On the Effective Dynamics of Impurity Particles in Interacting Fermionic Many-Body Systems

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To Oliver Schön.

Abstract

We investigate the dynamics of a small number of impurity particles immersed in a high-density Fermi gas confined to a d-dimensional box, with $d \in \{2, 3\}$, of side length L and periodic boundary conditions. The fermionic system is characterized by its Fermi momentum k_F. While previous works on this problem have focused on the case of a non-interacting Fermi gas, we allow for an additional two-body interaction $\kappa\, w(x)$ between the fermions themselves, in addition to the impurity–fermion coupling $\lambda\, v(x)$. Here, $\kappa, \lambda > 0$ are coupling constants scaling the strength of the interactions, and we require the Fourier transforms of the functions v and w to be compactly supported. This generalization requires a more delicate analysis of the thermodynamic limit ($L \to \infty$) at high density ($k_F \gg 1$): in contrast to the non-interacting case, the limit $L \to \infty$ at fixed density is no longer solely advantageous, since fermion–fermion correlations generate volume-enhanced terms that can dominate the dynamics. We show that, under the scaling $\lambda^2 = k_F^{2-d}$ and $\kappa^2 = L^{-2d}$, these interaction-induced terms become subleading, and the impurities effectively decouple from the fermions at leading order. Nevertheless, their motion is governed by an induced pairwise interaction between the impurities, generated by particle-hole excitations of the Fermi gas. Our results extend the derivation of effective impurity dynamics from the ideal Fermi gas to weakly interacting fermions, and clarify the role of the system size in controlling the asymptotic behaviour.

Zusammenfassung

Wir untersuchen die Dynamik einer kleinen Anzahl von Fremdteilchen (impurity particles), die an ein Fermi-Gas mit hoher Dichte gekoppelt sind, das in einer d-dimensionalen Box der Seitenlänge L mit periodischen Randbedingungen eingeschlossen ist, wobei $d \in \{2, 3\}$ gilt. Das fermionische System wird durch seinen Fermi-Impuls k_F charakterisiert. Während sich frühere Arbeiten zu diesem Problem auf den Fall eines nichtwechselwirkenden Fermi-Gases konzentriert haben, berücksichtigen wir, zusätzlich zu der Kopplung $\lambda\, v(x)$ zwischen Verunreinigungsteilchen und Fermionen, eine Zweiteilchen-Wechselwirkung $\kappa\, w(x)$ zwischen den Fermionen selbst. Dabei besitzen die Fouriertransformierten der Funktionen v und w kompakten Träger, und $\kappa, \lambda > 0$ sind die Kopplungsparameter, welche die Stärke der Wechselwirkungen festlegen. Diese Verallgemeinerung erfordert eine subtilere Analyse des thermodynamischen Limes ($L \to \infty$) für hohe Dichte ($k_F \gg 1$): Im Gegensatz zum nicht-wechselwirkenden Fall ist der Grenzwert $L \to \infty$ bei fester Dichte nicht mehr ausschließlich vorteilhaft, da Fermion-Fermion-Korrelationen volumenvergrößerte Terme erzeugen, die die Dynamik dominieren können. Wir zeigen, dass unter der Skalierung $\lambda^2 = k_F^{2-d}$ und $\kappa^2 = L^{-2d}$ diese wechselwirkungsinduzierten Terme subdominant werden und die Fremdteilchen in führender Ordnung effektiv von den Fermionen entkoppeln. Dennoch wird ihre Bewegung durch eine induzierte paarweise Wechselwirkung zwischen den Fremdteilchen be-

stimmt, die durch Teilchen-Loch-Anregungen des Fermi-Gases erzeugt wird.Unsere Ergebnisse erweitern die Herleitung effektiver Fremdteilchen-Dynamik vom idealen Fermi-Gas auf die von schwach wechselwirkenden Fermionen und verdeutlichen die Rolle der Systemgröße bei der Kontrolle des asymptotischen Verhaltens.

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Chapter 1

Introduction

1.1 The Need for Effective Many-Body Theories

Understanding the macroscopic behavior of physical systems through their microscopic constituents represents one of the most profound achievements of modern physics. This reductionist approach, pioneered in the 19th century, has become fundamental across diverse fields, from quantum gases and condensed matter to biological and astrophysical systems, and has motivated the development of rigorous mathematical methods for connecting microscopic and macroscopic descriptions. However, this perspective presents a fundamental challenge: the mathematical description of microscopic systems grows exponentially in complexity with the number of constituents. A complete analysis requires tracking an enormous number of degrees of freedom and their intricate interactions, making exact solutions generally intractable for realistic systems.

This complexity necessitates the development of *effective theories*, simplified descriptions that capture essential physics while eliminating irrelevant microscopic details. The central insight is that macroscopic phenomena often depend only on a subset of microscopic degrees of freedom, allowing systematic approximations that preserve the relevant physics while dramatically reducing computational complexity.

In quantum many-body systems, this framework acquires particular mathematical rigor. The quantum mechanical description involves wave functions $\Psi_N \in L^2(\mathbb{R}^{3N})$ whose Hilbert space dimension scales exponentially with particle number N. The time evolution, governed by the Schrödinger equation, becomes computationally prohibitive for large N. Effective theories address this challenge through systematic procedures that integrate out high-energy degrees of freedom, yielding reduced descriptions in terms of the remaining low-energy modes.

1.2 Fermionic Many-Body Systems

Among quantum many-body systems, fermionic gases provide a particularly compelling example of the challenges outlined above. The antisymmetric nature of fermionic wave functions creates a complex interplay between quantum statistics and inter-particle interactions that defies simple approximation schemes. Yet these same systems exhibit a natural hierarchy of energy scales that makes them ideal candidates for systematic effective theory approaches.

Consider an N-body system of interacting fermions described by the Hamiltonian

$$H_N = \sum_{i=1}^{N} \frac{1}{2m}(-\Delta_{x_i}) + \kappa \sum_{1\leq i<j\leq N} w(x_i - x_j)\,, \tag{1.1}$$

where the coupling parameter $\kappa \geq 0$ determines the relative importance of kinetic and interaction energies.

1.2.1 Structure of the Ideal Fermi Gas

To develop an intuitive understanding, we begin by considering the case of non-interacting fermions (i.e. $\kappa = 0$), often referred to as the *ideal Fermi gas.* Even without interactions, this system displays a rich quantum structure that arises solely from the Pauli exclusion principle and forms the basis for understanding more complex interacting systems.

The free Hamiltonian for N fermions takes the form

$$H_0 = \sum_{i=1}^{N} \frac{1}{2m}(-\Delta_{x_i})\,. \tag{1.2}$$

We restrict the spatial domain to a d-dimensional cube of side length $L > 0$, where $d \in \{2, 3\}$, and impose periodic boundary conditions. This allows us to expand the eigenfunctions in plane waves

$$\varphi_k(x) = e^{ikx}/L^{d/2} \tag{1.3}$$

corresponding to discrete momenta

$$k \in \left(\frac{2\pi}{L}\right)\mathbb{Z}^d\,. \tag{1.4}$$

Under this spatial constraint, the system approaches a dense Fermi gas in the limit $N \to \infty$. The antisymmetric nature of fermionic wave functions requires that each fermion occupies a distinct quantum state since otherwise

$$\Psi_N(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N) = -\Psi_N(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_N) = 0\,. \tag{1.5}$$

The ground state Ω_0 is therefore constructed by filling all single-particle momentum states up to the Fermi momentum k_F

$$\Omega_0 = \bigwedge_{k\in B_F} \varphi_k \,, \tag{1.6}$$

$$B_F = \{k \in (2\pi/L)\mathbb{Z}^d : |k| \leq k_F\} \,, \tag{1.7}$$

$$N = |B_F| \,, \tag{1.8}$$

which results in the so-called Fermi ball or Fermi sea. This creates a fundamental energy hierarchy in the system. The ground state energy

$$E_0 = \sum_{k\in B_F} |k|^2 \tag{1.9}$$

involves only occupied states, while all excitations require promoting fermions across the Fermi surface. The number of particles N is directly related to the volume of the Fermi ball, and one obtains the characteristic scaling relation

$$k_F \sim \left(\frac{N}{L^d}\right)^{1/d} , \tag{1.10}$$

where the number of particles N is directly linked to the Fermi momentum k_F. This establishes the crucial connection between particle density $\rho = N/L^d$ and the characteristic momentum scale k_F, showing that the density naturally sets the relevant microscopic momentum scale, and in the high-density regime k_F becomes large. The large value of k_F is essential: excitations typically occur near the Fermi surface, while fermions deep inside the Fermi sea are blocked from scattering due to the Pauli principle.

1.3 Impurity Problems in Many-Body Systems

Among the various problems in many-body physics, the dynamics of impurity particles immersed in a quantum medium has emerged as a particularly rich and experimentally relevant area of research. The impurity problem serves as a paradigmatic model for understanding how a small number of distinguishable particles interact with and are influenced by a much larger fermionic or bosonic environment.

The behavior of quantum particles can be significantly influenced by a surrounding medium, which can modify intrinsic properties such as mass and charge and modify the dispersion relation of the quantum particles [4]. It can also introduce new interactions mediated by the medium itself. A classic example of this phenomenon is the phonon-mediated attraction between two repulsive polarons. This attractive force can potentially outweigh the inherent repulsion between polarons, leading to the formation of a novel quasi-particle known as a bipolaron [3, 8, 11].

Similar effects are observed in the realm of ultracold atoms, where Casimir-like forces occur between heavy fermions immersed in a Fermi sea [25], and effective interactions arise among angulons [22] and Fermi polarons [10, 23] . Additionally, fermion-mediated interactions have been identified in dilute mixtures of Bose-Fermi gases [17, 21, 26], with experimental evidence documented in studies [26].

Consider a single impurity particle immersed in the fermionic background described in the previous section. The total Hamiltonian takes the form

$$H = \frac{1}{2m_y}(-\Delta_y) + H_0 + \lambda \sum_{i=1}^{N} v(y - x_i)\,, \tag{1.11}$$

where y denotes the impurity coordinate, H_0 is the free fermionic Hamiltonian (1.2), and the coupling strength λ controls the impurity-fermion interaction.
Assuming interactions are realized as momentum shifts, fermions deep inside of the Fermi sea are prevented from such shifts, as all nearby momentum states are already filled. Only fermions close to the Fermi surface have accessible unoccupied states into which they can scatter. This means interactions primarily influence particles on the Fermi surface, while the bulk remains largely uneffected. Given that only a small minority of fermions, specifically those at the surface, are able to engage in momentum shifts induced by interactions, the overall reaction of the Fermi sea is severely limited, suggesting that the Fermi ball remains rigid under weak impurity interactions, meaning that $\lambda \ll 1$.
When multiple impurities are present, the fermionic medium can mediate effective interactions between the impurities themselves as we can see .
The systematic derivation of such effective descriptions, particularly in the presence of fermion-fermion interactions (i.e. $\kappa > 0$) within the host medium, forms the central challenge addressed in this work.

We now review previous key results that form the foundation for the advances presented in this work.

1.3.1 Previous Results

Our approach and the work of this dissertation builds upon and extends previous findings in [18], [20], [19] and [24], which collectively demonstrate the emergence of effective dynamics for impurity particles in fermionic environments.

Single Impurity Dynamics in One Dimension

The simplest non-trivial setting for studying impurity dynamics in a Fermi gas is the one-dimensional system. Although lower-dimensional models may appear as idealized toy systems at first sight, the one-dimensional case exhibits remarkable structural and physical features that make it both mathematically tractable and conceptually illuminating.

From a mathematical perspective, the reduction to one spatial dimension simplifies the geometry of the Fermi surface dramatically: it reduces to just two points, corresponding to the Fermi edges at $\pm k_F$. This reduction enables explicit control of the many-body dynamics and allows rigorous estimates while retaining the essential features of fermionic statistics, such as antisymmetry, Pauli blocking, and the existence of a sharp Fermi edge.

The foundational work on this setting was carried out by Jeblick in his 2013 Master's thesis [18] and later discussed in [20]. In this model, a single tracer particle interacts with a one-dimensional ideal Fermi gas through an interaction potential, with coupling constant $\lambda = \mathcal{O}(1)$. The author rigorously proved that in the high-density limit $\rho \to \infty$, the tracer effectively decouples from the surrounding Fermi sea and evolves freely, up to a constant mean-field energy shift. Quantitatively, convergence to the free evolution was established in the L^2-norm with error bounds scaling as $\rho^{-1/2}$ for macroscopic times.

The physical mechanism underlying this result is deeply connected to the kinematic constraints imposed by energy and momentum conservation in one dimension. In contrast to higher dimensions, there exist no energetically allowed scattering processes that can transfer finite momentum between the impurity and the fermions while preserving total energy and momentum. Any potential scattering process that would transfer momentum q necessarily changes the kinetic energy of a Fermi particle by an amount of order $k_F q$, which becomes prohibitively large as $k_F \to \infty$. Consequently, impurity–fermion collisions that would lead to energy dissipation are kinematically forbidden.

In summary, the one-dimensional case provides an explicit realization of the counterintuitive phenomenon that dense fermionic environments can stabilize, rather than disturb, the motion of an impurity.

Single Impurity Dynamics in Two Dimensions

The first rigorous result was established in [20], where the authors consider a single impurity coupled to a dense ideal Fermi gas in two spatial dimensions. This foundational work demonstrates that for strong coupling (i.e. $\lambda = 1$), the impurity effectively decouples from the fermions while evolving according to a simple mean-field Hamiltonian, a simplified version of a many-body Hamiltonian where particle-particle interactions are replaced by each particle interacting with an average "mean field" created by all other particles. The analysis provides an L^2-norm approximation between the microscopic many-body evolution and an effective free dynamics in the so-called thermodynamic limit, where $N, L \to \infty$, with $\rho = N/L^d = \text{const.}$

More specifically, assuming an initial factorized state $\Psi_0 = \xi_0 \otimes \Omega_0$, where Ω_0 denotes the filled Fermi ball and ξ_0 is adequately regular, they established that the microscopic evolution remains close to the mean-field dynamics with error bounds of order $(1+t)^{3/2}\rho^{-1/8+\varepsilon}$ for arbitrarily small $\varepsilon > 0$. This demonstrates that the approximation quality improves polynomially with density and remains valid for macroscopic times.

The effective dynamics is governed by the decoupled Hamiltonian

$$H^{\text{eff}} = \frac{1}{2m_y}(-\Delta_y) - \sum_{i=1}^{N} \frac{1}{2m_{x_i}}(\Delta_{x_i}) + \rho F[v](0) - E_{re}(\rho) \tag{1.12}$$

where the mean-field energy $\rho F[v](0)$ captures the leading-order effect of the medium, while $E_{re}(\rho)$ represents a next-to-leading order correction arising from immediate recollision processes and $v \in C_0^\infty$.

The physical mechanism underlying this decoupling can be understood as an effective mean-field behavior arising from the fermionic structure. In a large Fermi ball, momentum transfers induced by interactions typically occur near the Fermi surface, where large momentum differences cause strong phase cancellations of excitations. Additionally, particles near the Fermi surface have large velocities ($\propto \sqrt{\rho}$), resulting in short interaction times and effectively small momentum transfer. This mechanism, together with the Pauli exclusion principle's suppression of major excitations, maintains the stability of the initial state under perturbation. This is a fundamental difference from bosonic and classical systems, where the absence of a Fermi ball structure allows the mean free path to shorten as density increases, resulting in frequent scattering.

This result stands in great contrast to the behavior observed in bosonic and classical systems, where increased density typically leads to enhanced scattering and reduced mean free paths [7]. The fundamental difference arises from the fermionic statistics and the resulting Fermi ball structure: while bosons can occupy the same quantum state without restriction, fermions are subject to the Pauli exclusion principle, which dramatically alters the scattering dynamics in dense systems.

In bosonic systems, the mean-field regime coincides with weak coupling limits, and increased density generally leads to more frequent collisions and shorter coherence times [2, 12, 13]. The absence of exclusion effects means that all particles in the medium can participate in scattering processes, leading to a mean free path that decreases as $\rho^{-\delta}$ for some $\delta > 0$. Consequently, an impurity particle in a dense Bose gas would experience significant disturbances on short time scales, preventing the emergence of effective decoupling [13].

For fermionic systems, however, the filled Fermi ball structure fundamentally restricts which scattering processes are kinematically allowed as explained above. This restriction, combined with the large energy gaps that suppress excitations, leads to the counterintuitive result that the mean-field approximation becomes more accurate at stronger coupling and higher densities, a regime where bosonic and classical descriptions would fail entirely [1, 9].

The mathematical manifestation of this difference appears in the scaling of fluctuations: while bosonic systems typically require weak coupling for mean-field validity, fermionic systems can maintain mean-field behavior even at strong coupling, provided the density is sufficiently large. This represents a fundamental advantage of fermionic statistics in enabling controllable many-body dynamics in dense quantum systems.

Related approaches to deriving effective dynamics in fermionic many-body systems have been pursued using different mathematical frameworks. In [15], the authors derived time-dependent Hartree equations for strongly interacting dense fermionic systems, while [16] studied effective polaron dynamics for impurity particles in Fermi gases. These results, together with a comprehensive treatment of both regimes, are presented in the dissertation [14], which establishes effective descriptions for interacting fermionic many-body systems including the derivation of

Fröhlich-type polaron Hamiltonians in the high-density limit and the computation of response functions. Although these works employ different techniques than the methods used here, they provide complementary perspectives on the emergence of effective descriptions in dense fermionic environments.

Two Impurity Case and Mediated Interactions

The framework was generalized in [19] to the case of two impurity particles, revealing a qualitatively new phenomenon: an additional effective interaction mediated by the Fermi gas. This interaction represents a nontrivial consequence of the system's fermionic nature, where the medium itself becomes a source of effective coupling between the impurities. The derivation demonstrates that even when impurities do not interact directly, the environment can induce fermion-mediated correlations.

Multiple Impurities and Mediated Interactions

Building on this foundation, [24] extended the framework to $n \geq 2$ impurities in two and three spatial dimensions with Coulomb-type interaction potentials. This work reveals a qualitatively new phenomenon: while individual impurities still decouple from the fermions, they develop attractive effective pair interactions mediated by the Fermi sea.

The key insight is identifying the appropriate coupling regime where non-trivial mediated interactions emerge while maintaining mathematical control over error terms. The analysis employs weak coupling scaling $\lambda^2 = k_F^{(2-d)}$, which leads to different regimes depending on dimension: In $d = 2$, this gives $|\lambda| = \mathcal{O}(1)$, corresponding to order-one coupling that remains consistent with the single impurity case. For $d = 3$, the scaling yields $|\lambda| = k_F^{-1/2}$, requiring genuinely weak coupling due to enhanced phase space effects in higher dimensions. Under this scaling, the impurities evolve according to an effective many-body Hamiltonian

$$h_n = h_n^0 - \sum_{i<j} \lambda^2 W_{k_F}(|y_i - y_j|) - n\lambda^2 W_{k_F}(0) \tag{1.13}$$

where $W_{k_F}(r)$ represents the fermion-mediated pair potential. This interaction arises from the same phase cancellation mechanism as in the single impurity case, but now intermediate fermionic excitations created by one impurity can be annihilated by another, leading to attractive correlations.

The approximation holds for macroscopic times $t \ll \sqrt{k_F}(\ln k_F)^{-3}$ with error bounds:

$$\|\Psi(t) - \Psi^{\text{eff}}(t)\| \leq C\,\frac{(1+|t|)(\ln k_F)^3}{\sqrt{k_F}} \tag{1.14}$$

The dimensional dependence reflects fundamental differences in the availability of phase space for virtual processes. The scaling $\lambda^2 = k_F^{(2-d)}$ ensures the effective interaction remains of order one as $k_F \to \infty$, but requires genuinely weak coupling in three dimensions where phase space effects are more pronounced.

This establishes that dense fermionic media can generate long-range attractive interactions between impurities, even when the underlying impurity-fermion interaction has short range.

The progression of these results, from single impurity decoupling to mediated multi-impurity interactions, and the emergence of dimensional constraints, establishes the theoretical foundation upon which the present work builds. The challenge that remains is to extend these findings to more complex scenarios involving interactions among the fermions themselves, which introduces qualitatively new correlation effects and mathematical complexities.

The Perturbative Resolvent Method

In the following we want to give an overview on the perturbative resolvent method, used in [24]. We build on this idea but modify it to accommodate our more general setting. The key physical insight is that when interactions excite fermions from occupied states $k \in B_F = \{k \in (2\pi/L)\mathbb{Z}^d : |k| \leq k_F\}$ to unoccupied states $l \notin B_F$, the resulting energy difference $l^2 - k^2 \sim k_F$ becomes arbitrarily large as the density increases. These large energy differences generate rapid phase oscillations $e^{i(l^2-k^2)t}$ in the time evolution, which lead to destructive interference when integrated over time, the same mechanism responsible for the phase cancellations discussed in both previous results.

Rather than rigorously detailing every calculation, we will provide a heuristic explanation for the results of the key estimates. We will provide detailed computational analysis when presenting our results.

One examines the Hamiltonian (1.11) in second quantization

$$\mathbb{H} = \frac{1}{2m_y}(-\Delta_y) + \underbrace{\sum_{k \in (2\pi/L)\mathbb{Z}^d} k^2 a_k^* a_k}_{=:\mathbb{T}} + \underbrace{\lambda L^{-d} \sum_{k,l \in (2\pi/L)\mathbb{Z}^d} \hat{v}(l-k) e^{-i(l-k)y_i} \; a_l^* a_k}_{=:\mathbb{V}} \,, \tag{1.15}$$

where $\hat{v} \geq 0$ is compactly supported. The initial state is a product of the Fermi sea Ω_0 and a general tracer wave function ξ_0. Mathematically, the method begins with Duhamel's formula applied to the difference between microscopic and effective dynamics

$$\left\| \left(e^{-i\mathbb{H}t} - e^{-iH^{\text{eff}}t} \right) \xi_0 \otimes \Omega_0 \right\| = \left\| \int_0^t \mathrm{d}s \, e^{i\mathbb{H}s} \mathbb{V} e^{-iH^{\text{eff}}s} (\xi_0 \otimes \Omega_0) \right\| , \tag{1.16}$$

where $\mathbb{V}$ represents the interaction terms that create excitations away from the Fermi sea ground state. The crucial step involves introducing a resolvent operator $R = (\mathbb{T} - E_0)^{-1}$ that inverts the kinetic energy operator on the subspace orthogonal to Ω_0 and E_0 is the ground state energy given by (1.9).

Through systematic integration by parts using the identity

$$e^{-iR^{-1}s} \left(\frac{\mathrm{d}}{\mathrm{d}s} e^{iR^{-1}s} \right) R = \mathrm{id} \, , \tag{1.17}$$

the method generates a perturbative expansion where each term involves increasingly higher powers of $R\mathbb{V}$. Since the resolvent scales as $R \sim k_F^{-1}$ (due to the large kinetic energy differences) while interaction terms scale as $\mathbb{V} \sim \lambda k_F^{(d-1)/2}$, the expansion parameter becomes $\lambda k_F^{(d-3)/2}$.

The method distinguishes between two types of processes: *bubble processes* that return the system to the original Fermi sea state (contributing to the effective Hamiltonian) and *excitation processes* that leave the system in excited states (contributing to error terms). The orthogonality of fermionic states with different particle numbers ensures these contributions can be separated cleanly: The bubble processes can be computed as expectation values with respect to Ω_0, while the excitation processes can be estimated by utilizing the resolvent.

The key observation is that destructive interference suppresses excitation processes, and only a limited number of iterations in the perturbative expansion contribute meaningfully. This leads to explicit error bounds for the effective evolution, which depend on powers of k_F and reveal the time scales on which the impurity can be treated as effectively free. For the appropriate coupling scaling $\lambda^2 = k_F^{(2-d)}$ we have the bubble processes yielding the effective interactions $W_{k_F}(r)$ at leading order and excitation processes producing error bounds that vanish as $k_F \to \infty$. This systematic exploitation of phase cancellations through kinetic energy dominance represents a powerful technique for analyzing dense fermionic systems, providing mathematical control over the complex many-body dynamics while revealing the emergent effective physics.

This approach has recently been applied to several areas, including deriving an effective interaction between multiple impurity particles mediated by a dense Fermi gas [24], analyzing radiative corrections to the dynamics of an impurity particle interacting with a bosonic scalar field [4], obtaining the two-body Coulomb potential from the renormalized Nelson model [5], and performing a stability analysis of Bose-Fermi mixtures [6].

1.4 From Non-Interacting to Interacting

Previous theoretical work on impurity dynamics has primarily focused on non-interacting Fermi gas environments, where the mathematical treatment is significantly simplified due to the absence of fermion-fermion correlations. However, real physical systems invariably involve interactions among the fermions themselves, introducing qualitatively new physics and mathematical challenges that can fundamentally alter the effective impurity dynamics, since the interacting fermion medium becomes a different environment.

The key difficulty arises from volume-enhanced contributions generated by fermion-

fermion correlations. While in non-interacting systems the thermodynamic limit $L \to \infty$ at fixed density is typically advantageous (sums over fermionic modes become integrals, and no dangerous volume factors remain), interactions can generate terms scaling with positive powers of the volume L^d. Without appropriate scaling of the coupling parameters, these contributions can dominate the dynamics and invalidate the effective theory approach, necessitating a coordinated treatment of both the high-density limit $k_F \to \infty$ and the thermodynamic limit $L \to \infty$.

1.4.1 Contribution of This Work

This work addresses the theoretical challenge of deriving effective impurity dynamics in the presence of weakly interacting fermions. We build upon and expand the work presented in [24]. Building directly on the perturbative resolvent method established for non-interacting systems, we extend the rigorous mathematical framework to include fermion-fermion interactions while carefully controlling volume-enhanced terms through appropriate parameter scaling.

We investigate the dynamics of multiple impurity particles immersed in a high-density, weakly interacting Fermi gas, meaning that we control the interaction strength by the coupling constant κ. The central question is whether the effective decoupling and mediated interactions observed in ideal Fermi gases survive the inclusion of fermion-fermion correlations, or if these correlations qualitatively alter the impurity dynamics.

To achieve this, we begin by constructing the Hamiltonian for the system, which includes both impurity-fermion and fermion-fermion interactions. Using a perturbative approach, we then derive the effective interaction potential between the impurity particles, accounting for the alterations brought about by the interacting

Fermi gas. Our main results demonstrate that under the scaling $\lambda^2 = k_F^{2-d}$ for impurity-fermion coupling and $\kappa^2 = L^{-2d}$ for fermion-fermion interactions, the correlation-induced terms become subleading. The impurities effectively decouple from the fermions at leading order, with their motion governed by the same type of induced pairwise attraction as in the non-interacting case, but now mediated by particle-hole excitations in the interacting Fermi sea, with $\kappa > 0$.

This work extends the mathematical rigor of the resolvent method to a previously inaccessible regime, providing the foundation for understanding impurity dynamics in realistic quantum many-body systems relevant to ultracold atomic gases and quantum materials. The results establish the robustness of fermion-mediated interactions against weak correlations in the medium, confirming that the fundamental physics identified in ideal systems persists in more realistic settings.

Chapter 2

Main Project

2.1 Setting, Effective Model and Main Result

2.1.1 The Model

In our model, we consider N fermions and $n \geq 2$ tracer particles in $\Lambda = [0, L]^d$, the d-dimensional cube with periodic boundary conditions. As our Hilbert space we allocate $\mathcal{H}_n \otimes \mathcal{H}_N^-$, where

$$\mathcal{H}_n = L^2(\Lambda)^{\otimes n} \quad \text{is the state space of the tracer particles,} \tag{2.1}$$

$$\mathcal{H}_N^- = \bigwedge^N L^2(\Lambda) \quad \text{is the state space for the fermions.} \tag{2.2}$$

With $\mathcal{H}_N^-$ we denote the subspace of all anti-symmetric wave functions in $\mathcal{H}_N$. The tracer particles have coordinates $y_1, y_2, \ldots, y_n$ and the fermions have coordinates

$x_1, x_2, \dots, x_N$. The associated Hamiltonian is given by

$$H = \underbrace{\sum_{i=1}^{n}(-\Delta_{y_i}) + \sum_{i<j}^{n} u(y_i - y_j)}_{=:h_n^0} + \underbrace{\sum_{i=1}^{N}(-\Delta_{x_i})}_{=:T_N} + \kappa \underbrace{\sum_{i<j}^{N} w(x_i - x_j)}_{:=W_N} + \underbrace{\lambda \sum_{i=1}^{n}\sum_{j=1}^{N} v\left(y_i - x_j\right)}_{:=V_{N+n}} \tag{2.3}$$

with $u(y_i - y_j)$ the interaction between the impurities, $v\left(y_i - x_j\right)$ the interaction between fermions and impurities, and $w(x_i - x_j)$ the interaction between the fermions.

We choose u to be a real-valued, even function on the d-dimensional cube that satisfies

$$u^2 \le c(1 - \Delta) \tag{2.4}$$

for some constant $c \in [0, 1)$ as an operator inequality on $L^2(\Lambda)$. Additionally, we require that $\hat{v}$ and $\hat{w}$ have compact support, i.e.

$$\hat{v}(k) = 0 = \hat{w}(k') \quad \text{for all } k, k' \notin [-D, D]^d\,, \tag{2.5}$$

and

$$|\hat{v}(k)| \le C\,, \quad \text{for } k \in [-D, D]^d\,, \quad |\hat{w}(k')| \le C'\,, \quad \text{for } k' \in [-D, D]^d\,, \tag{2.6}$$

for some $D > 0$ and constants $C, C' > 0$. These conditions imply that the interactions among fermions and between fermions and impurities are confined within a compact region. We also require $\hat{v}$ and $\hat{w}$ to be rotationally invariant functions. Concerning the different parts of the Hamiltonian the operator h_n^0 acts only on the tensor component $\mathcal{H}_n$, while T_N, W_N act on the tensor component $\mathcal{H}_N^-$.

In the following we are analyzing the solution of the time-dependent Schrödinger equation

$$i\frac{d}{dt}\Psi(t) = H\Psi(t)\,, \tag{2.7}$$

with $\Psi(0) = \Psi_0$ for initial states Ψ_0 of the form

$$\Psi_0(y_1,\ldots,y_n,x_1,\ldots,x_N) = \xi_0(y_1,\ldots,y_n)\otimes\Omega_0(x_1,\ldots,x_N)\in\mathcal{H}_n\otimes\mathcal{H}_N^-\,. \tag{2.8}$$

We assume that the fermions initially occupy the ground state Ω_0 of the Fermi gas, also known as the Fermi sea, which corresponds to the ground state of the kinetic energy operator T_N. The n-body wave function $\xi_0 \in \mathcal{H}_n$ can be selected more generally, with the only condition that its kinetic energy remains of order one with density $\rho \gg 1$. This ensures a separation of scales between the impurity particles and the fast-moving fermions. We choose a Fermi momentum $k_F > 0$ and fix N in terms of k_F and the length of our d-dimensional cube L by

$$N = |B_F| \quad \text{with} \quad B_F = \{k \in (2\pi/L)\mathbb{Z}^d \colon |k| \le k_F\}\,, \tag{2.9}$$

see (2.1). This implies the non-degeneracy of the free fermionic ground state, given by the anti-symmetric product of all plane waves with momenta inside the fermi ball B_F,

$$\Omega_0 = \bigwedge_{k\in B_F}\varphi_k \in \mathcal{H}^-_{N(k_F,L)}\,,\quad \varphi_k(x) = \frac{\exp(ikx)}{L^{d/2}} \in L^2(\Lambda)\,. \tag{2.10}$$

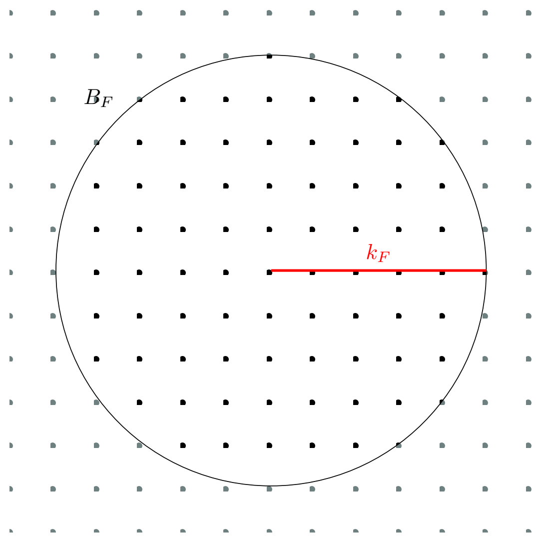

Figure 2.1: Fermi ball B_F with Fermi momentum k_F. The black dots represent occupied states in the Fermi ball, while the grey dots picture unoccupied ones.

Hence, we simplify the problem by initially ignoring the interactions between fermions. Our approach provides a clear, foundational state from which we can systematically introduce interactions and study their effects on the system's behavior. Clearly,

$$T_N \Omega_0 = E^0(k_F, L)\Omega_0 \,, \tag{2.11}$$

where $E^0(k_F, L)$ is an eigenvalue with

$$E^0(k_F, L) = \sum_{k \in B_F} k^2 \,. \tag{2.12}$$

A relation between the Fermi momentum and the average density in two and three dimensions can be found by replacing the sum by its Riemann integral

$$\frac{N(k_F, L)}{L^2} = V_d \, k_F^d + o(1) \,. \tag{2.13}$$

For $d \in \{2, 3\}$, the constants equal $V_2 = 1/(4\pi)$ and $V_3 = 1/(6\pi^2)$. As $L \to \infty$, $o(1)$ vanishes and we obtain from the relation above

$$\frac{N(k_F, L)}{L^d} \gg 1 \quad \Longleftrightarrow \quad k_F \gg 1\,. \tag{2.14}$$

In the perturbative expansion carried out later, both L and k_F are kept fixed. The asymptotic relations above are used only after the expansion, when we investigate the large-volume limit $L \to \infty$ and the high-density limit $k_F \gg 1$ under the scalings chosen in the main theorem.

2.1.2 Effective Model and Main Result

Effective n-body Model

In contrast to the initial state $\Psi_0 = \xi_0 \otimes \Omega_0$, the time-evolved wave-function $\Psi(t) = e^{-iHt}\Psi_0$ does not maintain its product form when there is a non-vanishing interaction v between the fermions and the tracer particles. However, assuming sufficiently small coupling and large k_F, the leading order of the interaction can be replaced by its mean-field interaction which only depends on the collective behavior of particles and does not generate correlations. We thus find that the product form is approximately preserved in that situation. In order to see this we compare $\Psi(t)$ with

$$\Psi^{\mathrm{eff}}(t) = e^{-ih_n^{\mathrm{eff}}t}\xi_0 \otimes e^{-iE(k_F,L)t}\Omega_0\ . \tag{2.15}$$

With

$$E(k_F, L) = E^0(k_F, L) + n\lambda\hat{v}(0)\frac{N(k_F, L)}{L^d} + \kappa\hat{w}(0)\frac{N(k_F, L)^2}{L^{2d}} \tag{2.16}$$

being the energy shift and the operator

$$h_n^{\text{eff}} = h_n^0 - \sum_{i<j}^{n} \lambda^2 W_{k_F} \left(|y_i - y_j| \right) - n\lambda^2 W_{k_F}(0) \tag{2.17}$$

defined on the Hilbert space $\mathcal{H}_n$, where

$$W_{k_F}(r) = V_d^2 \int\limits_{|m|\le k_F} \mathrm{d}^d m \int\limits_{|n|\ge k_F} \mathrm{d}^d n \; \frac{|\hat{v}(n-m)|^2}{n^2 - m^2 + (n-m)^2 + 1} \cos\left((n-m)\cdot r\hat{a}\right) \tag{2.18}$$

defines the effective interaction potential for $r \ge 0$ and $\hat{a} \in \mathbb{R}^d$ an arbitrary vector of length one. Here, V_d denotes a volume factor arising from the continuum limit of momentum sums with explicit values $V_2 = 1/(4\pi)$ and $V_3 = 1/(6\pi^2)$. Looking at the effective dynamics defined in (2.15) we see, that there is no interaction between the fermions and the tracer particles. While the tracer particles evolve with an extra pair interaction, the time evolution of the fermions is stationary. Heuristically one can think of the effective interaction as particle-hole excitations on the Fermi sea. One of the tracer particles causes a particle-hole excitation and then a different one annihilates it again. After the process (which is of second order) these two tracer particles are correlated to one another but not to the fermions. In the proof one can see, that similar processes with more than two tracer as well as processes caused by the interaction between the fermions and combinations of those two are possible and lead to more complicated interactions. However, in our setting, those are subleading and therefore do not need to be included in the effective dynamics.

We choose our coupling constants κ and λ such that

$$\lambda^2 = k_F^{(2-d)} \quad \text{and} \quad \kappa^2 = L^{-2d} \,. \tag{2.19}$$

With this choice the effective potential $\lambda^2 W_{k_F}(r)$ is of order one and the processes induced by the interaction of the fermions are subleading.

Main Result

We are now prepared to present our main result, which offers an estimate for the large-volume limit of the norm of $\Psi(t) - \Psi^{\text{eff}}(t)$. The norm and scalar product on the Hilbert space $\mathcal{H}_n \otimes \mathcal{H}_N^-$ are represented by $||\cdot||$ and $\langle \cdot, \cdot \rangle$, respectively. For the individual spaces $\mathcal{H}_n$ and $\mathcal{H}_N^-$, we use additional subscripts to specify them.

Theorem 1. *Let $d \in \{2, 3\}$, $n \geq 2$ and let u be a real-valued function on Λ that satisfies $u^2 \leq c(1 - \Delta)$ for some constant $c \in [0, 1)$. Additionally let*

$$|\hat{v}(k)| \leq C \,, \quad \textit{for } k \in [-D, D]^d \,, \quad |\hat{w}(k')| \leq C' \,, \quad \textit{for } k' \in [-D, D]^d \tag{2.20}$$

for some $D > 0$ and constants $C, C' > 0$. For $L, k_F > 0$ choose the number of fermions N by (2.9) and let the free fermionic ground state Ω_0 defined as in (2.10). Moreover, let $\xi_0 \in \mathcal{H}_n$ such that

$$\|\xi_0\|_{\mathcal{H}_n} = 1 \quad \textit{and} \quad \alpha_{\xi_0} := \sup_{k_F \geq 1} \sup_{L > 0} \sum_{i=1}^{n} \langle \xi_0, (-\Delta_{y_i}) \, \xi_0 \rangle_{\mathcal{H}_n} < \infty \,. \tag{2.21}$$

Given the energy shift operator $E(k_F, L)$ as defined by (2.16) and h_n^{eff} by (2.17), let $|\lambda| = k_F^{(2-d)/2}$ and $|\kappa| = L^{-d}$.

Then there exists a constant $C\left(R,n,\alpha_{\xi_0}\right)>0$ *independent of* L *and* k_F *such that*

$$\lim_{L\to\infty}\left\|e^{-iHt}\xi_0\otimes\Omega_0-e^{-ih_n^{\text{eff}}t}\xi_0\otimes e^{-iE(k_F,L)t}\Omega_0\right\|\leq C\left(R,n,\alpha_{\xi_0}\right)\frac{(1+|t|)\ln\left(k_F\right)^2}{k_F}\,, \tag{2.22}$$

for all $k_F\geq 2$ *and* $t\in\mathbb{R}$.

This bound is meaningful in the case of $k_F\gg 1$ as long as $|t|\ll k_F\ln\left(k_F\right)^{-2}$, since the wave functions on the left-hand side are normalized.

Remark 2. *Our proof and calculations yield a statement that is even more general than the one formulated in Theorem* (1). *A detailed summary of all term estimates can be found in* (3.141).
Setting $|\lambda|=k_F^{(2-d)/2}$ *and* $|\kappa|=L^{-d}$ *then leads to* (2.22).

Remark 3. *The theorem is formulated in terms of the large-volume limit* $L\to\infty$ *with fixed* k_F *and only later take the limit* $k_F\to\infty$. *However, the way our proof is structured, one can also consider a simultaneous limit* $L,k_F\to\infty$, *provided the growth rates of* L *and* k_F *are suitably balanced. In particular, as long as* L *increases sufficiently fast compared to* k_F, *the error estimate remains valid. A crucial point in our estimates is the interplay between the system size* L *and the Fermi momentum* k_F.
The correct balance is therefore achieved by choosing k_F *as a polynomial function of* L,

$$k_F=L^{\beta},\qquad 0<\beta<1/2\,, \tag{2.23}$$

which ensures that the terms decrease in a controlled way.
The explicit structure of our estimates, summarized in (3.141), *allows us to*

precisely fix how k_F must depend on L, then the same effective Hamiltonian emerges in the limit.

2.2 Proof

We begin by outlining the main idea of the proof, followed by the introduction of the second quantization formalism. Next, we estimate the norm difference within the second quantization framework and apply a form of perturbation theory expansion. Finally, we individually estimate all the resulting terms.

2.2.1 Heuristics and Idea of the Proof

Let us first start with the idea and sketch of the proof. We want to derive the bound stated in (1) by estimating the norm difference

$$\left\| e^{-i(H-E(k_F,L))t}\xi_0 \otimes \Omega_0 - e^{-ih_n^{\text{eff}}t}\xi_0 \otimes \Omega_0 \right\| . \tag{2.24}$$

The fundamental theorem of calculus gives us

$$\begin{aligned}
&\left(e^{i(H-E(k_F,L))\cdot 0}e^{-i(H-E(k_F,L))\cdot 0} - e^{i(H-E(k_F,L))t}e^{-ih_n^{\text{eff}}t}\right)\xi_0 \otimes \Omega_0 \qquad (2.25)\\
=&\left(1 - e^{i(H-E(k_F,L))t}e^{-ih_n^{\text{eff}}t}\right)\xi_0 \otimes \Omega_0\\
=&-\int_0^t \mathrm{d}s\ \frac{d}{ds}\left(e^{i(H-E(k_F,L))s}e^{-ih_n^{\text{eff}}s}\right)\xi_0 \otimes \Omega_0\\
=&-\int_0^t \mathrm{d}s\ \left(i(H-E(k_F,L))e^{i(H-E(k_F,L))s}e^{-ih_n^{\text{eff}}s} - e^{i(H-E(k_F,L))s}ih_n e^{-ih_n^{\text{eff}}s}\right)\xi_0 \otimes \Omega_0\\
=&-i\int_0^t \mathrm{d}s\ \left(e^{i(H-E(k_F,L))s}\left((H-E(k_F,L)) - h_n^{\text{eff}}\right)e^{-ih_n^{\text{eff}}s}\right)\xi_0 \otimes \Omega_0\\
=&-i\int_0^t \mathrm{d}s\ \Bigg(e^{i(H-E(k_F,L))s}\Bigg(\underbrace{T_N - E^0(k_F,L)}_{=:T_N^{\text{exc}}}
\end{aligned}$$

$$+\underbrace{V_{N+n}-n\lambda\hat{v}(0)\frac{N(k_F,L)}{L^d}}_{=:V^{\mathrm{exc}}_{N+n}}+\underbrace{W_N-\kappa\hat{w}(0)\frac{N(k_F,L)^2}{L^{2d}}}_{=:W^{\mathrm{exc}}_N}+h^0_n-h^{\mathrm{eff}}_n\Bigg)\Bigg)\,\xi(t)\otimes\Omega_0$$

$$= -i\int_0^t \mathrm{d}s\ \left(e^{i(H-E(k_F,L))s}\left(V_{N+n}-n\lambda\hat{v}(0)\frac{N(k_F,L)}{L^d}\right.\right.$$
$$\left.\left.+W_N-\kappa\hat{w}(0)\frac{N(k_F,L)^2}{L^{2d}}+h^0_n-h^{\mathrm{eff}}_n\right)\right)\xi(t)\otimes\Omega_0\,.$$

We recall that, in the present proof strategy, both L and k_F are kept fixed during the expansion and the scaling behaviour in L and k_F is analyzed only afterwards. We want to estimate the norm of the part with V^{exc}_{N+n} and W^{exc}_N by using the unitary $e^{i(H-E(k_F,L))s}$. We find

$$H-E(k_F,L)=h^0_n+T^{\mathrm{exc}}_N+V^{\mathrm{exc}}_{N+n}+W^{\mathrm{exc}}_N\,. \tag{2.26}$$

Heuristically the idea is, that, applying the operator T^{exc}_N to states orthogonal to Ω_0, produces a large energy shift. Those then lead to phase cancellations and suppress the value of the norm. This will later be done by introducing a resolvent type operator, which, after inserting it with an identity and after using again integration by parts, leads to a perturbation type expansion for V^{exc}_{N+n} and W^{exc}_N. We therefore sort the terms into two different categories: The ones parallel and the ones orthogonal to Ω_0. Since $T^{\mathrm{exc}}_N\Omega_0=0$, the are no more phase cancellations in the terms parallel to Ω_0. One of those terms is the one, which determines the choice of h^{eff}_n. For some of the terms orthogonal to Ω_0 one can prove their smallness rather directly, for some a second integration by parts will be used. All of the resulting terms are going to be estimated and computed separately.

Before doing these estimates we first introduce the formalism of second quantization. This formalism, while not strictly necessary for our proofs, simplifies many computations.

2.2.2 Second Quantization

In second quantization the Hilbert space $\mathcal{H}_n \otimes \mathcal{H}_N^-$ can be understood as the N-particle sector of $\mathcal{H}_n \otimes \mathcal{F}$, where $\mathcal{F}$ denotes the fermionic Fock space

$$\mathcal{F} = \bigoplus_{m=0}^{\infty} \mathcal{H}_m^- \,, \quad \mathcal{H}_m^- = \bigwedge^m L^2(\Lambda)\,. \tag{2.27}$$

For plane waves $\varphi_k(x)$, with $k \in (2\pi/L)\mathbb{Z}^d$, we define the annihilation and creation operators

$$a_k \colon \mathcal{H}_y \otimes \mathcal{F} \to \mathcal{H}_y \otimes \mathcal{F}\,, \tag{2.28}$$

$$a_k^* \colon \mathcal{H}_y \otimes \mathcal{F} \to \mathcal{H}_y \otimes \mathcal{F} \tag{2.29}$$

through

$$\begin{aligned} &(a_k\Psi)^{(m)}(y_1,\dots,y_n,x_1,\dots,x_m) \\ &= \sqrt{m+1}\int_\Lambda \mathrm{d}x\, \overline{\varphi_k(x)}\Psi^{(m+1)}(y_1,\dots,y_n,x_1,\dots,x_m,x)\,, \end{aligned} \tag{2.30}$$

$$\begin{aligned} &(a_k^*\Psi)^{(m)}(y_1,\dots,y_n,x_1,\dots,x_m) \\ &= \sum_{j=1}^m \frac{(-1)^j}{\sqrt{m}}\varphi_k(x_j)\Psi^{(n-1)}(y_1,\dots,y_n,x_1,\dots,x_{j-1},x_{j+1},\dots,x_m)\,, \end{aligned} \tag{2.31}$$

where $\Psi = \left(\Psi^{(m)}\right)_{m\geq 0}$ with $\Psi^{(m)} \in \mathcal{H}_n \otimes \mathcal{H}_m^-$. The creation and annihilation operator satisfy the anticommutation relations for all $k, l \in (2\pi/L)\mathbb{Z}^d$

$$a_k^* a_l + a_l a_k^* = \delta_{kl}\,, \quad a_k a_l + a_l a_k = 0\,. \tag{2.32}$$

Now we can write our Hamiltonian in terms of second quantization on the Hilbert space $\mathcal{H} \otimes \mathcal{F}$. We define the Hamiltonian in second quantization

$$\mathbb{H} = h_n^0 + \mathbb{T} + \mathbb{V} + \mathbb{W} - E^0(k_F, L)\,, \tag{2.33}$$

where the operators T_N, V_N and W_N are written in terms of creation and annihilation operators with

$$\mathbb{T} = \sum_k k^2 a_k^* a_k\,, \tag{2.34}$$

$$\mathbb{V} = \sum_{i=1}^n \mathbb{V}^{(i)}\,, \quad \text{where} \quad \mathbb{V}^{(i)} = \lambda L^{-d} \sum_{k \neq l} \hat{v}(l-k) e^{-i(l-k)y_i}\ a_l^* a_k\,, \tag{2.35}$$

$$\mathbb{W} = \kappa L^{-d} \sum_{p,q} \sum_k \hat{w}(k)\ a_p^* a_q^* a_{p-k} a_{q+k}\,. \tag{2.36}$$

Note, that the operator $\mathbb{W}$ ensures conservation of momentum.

Restricting the Hamiltonian $\mathbb{H}$ to the $N(k_F, L)$-particle sector we find

$$\mathbb{H}_{\ |\mathcal{H}_n \otimes \mathcal{H}^-_{N(k_F,L)}} = H - E(k_F, L)\,. \tag{2.37}$$

We subtracted the energy

$$E^0(k_F, L) = \sum_{k \in B_F} k^2 \tag{2.38}$$

and the momentum-conserving part of the interaction between fermions and tracer

$$\sum_{i=1}^n \lambda L^{-d} \sum_{k,l} \hat{v}(l-k) e^{-i(l-k)y_i}\ \delta_{lk} a_l^* a_k = \frac{n\lambda \hat{v}(0)}{L^d} \sum_k a_k^* a_k\,. \tag{2.39}$$

and the momentum-conserving part of the interaction between fermions

$$\kappa L^{-2d} \sum_{p,q} \sum_{k} \hat{w}(k)\ a_p^* a_q^* a_{p-k} a_{q+k} = \frac{\kappa \hat{w}(0)}{L^{2d}} \sum_{p,q} a_p^* a_q^* a_p a_q \,. \tag{2.40}$$

The Fermi sea Ω_0 in terms of the Fock space vacuum $|0\rangle = (1, 0, 0, \dots)$ is given by

$$\Omega_0 = \left(\prod_{k \in B_F} a_k^* \right) |0\rangle \,. \tag{2.41}$$

It satisfies

$$a_k \Omega_0 = 0\,, \qquad \text{for all } k \in B_F^C\,, \tag{2.42}$$

$$a_k^* \Omega_0 = 0\,, \qquad \text{for all } k \in B_F\,. \tag{2.43}$$

2.2.3 Estimation of the norm difference

All computations and rigorous steps are now done in terms of second quantization. Using (2.37) and again the fundamental theorem of calculus, we receive

$$\begin{aligned}
\left(e^{i\mathbb{H}\cdot 0} e^{-ih_n^{\text{eff}}\cdot 0} - e^{i\mathbb{H}t} e^{-ih_n^{\text{eff}}t} \right) \xi_0 \otimes \Omega_0 =&\ \left(1 - e^{i\mathbb{H}t} e^{-ih_n^{\text{eff}}t} \right) \xi_0 \otimes \Omega_0 \\
=&\ - \int_0^t \mathrm{d}s\ \frac{d}{ds} \left(e^{i\mathbb{H}s} e^{-ih_n^{\text{eff}}s} \right) \xi_0 \otimes \Omega_0 \\
=&\ - \int_0^t \mathrm{d}s\ \left(i\mathbb{H} e^{i\mathbb{H}s} e^{-ih_n^{\text{eff}}s} - e^{i\mathbb{H}s} i h_n^{\text{eff}} e^{-ih_n^{\text{eff}}s} \right) \xi_0 \otimes \Omega_0 \\
=&\ - i \int_0^t \mathrm{d}s\ \left(e^{i\mathbb{H}s} \left(\mathbb{H} - h_n^{\text{eff}} \right) e^{-ih_n^{\text{eff}}s} \right) \xi_0 \otimes \Omega_0 \\
=:&\ \widetilde{\Phi}(t) + \Phi(t) + \phi(t)
\end{aligned}$$

with

$$\widetilde{\Phi}(t) = i \int_0^t \mathrm{d}s \; e^{i\mathbb{H}s} \mathbb{W} \xi(s) \otimes \Omega_0 \,, \tag{2.44}$$

$$\Phi(t) = -i \int_0^t \mathrm{d}s \; e^{i\mathbb{H}s} \mathbb{V} \xi(s) \otimes \Omega_0 \,, \tag{2.45}$$

$$\phi(t) = i \int_0^t \mathrm{d}s \; e^{i\mathbb{H}s} \left(h_n^{\mathrm{eff}} - h_n^0 \right) \xi(s) \otimes \Omega_0 \,, \tag{2.46}$$

where $e^{-ih_n^{\mathrm{eff}} t} \xi_0 = \xi(t)$.

In the following, we use integration by parts in order to expand the states $\Phi(t)$ and $\widetilde{\Phi}(t)$ into several contributions.

Decomposition of Φ

We define a resolvent type operator

$$R = (\mathbb{T} - E^0 + P_f^2 + 1)^{-1} \mid \mathcal{H} \otimes \mathcal{H}_N^- \,, \tag{2.47}$$

with

$$P_f = \sum_k k a_k^* a_k \tag{2.48}$$

the momentum operator. As explained above, the operator R is defined such that it measures the energy-difference. P_f is included for important cancellations in the decomposition of Φ, while it does not make any difference in the decomposition of $\widetilde{\Phi}$. The plus one is needed to avoid singularities. Now we can rewrite Φ by

$$\begin{aligned} \Phi(t) &= -i \int_0^t \mathrm{d}s \; e^{i\mathbb{H}s} e^{-iR^{-1}s} R^{-1} e^{iR^{-1}s} R \mathbb{V} \xi(s) \otimes \Omega_0 \\ &= -i \int_0^t \mathrm{d}s \; e^{i\mathbb{H}s} e^{-iR^{-1}s} \left(\frac{d}{ds} e^{iR^{-1}s} \right) R \mathbb{V} \xi(s) \otimes \Omega_0 \,. \end{aligned} \tag{2.49}$$

In the following we use

$$\frac{d}{ds}\xi(s) = \frac{d}{ds}e^{-ih_n^{\mathrm{eff}}s}\xi_0 = -ih_n^{\mathrm{eff}}e^{-ih_n^{\mathrm{eff}}s}\xi_0 = -ih_n^{\mathrm{eff}}\xi(s) \tag{2.50}$$

as well as

$$\begin{aligned}
&\frac{d}{ds}e^{i\mathbb{H}s}e^{-iR^{-1}s} \qquad (2.51)\\
&= i\mathbb{H}e^{i\mathbb{H}s}e^{-iR^{-1}s} - e^{i\mathbb{H}s}iR^{-1}e^{-iR^{-1}s}\\
&= i\left(h_n^0 + \mathbb{T} + \mathbb{V} + \mathbb{W} + E^0\right)e^{i\mathbb{H}s}e^{-iR^{-1}s} - e^{i\mathbb{H}s}i\left(\mathbb{T} - E^0 + P_f^2 + 1\right)e^{-iR^{-1}s}\\
&= ie^{i\mathbb{H}s}\left[\left(h_n^0 + \mathbb{T} + \mathbb{V} + \mathbb{W} + E^0\right) - \left(\mathbb{T} - E^0 + P_f^2 + 1\right)\right]e^{-iR^{-1}s}\\
&= ie^{i\mathbb{H}s}\left(h_n^0 + \mathbb{V} + \mathbb{W} - P_f^2 - 1\right)e^{-iR^{-1}s}.
\end{aligned}$$

By integration by parts we find $\Phi(t) = \sum_{i=0}^4 \Phi_i(t)$ with

$$\Phi_0(t) = -\left[e^{i\mathbb{H}s}R\mathbb{V}\xi(s)\otimes\Omega_0\right]_0^t, \tag{2.52a}$$

$$\Phi_1(t) = i\int_0^t \mathrm{d}s\; e^{i\mathbb{H}s}R\mathbb{V}\left(-1 + h_n^0 - h_n^{\mathrm{eff}}\right)\xi(s)\otimes\Omega_0\,, \tag{2.52b}$$

$$\Phi_2(t) = i\int_0^t \mathrm{d}s\; e^{i\mathbb{H}s}R\{[h_n^0,\mathbb{V}] - P_f^2\mathbb{V}\}\xi(s)\otimes\Omega_0\,, \tag{2.52c}$$

$$\Phi_3(t) = i\int_0^t \mathrm{d}s\; e^{i\mathbb{H}s}\mathbb{V}R\mathbb{V}\xi(s)\otimes\Omega_0\,, \tag{2.52d}$$

$$\Phi_4(t) = i\int_0^t \mathrm{d}s\; e^{i\mathbb{H}s}\mathbb{W}R\mathbb{V}\xi(s)\otimes\Omega_0\,. \tag{2.52e}$$

Examining $\Phi_3(t)$, we observe that the state $\mathbb{V}R\mathbb{V}\xi(s)\otimes\Omega_0$ generates zero, one, or two holes in the Fermi sea, while in $\Phi_4(t)$, the state $\mathbb{W}R\mathbb{V}\xi(s)\otimes\Omega_0$ creates one, two, or three holes, as shown in (2.2) and (2.3).

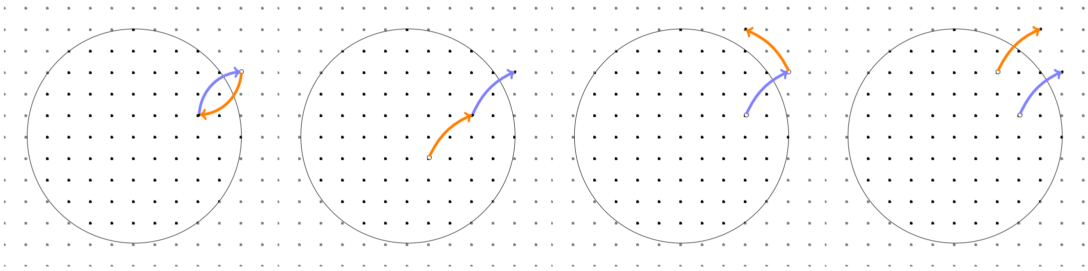

Figure 2.2: Pictures of zero, one and two holes in the Fermi sea, produced by term $\mathbb{V}R\mathbb{V}\xi(s)\otimes\Omega_0$. Since the operator $\mathbb{V}$ is applied first, it always produces one hole. The operator $\mathbb{V}$, which is applied second, then determines the number of holes.

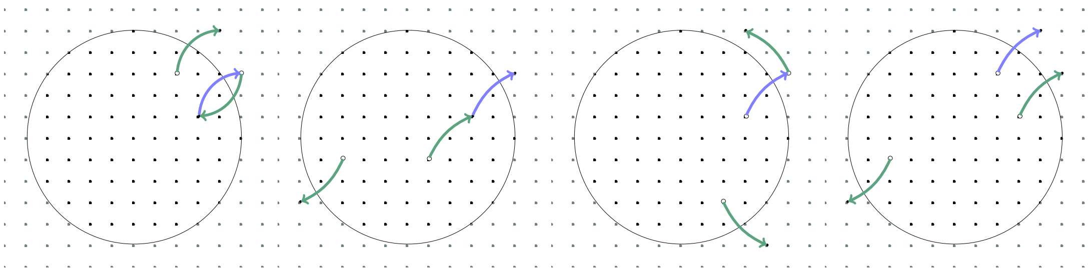

Figure 2.3: Pictures of one, two and three holes in the Fermi sea, produced by term $\mathbb{W}R\mathbb{V}\xi(s)\otimes\Omega_0$. The operator $\mathbb{V}$ first creates one hole in the Fermi sea, while the operator $\mathbb{W}$ then determines the number of holes. Notice, that since we have conservation of momentum in the operator $\mathbb{W}$ the arrows drawn always need to be of same length, but pointing in the exact opposite direction.

In order to decompose $\Phi_3(t)$ and $\Phi_4(t)$ into the number of hole we define the orthogonal projector $P^{(m)}$ in $\mathcal{H}_n\otimes\mathcal{H}_N^-$ acting in $\mathcal{H}_n$, that projects on the subspace of exactly m holes

$$\operatorname{ran}P^{(m)}=\{\Psi\in\mathcal{H}_n\otimes\mathcal{H}_N^-\colon\sum_{k\in B_F}\|a_k^*\Psi\|^2=m\,\|\Psi\|^2\}\,. \tag{2.53}$$

This helps us to sort the terms along Ω_0 and orthogonal to Ω_0. We obtain

$$\Phi_3(t) = \sum_{m=0}^{2} i \int_0^t e^{i\mathbb{H}s} P^{(m)} \mathbb{V} R \mathbb{V} \xi(s) \otimes \Omega_0 \tag{2.54}$$

$$:= \sum_{m=0}^{2} \Phi_{3m}(t) \,, \tag{2.55}$$

as well as

$$\Phi_4(t) = \sum_{n=1}^{3} i \int_0^t e^{i\mathbb{H}s} P^{(n)} \mathbb{W} R \mathbb{V} \xi(s) \otimes \Omega_0 \tag{2.56}$$

$$:= \sum_{n=1}^{3} \Phi_{4n}(t) \,. \tag{2.57}$$

We expand the term $\Phi_{32}(t)$ with two holes a second time in the same way as before, using integration by parts. Similarly as before one finds

$$\Phi_{32}(t) = \int_0^t e^{i\mathbb{H}s} e^{-iR^{-1}s} \left(\frac{d}{ds} e^{iR^{-1}s} \right) R P^{(2)} \mathbb{V} R \mathbb{V} \xi(s) \otimes \Omega_0 \tag{2.58}$$

$$:= \sum_{i=0}^{4} \Phi_{32;i}(t) \,, \tag{2.59}$$

with

$$\Phi_{32;0}(t) = \left[e^{i\mathbb{H}s} R P^{(2)} \mathbb{V} R \mathbb{V} \xi(s) \otimes \Omega_0 \right]_0^t \,, \tag{2.60a}$$

$$\Phi_{32;1}(t) = -i \int_0^t \mathrm{d}s \; e^{i\mathbb{H}s} R P^{(2)} \mathbb{V} R \mathbb{V} \left(-1 + h_n^0 - h_n^{\mathrm{eff}} \right) \xi(s) \otimes \Omega_0 \,, \tag{2.60b}$$

$$\Phi_{32;2}(t) = -i \int_0^t \mathrm{d}s \; e^{i\mathbb{H}s} R P^{(2)} \{ [h_n^0, \mathbb{V} R \mathbb{V}] - P_f^2 \mathbb{V} R \mathbb{V} \} \xi(s) \otimes \Omega_0 \,, \tag{2.60c}$$

$$\Phi_{32;3}(t) = -i \int_0^t \mathrm{d}s \; e^{i\mathbb{H}s} \mathbb{V} R P^{(2)} \mathbb{V} R \mathbb{V} \xi(s) \otimes \Omega_0 \,, \tag{2.60d}$$

$$\Phi_{32;4}(t) = -i \int_0^t \mathrm{d}s \; e^{i\mathbb{H}s} \mathbb{W} R P^{(2)} \mathbb{V} R \mathbb{V} \xi(s) \otimes \Omega_0 \,. \tag{2.60e}$$

We again decompose the last two lines in terms of the number of holes:

$$\Phi_{32;3m}(t) = -i\int_0^t \mathrm{d}s\; e^{i\mathbb{H}s} P^{(m)} \mathbb{V} R P^{(2)} \mathbb{V} R \mathbb{V} \xi(s) \otimes \Omega_0 \tag{2.61}$$

$$= \sum_{m=1}^{3} \Phi_{32;3m}(t)\,, \tag{2.62}$$

and

$$\Phi_{32;4n}(t) = -i\int_0^t \mathrm{d}s\; e^{i\mathbb{H}s} P^{(n)} \mathbb{W} R P^{(2)} \mathbb{V} R \mathbb{V} \xi(s) \otimes \Omega_0 \tag{2.63}$$

$$= \sum_{n=0}^{4} \Phi_{32;4n}(t)\,. \tag{2.64}$$

In summary we obtain the decomposition

$$\begin{aligned}\Phi(t) =& \sum_{i=0}^{2} \left(\Phi_i(t) + \Phi_{32;i}(t)\right) + \sum_{j=1}^{3} \Phi_{4j}(t) + \Phi_{30}(t)\\ &+ \; \Phi_{31}(t) + \sum_{m=1}^{4} \Phi_{32;3m}(t) + \sum_{n=0}^{4} \Phi_{32;4n}(t)\,,\end{aligned} \tag{2.65}$$

which is illustrated in a mindmap in (2.4).

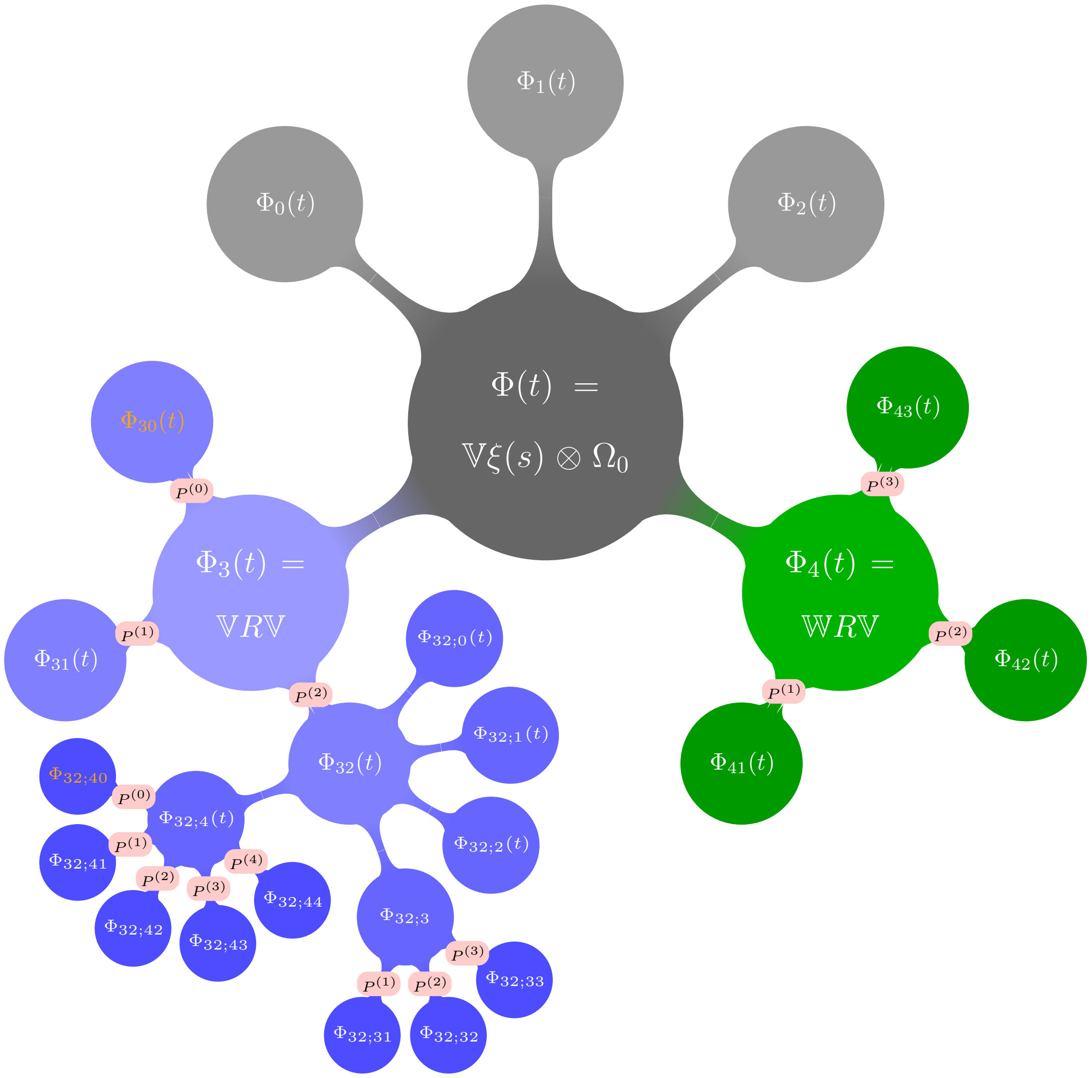

Figure 2.4: This illustration represents the decomposition of $\Phi(t)$ structured as a finite series of alternating steps. The first step involves an expansion of $\Phi(t)$ by inserting the resolvent operator R and then performing integration by parts. This structured expansion and simplification of terms is followed by a second step, where the decomposed terms are categorized according to the number of holes via the orthogonal projector $P^{(m)}$, that projects onto the subspace with exactly m holes.

Decomposition of $\widetilde{\Phi}$

Similarly to the decomposition of $\Phi(t)$, we aim to apply the same process to $\widetilde{\Phi}$, utilizing again the resolvent-type operator R. To do so, we first insert an identity and then proceed with integration by parts. We have

$$\begin{aligned}\widetilde{\Phi}(t) &= -i\int_0^t \mathrm{d}s\ e^{i\mathbb{H}s}e^{-iR^{-1}s}R^{-1}e^{iR^{-1}s}R\mathbb{W}\xi(t)\otimes\Omega_0 \\ &= -i\int_0^t \mathrm{d}s\ e^{i\mathbb{H}s}e^{-iR^{-1}s}\left(\frac{d}{ds}e^{iR^{-1}s}\right)R\mathbb{W}\xi(t)\otimes\Omega_0\,.\end{aligned} \tag{2.66}$$

This allows us to systematically break down the expression via integration by parts, leading to $\widetilde{\Phi}(t) = \sum_{i=0}^4 \widetilde{\Phi}_i(t)$ with

$$\widetilde{\Phi}_0(t) = -\left[e^{i\mathbb{H}s}R\mathbb{W}\xi(s)\otimes\Omega_0\right]_0^t \tag{2.67a}$$

$$\widetilde{\Phi}_1(t) = i\int_0^t \mathrm{d}s\ e^{i\mathbb{H}s}R\mathbb{W}\left(-1+h_n^0-h_n^{\mathrm{eff}}\right)\xi(s)\otimes\Omega_0 \tag{2.67b}$$

$$\widetilde{\Phi}_2(t) = i\int_0^t \mathrm{d}s\ e^{i\mathbb{H}s}R\{[h_n^0,\mathbb{W}]-P_f\mathbb{W}\}\xi(s)\otimes\Omega_0 \tag{2.67c}$$

$$\widetilde{\Phi}_3(t) = i\int_0^t \mathrm{d}s\ e^{i\mathbb{H}s}\mathbb{W}R\mathbb{W}\xi(s)\otimes\Omega_0 \tag{2.67d}$$

$$\widetilde{\Phi}_4(t) = i\int_0^t \mathrm{d}s\ e^{i\mathbb{H}s}\mathbb{V}R\mathbb{W}\xi(s)\otimes\Omega_0\,. \tag{2.67e}$$

To decompose $\widetilde{\Phi}_3(t)$ and $\widetilde{\Phi}_4(t)$ based on the number of holes in the Fermi sea, we once again make use of the orthogonal projector $P^{(m)}$, which projects onto the subspace corresponding to exactly m holes. By applying this projection operator, we can isolate the contributions of the states with different numbers of holes, allowing us to express $\widetilde{\Phi}_3(t)$ and $\widetilde{\Phi}_4(t)$ as sums over these distinct subspaces. We obtain

$$\widetilde{\Phi}_3(t) = \sum_{m=0}^4 i\int_0^t e^{i\mathbb{H}s}P^{(m)}\mathbb{W}R\mathbb{W}\xi(s)\otimes\Omega_0 \tag{2.68}$$

$$:= \sum_{m=0}^{2} \widetilde{\Phi}_{3m}(t)\,, \tag{2.69}$$

as well as

$$\widetilde{\Phi}_4(t) = \sum_{n=1}^{3} i \int_0^t e^{i\mathbb{H}s} P^{(n)} \mathbb{V} R \mathbb{W} \xi(s) \otimes \Omega_0 \tag{2.70}$$

$$:= \sum_{n=1}^{3} \widetilde{\Phi}_{4n}(t)\,. \tag{2.71}$$

In summary this gets us the decomposition

$$\widetilde{\Phi}(t) = \sum_{i=0}^{2} \widetilde{\Phi}_i(t) + \sum_{j=1}^{4} \widetilde{\Phi}_{3j}(t) + \widetilde{\Phi}_{30}(t) + \sum_{l=1}^{3} \widetilde{\Phi}_{4l}(t)\,, \tag{2.72}$$

which is illustrated in a mindmap in (2.5).

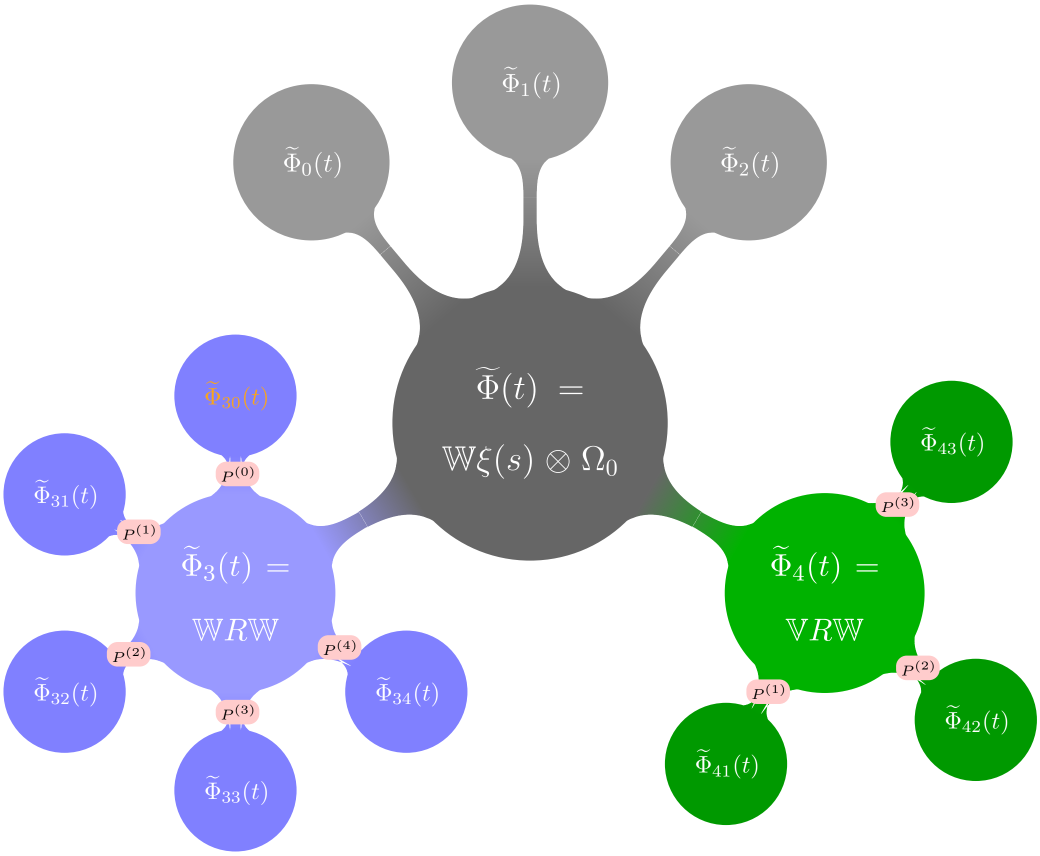

Figure 2.5: This illustration represents the decomposition of $\widetilde{\Phi}(t)$, structured follows: The first step involves an expansion of $\widetilde{\Phi}(t)$ by inserting the resolvent operator R and then performing integration by parts. The second step focuses on decomposing the resulting terms according to the number of holes. $P^{(m)}$ is an orthogonal projector, that projects onto the subspace with exactly m holes.

Let us clarify the motivation behind this expansion. When evaluating individual contributions, the idea is that each R introduces a factor of k_F^{-1}, while each $\mathbb{V}$ contributes a factor of $k_F^{(d-1)/2}$ and a $|\lambda|$. Each $\mathbb{W}$ gives a factor k_F^{d-1} and a $|\kappa|$. Very roughly and generally speaking, for any number $n \in \mathbb{N}$ of $\mathbb{V}$ in the term, as well as any number $m \in \mathbb{N}$ of $\mathbb{W}$, we find

$$\left(\left\|\underbrace{\mathbb{V}R\ldots\mathbb{V}}_{n-\text{times}}R\underbrace{\mathbb{W}R\ldots\mathbb{W}}_{m-\text{times}}\right\|^2\right)^{1/2} \lesssim \frac{|\kappa|^m|\lambda|^n}{k_F^{(n+m)-1}}\left(k_F^{(d-1)}\right)^{n/2}\left(k_F^{(2d-2)}\right)^{m/2} = |\kappa|^m|\lambda|^n \begin{cases} k_F^{(-n/2)+1}, & \text{for } d=2 \\ k_F^{(m+1)}, & \text{for } d=3\,. \end{cases} \tag{2.73}$$

Here, $(n+m)-1$ represents the total number of resolvents R in the term. This bound remains valid for all possible commutations of $\mathbb{V}$ and $\mathbb{W}$ within the norm. This helps explain why $\|\Psi_0(t)\|$ can be bounded by a constant time $|\lambda|\, k_F^{(d-3)/2}$, whereas $\|\Psi_{32;44}(t)\|$ can be bounded by a constant times $|t|\,|\kappa|\,\lambda^2\, k_F^{(2d-4)}$. Although this straightforward rule offers the correct intuition for all terms with the maximal possible number of holes, it does somewhat oversimplify the scenario, as it does not apply to every term in the expansion. For example, it fails when a $\mathbb{V}$ or $\mathbb{W}$ does not change the number of holes in the state it acts upon, such as in $\Psi_{31}(t)$.

The above counting rule captures only the dependence on the Fermi momentum k_F, it does not include the possible volume factors that can appear, especially in the interacting case through the fermion–fermion term $\mathbb{W}$. In our setting, these L–dependent factors also require careful control, which will be a crucial part of the computations later on. Unlike in the non-interacting Fermi gas, certain sums over momentum states can scale with positive powers of L, and without an appropriate L–dependence in the coupling κ these contributions may remain leading in the large-volume limit. Therefore we fix the scaling $\kappa^2 = L^{-2d}$ to ensure that such volume-enhanced terms become subleading in the high-density regime.

We now evaluate the contributions of the various terms in the decomposition. For this purpose, we define the set of momentum pairs

$$T_F := \{(l,k) \in B_F^C \times B_F\} \subset (2\pi/L)\,\mathbb{Z}^d \times (2\pi/L)\,\mathbb{Z}^d\,, \tag{2.74}$$

as well as the crescent moon inside and outside of B_F, see (2.6),

$$M_k^F = B_F^C \cap (B_F + k) = \{p \in (2\pi/L)\,\mathbb{Z}^d \mid |p-k| \le k_F < |p|\}\,, \tag{2.75}$$

$$\widetilde{M}_k^F = B_F \setminus (B_F - k)\ , \tag{2.76}$$

and the union of two crescent moons, see (2.7),

$$N_k^F := \{(p,q) \in M_{-k}^F \times M_k^F\}\,, \tag{2.77}$$

$$\widetilde{N}_k^F := \{(p-k, q+k) \in \widetilde{M}_{-k}^F \times \widetilde{M}_k^F\} \tag{2.78}$$

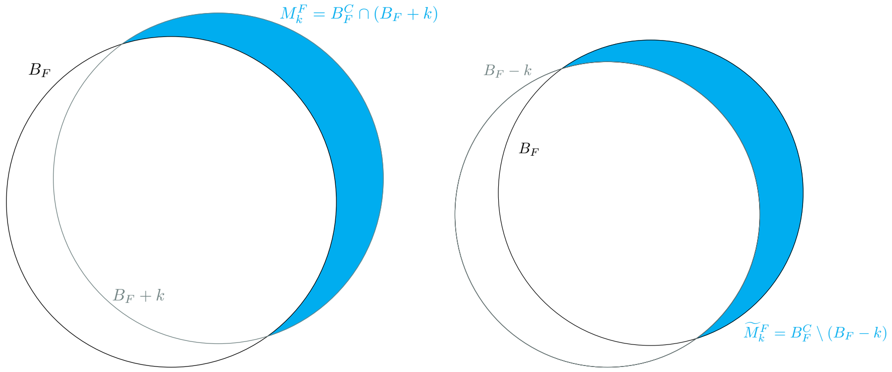

Figure 2.6: Pictures of M_k^F and $\widetilde{M}_k^F$.

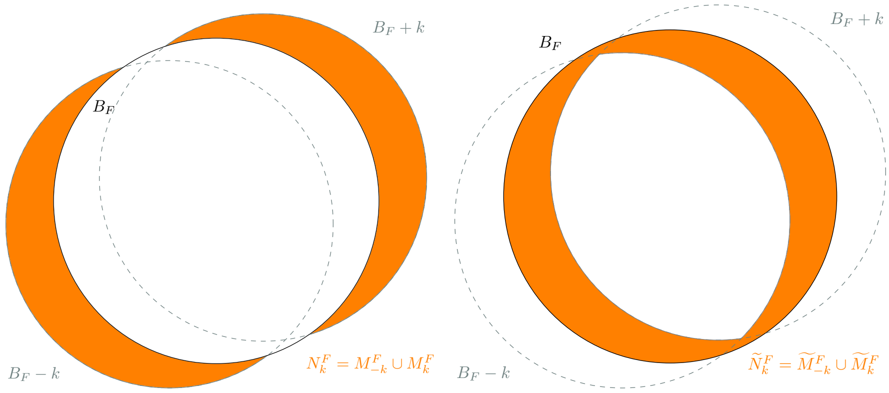

Figure 2.7: Pictures of N_k^F and $\widetilde{N}_k^F$.

We sometimes use the following notation in order to abbreviate the terms

$$\hat{v}_{lk} = \hat{v}(l-k)\,, \qquad \hat{w}_k = \hat{w}(k)\,, \qquad \varepsilon_{lk} = l^2 - k^2 + (l-k)^2 \tag{2.79}$$

and for $i, j, u \in \{1, ..., n\}$ we set

$$K_{lk}^{(i)} = e^{-i(l-k)y_i}, \quad K_{nm,lk}^{(i,j)} = K_{nm}^{(i)} K_{lk}^{(j)}, \quad K_{sr,nm,lk}^{(i,j,u)} = K_{sr}^{(i)} K_{nm}^{(j)} K_{lk}^{(u)}. \tag{2.80}$$

Our aim in the following chapter is to estimate each term individually in order to compare their respective contributions.

Chapter 3

Decomposition

3.1 Estimation of the terms

In the subsequent analysis, we will rely on several auxiliary bounds, given in (8), to estimate the various terms that arise from the decomposition. Detailed statements of these bounds, along with their full proofs, are provided in the Appendix for reference.

3.1.1 Φ-terms

Term $\Phi_0(t)$: We find that $a_l^* a_k \Omega_0$ is an eigenstate of $\mathbb{T}$ and P_f^2. Using the anticommutation relation for fermions we have

$$\begin{aligned} a_{k'}^* a_{k'} a_l^* a_k &= -a_{k'}^* a_l^* a_{k'} a_k + \delta_{lk'} a_{k'}^* a_k \\ &= -a_l^* a_{k'}^* a_k a_{k'} + \delta_{lk'} a_{k'}^* a_k \\ &= a_l^* a_k a_{k'}^* a_{k'} - \delta_{k'k} a_l^* a_{k'} + \delta_{lk'} a_{k'}^* a_k \,, \end{aligned} \tag{3.1}$$

and find with $k \in B_F$ and $k \neq l$

$$\begin{aligned}
\mathbb{T} a_l^* a_k \Omega_0 &= \sum_{k'=1}^{\infty} k'^2 a_{k'}^* a_{k'} a_l^* a_k \Omega_0 \qquad (3.2) \\
&= a_l^* a_k \sum_{k'=1}^{\infty} k'^2 a_{k'}^* a_{k'} \Omega_0 - \sum_{k'=1}^{\infty} \delta_{k'k} k'^2 a_l^* a_{k'} \Omega_0 + \sum_{k'=1}^{\infty} \delta_{lk'} k'^2 a_{k'}^* a_k \Omega_0 \\
&= \sum_{k' \in B_F} k'^2 a_l^* a_k \Omega_0 - k^2 a_l^* a_k \Omega_0 + l^2 a_l^* a_k \Omega_0 \\
&= \left(E^0 + l^2 - k^2 \right) a_l^* a_k \Omega_0
\end{aligned}$$

and

$$\begin{aligned}
P_f a_l^* a_k \Omega_0 &= \sum_{k'=1}^{\infty} k' a_{k'}^* a_{k'} a_l^* a_k \Omega_0 \qquad (3.3) \\
&= a_l^* a_k \sum_{k'=1}^{\infty} k' a_{k'}^* a_{k'} \Omega_0 - \sum_{k'=1}^{\infty} \delta_{k'k} k' a_l^* a_{k'} \Omega_0 + \sum_{k'=1}^{\infty} \delta_{lk'} k' a_{k'}^* a_k \Omega_0 \\
&= \underbrace{\sum_{k' \in B_F} k' a_l^* a_k \Omega_0}_{=\,0} - k a_l^* a_k \Omega_0 + l a_l^* a_k \Omega_0 \\
&= (l - k)\, a_l^* a_k \Omega_0 \,.
\end{aligned}$$

Hence,

$$(\mathbb{T} - E^0) a_l^* a_k \Omega_0 = (l^2 - k^2) a_l^* a_k \Omega_0 \,, \quad P_f^2 a_l^* a_k \Omega_0 = (l-k)^2 a_l^* a_k \Omega_0 \qquad (3.4)$$

and thus also

$$R a_l^* a_k \Omega_0 = \left(l^2 - k^2 + (l-k)^2 + 1 \right)^{-1} a_l^* a_k \Omega_0 \,. \qquad (3.5)$$

Using the Wick rule and

$$\left\| e^{i(l-k)y}\xi(s) \right\|_{\mathcal{H}_n} = \|\xi(s)\|_{\mathcal{H}_n} = 1 \tag{3.6}$$

one finds

$$\begin{aligned}
&\left\| R\mathbb{V}^{(i)}\xi(s) \otimes \Omega_0 \right\|^2 \qquad (3.7)\\
&= \left\| R\lambda L^{-d} \sum_{(l,k)\in T_F} \hat{v}_{lk} K^{(i)}_{lk} a^*_l a_k \xi(s) \otimes \Omega_0 \right\|^2 \\
&= \lambda^2 L^{-2d} \langle R \sum_{(l,k)\in T_F} \hat{v}_{lk} K^{(i)}_{lk} a^*_l a_k \xi(s) \otimes \Omega_0, R \sum_{(l,k)\in T_F} \hat{v}_{l'k'} K^{(i)}_{l'k'} a^*_{l'} a_{k'} \xi(s) \otimes \Omega_0 \rangle \\
&= \lambda^2 L^{-2d} \sum_{(l,k)\in T_F} \sum_{(l',k')\in T_F} \frac{\overline{\hat{v}_{lk}} \hat{v}_{l'k'} \langle K^{(i)}_{lk} \xi(s), K_{l'k'} \xi(s)^{(i)} \rangle_{\mathcal{H}_n} \langle a^*_l a_k \Omega_0, a^*_{l'} a_{k'} \Omega_0 \rangle_{\mathcal{H}^-_N}}{(l^2 - k^2 + (l-k)^2 + 1)\,(l'^2 - k'^2 + (l'-k')^2 + 1)} \\
&\leq \lambda^2 L^{-2d} \sum_{(l,k)\in T_F} \frac{|\hat{v}_{lk}|^2}{\left(l^2 - k^2 + (l-k)^2 + 1\right)^2} \\
&\leq \lambda^2 k_F^{-2} L^{-2d} \sum_{(l,k)\in T_F} \frac{|\hat{v}_{lk}|^2}{(|l| - |k| + k_F^{-1})^2} \,,
\end{aligned}$$

where we used the inequality

$$(|l| + |k|)\,(|l| - |k|) \geq k_F\,(|l| - |k|) \,, \tag{3.8}$$

which holds, since $k \in B_F$ and $l \in B_F^C$. Then we have with (A.43c)

$$\|\Phi_0(t)\| \lesssim |\lambda| \left(L^{1/2} \cdot \frac{1}{\left(1 + \frac{L}{k_F}\right)} k_F^{(d-3)/2} + \mathcal{O}\left(\frac{\ln(k_F)^{1/2}}{k_F} \right) \right) \,. \tag{3.9}$$

Term $\Phi_1(t)$: We find

$$\begin{aligned}
&\left\| R\mathbb{V}^{(i)}(-1 + h_n^0 - h_n^{\text{eff}})\xi(s) \otimes \Omega_0 \right\|^2 &\quad (3.10)\\
&\lesssim \lambda^2 L^{-2d} \sum_{(l,k)\in T_F} \frac{|\hat{v}(l-k)|^2}{\left(l^2 - k^2 + (l-k)^2 + 1\right)^2} \left(1 + \left\| (h_n^0 - h_n^{\text{eff}})\xi(s) \right\|_{\mathcal{H}_n}^2 \right) .
\end{aligned}$$

To bound the norm, we use

$$\begin{aligned}
h_n^0 - h_n^{\text{eff}} &= h_n^0 - \left(h_n^0 - \sum_{i<j}^{n} \lambda^2 W_{k_F}(y_i - y_j) + n\lambda^2 W_{k_F}(0) \right) &\quad (3.11)\\
&= \sum_{i<j}^{n} \lambda^2 W_{k_F}(y_i - y_j) + n\lambda^2 W_{k_F}(0)
\end{aligned}$$

and with (9)

$$|W_{k_F}(r)| \leq W_{k_F}(0) \lesssim k_F^{(d-2)} . \quad (3.12)$$

Together with (A.43c) we have

$$\begin{aligned}
&\|\Phi_1(t)\| &\quad (3.13)\\
&\lesssim |t|\,|\lambda| \left(L^{1/2} \cdot \frac{1}{\left(1 + \frac{L}{k_F}\right)} k_F^{(d-3)/2} + \mathcal{O}\left(\frac{\ln(k_F)^{1/2}}{k_F} \right) \right) \cdot \left(1 + \lambda^2 k_F^{(d-2)} \right) .
\end{aligned}$$

Term $\Phi_2(t)$: Here we need to evaluate two different terms. We have

$$P_f^2 \mathbb{V}^{(i)} \xi(s) \otimes \Omega_0 = L^{-d} \sum_{(l,k)\in T_F} \hat{v}(l-k)(l-k)^2 e^{-i(l-k)y_i} \xi(s) \otimes a_l^* a_k \Omega_0 \quad (3.14)$$

and

$$[h_n^0, \mathbb{V}^{(i)}]\xi(s) \otimes \Omega_0 = L^{-d} \sum_{(l,k)\in T_F} \hat{v}_{lk} \cdot (l-k)^2 K_{lk}^{(i)} \xi(s) \otimes a_l^* a_k \Omega_0 \\ + L^{-d} \sum_{(l,k)\in T_F} \hat{v}_{lk} K_{lk}^{(i)} (k-l) \cdot (-2i\nabla_{y_i})\xi(s) \otimes a_l^* a_k \Omega_0 \,, \tag{3.15}$$

which, computing the difference, leads to

$$R\{[h_n^0, \mathbb{V}^{(i)}] - P_f^2 \mathbb{V}\}\xi(s) \otimes \Omega_0 = L^{-d} \sum_{(l,k)\in T_F} \hat{v}_{lk} K_{lk}^{(i)} (k-l) \\ \cdot (-2i\nabla_{y_i})\xi(s) \otimes a_l^* a_k \Omega_0 \,. \tag{3.16}$$

Hence,

$$\left\| R\{[h_n^0, \mathbb{V}^{(i)}] - P_f^2 \mathbb{V}^{(i)}\}\xi(s) \otimes \Omega_0 \right\|^2 \\ \lesssim \left(L^{-2d} \sum_{(l,k)\in T_F} \frac{\lambda^2 |\hat{v}_{lk}|^2 (l-k)^2}{(\varepsilon_{lk}+1)^2} \right) \|\nabla_{y_i}\xi(s)\|_{\mathcal{H}_n}^2 \,. \tag{3.17}$$

To bound the norm involving the gradient, we use $u^2 \leq c(1-\Delta)$ for some $c \in [0,1)$, therefore

$$\sum_{i=1}^{n} \langle \xi(s), (-\Delta_{y_i})\, \xi(s) \rangle_{\mathcal{H}_n} \lesssim 1 + \langle \xi(s), h_n^0 \xi(s) \rangle_{\mathcal{H}_n} \,, \tag{3.18}$$

and using again

$$|W_{k_F}(r)| \leq W_{k_F}(0) \lesssim k_F^{(d-2)} \,, \tag{3.19}$$

it follows that

$$\begin{aligned}\langle\xi(s), h_n^0\xi(s)\rangle &\lesssim \lambda^2 k_F^{(d-2)} + \langle\xi(0), h_n^0\xi(0)\rangle_{\mathcal{H}_n} \\ &\lesssim \lambda^2 k_F^{(d-2)} + \sum_{i=1}^{n}\langle\xi(0), (-\Delta_{y_i})\,\xi(0)\rangle_{\mathcal{H}_n} \\ &\lesssim \lambda^2 k_F^{(d-2)} + 1\,.\end{aligned} \tag{3.20}$$

The last step follows from (2.21). In combination with (A.43d) we find

$$\begin{aligned}&\|\Phi_2(t)\| \\ &\lesssim |t|\,|\lambda|\left(L^{1/2}\cdot\frac{1}{\left(1+\frac{L}{k_F}\right)}\,k_F^{(d-3)/2} + \mathcal{O}\left(\frac{\ln(k_F)^{1/2}}{k_F}\right)\right)\cdot\left(1+\lambda^2 k_F^{(d-2)}\right)\,.\end{aligned} \tag{3.21}$$

Given that the state $\mathbb{V}R\mathbb{V}\xi(s)\otimes\Omega_0$ generates at most two holes in the Fermi sea, and we are decomposing Φ_{32}, we need to evaluate the contributions of Φ_{30} and Φ_{31}:

Term $\Phi_{30}(t)$: With $P^{(0)} = \mathbb{1}\otimes|\Omega_0\rangle\langle\Omega_0|$ and using the identity

$$P^{(0)}a_l^* a_k a_n^* a_m\Omega_0 = \delta_{m,l}\delta_{n,k}\Omega_0\,, \tag{3.22}$$

for all $(n,m)\in T_F$, as well as

$$\begin{aligned}&\langle\Omega_0, \mathbb{V}^{(i)}R\mathbb{V}^{(j)}\Omega_0\rangle_{\mathcal{H}_N^-} \\ &= \lambda^2 L^{-2d}\sum_{(n,m)\in T_F}\frac{|\hat{v}(n-m)|^2}{n^2-m^2+(n-m)^2+1}e^{i(n-m)(y_j-y_i)}\,,\end{aligned} \tag{3.23}$$

we find

$$\Phi_{30}(t) \tag{3.24}$$

$$
\begin{aligned}
&= i \sum_{i,j=1}^{n} \int_0^t \mathrm{d}s \; e^{i\mathbb{H}s} P^{(0)} \mathbb{V}^{(i)} R \mathbb{V}^{(j)} \xi(s) \otimes \Omega_0 \\
&= i \sum_{i=1}^{n} \int_0^t \mathrm{d}s \; e^{i\mathbb{H}s} \left(\lambda^2 L^{-2d} \sum_{(n,m)\in T_F} \frac{|\hat{v}(n-m)|^2}{n^2 - m^2 + (n-m)^2 + 1} \right) \xi(s) \otimes \Omega_0 \\
&\quad + i \sum_{i<j}^{n} \int_0^t \mathrm{d}s \; e^{i\mathbb{H}s} \Bigg(\lambda^2 L^{-2d} \sum_{(n,m)\in T_F} \frac{|\hat{v}(n-m)|^2}{n^2 - m^2 + (n-m)^2 + 1} \\
&\qquad\qquad \cdot \cos\left((n-m)(y_i - y_j)\right) \Bigg) \xi(s) \otimes \Omega_0 \, .
\end{aligned}
$$

This contribution is the one determining the effective Hamiltonian h_n. Since the Riemann sums converge to the corresponding integrals, we find that

$$
\begin{aligned}
&\lim_{L\to\infty} \lambda^2 L^{-2d} \sum_{(n,m)\in T_F} \frac{|\hat{v}(n-m)|^2}{n^2 - m^2 + (n-m)^2 + 1} \cos\left((n-m)(y_i - y_j)\right) \\
&= \lambda^2 W_{k_F}\left(|y_i - y_j|\right) ,
\end{aligned}
\tag{3.25}
$$

where we used the rotational invariance of $\hat{v}(k)$ in order to replace the argument in the cosine by $(n-m)\cdot \hat{a}\left(|y_i - y_j|\right)$ for any unit vector $\hat{a} \in \mathbb{R}^d$. Furthermore

$$
\lim_{L\to\infty} \lambda^2 L^{-2d} \sum_{(n,m)\in T_F} \frac{|\hat{v}(n-m)|^2}{n^2 - m^2 + (n-m)^2 + 1} = \lambda^2 W_{k_F}\left(0\right) \, . \tag{3.26}
$$

The prefactor V_d^2 in the definition of W_{k_F} in (2.18) emerges from the normalization that arises when replacing the two discrete momentum sums in finite volume by integrals in the thermodynamic limit, i.e., each discrete sum contributes a factor V_d, so that their product yields the overall volume–normalization factor V_d^2. Using (9) we notice that $\Phi_{30}(t)$ is of order $\lambda^2 k_F^{(d-2)}$. We then have a complete cancellation

$$
\lim_{L\to\infty} \|\phi(t) + \Phi_{30}(t)\| = 0 \, , \tag{3.27}
$$

since,

$$\phi(t) = i \int_0^t \mathrm{d}s\ e^{i\mathbb{H}s} \left(-\sum_{i<j}^{n} \lambda^2 W_{k_F}\left(|y_i - y_j|\right) - n\lambda^2 W_{k_F}(0) \right) \xi(s) \otimes \Omega_0 \,. \tag{3.28}$$

Everything together results in the choice $\lambda^2 = k_F^{(2-d)}$.

Term $\Phi_{31}(t)$: Using the identity

$$P^{(1)} a_l^* a_k a_n^* a_m \Omega_0 = \delta_{m,l} \chi_{B_F}(k) a_k a_n^* \Omega_0 + \delta_{k,n} \chi_{B_F^C}(l) a_l^* a_m \Omega_0 \tag{3.29}$$

for all $(n,m) \in B_F^C \times B_F$ we find

$$P^{(1)} \mathbb{V}^{(i)} R \mathbb{V}^{(j)} \xi(s) \otimes \Omega_0$$

$$= \lambda^4 L^{-4d} \sum_{k \in B_F} \sum_{(n,m) \in T_F} \frac{\hat{v}(m-k)\hat{v}(n-m)}{n^2 - m^2 + (n-m)^2 + 1} e^{(m-k)y_i} e^{(n-m)y_j} \xi(s) \otimes a_k a_n^* \Omega_0 \tag{3.30a}$$

$$+ \lambda^4 L^{-4d} \sum_{l \in B_F^C} \sum_{(n,m) \in T_F} \frac{\hat{v}(l-n)\hat{v}(n-m)}{n^2 - m^2 + (n-m)^2 + 1} e^{(l-n)y_i} e^{(n-m)y_j} \xi(s) \otimes a_l^* a_m \Omega_0 \,. \tag{3.30b}$$

Therefore

$$\left\| P^{(1)} \mathbb{V}^{(i)} R \mathbb{V}^{(i)} \xi(s) \otimes \Omega_0 \right\|^2 \tag{3.31}$$

$$\leq \lambda^4 L^{-4d} \sum_{k,k' \in B_F} \sum_{(n,m) \in T_F} \sum_{(n',m') \in T_F} \frac{\overline{\hat{v}_{mk}}\, \hat{v}_{m'k'}\, \overline{\hat{v}_{nm}}\, \hat{v}_{n'm'}}{(\varepsilon_{nm} + 1)\,(\varepsilon_{n'm'} + 1)} \ \left| \langle a_{k'} a_{n'}^* \Omega_0, a_k a_n^* \Omega_0 \rangle_{\mathcal{H}_N^-} \right|$$

$$+ \lambda^4 L^{-4d} \sum_{l,l' \in B_F^C} \sum_{(n,m) \in T_F} \sum_{(n',m') \in T_F} \frac{\overline{\hat{v}_{ln}}\, \hat{v}_{l'n'}\, \overline{\hat{v}_{nm}}\, \hat{v}_{n'm'}}{(\varepsilon_{nm} + 1)\,(\varepsilon_{n'm'} + 1)} \ \left| \langle a_{l'}^* a_{m'} \Omega_0, a_l^* a_m \Omega_0 \rangle_{\mathcal{H}_N^-} \right|$$

$$\leq \lambda^4 \left(L^{-2d} \sum_{(n,k)\in T_F} \left(L^{-2d} \sum_{m\in B_F} \frac{|\hat{v}_{mk}|\ |\hat{v}_{nm}|}{(n^2 - m^2 + (n-m)^2 + 1)} \right)^2 \right)$$
$$+ \lambda^4 \left(L^{-2d} \sum_{(l,m)\in T_F} \left(L^{-2d} \sum_{n\in B_F^C} \frac{|\hat{v}_{ln}|\ |\hat{v}_{nm}|}{(n^2 - m^2 + (n-m)^2 + 1)} \right)^2 \right) .$$

Then with (A.43g) and (A.43h)

$$\|\Phi_{31}(t)\| \lesssim |t|\, \lambda^2 k_F^{(d-3)/2} \ln (k_F) \ . \tag{3.32}$$

Furthermore we find for the decomposition of $\Phi_{4;n}$, $n \in \{1, 2, 3\}$:

Term $\Phi_{41}(t)$: The identity

$$\begin{aligned} P^{(2)} a_p^* a_q^* a_{p-k} a_{q+k} a_n^* a_m \Omega_0 &= \delta_{n,p-k} \delta_{m,p} \chi_{\widetilde{M}_k^F}(q+k) a_q^* a_{q+k} \Omega_0 \\ &\quad + \delta_{n,q+k} \delta_{m,q} \chi_{\widetilde{M}_k^F}(p-k) a_p^* a_{p-k} \Omega_0 \end{aligned} \tag{3.33}$$

for all (n, m) in $B_F^C \times B_F$ yields

$$P^{(1)} \mathbb{W} R \mathbb{V}^{(i)} \xi(s) \otimes \Omega_0$$
$$= \kappa\lambda L^{-2d} \sum_{q\in M_k^F} \sum_{k\in[-D,D]^d} \sum_{(n,m)\in T_F} \frac{\hat{w}(k)\ \hat{v}(n-m)}{(n^2 - m^2 + (n-m)^2 + 1)}\ K_{nm}^{(i)}\, \xi(s)$$
$$\otimes\, a_q^* a_{q+k} \Omega_0 \tag{3.34a}$$
$$+ \kappa\lambda L^{-2d} \sum_{q\in M_k^F} \sum_{k\in[-D,D]^d} \sum_{(n,m)\in T_F} \frac{\hat{w}(k)\ \hat{v}(n-m)}{(n^2 - m^2 + (n-m)^2 + 1)}\ K_{nm}^{(i)}\, \xi(s)$$
$$\otimes\, a_p^* a_{p-k} \Omega_0 \tag{3.34b}$$

and therefore we find

$$\begin{aligned} &\left\|(3.34\mathrm{a})\right\|^2 \qquad (3.35)\\ &\lesssim \kappa^2\lambda^2 \left(L^{-2d} \sum_{q\in M_k^F} \sum_{k\in[-D,D]^d} |\hat{w}(k)|^2 \right) L^{2d} \left(L^{-2d} \sum_{(n,m)\in T_F} \frac{|\hat{v}(n-m)|}{(n^2-m^2+1)} \right)^2 . \end{aligned}$$

Since (3.34b) is of the same order of magnitude, we have with (A.43b) and (A.43n)

$$\begin{aligned} &\|\Phi_{41}(t)\| \qquad (3.36)\\ &\lesssim |t|\,|\kappa|\,|\lambda| \left(\frac{1}{L^{1/2}}\, k_F^{(d-1)/2} + \mathcal{O}(1) \right) L^d \left(L^{1/2} \cdot \frac{1}{\left(1+\frac{L}{k_F}\right)}\, k_F^{d-2} + \mathcal{O}\left(\frac{\ln(k_F)}{k_F} \right) \right) . \end{aligned}$$

Term $\Phi_{42}(t)$: The identity

$$\begin{aligned} P^{(2)} a_p^* a_q^* a_{p-k} a_{q+k} a_n^* a_m \Omega_0 &= \delta_{n,p-k} \chi_{\widetilde{M}_k^F}(q+k) a_q^* a_p^* a_{q+k} a_m \Omega_0 \qquad (3.37)\\ &+ \delta_{n,q+k} \chi_{\widetilde{M}_k^F}(p-k) a_q^* a_p^* a_{p-k} a_m \Omega_0\\ &+ \delta_{m,p} \chi_{\widetilde{N}_k^F}\big((p-k,q+k)\big) a_q^* a_{p-k} a_{q+k} a_n^* \Omega_0\\ &+ \delta_{m,q} \chi_{\widetilde{N}_k^F}\big((q+k,p-k)\big) a_p^* a_{p-k} a_{q+k} a_n^* \Omega_0 \end{aligned}$$

for all (n,m) in $B_F^C \times B_F$. We find

$$\begin{aligned} &P^{(2)} \mathbb{W} R \mathbb{V}^{(i)} \xi(s) \otimes \Omega_0\\ &= \kappa\lambda L^{-2d} \sum_{(p,q)\in N_k^F} \sum_{k\in[-D,D]^d} \sum_{(n,m)\in T_F} \frac{\hat{w}_k\ \hat{v}_{nm}}{(\varepsilon_{nm}+1)}\ e^{-i(n-m)y_i} \xi(s)\\ &\qquad \otimes a_q^* a_p^* a_{q+k} a_m \Omega_0 \qquad (3.38\mathrm{a})\\ &+ \kappa\lambda L^{-2d} \sum_{(p,q)\in N_k^F} \sum_{k\in[-D,D]^d} \sum_{(n,m)\in T_F} \frac{\hat{w}_k\ \hat{v}_{nm}}{(\varepsilon_{nm}+1)}\ e^{-i(n-m)y_i} \xi(s) \end{aligned}$$

$$\otimes\, a_q^* a_p^* a_{p-k} a_m \Omega_0 \tag{3.38b}$$

$$+\,\kappa\lambda L^{-2d} \sum_{(p,q)\in N_k^F} \sum_{k\in[-D,D]^d} \sum_{(n,m)\in T_F} \frac{\hat{w}_k\ \hat{v}_{nm}}{(\varepsilon_{nm}+1)}\ e^{-i(n-m)y_i}\xi(s)$$

$$\otimes\, a_q^* a_{p-k} a_{q+k} a_n^* \Omega_0 \tag{3.38c}$$

$$+\,\kappa\lambda L^{-2d} \sum_{(p,q)\in N_k^F} \sum_{k\in[-D,D]^d} \sum_{(n,m)\in T_F} \frac{\hat{w}_k\ \hat{v}_{nm}}{(\varepsilon_{nm}+1)}\ e^{-i(n-m)y_i}\xi(s)$$

$$\otimes\, a_p^* a_{p-k} a_{q+k} a_n^* \Omega_0\,. \tag{3.38d}$$

This yields

$$\|(3.38\text{a})\|^2 \tag{3.39}$$

$$\lesssim \kappa^2\lambda^2 \left(L^{-2d} \sum_{(p,q)\in N_k^F} \sum_{k\in[-D,D]^d} |\hat{w}(k)|^2 \right) L^d \left(L^{-d} \sum_{m\in B_F} \left(L^{-d} \sum_{n\in B_F^C} \frac{|\hat{v}(n-m)|}{(n^2-m^2+1)} \right)^2 \right).$$

Because of the compact support of the functions $\hat{v}$ and $\hat{w}$ and conservation of momentum of the operator $\mathbb{W}$, the scalar product only gives this single term. Similarly we have the same bounds for (3.38b)-(3.38d), hence, with (A.43e) and (A.43m),

$$\|\Phi_{42}(t)\| \lesssim |t|\,|\kappa|\,|\lambda| \left(L^{d-1}k_F^{d-1} + L^d\cdot\mathcal{O}(1) \right) \left(k_F^{(d-3)/2}\ln(k_F) \right). \tag{3.40}$$

Term $\Phi_{43}(t)$: We have

$$P^{(3)}\mathbb{W}R\mathbb{V}^{(i)}\xi(s)\otimes\Omega_0 \tag{3.41}$$

$$=\kappa\lambda L^{-2d} \sum_{(p,q)\in N_k^F} \sum_{k\in[-D,D]^d} \sum_{(n,m)\in T_F} \frac{\hat{w}_k\ \hat{v}_{nm}}{(\varepsilon_{nm}+1)}\ e^{-i(n-m)y_i}\xi(s)$$

$$\otimes\, a_p^* a_q^* a_{p-k} a_{q+k} a_n^* a_m \Omega_0\,,$$

following from the identity

$$P^{(3)} a_p^* a_q^* a_{p-k} a_{q+k} a_n^* a_m \Omega_0 = \chi_{N_k^F}\Big((p,q)\Big) a_p^* a_q^* a_{p-k} a_{q+k} a_n^* a_m \Omega_0 \tag{3.42}$$

for all (n,m) in $B_F^C \times B_F$. One can easily check that

$$\begin{aligned}&\left|\frac{\hat{w}(k)\hat{v}(n-m)}{(n^2-m^2+(n-m)^2+1)} \cdot \frac{\hat{w}(k')\hat{v}(n'-m')}{(n'^2-m'^2+(n'-m')^2+1)}\right| \\ &\leq \frac{|\hat{w}(k)\hat{v}(n-m)|^2}{2\left(n^2-m^2+(n-m)^2+1\right)^2} + \frac{|\hat{w}(k')\hat{v}(n'-m')|^2}{2\left(n'^2-m'^2+(n'-m')^2+1\right)^2}\,.\end{aligned} \tag{3.43}$$

Then

$$\begin{aligned}&\left\|P^{(3)} \mathbb{W} R \mathbb{V}^{(i)} \xi(s) \otimes \Omega_0\right\|^2 \\ &\leq \kappa\lambda L^{-4d} \sum_{(p,q)\in N_k^F} \sum_{(p',q')\in N_{k'}^F} \sum_{k,k'\in[-D,D]^d} \sum_{(n,m)\in T_F} \sum_{(n',m')\in T_F} \frac{|\hat{w}(k)\hat{v}(n-m)|^2}{\left(n^2-m^2+(n-m)^2+1\right)^2} \\ &\quad\cdot \left\|e^{-i(n-m)y_i}\xi(s)\right\|_{\mathcal{H}_n} \times |\langle a_p^* a_q^* a_{p-k} a_{q+k} a_n^* a_m \Omega_0, a_{p'}^* a_{q'}^* a_{p'-k'} a_{q'+k'} a_{n'}^* a_{m'} \Omega_0\rangle_{\mathcal{H}_N^-}| \\ &\lesssim \kappa\lambda \left(L^{-2d} \sum_{(p,q)\in N_k^F} \sum_{k\in[-D,D]^d} |\hat{w}(k)|^2\right) \left(L^{-2d} \sum_{(n,m)\in T_F} \frac{|\hat{v}(n-m)|^2}{\left(n^2-m^2+(n-m)^2+1\right)^2}\right),\end{aligned} \tag{3.44}$$

where we used $\left\|e^{-i(n-m)y_i}\xi(s)\right\| = 1$ as well as the fact that the scalar product provides 36 summations. This and (A.43c), (A.43m) yields to

$$\begin{aligned}&\|\Phi_{43}(t)\| \\ &\lesssim |t|\,|\kappa|\,|\lambda|\, \left(L^{d-1}k_F^{d-1} + \ L^d \cdot \mathcal{O}(1)\right) \left(L^{1/2} \cdot \frac{1}{\left(1+\frac{L}{k_F}\right)}\, k_F^{(d-3)/2} + \mathcal{O}\left(\frac{\ln(k_F)^{1/2}}{k_F}\right)\right).\end{aligned} \tag{3.45}$$

Term $\Phi_{32;0}(t)$: We have

$$P^{(2)} a_l^* a_k a_n^* a_m \Omega_0 = \chi_{T_F}(l,k) a_l^* a_k a_n^* a_m \Omega_0 \tag{3.46}$$

for all $(n,m) \in T_F$. Therefore

$$\begin{aligned} &P^{(2)} R \mathbb{V}^{(i)} R \mathbb{V}^{(j)} \xi(s) \otimes \Omega_0 \\ &= \lambda^2 L^{-2d} \sum_{(l,k)\in T_F} \sum_{(n,m)\in T_F} \frac{\hat{v}_{lk}\ \hat{v}_{nm}}{(\varepsilon_{lk} + \varepsilon_{nm} + 1)(\varepsilon_{nm} + 1)} K^{(i,j)}_{lk,nm} \xi(s) \otimes a_l^* a_k a_n^* a_m \Omega_0 \end{aligned} \tag{3.47}$$

and using

$$\begin{aligned} &\left| \frac{\hat{v}_{lk}\ \hat{v}_{nm}}{(\varepsilon_{lk} + \varepsilon_{nm} + 1)(\varepsilon_{nm} + 1)} \cdot \frac{\hat{v}_{l'k'}\ \hat{v}_{n'm'}}{(\varepsilon_{l'k'} + \varepsilon_{n'm'} + 1)(\varepsilon_{n'm'} + 1)} \right| \\ &\leq \frac{|\hat{v}_{lk}\ \hat{v}_{nm}|^2}{2(\varepsilon_{lk} + 1)^2 (\varepsilon_{nm} + 1)^2} + \frac{|\hat{v}_{l'k'}\ \hat{v}_{n'm'}|^2}{2(\varepsilon_{l'k'} + 1)^2 (\varepsilon_{n'm'} + 1)^2}, \end{aligned} \tag{3.48}$$

we find

$$\begin{aligned} &\left\| P^{(2)} R \mathbb{V}^{(i)} R \mathbb{V}^{(j)} \xi(s) \otimes \Omega_0 \right\|^2 \\ &\leq \lambda^4 L^{-4d} \sum_{\substack{(l,k)\in T_F \\ (l',k')\in T_F}} \sum_{\substack{(n,m)\in T_F \\ (n'm')\in T_F}} \frac{|\hat{v}_{lk}\ \hat{v}_{nm}|^2}{(\varepsilon_{lk} + 1)^2 (\varepsilon_{nm} + 1)^2} \left| \langle a_l^* a_k a_n^* a_m \Omega_0, a_{l'}^* a_{k'} a_{n'}^* a_{m'} \Omega_0 \rangle_{\mathcal{H}_N^-} \right| \\ &\leq 4 \left(\lambda^2 L^{-2d} \sum_{(n,m)\in T_F} \frac{|\hat{v}_{nm}|^2}{(n^2 - m^2 + 1)^2} \right)^2, \end{aligned} \tag{3.49}$$

where we used that the scalar product provides four different possibilities. With (A.43c) this yields

$$\|\Phi_{32;0}(t)\| \lesssim \lambda^2 \left(L \cdot \frac{1}{\left(1 + \frac{L}{k_F}\right)^2} k_F^{d-3} + \mathcal{O}\left(\frac{\ln(k_F)}{k_F^2} \right) \right). \tag{3.50}$$

Term $\Phi_{32;1}(t)$: Similarly to the previous term we have

$$\begin{aligned}&\left\|RP^{(2)}\mathbb{V}R\mathbb{V}\left(-1+h_n^0-h_n^{\text{eff}}\right)\xi(s)\otimes\Omega_0\right\|^2 \\ \leq\ &\lambda^4L^{-4d}\sum_{(l,k)\in T_F}\sum_{(n,m)\in T_F}\left|\frac{\hat{v}_{lk}\ \hat{v}_{nm}}{(\varepsilon_{lk}+1)\,(\varepsilon_{nm}+1)}\right|^2\left(1+\left\|(h_n^{\text{eff}}-h_n^0)\xi_0\right\|^2_{\mathcal{H}_n}\right)\end{aligned} \tag{3.51}$$

and then as above

$$\begin{aligned}&\|\Phi_{32;1}(t)\| \\ \lesssim\ &|t|\,\lambda^2\left(L\cdot\frac{1}{\left(1+\frac{L}{k_F}\right)^2}k_F^{d-3}+\mathcal{O}\left(\frac{\ln(k_F)}{k_F^2}\right)\right)\cdot\left(1+\lambda^2k_F^{(d-2)}\right).\end{aligned} \tag{3.52}$$

Term $\Phi_{32;2}(t)$: Here we again need to compute the two different contributions

$$\begin{aligned}&P^{(2)}[h_n^0,\mathbb{V}R\mathbb{V}]\Omega_0 \\ =&\sum_{i,j=1}^{n}\mathbb{V}^{(i)}R[h_n^0,\mathbb{V}^{(j)}]\Omega_0+R[h_n^0,\mathbb{V}^{(i)}]\mathbb{V}^{(j)}\Omega_0 \\ =&\sum_{i,j=1}^{n}L^{-2d}\sum_{(n,m)\in T_F}\sum_{(l,k)\in T_F}\hat{v}_{lk}\hat{v}_{nm}e^{-i(l-k)y_i}e^{-i(n-m)y_j}(n-m+l-k)^2a_l^*a_kRa_n^*a_m\Omega_0 \\ &+\sum_{i,j=1}^{n}L^{-2d}\sum_{(n,m)\in T_F}\sum_{(l,k)\in T_F}\hat{v}_{lk}\hat{v}_{nm}e^{-i(l-k)y_i}e^{-i(n-m)y_j} \\ &\left((2(l-k)\cdot(-i\nabla_{y_i})+2(n-m)\cdot(-i\nabla_{y_j})\right)a_l^*a_kRa_n^*a_m\Omega_0\end{aligned} \tag{3.53}$$

and

$$\begin{aligned}&P_f^2P^{(2)}\mathbb{V}R\mathbb{V}\Omega_0 \\ =&\sum_{i,j=1}^{n}L^{-2d}\sum_{(n,m)\in T_F}\sum_{(l,k)\in T_F}\hat{v}_{lk}\hat{v}_{nm}e^{-i(l-k)y_i}e^{-i(n-m)y_j}\end{aligned} \tag{3.54}$$

$$\cdot\,(n-m+l-k)^2 a_l^* a_k R a_n^* a_m \Omega_0\,.$$

Then, computing the difference we find

$$\begin{aligned}
&\left\| R\{P^{(2)}[h_n^0, \mathbb{V}R\mathbb{V}] - P_f^2 \mathbb{V}R\mathbb{V}\}\xi(s)\otimes\Omega_0 \right\|^2 \\
&\lesssim \lambda^4 L^{-4d} \sum_{(n,m)\in T_F}\sum_{(l,k)\in T_F} \frac{|\hat{v}_{lk}\hat{v}_{nm}|^2}{(\varepsilon_{lk}+\varepsilon_{nm}+1)^2(\varepsilon_{nm}+1)^2} \\
&\qquad \cdot\Big((l-k)^2+(n-m)^2\Big)\sum_{i=1}^{n}\|\nabla_{y_i}\xi(s)\|^2_{\mathcal{H}_n} \\
&\lesssim \lambda^4\left(L^{-2d}\sum_{(n,m)\in T_F}\frac{|\hat{v}_{nm}|^2(n-m)^2}{(n^2-m^2+1)^2}\right)\left(L^{-2d}\sum_{(n,m)\in T_F}\frac{|\hat{v}_{nm}|^2}{(n^2-m^2+1)^2}\right) \\
&\qquad \cdot\left(1+\lambda^2\left(\frac{1}{1+\frac{L}{k_F}}k_F^{(d-2)}+\mathcal{O}\left(\frac{\ln(k_F)}{k_F^2}\right)\right)\right)
\end{aligned} \tag{3.55}$$

Therefore, with (A.43c) and (A.43d),

$$\begin{aligned}
&\|\Phi_{32;2}(t)\| \\
&\lesssim |t|\,\lambda^2\left(L\cdot\frac{1}{\left(1+\frac{L}{k_F}\right)^2}k_F^{d-3}+\mathcal{O}\left(\frac{\ln(k_F)}{k_F^2}\right)\right)\cdot\left(1+\lambda^2 k_F^{(d-2)}\right)\,.
\end{aligned} \tag{3.56}$$

Term $\Phi_{32;31}(t)$: Now we come to the term with one hole. Here we have

$$\begin{aligned}
P^{(1)}a_h^* a_g a_l^* a_k a_n^* a_m\Omega_0 &= \delta_{m,h}\delta_{n,g}a_l^* a_k\Omega_0 + \delta_{k,h}\delta_{l,g}a_n^* a_m\Omega_0 \\
&\quad + \delta_{k,h}\delta_{n,g}a_l^* a_m\Omega_0 + \delta_{m,h}\delta_{l,g}a_n^* a_k\Omega_0
\end{aligned} \tag{3.57}$$

for all $(n,m)\in T_F$ and $(l,k)\in T_F$. Therefore

$$P^{(1)}\mathbb{V}^{(u)}P^{(2)}R\mathbb{V}^{(i)}R\mathbb{V}^{(j)}\xi(s)\otimes\Omega_0$$

$$= 2\lambda^3 L^{-3d} \sum_{(l,k)\in T_F} \sum_{(n,m)\in T_F} \frac{|\hat{v}_{nm}|^2 \ \hat{v}_{lk}}{(\varepsilon_{lk} + \varepsilon_{nm} + 1)\,(\varepsilon_{nm} + 1)} K^{(u,i,j)}_{mn,lk,nm} \xi(s)$$
$$\otimes a_l^* a_k \Omega_0 \tag{3.58a}$$
$$+ 2\lambda^3 L^{-3d} \sum_{(l,k)\in T_F} \sum_{(n,m)\in T_F} \frac{\hat{v}_{nm} \ |\hat{v}_{lk}|^2}{(\varepsilon_{lk} + \varepsilon_{nm} + 1)\,(\varepsilon_{nm} + 1)} K^{(u,i,j)}_{kl,lk,nm} \xi(s)$$
$$\otimes a_n^* a_m \Omega_0 \tag{3.58b}$$
$$+ 2\lambda^3 L^{-3d} \sum_{(l,k)\in T_F} \sum_{(n,m)\in T_F} \frac{\hat{v}_{lk} \ \hat{v}_{nm} \ \hat{v}_{kn}}{(\varepsilon_{lk} + \varepsilon_{nm} + 1)\,(\varepsilon_{nm} + 1)} K^{(u,i,j)}_{kn,lk,nm} \xi(s)$$
$$\otimes a_l^* a_m \Omega_0 \tag{3.58c}$$
$$+ 2\lambda^3 L^{-3d} \sum_{(l,k)\in T_F} \sum_{(n,m)\in T_F} \frac{\hat{v}_{lk} \ \hat{v}_{lk} \ \hat{v}_{ml}}{(\varepsilon_{lk} + \varepsilon_{nm} + 1)\,(\varepsilon_{nm} + 1)} K^{(u,i,j)}_{ml,lk,nm} \xi(s)$$
$$\otimes a_n^* a_k \Omega_0 \,. \tag{3.58d}$$

Since we require compact support for $\hat{v}(k)$ for $k \in [-D, D]^d$, we find that $\hat{v}(k-n) \leq C$ and $\hat{v}(m-l) \leq C$ for some constant C. Then

$$\left\| P^{(1)} \mathbb{V}^{(u)} P^{(2)} R \mathbb{V}^{(i)} R \mathbb{V}^{(j)} \xi(s) \otimes \Omega_0 \right\|^2 \tag{3.59}$$
$$\lesssim \lambda^6 \left(L^{-2d} \sum_{(n,m)\in T_F} \frac{|\hat{v}(n-m)|^2}{n^2 - m^2 + 1} \right)^2 \left(L^{-2d} \sum_{(l,k)\in T_F} \frac{|\hat{v}(l-k)|^2}{(l^2 - k^2 + 1)^2} \right)$$
$$+ \lambda^6 \left(L^{-2d} \sum_{(n,m)\in T_F} \frac{|\hat{v}(n-m)|^2}{(n^2 - m^2 + 1)^2} \right) \left(L^{-2d} \sum_{(l,k)\in T_F} \frac{|\hat{v}(l-k)|^2}{l^2 - k^2 + 1} \right)^2$$
$$+ C\lambda^6 \left(L^{-2d} \sum_{(l,k)\in T_F} \frac{|\hat{v}(l-k)|^2}{(l^2 - k^2 + 1)^2} \right)^2 + C\lambda^6 \left(L^{-2d} \sum_{(l,k)\in T_F} \frac{|\hat{v}(l-k)|^2}{(l^2 - k^2 + 1)^2} \right)^2 ,$$

which in total with (A.43b) and (A.43c) gets us

$$\|\Phi_{32;31}(t)\| \lesssim |t|\,|\lambda|^3 \left(\frac{1}{1 + \frac{L}{k_F}} k_F^{(d-2)} + \mathcal{O}\left(\frac{\ln(k_F)}{k_F^2} \right) \right) \tag{3.60}$$

$$\cdot \left(L^{1/2} \cdot \frac{1}{\left(1+\frac{L}{k_F}\right)} k_F^{(d-3)/2} + \mathcal{O}\left(\frac{\ln(k_F)^{1/2}}{k_F} \right) \right) .$$

Term $\Phi_{32;32}(t)$: Next we consider the contribution with two holes. Using the identity

$$P^{(1)} a_h^* a_g a_l^* a_k a_n^* a_m \Omega_0 = \chi_{B_F^C}(h) \left(\delta_{n,g} a_h^* a_l^* a_k a_m \Omega_0 + \delta_{l,g} a_h^* a_k a_n^* a_m \Omega_0 \right) + \chi_{B_F}(g) \left(\delta_{m,h} a_g a_l^* a_k a_n^* \Omega_0 + \delta_{k,h} a_g a_l^* a_n^* a_m \Omega_0 \right) \tag{3.61}$$

we find

$$P^{(2)} \mathbb{V}^{(u)} P^{(2)} R \mathbb{V}^{(i)} R \mathbb{V}^{(j)} \xi(s) \otimes \Omega_0$$

$$= \lambda^3 L^{-3d} \sum_{h \in B_F^C} \sum_{(l,k) \in T_F} \sum_{(n,m) \in T_F} \frac{\hat{v}_{hn} \; \hat{v}_{lk} \; \hat{v}_{nm}}{(\varepsilon_{lk} + \varepsilon_{nm} + 1)(\varepsilon_{nm} + 1)} K^{(u,i,j)}_{hn,lk,nm} \xi(s) \otimes a_h^* a_l^* a_k a_m \Omega_0 \tag{3.62a}$$

$$+ \lambda^3 L^{-3d} \sum_{h \in B_F^C} \sum_{(l,k) \in T_F} \sum_{(n,m) \in T_F} \frac{\hat{v}_{hl} \; \hat{v}_{lk} \; \hat{v}_{nm}}{(\varepsilon_{lk} + \varepsilon_{nm} + 1)(\varepsilon_{nm} + 1)} K^{(u,i,j)}_{hl,lk,nm} \xi(s) \otimes a_h^* a_k a_n^* a_m \Omega_0 \tag{3.62b}$$

$$+ \lambda^3 L^{-3d} \sum_{g \in B_F} \sum_{(l,k) \in T_F} \sum_{(n,m) \in T_F} \frac{\hat{v}_{mg} \; \hat{v}_{lk} \; \hat{v}_{nm}}{(\varepsilon_{lk} + \varepsilon_{nm} + 1)(\varepsilon_{nm} + 1)} K^{(u,i,j)}_{mg,lk,nm} \xi(s) \otimes a_g a_l^* a_k a_n^* \Omega_0 \tag{3.62c}$$

$$+ \lambda^3 L^{-3d} \sum_{g \in B_F} \sum_{(l,k) \in T_F} \sum_{(n,m) \in T_F} \frac{\hat{v}_{kg} \; \hat{v}_{lk} \; \hat{v}_{nm}}{(\varepsilon_{lk} + \varepsilon_{nm} + 1)(\varepsilon_{nm} + 1)} K^{(u,i,j)}_{kg,lk,nm} \xi(s) \otimes a_g a_l^* a_n^* a_m \Omega_0 \, . \tag{3.62d}$$

We now estimate the first summand of the term

$$\begin{aligned}
&\|(3.62\text{a})\|^2 \qquad (3.63)\\
&\leq \lambda^6 \left(L^{-2d} \sum_{(n,m)\in T_F} \frac{|\hat{v}(n-m)|^2}{\left(n^2-m^2+1\right)^2} \right)\\
&\quad + \left(L^{-2d} \sum_{(h,k)\in T_F} \left(L^{-d} \sum_{l\in B_F^C} \frac{|\hat{v}(l-k)|\,|\hat{v}(l-h)|}{\left(n^2-m^2+1\right)^2} \right)^2 \right)\\
&\quad + \lambda^6 L^{-2d} \sum_{h,n\in B_F^C} \left(L^{-2d} \sum_{(l,k)\in T_F} \frac{|\hat{v}(n-k)|\,|\hat{v}(l-k)|\,|\hat{v}(h-l)|}{(n^2-k^2+1)\,(l^2-k^2+1)} \right)^2\\
&\quad + \lambda^6 L^{-2d} \sum_{m,k\in B_F} \left(L^{-2d} \sum_{n,l\in B_F^C} \frac{|\hat{v}(n-m)|\,|\hat{v}(l-k)|\,|\hat{v}(n-l)|}{(n^2-m^2+1)\,(l^2-k^2+1)} \right)^2\\
&\quad + \lambda^6 \left(L^{-3d} \sum_{(n,m)\in T_F} \sum_{l\in B_F} \frac{|\hat{v}(n-m)|\,|\hat{v}(l-m)|\,|\hat{v}(n-l)|}{(n^2-m^2+1)\,(l^2-m^2+1)} \right)^2 .
\end{aligned}$$

Analogously we find the same bounds for the other summands (3.62b)-(3.62d). Then with (A.43c), (A.43h), (A.43i), (A.43j) and (A.43k) This yields

$$\begin{aligned}
&\|\Phi_{32;32}(t)\| \qquad (3.64)\\
&\lesssim |t|\,|\lambda|^3 \left(L^{1/2} \cdot \frac{1}{\left(1+\frac{L}{k_F}\right)} k_F^{(d-3)/2} + \mathcal{O}\left(\frac{\ln(k_F)^{1/2}}{k_F} \right) \right) \left(k_F^{(d-3)/2} \ln(k_F) \right) .
\end{aligned}$$

Term $\Phi_{32;33}(t)$: We have

$$P^{(3)} a_h^* a_g a_l^* a_k a_n^* a_m \Omega_0 = \chi_{T_F}(h,g) a_h^* a_g a_l^* a_k a_n^* a_m \Omega_0 \,, \qquad (3.65)$$

for all $(n, m) \in T_F$ and $(l, k) \in T_F$. Therefore

$$\begin{aligned} &P^{(3)}\mathbb{V}^{(u)}P^{(2)}R\mathbb{V}^{(i)}R\mathbb{V}^{(j)}\xi(s)\otimes\Omega_0 \\ &= \lambda^3 L^{-3d} \sum_{(h,g)\in T_F}\sum_{(l,k)\in T_F}\sum_{(n,m)\in T_F} \frac{\hat{v}_{hg}\ \hat{v}_{lk}\ \hat{v}_{nm}}{(\varepsilon_{lk}+\varepsilon_{nm}+1)\,(\varepsilon_{nm}+1)} K^{(u,i,j)}_{hg,lk,nm}\xi(s) \\ &\qquad \otimes a_h^* a_g a_l^* a_k a_n^* a_m \Omega_0 \end{aligned} \tag{3.66}$$

and using

$$\begin{aligned} &\left|\frac{\hat{v}_{hg}\ \hat{v}_{lk}\ \hat{v}_{nm}}{(\varepsilon_{lk}+\varepsilon_{nm}+1)\,(\varepsilon_{nm}+1)}\cdot\frac{\hat{v}_{h'g'}\ \hat{v}_{l'k'}\ \hat{v}_{n'm'}}{(\varepsilon_{l'k'}+\varepsilon_{n'm'}+1)\,(\varepsilon_{n'm'}+1)}\right| \\ &\le \frac{|\hat{v}_{hg}\ \hat{v}_{lk}\ \hat{v}_{nm}|^2}{2\,(\varepsilon_{lk}+1)^2\,(\varepsilon_{nm}+1)^2} + \frac{|\hat{v}_{h'g'}\ \hat{v}_{l'k'}\ \hat{v}_{n'm'}|^2}{2\,(\varepsilon_{l'k'}+1)^2\,(\varepsilon_{n'm'}+1)^2}\,, \end{aligned} \tag{3.67}$$

we find

$$\begin{aligned} &\left\|P^{(3)}\mathbb{V}^{(u)}P^{(2)}R\mathbb{V}^{(i)}R\mathbb{V}^{(j)}\xi(s)\otimes\Omega_0\right\|^2 \\ &\le \lambda^6 L^{-6d} \sum_{\substack{(h,g)\in T_F\\(h',g')\in T_F}}\sum_{\substack{(l,k)\in T_F\\(l',k')\in T_F}}\sum_{\substack{(n,m)\in T_F\\(n'm')\in T_F}} \frac{|\hat{v}_{hg}\ \hat{v}_{lk}\ \hat{v}_{nm}|^2}{(\varepsilon_{lk}+1)^2\,(\varepsilon_{nm}+1)^2} \\ &\quad \times \left|\langle a_h^* a_g a_l^* a_k a_n^* a_m\Omega_0,\, a_{h'}^* a_{g'} a_{l'}^* a_{k'} a_{n'}^* a_{m'}\Omega_0\rangle_{\mathcal{H}_N^-}\right| \\ &\le 36\ \lambda^6 \left(L^{-2d}\sum_{(h,g)\in T_F}|\hat{v}_{hg}|^2\right)\left(L^{-2d}\sum_{(l,k)\in T_F}\frac{|\hat{v}_{lk}|^2}{(\varepsilon_{lk}+1)^2}\right)\left(L^{-2d}\sum_{(n,m)\in T_F}\frac{|\hat{v}_{nm}|^2}{(\varepsilon_{nm}+1)^2}\right). \end{aligned} \tag{3.68}$$

Therefore with (A.43a) and (A.43c) we find

$$\begin{aligned} &\|\Phi_{32;33}(t)\| \\ &\lesssim |t|\,|\lambda|^3\left(\frac{1}{L^{1/2}}\,k_F^{(d-1)/2}+\mathcal{O}(1)\right)\left(L\cdot\frac{1}{\left(1+\frac{L}{k_F}\right)^2}\,k_F^{d-3}+\mathcal{O}\left(\frac{\ln(k_F)}{k_F^2}\right)\right) \end{aligned} \tag{3.69}$$

Term $\Phi_{32;40}(t)$: Using the identity

$$P^{(0)} a_p^* a_q^* a_{p-k} a_{q+k} a_h^* a_g a_n^* a_m \Omega_0 = \delta_{g,p}\delta_{h,p-k}\delta_{m,q}\delta_{n,q+k}\Omega_0 + \delta_{g,q}\delta_{h,q+k}\delta_{m,p}\delta_{n,p-k}\Omega_0 \,, \tag{3.70}$$

for all $(n,m) \in T_F$ and $(h,g) \in T_F$, we find

$$\begin{aligned} &\langle \Omega_0, P^{(0)} \mathbb{W} R P^{(2)} \mathbb{V}^{(i)} R \mathbb{V}^{(j)} \Omega_0 \rangle \qquad (3.71)\\ &= \kappa \lambda^2 L^{-3d} \sum_{(n,m)\in T_F} \sum_{(h,g)\in T_F} \frac{|\hat{w}(n-m)|\ |\hat{v}(n-m)|\ |\hat{v}(h-g)|}{(\varepsilon_{hg} + \varepsilon_{nm} + 1)\,(\varepsilon_{nm}+1)} e^{-i(n-m)y_i - i(h-g)y_j} \\ &= \kappa \lambda^2 L^{-3d} \sum_{(n,m)\in T_F} \sum_{(h,g)\in T_F} \frac{|\hat{w}(n-m)|\ |\hat{v}(n-m)|^2}{(\varepsilon_{hg} + \varepsilon_{nm} + 1)\,(\varepsilon_{nm}+1)} e^{-i(n-m)(y_i+y_j)} \,. \end{aligned}$$

This yields

$$\begin{aligned} &\Phi_{32;40} \\ &= -\, i \sum_{i,j=1}^{n} \int_0^t \mathrm{d}s\ e^{i\mathbb{H}s} P^{(0)} \mathbb{W} R P^{(2)} \mathbb{V}^{(i)} R \mathbb{V}^{(j)} \xi(s) \otimes \Omega_0 \qquad (3.72)\\ &= -\, i \sum_{i=1}^{n} \int_0^t \mathrm{d}s\ e^{i\mathbb{H}s} \left(\kappa \lambda^2 L^{-3d} \sum_{(n,m)\in T_F} \sum_{(h,g)\in T_F} \frac{|\hat{w}_{nm}|\ |\hat{v}_{nm}|\ |\hat{v}_{hg}|}{(\varepsilon_{hg} + \varepsilon_{nm} + 1)\,(\varepsilon_{nm}+1)} \right) \xi(s) \otimes \Omega_0 \\ &\quad - i \sum_{i<j}^{n} \int_0^t \mathrm{d}s\ e^{i\mathbb{H}s} \Bigg(\kappa \lambda^2 L^{-3d} \sum_{(n,m)\in T_F} \sum_{(h,g)\in T_F} \frac{|\hat{w}_{nm}|\ |\hat{v}_{nm}|\ |\hat{v}_{hg}|}{(\varepsilon_{hg} + \varepsilon_{nm} + 1)\,(\varepsilon_{nm}+1)} \\ &\qquad \cdot \cos\left((n-m)(y_i+y_j)\right) \Bigg) \xi(s) \otimes \Omega_0 \,. \end{aligned}$$

This contribution represents the second component, following Φ_{30}, that aligns with Ω_0 and produces zero holes in the Fermi sea. However, due to the specific choice of κ, this term remains small in comparison to Φ_{30}, making it negligible in determining the effective Hamiltonian h_n^{eff}, which we can see by computing the norm

$$\left\|P^{(0)}\mathbb{W}RP^{(2)}\mathbb{V}^{(i)}R\mathbb{V}^{(j)}\xi(t)\otimes\Omega_0\right\| \leq \kappa\lambda^2 L^d \left(L^{-2d}\sum_{(n,m)\in T_F}\frac{|\hat{v}(n-m)|}{(\varepsilon_{nm}+1)}\right)^2 . \quad (3.73)$$

Then we have with (A.43b),

$$\|\Phi_{32;40}(t)\| \lesssim |t|\,|\kappa|\,\lambda^2\left(L^d\cdot\frac{1}{1+\frac{L}{k_F}}\,k_F^{(d-2)} + L^d\cdot\mathcal{O}\left(\frac{\ln(k_F)}{k_F^2}\right)\right). \quad (3.74)$$

Term $\Phi_{32;41}(t)$: We find

$$\begin{aligned}P^{(1)}a_p^*a_q^*a_{p-k}a_{q+k}a_h^*a_g a_n^*a_m\Omega_0 &= \delta_{n,p-k}\delta_{h,q+k}\delta_{g,q}\chi_{M_k^F}(p)a_p^*a_m\Omega_0 \quad (3.75)\\ &+ \delta_{n,q+k}\delta_{h,p-k}\delta_{g,p}\chi_{M_k^F}(q)a_q^*a_m\Omega_0\\ &+ \delta_{h,p-k}\delta_{m,q}\delta_{n,q+k}\chi_{M_k^F}(p)a_p^*a_g\Omega_0\\ &+ \delta_{h,q+k}\delta_{m,p}\delta_{n,p-k}\chi_{M_k^F}(q)a_q^*a_g\Omega_0\\ &+ \delta_{m,p}\delta_{h,q+k}\delta_{g,q}\chi_{\widetilde{M}_k^F}(p)a_{p-k}a_n^*\Omega_0\\ &+ \delta_{m,q}\delta_{h,p-k}\delta_{g,p}\chi_{\widetilde{M}_k^F}(q)a_{q+k}a_n^*\Omega_0\\ &+ \delta_{g,p}\delta_{m,q}\delta_{n,q+k}\chi_{\widetilde{M}_k^F}(p)a_{p-k}a_h^*\Omega_0\\ &+ \delta_{g,q}\delta_{m,p}\delta_{n,p-k}\chi_{\widetilde{M}_k^F}(q)a_{q+k}a_h^*\Omega_0\,,\end{aligned}$$

for $(n,m)\in T_F$ and for $(n,m)\in T_F$. Therefore

$$\begin{aligned}&P^{(1)}\mathbb{W}RP^{(2)}\mathbb{V}^{(i)}R\mathbb{V}^{(j)}\xi(t)\otimes\Omega_0\\ &=\kappa\lambda^2L^{-3d}\sum_{p\in M_k^F}\sum_{k\in[-D,D]^d}\sum_{(h,g)\in T_F}\sum_{(n,m)\in T_F}\frac{\hat{w}_{pn}\ \hat{v}_{hg}\ \hat{v}_{nm}}{(\varepsilon_{hg}+\varepsilon_{nm}+1)\,(\varepsilon_{nm}+1)}\end{aligned}$$

$$\cdot K^{(i,j)}_{hg,nm}\xi(t) \otimes a^*_p a_m \Omega_0 \tag{3.76a}$$

$$+ \kappa\lambda^2 L^{-3d} \sum_{q\in M_k^F} \sum_{k\in[-D,D]^d} \sum_{(h,g)\in T_F} \sum_{(n,m)\in T_F} \frac{\hat{w}_{pn}\ \hat{v}_{hg}\ \hat{v}_{nm}}{(\varepsilon_{hg}+\varepsilon_{nm}+1)(\varepsilon_{nm}+1)}$$

$$\cdot K^{(i,j)}_{hg,nm}\xi(t) \otimes a^*_q a_m \Omega_0 \tag{3.76b}$$

$$+ \kappa\lambda^2 L^{-3d} \sum_{p\in M_k^F} \sum_{k\in[-D,D]^d} \sum_{(h,g)\in T_F} \sum_{(n,m)\in T_F} \frac{\hat{w}_{pn}\ \hat{v}_{hg}\ \hat{v}_{nm}}{(\varepsilon_{hg}+\varepsilon_{nm}+1)(\varepsilon_{nm}+1)}$$

$$\cdot K^{(i,j)}_{hg,nm}\xi(t) \otimes a^*_p a_g \Omega_0 \tag{3.76c}$$

$$+ \kappa\lambda^2 L^{-3d} \sum_{q\in M_k^F} \sum_{k\in[-D,D]^d} \sum_{(h,g)\in T_F} \sum_{(n,m)\in T_F} \frac{\hat{w}_{pn}\ \hat{v}_{hg}\ \hat{v}_{nm}}{(\varepsilon_{hg}+\varepsilon_{nm}+1)(\varepsilon_{nm}+1)}$$

$$\cdot K^{(i,j)}_{hg,nm}\xi(t) \otimes a^*_q a_g \Omega_0 \tag{3.76d}$$

$$+ \kappa\lambda^2 L^{-3d} \sum_{p\in M_k^F} \sum_{k\in[-D,D]^d} \sum_{(h,g)\in T_F} \sum_{(n,m)\in T_F} \frac{\hat{w}_{pn}\ \hat{v}_{hg}\ \hat{v}_{nm}}{(\varepsilon_{hg}+\varepsilon_{nm}+1)(\varepsilon_{nm}+1)}$$

$$\cdot K^{(i,j)}_{hg,nm}\xi(t) \otimes a_{p-k} a^*_n \Omega_0 \tag{3.76e}$$

$$+ \kappa\lambda^2 L^{-3d} \sum_{q\in M_k^F} \sum_{k\in[-D,D]^d} \sum_{(h,g)\in T_F} \sum_{(n,m)\in T_F} \frac{\hat{w}_{pn}\ \hat{v}_{hg}\ \hat{v}_{nm}}{(\varepsilon_{hg}+\varepsilon_{nm}+1)(\varepsilon_{nm}+1)}$$

$$\cdot K^{(i,j)}_{hg,nm}\xi(t) \otimes a_{q+k} a^*_n \Omega_0 \tag{3.76f}$$

$$+ \kappa\lambda^2 L^{-3d} \sum_{p\in M_k^F} \sum_{k\in[-D,D]^d} \sum_{(h,g)\in T_F} \sum_{(n,m)\in T_F} \frac{\hat{w}_{pn}\ \hat{v}_{hg}\ \hat{v}_{nm}}{(\varepsilon_{hg}+\varepsilon_{nm}+1)(\varepsilon_{nm}+1)}$$

$$\cdot K^{(i,j)}_{hg,nm}\xi(t) \otimes a_{p-k} a^*_h \Omega_0 \tag{3.76g}$$

$$+ \kappa\lambda^2 L^{-3d} \sum_{q\in M_k^F} \sum_{k\in[-D,D]^d} \sum_{(h,g)\in T_F} \sum_{(n,m)\in T_F} \frac{\hat{w}_{pn}\ \hat{v}_{hg}\ \hat{v}_{nm}}{(\varepsilon_{hg}+\varepsilon_{nm}+1)(\varepsilon_{nm}+1)}$$

$$\cdot K^{(i,j)}_{hg,nm}\xi(t) \otimes a_{q+k} a^*_h \Omega_0 \,. \tag{3.76h}$$

Looking at the norm we have

$$\|(3.76a)\|^2 \tag{3.77}$$

$$\lesssim \kappa^2\lambda^4 \left(L^{-2d} \sum_{p\in M_k^F} \sum_{k\in[-D,D]^d} \right) \tag{3.78}$$

$$\cdot \left(L^{-2d} \sum_{m \in B_F} \left(L^{-d} \sum_{h,g \in T_F} \sum_{n \in B_F^C} \frac{|\hat{v}_{hg}| \, |\hat{v}_{nm}|}{(\varepsilon_{hg}+1)\,(\varepsilon_{nm}+1)} \right)^2 \right) .$$

Analogously we derive at the same bounds for the other summands (3.76b)-(3.76h). Therefore with, (A.43p) and (A.43l),

$$\|\Phi_{32;41}(t)\| \lesssim |t|\,|\kappa|\,\lambda^2 \left(\frac{1}{L^{(d-1)/2}} \cdot \left(k_F + \frac{\sqrt{d}}{2L} \right)^{(d-1)/2} \right) \cdot \left(k_F^{(d-1)/2} \ln(k_F) \cdot \left(\frac{1}{1+\frac{L}{k_F}} k_F^{(d-3)} + \mathcal{O}\left(\frac{\ln(k_F)}{k_F^3} \right) \right) \right) . \tag{3.79}$$

Term $\Phi_{32;42}(t)$: Here we have

$$\begin{aligned} P^{(2)} a_p^* a_q^* a_{p-k} a_{q+k} a_h^* a_g a_n^* a_m \Omega_0 = {} & \delta_{n,p-k}\delta_{h,q+k}\chi_{N_k^F}\Big((q,p)\Big) a_p^* a_q^* a_g a_m \Omega_0 \\ & + \delta_{n,q+k}\delta_{h,p-k}\chi_{N_k^F}\Big((p,q)\Big) a_p^* a_q^* a_g a_m \Omega_0 \\ & + \delta_{m,p}\delta_{g,q}\chi_{\widetilde{N}_k^F}\Big((p-k,q+k)\Big) a_{p-k} a_{q+k} a_h^* a_n^* \Omega_0 \\ & + \delta_{m,q}\delta_{g,p}\chi_{\widetilde{N}_k^F}\Big((q+k,p-k)\Big) a_{p-k} a_{q+k} a_h^* a_n^* \Omega_0 \\ & + \delta_{m,q}\delta_{n,q+k}\chi_{T_F}\Big((p,p-k)\Big) a_p^* a_{p-k} a_h^* a_g \Omega_0 \\ & + \delta_{m,p}\delta_{n,p-k}\chi_{T_F}\Big((q,q+k)\Big) a_q^* a_{q+k} a_h^* a_g \Omega_0 \\ & + \delta_{g,q}\delta_{h,q+k}\chi_{T_F}\Big((p,p-k)\Big) a_p^* a_{p-k} a_n^* a_m \Omega_0 \\ & + \delta_{g,p}\delta_{h,p-k}\chi_{T_F}\Big((q,q+k)\Big) a_q^* a_{q+k} a_n^* a_m \Omega_0 \, , \end{aligned} \tag{3.80}$$

for all $(n,m) \in T_F$ and $(h,g) \in T_F$. Therefore

$$P^{(2)} \mathbb{W} R P^{(2)} \mathbb{V}^{(i)} R \mathbb{V}^{(j)} \xi(t) \otimes \Omega_0$$

$$= \kappa\lambda^2 L^{-3d} \sum_{(q,p)\in N_k^F} \sum_{k\in[-D,D]^d} \sum_{(h,g)\in T_F} \sum_{(n,m)\in T_F} \frac{\hat{w}_{pn}\ \hat{v}_{hg}\ \hat{v}_{nm}}{(\varepsilon_{hg}+\varepsilon_{nm}+1)(\varepsilon_{nm}+1)} \cdot K^{(i,j)}_{hg,nm}\xi(t)\otimes a_p^* a_q^* a_g a_m \Omega_0 \tag{3.81a}$$

$$+ \kappa\lambda^2 L^{-3d} \sum_{(q,p)\in N_k^F} \sum_{k\in[-D,D]^d} \sum_{(h,g)\in T_F} \sum_{(n,m)\in T_F} \frac{\hat{w}_{nq}\ \hat{v}_{hg}\ \hat{v}_{nm}}{(\varepsilon_{hg}+\varepsilon_{nm}+1)(\varepsilon_{nm}+1)} \cdot K^{(i,j)}_{hg,nm}\xi(t)\otimes a_p^* a_q^* a_g a_m \Omega_0 \tag{3.81b}$$

$$+ \kappa\lambda^2 L^{-3d} \sum_{(q,p)\in N_k^F} \sum_{k\in[-D,D]^d} \sum_{(h,g)\in T_F} \sum_{(n,m)\in T_F} \frac{\hat{w}_{k}\ \hat{v}_{hg}\ \hat{v}_{nm}}{(\varepsilon_{hg}+\varepsilon_{nm}+1)(\varepsilon_{nm}+1)} \cdot K^{(i,j)}_{hg,nm}\xi(t)\otimes a_{p-k} a_{q+k} a_h^* a_n^* \Omega_0 \tag{3.81c}$$

$$+ \kappa\lambda^2 L^{-3d} \sum_{(q,p)\in N_k^F} \sum_{k\in[-D,D]^d} \sum_{(h,g)\in T_F} \sum_{(n,m)\in T_F} \frac{\hat{w}_{k}\ \hat{v}_{hg}\ \hat{v}_{nm}}{(\varepsilon_{hg}+\varepsilon_{nm}+1)(\varepsilon_{nm}+1)} \cdot K^{(i,j)}_{hg,nm}\xi(t)\otimes a_{p-k} a_{q+k} a_h^* a_n^* \Omega_0 \tag{3.81d}$$

$$+ \kappa\lambda^2 L^{-3d} \sum_{p\in M_k^F} \sum_{k\in[-D,D]^d} \sum_{(h,g)\in T_F} \sum_{(n,m)\in T_F} \frac{\hat{w}_{k}\ \hat{v}_{hg}\ \hat{v}_{nm}}{(\varepsilon_{hg}+\varepsilon_{nm}+1)(\varepsilon_{nm}+1)} \cdot K^{(i,j)}_{hg,nm}\xi(t)\otimes a_p^* a_{p-k} a_h^* a_g \Omega_0 \tag{3.81e}$$

$$+ \kappa\lambda^2 L^{-3d} \sum_{q\in M_k^F} \sum_{k\in[-D,D]^d} \sum_{(h,g)\in T_F} \sum_{(n,m)\in T_F} \frac{\hat{w}_{k}\ \hat{v}_{hg}\ \hat{v}_{nm}}{(\varepsilon_{hg}+\varepsilon_{nm}+1)(\varepsilon_{nm}+1)} \cdot K^{(i,j)}_{hg,nm}\xi(t)\otimes a_q^* a_{q+k} a_h^* a_g \Omega_0 \tag{3.81f}$$

$$+ \kappa\lambda^2 L^{-3d} \sum_{p\in M_k^F} \sum_{k\in[-D,D]^d} \sum_{(h,g)\in T_F} \sum_{(n,m)\in T_F} \frac{\hat{w}_{k}\ \hat{v}_{hg}\ \hat{v}_{nm}}{(\varepsilon_{hg}+\varepsilon_{nm}+1)(\varepsilon_{nm}+1)} \cdot K^{(i,j)}_{hg,nm}\xi(t)\otimes a_p^* a_{p-k} a_n^* a_m \Omega_0 \tag{3.81g}$$

$$+ \kappa\lambda^2 L^{-3d} \sum_{q\in M_k^F} \sum_{k\in[-D,D]^d} \sum_{(h,g)\in T_F} \sum_{(n,m)\in T_F} \frac{\hat{w}_{k}\ \hat{v}_{hg}\ \hat{v}_{nm}}{(\varepsilon_{hg}+\varepsilon_{nm}+1)(\varepsilon_{nm}+1)} \cdot K^{(i,j)}_{hg,nm}\xi(t)\otimes a_q^* a_{q+k} a_n^* a_m \Omega_0 \,. \tag{3.81h}$$

Then

$$\|(3.81a)\|^2 \tag{3.82}$$

$$\lesssim \kappa^2\lambda^4 \left(L^{-2d} \sum_{(q,p)\in N_k^F} \sum_{k\in[-D,D]^d} \right) \cdot L^{2d} \left(L^{-2d} \sum_{g,m\in B_F} \left(L^{-2d} \sum_{h,n\in B_F^C} \frac{|\hat{v}_{hg}|\ |\hat{v}_{nm}|}{(\varepsilon_{hg}+1)(\varepsilon_{nm}+1)} \right)^2 \right) \tag{3.83}$$

and in close analogy we have the same bound for (3.81b)-(3.81d). Here, the scalarproduct does not lead to more summands, because of the compact support of $\hat{v}$ and $\hat{w}$. Similarly

$$\|(3.81e)\|^2 \tag{3.84}$$

$$\lesssim 2\kappa^2\lambda^4 \left(L^{-2d} \sum_{p\in M_k^F} \sum_{k\in[-D,D]^d} |\hat{w}_k|^2 \right) \left(L^{-2d} \sum_{(h,g)\in T_F} \frac{|\hat{v}_{hg}|^2}{(\varepsilon_{hg}+1)^2} \right) \cdot L^{2d} \left(L^{-2d} \sum_{(n,m)\in T_F} \frac{|\hat{v}_{nm}|}{(\varepsilon_{nm}+1)} \right)^2 \tag{3.85}$$

$$+\ 2\kappa^2\lambda^4 \left(L^{-2d} \sum_{k\in[-D,D]^d} \sum_{(p,g)\in T_F} \sum_{h\in B_F^C} \frac{|\hat{w}_{h(p-k)}|\ |\hat{v}_{pg}|}{(\varepsilon_{pg}+1)} \right)^2 \cdot L^{2d} \left(L^{-2d} \sum_{(n,m)\in T_F} \frac{|\hat{v}_{nm}|}{(\varepsilon_{nm}+1)} \right)^2, \tag{3.86}$$

which we also find similarly for (3.81f)-(3.81h). Together with (A.43q) and (A.43j), as well as (A.43b), (A.43c), (A.43n) and (A.43t), this yields

$$\|\Phi_{32;42}(t)\| \tag{3.87}$$

$$\lesssim |t|\,|\kappa|\,\lambda^2 \left(L^{-1/2}k_F^{(d-1)/2}\right) L^d \left(L^{1/2}\frac{1}{\left(1+\frac{L}{k_F}\right)}k_F^{(d-3)/2} + \mathcal{O}\left(\frac{\ln(k_F)^{1/2}}{k_F}\right)\right) \cdot \left(\frac{1}{1+\frac{L}{k_F}}k_F^{(d-2)} + \mathcal{O}\left(\frac{\ln(k_F)}{k_F^2}\right)\right)$$

Term $\Phi_{32;43}(t)$: Here we have

$$
\begin{aligned}
P^{(3)} a_p^* a_q^* a_{p-k} a_{q+k} a_g^* a_h a_n^* a_m \Omega_0 = {} & \delta_{n,q+k} \chi_{\widetilde{M}_k^F}(p-k)) a_p^* a_q^* a_{p-k} a_h^* a_g a_m \Omega_0 \\
& + \delta_{n,p-k} \chi_{\widetilde{M}_k^F}(q+k) a_p^* a_q^* a_{q+k} a_h^* a_g a_m \Omega_0 \\
& + \delta_{h,q+k} \chi_{\widetilde{M}_k^F}(p-k) a_p^* a_q^* a_{p-k} a_g a_n^* a_m \Omega_0 \\
& + \delta_{h,p-k} \chi_{\widetilde{M}_k^F}(q+k) a_p^* a_q^* a_{q+k} a_g a_n^* a_m \Omega_0 \\
& + \delta_{m,q} \chi_{\widetilde{N}_k^F}\Big((p-k, q+k)\Big) a_p^* a_{p-k} a_{q+k} a_h^* a_g a_n^* \Omega_0 \\
& + \delta_{m,p} \chi_{\widetilde{N}_k^F}\Big((q+k, p-k)\Big) a_q^* a_{p-k} a_{q+k} a_h^* a_g a_n^* \Omega_0 \\
& + \delta_{g,q} \chi_{\widetilde{N}_k^F}\Big((p-k, q+k)\Big) a_p^* a_{p-k} a_{q+k} a_h^* a_n^* a_m \Omega_0 \\
& + \delta_{g,p} \chi_{\widetilde{N}_k^F}\Big((q+k, p-k)\Big) a_q^* a_{p-k} a_{q+k} a_h^* a_n^* a_m \Omega_0 \,,
\end{aligned} \tag{3.88}
$$

for all $(n,m) \in T_F$ and $(h,g) \in T_F$. Then

$$
\begin{aligned}
& P^{(3)} \mathbb{W} R P^{(2)} \mathbb{V}^{(i)} R \mathbb{V}^{(j)} \xi(t) \otimes \Omega_0 \\
= {} & \kappa \lambda^2 L^{-3d} \sum_{(q,p) \in N_k^F} \sum_{k \in [-D,D]^d} \sum_{(h,g) \in T_F} \sum_{(n,m) \in T_F} \frac{\hat{w}_k \ \hat{v}_{hg} \ \hat{v}_{nm}}{(\varepsilon_{hg} + \varepsilon_{nm} + 1)\,(\varepsilon_{nm} + 1)} \\
& \qquad \cdot K_{hg,nm}^{(i,j)} \xi(t) \otimes a_p^* a_q^* a_{p-k} a_h^* a_g a_m \Omega_0 \qquad (3.89a) \\
& + \kappa \lambda^2 L^{-3d} \sum_{(q,p) \in N_k^F} \sum_{k \in [-D,D]^d} \sum_{(h,g) \in T_F} \sum_{(n,m) \in T_F} \frac{\hat{w}_k \ \hat{v}_{hg} \ \hat{v}_{nm}}{(\varepsilon_{hg} + \varepsilon_{nm} + 1)\,(\varepsilon_{nm} + 1)} \\
& \qquad \cdot K_{hg,nm}^{(i,j)} \xi(t) \otimes a_p^* a_q^* a_{q+k} a_h^* a_g a_m \Omega_0 \qquad (3.89b) \\
& + \kappa \lambda^2 L^{-3d} \sum_{(q,p) \in N_k^F} \sum_{k \in [-D,D]^d} \sum_{(h,g) \in T_F} \sum_{(n,m) \in T_F} \frac{\hat{w}_k \ \hat{v}_{hg} \ \hat{v}_{nm}}{(\varepsilon_{hg} + \varepsilon_{nm} + 1)\,(\varepsilon_{nm} + 1)} \\
& \qquad \cdot K_{hg,nm}^{(i,j)} \xi(t) \ \otimes a_p^* a_q^* a_{p-k} a_g a_n^* a_m \Omega_0 \qquad (3.89c) \\
& + \kappa \lambda^2 L^{-3d} \sum_{(q,p) \in N_k^F} \sum_{k \in [-D,D]^d} \sum_{(h,g) \in T_F} \sum_{(n,m) \in T_F} \frac{\hat{w}_k \ \hat{v}_{hg} \ \hat{v}_{nm}}{(\varepsilon_{hg} + \varepsilon_{nm} + 1)\,(\varepsilon_{nm} + 1)}
\end{aligned}
$$

$$\cdot K^{(i,j)}_{hg,nm}\xi(t) \otimes a^*_p a^*_q a_{q+k} a_g a^*_n a_m \Omega_0 \tag{3.89d}$$

$$+ \kappa\lambda^2 L^{-3d} \sum_{(q,p)\in N^F_k} \sum_{k\in[-D,D]^d} \sum_{(h,g)\in T_F} \sum_{(n,m)\in T_F} \frac{\hat{w}_k \ \hat{v}_{hg} \ \hat{v}_{nm}}{(\varepsilon_{hg} + \varepsilon_{nm} + 1)\,(\varepsilon_{nm} + 1)}$$

$$\cdot K^{(i,j)}_{hg,nm}\xi(t) \otimes a^*_p a_{p-k} a_{q+k} a^*_h a_g a^*_n \Omega_0 \tag{3.89e}$$

$$+ \kappa\lambda^2 L^{-3d} \sum_{(q,p)\in N^F_k} \sum_{k\in[-D,D]^d} \sum_{(h,g)\in T_F} \sum_{(n,m)\in T_F} \frac{\hat{w}_k \ \hat{v}_{hg} \ \hat{v}_{nm}}{(\varepsilon_{hg} + \varepsilon_{nm} + 1)\,(\varepsilon_{nm} + 1)}$$

$$\cdot K^{(i,j)}_{hg,nm}\xi(t) \otimes a^*_q a_{p-k} a_{q+k} a^*_h a_g a^*_n \Omega_0 \tag{3.89f}$$

$$+ \kappa\lambda^2 L^{-3d} \sum_{(q,p)\in N^F_k} \sum_{k\in[-D,D]^d} \sum_{(h,g)\in T_F} \sum_{(n,m)\in T_F} \frac{\hat{w}_k \ \hat{v}_{hg} \ \hat{v}_{nm}}{(\varepsilon_{hg} + \varepsilon_{nm} + 1)\,(\varepsilon_{nm} + 1)}$$

$$\cdot K^{(i,j)}_{hg,nm}\xi(t) \otimes a^*_p a_{p-k} a_{q+k} a^*_h a^*_n a_m \Omega_0 \tag{3.89g}$$

$$+ \kappa\lambda^2 L^{-3d} \sum_{(q,p)\in N^F_k} \sum_{k\in[-D,D]^d} \sum_{(h,g)\in T_F} \sum_{(n,m)\in T_F} \frac{\hat{w}_k \ \hat{v}_{hg} \ \hat{v}_{nm}}{(\varepsilon_{hg} + \varepsilon_{nm} + 1)\,(\varepsilon_{nm} + 1)}$$

$$\cdot K^{(i,j)}_{hg,nm}\xi(t) \otimes a^*_q a_{p-k} a_{q+k} a^*_h a^*_n a_m \Omega_0 \tag{3.89h}$$

Then

$$\left\|(3.89\text{a})\right\|^2 \tag{3.90}$$

$$\lesssim \kappa^2\lambda^4 \left(L^{-2d} \sum_{(q,p)\in N^F_k} \sum_{k\in[-D,D]^d} |\hat{w}_k|^2 \right)$$

$$\cdot L^{2d} \left(L^{-2d} \sum_{m\in B_F} \sum_{(h,g)\in T_F} \left(L^{-2d} \sum_{n\in B^C_F} \frac{|\hat{v}_{hg}| \ |\hat{v}_{nm}|}{(\varepsilon_{hg} + 1)\,(\varepsilon_{nm} + 1)} \right)^2 \right)$$

$$+ \kappa^2\lambda^4 \left(L^{-2d} \sum_{p\in M^F_k} \sum_{k\in[-D,D]^d} |\hat{w}_k|^2 \right)$$

$$\cdot L^{2d} \left(L^{-2d} \sum_{q,h\in B^C_F} \left(L^{-2d} \sum_{(n,m)\in T_F} \frac{|\hat{v}_{hm}| \ |\hat{v}_{nm}| \ |\hat{v}_{qn}|}{(\varepsilon_{hm} + 1)\,(\varepsilon_{nm} + 1)} \right)^2 \right)$$

$$+ \kappa^2\lambda^4 \left(L^{-2d} \sum_{p\in M^F_k} \sum_{k\in[-D,D]^d} |\hat{w}_k|^2 \right)$$

$$\cdot L^{2d}\left(L^{-2d}\sum_{g,m\in B_F}\left(L^{-2d}\sum_{h,n\in B_F^C}\frac{|\hat{v}_{hg}|\;|\hat{v}_{nm}|\;|\hat{v}_{hn}|}{(\varepsilon_{hg}+1)\,(\varepsilon_{nm}+1)}\right)^2\right)$$
$$+\kappa^2\lambda^4\left(L^{-2d}\sum_{p\in M_k^F}\sum_{k\in[-D,D]^d}|\hat{w}_k|^2\right)\left(L^{-2d}\sum_{(h,g)\in T_F}\sum_{n\in B_F}\frac{|\hat{v}_{hg}|\;|\hat{v}_{ng}|\;|\hat{v}_{hn}|}{(\varepsilon_{hg}+1)\,(\varepsilon_{nm}+1)}\right)^2.$$

Analogously we find the same bounds for the other summands (3.89b)-(3.89h). Then with (A.43i), (A.43j), (A.43k), (A.43m), (A.43n) and (A.43u) this yields

$$\|\Phi_{32;43}(t)\|\lesssim|t|\,|\kappa|\,\lambda^2\left(L^{-1/2}k_F^{(d-1)/2}+\mathcal{O}(1)\right)\cdot L^d\left(k_F^{(d-3)}\ln(k_F)^2\right)$$

Term $\Phi_{32;44}(t)$: We have

$$P^{(4)}a_p^*a_q^*a_{p-k}a_{q+k}a_h^*a_ga_n^*a_m\Omega_0=\chi_{N_k^F}\Big((p,q)\Big)a_p^*a_q^*a_{p-k}a_{q+k}a_h^*a_ga_n^*a_m\Omega_0 \qquad (3.91)$$

for all $(n,m)\in T_F$ and $(h,g)\in T_F$. Using the inequality

$$\left|\frac{\hat{w}_k\;\hat{v}_{hg}\;\hat{v}_{nm}}{(\varepsilon_{hg}+\varepsilon_{nm}+1)\,(\varepsilon_{nm}+1)}\cdot\frac{\hat{w}_{k'}\;\hat{v}_{h'g'}\;\hat{v}_{n'm'}}{(\varepsilon_{h'g'}+\varepsilon_{n'm'}+1)\,(\varepsilon_{n'm'}+1)}\right| \qquad (3.92)$$
$$\leq\frac{|\hat{w}_k\;\hat{v}_{hg}\;\hat{v}_{nm}|^2}{2\,(\varepsilon_{hg}+1)^2\,(\varepsilon_{nm}+1)^2}+\frac{|\hat{w}_{k'}\;\hat{v}_{h'g'}\;\hat{v}_{n'm'}|^2}{2\,(\varepsilon_{h'g'}+1)^2\,(\varepsilon_{n'm'}+1)^2}.$$

we find

$$\left\|P^{(4)}\mathbb{W}RP^{(2)}\mathbb{V}^{(i)}R\mathbb{V}^{(j)}\xi(t)\otimes\Omega_0\right\|^2 \qquad (3.93)$$
$$\leq\kappa^2\lambda^4L^{-6d}\sum_{(p,q)\in N_k^F}\sum_{k\in[-D,D]^d}\sum_{(n,m)\in T_F}\sum_{(h,g)\in T_F}\frac{|\hat{w}_k\;\hat{v}_{hg}\;\hat{v}_{nm}|^2}{(\varepsilon_{hg}+1)^2\,(\varepsilon_{nm}+1)^2}\xi(t)$$
$$\times\left|\langle a_p^*a_q^*a_{p-k}a_{q+k}a_h^*a_ga_n^*a_m\Omega_0,\,a_{p'}^*a_{q'}^*a_{p'-k'}a_{q'+k'}a_{h'}^*a_{g'}a_{n'}^*a_{m'}\Omega_0\rangle_{\mathcal{H}_N^-}\right|$$

$$\lesssim \kappa^2 \lambda^4 \left(L^{-2d} \sum_{(p,q)\in N_k^F} \sum_{k\in[-D,D]^d} |\hat{w}_k|^2 \right)$$
$$\cdot \left(L^{-2d} \sum_{(h,g)\in T_F} \frac{|\hat{v}_{hg}|^2}{(\varepsilon_{hg}+1)^2} \right) \left(L^{-2d} \sum_{(n,m)\in T_F} \frac{|\hat{v}_{nm}|^2}{(\varepsilon_{nm}+1)^2} \right) .$$

Therefore with (A.43c) and (A.43m)

$$\|\Phi_{32;44}(t)\| \tag{3.94}$$
$$\lesssim |t|\,|\kappa|\,\lambda^2 \left(L^{d-1} k_F^{(d-1)} + \; L^d \cdot \mathcal{O}(1) \right) \left(L \cdot \frac{1}{\left(1+\frac{L}{k_F}\right)^2} \, k_F^{(d-3)} + \mathcal{O}\left(\frac{\ln(k_F)}{k_F^2} \right) \right) .$$

In summary we obtain with (2.65) the decomposition

$$\|\Phi(t)\| \tag{3.95}$$
$$\lesssim |\lambda| \left(L^{1/2} \cdot \frac{1}{\left(1+\frac{L}{k_F}\right)} \, k_F^{(d-3)/2} + \mathcal{O}\left(\frac{\ln(k_F)^{1/2}}{k_F} \right) \right)$$
$$+ |t|\,|\lambda| \left(L^{1/2} \cdot \frac{1}{\left(1+\frac{L}{k_F}\right)} \, k_F^{(d-3)/2} + \mathcal{O}\left(\frac{\ln(k_F)^{1/2}}{k_F} \right) \right) \cdot \left(1+\lambda^2 k_F^{(d-2)}\right) \tag{3.96}$$
$$+ |t|\,\lambda^2 k_F^{(d-3)/2} \ln\left(k_F\right)$$
$$+ |t|\,|\kappa|\,|\lambda| \left(L^{-1/2} k_F^{(d-1)/2} + \mathcal{O}(1) \right) L^d \left(L^{1/2} \cdot \frac{1}{\left(1+\frac{L}{k_F}\right)} \, k_F^{d-2} + \mathcal{O}\left(\frac{\ln(k_F)}{k_F} \right) \right)$$
$$+ |t|\,|\kappa|\,|\lambda| \left(L^{d-1} k_F^{d-1} + L^d \cdot \mathcal{O}(1) \right) \left(k_F^{(d-3)/2} \ln(k_F) \right)$$
$$+ |t|\,|\kappa|\,|\lambda| \left(L^{d-1} k_F^{d-1} + L^d \cdot \mathcal{O}(1) \right) \left(L^{1/2} \cdot \frac{1}{\left(1+\frac{L}{k_F}\right)} \, k_F^{(d-3)/2} + \mathcal{O}\left(\frac{\ln(k_F)^{1/2}}{k_F} \right) \right)$$
$$+ \lambda^2 \left(L \cdot \frac{1}{\left(1+\frac{L}{k_F}\right)^2} \, k_F^{d-3} + \mathcal{O}\left(\frac{\ln(k_F)}{k_F^2} \right) \right)$$

$$
\begin{aligned}
&+ |t|\,\lambda^2 \left(L \cdot \frac{1}{\left(1+\frac{L}{k_F}\right)^2}\, k_F^{d-3} + \mathcal{O}\left(\frac{\ln(k_F)}{k_F^2}\right)\right) \cdot \left(1 + \lambda^2 k_F^{(d-2)}\right) \\
&+ |t|\,|\lambda|^3 \left(\frac{1}{1+\frac{L}{k_F}} k_F^{(d-2)} + \mathcal{O}\left(\frac{\ln(k_F)}{k_F^2}\right)\right) \\
&\qquad\qquad \cdot \left(L^{1/2} \cdot \frac{1}{\left(1+\frac{L}{k_F}\right)}\, k_F^{(d-3)/2} + \mathcal{O}\left(\frac{\ln(k_F)^{1/2}}{k_F}\right)\right) \\
&+ |t|\,|\lambda|^3 \left(L^{1/2} \cdot \frac{1}{\left(1+\frac{L}{k_F}\right)}\, k_F^{(d-3)/2} + \mathcal{O}\left(\frac{\ln(k_F)^{1/2}}{k_F}\right)\right) \left(k_F^{(d-3)/2} \ln(k_F)\right) \\
&+ |t|\,|\lambda|^3 \left(\frac{1}{L^{1/2}} \cdot k_F^{(d-1)/2} + \mathcal{O}(1)\right) \left(L \cdot \frac{1}{\left(1+\frac{L}{k_F}\right)^2}\, k_F^{d-3} + \mathcal{O}\left(\frac{\ln(k_F)}{k_F^2}\right)\right) \\
&+ |t|\,|\kappa|\,\lambda^2 L^d \left(\frac{1}{1+\frac{L}{k_F}}\, k_F^{(d-2)} + \mathcal{O}\left(\frac{\ln(k_F)}{k_F^2}\right)\right) \\
&+ |t|\,|\kappa|\,\lambda^2 \left(\frac{1}{L^{(d-1)/2}} \cdot \left(k_F + \frac{\sqrt{d}}{2L}\right)^{(d-1)/2}\right) \\
&\qquad\qquad \cdot \left(k_F^{(d-1)/2} \ln(k_F) \cdot \left(\frac{1}{1+\frac{L}{k_F}}\, k_F^{(d-3)} + \mathcal{O}\left(\frac{\ln(k_F)}{k_F^3}\right)\right)\right) \\
&+ |t|\,|\kappa|\,\lambda^2 \left(L^{-1/2} k_F^{(d-1)/2}\right) L^d \left(L^{1/2} \frac{1}{\left(1+\frac{L}{k_F}\right)} k_F^{(d-3)/2} + \mathcal{O}\left(\frac{\ln(k_F)^{1/2}}{k_F}\right)\right) \\
&\qquad\qquad \cdot \left(\frac{1}{1+\frac{L}{k_F}} k_F^{(d-2)} + \mathcal{O}\left(\frac{\ln(k_F)}{k_F^2}\right)\right) \\
&+ |t|\,|\kappa|\,\lambda^2 \left(L^{-1/2} k_F^{(d-1)/2} + \mathcal{O}(1)\right) \cdot L^d \left(k_F^{(d-3)} \ln(k_F)^2\right) \\
&+ |t|\,|\kappa|\,\lambda^2 \left(L^{d-1} k_F^{(d-1)} + L^d \cdot \mathcal{O}(1)\right) \left(L \cdot \frac{1}{\left(1+\frac{L}{k_F}\right)^2}\, k_F^{(d-3)} + \mathcal{O}\left(\frac{\ln(k_F)}{k_F^2}\right)\right)
\end{aligned}
$$

3.1.2 $\widetilde{\Phi}$-terms

In the following step, we again proceed by estimating each of the resulting terms individually.

Term $\widetilde{\Phi}_0(t)$: We have

$$\begin{aligned} &\|R\mathbb{W}\xi(s)\otimes\Omega_0\|^2 && (3.97)\\ &\le 2\kappa^2 L^{-2d}\sum_{(p,q)\in\widetilde{N}_k^F}\sum_{k\in[-D,D]^d}\frac{|\hat{w}(k)|^2}{\left(p^2-(p-k)^2+q^2-(q+k)^2+1\right)^2}\\ &\le 2\kappa^2 k_F^{-2} L^{-2d}\sum_{(p,q)\in\widetilde{N}_k^F}\sum_{k\in[-D,D]^d}\frac{|\hat{w}(k)|^2}{\left(|p|-|p-k|+|q|-|q+k|+k_F^{-1}\right)^2}\,. \end{aligned}$$

We utilized the identities

$$p^2-(p-k)^2=(|p|-|p-k|)\,(|p|+|p-k|)\ge(|p|-|p-k|)\,k_F \quad \text{and} \tag{3.98}$$

$$q^2-(q+k)^2=(|q|-|q+k|)\,(|q|+|q+k|)\ge(|q|-|q+k|)\,k_F\,, \tag{3.99}$$

since $p-k, q+k\in B_F$ and $p,q\in B_F^C$, alongside Wick's theorem. This initially suggested the presence of four different cases. However, due to the assumption of compact support for $\hat{w}(k)$, only two cases remain, see (3.1). The compact support effectively reduces the complexity of the scenario by eliminating contributions from terms outside the support of $\hat{w}(k)$, allowing us to focus on these two cases.

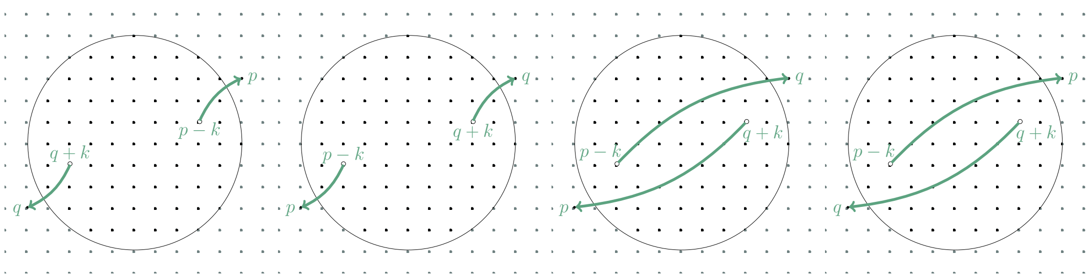

Figure 3.1: Pictures of the four different cases, suggested by $\langle a_p^*a_q^*a_{p-k}a_{p+k}\Omega_0, a_{p'}^*a_{q'}^*a_{p'-k'}a_{p'+k'}\Omega_0\rangle_{\mathcal{H}_N^-}$, produced by term $\mathbb{W}\Omega_0$. Due to the assumption of compact support for $\hat{w}(k)$, only the first two cases are possible.

This yields with (A.43o)

$$\left\|\widetilde{\Phi}_0(t)\right\| \lesssim |\kappa| \left(L^d \frac{1}{\left(2+\frac{L}{k_F}\right)} k_F^{(d-2)} + \ L^d \cdot \mathcal{O}\left(\frac{1}{k_F}\right)\right) . \tag{3.100}$$

Term $\widetilde{\Phi}_1(t)$:

$$\begin{aligned} &\left\| R\mathbb{W}(-1+h_n^0-h_n^{\text{eff}})\xi(s)\otimes\Omega_0\right\|^2 \\ \lesssim\ & \kappa^2 L^{-2d} \sum_{(p,q)\in\widetilde{N}_k^F} \sum_{k\in[-D,D]^d} \frac{|\hat{w}(k)|^2}{\left(p^2+q^2-(p-k)^2-(q+k)^2+1\right)^2} \\ & \cdot \left(1+\left\|(h_n^0-h_n^{\text{eff}})\xi(s)\right\|^2_{\mathcal{H}_n}\right) \end{aligned} \tag{3.101}$$

To bound the norm, we again use $h_n^0 - h_n^{\text{eff}} = \lambda^2 \sum_{i<j}^n W_{k_F}(y_i-y_j) + n\lambda^2 W_{k_F}(0)$. Then

$$\begin{aligned} &\left\|\widetilde{\Phi}_1(t)\right\| \\ \lesssim\ & |t|\,|\kappa| \left(L^d \frac{1}{\left(2+\frac{L}{k_F}\right)} k_F^{(d-2)} + \ L^d \cdot \mathcal{O}\left(\frac{1}{k_F}\right)\right) \cdot \left(1+\lambda^2 k_F^{(d-2)}\right) . \end{aligned} \tag{3.102}$$

Term $\widetilde{\Phi}_2(t)$: Here we need to evaluate two different terms. Since

$$\begin{aligned} P_f^2 \mathbb{W}\xi(s)\otimes\Omega_0 &= L^{-d} \sum_{(p,q)\in\widetilde{N}_k^F} \sum_{k\in[-D,D]^d} \hat{w}(k)\xi(s)\otimes P_f^2 a_p^* a_q^* a_{p-k} a_{p+k}\Omega_0 \\ &= 0 \end{aligned} \tag{3.103}$$

and

$$[h_n^0, \mathbb{W}]\xi(s)\otimes\Omega_0 = h_n^0 L^{-d} \sum_{(p,q)\in\widetilde{N}_k^F} \sum_{k\in[-D,D]^d} \hat{w}(k)\xi(s)\otimes a_p^* a_q^* a_{p-k} a_{p+k}\Omega_0 \tag{3.104}$$

$$- L^{-d} \sum_{(p,q)\in \widetilde{N}_k^F} \sum_{k\in[-D,D]^d} \hat{w}(k)\xi(s) \otimes a_p^* a_q^* a_{p-k} a_{p+k} \Omega_0 h_n^0$$

$$= 0\,,$$

we find

$$\left\| R\{[h_n^0, \mathbb{W}] - P_f^2 \mathbb{W}\}\xi(s) \otimes \Omega_0 \right\|^2 = 0\,, \tag{3.105}$$

and therefore

$$\left\| \widetilde{\Phi}_2(t) \right\| = 0\,. \tag{3.106}$$

Term $\widetilde{\Phi}_{30}(t)$: Here, we also obtain a term aligned with Ω_0 that generates no holes:

$$P^{(0)} a_p^* a_q^* a_{p-k} a_{q+k} a_r^* a_s^* a_{r-t} a_{s+t} \Omega_0 = \delta_{p-k,s}\delta_{p,s+t}\delta_{q+k,r}\delta_{q,r-t}\Omega_0 + \delta_{p-k,r}\delta_{p,r-t}\delta_{q+k,s}\delta_{q,s+t}\Omega_0\,, \tag{3.107}$$

for all $(r,s) \in N_t^F$ and we have

$$\langle \Omega, P^{(0)} \mathbb{W} R \mathbb{W} \Omega_0 \rangle_{\mathcal{H}_N^-} \tag{3.108}$$

$$= \kappa^2 L^{-2d} \sum_{(r,s)\in N_t^F} \sum_{t\in[-D,D]^d} \frac{|\hat{w}(t)|^2}{r^2 - (r-t)^2 + s^2 - (s+t)^2 + 1}\,,$$

Therefore

$$\widetilde{\Phi}_{30}(t) \tag{3.109}$$

$$= i \int_0^t \mathrm{d}s\; e^{i\mathbb{H}s} P^{(0)} \mathbb{W} R \mathbb{W} \xi(s) \otimes \Omega_0 \tag{3.110}$$

$$= i\int_0^t \mathrm{d}s\; e^{i\mathbb{H}s}\left(\kappa^2 L^{-2d} \sum_{(r,s)\in N_t^F}\ \sum_{t\in[-D,D]^d} \frac{|\hat{w}(t)|^2}{r^2-(r-t)^2+s^2-(s+t)^2+1}\right)\xi(s)\otimes\Omega_0\,.$$

In terms of the norm, we have

$$\left\|\widetilde{\Phi}_{30}(t)\right\| \lesssim \kappa^2\left(L^{2d}\frac{1}{\left(2L+\frac{L^2}{k_F}\right)^2}k_F^{2d-3} +\ L^{2d}\cdot\mathcal{O}(1)\right) \tag{3.111}$$

Due to the specific choice of κ, this term is once again too small to influence the determination of h_n^{eff}.

Term $\widetilde{\Phi}_{31}(t)$: Producing one hole is impossible with two operators $\mathbb{W}$ due to conservation of momentum, hence

$$P^{(1)}\mathbb{W}R\mathbb{W}\xi(s)\otimes\Omega_0 = 0\,. \tag{3.112}$$

Therefore

$$\left\|\widetilde{\Phi}_{31}(t)\right\| = 0\,. \tag{3.113}$$

Term $\widetilde{\Phi}_{32}(t)$: We find

$$\begin{aligned} &P^{(2)}a_p^*a_q^*a_{p-k}a_{q+k}a_r^*a_s^*a_{r-t}a_{s+t}\Omega_0 \qquad (3.114)\\ &= \delta_{r,p-k}\delta_{s,q+k}\chi_{N_k^F}(p,q)a_p^*a_q^*a_{r-t}a_{s+t}\Omega_0\\ &\quad+\delta_{r,q+k}\delta_{s,p-k}\chi_{N_k^F}(p,q)a_p^*a_q^*a_{r-t}a_{s+t}\Omega_0\\ &\quad+\delta_{r-t,p}\delta_{s+t,q}\chi_{\widetilde{N}_k^F}\Big((p-k,q+k)\Big)a_{p-k}a_{q+k}a_r^*a_s^*\Omega_0\\ &\quad+\delta_{r-t,q}\delta_{s+t,p}\chi_{\widetilde{N}_k^F}\Big((p-k,q+k)\Big)a_{p-k}a_{q+k}a_r^*a_s^*\Omega_0\,, \end{aligned}$$

for all $(r,s) \in N_t^F$. Then

$$
\begin{aligned}
&P^{(2)} \mathbb{W} R \mathbb{W} \xi(s) \otimes \Omega_0 \\
&= \kappa^2 L^{-2d} \sum_{(p,q)\in N_k^F} \sum_{k\in[-D,D]^d} \sum_{(r,s)\in N_t^F} \sum_{t\in[-D,D]^d} \frac{\hat{w}(p-r)\hat{w}(t)}{(\varepsilon_{r,r-t}+\varepsilon_{s,s+t}+1)} \xi(s) \\
&\qquad \otimes a_p^* a_q^* a_{r-t} a_{s+t} \Omega_0 \qquad (3.115a) \\
&+ \kappa^2 L^{-2d} \sum_{(p,q)\in N_k^F} \sum_{k\in[-D,D]^d} \sum_{(r,s)\in N_t^F} \sum_{t\in[-D,D]^d} \frac{\hat{w}(r-q)\hat{w}(t)}{(\varepsilon_{r,r-t}+\varepsilon_{s,s+t}+1)} \xi(s) \\
&\qquad \otimes a_p^* a_q^* a_{r-t} a_{s+t} \Omega_0 \qquad (3.115b) \\
&+ \kappa^2 L^{-2d} \sum_{(p,q)\in \widetilde{N}_k^F} \sum_{k\in[-D,D]^d} \sum_{(r,s)\in N_t^F} \sum_{t\in[-D,D]^d} \frac{\hat{w}(k)\hat{w}(r-p)}{(\varepsilon_{r,r-t}+\varepsilon_{s,s+t}+1)} \xi(s) \\
&\qquad \otimes a_{p-k} a_{q+k} a_r^* a_s^* \Omega_0 \qquad (3.115c) \\
&+ \kappa^2 L^{-2d} \sum_{(p,q)\in \widetilde{N}_k^F} \sum_{k\in[-D,D]^d} \sum_{(r,s)\in N_t^F} \sum_{t\in[-D,D]^d} \frac{\hat{w}(k)\hat{w}(r-q)}{(\varepsilon_{r,r-t}+\varepsilon_{s,s+t}+1)} \xi(s) \\
&\qquad \otimes a_{p-k} a_{q+k} a_r^* a_s^* \Omega_0 \,, \qquad (3.115d)
\end{aligned}
$$

and we find

$$
\begin{aligned}
&\left\| (3.115a) \right\|^2 \qquad (3.116) \\
&\lesssim \kappa^4 \left(L^{-2d} \sum_{(p,q)\in N_k^F} \sum_{k\in[-D,D]^d} \right) \left(L^{-2d} \sum_{(r,s)\in N_t^F} \sum_{t\in[-D,D]^d} \frac{|\hat{w}_t|^2}{(\varepsilon_{r,r-t}+\varepsilon_{s,s+t}+1)^2} \right) .
\end{aligned}
$$

Similarly we derive with the same bounds for (3.115b)-(3.115d). Then, with (A.43o),

$$
\begin{aligned}
&\left\| \widetilde{\Phi}_{32}(t) \right\| \qquad (3.117) \\
&\lesssim |t|\, \kappa^2 \left(\frac{L^d}{\left(2L+\frac{L^2}{k_F}\right)} k_F^{d-1} + \; L^d \cdot \mathcal{O}(1) \right) \left(\frac{L^d}{\left(2+\frac{L}{k_F}\right)} k_F^{d-2} + \; L^d \cdot \mathcal{O}\left(\frac{1}{k_F}\right) \right) .
\end{aligned}
$$

Term $\widetilde{\Phi}_{33}(t)$: Here,

$$
\begin{aligned}
P^{(2)} a_p^* a_q^* a_{p-k} a_{q+k} a_r^* a_s^* a_{r-t} a_{s+t} \Omega_0 = {} & \delta_{r,q+k} \chi_{M_k^F}(q) \chi_{M_{-k}^F}(p) a_p^* a_q^* a_{p-k} a_s^* a_{r-t} a_{s+t} \Omega_0 \\
& + \delta_{r,p-k} \chi_{M_k^F}(p) \chi_{M_{-k}^F}(q) a_p^* a_q^* a_{q+k} a_s^* a_{r-t} a_{s+t} \Omega_0 \\
& + \delta_{s,q+k} \chi_{M_k^F}(q) \chi_{M_{-k}^F}(p) a_p^* a_q^* a_{p-k} a_r^* a_{r-t} a_{s+t} \Omega_0 \\
& + \delta_{s,p-k} \chi_{M_k^F}(p) \chi_{M_{-k}^F}(q) a_p^* a_q^* a_{q+k} a_r^* a_{r-t} a_{s+t} \Omega_0 \\
& + \delta_{r-t,q} \chi_{\widetilde{M}_k^F}(q) \chi_{M_{-k}^F}(p) a_p^* a_{p-k} a_{q+k} a_r^* a_s^* a_{s+t} \Omega_0 \\
& + \delta_{r-t,p} \chi_{\widetilde{M}_k^F}(p) \chi_{M_{-k}^F}(q) a_q^* a_{p-k} a_{q+k} a_r^* a_s^* a_{s+t} \Omega_0 \\
& + \delta_{s+t,p} \chi_{\widetilde{M}_k^F}(q) \chi_{M_{-k}^F}(p) a_q^* a_{p-k} a_{q+k} a_r^* a_s^* a_{r-t} \Omega_0 \\
& + \delta_{s+t,p} \chi_{\widetilde{M}_k^F}(p) \chi_{M_{-k}^F}(q) a_q^* a_{p-k} a_{q+k} a_r^* a_s^* a_{r-t} \Omega_0 \, ,
\end{aligned} \tag{3.118}
$$

for all $(r,s) \in N_t^F$. This yields

$$
\begin{aligned}
& P^{(3)} \mathbb{W} R \mathbb{W} \xi(s) \otimes \Omega_0 \\
&= \kappa^2 L^{-2d} \sum_{(p,q)\in N_k^F} \sum_{k\in[-D,D]^d} \sum_{(r,s)\in N_t^F} \sum_{t\in[-D,D]^d} \frac{\hat{w}(k)\hat{w}(t)}{(\varepsilon_{r,r-t} + \varepsilon_{s,s+t} + 1)} \xi(s) \\
&\qquad \otimes a_p^* a_q^* a_{p-k} a_s^* a_{r-t} a_{s+t} \Omega_0 \qquad (3.119a) \\
&\quad + \kappa^2 L^{-2d} \sum_{(p,q)\in N_k^F} \sum_{k\in[-D,D]^d} \sum_{(r,s)\in N_t^F} \sum_{t\in[-D,D]^d} \frac{\hat{w}(k)\hat{w}(t)}{(\varepsilon_{r,r-t} + \varepsilon_{s,s+t} + 1)} \xi(s) \\
&\qquad \otimes a_p^* a_q^* a_{q+k} a_s^* a_{r-t} a_{s+t} \Omega_0 \qquad (3.119b) \\
&\quad + \kappa^2 L^{-2d} \sum_{(p,q)\in N_k^F} \sum_{k\in[-D,D]^d} \sum_{(r,s)\in N_t^F} \sum_{t\in[-D,D]^d} \frac{\hat{w}(k)\hat{w}(t)}{(\varepsilon_{r,r-t} + \varepsilon_{s,s+t} + 1)} \xi(s) \\
&\qquad \otimes a_p^* a_q^* a_{p-k} a_r^* a_{r-t} a_{s+t} \Omega_0 \qquad (3.119c) \\
&\quad + \kappa^2 L^{-2d} \sum_{(p,q)\in N_k^F} \sum_{k\in[-D,D]^d} \sum_{(r,s)\in N_t^F} \sum_{t\in[-D,D]^d} \frac{\hat{w}(k)\hat{w}(t)}{(\varepsilon_{r,r-t} + \varepsilon_{s,s+t} + 1)} \xi(s)
\end{aligned}
$$

$$\otimes\, a_p^* a_q^* a_{q+k} a_r^* a_{r-t} a_{s+t} \Omega_0 \tag{3.119d}$$

$$+ \;\kappa^2 L^{-2d} \sum_{(p,q)\in \widetilde{N}_k^F} \sum_{k\in[-D,D]^d} \sum_{(r,s)\in N_t^F} \sum_{t\in[-D,D]^d} \frac{\hat{w}(k)\hat{w}(t)}{(\varepsilon_{r,r-t}+\varepsilon_{s,s+t}+1)} \xi(s)$$

$$\otimes\, a_p^* a_{p-k} a_{q+k} a_r^* a_s^* a_{s+t} \Omega_0 \tag{3.119e}$$

$$+ \;\kappa^2 L^{-2d} \sum_{(p,q)\in \widetilde{N}_k^F} \sum_{k\in[-D,D]^d} \sum_{(r,s)\in N_t^F} \sum_{t\in[-D,D]^d} \frac{\hat{w}(k)\hat{w}(t)}{(\varepsilon_{r,r-t}+\varepsilon_{s,s+t}+1)} \xi(s)$$

$$\otimes\, a_q^* a_{p-k} a_{q+k} a_r^* a_s^* a_{s+t} \Omega_0 \tag{3.119f}$$

$$+ \;\kappa^2 L^{-2d} \sum_{(p,q)\in \widetilde{N}_k^F} \sum_{k\in[-D,D]^d} \sum_{(r,s)\in N_t^F} \sum_{t\in[-D,D]^d} \frac{\hat{w}(k)\hat{w}(t)}{(\varepsilon_{r,r-t}+\varepsilon_{s,s+t}+1)} \xi(s)$$

$$\otimes\, a_q^* a_{p-k} a_{q+k} a_r^* a_s^* a_{r-t} \Omega_0 \tag{3.119g}$$

$$+ \;\kappa^2 L^{-2d} \sum_{(p,q)\in \widetilde{N}_k^F} \sum_{k\in[-D,D]^d} \sum_{(r,s)\in N_t^F} \sum_{t\in[-D,D]^d} \frac{\hat{w}(k)\hat{w}(t)}{(\varepsilon_{r,r-t}+\varepsilon_{s,s+t}+1)} \xi(s)$$

$$\otimes\, a_q^* a_{p-k} a_{q+k} a_r^* a_s^* a_{r-t} \Omega_0\,. \tag{3.119h}$$

We find

$$\begin{aligned} &\|(3.119\text{a})\|^2 \\ &\leq 2\kappa^4 \left(L^{-2d} \sum_{(p,q)\in N_k^F} \sum_{k\in[-D,D]^d} |\hat{w}_k|^2 \right) \left(L^{-2d} \sum_{(r,s)\in N_t^F} \sum_{t\in[-D,D]^d} \frac{|\hat{w}_t|^2}{(\varepsilon_{r,r-t}+\varepsilon_{s,s+t}+1)^2} \right) \end{aligned} \tag{3.120}$$

The same bounds can be found analogously for (3.119b)-(3.119h). Therefore, using (A.43m) and (A.43o),

$$\begin{aligned} &\left\| \widetilde{\Phi}_{33}(t) \right\| \\ &\lesssim |t|\, \kappa^2 \left(L^{d-1} k_F^{(d-1)} + L^d \cdot \mathcal{O}(1) \right) \left(\frac{L^d}{\left(2+\frac{L}{k_F}\right)} k_F^{(d-2)} + \; L^d \cdot \mathcal{O}\left(\frac{1}{k_F}\right) \right) . \end{aligned} \tag{3.121}$$

Term $\widetilde{\Phi}_{34}(t)$: We have

$$
\begin{aligned}
&P^{(4)}\mathbb{W}R\mathbb{W}\xi(s)\otimes\Omega_0 \qquad (3.122)\\
&= \kappa^2 L^{-2d}\sum_{(p,q)\in N_k^F}\sum_{k\in[-D,D]^d}\sum_{(r,s)\in N_t^F}\sum_{t\in[-D,D]^d}\frac{\hat{w}(k)\hat{w}(t)}{(\varepsilon_{r,r-t}+\varepsilon_{s,s+t}+1)}\xi(s)\\
&\qquad\otimes a_p^* a_q^* a_{p-k}a_{q+k}a_r^* a_s^* a_{r-t}a_{s+t}\Omega_0
\end{aligned}
$$

following from the identity

$$
P^{(4)}a_p^* a_q^* a_{p-k}a_{q+k}a_r^* a_s^* a_{r-t}a_{s+t}\Omega_0 = \chi_{N_k^F}\Big((p,q)\Big)a_p^* a_q^* a_{p-k}a_{q+k}a_r^* a_s^* a_{r-t}a_{s+t}\Omega_0 \qquad (3.123)
$$

for all (r,s) in N_t^F. One can easily check that

$$
\begin{aligned}
&\left|\frac{\hat{w}(k)\hat{w}(t)}{(r^2-(r-t)^2+s^2-(s+t)^2+1)}\cdot\frac{\hat{w}(k')\hat{w}(t')}{(r'^2-(r'-t')^2+s'^2-(s'+t')^2+1)}\right|\\
&\leq \frac{|\hat{w}(k)\hat{w}(t)|^2}{2\left(r^2-(r-t)^2+s^2-(s+t)^2+1\right)^2}+\frac{|\hat{w}(k')\hat{w}(t')|^2}{2\left(r'^2-(r'-t')^2+s'^2-(s'+t')^2+1\right)^2}\,.
\end{aligned} \qquad (3.124)
$$

Then

$$
\begin{aligned}
&\left\|P^{(4)}\mathbb{W}R\mathbb{W}\xi(s)\otimes\Omega_0\right\|^2 \qquad (3.125)\\
&\leq \kappa^4 L^{-4d}\sum_{(p,q)\in N_k^F}\sum_{(r,s)\in N_t^F}\sum_{(p',q')\in N_{k'}^F}\sum_{(r',s')\in N_{t'}^F}\sum_{\substack{t,t'\in[-D,D]^d\\ k,k'\in[-D,D]^d}}\frac{|\hat{w}(k)\ \hat{w}(t)|^2}{\left(\varepsilon_{r,r-t}+\varepsilon_{s,s+t}+1\right)^2}\\
&\quad\times\left|\langle a_p^* a_q^* a_{p-k}a_{q+k}a_r^* a_s^* a_{r-t}a_{s+t}\Omega_0, a_{p'}^* a_{q'}^* a_{p'-k'}a_{q'+k'}a_{r'}^* a_{s'}^* a_{r'-t'}a_{s'+t'}\Omega_0\rangle_{\mathcal{H}_N^-}\right|\\
&\lesssim \kappa^4\left(L^{-2d}\sum_{(p,q)\in N_k^F}\sum_{k\in[-D,D]^d}|\hat{w}_k|^2\right)\left(L^{-2d}\sum_{(r,s)\in N_t^F}\sum_{t\in[-D,D]^d}\frac{|\hat{w}_t|^2}{\left(\varepsilon_{r,r-t}+\varepsilon_{s,s+t}+1\right)^2}\right),
\end{aligned}
$$

which, using (A.43m) and (A.43o), yields

$$\left\|\widetilde{\Phi}_{34}(t)\right\| \tag{3.126}$$
$$\lesssim |t|\,\kappa^2\left(L^{d-1}k_F^{(d-1)}+L^d\cdot\mathcal{O}(1)\right)\left(\frac{L^d}{\left(2+\frac{L}{k_F}\right)}k_F^{(d-2)}+\ L^d\cdot\mathcal{O}\left(\frac{1}{k_F}\right)\right)$$

Term $\widetilde{\Phi}_{41}(t)$: Using

$$P^{(1)}a_n^*a_ma_p^*a_q^*a_{p-k}a_{q+k}\Omega_0=\delta_{p,m}\delta_{p-k,n}\chi_{T_F}\left((m,n)\right)a_q^*a_{q+k}\Omega_0 \tag{3.127}$$
$$+\ \delta_{q,m}\delta_{q+k,n}\chi_{T_F}\left((m,n)\right)a_p^*a_{p-k}\Omega_0$$

for all (p,q) in N_k^F, we find

$$P^{(1)}\mathbb{V}^{(i)}R\mathbb{W}\xi(s)\otimes\Omega_0$$
$$=\kappa\lambda L^{-2d}\sum_{p\in M_{-k}^F}\sum_{q\in M_k^F}\sum_{k\in[-D,D]^d}\frac{\hat{v}(k)\ \hat{w}(k)}{(\varepsilon_{p,p-k}+\varepsilon_{q,q+k}+1)}\ e^{-iky_i}\xi(s)$$
$$\otimes\ a_q^*a_{q+k}\Omega_0 \tag{3.128a}$$
$$+\ \kappa\lambda L^{-2d}\sum_{p\in M_{-k}^F}\sum_{q\in M_k^F}\sum_{k\in[-D,D]^d}\frac{\hat{v}(k)\ \hat{w}(k)}{(\varepsilon_{p,p-k}+\varepsilon_{q,q+k}+1)}\ e^{-iky_i}\xi(s)$$
$$\otimes\ a_p^*a_{p-k}\Omega_0\,. \tag{3.128b}$$

This yields

$$\left\|P^{(1)}\mathbb{V}^{(i)}R\mathbb{W}\xi(s)\otimes\Omega_0\right\|^2 \tag{3.129}$$
$$\lesssim\kappa^2\lambda^2\left(L^{-d}\sum_{p\in M_{-k}^F}\right)^2\left(L^{-2d}\sum_{q\in M_k^F}\sum_{k\in[-D,D]^d}\frac{|\hat{v}(k)|^2\ |\hat{w}(k)|^2}{(\varepsilon_{q,q+k}+1)^2}\right)$$

$$+\ \kappa^2\lambda^2\left(L^{-d}\sum_{q\in M_k^F}\right)^2\left(L^{-2d}\sum_{p\in M_{-k}^F}\sum_{k\in[-D,D]^d}\frac{|\hat{v}(k)|^2\ |\hat{w}(k)|^2}{\left(\varepsilon_{p,p-k}+1\right)^2}\right).$$

Therefore, using (A.43p) and (A.176), we find

$$\left\|\widetilde{\Phi}_{41}(t)\right\| \lesssim |t|\,|\kappa|\,|\lambda|\left(\left(k_F+\frac{\sqrt{d}}{2L}\right)^{d-1}\cdot\frac{1}{L}\right) \cdot\left((2DL+1)^{d/2}\cdot\frac{1}{\left(1+\frac{L}{k_F}\right)^{1/2}}k_F^{(d-3)/2}+\mathcal{O}\left(\frac{\ln\left(k_F\right)}{k_F}\right)\right). \tag{3.130}$$

Term $\widetilde{\Phi}_{42}(t)$: The identity

$$\begin{aligned} P^{(2)}a_n^*a_ma_p^*a_q^*a_{p-k}a_{q+k}\Omega_0 &= \delta_{n,p-k}\chi_{B_F}(m)\,a_q^*a_p^*a_{q+k}a_m\Omega_0 \\ &+\delta_{n,q+k}\chi_{B_F}(m)\,a_q^*a_p^*a_{p-k}a_m\Omega_0 \\ &+\delta_{m,p}\chi_{B_F^C}(n)\,a_q^*a_{p-k}a_{q+k}a_n^*\Omega_0 \\ &+\delta_{m,q}\chi_{B_F^C}(n)\,a_p^*a_{p-k}a_{q+k}a_n^*\Omega_0 \end{aligned} \tag{3.131}$$

for all $(p,q)\in N_k^F$ and $(p-k,q+k)\in\widetilde{N_k^F}$, yields

$$\begin{aligned} &P^{(2)}\mathbb{V}^{(i)}R\mathbb{W}\xi(s)\otimes\Omega_0 \\ &=\kappa\lambda L^{-2d}\sum_{m\in B_F}\sum_{(p,q)\in N_k^F}\sum_{k\in[-D,D]^d}\frac{\hat{v}(p-k-m)\ \hat{w}(k)}{(\varepsilon_{p,p-k}+\varepsilon_{q,q+k}+1)}\ e^{-i(p-k-m)y_i}\xi(s) \\ &\qquad\otimes a_q^*a_p^*a_{q+k}a_m\Omega_0 \qquad (3.132a) \\ &+\kappa\lambda L^{-2d}\sum_{m\in B_F}\sum_{(p,q)\in N_k^F}\sum_{k\in[-D,D]^d}\frac{\hat{v}(q+k-m)\ \hat{w}(k)}{(\varepsilon_{p,p-k}+\varepsilon_{q,q+k}+1)}\ e^{-i(q+k-m)y_i}\xi(s) \\ &\qquad\otimes a_q^*a_p^*a_{p-k}a_m\Omega_0 \qquad (3.132b) \\ &+\kappa\lambda L^{-2d}\sum_{n\in B_F^C}\sum_{(p,q)\in N_k^F}\sum_{k\in[-D,D]^d}\frac{\hat{v}(n-p)\ \hat{w}(k)}{(\varepsilon_{p,p-k}+\varepsilon_{q,q+k}+1)}\ e^{-i(n-p)y_i}\xi(s) \end{aligned}$$

$$\otimes\, a_q^* a_{p-k} a_{q+k} a_n^* \Omega_0 \tag{3.132c}$$

$$+ \kappa\lambda L^{-2d} \sum_{n\in B_F^C} \sum_{(p,q)\in N_k^F} \sum_{k\in[-D,D]^d} \frac{\hat{v}(n-q)\ \hat{w}(k)}{(\varepsilon_{p,p-k}+\varepsilon_{q,q+k}+1)}\ e^{-i(n-q)y_i}\xi(s)$$

$$\otimes\, a_p^* a_{p-k} a_{q+k} a_n^* \Omega_0\,. \tag{3.132d}$$

Then

$$\begin{aligned} &\|(3.132\text{a})\|^2 \\ \lesssim\ &\kappa^2\lambda^2 \left(L^{-2d}\sum_{m\in B_F}\right)\left(L^{-2d}\sum_{(p,q)\in N_k^F}\sum_{k\in[-D,D]^d}\frac{|\hat{w}_k|^2}{(\varepsilon_{p,p-k}+\varepsilon_{q,q+k}+1)^2}\right) \end{aligned} \tag{3.133}$$

Analogously we find the same bounds for (3.132b)-(3.132d). Therefore with (A.43o) and (A.43s)

$$\begin{aligned} \left\|\widetilde{\Phi}_{42}(t)\right\| &\lesssim |t|\,|\kappa|\,|\lambda|\ k_F^{(3d-5)/2}\,. \\ &\lesssim |t|\,|\kappa|\,|\lambda| \left(L^{-d}\cdot\left(k_F+\frac{\sqrt{d}}{2L}\right)^d\right)\left(\frac{L^d}{\left(2+\frac{L}{k_F}\right)}k_F^{(d-2)} +\ L^d\cdot\mathcal{O}\left(\frac{1}{k_F}\right)\right)\,. \end{aligned} \tag{3.134}$$

Term $\widetilde{\Phi}_{43}(t)$: We have

$$\begin{aligned} &P^{(3)}\mathbb{V}^{(i)}R\mathbb{W}\xi(s)\otimes\Omega_0 \\ &= \kappa\lambda L^{-2d}\sum_{(n,m)\in T_F}\sum_{(p,q)\in \widetilde{N}_k^F}\sum_{k\in[-D,D]^d}\frac{\hat{v}(n-m)\ \hat{w}(k)}{(\varepsilon_{p,p-k}+\varepsilon_{q,q+k}+1)}e^{-i(n-m)y_i}\xi(s) \\ &\qquad \otimes\, a_n^* a_m a_p^* a_q^* a_{p-k} a_{q+k}\Omega_0 \end{aligned} \tag{3.135}$$

following from the identity

$$P^{(3)} a_n^* a_m a_p^* a_q^* a_{p-k} a_{q+k}\Omega_0 = \chi_{T_F}\Big((n,m)\Big) a_n^* a_m a_p^* a_q^* a_{p-k} a_{q+k}\Omega_0 \tag{3.136}$$

for all (p, q) in N_k^F. One can easily check that

$$\begin{aligned} &\left|\frac{\hat{v}(n-m)\ \hat{w}(k)}{(\varepsilon_{p,p-k}+\varepsilon_{q,q+k}+1)}\cdot\frac{\hat{v}(n'-m')\ \hat{w}(k')}{(\varepsilon_{p',p'-k'}+\varepsilon_{q',q'+k'}+1)}\right| \\ &\leq \frac{|\hat{v}(n-m)\ \hat{w}(k)|^2}{2\left(\varepsilon_{p,p-k}+\varepsilon_{q,q+k}+1\right)^2}+\frac{|\hat{v}(n'-m')\ \hat{w}(k')|^2}{2\left(\varepsilon_{p',p'-k'}+\varepsilon_{q',q'+k'}+1\right)^2}. \end{aligned} \tag{3.137}$$

Then

$$\begin{aligned} &\left\|P^{(3)}\mathbb{V}^{(i)}R\mathbb{W}\xi(s)\otimes\Omega_0\right\|^2 \\ &\leq \kappa\lambda L^{-2d}\sum_{(n,m)\in T_F}\ \sum_{(n',m')\in T_F}\ \sum_{(p,q)\in\widetilde{N}_k^F}\ \sum_{(p',q')\in\widetilde{N}_{k'}^F}\ \sum_{k,k'\in[-D,D]^d}\frac{|\hat{w}(k)\hat{v}(n-m)|^2}{\left(\varepsilon_{p,p-k}+\varepsilon_{q,q+k}+1\right)^2} \\ &\quad\cdot\left\|e^{-i(n-m)y}\xi(s)\right\|_{\mathcal{H}_n}\times\left|\langle a_n^*a_ma_p^*a_q^*a_{p-k}a_{q+k}\Omega_0,\, a_{n'}^*a_{m'}a_{p'}^*a_{q'}^*a_{p'-k'}a_{q'+k'}\Omega_0\rangle_{\mathcal{H}_N^-}\right| \\ &\lesssim \kappa\lambda\left(L^{-2d}\sum_{(n,m)\in T_F}|\hat{v}(n-m)|^2\right)\left(L^{-2d}\sum_{(p,q)\in N_k^F}\ \sum_{k\in[-D,D]^d}\frac{|\hat{w}(k)|^2}{\left(\varepsilon_{p,p-k}+\varepsilon_{q,q+k}+1\right)^2}\right), \end{aligned} \tag{3.138}$$

where we used $\left\|e^{-i(n-m)y_i}\xi(s)\right\| = 1$ as well as the fact that the scalar product provides 36 summations. This yields to

$$\begin{aligned} &\left\|\widetilde{\Phi}_{43}(t)\right\| \\ &\lesssim |t|\,|\kappa|\,|\lambda|\left(\frac{1}{L^{1/2}}k_F^{(d-1)/2}+\mathcal{O}\left(1\right)\right)\left(\frac{L^d}{\left(2+\frac{L}{k_F}\right)}k_F^{(d-2)}+\ L^d\cdot\mathcal{O}\left(\frac{1}{k_F}\right)\right). \end{aligned} \tag{3.139}$$

Together with (2.72), we obtain

$$\begin{aligned} &\left\|\widetilde{\Phi}(t)\right\| \\ &\lesssim |\kappa|\left(L^d\frac{1}{\left(2+\frac{L}{k_F}\right)}k_F^{(d-2)}+\ L^d\cdot\mathcal{O}\left(\frac{1}{k_F}\right)\right) \\ &\quad+|t|\,|\kappa|\left(L^d\frac{1}{\left(2+\frac{L}{k_F}\right)}k_F^{(d-2)}+\ L^d\cdot\mathcal{O}\left(\frac{1}{k_F}\right)\right)\left(1+\lambda^2k_F^{(d-2)}\right) \end{aligned} \tag{3.140}$$

$$+\kappa^2\left(L^{2d}\frac{1}{\left(2L+\frac{L^2}{k_F}\right)^2}k_F^{2d-3}+\ L^{2d}\cdot\mathcal{O}(1)\right)$$
$$+\left|t\right|\kappa^2\left(\frac{L^d}{\left(2L+\frac{L^2}{k_F}\right)}k_F^{d-1}+\ L^d\cdot\mathcal{O}(1)\right)\left(\frac{L^d}{\left(2+\frac{L}{k_F}\right)}k_F^{d-2}+\ L^d\cdot\mathcal{O}\left(\frac{1}{k_F}\right)\right)$$
$$+\left|t\right|\kappa^2\left(L^{d-1}k_F^{(d-1)}+L^d\cdot\mathcal{O}(1)\right)\left(\frac{L^d}{\left(2+\frac{L}{k_F}\right)}k_F^{(d-2)}+\ L^d\cdot\mathcal{O}\left(\frac{1}{k_F}\right)\right)$$
$$+\left|t\right|\left|\kappa\right|\left|\lambda\right|\left(\left(k_F+\frac{\sqrt{d}}{2L}\right)^{d-1}\cdot\frac{1}{L}\right)$$
$$\cdot\left((2DL+1)^{d/2}\cdot\frac{1}{\left(1+\frac{L}{k_F}\right)^{1/2}}k_F^{(d-3)/2}+\mathcal{O}\left(\frac{\ln\left(k_F\right)}{k_F}\right)\right)$$
$$+\left|t\right|\left|\kappa\right|\left|\lambda\right|\left(L^{-d}\cdot\left(k_F+\frac{\sqrt{d}}{2L}\right)^d\right)\left(\frac{L^d}{\left(2+\frac{L}{k_F}\right)}k_F^{(d-2)}+\ L^d\cdot\mathcal{O}\left(\frac{1}{k_F}\right)\right)$$
$$+\left|t\right|\left|\kappa\right|\left|\lambda\right|\left(\frac{1}{L^{1/2}}k_F^{(d-1)/2}+\mathcal{O}\left(1\right)\right)\left(\frac{L^d}{\left(2+\frac{L}{k_F}\right)}k_F^{(d-2)}+\ L^d\cdot\mathcal{O}\left(\frac{1}{k_F}\right)\right)$$

Summary of Term Estimations

We showed, that for $d\in\{2,3\}$, $\lambda,\kappa\in\mathbb{R}$ and $k_F\geq 2$, combining all bounds of the decomposition of Φ and $\widetilde{\Phi}$, that we end up with

$$\left\|\Psi(t)-\xi(t)\otimes e^{-iE(k_F,L)t}\Omega_0\right\| \tag{3.141}$$
$$\lesssim\left|\lambda\right|\left(L^{1/2}\cdot\frac{1}{\left(1+\frac{L}{k_F}\right)}k_F^{(d-3)/2}+\mathcal{O}\left(1\right)\right)$$
$$+\left|t\right|\left|\lambda\right|\left(L^{1/2}\cdot\frac{1}{\left(1+\frac{L}{k_F}\right)}k_F^{(d-3)/2}+\mathcal{O}\left(1\right)\right)\cdot\left(1+\lambda^2k_F^{(d-2)}\right)$$
$$+\left|t\right|\lambda^2k_F^{(d-3)/2}\ln\left(k_F\right)$$
$$+\left|t\right|\left|\kappa\right|\left|\lambda\right|L^d\cdot\left(L^{-1/2}k_F^{(d-1)/2}+\mathcal{O}(1)\right)\left(L^{1/2}\cdot\frac{1}{\left(1+\frac{L}{k_F}\right)}k_F^{d-2}+\mathcal{O}\left(1\right)\right)$$
$$+\left|t\right|\left|\kappa\right|\left|\lambda\right|L^d\cdot\left(L^{-1}k_F^{d-1}+\mathcal{O}(1)\right)\left(k_F^{(d-3)/2}\ln(k_F)\right)$$

$$+ |t|\,|\kappa|\,|\lambda|\,L^d \cdot \left(L^{-1}k_F^{d-1} + \ \mathcal{O}(1)\right)\left(L^{1/2} \cdot \frac{1}{\left(1+\frac{L}{k_F}\right)} k_F^{(d-3)/2} + \mathcal{O}\left(1\right)\right)$$

$$+ \lambda^2 \left(L \cdot \frac{1}{\left(1+\frac{L}{k_F}\right)^2} k_F^{d-3} + \mathcal{O}\left(1\right)\right)$$

$$+ |t|\,\lambda^2 \left(L \cdot \frac{1}{\left(1+\frac{L}{k_F}\right)^2} k_F^{d-3} + \mathcal{O}\left(1\right)\right) \cdot \left(1 + \lambda^2 k_F^{(d-2)}\right)$$

$$+ |t|\,|\lambda|^3 \left(\frac{1}{1+\frac{L}{k_F}} k_F^{(d-2)} + \mathcal{O}\left(1\right)\right) \cdot \left(L^{1/2} \cdot \frac{1}{\left(1+\frac{L}{k_F}\right)} k_F^{(d-3)/2} + \mathcal{O}\left(1\right)\right)$$

$$+ |t|\,|\lambda|^3 \left(L^{1/2} \cdot \frac{1}{\left(1+\frac{L}{k_F}\right)} k_F^{(d-3)/2} + \mathcal{O}\left(1\right)\right)\left(k_F^{(d-3)/2} \ln(k_F)\right)$$

$$+ |t|\,|\lambda|^3 \left(L^{-1/2} \cdot k_F^{(d-1)/2} + \mathcal{O}(1)\right)\left(L \cdot \frac{1}{\left(1+\frac{L}{k_F}\right)^2} k_F^{d-3} + \mathcal{O}\left(1\right)\right)$$

$$+ |t|\,|\kappa|\,\lambda^2\,L^d \cdot \left(\frac{1}{1+\frac{L}{k_F}} k_F^{(d-2)} + \mathcal{O}\left(1\right)\right)$$

$$+ |t|\,|\kappa|\,\lambda^2 \left(\frac{1}{L^{(d-1)/2}} \cdot \left(k_F + \frac{\sqrt{d}}{2L}\right)^{(d-1)/2}\right)$$

$$\cdot \left(k_F^{(d-1)/2} \ln\left(k_F\right) \cdot \left(\frac{1}{1+\frac{L}{k_F}} k_F^{(d-3)} + \mathcal{O}\left(1\right)\right)\right)$$

$$+ |t|\,|\kappa|\,\lambda^2\,L^d \cdot \left(L^{-1/2} k_F^{(d-1)/2}\right)$$

$$\cdot \left(L^{1/2} \frac{1}{\left(1+\frac{L}{k_F}\right)} k_F^{(d-3)/2} + \mathcal{O}\left(1\right)\right) \cdot \left(\frac{1}{1+\frac{L}{k_F}} k_F^{(d-2)} + \mathcal{O}\left(1\right)\right)$$

$$+ |t|\,|\kappa|\,\lambda^2\,L^d \cdot \left(L^{-1/2} k_F^{(d-1)/2} + \mathcal{O}\left(1\right)\right)\left(k_F^{(d-3)} \ln\left(k_F\right)^2\right)$$

$$+ |t|\,|\kappa|\,\lambda^2\,L^d \cdot \left(L^{-1} k_F^{(d-1)} + \mathcal{O}(1)\right)\left(L \cdot \frac{1}{\left(1+\frac{L}{k_F}\right)^2} k_F^{(d-3)} + \mathcal{O}\left(1\right)\right)$$

$$+ |\kappa|\,L^d \cdot \left(\frac{1}{\left(2+\frac{L}{k_F}\right)} k_F^{(d-2)} + \ \mathcal{O}\left(1\right)\right)$$

$$+ |t|\,|\kappa|\,L^d \cdot \left(\frac{1}{\left(2+\frac{L}{k_F}\right)} k_F^{(d-2)} + \mathcal{O}\left(1\right)\right)\left(1 + \lambda^2 k_F^{(d-2)}\right)$$

$$
\begin{aligned}
&+ \kappa^2 L^{2d} \cdot \left(L^{-2} \cdot \frac{1}{\left(2 + \frac{L}{k_F}\right)^2} k_F^{2d-3} + \mathcal{O}(1) \right) \\
&+ |t| \, \kappa^2 L^{2d} \cdot \left(L^{-1} \cdot \frac{1}{\left(2 + \frac{L}{k_F}\right)} k_F^{d-1} + \mathcal{O}(1) \right) \left(\frac{1}{\left(2 + \frac{L}{k_F}\right)} k_F^{d-2} + \mathcal{O}\left(1\right) \right) \\
&+ |t| \, \kappa^2 L^{2d} \cdot \left(L^{-1} k_F^{(d-1)} + \mathcal{O}(1) \right) \left(\frac{1}{\left(2 + \frac{L}{k_F}\right)} k_F^{(d-2)} + \mathcal{O}\left(1\right) \right) \\
&+ |t| \, |\kappa| \, |\lambda| \left(\left(k_F + \frac{\sqrt{d}}{2L} \right)^{d-1} \cdot \frac{1}{L} \right) \\
&\qquad \cdot \left((2DL+1)^{d/2} \cdot \frac{1}{\left(1 + \frac{L}{k_F}\right)^{1/2}} k_F^{(d-3)/2} + \mathcal{O}\left(1\right) \right) \\
&+ |t| \, |\kappa| \, |\lambda| \, L^d \cdot \left(L^{-d} \cdot \left(k_F + \frac{\sqrt{d}}{2L} \right)^d \right) \left(\frac{1}{\left(2 + \frac{L}{k_F}\right)} k_F^{(d-2)} + \mathcal{O}\left(1\right) \right) \\
&+ |t| \, |\kappa| \, |\lambda| \, L^d \cdot \left(L^{-1} k_F^{(d-1)/2} + \mathcal{O}\left(1\right) \right) \left(\frac{1}{\left(2 + \frac{L}{k_F}\right)} k_F^{(d-2)} + \mathcal{O}\left(1\right) \right)
\end{aligned}
$$

Setting now $\lambda^2 = k_F^{2-d}$ and $\kappa^2 = L^{-2d}$, we find the statement in (1)

$$
\limsup_{L \to \infty} \left\| \Psi(t) - \xi(t) \otimes e^{-iE(k_F, L)t} \Omega_0 \right\| \leq C\left(n, R, \alpha_{\xi_0}\right) \frac{\left(1 + |t|\right) \ln \left(k_F\right)^2}{k_F} . \quad (3.142)
$$

In conclusion, we have successfully extended the study of impurity particles in a dense Fermi gas by incorporating internal interactions into the theoretical framework. By employing a perturbative approach, we evaluated the various terms that arise due to these interactions. Our analysis shows that, consistent with the findings in [24], the leading-order term responsible for restoring the zero-hole state in the Fermi sea is of order one. All other terms, as expected, are smaller, given the choices of coupling constants $\kappa^2 = L^{-2d}$ for the internal fermion interaction and $\lambda^2 = k_F^{2-d}$ for the impurity-fermion interaction.

This careful selection of parameters allows us to reproduce the same effective interaction among impurities as seen in the non-interacting case. Furthermore, our results demonstrate that the product structure of the many-body wave function, as assumed in earlier works, is approximately preserved in the limit of large Fermi momentum k_F. This preservation of structure validates the robustness of previous models, even in the presence of weak fermion-fermion interactions.

On the validity of the large-volume limit.

In the setting without fermion–fermion interaction (as in [24]), the thermodyy-namic limit $L \to \infty$ at fixed density (i.e. fixed k_F) is harmless: sums over fermionic modes become integrals and no dangerous volume factors remain. In our extended model with an additional two-body interaction $\mathbb{W}$ between the fermions, the situation is more delicate. The summations in the $\mathbb{W}$-terms generate factors of order L^d or L^{2d}. If one were to take $L \to \infty$ without adapting the coupling constant κ, the fermion–fermion contribution would remain *leading* and the effective dynamics would no longer coincide with the one derived in the non-interacting Fermi gas case. This is precisely why, in our proof, we choose

$$\kappa^2 = L^{-2d}\,, \tag{3.143}$$

which suppresses these volume factors before passing to the high-density regime $k_F \gg 1$. With this scaling, the $\mathbb{W}$-terms become subleading, and the large-volume limit can be taken either in two stages (first $L \to \infty$ at fixed k_F, then $k_F \gg 1$) or jointly with $k_F = O(L^\beta)$ for $0 < \beta < 1/2$. We emphasize that without such a scaling of κ, the argument from the ideal Fermi gas case does not carry over directly.

Chapter 4

Discussion

4.1 Challenges in the Large Volume Limit

Despite the clarity of our results, the current theoretical framework encounters significant challenges when considering large system volumes, particularly at the boundaries of the confining box. In our setting, the influence of the system's finite size, even with periodic boundary conditions often assumed in such models, becomes pronounced as $L \to \infty$. The effective decoupling we observe is contingent on specific scaling of interaction parameters.

4.1.1 Limitations of the Current Method

A key limitation of our current approach lies in its ability to control excitations within the fermionic system. The only way we have been able to manage these excitations and ensure the validity of our approximations is by carefully scaling the fermion-fermion coupling constant κ in relation to the system volume L. Specifically, by making κ sufficiently small in a volume-dependent manner, we effectively suppress problematic terms arising from fermion-fermion interactions that would otherwise destabilize our analysis or obscure the impurity dynamics. This choice of κ essentially acts as a compensatory mechanism, allowing us to tame the influence of the fermionic background. However, this method is somewhat restrictive; it implies that our results are most robust in a regime where fermion-fermion interactions are inherently weak or effectively screened by the large volume. It does not offer a direct method to incorporate or analyze strongly interacting fermionic systems where such suppression of κ would be physically unrealistic. This means that if we were to venture into regimes of stronger κ, our current methode would likely break down, necessitating a different approach to handle the more complex interplay of fermionic excitations.

4.1.2 Future Directions and New Theoretical Approaches

To overcome the challenges associated with large volumes and to achieve a more comprehensive understanding of these systems, new theoretical approaches are essential.

One idea for future research is to focus and perform calculations directly on the ground state of the system. A rigorous ground-state approach could offer a more stable and accurate foundation for understanding the effective interactions between impurities. Such an approach might provide a more natural way to account for the collective behavior of the fermions and their influence on the impurities without being overly constrained by the need to explicitly control excited states.

A second direction involves rethinking our definition of „proximity“ within the system. Instead of relying on global norms, which can be sensitive to boundary effects and long-range correlations in large volumes, we could explore a more local concept, perhaps by considering a form of scalar product rather than a traditional norm. This would entail focusing on how the impurities interact with localized fermionic densities or fluctuations, rather than assuming a uniform background.

Chapter 5

Conclusion and Outlook

Our research investigates the intriguing behavior of a small number of impurity particles immersed in a system of fermions within a d-dimensional box. We have shown that in the thermodynamic limit for high-density and under specific scaling of coupling constants, these impurities effectively decouple from the fermionic system. Their dynamics are dictated by an induced pairwise attraction, mediated by fluctuations within the Fermi gas. This finding provides valuable insights into the emergent properties of impurity-fermion systems in a regime where quantum many-body effects are paramount.

5.1 Future Directions

While this work establishes the validity of effective impurity dynamics in weakly interacting Fermi gases, several natural extensions and open questions remain:

1) In our analysis, we have assumed that the fermionic system is initially prepared in the non-interacting ground state Ω_0, corresponding to a filled Fermi ball. A natural question is: How would the situation change if one

considers an initial state that is the true ground state of the interacting Hamiltonian? The resolvent method could potentially be adapted to this setting by working in the interaction picture with respect to the interacting ground state, though this would require controlling additional correlation effects.

2) Our main result provides convergence in the L^2-norm, which is natural from a quantum mechanical perspective but comes at the cost of requiring the limit $L \to \infty$. One might ask: Can different results be obtained by working directly at finite L? Working at fixed L would eliminate the need to take the large volume limit and might yield effective Hamiltonians involving discrete sums rather than the integrals appearing in $W_{k_F}(r)$.

3) Our approach relies on weak coupling, which depends on the system size L. Using the resolvent method and considering the norm, it may be very difficult, to improve κ. A major challenge, therefore, is to extend these results to more strongly interacting Fermi gases by means of different techniques. One possible idea would be to adopt a different notion of proximity and to work with the scalar product instead of the norm, as discussed above.

4) While we have considered $d \in \{2, 3\}$, the dependence of our results on spatial dimension is non-trivial, as reflected in the scaling $\lambda^2 = k_F^{2-d}$. Exploring the behavior in one dimension or in higher dimensions could provide a more complete understanding of how dimensionality affects fermion-mediated interactions.

Appendix A

Bounds

A.1 Bounds and Computations

The following Lemma presents and proves a statement related to the summation of an indicator function over lattice points within an annulus. To simplify notation and computations, we have chosen a lattice distance of $1/L$ instead of $2\pi/L$. This does not affect the validity of the subsequent calculations.

Notation 4. *The following notations are used:*

- $d \in \{2, 3\}$ *denotes the dimension.*
- $L > 0$ *is a scaling factor for the lattice.*
- $B_R(u) := \{x \in \mathbb{R}^d : |x - u| < R\}$ *and* $B_R \equiv B_R(0)$.
- $B_R^L(u) := \{k \in \frac{1}{L}\mathbb{Z}^d : |k - u| < R\}$ *and* $B_R^L \equiv B_R^L(0)$.
- $B_{(R_1,R_2]}^L(u) := \{k \in \frac{1}{L}\mathbb{Z}^d : R_1 < |k - u| \leq R_2\}$ *and* $B_{(R_1,R_2]}^L \equiv B_{(R_1,R_2]}^L(0)$.
- $\mathbb{1}_A(x) = 1$ *if* $x \in A$ *and* 0 *otherwise.*

- *Angular coordinates for integration in* $\mathbb{R}^d$ *are denoted by* ω*.*

Lemma 5. *For* $m \in \frac{1}{L}\mathbb{Z}^d$*, and real numbers* $r > 0, R > 0, \delta > 0$ *such that* $r - \frac{\sqrt{d}}{2L} > 0$ *and* $|m| - R > 0$*, the following inequality holds:*

$$\sum_{n \in B^L_{(r,r+\delta]}} \mathbb{1}_{B_R(m)}(n) \tag{A.1}$$

$$\leq \mathbb{1}_{(|m|-R-\delta,|m|+R)}(r) \cdot L^d \left(r + \delta + \frac{\sqrt{d}}{2L} \right)^{d-1} \left(\delta + \frac{\sqrt{d}}{L} \right) \left(2 \arcsin \left(\frac{R + \frac{\sqrt{d}}{2L}}{|m| - R} \right) \right)^{d-1}$$

Proof. The sum on the left-hand side counts the number of lattice points $n \in \frac{1}{L}\mathbb{Z}^d$ that satisfy both $r < |n| \leq r + \delta$ and $|n - m| < R$. Let this number be

$$N' = |B^L_R(m) \cap B^L_{(r,r+\delta]}| \,. \tag{A.2}$$

The indicator function $\mathbb{1}_{(|m|-R-\delta,|m|+R)}(r)$ accounts for the necessary condition for this intersection to be non-empty. If r is outside the interval $(|m|-R-\delta, |m|+R)$, the annulus $B^L_{(r,r+\delta]}$ and the ball $B^L_R(m)$ are disjoint, and the sum is zero.

To bound N', we consider the continuous regions

$$B_R(m) = \{x \in \mathbb{R}^d : |x - m| < R\} \quad \text{and} \quad A = \{x \in \mathbb{R}^d : r < |x| \leq r + \delta\} \,. \tag{A.3}$$

The intersection of these continuous regions is $S_0 = B_R(m) \cap A$. A lattice point $n \in \frac{1}{L}\mathbb{Z}^d$ is counted in N' if and only if $n \in S_0$.

Each lattice point $n \in \frac{1}{L}\mathbb{Z}^d$ can be associated with a fundamental cell, for instance, the hypercube

$$\left[n_1 - \frac{1}{2L}, n_1 + \frac{1}{2L}\right) \times \cdots \times \left[n_d - \frac{1}{2L}, n_d + \frac{1}{2L}\right) \tag{A.4}$$

centered at n, which has volume $(1/L)^d$. If a lattice point n lies within S_0, some parts of its fundamental cell might fall outside S_0, and conversely, S_0 might contain parts of fundamental cells of lattice points not in S_0.

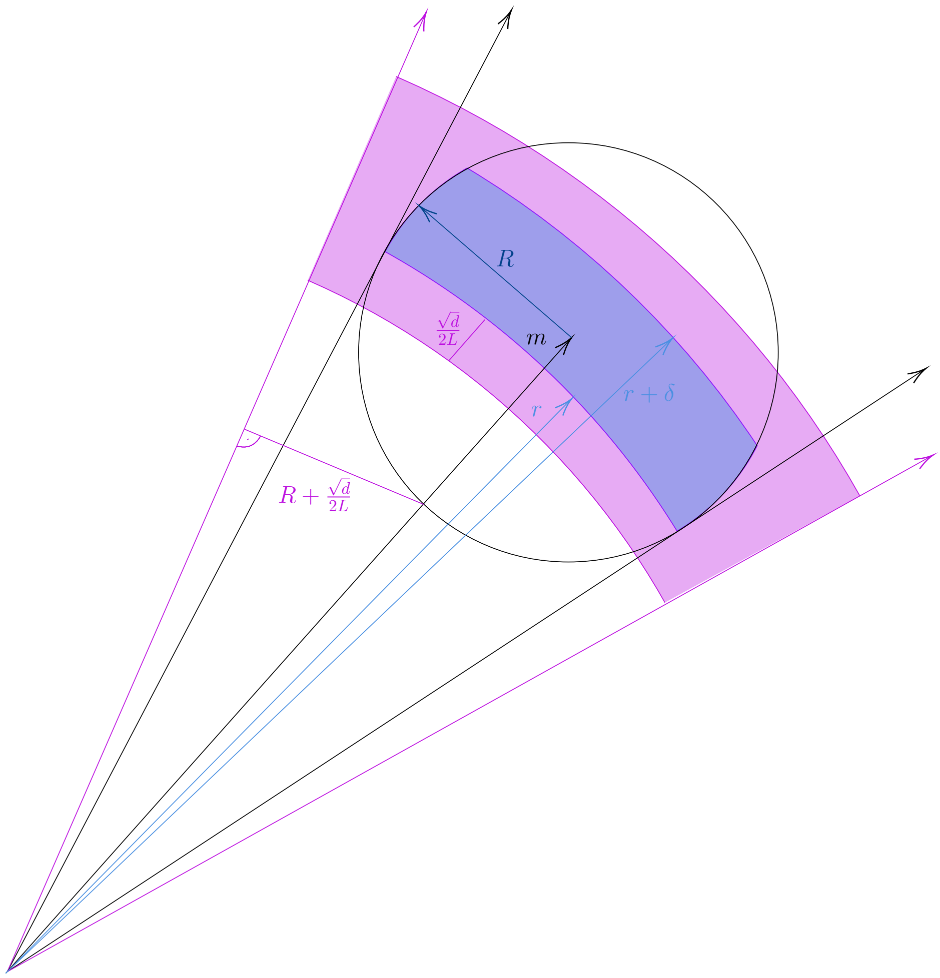

Figure A.1: The blue area denotes the intersection of the continuous regions, $S_0 = B_R(m) \cap A$. The pink area shows the enlarged region S', which contains all fundamental cells of lattice points in S_0.

To obtain a reliable upper bound on the number of lattice points in S_0, we consider an enlarged region S' such that if a lattice point n is in S_0, then its entire fundamental cell is contained within S'. The maximum distance from any point in the fundamental cell to its center n is half the length of the space diagonal, which is

$$\frac{1}{2}\sqrt{\left(\frac{1}{L}\right)^2 + \cdots + \left(\frac{1}{L}\right)^2} = \frac{\sqrt{d}}{2L}\,. \tag{A.5}$$

Therefore, if a lattice point n is in S_0, then n is within distance $\frac{\sqrt{d}}{2L}$ of the enlarged region S' defined as:

$$S' = B_{R+\frac{\sqrt{d}}{2L}}(m) \cap \left\{x \in \mathbb{R}^d : r - \frac{\sqrt{d}}{2L} < |x| < r + \delta + \frac{\sqrt{d}}{2L}\right\} \tag{A.6}$$

The number of lattice points N' in S_0 is then bounded by the volume of this enlarged region S' multiplied by the density of lattice points L^d

$$N' \leq L^d \cdot \mathrm{Vol}(S')\,. \tag{A.7}$$

The volume of S' can be estimated by integrating in spherical coordinates. The radial part of the integration is over $\rho \in (r - \frac{\sqrt{d}}{2L}, r + \delta + \frac{\sqrt{d}}{2L})$. The angular part is bounded by the solid angle subtended by $B_{R+\frac{\sqrt{d}}{2L}}(m)$ as viewed from the origin. An upper bound for the $(d-1)$-dimensional surface area of the intersection of the sphere of radius ρ with the enlarged ball is ρ^{d-1} times an upper bound for the solid angle, which is given by $\left(2\arcsin\left(\frac{R+\frac{\sqrt{d}}{2L}}{|m|-R}\right)\right)^{d-1}$. This bound holds under the condition $|m| - R > 0$.

Thus,

$$\begin{aligned}\operatorname{Vol}(S') &\leq \int_{r-\frac{\sqrt{d}}{2L}}^{r+\delta+\frac{\sqrt{d}}{2L}} \rho^{d-1}\left(2\arcsin\left(\frac{R+\frac{\sqrt{d}}{2L}}{|m|-R}\right)\right)^{d-1}\mathrm{d}\rho \\ &= \left(\int_{r-\frac{\sqrt{d}}{2L}}^{r+\delta+\frac{\sqrt{d}}{2L}} \rho^{d-1}\,\mathrm{d}\rho\right)\left(2\arcsin\left(\frac{R+\frac{\sqrt{d}}{2L}}{|m|-R}\right)\right)^{d-1}.\end{aligned} \tag{A.8}$$

The radial integral evaluates to

$$\int_{r-\frac{\sqrt{d}}{2L}}^{r+\delta+\frac{\sqrt{d}}{2L}} \rho^{d-1}\,\mathrm{d}\rho = \frac{1}{d}\left(\left(r+\delta+\frac{\sqrt{d}}{2L}\right)^{d}-\left(r-\frac{\sqrt{d}}{2L}\right)^{d}\right). \tag{A.9}$$

Using the inequality

$$b^d - a^d \leq d\cdot b^{d-1}(b-a) \tag{A.10}$$

for $b>a>0$, with $b=r+\delta+\frac{\sqrt{d}}{2L}$ and $a=r-\frac{\sqrt{d}}{2L}$, and noting that

$$b-a=\delta+\frac{\sqrt{d}}{L}, \tag{A.11}$$

we get

$$\frac{1}{d}\left(\left(r+\delta+\frac{\sqrt{d}}{2L}\right)^{d}-\left(r-\frac{\sqrt{d}}{2L}\right)^{d}\right) \leq \left(r+\delta+\frac{\sqrt{d}}{2L}\right)^{d-1}\left(\delta+\frac{\sqrt{d}}{L}\right). \tag{A.12}$$

Combining these bounds and including the indicator function, we obtain the stated inequality.

$$
\begin{aligned}
&\sum_{n \in B^L_{(r,r+\delta]}} \mathbb{1}_{B_R(n)}(m) && \text{(A.13)}\\
&\leq \mathbb{1}_{(|m|-R-\delta,|m|+R)}(r) \cdot L^d \left(r + \delta + \frac{\sqrt{d}}{2L}\right)^{d-1} \left(\delta + \frac{\sqrt{d}}{L}\right) \left(2 \arcsin\left(\frac{R + \frac{\sqrt{d}}{2L}}{|m| - R}\right)\right)^{d-1}
\end{aligned}
$$

□

Lemma 6. *Let $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function that is anti-monotone in the first and monotone in the second argument. Let further*

$$
f^{sup(L)} := \sup_{\substack{n,m \in \frac{1}{L}\mathbb{Z}^d \\ n \neq m}} f\left(|n|, |m|\right) . \tag{A.14}
$$

Then the following inequality holds

$$
\begin{aligned}
&L^{-2d} \sum_{\substack{n \in B^L_{[k_F,\infty)} \\ m \in B^L_{k_F}}} f\left(|n|, |m|\right) \mathbb{1}_{B_R(m)}(n) && \text{(A.15)}\\
&\leq f^{sup(L)} \frac{1}{L} \left(C + \widetilde{C}\right) k_F^{d-1} + \hat{C} \int_{k_F - R}^{k_F} \int_{k_F}^{k_F + R} f\left(\omega', \mu'\right) \, \mathrm{d}\omega' \, \mathrm{d}\mu' ,
\end{aligned}
$$

where

$$
\begin{aligned}
C &:= \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2} + 1\right)} \left(R + \frac{\sqrt{d}}{L}\right) \left(1 + \sqrt{d}\right) && \text{(A.16a)}\\
&\quad \cdot \left(\frac{\sqrt{d}}{2L}\right)^{d-1} \left(k_F + \frac{2 + \sqrt{d}}{2L}\right)^{d-1} \left(2 \arcsin\left(\frac{R + \frac{\sqrt{d}}{2L}}{k_F - 2R}\right)\right)^{d-1} , \\
\widetilde{C} &:= \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2} + 1\right)} \left(R + \frac{\sqrt{d}}{L}\right) \left(1 + \sqrt{d}\right) && \text{(A.16b)}
\end{aligned}
$$

$$\cdot \left(\frac{\sqrt{d}}{2L}\right)^{d-1} \left(k_F + R + \frac{\sqrt{d}}{L}\right)^{d-1} \left(2 \arcsin\left(\frac{R + \frac{\sqrt{d}}{2L}}{k_F - 2R}\right)\right)^{d-1},$$

$$\hat{C} := \left(1 + \sqrt{d}\right) \left(k_F + R + \frac{2 + \sqrt{d}}{2L}\right)^{d-1} \left(2 \arcsin\left(\frac{R + \frac{\sqrt{d}}{2L}}{k_F - 2R}\right)\right)^{d-1}. \tag{A.16c}$$

Proof. We aim to split the sum into parts: a critical region within an annulus around the Fermi momentum k_F, and the remaining less critical regions. We find

$$\begin{aligned} &L^{-2d} \sum_{\substack{n \in B^L_{[k_F, \infty)} \\ m \in B^L_{k_F}}} f\left(|n|, |m|\right) \mathbb{1}_{B_R(m)}(n) \\ &\leq L^{-2d} \Bigg(f^{\sup(L)} \Bigg(\underbrace{\sum_{\substack{n \in B^L_{[k_F, k_F + \frac{1}{L})} \\ m \in B^L_{[k_F - R, k_F)}}} \mathbb{1}_{B_R(m)}(n)}_{=:(1)} + \underbrace{\sum_{\substack{n \in B^L_{[k_F, k_F + R)} \\ m \in B^L_{[k_F - \frac{1}{L}, k_F)}}} \mathbb{1}_{B_R(m)}(n)}_{=:(2)} \Bigg) \\ &\quad + \sum_{\substack{n \in B^L_{[k_F + \frac{1}{L}, k_F + R)} \\ m \in B^L_{[k_F - R, k_F - \frac{1}{L})}}} f\left(|n|, |m|\right) \mathbb{1}_{B_R(m)}(n) \Bigg). \end{aligned} \tag{A.17}$$

In the following we are going to estimate the three sums separately. Starting with the first one we get

$$\begin{aligned} &(1) \\ &\leq L^d \sum_{m \in B^L_{[k_F - R, k_F)}} \left(k_F + \frac{2 + \sqrt{d}}{2L}\right)^{d-1} \left(\frac{1 + \sqrt{d}}{L}\right) \left(2 \arcsin\left(\frac{R + \frac{\sqrt{d}}{2L}}{k_F - 2R}\right)\right)^{d-1} \\ &\leq L^{2d} \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2} + 1\right)} \left(k_F + \frac{\sqrt{d}}{2L}\right)^{d-1} \left(R + \frac{\sqrt{d}}{L}\right) \left(k_F + \frac{2 + \sqrt{d}}{2L}\right)^{d-1} \\ &\qquad \cdot \left(\frac{1 + \sqrt{d}}{L}\right) \left(2 \arcsin\left(\frac{R + \frac{\sqrt{d}}{2L}}{k_F - 2R}\right)\right)^{d-1}, \end{aligned} \tag{A.18}$$

where in the first step we used (5) and in the second one we used similar estimates to count the number of points in the annulus than before. This yields

$$L^{-2d}\, f^{\sup(L)}\,(1) \leq f^{\sup(L)} \frac{C}{L}\, k_F^{d-1}\,, \tag{A.19}$$

where

$$\begin{aligned} C =& \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} \left(\frac{\sqrt{d}}{2L}\right)^{d-1} \left(R + \frac{\sqrt{d}}{L}\right) \left(1+\sqrt{d}\right) \\ &\cdot \left(k_F + \frac{2+\sqrt{d}}{2L}\right)^{d-1} \left(2 \arcsin\left(\frac{R+\frac{\sqrt{d}}{2L}}{k_F - 2R}\right)\right)^{d-1} . \end{aligned} \tag{A.20}$$

For the second sum we analogously obtain

$$L^{-2d}\, f^{\sup(L)}\,(2) \leq f^{\sup(L)} \frac{\widetilde{C}}{L}\, k_F^{d-1}\,, \tag{A.21}$$

where

$$\begin{aligned} \widetilde{C} =& \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} \left(\frac{\sqrt{d}}{2L}\right)^{d-1} \left(R + \frac{\sqrt{d}}{L}\right) \left(1+\sqrt{d}\right) \\ &\cdot \left(k_F + R + \frac{\sqrt{d}}{L}\right)^{d-1} \left(2 \arcsin\left(\frac{R+\frac{\sqrt{d}}{2L}}{k_F - 2R}\right)\right)^{d-1} . \end{aligned} \tag{A.22}$$

Now we estimate the third sum. Let m be fixed with $|m| \geq k_F - R$ and further assume that $k_F \cdot L \in \mathbb{N}$ for simplicity. We want to split the sum into rings of thickness $1/L$ to find

$$\sum_{n \in B^L_{[k_F + \frac{1}{L}, k_F + R)}} f\left(|n|, |m|\right) \mathbb{1}_{B_R(m)}(n) \tag{A.23}$$

$$= \sum_{\substack{r\in\mathbb{N} \\ L(k_F+R)\geq r\geq k_F\cdot L+1}} \sum_{n\in B^L_{\left(\frac{r}{L},\frac{r+1}{L}\right]}} f\left(|n|,|m|\right) \mathbb{1}_{B_{R(m)}}(n)$$

$$\leq \sum_{\substack{r\in\mathbb{N} \\ L(k_F+R)\geq r\geq k_F\cdot L+1}} f\left(\frac{r}{L},|m|\right) \sum_{n\in B^L_{\left(\frac{r}{L},\frac{r+1}{L}\right]}} \mathbb{1}_{B_{R(m)}}(n)\,,$$

where we used the anti-monotony of f in n, i.e. $|n| \geq r/L$. Using (5) we find

$$\leq \sum_{\substack{r\in\mathbb{N} \\ L(k_F+R)\geq r\geq k_F\cdot L+1}} f\left(\frac{r}{L},|m|\right)\left(k_F+R+\frac{2+\sqrt{d}}{2L}\right)^{d-1} \cdot \left(\frac{1+\sqrt{d}}{L}\right)\left(2\arcsin\left(\frac{R+\frac{\sqrt{d}}{2L}}{k_F-2R}\right)\right)^{d-1}$$

$$= L^{d-1}\,\hat{C} \sum_{\substack{r\in\mathbb{N} \\ L(k_F+R)\geq r\geq k_F\cdot L+1}} f\left(\frac{r}{L},|m|\right)\,. \tag{A.24}$$

Here we defined

$$\hat{C} = \left(k_F+R+\frac{2+\sqrt{d}}{2L}\right)^{d-1}\left(1+\sqrt{d}\right)\left(2\arcsin\left(\frac{R+\frac{\sqrt{d}}{2L}}{k_F-2R}\right)\right)^{d-1}\,. \tag{A.25}$$

Since f is anti-monotone in n, we can use the inequality

$$f\left(\frac{k_F\cdot L+1}{L},|m|\right) = \int_{k_F\cdot L}^{k_F\cdot L+1} f\left(\frac{k_F\cdot L+1}{L},|m|\right)\mathrm{d}\omega \leq \int_{k_F\cdot L}^{k_F\cdot L+1} f\left(\frac{\omega}{L},|m|\right)\mathrm{d}\omega\,, \tag{A.26}$$

which allows us to write

$$L^{d-1}\,\hat{C} \sum_{\substack{r\in\mathbb{N} \\ L(k_F+R)\geq r\geq k_F\cdot L+1}} f\left(\frac{r}{L},|m|\right) \tag{A.27}$$

$$
\begin{aligned}
&= L^{d-1}\,\hat{C} \sum_{\substack{r\in\mathbb{N} \\ L(k_F+R)\geq r\geq k_F\cdot L+1}} \int_{r-1}^{r} f\left(\frac{r}{L}, |m|\right)\,\mathrm{d}\omega \\
&\leq L^{d-1}\,\hat{C} \int_{k_F\cdot L}^{L(k_F+R)} f\left(\frac{\omega}{L}, |m|\right)\,\mathrm{d}\omega \\
&= L^{d}\,\hat{C} \int_{k_F}^{k_F+R} f\left(\omega', |m|\right)\,\mathrm{d}\omega'\,,
\end{aligned}
$$

where the last step uses the substitution $\omega \mapsto \omega' = \omega/L$. Now we need to use the exact method again for m backwards by using that f is monotone in m, to obtain

$$
\sum_{\substack{n\in B^L_{[k_F+\frac{1}{L},k_F+R)} \\ m\in B^L_{[k_F-R,k_F-\frac{1}{L})}}} f\left(|n|,|m|\right)\mathbb{1}_{B_R(m)}(n) \leq L^{2d}\,\hat{C} \int_{k_F-R}^{k_F} \int_{k_F}^{k_F+R} f\left(\omega',\mu'\right)\,\mathrm{d}\omega'\,\mathrm{d}\mu' \quad \text{(A.28)}
$$

Combining the bounds, we obtain the stated inequality. □

We now establish analogous results in the continuous setting, obtained by taking the limit $L\to\infty$, whereby the sums are replaced by Riemann integrals.

Lemma 7. *Let $d\in\{2,3\}$, $D>0$ and $k\in\mathbb{R}^3$ with $|k|\geq 3D$. Then*

$$
\int_\Omega \mathrm{d}\Omega\;\mathbb{1}_{B_D(k)}\left(l\right) \leq \mathbb{1}_{[|k|-D,|k|+D]}\left(|l|\right)\cdot\left(2\arcsin\left(\frac{D}{|k|}\right)\right)^{d-1}. \quad \text{(A.29)}
$$

Here, we denote with $B_D\left(k\right)$ the sphere around k with radius D and $\mathrm{d}\Omega$ is the integral over the space angle(s) and $l=|l|\cdot e_\Omega$, with e being the unit vector.

Proof. We split the proof into two parts and prove the two- and three-dimensional part separately.

dim = 2: We can w.l.o.g. assume that

$$k = |k| \begin{pmatrix} \cos(\varphi_k) \\ \sin(\varphi_k) \end{pmatrix} \tag{A.30}$$

lies along the y-axis, and otherwise we can perform a rotation $\varphi \to \varphi - \varphi_k + \pi/2$. In two dimensions it holds, that

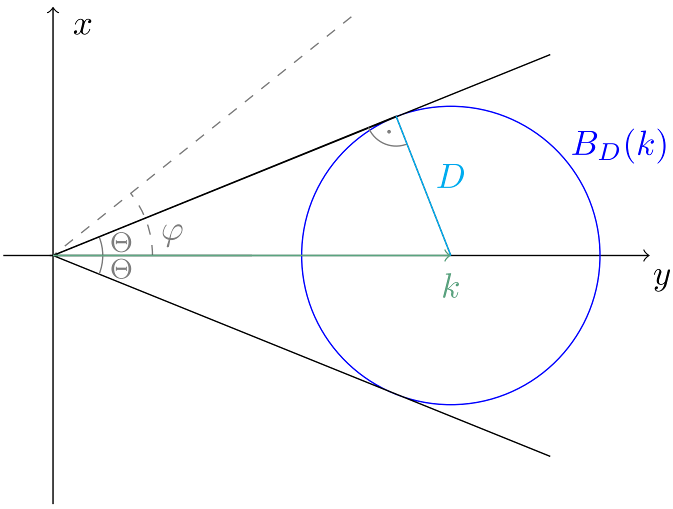

Figure A.2: Picture of the proof in two dimensions.

$$\int_\Omega \mathrm{d}\Omega \; \mathbb{1}_{B_D(k)}(l) = \int_{-\pi}^{\pi} \mathrm{d}\varphi \; \mathbb{1}_{B_D(k)} \left(|l| \cdot \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \end{pmatrix} \right) . \tag{A.31}$$

The angle φ is taken against the y-axis (see picture). It is clear that, independent on the additional constraint on l,

$$\mathbb{1}_{B_D(k)} \left(|l| \cdot \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \end{pmatrix} \right) \tag{A.32}$$

evaluates to zero, if $\varphi \notin [-\Theta, \Theta]$. Independent of φ it further evaluates to zero, if

$|k| - D \leq |l| \leq |k| + D$. This clearly means

$$\mathbb{1}_{B_D(k)} \left(|l| \cdot \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \end{pmatrix} \right) \leq \mathbb{1}_{[-\Theta,\Theta]}(\varphi)\, \mathbb{1}_{[|k|-D,|k|+D]}(|l|) \;. \tag{A.33}$$

Trigonometry gives for $|k| \geq 3D$

$$\begin{aligned} \sin(\Theta) &= \frac{D}{|k|} \qquad \text{(A.34)} \\ \Longleftrightarrow \qquad \Theta &= \arcsin\left(\frac{D}{|k|}\right) \;. \end{aligned}$$

Therefore

$$\begin{aligned} \int_\Omega \mathrm{d}\Omega\ \mathbb{1}_{B_D(k)}(l) &= \int_{-\pi}^{\pi} \mathrm{d}\varphi\ \mathbb{1}_{B_D(k)} \left(|l| \cdot \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \end{pmatrix} \right) \qquad \text{(A.35)} \\ &\leq \int_{-\pi}^{\pi} \mathrm{d}\varphi\ \mathbb{1}_{[-\arcsin(D/|k|),\arcsin(D/|k|)]}(\varphi)\, \mathbb{1}_{[|k|-D,|k|+D]}(|l|) \\ &= 2\arcsin\left(\frac{D}{|k|}\right) \mathbb{1}_{[|k|-D,|k|+D]}(|l|) \;. \end{aligned}$$

$dim = 3$: We assume w.l.o.g. that

$$k = \begin{pmatrix} |k| \\ 0 \\ 0 \end{pmatrix} \;. \tag{A.36}$$

We have

$$\int_\Omega \mathrm{d}\Omega\ \mathbb{1}_{B_D(k)}(l) \tag{A.37}$$

$$= \int_{-\pi}^{\pi} \mathrm{d}\varphi \int_{0}^{\pi} \sin(\Theta)\ \mathrm{d}\lambda\ \mathbb{1}_{B_D(k)} \left(|l| \cdot \begin{pmatrix} \sin(\lambda)\cos(\varphi) \\ \sin(\lambda)\sin(\varphi) \\ \cos(\lambda) \end{pmatrix} \right).$$

In the x-y-plane, we get Hence, we can conclude with all the same steps as in the

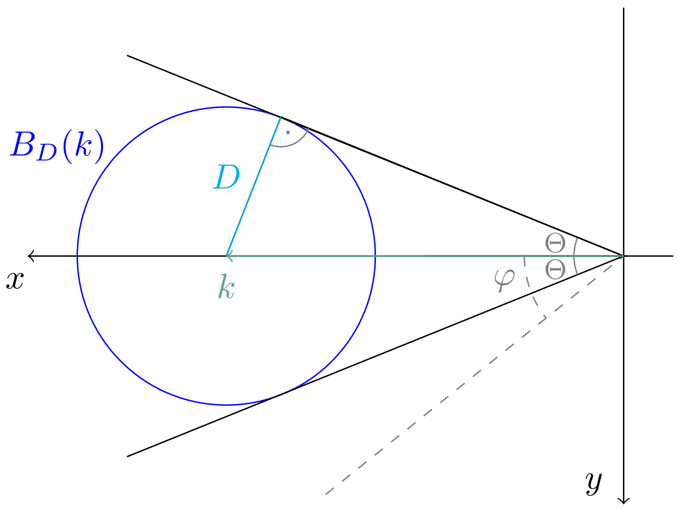

Figure A.3: Picture of the proof in three dimensions, looking at the x-y-plane.

two-dimensional case, that

$$\mathbb{1}_{B_D(\vec{k})} \left(l \cdot \begin{pmatrix} \sin(\lambda)\cos(\varphi) \\ \sin(\lambda)\sin(\varphi) \\ \cos(\lambda) \end{pmatrix} \right) \tag{A.38}$$

is zero, if $\varphi \notin [\arcsin(D/|k|), \arcsin(D/|k|)]$. In the x-z-plane we get Hence,

$$\mathbb{1}_{B_D(k)} \left(|l| \cdot \begin{pmatrix} \sin(\lambda)\cos(\varphi) \\ \sin(\lambda)\sin(\varphi) \\ \cos(\lambda) \end{pmatrix} \right) \tag{A.39}$$

is zero, if $\lambda \notin [\pi/2 - \arcsin(D/|k|), \pi/2 + \arcsin(D/|k|)]$. With the constraint

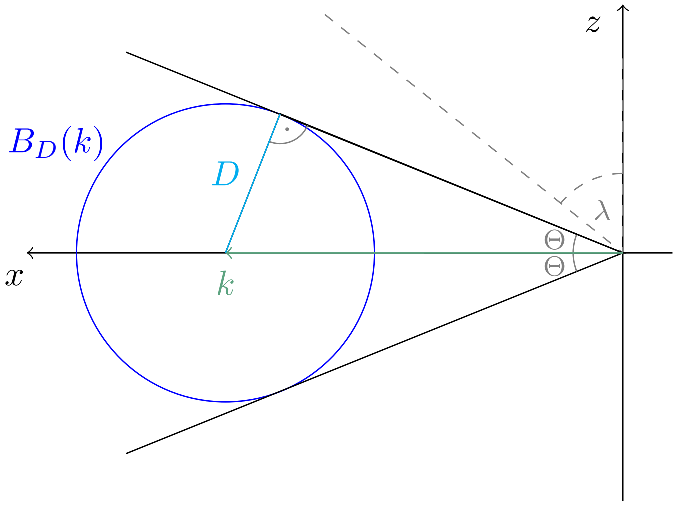

Figure A.4: Picture of the proof in three dimensions, looking at the x-z-plane.

on l we finally get

$$\mathbb{1}_{B_D(k)}\left(|l|\cdot\begin{pmatrix}\sin(\lambda)\cos(\varphi)\\ \sin(\lambda)\sin(\varphi)\\ \cos(\lambda)\end{pmatrix}\right) \tag{A.40}$$

$$\leq \mathbb{1}_{[\pi/2-\arcsin(D/|k|),\pi/2+\arcsin(D/|k|)]}(\lambda)\,\mathbb{1}_{[-\arcsin(D/|k|),\arcsin(D/|k|)]}(\varphi)\,\mathbb{1}_{[|k|-D,|k|+D]}(|l|)\ .$$

Therefore, with $\sin(\lambda)\leq 1$,

$$\begin{aligned}&\int_\Omega \mathrm{d}\Omega\ \mathbb{1}_{B_D(k)}(l) &\text{(A.41)}\\ &=\int_{-\pi}^{\pi}\mathrm{d}\varphi\int_0^{\pi}\sin(\Theta)\ \mathrm{d}\lambda\ \mathbb{1}_{B_D(k)}\left(|l|\cdot\begin{pmatrix}\sin(\lambda)\cos(\varphi)\\ \sin(\lambda)\sin(\varphi)\\ \cos(\lambda)\end{pmatrix}\right)\\ &\leq\int_{-\pi}^{\pi}\mathrm{d}\varphi\int_0^{\pi}\mathrm{d}\lambda\ \mathbb{1}_{[\pi/2-\arcsin(D/|k|),\pi/2+\arcsin(D/|k|)]}(\lambda)\\ &\qquad\qquad\mathbb{1}_{[-\arcsin(D/|k|),\arcsin(D/|k|)]}(\varphi)\,\mathbb{1}_{[|k|-D,|k|+D]}(|l|)\end{aligned}$$

$$\leq 4\left(\arcsin\left(\frac{D}{|k|}\right)\right)^2 \mathbb{1}_{[|k|-D,|k|+D]}\left(|l|\right) .$$

□

Lemma 8. *Let $d \in \{2,3\}$. Suppose the functions $\hat{v},\hat{w}$ are compactly supported in Fourier space, i.e.,*

$$\mathrm{supp}(\hat{v}) \subseteq [-D,D]^d, \quad \mathrm{supp}(\hat{w}) \subseteq [-D',D']^d \tag{A.42}$$

for some constants $D, D' > 0$. There exist constants $C, \widetilde{C}, \hat{C} > 0$, defined in (A.16a)–(A.16c)*, such that for $L, k_F \gg 1$, the following bounds hold:*

a)

$$L^{-2d} \sum_{(n,m)\in T_F} |\hat{v}(n-m)|^2 \leq \frac{c}{L}\left(C+\widetilde{C}\right) k_F^{(d-1)} + c\hat{C}D^2 \tag{A.43a}$$

b)

$$L^{-2d} \sum_{(n,m)\in T_F} \frac{|\hat{v}\,(n-m)|^2}{(|n|-|m|+k_F^{-1})} \leq \frac{1}{1+\frac{L}{k_F}}\left(C+\widetilde{C}\right) k_F^{(d-1)} + \hat{C}\mathcal{O}\left(\frac{\ln\left(k_F\right)}{k_F}\right) \tag{A.43b}$$

c)

$$L^{-2d} \sum_{(n,m)\in T_F} \frac{|\hat{v}\,(n-m)|^2}{(|n|-|m|+k_F^{-1})^2} \leq \frac{1}{\left(L^{-1/2}+\frac{L^{1/2}}{k_F}\right)^2}\left(C+\widetilde{C}\right) k_F^{d-1} + \hat{C}\mathcal{O}\left(\ln(k_F)\right) \tag{A.43c}$$

d)

$$L^{-2d} \sum_{(n,m)\in T_F} \frac{|\hat{v}\,(n-m)|^2\,(n-m)^2}{(|n|-|m|+k_F^{-1})^2} \tag{A.43d}$$
$$\leq \frac{1}{\left(L^{-1/2}+\frac{L^{1/2}}{k_F}\right)^2}\left(C+\widetilde{C}\right)k_F^{d-1}+\hat{C}\mathcal{O}\left(\ln(k_F)\right)$$

e)

$$L^{-d}\sum_{m\in B_F}\left(L^{-d}\sum_{n\in B_F^C}\frac{|\hat{v}(n-m)|}{\left(|n|-|m|+k_F^{-1}\right)}\right)^2 \leq k_F^{(d-1)}\ln\left(k_F\right)^2\cdot C^{e2)}\,, \tag{A.43e}$$

where

$$C^{e2)} := \left(\frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}\left(\frac{\sqrt{d}}{2L}\right)^{d-1}\left(R+\frac{\sqrt{d}}{L}\right)\right)\cdot\left(\frac{C^{e)}}{1+\frac{L}{k_F}}+\hat{C}^2\,\mathbb{1}_{[-D-(1/L),0)}(|m|-k_F)\right)^2$$
$$\cdot\left(\ln\left(k_F\right)^{-2}+\ln\left(k_F\right)^{-1}+1\right)\,,$$
$$C^{e)} := \left(k_F+\frac{2+\sqrt{d}}{2L}\right)^{d-1}\left(1+\sqrt{d}\right)\left(2\arcsin\left(\frac{D+\frac{\sqrt{d}}{2L}}{k_F-2D}\right)\right)^{d-1}\,.$$

f)

$$L^{-d}\sum_{n\in B_F^C}\left(L^{-d}\sum_{m\in B_F}\frac{|\hat{v}(n-m)|}{\left(|n|-|m|+k_F^{-1}\right)}\right)^2 \leq k_F^{(d-1)}\ln\left(k_F\right)^2\cdot C^{f2)}\,, \tag{A.43f}$$

where

$$C^{f2)} := \left(\frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}\left(\frac{\sqrt{d}}{2L}\right)^{d-1}\left(R+\frac{\sqrt{d}}{L}\right)\right)\cdot\left(\frac{C^{f)}}{1+\frac{L}{k_F}}+\hat{C}^2\,\mathbb{1}_{[0,D)}(|n|-k_F)\right)^2$$
$$\cdot\left(\ln\left(k_F\right)^{-2}+\ln\left(k_F\right)^{-1}+1\right)\,,$$

$$C^{f)} := \left(k_F + \frac{\sqrt{d}}{2L}\right)^{d-1} \left(R + \sqrt{d}\right) \left(2 \arcsin\left(\frac{D + \frac{\sqrt{d}}{2L}}{k_F - 2D}\right)\right)^{d-1} .$$

g)

$$L^{-2d} \sum_{(n,k)\in T_F} \left(L^{-d} \sum_{m\in B_F} \frac{|\hat{v}(m-k)|\ |\hat{v}(n-m)|}{\left(|n| - |m| + k_F^{-1}\right)}\right)^2 \tag{A.43g}$$
$$\leq k_F^{d-1} \ln\left(k_F\right)^2 \cdot C^{f2)}$$

h)

$$L^{-2d} \sum_{(l,m)\in T_F} \left(L^{-d} \sum_{n\in B_F^C} \frac{|\hat{v}(l-n)|\ |\hat{v}(n-m)|}{\left(|n| - |m| + k_F^{-1}\right)}\right)^2 \tag{A.43h}$$
$$\leq k_F^{d-1} \ln\left(k_F\right)^2 \cdot C^{e2)}$$

i)

$$L^{-2d} \sum_{h,n\in B_F^C} \left(L^{-2d} \sum_{(l,k)\in T_F} \frac{|\hat{v}(n-k)|\ |\hat{v}(l-k)|\ |\hat{v}(h-l)|}{\left(|n| - |k| + k_F^{-1}\right)\left(|l| - |k| + k_F^{-1}\right)}\right)^2$$
$$\leq k_F^{2d-2} \ln\left(k_F\right)^4 \cdot \left(C^{e2)}\right)^2 \tag{A.43i}$$

j)

$$L^{-2d} \sum_{m,k\in B_F} \left(L^{-2d} \sum_{n,l\in B_F^C} \frac{|\hat{v}(n-m)|\ |\hat{v}(l-k)|\ |\hat{v}(n-l)|}{\left(|n| - |m| + k_F^{-1}\right)\left(|l| - |k| + k_F^{-1} + 1\right)}\right)^2$$
$$\leq k_F^{2d-2} \ln\left(k_F\right)^4 \cdot \left(C^{e2)}\right)^2 \tag{A.43j}$$

k)

$$L^{-3d}\sum_{(n,m)\in T_F}\sum_{l\in B_F}\frac{|\hat{v}(n-m)|\,|\hat{v}(l-m)|\,|\hat{v}(n-l)|}{\left(|n|-|m|+k_F^{-1}\right)\left(|l|-|m|+k_F^{-1}\right)} \tag{A.43k}$$
$$\leq k_F^{d-1}\ln\left(k_F\right)^2\cdot C^{e2)}$$

l)

$$L^{-2d}\sum_{m\in B_F}\left(L^{-d}\sum_{h,g\in T_F}\sum_{n\in B_F^C}\frac{|\hat{v}(h-g)|\,|\hat{v}(n-m)|}{\left(|h|-|g|+k_F^{-1}\right)\left(|n|-|m|+k_F^{-1}\right)}\right)^2$$
$$\leq L^{2d}\left(\frac{1}{1+\frac{L}{k_F}}\left(C+\widetilde{C}\right)k_F^{(d-1)}+\hat{C}\mathcal{O}\left(\frac{\ln\left(k_F\right)}{k_F}\right)\right)^2\cdot L^d k_F^{d-1}\ln\left(k_F\right)^2\cdot C^{e2)} \tag{A.43l}$$

m)

$$L^{-2d}\sum_{(p,q)\in N_k^F}\sum_{k\in[-D,D]^d}|\hat{w}(k)|^2 \tag{A.43m}$$
$$\leq L^{2d-2}k_F^{2d-2}c'\left(C^2+\widetilde{C}^2+2\widetilde{C}C\right)+L^{2d}\hat{C}^2D^4$$

n)

$$L^{-2d}\sum_{p\in M_k^F}\sum_{k\in[-D,D]^d}|\hat{w}(k)|^2\leq\frac{c''}{L}\left(2C+2\widetilde{C}\right)k_F^{d-1}+\hat{C}D^2 \tag{A.43n}$$

o)

$$L^{-2d}\sum_{(p,q)\in N_k^F}\sum_{k\in[-D,D]^d}\frac{|\hat{w}(k)|^2}{\left(|p|-|p-k|+|q|-|q+k|+k_F^{-1}\right)^2}$$

$$\leq L^{2d} \frac{1}{\left(2+\frac{L}{k_F}\right)^2} \left(C^2 + \widetilde{C}^2 + 2C\widetilde{C}\right) k_F^{2d-2} + \ L^{2d} \hat{C}^2 \cdot \mathcal{O}(1) \tag{A.43o}$$

p)

$$L^{-2d} \sum_{p \in M_k^F} \sum_{k \in [-D,D]^d} \tag{A.43p}$$

$$\leq \ \left(2D + \frac{1}{L}\right)^d \cdot \frac{d\sqrt{d}\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} \left(k_F + \sqrt{d}D + \frac{\sqrt{d}}{2L}\right)^{d-1} \left(D + \frac{1}{L}\right)$$

q)

$$L^{-2d} \sum_{(q,p) \in N_k^F} \sum_{k \in [-D,D]^d} \tag{A.43q}$$

$$\leq \ \left(2D + \frac{1}{L}\right)^d \cdot L^d \cdot \left(\frac{d\sqrt{d}\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}\right)^2 \left(k_F + \sqrt{d}D + \frac{\sqrt{d}}{2L}\right)^{2d-2} \left(D + \frac{1}{L}\right)^2$$

r)

$$L^{-2d} \sum_{(q,p) \in N_k^F} \sum_{k \in [-D,D]^d} \frac{|\hat{w}(k)|^2}{\left(|p| - |p-k| + |q| - |q+k| + k_F^{-1}\right)} \tag{A.43r}$$

$$\leq L^{2d} \frac{1}{\left(2L+\frac{L^2}{k_F}\right)^2} \left(C^2 + \widetilde{C}^2 + 2C\widetilde{C}\right) k_F^{2d-2} + \ L^{2d} \hat{C}^2 \cdot \mathcal{O}(1)$$

s)

$$L^{-2d} \sum_{m \in B_F} \leq L^{-d} \cdot \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} \left(k_F + \frac{\sqrt{d}}{2L}\right)^d \tag{A.43s}$$

t)

$$L^{-2d} \sum_{k\in[-D,D]^d} \sum_{(p,g)\in T_F} \sum_{h\in B_F^C} \frac{|\hat{w}(h-(p-k))||\hat{v}(p-g)|}{|p|-|g|+k_F^{-1}} \tag{A.43t}$$
$$\leq \left(\frac{1}{1+\frac{L}{k_F}} \left(C+\widetilde{C}\right) k_F^{(d-1)} + \hat{C}\mathcal{O}\left(\frac{\ln(k_F)}{k_F}\right)\right) (2DL+1)^d L^d C',$$

where

$$C' := \left(D+\frac{\sqrt{d}}{L}\right)\left(1+\sqrt{d}\right)\cdot\left(k_F+D+\frac{2+\sqrt{d}}{2L}\right)^{d-1}\left(2\arcsin\left(\frac{D+\frac{\sqrt{d}}{2L}}{k_F-2D}\right)\right)^{d-1}$$

u)

$$L^{-2d} \sum_{m\in B_F} \sum_{(h,g)\in T_F} \left(L^{-d} \sum_{n\in B_F^C} \frac{|\hat{v}(h-g)||\hat{v}(n-m))|}{\left(|h|-|g|+k_F^{-1}\right)\left(|n|-|m|+k_F^{-1}\right)}\right)^2$$
$$\leq \left(\frac{1}{\left(L^{-1/2}+\frac{L^{1/2}}{k_F}\right)^2} \left(C+\widetilde{C}\right) k_F^{d-1} + \hat{C}\mathcal{O}\left(\ln(k_F)\right)\right) \cdot L^d k_F^{d-1} \ln(k_F)^2 \cdot C^{e2)} \tag{A.43u}$$

v)

$$L^{-2d} \sum_{k\in[-D,D]^d} \sum_{q\in M_k^F} \frac{|\hat{v}(k)|^2 \ |\hat{w}(k)|^2}{\left(|q|-|q+k|+k_F^{-1}\right)^2} \tag{A.43v}$$
$$\leq (2DL+1)^d \cdot \frac{1}{1+\frac{L}{k_F}} \left(C+\widetilde{C}\right) k_F^{(d-1)} + \hat{C}\mathcal{O}\left(\frac{\ln(k_F)}{k_F}\right)$$

Proof. For all the following computations and bounds we use (5) and (6).

a) To bound the sum

$$L^{-2d} \sum_{(n,m)\in T_F} |\hat{v}(n-m)|^2 \tag{A.44}$$

let $f(|n|,|m|) = c$ be a constant function with $c > 0$, which is anti-monotone and monotone in both coordinates. Then

$$\begin{aligned} L^{-2d} \sum_{(n,m)\in T_F} |\hat{v}(n-m)|^2 &\leq cL^{-2d} \sum_{\substack{n\in B^L_{[k_F,\infty)} \\ m\in B^L_{k_F}}} \mathbb{1}_{B_D(m)}(n) \\ &\leq \frac{c}{L}\left(C+\widetilde{C}\right) k_F^{(d-1)} + c\hat{C} \int\limits_{k_F-D}^{k_F} \int\limits_{k_F}^{k_F+D} \mathrm{d}\omega' \, \mathrm{d}\mu' \\ &\leq \frac{c}{L}\left(C+\widetilde{C}\right) k_F^{(d-1)} + c\hat{C}D^2 \, , \end{aligned} \tag{A.45}$$

where the constants $C, \widetilde{C}$ and $\hat{C}$ are defined in (A.16a)-(A.16c) above. For all constants we use the fact that the function $x \mapsto x \cdot \arcsin(1/x)$ is monotonically decreasing. Therefore, for increasing values of L and k_F, the constants above contribute only marginally and we have

$$L^{-2d} \sum_{(n,m)\in T_F} |\hat{v}(n-m)|^2 \leq \frac{c}{L}\left(C+\widetilde{C}\right) k_F^{(d-1)} + \mathcal{O}(1) \tag{A.46}$$

b) Similar to the computations before we bound the sum

$$L^{-2d} \sum_{(n,m)\in T_F} \frac{|\hat{v}\,(n-m)|^2}{(|n|-|m|+k_F^{-1})} \leq L^{-2d} \sum_{\substack{n\in B^L_{[k_F,\infty)} \\ m\in B^L_{k_F}}} \frac{1}{(|n|-|m|+k_F^{-1})} \mathbb{1}_{B_D(m)}(n) \tag{A.47}$$

Then we find, by using (6),

$$
\begin{aligned}
&L^{-2d} \sum_{\substack{n\in B^L_{[k_F,\infty)}\\ m\in B^L_{k_F}}} f\left(|n|,|m|\right) \mathbb{1}_{B_D(m)}(n) \qquad \text{(A.48)}\\
&\leq f^{\sup(L)} \frac{1}{L}\left(C+\widetilde{C}\right)\left(k_F+\frac{\sqrt{d}}{2L}\right)^{d-1} + \hat{C}\int\limits_{k_F-D}^{k_F}\int\limits_{k_F}^{k_F+D} f(\omega',\mu')\,\mathrm{d}\omega'\,\mathrm{d}\mu'\,,
\end{aligned}
$$

where

$$
f\left(|n|,|m|\right) = \frac{1}{\left(|n|-|m|+k_F^{-1}\right)}\,, \qquad \text{(A.49)}
$$

and $C, \widetilde{C}$ and $\hat{C}$ as defined in (A.16a)-(A.16c). We again use te fact that $x \mapsto x\cdot \arcsin\left(1/x\right)$ is monotonically decreasing. Therefore, for increasing values of L and k_F, the constants above contribute only marginally. With

$$
\begin{aligned}
f^{\sup(L)} &:= \sup_{\substack{n,m\in\frac{1}{L}\mathbb{Z}^d\\ n\neq m}} f\left(|n|,|m|\right) \qquad \text{(A.50)}\\
&= \sup_{\substack{n,m\in\frac{1}{L}\mathbb{Z}^d\\ n\neq m}} \frac{1}{\left(|n|-|m|+k_F^{-1}\right)}\\
&= \frac{1}{\frac{1}{L}+k_F^{-1}}\,,
\end{aligned}
$$

we then have

$$
\begin{aligned}
&L^{-2d}\sum_{(n,m)\in T_F} \frac{|\hat{v}\left(n-m\right)|^2}{\left(|n|-|m|+k_F^{-1}\right)} \qquad \text{(A.51)}\\
&\leq \frac{1}{1+\frac{L}{k_F}}\left(C+\widetilde{C}\right)k_F^{(d-1)} + \hat{C}\int\limits_{k_F-D}^{k_F}\int\limits_{k_F}^{k_F+D}\frac{1}{\omega'-\mu'+k_F^{-1}}\,\mathrm{d}\omega'\,\mathrm{d}\mu'\,.
\end{aligned}
$$

Lets have a look and the integrals. Using $\omega' \mapsto r-k_F$ and $\mu' \mapsto s-k_F$, we find

$$\begin{aligned}
&\hat{C} \int_{k_F-D}^{k_F} \int_{k_F}^{k_F+D} \frac{1}{\omega' - \mu' + k_F^{-1}} \,\mathrm{d}\omega' \,\mathrm{d}\mu' \\
&= \hat{C} \int_{-D}^{0} \int_{0}^{D} \frac{1}{r - k_F - (s - k_F) + k_F^{-1}} \,\mathrm{d}r \,\mathrm{d}s \\
&= \hat{C} \int_{-D}^{0} \int_{0}^{D} \frac{1}{r - s + k_F^{-1}} \,\mathrm{d}r \,\mathrm{d}s
\end{aligned} \tag{A.52}$$

Computing the integral, we find

$$\begin{aligned}
\int_{-D}^{0} \mathrm{d}s \int_{0}^{D} \mathrm{d}r \frac{1}{\left(r - s + k_F^{-1}\right)} &= \int_{-B}^{0} \mathrm{d}s \ \left[\ln\left(r - s + k_F^{-1}\right)\right]_0^D \\
&= \int_{-D}^{0} \mathrm{d}s \ \ln\left(D - s + k_F^{-1}\right) - \ln\left(-s + k_F^{-1}\right) \\
&= \int_{-D}^{0} \mathrm{d}s \ \ln\left(D - s + k_F^{-1}\right) - \int_{-D}^{0} \mathrm{d}s \ \ln\left(-s + k_F^{-1}\right) \ ,
\end{aligned} \tag{A.53}$$

where, using $D - s + k_F^{-1} \mapsto x$,

$$\begin{aligned}
\int_{-D}^{0} \mathrm{d}s \ \ln\left(D - s + k_F^{-1}\right) &= \int_{2D+k_F^{-1}}^{D+k_F^{-1}} \mathrm{d}x \ \ln(x) = [x\ln(x) - x]_{2D+k_F^{-1}}^{D+k_F^{-1}} \\
&= \left(D + k_F^{-1}\right) \ln\left(D + k_F^{-1}\right) - \left(D + k_F^{-1}\right) \\
&\quad - \left(\left(2D + k_F^{-1}\right) \ln\left(2D + k_F^{-1}\right) - \left(2D + k_F^{-1}\right)\right) \\
&= \left(D + k_F^{-1}\right) \ln\left(D + k_F^{-1}\right) - \left(2D + k_F^{-1}\right) \\
&\quad \cdot \ln\left(2D + k_F^{-1}\right) + D \,,
\end{aligned} \tag{A.54}$$

and, using $-s + k_F^{-1} \mapsto y$,

$$\int_{-D}^{0} \mathrm{d}s \ \ln\left(-s + k_F^{-1}\right) = \int_{D+k_F^{-1}}^{k_F^{-1}} \mathrm{d}y \ \ln(y) = \left[y\ln(y) - y\right]_{D+k_F^{-1}}^{k_F^{-1}} \tag{A.55}$$
$$= k_F^{-1}\ln\left(k_F^{-1}\right) - k_F^{-1}$$
$$-\left(\left(D + k_F^{-1}\right)\ln\left(D + k_F^{-1}\right) - \left(D + k_F^{-1}\right)\right)$$
$$= -\left(D + k_F^{-1}\right)\ln\left(D + k_F^{-1}\right) - k_F^{-1}\ln\left(k_F^{-1}\right) + D\,.$$

Therefore

$$\int_{-D}^{0} \mathrm{d}s \int_{0}^{D} \mathrm{d}r \ \frac{1}{\left(r - s + k_F^{-1}\right)} \tag{A.56}$$
$$= 2(D + k_F^{-1})\ln(D + k_F^{-1}) - (2D + k_F^{-1})\ln(2D + k_F^{-1}) + k_F^{-1}\ln(k_F^{-1})$$
$$= \mathcal{O}\left(\frac{\ln(k_F)}{k_F}\right).$$

Combining all computations, we have

$$L^{-2d} \sum_{(n,m)\in T_F} \frac{|\hat{v}(n-m)|^2}{(|n| - |m| + k_F^{-1})} \leq \frac{1}{1 + \frac{L}{k_F}} \left(C + \widetilde{C}\right) k_F^{(d-1)} + \hat{C}\mathcal{O}\left(\frac{\ln(k_F)}{k_F}\right) \tag{A.57}$$

c) The sum

$$L^{-2d} \sum_{(n,m)\in T_F} \frac{|\hat{v}(n-m)|^2}{(|n| - |m| + k_F^{-1})^2} \leq L^{-2d} \sum_{\substack{n\in B^L_{[k_F,\infty)} \\ m\in B^L_{k_F}}} \frac{1}{(|n| - |m| + k_F^{-1})^2} \mathbb{1}_{B_D(m)}(n)\,, \tag{A.58}$$

can be evaluated and bounded in essentially the same way as the previous one, using

$$f^{\sup(L)} := \sup_{\substack{n,m\in\frac{1}{L}\mathbb{Z}^d \\ n\neq m}} f\left(|n|,|m|\right) = \sup_{\substack{n,m\in\frac{1}{L}\mathbb{Z}^d \\ n\neq m}} \frac{1}{(|n|-|m|+k_F^{-1})^2} = \frac{1}{\left(\frac{1}{L}+k_F^{-1}\right)^2}\,. \tag{A.59}$$

Then

$$L^{-2d}\sum_{(n,m)\in T_F}\frac{|\hat{v}\left(n-m\right)|^2}{(|n|-|m|+k_F^{-1})^2} \leq \frac{1}{\left(L^{-1/2}+\frac{L^{1/2}}{k_F}\right)^2}\left(C+\widetilde{C}\right)k_F^{d-1}+\hat{C}\int_{-D}^{0}\int_{0}^{D}\frac{1}{r-s+k_F^{-1}}\,\mathrm{d}r\,\mathrm{d}s\,. \tag{A.60}$$

Similarly to the computation above one can check

$$\begin{aligned}\int_{-D}^{0}\mathrm{d}s\int_{0}^{D}\mathrm{d}r\frac{1}{\left(r-s+k_F^{-1}\right)^2} &= \int_{-D}^{0}\mathrm{d}s\left[-\frac{1}{r-s+k_F^{-1}}\right]_0^D \\ &= \int_{-D}^{0}\mathrm{d}s\ \frac{-1}{D-s+k_F^{-1}}+\frac{1}{-s+k_F^{-1}} \\ &= \left[\ln\left(D-s+k_F^{-1}\right)-\ln\left(k_F^{-1}-s\right)\right]_{-D}^{0} \\ &= \ln\left(D+k_F^{-1}\right)-\ln\left(k_F^{-1}\right) \\ &\quad -\left(\ln\left(2D+k_F^{-1}\right)-\ln\left(D+k_F^{-1}\right)\right) \\ &= 2\ln\left(D+k_F^{-1}\right)-\ln\left(k_F^{-1}\right)-\ln\left(2D+k_F^{-1}\right)\,,\end{aligned} \tag{A.61}$$

where

$$2\ln\left(D+k_F^{-1}\right)-\ln\left(k_F^{-1}\right)-\ln\left(2D+k_F^{-1}\right)=\mathcal{O}\left(\ln(k_F)\right) \tag{A.62}$$

for large k_F. Therefore we have

$$\begin{aligned} &L^{-2d}\sum_{(n,m)\in T_F}\frac{|\hat{v}\,(n-m)|^2}{(|n|-|m|+k_F^{-1})^2} \\ \leq\ &\frac{1}{\left(L^{-1/2}+\frac{L^{1/2}}{k_F}\right)^2}\left(C+\widetilde{C}\right)k_F^{d-1}+\hat{C}\mathcal{O}\left(\ln(k_F)\right)\ . \end{aligned} \tag{A.63}$$

d) The sum

$$\sum_{(n,m)\in T_F}\frac{|\hat{v}\,(n-m)|^2\,(n-m)^2}{(|n|-|m|+k_F^{-1})^2}\,, \tag{A.64}$$

can be bounded the exact same way as in c), since the difference $n-m$ is of order one. Therefore

$$\begin{aligned} &\sum_{(n,m)\in T_F}\frac{|\hat{v}\,(n-m)|^2\,(n-m)^2}{(|n|-|m|+k_F^{-1})^2} \\ \leq\ &\frac{1}{\left(L^{-1/2}+\frac{L^{1/2}}{k_F}\right)^2}\left(C+\widetilde{C}\right)k_F^{d-1}+\hat{C}\mathcal{O}\left(\ln(k_F)\right)\ . \end{aligned} \tag{A.65}$$

e) The sum

$$L^{-d}\sum_{m\in B_F}\left(L^{-d}\sum_{n\in B_F^C}\frac{|\hat{v}(n-m)|}{\left(|n|-|m|+k_F^{-1}\right)}\right)^2\,, \tag{A.66}$$

can be computed and bounded in the following way. Using (5) and parts of the proof of (6), we find

$$
\begin{aligned}
&L^{-d} \sum_{n \in B_F^C} \frac{|\hat{v}(n-m)|}{(|n|-|m|+k_F^{-1})} \qquad \text{(A.67)}\\
&\leq L^{-d} \sum_{n \in B^L_{[k_F,\infty)}} \frac{1}{(|n|-|m|+k_F^{-1})} \mathbb{1}_{B_D(m)}(n)\\
&\leq L^{-d} \left(f^{\sup(L)} \sum_{n \in B^L_{[k_F, k_F+\frac{1}{L})}} \mathbb{1}_{B_D(m)}(n) + \sum_{n \in B^L_{[k_F+\frac{1}{L}, k_F+R)}} f(|n|,|m|) \, \mathbb{1}_{B_D(m)}(n) \right)\\
&\leq \frac{1}{1+\frac{L}{k_F}} C^{e)} + \hat{C} \int_{k_F}^{k_F+D} f(\omega', |m|) \, \mathrm{d}\omega' \mathbb{1}_{(k_F-D-(1/L),0)}(|m|-k_F) \,,
\end{aligned}
$$

where

$$
C^{e)} := \left(k_F + \frac{2+\sqrt{d}}{2L} \right)^{d-1} \left(1+\sqrt{d}\right) \left(2 \arcsin \left(\frac{D+\frac{\sqrt{d}}{2L}}{k_F - 2D} \right) \right)^{d-1} = \mathcal{O}(1) \,. \qquad \text{(A.68)}
$$

Then, with

$$
\begin{aligned}
&\hat{C} \int_0^D f(r,|m|) \, \mathrm{d}r \; \mathbb{1}_{(k_F-D-(1/L),0)}(|m|-k_F) \qquad \text{(A.69)}\\
&= \hat{C} \int_0^D \frac{1}{(r-|m|+k_F^{-1})} \mathrm{d}r \; \mathbb{1}_{(k_F-D-(1/L),0)}(|m|-k_F)\\
&\leq \hat{C} \left(\ln \left(k_F + D - |m| + k_F^{-1} \right) - \ln \left(k_F - |m| + k_F^{-1} \right) \right)\\
&\quad \mathbb{1}_{[-D-(1/L),0)}(|m|-k_F)\\
&\leq \hat{C} \ln(k_F) \mathbb{1}_{[-D-(1/L),0)}(|m|-k_F) \,,
\end{aligned}
$$

we have

$$L^{-d} \sum_{n \in B_F^C} \frac{|\hat{v}(n-m)|}{(|n|-|m|+k_F^{-1})} \tag{A.70}$$
$$\leq \frac{1}{1+\frac{L}{k_F}} C^{e)} + \hat{C} \ln(k_F) \mathbb{1}_{[(-D-(1/L),0)}(|m|-k_F)\,.$$

This yields

$$\begin{aligned}
(\text{A.66}) &\leq L^{-d} \sum_{m \in B_F} \left(\frac{1}{1+\frac{L}{k_F}} C^{e)} + \hat{C} \ln(k_F) \mathbb{1}_{[[-D-(1/L),0)}(|m|-k_F) \right)^2 \\
&\leq L^{-d} \sum_{m \in B_F} \left(\frac{C^{e)}}{1+\frac{L}{k_F}} + \hat{C} \ln(k_F) \mathbb{1}_{(k_F-D-(1/L),0)}(|m|-k_F) \right)^2 \\
&\leq L^{-d} \sum_{m \in B_F} \cdot \left(\left(\frac{C^{e)}}{1+\frac{L}{k_F}} \right)^2 + \hat{C}^2 \ln(k_F)^2 \mathbb{1}_{[-D-(1/L),0)}(|m|-k_F) \right. \\
&\qquad \left. + \frac{2C^{e)}}{1+\frac{L}{k_F}} \cdot \hat{C} \ln(k_F) \mathbb{1}_{(k_F-D-(1/L),0)}(|m|-k_F) \right) \\
&\leq k_F^{d-1} \left(\frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} \left(\frac{\sqrt{d}}{2L} \right)^{d-1} \left(R + \frac{\sqrt{d}}{L} \right) \right) \\
&\quad \cdot \left(\left(\frac{C^{e)}}{1+\frac{L}{k_F}} \right)^2 + \hat{C}^2 \ln(k_F)^2 \mathbb{1}_{[-D-(1/L),0)}(|m|-k_F) \right. \\
&\qquad \left. + \frac{2C^{e)}}{1+\frac{L}{k_F}} \cdot \hat{C} \ln(k_F) \mathbb{1}_{[-D-(1/L),0)}(|m|-k_F) \right) \\
&\leq k_F^{d-1} \ln(k_F)^2 \cdot C^{e2)}\,,
\end{aligned} \tag{A.71}$$

with

$$C^{e2)} := \left(\frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} \left(\frac{\sqrt{d}}{2L} \right)^{d-1} \left(R + \frac{\sqrt{d}}{L} \right) \right) \tag{A.72}$$

$$\cdot \left(\frac{C^{e)}}{1+\frac{L}{k_F}} + \hat{C}^2 \, \mathbb{1}_{[-D-(1/L),0)}(|m|-k_F) \right)^2$$
$$\cdot \left(\ln\left(k_F\right)^{-2} + \ln\left(k_F\right)^{-1} + 1 \right) \,,$$

where

$$\left(\frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} \left(\frac{\sqrt{d}}{2L}\right)^{d-1} \left(R + \frac{\sqrt{d}}{L}\right) \right) \cdot \left(\frac{C^{e)}}{1+\frac{L}{k_F}} + \hat{C}^2 \, \mathbb{1}_{[-D-(1/L),0)}(|m|-k_F) \right)^2$$
$$\cdot \left(\ln\left(k_F\right)^{-2} + \ln\left(k_F\right)^{-1} + 1 \right)$$
$$\leq \mathcal{O}\left(\frac{1}{L^{d-1}}\right) \tag{A.73}$$

f) The sum

$$L^{-d} \sum_{n \in B_F^C} \left(L^{-d} \sum_{m \in B_F} \frac{|\hat{v}(n-m)|}{\left(|n|-|m|+k_F^{-1}\right)} \right)^2 \,, \tag{A.74}$$

can be computed and bounded equivalently to the previous one:

$$L^{-d} \sum_{m \in B_F} \frac{|\hat{v}(n-m)|}{\left(|n|-|m|+k_F^{-1}\right)} \tag{A.75}$$
$$\leq L^{-d} \sum_{n \in B_{k_F}^L} \frac{1}{(|n|-|m|+k_F^{-1})} \mathbb{1}_{B_D(m)}(n)$$
$$\leq \frac{C^{f)}}{1+\frac{L}{k_F}} + \hat{C} \int_{k_F-D}^{k_F} f\left(|n|,\mu'\right) \, \mathrm{d}\mu' \, \mathbb{1}_{[0,D)}(|n|-k_F) \,,$$

where

$$C^{f)} := \left(k_F + \frac{\sqrt{d}}{2L}\right)^{d-1} \left(R+\sqrt{d}\right) \left(2 \arcsin\left(\frac{D+\frac{\sqrt{d}}{2L}}{k_F-2D}\right)\right)^{d-1} = \mathcal{O}(1) \,. \tag{A.76}$$

Then,

$$\hat{C}\int_{-D}^{0} f\left(|n|, s\right)\,\mathrm{d}s\ \mathbb{1}_{[0,D)}(|n|-k_F) \leq \hat{C}\ln\left(k_F\right)\mathbb{1}_{[0,D)}(|n|-k_F)\,, \tag{A.77}$$

we have

$$\begin{aligned} &L^{-d}\sum_{m\in B_F}\frac{|\hat{v}\left(n-m\right)|}{\left(|n|-|m|+k_F^{-1}\right)} \\ &\leq \frac{1}{1+\frac{L}{k_F}}C^{f)}+\hat{C}\ln\left(k_F\right)\mathbb{1}_{[0,D]}(|n|-k_F)\,. \end{aligned} \tag{A.78}$$

This leads to

$$\begin{aligned} \text{(A.74)} &\lesssim L^{-d}\sum_{n\in B_F^C}\left(\frac{1}{1+\frac{L}{k_F}}C^{f)}+\hat{C}\ln\left(k_F\right)\mathbb{1}_{[0,D)}(|n|-k_F)\right)^2 \\ &\leq k_F^{d-1}\ln\left(k_F\right)^2\cdot C^{f2)}\,, \end{aligned} \tag{A.79}$$

with

$$\begin{aligned} C^{f2)} := &\left(\frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}\left(\frac{\sqrt{d}}{2L}\right)^{d-1}\left(R+\frac{\sqrt{d}}{L}\right)\right)\cdot\left(\frac{C^{f)}}{1+\frac{L}{k_F}}+\hat{C}^2\,\mathbb{1}_{[0,D)}(|n|-k_F)\right)^2 \\ &\cdot\left(\ln\left(k_F\right)^{-2}+\ln\left(k_F\right)^{-1}+1\right)\,, \end{aligned} \tag{A.80}$$

where

$$\begin{aligned} &\left(\frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}\left(\frac{\sqrt{d}}{2L}\right)^{d-1}\left(R+\frac{\sqrt{d}}{L}\right)\right)\cdot\left(\frac{C^{f)}}{1+\frac{L}{k_F}}+\hat{C}^2\,\mathbb{1}_{[0,D)}(|n|-k_F)\right)^2 \\ &\cdot\left(\ln\left(k_F\right)^{-2}+\ln\left(k_F\right)^{-1}+1\right) \\ \leq\ &\mathcal{O}\left(\frac{1}{L^{d-1}}\right)\,. \end{aligned} \tag{A.81}$$

g) The sum

$$L^{-2d} \sum_{(n,k)\in T_F} \left(L^{-d} \sum_{m\in B_F} \frac{|\hat{v}(m-k)|\ |\hat{v}(n-m)|}{\left(|n|-|m|+k_F^{-1}\right)} \right)^2 \tag{A.82}$$

can be bounded by

$$\begin{aligned} &L^{-2d} \sum_{(n,k)\in T_F} \left(L^{-d} \sum_{m\in B_F} \frac{|\hat{v}(m-k)|\ |\hat{v}(n-m)|}{\left(|n|-|m|+k_F^{-1}\right)} \right)^2 \\ &\leq L^{-d} \sum_{n\in B_F^C} \left(L^{-d} \sum_{m\in B_F} \frac{|\hat{v}(n-m)|}{\left(|n|-|m|+k_F^{-1}\right)} \right)^2 . \end{aligned} \tag{A.83}$$

Hence, with the computations before we have

$$(\text{A.82}) \leq k_F^{d-1} \ln (k_F)^2 \cdot C^{f2)} . \tag{A.84}$$

h) The sum

$$L^{-2d} \sum_{(l,m)\in T_F} \left(L^{-d} \sum_{n\in B_F^C} \frac{|\hat{v}(l-n)|\ |\hat{v}(n-m)|}{\left(|n|-|m|+k_F^{-1}\right)} \right)^2 \tag{A.85}$$

can be bounded by

$$\begin{aligned} &L^{-2d} \sum_{(l,m)\in T_F} \left(L^{-d} \sum_{n\in B_F^C} \frac{|\hat{v}(l-n)|\ |\hat{v}(n-m)|}{\left(|n|-|m|+k_F^{-1}\right)} \right)^2 \\ &\leq L^{-d} \sum_{m\in B_F} \left(L^{-d} \sum_{n\in B_F^C} \frac{|\hat{v}(n-m)|}{\left(|n|-|m|+k_F^{-1}\right)} \right)^2 . \end{aligned} \tag{A.86}$$

Hence, with the computations before we have

$$\text{(A.85)} \leq k_F^{d-1} \ln\left(k_F\right)^2 \cdot C^{e2)} . \tag{A.87}$$

i) The sum

$$L^{-2d} \sum_{h,n \in B_F^C} \left(L^{-2d} \sum_{(l,k) \in T_F} \frac{|\hat{v}(n-k)|\ |\hat{v}(l-k)|\ |\hat{v}(h-l)|}{\left(|n| - |k| + k_F^{-1}\right)\left(|l| - |k| + k_F^{-1}\right)} \right)^2 \tag{A.88}$$

can be bounded by

$$\begin{aligned} & L^{-2d} \sum_{h,n \in B_F^C} \left(L^{-2d} \sum_{(l,k) \in T_F} \frac{|\hat{v}(n-k)|\ |\hat{v}(l-k)|\ |\hat{v}(h-l)|}{\left(|n| - |k| + k_F^{-1}\right)\left(|l| - |k| + k_F^{-1}\right)} \right)^2 \\ \lesssim & \left(L^{-d} \sum_{k \in B_F} \left(L^{-d} \sum_{n \in B_F^C} \frac{|\hat{v}(n-k)|}{\left(|n| - |k| + k_F^{-1}\right)} \right)^2 \right)^2 \\ \leq & \left(k_F^{d-1} \ln\left(k_F\right)^2 \cdot C^{e2)} \right)^2 \\ = & k_F^{2d-2} \ln\left(k_F\right)^4 \cdot \left(C^{e2)}\right)^2 . \end{aligned} \tag{A.89}$$

j) The sum

$$L^{-2d} \sum_{m,k \in B_F} \left(L^{-2d} \sum_{n,l \in B_F^C} \frac{|\hat{v}(n-m)|\ |\hat{v}(l-k)|\ |\hat{v}(n-l)|}{\left(|n| - |m| + k_F^{-1}\right)\left(|l| - |k| + k_F^{-1} + 1\right)} \right)^2 \tag{A.90}$$

can be bounded by

$$L^{-2d} \sum_{m,k \in B_F} \left(L^{-2d} \sum_{n,l \in B_F^C} \frac{|\hat{v}(n-m)|\ |\hat{v}(l-k)|\ |\hat{v}(n-l)|}{\left(|n| - |m| + k_F^{-1}\right)\left(|l| - |k| + k_F^{-1} + 1\right)} \right)^2$$

$$\lesssim \left(L^{-d} \sum_{m \in B_F} \left(L^{-d} \sum_{n \in B_F^C} \frac{|\hat{v}(n-m)|}{\left(|n| - |m| + k_F^{-1}\right)} \right)^2 \right)^2$$
$$\leq \left(k_F^{d-1} \ln (k_F)^2 \cdot C^{e2)} \right)^2$$
$$= k_F^{2d-2} \ln (k_F)^4 \cdot \left(C^{e2)} \right)^2 \tag{A.91}$$

k) The sum

$$L^{-3d} \sum_{(n,m) \in T_F} \sum_{l \in B_F} \frac{|\hat{v}(n-m)| \; |\hat{v}(l-m)| \; |\hat{v}(n-l)|}{\left(|n| - |m| + k_F^{-1}\right) \left(|l| - |m| + k_F^{-1}\right)} \tag{A.92}$$

can be bounded by

$$L^{-3d} \sum_{(n,m) \in T_F} \sum_{l \in B_F} \frac{|\hat{v}(n-m)| \; |\hat{v}(l-m)| \; |\hat{v}(n-l)|}{\left(|n| - |m| + k_F^{-1}\right) \left(|l| - |m| + k_F^{-1}\right)} \tag{A.93}$$
$$\lesssim L^{-d} \sum_{m \in B_F} \left(L^{-d} \sum_{n \in B_F^C} \frac{|\hat{v}(n-m)|}{\left(|n| - |m| + k_F^{-1}\right)} \right)^2$$
$$\leq k_F^{d-1} \ln (k_F)^2 \cdot C^{e2)} .$$

l) The sum

$$L^{-d} \sum_{m \in B_F} \left(L^{-2d} \sum_{(h,g) \in T_F} \sum_{n \in B_F^C} \frac{|\hat{v}(h-g)| \; |\hat{v}(n-m)|}{\left(|h| - |g| + k_F^{-1}\right) \left(|n| - |m| + k_F^{-1}\right)} \right)^2 \tag{A.94}$$

can be bounded by

$$L^{-2d} \sum_{m \in B_F} \left(L^{-d} \sum_{(h,g) \in T_F} \sum_{n \in B_F^C} \frac{|\hat{v}(h-g)| \; |\hat{v}(n-m)|}{\left(|h| - |g| + k_F^{-1}\right) \left(|n| - |m| + k_F^{-1}\right)} \right)^2$$
$$\leq L^{2d} \left(L^{-2d} \sum_{(h,g) \in T_F} \frac{|\hat{v}(h-g)|}{\left(|h| - |g| + k_F^{-1}\right)} \right)^2$$

$$\cdot L^d \left(L^{-d} \sum_{m \in B_F} \left(L^{-d} \sum_{n \in B_F^C} \frac{|\hat{v}(n-m)|}{\left(|n| - |m| + k_F^{-1}\right)} \right)^2 \right), \tag{A.95}$$

which we can compute with the previous bounds.

m) The sum

$$L^{-2d} \sum_{(p,q) \in \widetilde{N}_k^F} \sum_{k \in [-D,D]^d} |\hat{w}(k)|^2 \tag{A.96}$$

can be bounded by

$$\begin{aligned} &L^{-2d} \sum_{\substack{p-k \in B_F^C \\ p \in B_F}} \sum_{\substack{q+k \in B_F^C \\ q \in B_F}} |\hat{w}(k)|^2 \\ &\leq c' L^{-2d} \sum_{\substack{p-k \in B_F^C \\ p \in B_F}} \sum_{\substack{q+k \in B_F^C \\ q \in B_F}} \mathbb{1}_{B_D(p)}(k)\ \mathbb{1}_{B_D(q)}(k), \end{aligned} \tag{A.97}$$

where $|\hat{w}(k)|^2 \leq c' \mathbb{1}_{B_D(p)}(k)\ \mathbb{1}_{B_D(q)}(k)$. Similar to the proof of (6) we aim to split the sum into parts: a critical region within an annulus around the Fermi momentum k_F, and the remaining less critical regions. We find

$$\begin{aligned} &L^{-2d} \sum_{\substack{p-k \in B^L_{[k_F,\infty)} \\ p \in B^L_{k_F}}} \sum_{\substack{q+k \in B^L_{[k_F,\infty)} \\ q \in B^L_{k_F}}} \mathbb{1}_{B_D(p)}(k) \mathbb{1}_{B_D(q)}(k) \\ &\leq L^{-2d} \Bigg(\underbrace{\sum_{\substack{p-k \in B^L_{[k_F, k_F + \frac{1}{L})} \\ p \in B^L_{[k_F - D, k_F)}}} \sum_{\substack{q+k \in B^L_{[k_F, k_F + \frac{1}{L})} \\ q \in B^L_{[k_F - D, k_F)}}} \mathbb{1}_{B_D(p)}(k) \mathbb{1}_{B_D(q)}(k)}_{=:(1m)} \end{aligned} \tag{A.98}$$

$$
\begin{aligned}
&+ \underbrace{\sum_{\substack{p-k\in B^L_{[k_F,k_F+D)} \\ p\in B^L_{[k_F-\frac{1}{L},k_F)}}} \sum_{\substack{q+k\in B^L_{[k_F,k_F+D)} \\ q\in B^L_{[k_F-\frac{1}{L},k_F)}}} \mathbb{1}_{B_D(p)}(k)\mathbb{1}_{B_D(q)}(k)}_{=:(2m)} \\
&+ \underbrace{\sum_{\substack{p-k\in B^L_{[k_F,k_F+\frac{1}{L})} \\ p\in B^L_{[k_F-D,k_F)}}} \sum_{\substack{q+k\in B^L_{[k_F,k_F+D)} \\ q\in B^L_{[k_F-\frac{1}{L},k_F)}}} \mathbb{1}_{B_D(p)}(k)\mathbb{1}_{B_D(q)}(k)}_{=:(3m)} \\
&+ \underbrace{\sum_{\substack{p-k\in B^L_{[k_F,k_F+R)} \\ p\in B^L_{[k_F-\frac{1}{L},k_F)}}} \sum_{\substack{q+k\in B^L_{[k_F,k_F+\frac{1}{L})} \\ q\in B^L_{[k_F-D,k_F)}}} \mathbb{1}_{B_D(p)}(k)\mathbb{1}_{B_D(q)}(k)}_{=:(4m)} \\
&+ \sum_{\substack{p-k\in B^L_{[k_F+\frac{1}{L},k_F+D)} \\ p\in B^L_{[k_F-D,k_F-\frac{1}{L})}}} \sum_{\substack{q+k\in B^L_{[k_F+\frac{1}{L},k_F+D)} \\ q\in B^L_{[k_F-D,k_F-\frac{1}{L})}}} \mathbb{1}_{B_D(p)}(k)\mathbb{1}_{B_D(q)}(k)\Bigg)
\end{aligned}
$$

In the following we a re going to estimate all sums separately. We get

$$
(1m) \leq L^{2d}\frac{C}{L}k_F^{d-1} \sum_{\substack{p-k\in B^L_{[k_F,k_F+D)} \\ p\in B^L_{[k_F-\frac{1}{L},k_F)}}} \mathbb{1}_{B_D(p)}(k) \leq L^{4d}\frac{C^2}{L^2}k_F^{2d-2}\,, \tag{A.99}
$$

as well as

$$
(2m) \leq L^{2d}\frac{\widetilde{C}}{L}k_F^{d-1} \sum_{\substack{p-k\in B^L_{[k_F,k_F+\frac{1}{L})} \\ p\in B^L_{[k_F-D,k_F)}}} \mathbb{1}_{B_D(p)}(k) \leq L^{4d}\frac{\widetilde{C}^2}{L^2}k_F^{2d-2}\,, \tag{A.100}
$$

and

$$(3m) \leq L^{2d}\frac{\widetilde{C}}{L}k_F^{d-1}\sum_{\substack{p-k\in B^L_{[k_F,k_F+D)}\\ p\in B^L_{[k_F-\frac{1}{L},k_F)}}}\mathbb{1}_{B_D(p)}(k) \leq L^{4d}\frac{\widetilde{C}C}{L^2}k_F^{2d-2}\,, \tag{A.101}$$

and

$$(4m) \leq L^{2d}\frac{C}{L}k_F^{d-1}\sum_{\substack{p-k\in B^L_{[k_F,k_F+\frac{1}{L})}\\ p\in B^L_{[k_F-D,k_F)}}}\mathbb{1}_{B_D(p)}(k) \leq L^{4d}\frac{C\widetilde{C}}{L^2}k_F^{2d-2}\,. \tag{A.102}$$

Now we estimate the last sum. Let p and q be fixed with $|p|,|q| \geq k_F - R$ and further assume that $k_F \cdot L \in \mathbb{N}$ for simplicity. We want to split the sum into rings of thickness $1/L$ to find

$$\sum_{p-k\in B^L_{[k_F+\frac{1}{L},k_F+D)}}\;\sum_{q+k\in B^L_{[k_F+\frac{1}{L},k_F+D)}}\mathbb{1}_{B_D(p)}(k)\mathbb{1}_{B_D(q)}(k) \tag{A.103}$$

$$= \sum_{\substack{r\in\mathbb{N}\\ L(k_F+D)\geq r\geq k_F\cdot L+1}}\;\sum_{\substack{r'\in\mathbb{N}\\ L(k_F+D)\geq r'\geq k_F\cdot L+1}}\;\sum_{p-k\in B^L_{\left(\frac{r}{L},\frac{r+1}{L}\right]}}\;\sum_{q+k\in B^L_{\left(\frac{r'}{L},\frac{r'+1}{L}\right]}}\mathbb{1}_{B_D(p)}(k)\mathbb{1}_{B_D(q)}(k)\,.$$

Using (5) and similar methods as in the proof of (6) we find

$$\leq L^{4d}\hat{C}^2\int\limits_{k_F-D}^{k_F}\int\limits_{k_F}^{k_F+D}\int\limits_{k_F-D}^{k_F}\int\limits_{k_F}^{k_F+D}\mathrm{d}\omega'\,\mathrm{d}\mu'\,\mathrm{d}\nu'\,\mathrm{d}\eta' = L^{4d}\hat{C}^2D^4\,. \tag{A.104}$$

Combining all computations we find

$$L^{-2d}\sum_{(p,q)\in\widetilde{N}^F_k}\;\sum_{k\in[-D,D]^d}|\hat{w}(k)|^2 \tag{A.105}$$

$$\leq L^{2d-2}k_F^{2d-2}c'\left(C^2+\widetilde{C}^2+2\widetilde{C}C\right)+L^{2d}\hat{C}^2D^4\,.$$

n) The sum

$$L^{-2d} \sum_{p \in M_k^F} \sum_{k \in [-D,D]^d} |\hat{w}(k)|^2 \tag{A.106}$$

can be bounded by

$$L^{-2d} \sum_{\substack{p-k \in B_F^C \\ p \in B_F}} |\hat{w}(k)|^2 \leq c'' L^{-2d} \sum_{\substack{p-k \in B_F^C \\ p \in B_F}} \mathbb{1}_{B_D(p)}(k)\,, \tag{A.107}$$

where $|\hat{w}(k)|^2 \leq c'' \mathbb{1}_{B_D(p)}(k)$. We again aim to split the sum into parts different parts to find

$$\begin{aligned} & L^{-2d} \sum_{\substack{p-k \in B^L_{[k_F,\infty)} \\ p \in B^L_{k_F}}} \mathbb{1}_{B_D(p)}(k) \\ & \leq L^{-2d} \Bigg(\underbrace{\sum_{\substack{p-k \in B^L_{[k_F, k_F+\frac{1}{L})} \\ p \in B^L_{[k_F-D, k_F)}}} \mathbb{1}_{B_D(p)}(k)}_{=:(1n)} + \underbrace{\sum_{\substack{p-k \in B^L_{[k_F, k_F+D)} \\ p \in B^L_{[k_F-\frac{1}{L}, k_F)}}} \mathbb{1}_{B_D(p)}(k)}_{=:(2n)} \\ & \qquad + \sum_{\substack{p-k \in B^L_{[k_F+\frac{1}{L}, k_F+D)} \\ p \in B^L_{[k_F-D, k_F-\frac{1}{L})}}} \mathbb{1}_{B_D(p)}(k) \Bigg)\,. \end{aligned} \tag{A.108}$$

Using the same computations as before we find

$$(1n) \leq L^{2d} \frac{C}{L} k_F^{d-1}\,, \tag{A.109}$$

and

$$(2n) \leq L^{2d} \frac{\widetilde{C}}{L} k_F^{d-1}\,. \tag{A.110}$$

The last sum can be computed via

$$\sum_{\substack{p-k\in B^L_{[k_F+\frac{1}{L},k_F+D)}\\ p\in B^L_{[k_F-D,k_F-\frac{1}{L})}}} \mathbb{1}_{B_D(p)}(k) \leq L^{2d}\hat{C}\int_{k_F-D}^{k_F}\int_{k_F}^{k_F+D} \mathrm{d}\omega'\,\mathrm{d}\mu' = L^{2d}\hat{C}D^2\,. \quad \text{(A.111)}$$

This yields

$$L^{-2d}\sum_{\substack{p-k\in B_F^C\\ p\in B_F}} |\hat{w}(k)|^2 \leq \frac{c''}{L}\left(C+\widetilde{C}\right)k_F^{d-1} + \hat{C}D^2\,. \quad \text{(A.112)}$$

o) To bound the sum

$$L^{-2d}\sum_{(p,q)\in\widetilde{N}^F_k}\sum_{k\in[-D,D]^d}\frac{|\hat{w}(k)|^2}{\left(|p|-|p-k|+|q|-|q+k|+k_F^{-1}\right)^2} \quad \text{(A.113)}$$

we use (5), (6) and the computations in the proof of m). We bound the sum by

$$\begin{aligned}
&L^{-2d}\sum_{\substack{p-k\in B_F^C\\ p\in B_F}}\sum_{\substack{q+k\in B_F^C\\ q\in B_F}}\frac{|\hat{w}(k)|}{\left(|p|-|p-k|+|q|-|q+k|+k_F^{-1}\right)^2} \quad \text{(A.114)}\\
&\leq L^{-2d}\sum_{\substack{p-k\in B_F^C\\ p\in B_F}}\sum_{\substack{q+k\in B_F^C\\ q\in B_F}}\frac{1}{\left(|p|-|p-k|+|q|-|q+k|+k_F^{-1}\right)^2}\mathbb{1}_{B_D(p)}(k)\mathbb{1}_{B_D(q)}(k)\,.
\end{aligned}$$

We consider the function

$$g\left(|p|,|p-k|,|q|,|q+k|\right) = \left(|p|-|p-k|+|q|-|q+k|+k_F^{-1}\right)^{-2}, \quad \text{(A.115)}$$

that is anti-monotone in the first and third component and monotone in the second and fourth. With

$$\begin{aligned} g^{\sup(L)} &:= \sup_{\substack{p,p-k,q,q+k\in\frac{1}{L}\mathbb{Z}^d \\ p\neq p-k\neq q\neq q+k}} g\left(|p|,|p-k|,|q|,|q+k|\right) \\ &= \sup_{\substack{p,p-k,q,q+k\in\frac{1}{L}\mathbb{Z}^d \\ p\neq p-k\neq q\neq q+k}} \frac{1}{(|p|-|p-k|+|q|-|q+k|+k_F^{-1})^2} \\ &= \frac{1}{\left(\frac{2}{L}+k_F^{-1}\right)^2}, \end{aligned} \tag{A.116}$$

we then have

$$\begin{aligned} & L^{-2d} \sum_{\substack{p-k\in B^L_{[k_F,\infty)} \\ p\in B^L_{k_F}}} \sum_{\substack{q+k\in B^L_{[k_F,\infty)} \\ q\in B^L_{k_F}}} g\left(|p|,|p-k|,|q|,|q+k|\right) \mathbb{1}_{B_D(p)}(k)\mathbb{1}_{B_D(q)}(k) \\ & \le L^{-2d}\Bigg(g^{\sup(L)}\Bigg(\underbrace{\sum_{\substack{p-k\in B^L_{[k_F,k_F+\frac{1}{L})} \\ p\in B^L_{[k_F-D,k_F)}}} \sum_{\substack{q+k\in B^L_{[k_F,k_F+\frac{1}{L})} \\ q\in B^L_{[k_F-D,k_F)}}} \mathbb{1}_{B_D(p)}(k)\mathbb{1}_{B_D(q)}(k)}_{=:(1o)} \\ & + \underbrace{\sum_{\substack{p-k\in B^L_{[k_F,k_F+D)} \\ p\in B^L_{[k_F-\frac{1}{L},k_F)}}} \sum_{\substack{q+k\in B^L_{[k_F,k_F+D)} \\ q\in B^L_{[k_F-\frac{1}{L},k_F)}}} \mathbb{1}_{B_D(p)}(k)\mathbb{1}_{B_D(q)}(k)}_{=:(2o)} \\ & + \underbrace{\sum_{\substack{p-k\in B^L_{[k_F,k_F+\frac{1}{L})} \\ p\in B^L_{[k_F-D,k_F)}}} \sum_{\substack{q+k\in B^L_{[k_F,k_F+D)} \\ q\in B^L_{[k_F-\frac{1}{L},k_F)}}} \mathbb{1}_{B_D(p)}(k)\mathbb{1}_{B_D(q)}(k)}_{=:(3o)} \end{aligned} \tag{A.117}$$

$$+ \underbrace{\sum_{\substack{p-k\in B^L_{[k_F,k_F+R)}\\ p\in B^L_{[k_F-\frac{1}{L},k_F)}}} \sum_{\substack{q+k\in B^L_{[k_F,k_F+\frac{1}{L})}\\ q\in B^L_{[k_F-D,k_F)}}} \mathbb{1}_{B_D(p)}(k)\mathbb{1}_{B_D(q)}(k) \Bigg)}_{=:(4o)}$$

$$+ \sum_{\substack{p-k\in B^L_{[k_F+\frac{1}{L},k_F+D)}\\ p\in B^L_{[k_F-D,k_F-\frac{1}{L})}}} \sum_{\substack{q+k\in B^L_{[k_F+\frac{1}{L},k_F+D)}\\ q\in B^L_{[k_F-D,k_F-\frac{1}{L})}}} g\left(|p|,|p-k|,|q|,|q+k|\right)\mathbb{1}_{B_D(p)}(k)\mathbb{1}_{B_D(q)}(k) \Bigg) .$$

We obtain

$$g^{\sup(L)}(1o) \leq g^{\sup(L)} L^{4d} \frac{C^2}{L^2} k_F^{2d-2} = L^{4d} \frac{C^2}{\left(2+\frac{L}{k_F}\right)^2} k_F^{2d-2} , \tag{A.118}$$

as well as

$$g^{\sup(L)}(2o) \leq g^{\sup(L)} L^{4d} \frac{\widetilde{C}^2}{L^2} k_F^{2d-2} = L^{4d} \frac{\widetilde{C}^2}{\left(2+\frac{L}{k_F}\right)^2} k_F^{2d-2} , \tag{A.119}$$

and

$$g^{\sup(L)}(3o) \leq g^{\sup(L)} L^{4d} \frac{\widetilde{C}C}{L^2} k_F^{2d-2} = L^{4d} \frac{\widetilde{C}C}{\left(2+\frac{L}{k_F}\right)^2} k_F^{2d-2} , \tag{A.120}$$

and

$$g^{\sup(L)}(4o) \leq g^{\sup(L)} L^{4d} \frac{C\widetilde{C}}{L^2} k_F^{2d-2} = L^{4d} \frac{C\widetilde{C}}{\left(2+\frac{L}{k_F}\right)^2} k_F^{2d-2} . \tag{A.121}$$

For the last sum, combining previous computations and using $\omega' \mapsto r -$

$k_F, \mu' \mapsto s - k_F, \nu' \mapsto t - k_F$ and $\eta' \mapsto u - k_F$, we get

$$\begin{aligned} &\leq L^{4d}\hat{C}^2 \int\limits_{k_F-D}^{k_F} \int\limits_{k_F}^{k_F+D} \int\limits_{k_F-D}^{k_F} \int\limits_{k_F}^{k_F+D} \frac{1}{\left(\omega' - \mu' + \nu' - \eta' + k_F^{-1}\right)^2} \mathrm{d}\omega'\,\mathrm{d}\mu'\,\mathrm{d}\nu'\,\mathrm{d}\eta' \\ &= L^{4d}\hat{C}^2 \int\limits_{-D}^{0} \mathrm{d}s \int\limits_{0}^{D} \mathrm{d}r \int\limits_{-D}^{0} \mathrm{d}u \int\limits_{0}^{D} \mathrm{d}t\ \frac{1}{\left(r - s + t - u + k_F^{-1}\right)^2}\,. \end{aligned} \tag{A.122}$$

Computing the integral we find

$$\begin{aligned} &\int\limits_{-D}^{0} \mathrm{d}s \int\limits_{0}^{D} \mathrm{d}r \int\limits_{-D}^{0} \mathrm{d}u \int\limits_{0}^{D} \mathrm{d}t\ \frac{1}{\left(r - s + t - u + k_F^{-1}\right)^2} \\ &= \int\limits_{-D}^{0} \mathrm{d}s \int\limits_{0}^{D} \mathrm{d}r \int\limits_{-D}^{0} \mathrm{d}u\ \left[-\frac{1}{r - s + t - u + k_F^{-1}}\right]_0^D \\ &= \int\limits_{-D}^{0} \mathrm{d}s \int\limits_{0}^{D} \mathrm{d}r \int\limits_{-D}^{0} \mathrm{d}u\ -\frac{1}{r - s + D - u + k_F^{-1}} + \frac{1}{r - s - u + k_F^{-1}} \\ &= \int\limits_{-D}^{0} \mathrm{d}s \int\limits_{0}^{D} \mathrm{d}r \left[\ln\left(r - s + D - u + k_F^{-1}\right) - \ln\left(r - s - u + k_F^{-1}\right)\right]_{-D}^{0} \\ &= \int\limits_{-D}^{0} \mathrm{d}s \int\limits_{0}^{D} \mathrm{d}r\ \ln\left(r - s + D + k_F^{-1}\right) - \ln\left(r - s + k_F^{-1}\right) \\ &\quad - \left(\ln\left(r - s + 2D + k_F^{-1}\right) - \ln\left(r - s + D + k_F^{-1}\right)\right) \\ &= \int\limits_{-D}^{0} \mathrm{d}s \int\limits_{0}^{D} \mathrm{d}r\ 2\ln\left(r - s + D + k_F^{-1}\right) \\ &\quad - \ln\left(r - s + k_F^{-1}\right) - \ln\left(r - s + 2D + k_F^{-1}\right) \\ &= 2\underbrace{\int\limits_{-D}^{0} \mathrm{d}s \int\limits_{0}^{D} \mathrm{d}r\ \ln\left(r - s + D + k_F^{-1}\right)}_{:=I} - \underbrace{\int\limits_{-D}^{0} \mathrm{d}s \int\limits_{0}^{D} \mathrm{d}r\ \ln\left(r - s + k_F^{-1}\right)}_{:=II} \\ &\quad - \underbrace{\int\limits_{-D}^{0} \mathrm{d}s \int\limits_{0}^{D} \mathrm{d}r\ \ln\left(r - s + 2D + k_F^{-1}\right)}_{:=III}, \end{aligned} \tag{A.123}$$

with, using $r - s + D + k_F^{-1} \mapsto x$, and later $2D - s + k_F^{-1} \mapsto y$ and $D - s + k_F^{-1} \mapsto z$,

$$\begin{aligned}
I &= \int_{-D}^{0} \mathrm{d}s \int_{0}^{D} \mathrm{d}r \;\ln\left(r - s + D + k_F^{-1}\right) \qquad \text{(A.124)}\\
&= \int_{-D}^{0} \mathrm{d}s \int_{-s+D+k_F^{-1}}^{2D-s+k_F^{-1}} \mathrm{d}x \;\ln(x)\\
&= \int_{-D}^{0} \mathrm{d}s \;\left[x\ln(x) - x\right]_{-s+D+k_F^{-1}}^{2D-s+k_F^{-1}}\\
&= \int_{-D}^{0} \mathrm{d}s \;\left(2D - s + k_F^{-1}\right)\ln\left(2D - s + k_F^{-1}\right) - \left(2D - s + k_F^{-1}\right)\\
&\quad - \left(\left(D - s + k_F^{-1}\right)\ln\left(D - s + k_F^{-1}\right) - \left(D - s + k_F^{-1}\right)\right)\\
&= \int_{-D}^{0} \mathrm{d}s \;\left(2D - s + k_F^{-1}\right)\ln\left(2D - s + k_F^{-1}\right)\\
&\quad - \left(D - s + k_F^{-1}\right)\ln\left(D - s + k_F^{-1}\right) - D\\
&= \int_{-D}^{0} \mathrm{d}s \;\left(2D - s + k_F^{-1}\right)\ln\left(2D - s + k_F^{-1}\right)\\
&\quad - \int_{-D}^{0} \mathrm{d}s \;\left(D - s + k_F^{-1}\right)\ln\left(D - s + k_F^{-1}\right) - \int_{-D}^{0} \mathrm{d}s \; D\\
&= \int_{D+k_F^{-1}}^{2D+k_F^{-1}} \mathrm{d}y \; y\ln(y) - \int_{2D+k_F^{-1}}^{D+k_F^{-1}} \mathrm{d}z \; z\ln(z) - \int_{-D}^{0} \mathrm{d}s \; D\\
&= \left[\frac{1}{2}y^2\ln(y) - \frac{1}{4}y^2\right]_{D+k_F^{-1}}^{2D+k_F^{-1}} - \left[\frac{1}{2}z^2\ln(z) - \frac{1}{4}z^2\right]_{2D+k_F^{-1}}^{D+k_F^{-1}} - \Big[Ds\Big]_{-D}^{0}\\
&= \frac{1}{2}\left(2D + k_F^{-1}\right)^2\ln\left(2D + k_F^{-1}\right) - \frac{1}{4}\left(2D + k_F^{-1}\right)^2\\
&\quad - \left(\frac{1}{2}\left(D + k_F^{-1}\right)^2\ln\left(D + k_F^{-1}\right) - \frac{1}{4}\left(D + k_F^{-1}\right)^2\right)\\
&\quad - \frac{1}{2}\left(D + k_F^{-1}\right)^2\ln\left(D + k_F^{-1}\right) - \frac{1}{4}\left(D + k_F^{-1}\right)^2
\end{aligned}$$

$$
\begin{aligned}
&- \left(\frac{1}{2}\left(2D + k_F^{-1}\right)^2 \ln\left(2D + k_F^{-1}\right) - \frac{1}{4}\left(2D + k_F^{-1}\right)^2\right) - D^2 \\
= &\frac{1}{2}\left(2D + k_F^{-1}\right)^2 \ln\left(2D + k_F^{-1}\right) - \frac{1}{4}\left(2D + k_F^{-1}\right)^2 \\
&- \frac{1}{2}\left(D + k_F^{-1}\right)^2 \ln\left(D + k_F^{-1}\right) + \frac{1}{4}\left(D + k_F^{-1}\right)^2 \\
&- \frac{1}{2}\left(D + k_F^{-1}\right)^2 \ln\left(D + k_F^{-1}\right) - \frac{1}{4}\left(D + k_F^{-1}\right)^2 \\
&- \frac{1}{2}\left(2D + k_F^{-1}\right)^2 \ln\left(2D + k_F^{-1}\right) + \frac{1}{4}\left(2D + k_F^{-1}\right)^2 - D^2 \\
= &\left(D + k_F^{-1}\right)^2 \ln\left(D + k_F^{-1}\right) - D^2
\end{aligned}
$$

and, using $r - s + k_F^{-1} \mapsto x$, and later $D - s + k_F^{-1} \mapsto y$ and $k_F^{-1} - s \mapsto z$,

$$
\begin{aligned}
II = &\int_{-D}^{0} \mathrm{d}s \int_{0}^{D} \mathrm{d}r \;\ln\left(r - s + k_F^{-1}\right) \qquad \text{(A.125)} \\
= &\int_{-D}^{0} \mathrm{d}s \int_{k_F^{-1}-s}^{D-s+k_F^{-1}} \mathrm{d}x \;\ln(x) \\
= &\int_{-D}^{0} \mathrm{d}s \;\left[x\ln(x) - x\right]_{k_F^{-1}-s}^{D-s+k_F^{-1}} \\
= &\int_{-D}^{0} \mathrm{d}s \;\left(D - s + k_F^{-1}\right)\ln\left(D - s + k_F^{-1}\right) - \left(D - s + k_F^{-1}\right) \\
&- \left(\left(k_F^{-1} - s\right)\ln\left(k_F^{-1} - s\right) - \left(k_F^{-1} - s\right)\right) \\
= &\int_{-D}^{0} \mathrm{d}s \;\left(D - s + k_F^{-1}\right)\ln\left(D - s + k_F^{-1}\right) \\
&- \left(k_F^{-1} - s\right)\ln\left(k_F^{-1} - s\right) + D \\
= &\int_{-D}^{0} \mathrm{d}s \;\left(D - s + k_F^{-1}\right)\ln\left(D - s + k_F^{-1}\right) \\
&- \int_{-D}^{0} \mathrm{d}s \;\left(k_F^{-1} - s\right)\ln\left(k_F^{-1} - s\right) + \int_{-D}^{0} \mathrm{d}s \; D
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{2D+k_F^{-1}}^{D+k_F^{-1}} \mathrm{d}y \; y \ln(y) - \int_{D+k_F^{-1}}^{k_F^{-1}} \mathrm{d}z \; z \ln(z) + \int_{-D}^{0} \mathrm{d}s \; D \\
&= \left[\frac{1}{2} y^2 \ln(y) - \frac{1}{4} y^2\right]_{2D+k_F^{-1}}^{D+k_F^{-1}} - \left[\frac{1}{2} z^2 \ln(z) - \frac{1}{4} z^2\right]_{D+k_F^{-1}}^{k_F^{-1}} + \Big[Ds\Big]_{-D}^{0} \\
&= \frac{1}{2}\left(D + k_F^{-1}\right)^2 \ln\left(D + k_F^{-1}\right) - \frac{1}{4}\left(D + k_F^{-1}\right)^2 \\
&\quad - \left(\frac{1}{2}\left(2D + k_F^{-1}\right)^2 \ln\left(2D + k_F^{-1}\right) - \frac{1}{4}\left(2D + k_F^{-1}\right)^2\right) \\
&\quad - \frac{1}{2}\left(k_F^{-1}\right)^2 \ln\left(k_F^{-1}\right) - \frac{1}{4}\left(k_F^{-1}\right)^2 \\
&\quad - \left(\frac{1}{2}\left(D + k_F^{-1}\right)^2 \ln\left(D + k_F^{-1}\right) - \frac{1}{4}\left(D + k_F^{-1}\right)^2\right) + D^2 \\
&= \frac{1}{2}\left(D + k_F^{-1}\right)^2 \ln\left(D + k_F^{-1}\right) - \frac{1}{4}\left(D + k_F^{-1}\right)^2 \\
&\quad - \frac{1}{2}\left(2D + k_F^{-1}\right)^2 \ln\left(2D + k_F^{-1}\right) + \frac{1}{4}\left(2D + k_F^{-1}\right)^2 \\
&\quad - \frac{1}{2}\left(k_F^{-1}\right)^2 \ln\left(k_F^{-1}\right) - \frac{1}{4}\left(k_F^{-1}\right)^2 \\
&\quad - \frac{1}{2}\left(D + k_F^{-1}\right)^2 \ln\left(D + k_F^{-1}\right) + \frac{1}{4}\left(D + k_F^{-1}\right)^2 + D^2 \\
&= -\frac{1}{2}\left(2D + k_F^{-1}\right)^2 \ln\left(2D + k_F^{-1}\right) + \frac{1}{4}\left(2D + k_F^{-1}\right)^2 \\
&\quad - \frac{1}{2} k_F^{-2} \ln\left(k_F^{-1}\right) - \frac{1}{4} k_F^{-2} + D^2
\end{aligned}
$$

and, using $r - s + 2D + k_F^{-1} \mapsto x$, and later $3D - s + k_F^{-1} \mapsto y$ and $2D - s + k_F^{-1} \mapsto z$,

$$
\begin{aligned}
III &= \int_{-D}^{0} \mathrm{d}s \int_{0}^{D} \mathrm{d}r \;\ln\left(r - s + 2D + k_F^{-1}\right) \qquad \text{(A.126)} \\
&= \int_{-D}^{0} \mathrm{d}s \int_{2D-s+k_F^{-1}}^{3D-s+k_F^{-1}} \mathrm{d}x \;\ln(x) \\
&= \int_{-D}^{0} \mathrm{d}s \;\left[x \ln(x) - x\right]_{2D-s+k_F^{-1}}^{3D-s+k_F^{-1}}
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{-D}^{0} \mathrm{d}s \, \left(3D - s + k_F^{-1}\right) \ln\left(3D - s + k_F^{-1}\right) - \left(3D - s + k_F^{-1}\right) \\
&\quad - \left(\left(2D - s + k_F^{-1}\right) \ln\left(2D - s + k_F^{-1}\right) - \left(2D - s + k_F^{-1}\right)\right) \\
&= \int_{-D}^{0} \mathrm{d}s \, \left(3D - s + k_F^{-1}\right) \ln\left(3D - s + k_F^{-1}\right) \\
&\quad - \left(2D - s + k_F^{-1}\right) \ln\left(2D - s + k_F^{-1}\right) + 5D \\
&= \int_{-D}^{0} \mathrm{d}s \, \left(3D - s + k_F^{-1}\right) \ln\left(3D - s + k_F^{-1}\right) \\
&\quad - \int_{-D}^{0} \mathrm{d}s \, \left(2D - s + k_F^{-1}\right) \ln\left(2D - s + k_F^{-1}\right) + \int_{-D}^{0} \mathrm{d}s \, 5D \\
&= \int_{4D+k_F^{-1}}^{3D+k_F^{-1}} \mathrm{d}y \, y \ln(y) - \int_{3D+k_F^{-1}}^{2D+k_F^{-1}} \mathrm{d}z \, z \ln(z) + \int_{-D}^{0} \mathrm{d}s \, 5D \\
&= \left[\frac{1}{2} y^2 \ln(y) - \frac{1}{4} y^2\right]_{4D+k_F^{-1}}^{3D+k_F^{-1}} - \left[\frac{1}{2} z^2 \ln(z) - \frac{1}{4} z^2\right]_{3D+k_F^{-1}}^{2D+k_F^{-1}} + \Big[5Ds\Big]_{-D}^{0} \\
&= \frac{1}{2}\left(3D + k_F^{-1}\right)^2 \ln\left(3D + k_F^{-1}\right) - \frac{1}{4}\left(3D + k_F^{-1}\right)^2 \\
&\quad - \left(\frac{1}{2}\left(4D + k_F^{-1}\right)^2 \ln\left(4D + k_F^{-1}\right) - \frac{1}{4}\left(4D + k_F^{-1}\right)^2\right) \\
&\quad - \frac{1}{2}\left(2D + k_F^{-1}\right)^2 \ln\left(2D + k_F^{-1}\right) - \frac{1}{4}\left(2D + k_F^{-1}\right)^2 \\
&\quad - \left(\frac{1}{2}\left(3D + k_F^{-1}\right)^2 \ln\left(3D + k_F^{-1}\right) - \frac{1}{4}\left(3D + k_F^{-1}\right)^2\right) + 5D^2 \\
&= \frac{1}{2}\left(3D + k_F^{-1}\right)^2 \ln\left(3D + k_F^{-1}\right) - \frac{1}{4}\left(3D + k_F^{-1}\right)^2 \\
&\quad - \frac{1}{2}\left(4D + k_F^{-1}\right)^2 \ln\left(4D + k_F^{-1}\right) + \frac{1}{4}\left(4D + k_F^{-1}\right)^2 \\
&\quad - \frac{1}{2}\left(2D + k_F^{-1}\right)^2 \ln\left(2D + k_F^{-1}\right) - \frac{1}{4}\left(2D + k_F^{-1}\right)^2 \\
&\quad - \frac{1}{2}\left(3D + k_F^{-1}\right)^2 \ln\left(3D + k_F^{-1}\right) + \frac{1}{4}\left(3D + k_F^{-1}\right)^2 + 5D^2 \\
&= -\frac{1}{2}\left(4D + k_F^{-1}\right)^2 \ln\left(4D + k_F^{-1}\right) + \frac{1}{4}\left(4D + k_F^{-1}\right)^2 \\
&\quad - \frac{1}{2}\left(2D + k_F^{-1}\right)^2 \ln\left(2D + k_F^{-1}\right) - \frac{1}{4}\left(2D + k_F^{-1}\right)^2 + 5D^2 \,.
\end{aligned}
$$

Putting I, II and II together we find

$$\int_{-D}^{0} \mathrm{d}s \int_{0}^{D} \mathrm{d}r \int_{-D}^{0} \mathrm{d}u \int_{0}^{D} \mathrm{d}t \; \frac{1}{\left(r-s+t-u+k_F^{-1}\right)^2} \tag{A.127}$$

$$\begin{aligned}
&= 2\left(\left(D+k_F^{-1}\right)^2 \ln\left(D+k_F^{-1}\right) - D^2\right) \\
&\quad - \left(-\frac{1}{2}\left(2D+k_F^{-1}\right)^2 \ln\left(2D+k_F^{-1}\right) + \frac{1}{4}\left(2D+k_F^{-1}\right)^2\right. \\
&\quad \left.-\frac{1}{2}k_F^{-2}\ln\left(k_F^{-1}\right) - \frac{1}{4}k_F^{-2} + D^2\right) \\
&\quad - \left(-\frac{1}{2}\left(2D+k_F^{-1}\right)^2 \ln\left(2D+k_F^{-1}\right) - \frac{1}{4}\left(2D+k_F^{-1}\right)^2 + 5D^2\right) \\
&= 2\left(D+k_F^{-1}\right)^2 \ln\left(D+k_F^{-1}\right) - 2D^2 \\
&\quad + \frac{1}{2}\left(2D+k_F^{-1}\right)^2 \ln\left(2D+k_F^{-1}\right) - \frac{1}{4}\left(2D+k_F^{-1}\right)^2 \\
&\quad + \frac{1}{2}k_F^{-2}\ln\left(k_F^{-1}\right) + \frac{1}{4}k_F^{-2} - D^2 \\
&\quad + \frac{1}{2}\left(2D+k_F^{-1}\right)^2 \ln\left(2D+k_F^{-1}\right) + \frac{1}{4}\left(2D+k_F^{-1}\right)^2 - 5D^2 \\
&= 2\left(D+k_F^{-1}\right)^2 \ln\left(D+k_F^{-1}\right) + \left(2D+k_F^{-1}\right)^2 \ln\left(2D+k_F^{-1}\right) \\
&\quad + \frac{1}{2}k_F^{-2}\ln\left(k_F^{-1}\right) + \frac{1}{4}k_F^{-2} - 8D^2 \\
&= \mathcal{O}(1)\,.
\end{aligned} \tag{A.128}$$

Now we can combine everything to end up with

$$L^{-2d} \sum_{\substack{p-k\in B_F^C \\ p\in B_F}} \sum_{\substack{q+k\in B_F^C \\ q\in B_F}} \frac{|\hat{w}(k)|}{\left(|p|-|p-k|+|q|-|q+k|+k_F^{-1}\right)^2} \tag{A.129}$$

$$\leq L^{2d} \frac{1}{\left(2+\frac{L}{k_F}\right)^2} \left(C^2 + \widetilde{C}^2 + 2C\widetilde{C}\right) k_F^{2d-2} + \; L^{2d}\hat{C}^2 \cdot \mathcal{O}(1)\,. \tag{A.130}$$

p) In order to bound the sum, we find

$$L^{-2d} \sum_{p \in M_k^F} \sum_{k \in [-D,D]^d} = L^{-2d} \sum_{k \in [-D,D]^d} |M_k^F| , \tag{A.131}$$

which means that we need to count the number of lattice points that are outside the Fermi ball but within distance k_F from the shifted center k. For fixed $k \in [-D, D]^d$, we need to estimate $|M_k^F|$, the number of lattice points p satisfying $|p - k| \leq k_F$ (inside the shifted Fermi ball centered at k) and $|p| > k_F$ (outside the original Fermi ball)

The set M_k^F is contained in the annular region between spheres of radius $k_F - |k|$ and $k_F + |k|$ centered at the origin, provided these radii are positive. More precisely, if $|p - k| \leq k_F$ and $|p| > k_F$, then by the triangle inequality:

$$|p| - |k| \leq |p - k| \leq k_F \quad \text{and} \quad |p| > k_F \tag{A.132}$$

This gives us $k_F < |p| \leq k_F + |k|$. Similarly, from $|p - k| \leq k_F$ and the reverse triangle inequality:

$$|p| \geq ||p - k| - |k|| \geq |k_F - |k|| \tag{A.133}$$

For $k \in [-D, D]^d$, we have $|k| \leq \sqrt{d}D$. The set M_k^F is therefore contained in the annular region:

$$\left\{ p \in \frac{1}{L}\mathbb{Z}^d : k_F < |p| \leq k_F + \sqrt{d}D \right\} \tag{A.134}$$

Using the same technique as in Lemma 5, each lattice point can be associated with a fundamental cell of volume L^{-d}, and we enlarge the region by $\frac{\sqrt{d}}{2L}$ to account for the discrete nature of the lattice.

The volume of the enlarged annular region is:

$$\mathrm{Vol} = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} \left[\left(k_F + \sqrt{d}D + \frac{\sqrt{d}}{2L} \right)^d - \left(k_F - \frac{\sqrt{d}}{2L} \right)^d \right] \tag{A.135}$$

Using the inequality $b^d - a^d \leq d \cdot b^{d-1}(b-a)$ for $b > a > 0$:

$$\begin{aligned} \mathrm{Vol} &\leq \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} \cdot d \cdot \left(k_F + \sqrt{d}D + \frac{\sqrt{d}}{2L} \right)^{d-1} \left(\sqrt{d}D + \frac{\sqrt{d}}{L} \right) \\ &= \frac{d\sqrt{d}\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} \left(k_F + \sqrt{d}D + \frac{\sqrt{d}}{2L} \right)^{d-1} \left(D + \frac{1}{L} \right) . \end{aligned} \tag{A.136}$$

Therefore we have

$$|M_k^F| \leq L^d \cdot \frac{d\sqrt{d}\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} \left(k_F + \sqrt{d}D + \frac{\sqrt{d}}{2L} \right)^{d-1} \left(D + \frac{1}{L} \right) . \tag{A.137}$$

The number of lattice points k in $[-D, D]^d$ can be bounded by

$$\sum_{k \in [-D,D]^d} \leq \frac{\left(2D + \frac{1}{L}\right)}{\left(\frac{1}{L}\right)^d} = (2DL+1)^d , \tag{A.138}$$

where we divide the volume of the enlarged cube of dimension d by the volume of the fundamental cell, similar to computations in (5). Thus

$$\begin{aligned} &L^{-2d} \sum_{k \in [-D,D]^d} |M_k^F| \\ \leq\ &L^{-2d} \cdot (2DL+1)^d \cdot L^d \cdot \frac{d\sqrt{d}\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} \left(k_F + \sqrt{d}D + \frac{\sqrt{d}}{2L} \right)^{d-1} \left(D + \frac{1}{L} \right) \end{aligned} \tag{A.139}$$

$$\leq (2D + L^{-1})^d \cdot \frac{d\sqrt{d}\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} \left(k_F + \sqrt{d}D + \frac{\sqrt{d}}{2L}\right)^{d-1} \left(D + \frac{1}{L}\right),$$

q)

$$L^{-2d} \sum_{(q,p)\in N_k^F} \sum_{k\in[-D,D]^d} \tag{A.140}$$

In order to bound the sum, we find

$$L^{-2d} \sum_{(q,p)\in N_k^F} \sum_{k\in[-D,D]^d} 1 = L^{-2d} \sum_{k\in[-D,D]^d} |N_k^F| \tag{A.141}$$

Since $N_k^F = \{(p-k, q+k) \in M_{-k}^F \times M_k^F\}$ is the Cartesian product of two annular regions, we have

$$|N_k^F| = |M_{-k}^F| \cdot |M_k^F| \tag{A.142}$$

From the previous bound, we know that for $k \in [-D, D]^d$

$$|M_k^F| \leq L^d \cdot \frac{d\sqrt{d}\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} \left(k_F + \sqrt{d}D + \frac{\sqrt{d}}{2L}\right)^{d-1} \left(D + \frac{1}{L}\right). \tag{A.143}$$

Since M_{-k}^F has the same structure as M_k^F (only with shifted center $-k$ instead of k), the same bound applies

$$|M_{-k}^F| \leq L^d \cdot \frac{d\sqrt{d}\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} \left(k_F + \sqrt{d}D + \frac{\sqrt{d}}{2L}\right)^{d-1} \left(D + \frac{1}{L}\right) \tag{A.144}$$

Therefore

$$|N_k^F| = |M_{-k}^F| \cdot |M_k^F| \tag{A.145}$$
$$\leq L^{2d} \cdot \left[\frac{d\sqrt{d}\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}\right]^2 \left(k_F + \sqrt{d}D + \frac{\sqrt{d}}{2L}\right)^{2(d-1)} \left(D + \frac{1}{L}\right)^2 .$$

The number of lattice points k in $[-D, D]^d$ can be bounded by

$$\sum_{k\in[-D,D]^d} 1 \leq (2DL+1)^d . \tag{A.146}$$

Thus

$$L^{-2d} \sum_{k\in[-D,D]^d} |N_k^F| \tag{A.147}$$
$$\leq L^{-2d} \cdot (2DL+1)^d \cdot L^{2d} \cdot \left(\frac{d\sqrt{d}\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}\right)^2$$
$$\cdot \left(k_F + \sqrt{d}D + \frac{\sqrt{d}}{2L}\right)^{2(d-1)} \left(D + \frac{1}{L}\right)^2$$
$$\leq \left(2D + \frac{1}{L}\right)^d \cdot L^d \cdot \left(\frac{d\sqrt{d}\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}\right)^2 \left(k_F + \sqrt{d}D + \frac{\sqrt{d}}{2L}\right)^{2(d-1)} \left(D + \frac{1}{L}\right)^2 .$$

This yields

$$L^{-2d} \sum_{k\in[-D,D]^d} |N_k^F| \tag{A.148}$$
$$\leq \left(2D + \frac{1}{L}\right)^d \cdot L^d \cdot \left(\frac{d\sqrt{d}\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}\right)^2 \left(k_F + \sqrt{d}D + \frac{\sqrt{d}}{2L}\right)^{2(d-1)} \left(D + \frac{1}{L}\right)^2 .$$

The key observation is that we get the L^d factor from the lattice structure and the $k_F^{2(d-1)} = k_F^{2d-2}$ scaling from the Cartesian product of the two annular regions.

r) To bound the sum

$$L^{-2d} \sum_{(q,p)\in N_k^F} \sum_{k\in[-D,D]^d} \frac{|\hat{w}(k)|^2}{\left(|p| - |p-k| + |q| - |q+k| + k_F^{-1}\right)} \tag{A.149}$$

we use the similar computations as in o). We find

$$\begin{aligned} & L^{-2d} \sum_{\substack{p-k\in B_F^C \\ p\in B_F}} \sum_{\substack{q+k\in B_F^C \\ q\in B_F}} \frac{|\hat{w}(k)|}{\left(|p| - |p-k| + |q| - |q+k| + k_F^{-1}\right)} \\ \leq\ & L^{-2d} \sum_{\substack{p-k\in B_F^C \\ p\in B_F}} \sum_{\substack{q+k\in B_F^C \\ q\in B_F}} \frac{1}{\left(|p| - |p-k| + |q| - |q+k| + k_F^{-1}\right)} \mathbb{1}_{B_D(p)}(k) \mathbb{1}_{B_D(q)}(k)\,. \end{aligned} \tag{A.150}$$

We consider the function

$$\begin{aligned} & h\left(|p|, |p-k|, |q|, |q+k|\right) \\ =\ & \left(|p| - |p-k| + |q| - |q+k| + k_F^{-1}\right)^{-1}\,, \end{aligned} \tag{A.151}$$

that is anti-monotone in the first and third component and monotone in the second and fourth. With

$$\begin{aligned} h^{\sup(L)} :=& \sup_{\substack{p,p-k,q,q+k\in\frac{1}{L}\mathbb{Z}^d \\ p\neq p-k\neq q\neq q+k}} h\left(|p|, |p-k|, |q|, |q+k|\right) \\ =& \sup_{\substack{p,p-k,q,q+k\in\frac{1}{L}\mathbb{Z}^d \\ p\neq p-k\neq q\neq q+k}} \frac{1}{(|p| - |p-k| + |q| - |q+k| + k_F^{-1})} \\ =& \frac{1}{\left(\frac{2}{L} + k_F^{-1}\right)}\,, \end{aligned} \tag{A.152}$$

we then have

$$L^{-2d} \sum_{\substack{p-k\in B^L_{[k_F,\infty)} \\ p\in B^L_{k_F}}} \sum_{\substack{q+k\in B^L_{[k_F,\infty)} \\ q\in B^L_{k_F}}} h\left(|p|,|p-k|,|q|,|q+k|\right) \mathbb{1}_{B_D(p)}(k)\mathbb{1}_{B_D(q)}(k) \tag{A.153}$$

$$\leq L^{-2d} \Bigg(g^{\sup(L)} \Bigg(\underbrace{\sum_{\substack{p-k\in B^L_{[k_F,k_F+\frac{1}{L})} \\ p\in B^L_{[k_F-D,k_F)}}} \sum_{\substack{q+k\in B^L_{[k_F,k_F+\frac{1}{L})} \\ q\in B^L_{[k_F-D,k_F)}}} \mathbb{1}_{B_D(p)}(k)\mathbb{1}_{B_D(q)}(k)}_{=:(1r)}$$

$$+ \underbrace{\sum_{\substack{p-k\in B^L_{[k_F,k_F+D)} \\ p\in B^L_{[k_F-\frac{1}{L},k_F)}}} \sum_{\substack{q+k\in B^L_{[k_F,k_F+D)} \\ q\in B^L_{[k_F-\frac{1}{L},k_F)}}} \mathbb{1}_{B_D(p)}(k)\mathbb{1}_{B_D(q)}(k)}_{=:(2r)}$$

$$+ \underbrace{\sum_{\substack{p-k\in B^L_{[k_F,k_F+\frac{1}{L})} \\ p\in B^L_{[k_F-D,k_F)}}} \sum_{\substack{q+k\in B^L_{[k_F,k_F+D)} \\ q\in B^L_{[k_F-\frac{1}{L},k_F)}}} \mathbb{1}_{B_D(p)}(k)\mathbb{1}_{B_D(q)}(k)}_{=:(3r)}$$

$$+ \underbrace{\sum_{\substack{p-k\in B^L_{[k_F,k_F+R)} \\ p\in B^L_{[k_F-\frac{1}{L},k_F)}}} \sum_{\substack{q+k\in B^L_{[k_F,k_F+\frac{1}{L})} \\ q\in B^L_{[k_F-D,k_F)}}} \mathbb{1}_{B_D(p)}(k)\mathbb{1}_{B_D(q)}(k)}_{=:(4r)} \Bigg)$$

$$+ \sum_{\substack{p-k\in B^L_{[k_F+\frac{1}{L},k_F+D)} \\ p\in B^L_{[k_F-D,k_F-\frac{1}{L})}}} \sum_{\substack{q+k\in B^L_{[k_F+\frac{1}{L},k_F+D)} \\ q\in B^L_{[k_F-D,k_F-\frac{1}{L})}}} h\left(|p|,|p-k|,|q|,|q+k|\right) \mathbb{1}_{B_D(p)}(k)\mathbb{1}_{B_D(q)}(k) \Bigg).$$

We obtain

$$h^{\sup(L)}(1r) \leq h^{\sup(L)} L^{4d} \frac{C^2}{L^2} k_F^{2d-2} = L^{4d} \frac{C^2}{\left(2L + \frac{L^2}{k_F}\right)} k_F^{2d-2}, \tag{A.154}$$

as well as

$$h^{\sup(L)}(2r) \leq h^{\sup(L)} L^{4d} \frac{\widetilde{C}^2}{L^2} k_F^{2d-2} = L^{4d} \frac{\widetilde{C}^2}{\left(2L + \frac{L^2}{k_F}\right)} k_F^{2d-2}, \tag{A.155}$$

and

$$h^{\sup(L)}(3r) \leq h^{\sup(L)} L^{4d} \frac{\widetilde{C}C}{L^2} k_F^{2d-2} = L^{4d} \frac{\widetilde{C}C}{\left(2L + \frac{L^2}{k_F}\right)} k_F^{2d-2}, \tag{A.156}$$

and

$$h^{\sup(L)}(4r) \leq h^{\sup(L)} L^{4d} \frac{C\widetilde{C}}{L^2} k_F^{2d-2} = L^{4d} \frac{C\widetilde{C}}{\left(2L + \frac{L^2}{k_F}\right)} k_F^{2d-2}. \tag{A.157}$$

For the last sum, combining previous computations and using $\omega' \mapsto r - k_F, \mu' \mapsto s - k_F, \nu' \mapsto t - k_F$ and $\eta' \mapsto u - k_F$, we get

$$\begin{aligned}
&\leq L^{4d} \hat{C}^2 \int_{k_F-D}^{k_F} \int_{k_F}^{k_F+D} \int_{k_F-D}^{k_F} \int_{k_F}^{k_F+D} \frac{1}{\left(\omega' - \mu' + \nu' - \eta' + k_F^{-1}\right)} \, \mathrm{d}\omega' \, \mathrm{d}\mu' \, \mathrm{d}\nu' \, \mathrm{d}\eta' \\
&= L^{4d} \hat{C}^2 \int_{-D}^{0} \mathrm{d}s \int_{0}^{D} \mathrm{d}r \int_{-D}^{0} \mathrm{d}u \int_{0}^{D} \mathrm{d}t \, \frac{1}{\left(r - s + t - u + k_F^{-1}\right)}.
\end{aligned} \tag{A.158}$$

Computing the integral we find

$$\begin{aligned}
&\int_{-D}^{0} \mathrm{d}s \int_{0}^{D} \mathrm{d}r \int_{-D}^{0} \mathrm{d}u \int_{0}^{D} \mathrm{d}t \, \frac{1}{\left(r - s + t - u + k_F^{-1}\right)} \qquad \text{(A.159)} \\
&= \int_{-D}^{0} \mathrm{d}s \int_{0}^{D} \mathrm{d}r \int_{-D}^{0} \mathrm{d}u \, \left[\ln\left(r - s + t - u + k_F^{-1}\right)\right]_0^D \\
&= \int_{-D}^{0} \mathrm{d}s \int_{0}^{D} \mathrm{d}r \int_{-D}^{0} \mathrm{d}u \, \left(\ln\left(r - s + D - u + k_F^{-1}\right)\right) - \left(\ln\left(r - s - u + k_F^{-1}\right)\right)
\end{aligned}$$

$$= \int_{-D}^{0} \mathrm{d}s \int_{0}^{D} \mathrm{d}r \, \Big[-\Big(r - s + D - u + k_F^{-1}\Big) \ln\Big(r - s + D - u + k_F^{-1}\Big)$$
$$+ \Big(r - s + D - u + k_F^{-1}\Big)\Big(r - s - u + k_F^{-1}\Big)$$
$$\cdot \ln\Big(r - s - u + k_F^{-1}\Big) - \Big(r - s - u + k_F^{-1}\Big)\Big]_{-D}^{0}$$
$$= \int_{-D}^{0} \mathrm{d}s \int_{0}^{D} \mathrm{d}r \, \Big(\Big(r - s + 2D + k_F^{-1}\Big) \ln\Big(r - s + 2D + k_F^{-1}\Big)$$
$$- 2\Big(r - s + D + k_F^{-1}\Big) \ln\Big(r - s + D + k_F^{-1}\Big)$$
$$+ \Big(r - s + k_F^{-1}\Big) \ln\Big(r - s + k_F^{-1}\Big)\Big)$$
$$= \underbrace{\int_{-D}^{0} \mathrm{d}s \int_{0}^{D} \mathrm{d}r \, \Big(r - s + 2D + k_F^{-1}\Big) \ln\Big(r - s + 2D + k_F^{-1}\Big)}_{:=A}$$
$$- 2 \underbrace{\int_{-D}^{0} \mathrm{d}s \int_{0}^{D} \mathrm{d}r \, \Big(r - s + D + k_F^{-1}\Big) \ln\Big(r - s + D + k_F^{-1}\Big)}_{:=B}$$
$$+ \underbrace{\int_{-D}^{0} \mathrm{d}s \int_{0}^{D} \mathrm{d}r \, \Big(r - s + k_F^{-1}\Big) \ln\Big(r - s + k_F^{-1}\Big)}_{:=C} .$$

Evaluation of A, using $r - s + 2D + k_F^{-1} \mapsto x$, and later $3D - s + k_F^{-1} \mapsto y$ and $2D - s + k_F^{-1} \mapsto z$:

$$A = \int_{-D}^{0} \mathrm{d}s \int_{0}^{D} \mathrm{d}r \, \Big(r - s + 2D + k_F^{-1}\Big) \ln\Big(r - s + 2D + k_F^{-1}\Big) \tag{A.160}$$
$$= \int_{-D}^{0} \mathrm{d}s \int_{2D-s+k_F^{-1}}^{3D-s+k_F^{-1}} \mathrm{d}x \; x \ln x$$
$$= \int_{-D}^{0} \mathrm{d}s \, \Big[\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2\Big]_{2D-s+k_F^{-1}}^{3D-s+k_F^{-1}}$$

$$\begin{aligned}
&= \int_{-D}^{0} \mathrm{d}s \, \Big(\frac{1}{2}(3D - s + k_F^{-1})^2 \ln(3D - s + k_F^{-1}) - \frac{1}{4}(3D - s + k_F^{-1})^2 \Big) \\
&\quad - \Big(\frac{1}{2}(2D - s + k_F^{-1})^2 \ln(2D - s + k_F^{-1}) - \frac{1}{4}(2D - s + k_F^{-1})^2 \Big) \\
&= \frac{1}{2}\int_{-D}^{0} \mathrm{d}s \, (3D - s + k_F^{-1})^2 \ln(3D - s + k_F^{-1}) \\
&\quad - \frac{1}{4}\int_{-D}^{0} \mathrm{d}s \, (3D - s + k_F^{-1})^2 \\
&\quad - \frac{1}{2}\int_{-D}^{0} \mathrm{d}s \, (2D - s + k_F^{-1})^2 \ln(2D - s + k_F^{-1}) \\
&\quad + \frac{1}{4}\int_{-D}^{0} \mathrm{d}s \, (2D - s + k_F^{-1})^2 \\
&= \frac{1}{2}\int_{3D+k_F^{-1}}^{4D+k_F^{-1}} \mathrm{d}y \; y^2 \ln y \; - \frac{1}{4}\int_{3D+k_F^{-1}}^{4D+k_F^{-1}} \mathrm{d}y \; y^2 \\
&\quad - \frac{1}{2}\int_{2D+k_F^{-1}}^{3D+k_F^{-1}} \mathrm{d}z \; z^2 \ln z + \frac{1}{4}\int_{2D+k_F^{-1}}^{3D+k_F^{-1}} \mathrm{d}z \; z^2 \\
&= \frac{1}{2}\Big[\frac{1}{3}y^3 \ln y - \frac{1}{9}y^3\Big]_{3D+k_F^{-1}}^{4D+k_F^{-1}} - \frac{1}{4}\Big[\frac{1}{3}y^3\Big]_{3D+k_F^{-1}}^{4D+k_F^{-1}} \\
&\quad - \frac{1}{2}\Big[\frac{1}{3}z^3 \ln z - \frac{1}{9}z^3\Big]_{2D+k_F^{-1}}^{3D+k_F^{-1}} + \frac{1}{4}\Big[\frac{1}{3}z^3\Big]_{2D+k_F^{-1}}^{3D+k_F^{-1}} \\
&= \Big[\frac{1}{6}y^3 \ln y - \frac{5}{36}y^3\Big]_{3D+k_F^{-1}}^{4D+k_F^{-1}} - \Big[\frac{1}{6}z^3 \ln z - \frac{5}{36}z^3\Big]_{2D+k_F^{-1}}^{3D+k_F^{-1}} .
\end{aligned}$$

Evaluation of B, using $r - s + D + k_F^{-1} \mapsto x$, and later $2D - s + k_F^{-1} \mapsto y$ and $D - s + k_F^{-1} \mapsto z$:

$$B = \int_{-D}^{0} \mathrm{d}s \int_{0}^{D} \mathrm{d}r \, \Big(r - s + D + k_F^{-1}\Big) \ln\Big(r - s + D + k_F^{-1}\Big) \tag{A.161}$$

$$
\begin{aligned}
&= \int_{-D}^{0} \mathrm{d}s \int_{D-s+k_F^{-1}}^{2D-s+k_F^{-1}} \mathrm{d}x \; x \ln x \\
&= \int_{-D}^{0} \mathrm{d}s \left[\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2\right]_{D-s+k_F^{-1}}^{2D-s+k_F^{-1}} \\
&= \frac{1}{2}\int_{-D}^{0} \mathrm{d}s \; (2D - s + k_F^{-1})^2 \ln(2D - s + k_F^{-1}) - \frac{1}{4}\int_{-D}^{0} \mathrm{d}s \; (2D - s + k_F^{-1})^2 \\
&\quad - \frac{1}{2}\int_{-D}^{0} \mathrm{d}s \; (D - s + k_F^{-1})^2 \ln(D - s + k_F^{-1}) + \frac{1}{4}\int_{-D}^{0} \mathrm{d}s \; (D - s + k_F^{-1})^2 \\
&= \frac{1}{2}\int_{2D+k_F^{-1}}^{3D+k_F^{-1}} \mathrm{d}y \; y^2 \ln y - \frac{1}{4}\int_{2D+k_F^{-1}}^{3D+k_F^{-1}} \mathrm{d}y \; y^2 \\
&\quad - \frac{1}{2}\int_{D+k_F^{-1}}^{2D+k_F^{-1}} \mathrm{d}z \; z^2 \ln z + \frac{1}{4}\int_{D+k_F^{-1}}^{2D+k_F^{-1}} \mathrm{d}z \; z^2 \\
&= \frac{1}{2}\left[\frac{1}{3}y^3 \ln y - \frac{1}{9}y^3\right]_{2D+k_F^{-1}}^{3D+k_F^{-1}} - \frac{1}{4}\left[\frac{1}{3}y^3\right]_{2D+k_F^{-1}}^{3D+k_F^{-1}} \\
&\quad - \frac{1}{2}\left[\frac{1}{3}z^3 \ln z - \frac{1}{9}z^3\right]_{D+k_F^{-1}}^{2D+k_F^{-1}} + \frac{1}{4}\left[\frac{1}{3}z^3\right] D + k_F^{-1}{}^{2D+k_F^{-1}} \\
&= \left[\frac{1}{6}y^3 \ln y - \frac{5}{36}y^3\right]_{2D+k_F^{-1}}^{3D+k_F^{-1}} - \left[\frac{1}{6}z^3 \ln z - \frac{5}{36}z^3\right]_{D+k_F^{-1}}^{2D+k_F^{-1}} .
\end{aligned}
$$

Evaluation of C (using $r - s + k_F^{-1} \mapsto x$, and later $D - s + k_F^{-1} \mapsto y$ and $k_F^{-1} - s \mapsto z$):

$$
\begin{aligned}
C &= \int_{-D}^{0} \mathrm{d}s \int_{0}^{D} \mathrm{d}r \; \left(r - s + k_F^{-1}\right) \ln\left(r - s + k_F^{-1}\right) \qquad \text{(A.162)} \\
&= \int_{-D}^{0} \mathrm{d}s \int_{k_F^{-1}-s}^{D-s+k_F^{-1}} \mathrm{d}x \; x \ln x \\
&= \int_{-D}^{0} \mathrm{d}s \left[\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2\right]_{k_F^{-1}-s}^{D-s+k_F^{-1}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{-D}^{0} \mathrm{d}s\ (D-s+k_F^{-1})^2 \ln(D-s+k_F^{-1}) - \frac{1}{4}\int_{-D}^{0} \mathrm{d}s\ (D-s+k_F^{-1})^2 \\
&\quad - \frac{1}{2}\int_{-D}^{0} \mathrm{d}s\ (k_F^{-1}-s)^2 \ln(k_F^{-1}-s) + \frac{1}{4}\int_{-D}^{0} \mathrm{d}s\ (k_F^{-1}-s)^2 \\
&= \frac{1}{2}\int_{D+k_F^{-1}}^{2D+k_F^{-1}} \mathrm{d}y\ y^2 \ln y - \frac{1}{4}\int_{D+k_F^{-1}}^{2D+k_F^{-1}} \mathrm{d}y\ y^2 - \frac{1}{2}\int_{k_F^{-1}}^{D+k_F^{-1}} \mathrm{d}z\ z^2 \ln z + \frac{1}{4}\int_{k_F^{-1}}^{D+k_F^{-1}} \mathrm{d}z\ z^2 \\
&= \frac{1}{2}\left[\frac{1}{3}y^3 \ln y - \frac{1}{9}y^3\right]_{D+k_F^{-1}}^{2D+k_F^{-1}} - \frac{1}{4}\left[\frac{1}{3}y^3\right]_{D+k_F^{-1}}^{2D+k_F^{-1}} \\
&\quad - \frac{1}{2}\left[\frac{1}{3}z^3 \ln z - \frac{1}{9}z^3\right]_{k_F^{-1}}^{D+k_F^{-1}} + \frac{1}{4}\left[\frac{1}{3}z^3\right]_{k_F^{-1}}^{D+k_F^{-1}} \\
&= \left[\frac{1}{6}y^3 \ln y - \frac{5}{36}y^3\right]_{D+k_F^{-1}}^{2D+k_F^{-1}} - \left[\frac{1}{6}z^3 \ln z - \frac{5}{36}z^3\right]_{k_F^{-1}}^{D+k_F^{-1}} .
\end{aligned}
$$

Putting A, B, and C together we find

$$
\begin{aligned}
&\int_{-D}^{0} \mathrm{d}s \int_{0}^{D} \mathrm{d}r \int_{-D}^{0} \mathrm{d}u \int_{0}^{D} \mathrm{d}t\ \frac{1}{\left(r-s+t-u+k_F^{-1}\right)} \qquad \text{(A.163)} \\
&= A - 2B + C \\
&= \Big(\frac{1}{6}(4D+k_F^{-1})^3 \ln(4D+k_F^{-1}) - 4\cdot\frac{1}{6}(3D+k_F^{-1})^3 \ln(3D+k_F^{-1}) \\
&\quad + 6\cdot\frac{1}{6}(2D+k_F^{-1})^3 \ln(2D+k_F^{-1}) - 4\cdot\frac{1}{6}(D+k_F^{-1})^3 \ln(D+k_F^{-1}) \\
&\quad + \frac{1}{6}(k_F^{-1})^3 \ln(k_F^{-1})\Big) \\
&= \mathcal{O}(1) .
\end{aligned}
$$

Now we can combine everything to end up with

$$
L^{-2d} \sum_{\substack{p-k\in B_F^C \\ p\in B_F}} \sum_{\substack{q+k\in B_F^C \\ q\in B_F}} \frac{|\hat{w}(k)|}{\left(|p|-|p-k|+|q|-|q+k|+k_F^{-1}\right)} \qquad \text{(A.164)}
$$

$$\leq L^{2d} \frac{1}{\left(2L + \frac{L^2}{k_F}\right)^2} \left(C^2 + \widetilde{C}^2 + 2C\widetilde{C}\right) k_F^{2d-2} + \ L^{2d} \hat{C}^2 \cdot \mathcal{O}(1)$$

s) In order to bound the sum, we find

$$L^{-2d} \sum_{m \in B_F} 1 = L^{-2d} \left| B_F \right| \tag{A.165}$$

To estimate $|B_F|$, we note that B_F is contained in the ball of radius k_F centered at the origin. Using the same technique as in the previous computations, each lattice point can be associated with a fundamental cell of volume L^{-d}, and we enlarge the region by $\frac{\sqrt{d}}{2L}$ to account for the discrete nature of the lattice. The volume of the enlarged ball is given by

$$\mathrm{Vol} = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} \left(k_F + \frac{\sqrt{d}}{2L}\right)^d . \tag{A.166}$$

Therefore, the number of lattice points in B_F can be bounded by

$$|B_F| \leq L^d \cdot \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} \left(k_F + \frac{\sqrt{d}}{2L}\right)^d . \tag{A.167}$$

Thus,

$$\begin{aligned} L^{-2d} \left| B_F \right| &\leq L^{-2d} \cdot L^d \cdot \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} \left(k_F + \frac{\sqrt{d}}{2L}\right)^d \\ &= L^{-d} \cdot \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} \left(k_F + \frac{\sqrt{d}}{2L}\right)^d . \end{aligned} \tag{A.168}$$

t) We find

$$L^{-2d} \sum_{k\in[-D,D]^d} \sum_{(p,g)\in T_F} \sum_{h\in B_F^C} \frac{|\hat{w}_{h(p-k)}||\hat{v}_{pg}|}{|p|-|g|+k_F^{-1}} \tag{A.169}$$

$$\leq L^{-2d} \sum_{(p,g)\in T_F} \frac{|\hat{v}_{pg}|}{|p|-|g|+k_F^{-1}} \sum_{k\in[-D,D]^d} \sum_{h\in B_F^C} |\hat{w}_{h(p-k)}| \,.$$

We want to estimate:

$$\sum_{k\in[-D,D]^d} \sum_{h\in B_F^C} |\hat{w}(h-p-k)| \tag{A.170}$$

Since $\hat{w}$ has compact support, we have $|\hat{w}(\cdot)| \leq \|\hat{w}\|_\infty =: C$ for some constant C. Moreover, $\hat{w}(k) \neq 0$ only for $|k| \leq D$. Therefore

$$\sum_{k\in[-D,D]^d} \sum_{h\in B_F^C} |\hat{w}(h-p-k)| \leq C \sum_{k\in[-D,D]^d} \sum_{h\in B_F^C} \mathbb{1}_{|h-p-k|\leq D} \tag{A.171}$$

$$\leq \sum_{k\in[-D,D]^d} \sum_{h\in B_F^C} \mathbb{1}_{|h-p-k|\leq D} \,.$$

We bound $\sum_{h\in B_F^C} \mathbf{1}_{|h-p-k|\leq D}$ using Lemma (5) to find

$$\sum_{h\in B_F^C} \mathbf{1}_{|h-p-k|\leq D} \leq L^d \left(D + \frac{\sqrt{d}}{L}\right)\left(1+\sqrt{d}\right) \tag{A.172}$$

$$\cdot \left(k_F + D + \frac{2+\sqrt{d}}{2L}\right)^{d-1} \left(2\arcsin\left(\frac{D+\frac{\sqrt{d}}{2L}}{k_F - 2D}\right)\right)^{d-1} .$$

With

$$\sum_{k\in[-D,D]^d} \leq (2DL+1)^d \,, \tag{A.173}$$

and using previous bounds, we find

$$
\begin{aligned}
&L^{-2d} \sum_{k\in[-D,D]^d} \sum_{(p,g)\in T_F} \sum_{h\in B_F^C} \frac{|\hat{w}_{h(p-k)}||\hat{v}_{pg}|}{|p|-|g|+k_F^{-1}} \qquad \text{(A.174)}\\
\leq &\left(\frac{1}{1+\frac{L}{k_F}}\left(C+\widetilde{C}\right)k_F^{(d-1)} + \hat{C}\mathcal{O}\left(\frac{\ln(k_F)}{k_F}\right)\right)(2DL+1)^d\\
&L^d\left(D+\frac{\sqrt{d}}{L}\right)\left(1+\sqrt{d}\right)\cdot\left(k_F+D+\frac{2+\sqrt{d}}{2L}\right)^{d-1}\\
&\cdot\left(2\arcsin\left(\frac{D+\frac{\sqrt{d}}{2L}}{k_F-2D}\right)\right)^{d-1}.
\end{aligned}
$$

u) To bound the sum, we find

$$
\begin{aligned}
&L^{-2d}\sum_{m\in B_F}\sum_{(h,g)\in T_F}\left(L^{-d}\sum_{n\in B_F^C}\frac{|\hat{v}_{hg}||\hat{v}_{nm}|}{\left(|h|-|g|+k_F^{-1}\right)\left(|n|-|m|+k_F^{-1}\right)}\right)^2\\
\leq &\left(L^{-2d}\sum_{(h,g)\in T_F}\frac{|\hat{v}_{hg}|^2}{\left(|h|-|g|+k_F^{-1}\right)^2}\right)\\
&\cdot L^d\left(L^{-d}\sum_{m\in B_F}\left(L^{-d}\sum_{n\in B_F^C}\frac{|\hat{v}_{nm}|}{\left(|n|-|m|+k_F^{-1}\right)}\right)^2\right), \qquad \text{(A.175)}
\end{aligned}
$$

which we can compute using previous bounds.

v) For fixed k we can use (6) and previous computations to find

$$
\begin{aligned}
&L^{-2d}\sum_{q\in M_k^F}\frac{|\hat{v}(k)|^2\ |\hat{w}(k)|^2}{\left(|q|-|q+k|+k_F^{-1}\right)^2} \qquad \text{(A.176)}\\
\leq &\frac{1}{\frac{1}{L}+k_F^{-1}}\cdot\frac{1}{L}k_F^{(d-1)}+\hat{C}\int\limits_{k_F-D}^{k_F}\int\limits_{k_F}^{k_F+D}\frac{1}{\omega'-\mu'+k_F^{-1}}\,\mathrm{d}\omega'\,\mathrm{d}\mu'\\
\leq &\frac{1}{1+\frac{L}{k_F}}\left(C+\widetilde{C}\right)k_F^{(d-1)}+\hat{C}\mathcal{O}\left(\frac{\ln(k_F)}{k_F}\right).
\end{aligned}
$$

Since the number of lattice points k in $[-D,D]^d$ can be bounded by

$$\sum_{k\in[-D,D]^d} 1 \leq (2DL+1)^d\,, \tag{A.177}$$

we find

$$\begin{aligned} &L^{-2d}\sum_{k\in[-D,D]^d}\sum_{q\in M_k^F}\frac{|\hat{v}(k)|^2\;|\hat{w}(k)|^2}{\left(|q|-|q+k|+k_F^{-1}\right)^2} \\ &\leq (2DL+1)^d\cdot\frac{1}{1+\frac{L}{k_F}}\left(C+\tilde{C}\right)k_F^{(d-1)}+\hat{C}\mathcal{O}\left(\frac{\ln\left(k_F\right)}{k_F}\right)\,. \end{aligned} \tag{A.178}$$

□

Lemma 9. *Let $d\in\{2,3\}$ and v,w compactly supported in Fourier space, then there exists a constant $C'>0$, such that for all $k_F\geq 2$ and $|k|\geq 3B$ the following bounds hold*

$$\lim_{L\to\infty} L^{-2d}\sum_{(n,m)\in T_F}\frac{|\hat{v}\left(n-m\right)|^2}{\left(|n|-|m|+k_F^{-1}\right)}\lesssim C'k_F^{(d-1)} \tag{A.179}$$

Proof. To derive the bounds, we replace the sum by its Riemann integral, which we arise with in the high volume limit $L\to\infty$.
Fix m with $|m|\leq k_F$. Since $|n|\geq k_F$, we have $n\in[k_F,k_F+D]$, hence

$$\int\limits_{|n|\geq k_F}\mathrm{d}^d n\;\chi_{[0,D]}\left(|n-m|\right)\leq\int\limits_{k_F}^{k_F+D}\mathrm{d}n\;n^{d-1}\left(2\arcsin\left(\frac{D}{k_F-D}\right)\right)^{d-1}\,, \tag{A.180}$$

where we used a coordinate transformation to use Lemma (7), as well as the fact that $x\mapsto\arcsin\left(D/x\right)$ is monotonically decreasing and $k_F-D\leq m$. Since

$n^{d-1} \leq (k_F + D)^{d-1}$, we have

$$\begin{aligned} &\int_{k_F}^{k_F+D} \mathrm{d}n\; n^{d-1} \left(2 \arcsin\left(\frac{D}{k_F - D}\right)\right)^{d-1} \\ \leq\; &(k_F + D)^{d-1} \left(2 \arcsin\left(\frac{D}{k_F - D}\right)\right)^{d-1} \int_{k_F}^{k_F+D} \mathrm{d}n\,. \end{aligned} \tag{A.181}$$

For $k_F \geq 3D$

$$\begin{aligned} (k_F + D)^{d-1} \left(2 \arcsin\left(\frac{D}{k_F - D}\right)\right)^{d-1} &\leq (4D)^{d-1} \left(2 \arcsin\left(\frac{1}{2}\right)\right)^{d-1} \\ &=: C \cdot D^{d-1}\,, \end{aligned} \tag{A.182}$$

therefore

$$\begin{aligned} &(k_F + D)^{d-1} \left(2 \arcsin\left(\frac{D}{k_F - D}\right)\right)^{d-1} \int_{k_F}^{k_F+D} \mathrm{d}n \\ \leq\; &C \cdot D^{d-1} \int_{k_F}^{k_F+D} \mathrm{d}n \\ =\; &C \cdot D^d\,. \end{aligned} \tag{A.183}$$

Plugging this into the integral over m and using $|n| \geq k_F$ and $|n - m| \leq D$, we find

$$\begin{aligned} &\int_{|m| \leq k_F} \mathrm{d}^d m \int_{|n| \geq k_F} \mathrm{d}^d n\; \chi_{[0,D]}\left(|n - m|\right) \\ \lesssim\; &C \cdot D^d \int_{|m| \leq k_F} \mathrm{d}^d m\; \chi_{[k_F - D, k_F]}\left(|m|\right) \\ =\; &C \cdot D^d \int_{\Omega} \mathrm{d}\Omega \int_{k_F - D}^{k_F} \mathrm{d}m\; m^{d-1} \end{aligned} \tag{A.184}$$

$$
\begin{aligned}
&\le C \cdot D^d k_F^{d-1} \int\limits_{k_F - D}^{k_F} \mathrm{d}m \\
&\le C \cdot (D k_F)^{d-1} \; .
\end{aligned}
$$

Therefore we bound the integral

$$
\int\limits_{|m| \le k_F} \mathrm{d}^d m \int\limits_{|n| \ge k_F} \mathrm{d}^d n \; \frac{|\hat{v}\,(n-m)|^2}{(|n| - |m| + k_F^{-1})} \, , \tag{A.185}
$$

by fixing m with $|m| \le k_F$ and using $|n| \ge k_F$ with $n \in [k_F, k_F + D]$. Then

$$
\begin{aligned}
&\int\limits_{|n| \ge k_F} \mathrm{d}^d n \; \frac{\chi_{[0,D]}\,(|n-m|)}{(|n| - |m| + k_F^{-1})} \qquad \text{(A.186)}\\
\le &\int\limits_{k_F}^{k_F + D} \mathrm{d}n \; n^{d-1} \left(2 \arcsin \left(\frac{D}{k_F - D} \right) \right)^{d-1} \frac{1}{\left(l - |m| + k_F^{-1} \right)} \\
\le &\; C \cdot D^{d-1} \int\limits_{k_F}^{k_F + D} \mathrm{d}n \; \frac{1}{\left(l - |m| + k_F^{-1} \right)} \, ,
\end{aligned}
$$

as before. Using $n \mapsto r - k_F$, we have

$$
\begin{aligned}
&C \cdot D^{d-1} \int\limits_{k_F}^{k_F + D} \mathrm{d}n \; \frac{1}{\left(n - |m| + k_F^{-1} \right)} \qquad \text{(A.187)}\\
&= C \cdot D^{d-1} \int\limits_{0}^{D} \mathrm{d}r \; \frac{1}{\left(r - k_F - |m| + k_F^{-1} \right)} \, .
\end{aligned}
$$

Plugging this into the integral over m and using again $|n| \ge k_F$, $|n-m| \le D$, we find

$$
\int\limits_{|m| \le k_F} \mathrm{d}^d m \int\limits_{|n| \ge k_F} \mathrm{d}^d n \frac{\chi_{[0,D]}\,(|n-m|)}{(|n| - |m| + k_F^{-1})} \tag{A.188}
$$

$$\lesssim C \cdot D^{d-1} \int\limits_{|m|\leq k_F} \mathrm{d}^d m \int\limits_0^D \mathrm{d}r \; \frac{1}{\left(r - k_F - |m| + k_F^{-1}\right)} \, \chi_{[k_F - D, k_F]}\left(|m|\right)$$

$$\lesssim C \cdot D^{d-1} \int\limits_\Omega \mathrm{d}\Omega \int\limits_{k_F - D}^{k_F} \mathrm{d}m \; m^{d-1} \int\limits_0^D \mathrm{d}r \; \frac{1}{\left(r - k_F - m + k_F^{-1}\right)}$$

$$\lesssim C \cdot \left(k_F D\right)^{d-1} \int\limits_{k_F - D}^{k_F} \mathrm{d}m \int\limits_0^D \mathrm{d}r \; \frac{1}{\left(r - k_F - m + k_F^{-1}\right)} \, .$$

Using $m \mapsto s - k_F$, we have

$$\begin{aligned} & C \cdot \left(k_F D\right)^{d-1} \int\limits_{k_F - D}^{k_F} \mathrm{d}m \int\limits_0^D \mathrm{d}r \; \frac{1}{\left(r - k_F - m + k_F^{-1}\right)} \\ = \; & C \cdot \left(k_F D\right)^{d-1} \int\limits_{-D}^{0} \mathrm{d}s \int\limits_0^D \mathrm{d}r \; \frac{1}{\left(r - s + k_F^{-1}\right)} \, . \end{aligned} \tag{A.189}$$

Therefore,

$$\begin{aligned} & \int\limits_{|m|\leq k_F} \mathrm{d}^d m \int\limits_{|n|\geq k_F} \mathrm{d}^d n \; \frac{\chi_{[0,D]}\left(|n-m|\right)}{\left(|n| - |m| + k_F^{-1}\right)} \\ \lesssim \; & \left(k_F D\right)^{(d-1)} \int\limits_{-D}^{0} \mathrm{d}s \int\limits_0^D \mathrm{d}r \; \frac{1}{\left(r - s + k_F^{-1}\right)} \, . \end{aligned} \tag{A.190}$$

Computing the integral, we find

$$\begin{aligned} \int\limits_{-D}^{0} \mathrm{d}s \int\limits_0^D \mathrm{d}r \; \frac{1}{\left(r - s + k_F^{-1}\right)} &= \int\limits_{-B}^{0} \mathrm{d}s \; \left[\ln\left(r - s + k_F^{-1}\right)\right]_0^D \\ &= \int\limits_{-D}^{0} \mathrm{d}s \; \ln\left(D - s + k_F^{-1}\right) - \ln\left(-s + k_F^{-1}\right) \\ &= \int\limits_{-D}^{0} \mathrm{d}s \; \ln\left(D - s + k_F^{-1}\right) - \int\limits_{-D}^{0} \mathrm{d}s \; \ln\left(-s + k_F^{-1}\right) \, , \end{aligned} \tag{A.191}$$

where, using $D - s + k_F^{-1} \mapsto x$,

$$\int\limits_{-D}^{0} \mathrm{d}s \;\ln\left(D - s + k_F^{-1}\right) = \int\limits_{2D+k_F^{-1}}^{D+k_F^{-1}} \mathrm{d}x \;\ln(x) = \left[x\ln(x) - x\right]_{2D+k_F^{-1}}^{D+k_F^{-1}} \tag{A.192}$$

$$\begin{aligned}
&= \left(D + k_F^{-1}\right)\ln\left(D + k_F^{-1}\right) - \left(D + k_F^{-1}\right) \\
&\quad - \left(\left(2D + k_F^{-1}\right)\ln\left(2D + k_F^{-1}\right) - \left(2D + k_F^{-1}\right)\right) \\
&= \left(D + k_F^{-1}\right)\ln\left(D + k_F^{-1}\right) \\
&\quad - \left(2D + k_F^{-1}\right)\ln\left(2D + k_F^{-1}\right) + D\,,
\end{aligned}$$

and, using $-s + k_F^{-1} \mapsto y$,

$$\int\limits_{-D}^{0} \mathrm{d}s \;\ln\left(-s + k_F^{-1}\right) = \int\limits_{D+k_F^{-1}}^{k_F^{-1}} \mathrm{d}y \;\ln(y) = \left[y\ln(y) - y\right]_{D+k_F^{-1}}^{k_F^{-1}} \tag{A.193}$$

$$\begin{aligned}
&= k_F^{-1}\ln\left(k_F^{-1}\right) - k_F^{-1} - \left(\left(D + k_F^{-1}\right)\ln\left(D + k_F^{-1}\right)\right. \\
&\quad \left. - \left(D + k_F^{-1}\right)\right) \\
&\quad - \left(D + k_F^{-1}\right)\ln\left(D + k_F^{-1}\right) - k_F^{-1}\ln\left(k_F^{-1}\right) + D\,.
\end{aligned}$$

Therefore

$$\int\limits_{-D}^{0} \mathrm{d}s \int\limits_{0}^{D} \mathrm{d}r \;\frac{1}{\left(r - s + k_F^{-1}\right)} \tag{A.194}$$

$$= 2(D + k_F^{-1})\ln(D + k_F^{-1}) - (2D + k_F^{-1})\ln(2D + k_F^{-1}) + k_F^{-1}\ln(k_F^{-1})\,,$$

which only is a very small contribution for large k_F and can be bounded by a constant, depending on D. □

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