our proof of Theorem 1.4, we use the concept of lattice width, which we now quickly recall. For a polygon $P \subseteq \mathbb{R}^2$ and a non-zero vector $w \in \mathbb{R}^2$, we define the *width* of P in direction w as

$$\mathrm{width}_w(P) \;:=\; \max_{v\in P}\langle w, v\rangle - \min_{v\in P}\langle w, v\rangle.$$

The *lattice width* $\mathrm{lw}(P)$ of P is defined to be the minimum over $\mathrm{width}_w(P)$ where $w \in \mathbb{Z}^2\setminus\{0\}$. By a *lattice width direction*, we mean a $w \in \mathbb{Z}^2\setminus\{0\}$ such that $\mathrm{lw}(P) = \mathrm{width}_w(P)$.

Proof of Theorem 1.4. After an affine unimodular transformation, we can assume that $(0,1)$ is a lattice width direction of P. Let n be the number of interior integral horizontal lines, i.e.

$$n \;:=\; |\{j \in \mathbb{Z} \mid P^\circ \cap (\mathbb{R}\times\{j\}) \neq \emptyset\}|.$$

We first treat $n=1$, where we distinguish two subcases: Either P can be realized in $\mathbb{R}\times[-1,\frac{1}{2}]$ or it can be realized in $\mathbb{R}\times[-1,1]$ but not in $\mathbb{R}\times[-1,\frac{1}{2}]$. In the former case, the assertion follows from the proof of Theorem 1.3, see also Remark 2.7. In particular, we get sharpness for our bound if $b \geq 1$. In the latter case, we refer to lemma 3.1, where we also obtain sharpness for $b = 0$.

Now we consider $n \geq 2$. We may assume $P \subseteq \mathbb{R} \times [0, n+1]$. For $0 \leq y \leq n+1$, let ℓ_y be the length of the line segment $P \cap (\mathbb{R} \times \{y\})$. Consider the numbers $i_j := |P^\circ \cap (\mathbb{Z} \times \{j\})|$ for $j = 1, \ldots, n$ as well as $b_j := |\partial P \cap (\mathbb{Z} \times \{j\})|$ for $j = 0, \ldots, n+1$. Note that we have

$$\ell_0 \leq b_0, \qquad \ell_{n+1} \leq b_{n+1}, \qquad \ell_j \leq i_j + 1 \text{ for } j = 1, \ldots, n.$$

Consider the subpolygons $P_j := P \cap (\mathbb{R} \times [j - \frac{1}{2}, j + \frac{1}{2}])$ for $j = 0, \ldots, n+1$. For $j = 1, \ldots, n$, we can estimate their area by

$$\mathrm{Area}_2(P_j) = 4\ell_j + 2(\ell_{j-\frac{1}{2}} + \ell_{j+\frac{1}{2}}) \leq 8\ell_j \leq 8(i_j + 1).$$

In particular, we obtain $\sum_{j=1}^{n} \mathrm{Area}_2(P_j) \leq 8i + 8n$. Moreover, we have $\mathrm{Area}_2(P_0) = 2(\ell_0 + \ell_{\frac{1}{2}})$ and $\mathrm{Area}_2(P_{n+1}) = 2(\ell_{n+\frac{1}{2}} + \ell_{n+1})$. By convexity, $\ell_{\frac{1}{2}} + \ell_{n+\frac{1}{2}} \leq \ell_k + \ell_{n-k+1}$ holds for all $k = 1, \ldots, \lfloor \frac{n}{2} \rfloor$. It follows

$$\mathrm{Area}_2(P_0) + \mathrm{Area}_2(P_{n+1}) \ \leq \ 2(b_0 + b_{n+1} + \ell_k + \ell_{n+k+1}) \ \leq \ 2(b + i_k + i_{n+k+1} + 2).$$

By choosing $k = 1, \ldots, \lfloor \frac{n}{2} \rfloor$ such that $i_k + i_{n+k+1}$ is minimal, we ensure $i_k + i_{n+k+1} \leq \frac{i}{\lfloor n/2 \rfloor}$. In total, we obtain

$$\mathrm{Area}_2(P) \ = \ \sum_{j=0}^{n+1} \mathrm{Area}_2(P_j) \ \leq \ 8i + 8n + 2b + \frac{2i}{\lfloor n/2 \rfloor} + 4.$$

For $n = 2$, this gives $\mathrm{Area}_2(P) \leq 10i + 2b + 20$, which strictly less than $12i + 2b + 6$ for $i \geq 8$. By our classification [4], we could verify that the claim holds for the finitely many polygons with $i \leq 7$. In particular, we found that the only maximizers with $n = 2$ are the finitely many polygons with $i \in \{1, 2\}$ drawn in figure 4. For $i = 1$ and $b \leq 6$, their area even exceeds the generic bound, which is the reason we list $i = 1$ seperately in the assertion.

For $n \geq 3$, we use the estimate $\lfloor n/2 \rfloor \geq \frac{n-1}{2}$. It is then enough to show

$$8i + 8n + \frac{4i}{n-1} + 2b + 4 \ < \ 12i + 2b + 6,$$

which is equivalent to

$$f(n) \ := \ \frac{4n^2 - 5n + 1}{2n - 4} \ < \ i.$$

By [2, Proposition 2.3], we get estimates $i_j \geq j - 1$ and $i_{n+1-j} \geq j - 1$ for $j = 1, \ldots, \lceil \frac{n}{2} \rceil$. Taking the sum, we obtain $i \geq \frac{n^2 - 2n}{4}$. For $n \geq 11$, this implies the claim. On the other hand, for $n \leq 7$, we have $f(n) < 17$. Again by our classification, we know the claim holds for $i \leq 16$, hence this case is done. For the remaining cases $n \in \{8, 9, 10\}$, we used our classification [4] of maximal half-integral polygons with up to 40 interior lattice points to compute their maximally attained lattice width $\mathrm{lw}_{\max}$. We obtained the following table:

i	1	2	3	4	5	6	7	8	9	10
$\mathrm{lw}_{\max}$	3	3	4	4	4	5	5	5	$11/2$	6
i	11	12	13	14	15	16	17	18	19	20
$\mathrm{lw}_{\max}$	$11/2$	6	6	6	7	7	7	$15/2$	8	$15/2$
i	21	22	23	24	25	26	27	28	29	30
$\mathrm{lw}_{\max}$	8	8	8	8	$17/2$	9	$17/2$	9	9	$19/2$
i	31	32	33	34	35	36	37	38	39	40
$\mathrm{lw}_{\max}$	10	$19/2$	$19/2$	$19/2$	10	10	10	10	10	11

We now treat $n = 8$. We have $f(8) < 19$, hence it suffices to show $i \geq 19$. Assume $i \leq 18$, then the table above implies $n \leq \mathrm{lw}(P) \leq \frac{15}{2} < 8$, a contradiction. The cases $n \in \{9, 10\}$ are settled analogously. □

Remark 3.2. For half-integral polygons with at least three interior integral horizontal lines, the proof of Theorem 1.4 shows that the area bound is strict. This means that there are no half-integral area maximizers with $n \geq 3$. Hence the half-integral area maximizers are precisely the following:

- All maximizers in $\mathbb{R} \times [-1, \frac{1}{2}]$, which are described in Theorem 1.3,
- maximizers in $\mathbb{R} \times [-1, 1]$ as described by Lemma 3.1 and
- the finitely many area maximizers with $n = 2$ for $i \in \{1, 2\}$, which are drawn in figure 4.

In the rest of this section, we show how our area bounds can be applied to Ehrhart theory of half-integral polygons. Recall that a rational polygon $P \subseteq \mathbb{R}^2$ admits an *Ehrhart quasipolynomial*, which counts lattice points in its integral dilations, i.e.

$$\mathrm{ehr}_P(t) \;:=\; |tP \cap \mathbb{Z}^2| \;=\; \mathrm{area}(P)t^2 + c_1(t)t + c_2(t), \qquad \text{for } t \in \mathbb{Z}_{\geq 1},$$

where $c_1, c_2 \colon \mathbb{Z} \to \mathbb{Q}$ are uniquely determined periodic functions whose period divides the denominator of P. For polygons of denominator two, the four coefficients $c_1(1), c_1(2), c_2(1)$ and $c_2(2)$ can be described in terms of the numbers i and b of interior and boundary lattice points, the area and the number $b(2P) = |\partial P \cap \frac{1}{2}\mathbb{Z}^2|$ of half-integral boundary points:

$$c_1(1) = \frac{b}{2}, \qquad c_2(1) = i + \frac{b}{2} - \mathrm{area}(P), \qquad c_1(2) = \frac{b(2P)}{4}, \qquad c_2(2) = 1.$$

We refer to [1] for more information on Ehrhart theory in general. The half-integral case was already studied in [6] with focus on the coefficients $c_1(1)$ and $c_2(1)$ for not necessarily convex polygons. Moreover, the cases with period collapse, i.e. $c_1(1) = c_1(2)$ and $c_2(1) = c_2(2)$, were studied in [3]. Restrictions on the coefficients of Ehrhart quasipolynomials of half-integral polygons, in particular for the case $i = 0$, were also given in [5].

Using our results, we contribute to the description of Ehrhart quasipolynomials for convex half-integral polygons with at least one interior point. Note that by the formulas for the coefficients above, the quadruple $(i, b, \mathrm{area}(P), b(2P))$ completely determines the Ehrhart quasipolynomial. To understand which Ehrhart quasipolynomials are possible, we look for sharp bounds between these four parameters: By Theorem 1.1, we have the sharp bound $0 \leq b \leq 3i+6$ and clearly every intermediate value is attained. Theorems 1.2 and 1.4 give sharp lower and upper bounds for the area in terms of i and b, while Remark 2.8 shows that all intermediate values are attained if $b \geq 3$. It remains to bound $b(2P)$ in terms of the first three parameters. We succeeded in proving the following sharp lower bound:

Proposition 3.3. *Let P be a half integral polygon with $i \geq 2$ interior lattice points and $b \geq 3$ boundary lattice points. Set $A := \mathrm{Area}_2(P)$. Then the number of half-integral boundary points of P is bounded sharply as follows:*

$$b(2P) \;\geq\; 2b + r + \begin{cases} 2, & \text{if } A = 12i + 2b + 8 \text{ and } b \neq 2i + 4 \\ 0, & \text{otherwise} \end{cases},$$

where r is 1 if A is odd and 0 if it is even.

Proof. Since there is always a half-integral point between two adjacent boundary lattice points, we get $b(2P) \geq 2b$. By Pick's formula, we have

$$A = \mathrm{Area}_1(2P) = 2i(2P) + b(2P) + 2.$$

This implies that $b(2P) - A$ is an even number, hence $b(2P) \geq 2b + r$. We now argue that this is a sharp inequality for $A \neq 12i + 2b + 8$ by giving examples. Note that by Theorem 1.2, we have $A \geq 6i + 4b - 6$. Consider $\tilde{A} := 12i + 2b - 7 - A$. Then for $A \neq 6i + 4b - 5$, the convex hull of

$$(0, 1/2), (0, -1), (b - 3 + r/2, -1), \left(2i + 2 - \frac{1}{2}\left\lfloor \tilde{A}/2 \right\rfloor, -1/2\right), (i + 1, 0)$$

satisfies $b(2P) = 2b + r$. For $A = 6i + 4b - 5$, the convex hull of

$$(0, 1/2), (1/2, 1/2), (i + 1, 0), (b - 3, -1), (0, -1)$$

works. Moreover, for $A = 12i + 2b + 8$ and $b = 2i + 4$, we can realize $b(2P) = 2b + r$ by the following triangle:

$$\mathrm{conv}((-1/2, -1), (2i + 3/2, -1), (1/2, 1)).$$

It remains to show that for $A = 12i + 2b + 8$ and $b \neq 2i + 4$, we have the sharp inequality $b(2P) \geq 2b + 2$ (note that $r = 0$ in this case). By Remark 3.2, there are three kinds of half-integral area maximizers: Those in $\mathbb{R} \times [-1, \frac{1}{2}]$ are described by Theorem 1.3 and we can check that $b(2P) = 2b + 2$ holds for all of them. The same is true for the finitely many area maximizers with two interior integral lines and $i = 2$ drawn in figure 4. The remaining ones are described by lemma 3.1. Here, $b \neq 2i + 4$ implies that we have at least one vertex with second coordinate $\pm\frac{1}{2}$. Hence there is at least one pair of adjacent non-intgral vertices with no lattice point betwen them, which implies $b(2P) \geq 2b + 2$. $\square$

It remains to bound $b(2P)$ from above in terms of i, b and $\mathrm{area}(P)$ and to understand which intermediate values are attained. Together with our other bounds, this would allow the complete description of the four-parameter family $(i(P), b(P), \mathrm{area}(P), b(2P))$ and hence all possible Ehrhart quasipolynomials of half-integral polygons having $i \geq 1$. In particular, this should provide a path to prove that the number of distinct Ehrhart quasipolynomial of half-integral polygons is given by the following polynomial, which we have already observed empirically from our classification:

Conjecture 3.4 (See also [4, Conjecture 6.4])**.** *There are exactly*

$$\frac{9}{2}i^3 + 36i^2 + \frac{175}{2}i + 53$$

Ehrhart quasi-polynomials of half-integral polygons with $i \in \mathbb{Z}_{\geq 2}$ interior lattice points and at least 3 boundary lattice points.

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List of Figures

Figure 1. All area minimizers with collinear lattice points as described by Theorem 1.2 for $(k, i) = (3, 1)$. On the left is the number b of boundary lattice points.

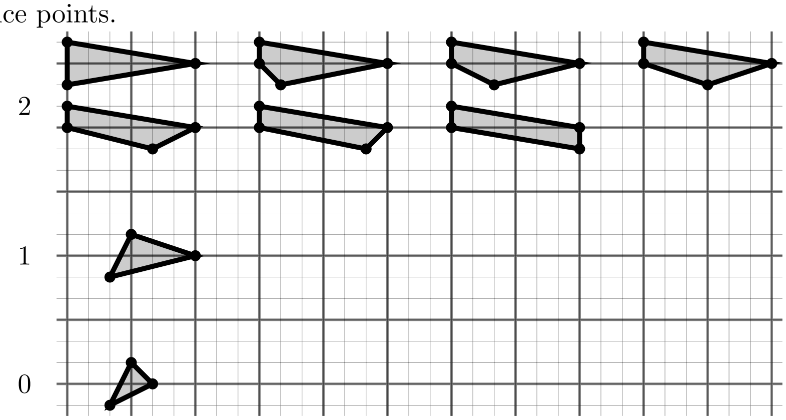


FIGURE 2. All area minimizers with two-dimensional integer hull as described by Theorem 1.2 for $(k, i) = (3, 1)$. On the left is the number b of boundary lattice points. Each polygon is labeled according to the Theorem's notation.

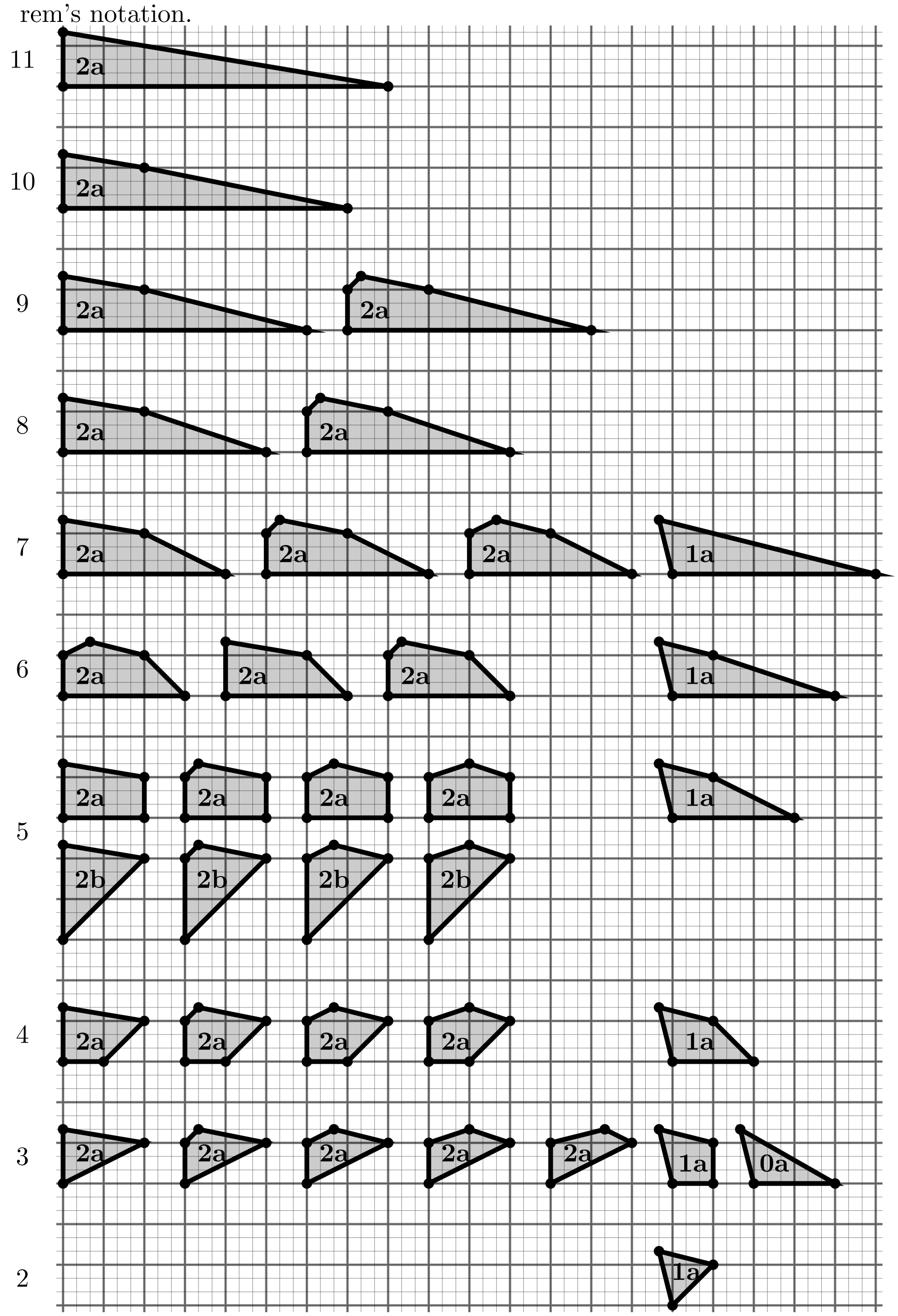


FIGURE 3. All area maximizers as described by Theorem 1.3 for $(k, i) = (3, 1)$. On the left is the number b of boundary lattice points. Each polygon is labeled according to the Theorem's notation.

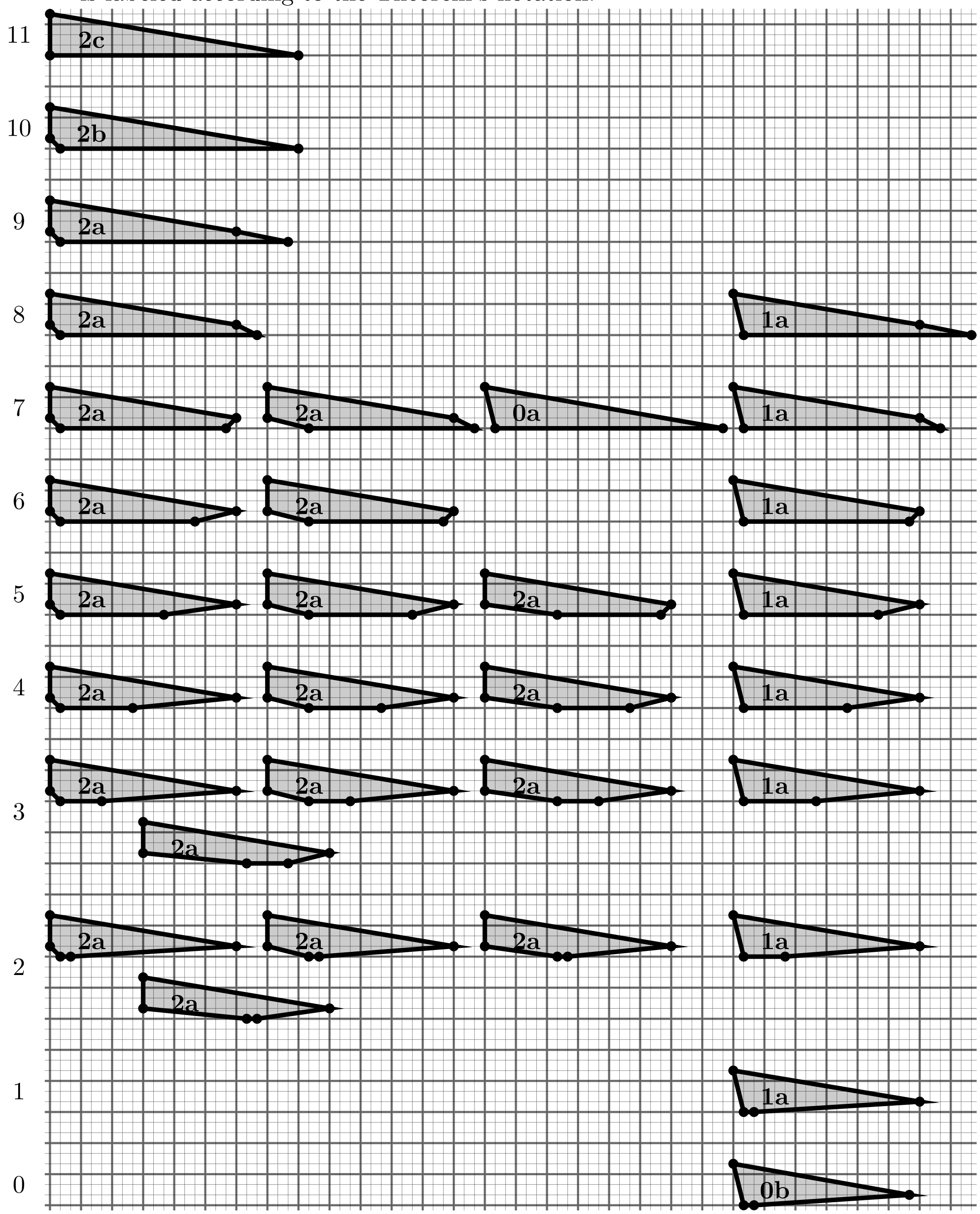


Figure 4. All half-integral area maximizers with two interior integral lines. On the left is the number b of boundary lattice points. The dashed line seperates polygons with $i = 1$ from polygons with $i = 2$.

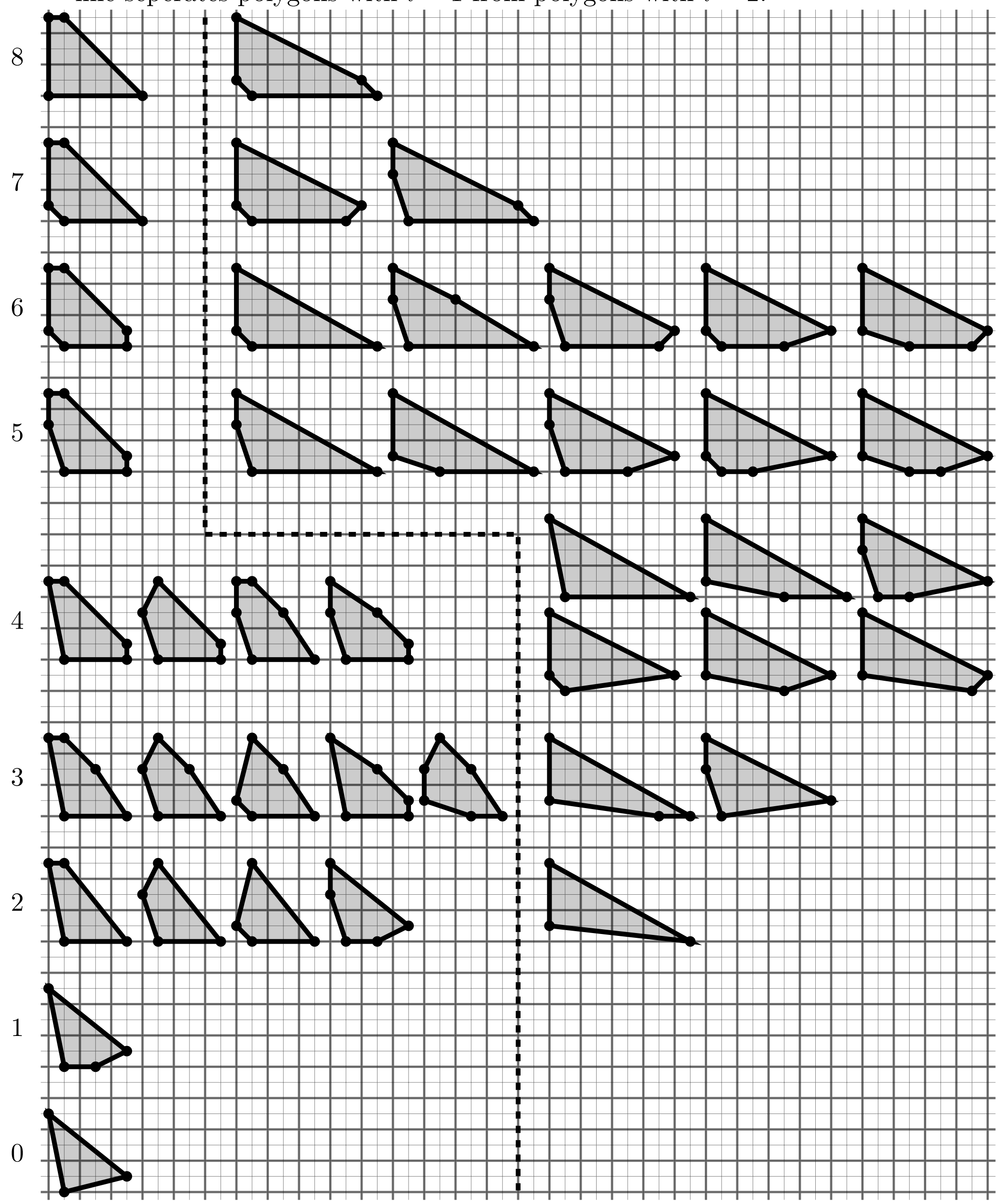


FIGURE 5. All half-integral area maximizers in $\mathbb{R} \times [-1, 1]$ with two interior lattice points and five boundary lattice points, which cannot be realized in $\mathbb{R} \times [-1, \frac{1}{2}]$, as described by Lemma 3.1. The upper two rows of polygons have $b_0 = 2$, the polygons in the lower two rows have $b_0 = 1$

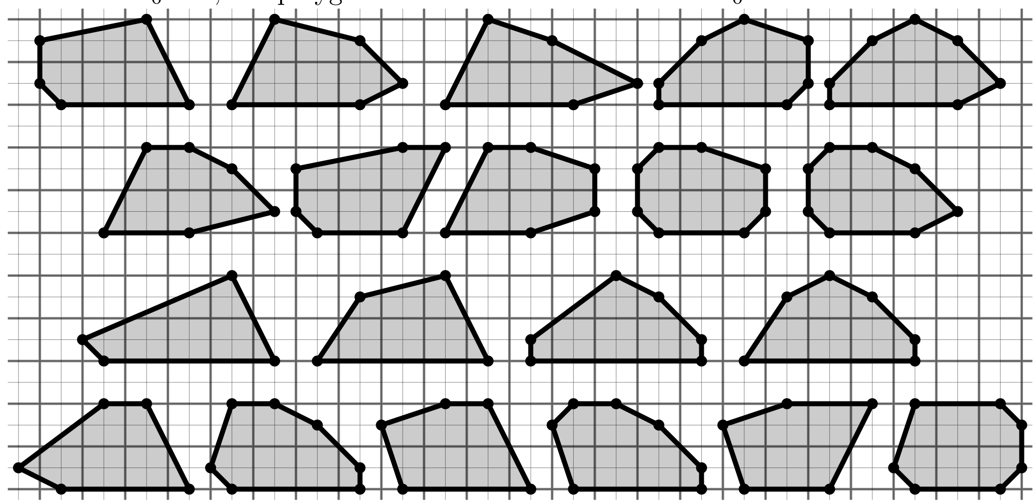
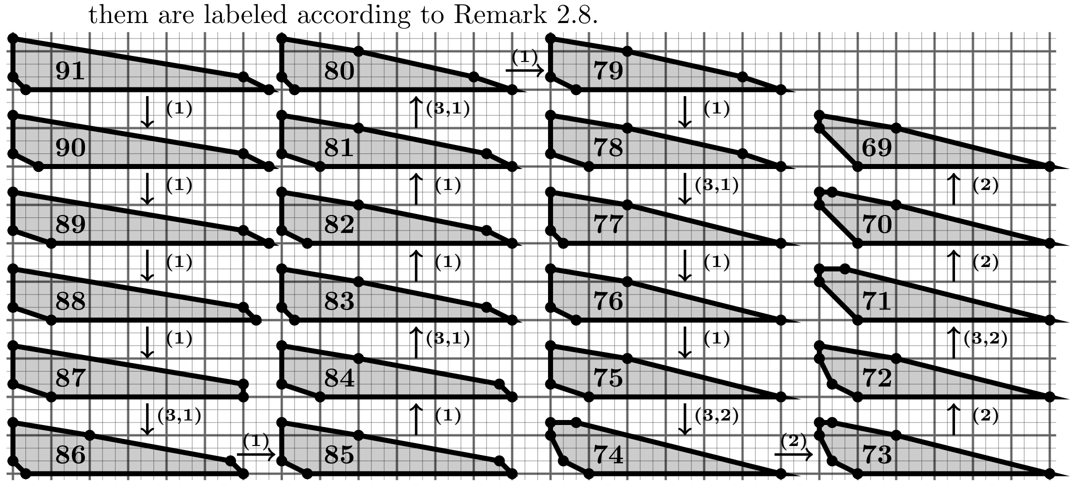


FIGURE 6. A polygon for each possible area between the lower and upper bounds of Theorems 1.2 and 1.3. Here, we have $(k, i, b) = (3, 1, 8)$. Each polygon is labeled according to its k-normalized area and the steps between them are labeled according to Remark 2.8.



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