On Tensor Multi-Scalar Theories in a Post-Newtonian Setting

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Oliver Schön
aus Albstadt-Ebingen

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1. Berichterstatterin:	Dr. Daniela D. Doneva
2. Berichterstatter:	Prof. Dr. Kostas D. Kokkotas

To Karla Elisabeth Schön.

Abstract

On Tensor Multi-Scalar Theories in a Post-Newtonian Setting

Oliver Schön

The first experimental detection of gravitational waves in 2015 [1] established a fundamentally new way of testing General Relativity 100 years after its inception in 1915 by Albert Einstein [2, 3]. This novel approach allows us to test General Relativity against alternative theories in the strong field regime. During the past century, numerous tests of Relativity have been conducted in the gravity regime of our solar system. Einstein's theory amazingly passed these tests with very high accuracy. Hence, testing General Relativity in regimes much more bound to gravity than our weak field solar system is essential [4, 5].

We establish the mathematical foundation of General Relativity and viable alternatives in the second chapter of this thesis after a brief introduction in Chapter 1. We explicitly detail various ways to append Relativity to more complex theories viably. The fact that the theory of Albert Einstein is already excellent in multiple tests, some of them explained in Chapter 2, of course, puts quite strong constraints on new theories, and most will have a specific limit towards General Relativity.

The protagonist theory discussed in this dissertation, known as Tensor Multi-Scalar Theory of Gravity, is a viable alternative that allows for differences to General Relativity in the strong field regime. The theory adds multiple scalar fields to Relativity already on the level of the action in a covariant way. Hence, we keep several geometrical properties of Relativity that we can utilize to our advantage later on.

We detail the mathematical machinery of *Direct Integration of the Relaxed Field Equations* [6–14] in Chapter 3. This toolkit, designed to ultimately end up with a family of gravitational wave templates via post-Newtonian analysis, is very well suited to our use case since it is theory agnostic in the sense that every theory motivated by an action can be analyzed using this specific framework. Approaches adding a single scalar field to the Einstein-Hilbert action of General Relativity have already utilized this mathematical setup to significant effect [15–18], and we aim to generalize this even further in the following chapters.

The bulk of this dissertation is the calculations in Chapter 4. We adapt the previously mentioned formalism to the generalized theory of Tensor Multi-Scalar Gravitation. Extra flat wave equations append the field equations for the multiple scalar fields, which need to be carefully evaluated at every step in the goal of calculating the post-Newtonian metric to some order accurate enough for the analysis in this dissertation. We find out that the geometry target space, a manifold equipped with a Riemannian metric, plays a crucial role in the near zone dynamics of compact objects as we calculate the equation of motion of a broad class Tensor Multi-Scalar Theories through 2.5 post-Newtonian order. This effect differs from pure General Relativity as no extra scalar fields are added, and hence no target space exists. As expected from the analysis of single Scalar Field Theories [15–18], we also find a 1.5 post-Newtonian contribution to the motion absent in classical Relativity.

We end the dissertation with an Outlook to future work and explain how this work fits in with the brighter goal of having an extensive library of gravitational wave templates to compare to experimental data to test General Relativity right down to its core indeed.

Zusammenfassung

On Tensor-Multi-Scalar Theories in a Post-Newtonian Setting

Oliver Schön

Mit dem ersten experimentellen Nachweis von Gravitationswellen im Jahr 2015 [1] wurde eine grundlegend neue Möglichkeit geschaffen, die Allgemeine Relativitätstheorie 100 Jahre nach ihrer Einführung im Jahr 1915 durch Albert Einstein zu testen [2, 3]. Dieser neuartige Ansatz ermöglicht es uns, die Allgemeine Relativitätstheorie gegen alternative Theorien im starken Gravitaionsfeld zu testen. Im vergangenen Jahrhundert wurden zahlreiche Tests der Relativitätstheorie im Gravitationsbereich unseres Sonnensystems durchgeführt. Einsteins Theorie hat diese Tests erstaunlicherweise mit sehr hoher Genauigkeit bestanden. Daher ist das Testen der Allgemeinen Relativitätstheorie in Regimen, die viel stärker an die Schwerkraft gebunden sind als unser Sonnensystem mit schwachem Feld, von wesentlicher Bedeutung [4, 5].

Nach einer kurzen Einführung in Kapitel 1 legen wir im zweiten Kapitel dieser Arbeit die mathematischen Grundlagen der Allgemeinen Relativitätstheorie und praktikable Alternativen fest. Wir zeigen explizit verschiedene Möglichkeiten auf,

die Relativitätstheorie zu komplexeren Modellen verallgemeinern. Die Tatsache, dass die Theorie von Albert Einstein bereits in mehreren Tests hervorragend abgeschnitten hat, von denen einige in Kapitel 2 erläutert werden, setzt neuen Theorien natürlich ziemlich starke Grenzen, und die meisten werden einen spezifische Limes zur Allgemeinen Relativitätstheorie haben. Die Haupttheorie, die in dieser Dissertation diskutiert wird und als Tensor Multi-Scalar Theory der Gravitation bekannt ist, ist eine mögliche Alternative, die Unterschiede zur Allgemeinen Relativitätstheorie im Starkfeldbereich zulässt. Die Theorie fügt der Relativitätstheorie bereits auf der Ebene der Wirkung mehrere Skalarfelder auf kovariante Weise hinzu. Dadurch bleiben einige geometrische Eigenschaften der Relativitätstheorie erhalten, die wir später zu unserem Vorteil nutzen können.

Wir erläutern die mathematische Maschinerie der *Direct Integration of the Relaxed Field Equations* [6–14] in Kapitel 3. Dieses Toolkit, das darauf abzielt, mittels post-Newtonscher Analyse eine Familie von Gravitationswellen-Templates zu erhalten, eignet sich sehr gut für unseren Anwendungsfall, da es theorieunabhängig ist, d. h. jede Theorie, die durch eine Wirkung motiviert ist, kann mit diesem spezifischen Rahmen analysiert werden. Ansätze, die der Einstein-Hilbert Wirkung der Allgemeinen Relativitätstheorie ein einzelnes Skalarfeld hinzufügen, haben diesen mathematischen Aufbau bereits mit großem Erfolg genutzt [15–18], und wir versuchen, dies in den folgenden Kapiteln noch weiter zu verallgemeinern.

Den Hauptteil dieser Dissertation bilden die Berechnungen in Kapitel 4. Wir passen den zuvor erwähnten Formalismus an die verallgemeinerte Theorie der Tensor Multi-Scalar Gravitation an. Die Feldgleichungen für die Mehrfachskalarfelder werden durch zusätzliche flache Wellengleichungen ergänzt, die in jedem Schritt sorgfältig ausgewertet werden müssen, um die post-Newtonsche Metrik mit einer Ordnung zu berechnen, die für die Analyse in dieser Dissertation genügend ist.

Wir finden heraus, dass die Geometrie des Zielraums, eine Mannigfaltigkeit mit einer Riemannschen Metrik, eine entscheidende Rolle in der Nahbereichsdynamik von kompakten Objekten spielt, wenn wir die Bewegungsgleichung einer breiten Klasse von Tensor Multi-Scalar Theorien bis zur 2.5 post-Newtonschen Ordnung berechnen. Dieser Effekt unterscheidet sich von der reinen Allgemeinen Relativitätstheorie, da hier keine zusätzlichen Skalarfelder hinzugefügt werden und somit kein Zielraum existiert. Wie von der Analyse einzelner Skalarfeldtheorien [15–18] erwartet, finden wir auch einen 1.5 post-Newtonschen Beitrag zur Bewegung, der in der klassischen Relativitätstheorie fehlt.

Wir beenden die Dissertation mit einem Ausblick auf künftige Arbeiten und erklären, wie diese Arbeit zu dem ehrgeizigeren Ziel passt, eine umfangreiche Bibliothek von Gravitationswellenvorlagen zu haben, die mit experimentellen Daten verglichen werden können, um die Allgemeine Relativitätstheorie wirklich bis auf ihren Kern zu testen.

Acknowledgments

First and foremost, I would like to thank my supervisor Daniela D. Doneva for allowing me to pursue the projects that eventually accumulated into this thesis. I am grateful that you never gave up on me in admittedly tricky times and always encouraged me to keep moving forward. I thoroughly enjoyed all our chats and discussions about physics, maths, or anything. I am pretty privileged to have learned physics from you, and many of your lessons will undoubtedly help me in my future endeavors. Thank you.

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Dedication

To my wife, Karla, the love of my life. I want to express my deepest gratitude for standing by my side throughout this journey and supporting me unwaveringly. Without you, I would not have been able to complete this thesis. I acknowledge that I wasn't always the easiest person to be around, especially during certain phases, and that's putting it mildly. However, your resilience, joyfulness, and empathetic nature have been instrumental in our relationship, and I am confident that together we can overcome any challenge life presents us. Starting the rest of our lives with you fills me with indescribable happiness. And I for sure cannot wait.

To my incredible parents, Diana and Jürgen, thank you for nurturing my curiosity and instilling a passion for scientific exploration. Growing up in a non-academic household never hindered me, primarily because of both of you. From the beginning, it was evident that you valued science and knowledge and never made me feel like an outsider in our home. Your moral and financial support means the world to me, and I will always cherish it.

To my extended family, Jennifer and Christopher, Barbara and Joachim, Freddy and Jana, Hannelore, Linda and Werner. I am immensely grateful for your assistance throughout this project. Your willingness to help, especially in shouldering my family responsibilities during busy periods, is not something I take for granted. I consider myself incredibly fortunate to be a part of this family.

To my in-laws, including but not limited to Susanne and Christian, Bernhard, Olga, and Emilia. I cannot express enough gratitude for embracing me as a member of your family. From the beginning, I felt welcomed with open arms. Spending time with all of you, whether at larger gatherings or intimate settings, quickly became second nature to me. I am looking forward to many festivities for our ever-growing family.

To my dear friends and colleagues, far too many to mention by name, thank you for our countless unforgettable moments. It has been both a pleasure and an honor to work and spend time with each and every one of you. The memories we have created and the insights we have gained could fill multiple lifetimes with joy and excitement. My only hope is that you feel the same way. Thank you from the bottom of my heart for lifting me during challenging times and continuously motivating me when the road seemed rough. Here is to our future.

To the community of Mathematical Physics, especially those dedicated to Mathematical Relativity at both the University of Tübingen and the University of Otago, I want to express my deep and heartfelt appreciation. You have introduced me to the fascinating world of General Relativity, and I consider myself fortunate to have learned about the theory's mathematical and physical beauty from all of you. I am immensely grateful for the valuable guidance and encouragement from my previous mentors, Carla Cederbaum and Florian Beyer. Thanks to your support, I was able to embark on captivating projects for both my Bachelor's and Master's theses, which have been instrumental in shaping my academic journey.

Last but certainly not least, I want to express my gratitude to the bicycle community of Tübingen. Thank you for advancing the cycling infrastructure and ensuring my safety during my commutes to and from work. Especially during the pandemic, my fascination with bicycles grew immensely, and the supportive community in this town continues to nurture and foster that passion to this day.

Declaration

I hereby certify that this thesis has been composed by me and is based on my own work unless stated otherwise. No other person's work has been used without due acknowledgment in this thesis. All references and verbatim extracts have been cited, and all sources of information, including graphs and data sets, have been specifically acknowledged.

Contribution Statement

This dissertation is largely based on the publication *„Tensor-multiscalar gravity: Equations of motion to 2.5 post-Newtonian order“*, published in 2022 in Phys. Rev. D, 105.064034[1] [19] authored by myself, Oliver Schön, and my supervisor Daniela D. Doneva. These parts include passages in Chapters 1, 3, 4, and 5. Each of these chapters starts with an additional small disclaimer.

Daniela D. Doneva proposed the research topic to me, and we jointly discussed how to engage with the project scientifically. I reported progress regularly, and discussions during those meetings helped to overcome problems and blockades. I did the technical adaption of the mathematical framework DIRE to TMSTs. All detailed calculations of this novel approach as well as 90% of literature research were also performed by me. Daniela D. Doneva wrote 10% of the final published manuscript, whereas I have written 90% of the paper. No data was generated, collected, analyzed, or interpreted in the published manuscript. All figures were created by myself.

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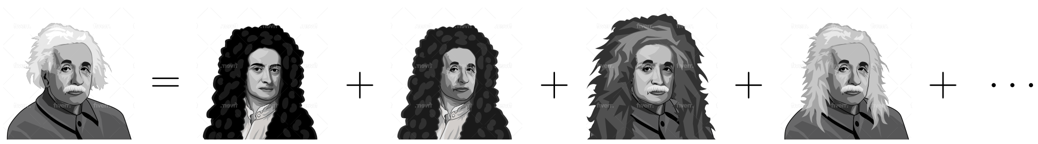
List of Acronyms

ADM	Arnowitt-Deser-Misner
BBH	Binary Black Hole
BH	Black Hole
BNS	Binary Neutron Star
DEF	Damour and Esposito-Farèse
DIRE	Direct Integration of the Relaxed Field Equations
EF	Einstein Frame
EFE	Einstein Field Equations
EFT	Effective Field Theory
EH	Einstein-Hilbert
EOM	Equation of Motion
ET	Einstein Telescope
GB	Gauss-Bonnet
GR	General Relativity
GW	Gravitational Wave
JF	Jordan Frame
LIGO	Laser Interferometer Gravitational Wave Observatory
MPM-PN	Multipolar-Post-Minkowskian Post-Newtonian

MW	Mirshekari and Will
NS	Neutron Star
NZ	Near-Zone
PDE	Partial Differential Equation
PN	Post-Newtonian
STT	Scalar Tensor Theory
TMST	Tensor Multi-Scalar Theory
TT	Transverse-Traceless
WZ	Wave-Zone



Post-Newtonian analysis in a nutshell.

Chapter 1

Introduction

Parts of our work in this chapter are based on the publication „Tensor-multiscalar gravity: Equations of motion to 2.5 post-Newtonian order", in Phys. Rev. D, 105.064034[1] *[19] by O. Schön, and D. D. Doneva. Please refer to our Contribution Statement at the beginning of this dissertation for more information.*

Testing General Relativity (GR) in the gravitational wave (GW) astrophysics era has advanced a lot. And GR is passing these tests remarkably well, especially in the weak-field regime. There are still viable alternatives, though, mainly generalizations of GR left to explore. One quite natural extension of GR is the introduction of a scalar field in addition to the metric tensor being an additional mediator of the gravitational interaction. We call those well-known generalizations of GR Scalar Tensor Theories (STT) [20, 21]. Due to its simplicity, the single scalar field case was much more widely considered in the literature. However, there is no particular reason for appending GR by only one scalar field. Even more, there are several motivations behind the idea that GR should be supplemented with multiple scalar fields related, e.g., to higher dimensional gravity, string theories, etc. (see,

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e.g., [22–25]). In addition, it is well known that different classes of alternative theories of gravity are mathematically equivalent to specific sectors of scalar-tensor theories that often offer the possibility of a more accessible and uniform treatment of these theories. Allowing for the existence of multiple scalar fields extends the possibilities for such analogies [26]. Those theories are mathematically well-defined and can pass all known experimental and observational tests, making the less explored class of Tensor Multi-Scalar Theories of gravitation (TMST) a viable and exciting class of modified gravity. Ones of the first seminal works on the topic of scalar-tensor theories by Damour and Esposito-Farése considered the possibility for multiple scalar fields [21, 27], and recently, the $3+1$ formulation of the theory was developed in [28].

What is very interesting about TMST from a theory point of view is that it is not just a mechanical addition of more scalar fields. Instead, these theories offer the possibility of entirely new phenomena and solutions unseen in any other modified gravity theory until now. The richness of the solution's spectrum is controlled by the choice of target space for the scalar fields equipped with a given metric and the choice of the map $\varphi\colon spacetime \to target\ space$. Compact objects in TMST, including black holes, neutron stars, and solitons, were studied in several papers [28–39]. It was demonstrated that if one chooses the target space metric and the map to the target space in a nontrivial way, new types of solutions can exist, such as topological neutron stars [34]. Scalarization, that is, a nonlinear scalar field development for sectors of the theory where the weak field regime coincides with GR [40], was also considered in the context of TSMT [28, 36]. Surprisingly it is possible to have scalarization with massless scalar fields leading to compact objects with zero scalar charges. This is contrary to all known scalarized solutions, both neutron stars and black holes [40–42], and suggests possible essential deviations from the standard treatment of scalarization. Moreover, contrary to standard STT, black holes with scalar hair can exist in TMST [39]. All this calls for further development of TMST to study possible astrophysical implications.

Among the most promising test beds for alternative theories of gravity are binary mergers, especially their inspiral phase that can produce a strong signal-to-noise ratio. These phenomena are most prominently studied via a post-Newtonian (PN) approach [43, 44] that attracted considerable attention during the past decades due to its ability to produce fast and accurate enough waveforms. The recent results in GR include an analysis of 4 PN order [45–55], including analysis of memory-tails and quasi-circular orbits, and 4.5 PN order [56–58]. The energy-flux was also studied in [47, 59–62] to 3.5 PN and 4.5 PN order. Gravitational waveforms were studied to 3.5 PN order [63–65]. Later the 5 PN order [66–70] and the 6 PN order [71, 72] were considered as well.

One common school of thought is to calculate the PN expansion via direct integration of the relaxed field equations (DIRE), pioneered by Epstein and Wagoner [6] and expanded by Thorne, Wiseman, Will, Pati, Wang, and Mitchell [8–14, 73]. This paper series establishes an equation of motion and radiation-reaction for binaries to 3.5 PN order in GR. Their method weakens the standard field equations into a relaxed form as studied by Landau and Lifshitz [74]. This framework is understood as post-Minkowskian theory. The formalism allows one to rewrite the exact field equations as a set of ten flat, i.e., Minkowskian, wave equations together with imposing harmonic gauge conditions. Of course, the source terms are not trivial as they are highly nonlinear and convoluted. Via an iteration process [9], one can expand the metric systematically and then integrate the wave equations using concepts such as retarded Green's functions.

This concept merges directly in the post-Newtonian formalism by incorporating both weak-field and slow-motion conditions $Gm/rc^2 \sim v^2/c^2 \ll 1$, where the characteristic mass, size, and velocity of the source are denoted by m, r, and v. Using geometric coordinate units, i.e. $G = c = 1$, these conditions enable us to use the expansion parameter $\varepsilon \sim m/r \sim v^2$ to expand the metric fields $h^{\alpha\beta} := \eta^{\alpha\beta} - \sqrt{-g}g^{\alpha\beta}$ for the Minkowski metric $\eta^{\alpha\beta}$ and spacetime metric $g^{\alpha\beta}$, $g = \det(g^{\alpha\beta})$. To integrate the wave equations for the $h^{\alpha\beta}$ fields, we decompose the past light-cone in a near-zone domain $\mathcal{N}$ and a wave-zone domain $\mathcal{W}$, such that $h^{\alpha\beta} = h^{\alpha\beta}_{\mathcal{N}} + h^{\alpha\beta}_{\mathcal{W}}$. The integration concepts change slightly depending on whether the field point is in the near- or the wave-zone. As a method, DIRE is theory agnostic and can be adapted to any theory of gravity provided a set of field equations. The present work is dedicated to calculating the near-zone expansion in TMST. This is canonically the first step in the process of a full post-Newtonian analysis accumulating in gravitational waveforms. Our final result will be an equation of motion accurate to 2.5 PN order. Later, calculations performed here can be used and furthered to obtain the tensorial gravitational waveform and, finally, the scalar flux. For STTs, this has been performed in that order in the paper series [15, 16, 18].

It should be mentioned that DIRE is not the only mathematical framework to study gravity in the context of post-Newtonian analysis and gravitational waves [5]. Similar methods of iterating the Einstein Field Equations utilizing harmonic coordinates have been used by Blanchet, Faye, Ponsot, de Andrade, Damour, and Esposito-Farese [75–78]. The equation of motion has also been calculated by the Hamiltonian Arnowitt-Deser-Misner (ADM) formalism [79–81]. Einstein himself briefly worked on the problem as well and pioneered a framework known as Einstein-Infeld-Hoffmann surface integral approach [82–85]. More recently, methods borrowed from Effective Field Theory (EFT) could also derive a 3 PN order equation of motion in GR [86, 87]. Although those frameworks work very differently and utilize distinct mathematical machinery, it has been shown that the resulting equation of motion is in perfect agreement between those approaches [5].

As mentioned before, STTs belong to the most studied generalizations of GR. Naturally, they have also been studied in the context of PN approximations. DIRE was already adapted to calculate a PN expansion for a broad class of single STT [15–18], including an equation of motion to 2.5 PN order by Mirshekari and Will [15] as well as an analysis of tensor gravitational waves to second PN order and a scalar waveform accurate to 1.5 PN order by Lang [16–18]. In addition, the metric sufficient to study light deflection at 2 PN order was examined in [88, 89]. At the same time, the generic structure of the 2 PN Lagrangian for TMST and N compact bodies was derived in [27]. Since the standard formalism does not work for STT admitting scalarization, which can be viewed as a second-order phase transition, generalizations of the PN expansions were developed in [90–93], modeling dynamical scalarization with a resumed PN expansion and obtaining gravitational waveforms in a class of STT to 2 PN relative order. More recently, Bernard studied in a series [94–96] the equations of motion in STT to 3 PN order, the resulting conserved quantities, and the dipolar tidal effects via the Multipolar-Post-Minkowskian Post-Newtonian formalism (MPM-PN) [43, 97–100]. Recently, waveforms accurate to 1.5 PN order beyond GR's standard quadrupole moment were generated [101]. The PN expansion and the corresponding gravitational waveforms up to different orders were also studied in other alternative theories of gravity, e.g., massive STT, Gauss-Bonnet gravity, and Chern-Simons theories [102–109].

The thesis is organized as follows: We introduce the core concepts of GR and the road map to its alternatives in Chapter 2. In there, we especially motivate TMSTs and how they fit into the modern context of GW astrophysics. This relies on a uniqueness result of the Einstien-Hilbert action allowing for an excellent classification of modified gravity. In addition to that, we briefly explain testing GR and give a short historical context. Chapter 3 presents the main framework called DIRE in detail. We will discuss its approach to generating waveforms and its advantages for our use case here. Most of our work is presented in Chapter 4. In Section 4.1, we adapt DIRE to multiple scalar fields needed in our analysis and explain the key differences to previous work. We continue in Section 4.2 with the formal structure of the near-zone fields and their underlying building blocks. In there, we introduce all the relevant potentials utilized in the equation of motion. Next, in Section 4.3, we iterate through the process of DIRE until each field is of the desired order to reach an accuracy of 2.5 PN order in the final ready-to-use equation of motion. We again highlight the main differences with the single scalar field theories. At the end of the lengthy 1 PN and 2 PN calculation, we present some mathematical techniques and in-between results to make it easier to follow along as we progress. We also explain why some potentials arising in TMST fundamentally differ from GR and STTs and how we handle these terms in our analysis. Finally, in Section 4.4, we explain our skeletonized matter model and expand all relevant associated fields. We perform a transformation of densities suited for our point particle model. This is followed by a brief derivation of the equation of motion in TMST, and the section ends by giving the full expansion of said equation. Our results are then analyzed in the following discussion, Chapter 5. We take a deeper look at the equation of motion and highlight the critical structures regarding the nontrivial target space and compact binaries. We end, as is usual, in a discussion and outlook part in Chapter 6. We emphasize the importance of continuing the analysis and discuss our expectations for future results. For that, we briefly introduce the Epstein-Wagoner moments and their significance for gravitational waveforms. The Appendix A already details some generalizations of wave-zone DIRE in TMST. More precisely, we perform needed calculations and hint at what challenges will arise in the future.

Chapter 2

General Relativity and its Alternatives

Since its inception in the early 19th century, General Relativity has been studied by many experimental and theoretical physicists and mathematicians. As with any physical theory, motivated scientists have tested it numerous times over the last 100 years. Although General Relativity stood the test of time remarkably well, some of those tests and analyses gave rise to a few prominent alternative theories of gravity. Most of those theories are, in fact, generalizations of the standard formulation of Relativity. Pure GR will be recovered as a particular case of a broader family of theories. In this chapter, we will explain the specific case of GR in the pool of viable alternatives and how to classify possible deviations.

2.1 The Theory of General Relativity

This section briefly introduces common textbook knowledge necessary for our work. If not noted otherwise, all relevant information might be gathered by a textbook of your choosing such as [110–113].

2.1.1 Mathematical Setup and Action

General Relativity is formulated in the language of Differential Geometry and Geometric Analysis. These two areas of mathematics are powerful concepts, and GR benefits greatly by using hundreds of years worth of mathematical research as a toolkit to do exciting physics. When we talk about a *spacetime*, we usually refer to a semi-Riemannian[1] manifold (M, g) consisting of a four-dimensional differentiable manifold M equipped with second rank, covariant, symmetric, nondegenerate, tensor field g called *spacetime metric*. The topology of the manifold M can differ greatly depending on the features one wants to study in depth. In our case, however, we generally assume a smooth manifold that usually can be covered by a single chart $(x^0, \ldots, x^3)$. Typically, the time coordinate is renamed as $t := x^0$. Since Differential Geometry is used as a primary tool in GR, we are also prone to all the conventions one can choose in its concepts. The signature of the spacetime metric that we set to $(- + ++)$ is relevant. A metric of that signature is also referred to as *Lorentz*[2] *metric* and the tuple (M, g) as *Lorentzian manifold*. We will frequently use the Greek indices $\{\alpha, \beta, \gamma, \mu, \nu, \ldots\}$ to denote the spacetime metric components $g_{\alpha\beta}$, meaning $\alpha, \beta = 0, \ldots, 3$. To mark the purely spatial components of said metric, we utilize the Latin letters $\{i, j, k, l, \ldots\}$ taking values in $1, \ldots, 3$. We start at the letter i for the spatial indices as the letters $\{a, b, c, d, \ldots\}$ are reserved for a different purpose further in this work to reduce the confusion that could arise by handling a larger class of gravitational theories. Throughout our

[1] Georg Friedrich Bernhard Riemann, September 17, 1826 – July 20, 1866, German mathematician.

[2] Hendrik Antoon Lorentz, July 18, 1853 – February 4, 1928, Dutch physicist.

calculation, we adopt the Einstein summation convention and generally follow the notation explained in, e.g., Wald [110]. All the standard formulation of important curvature quantities we use in this work is also found there.

A good way of introducing a field theory like GR is by motivating it with an action, in this case, the *Einstein-Hilbert*[3] *action* [114]

$$S = \frac{c^4}{16\pi G} \int_M R\sqrt{-g}\,\mathrm{d}^4 x\,, \tag{2.1}$$

where c denotes the speed of light in vacuum, G is the gravitational constant, and $g = \det(g_{\alpha\beta})$ is the determinant of the spacetime matrix representation appearing as part of the manifold volume element. Integrated over the whole spacetime (assuming convergences) is the *Ricci*[4] *scalar curvature* (also known as *Ricci scalar* or *scalar curvature*) $R \equiv R(g)$ defined via the contraction

$$R := g^{\alpha\beta} R_{\alpha\beta}\,, \tag{2.2}$$

for the contra variant metric components $g^{\alpha\beta}$ and the *Ricci curvature* $R_{\alpha\beta}$. The latter, in turn, is calculated as a contraction of the *Riemann curvature tensor*

$$R_{\alpha\beta} := R_{\alpha\beta\gamma}{}^{\delta}\,. \tag{2.3}$$

The Riemann tensor marks the highest entity in terms of curvature information for our purpose and generally in GR, as the Ricci curvature and scalar can be derived via direct contractions. It is defined purely from the metric tensor and the first and second derivatives. A convenient way to formulate the tensor is via

[3] David Hilbert, January 23, 1862 – February 14, 1943, German mathematician.
[4] Gregorio Ricci-Curbastro, January 12, 1853 – August 6, 1925, Italian mathematician

the *Christoffel*[5] *symbols*

$$\Gamma^{\gamma}_{\alpha\beta} := \frac{1}{2} g^{\gamma\mu} \left(g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu} \right) . \tag{2.4}$$

Here we introduced yet another convention in that a comma followed by a coordinate index refers to a derivative in that coordinate direction, hence $g_{\mu\alpha,\beta} = \partial/\partial x^{\beta}\, g_{\mu\alpha} = \partial_{\beta}\, g_{\mu\alpha}$. The full Riemann tensor ${R_{\alpha\beta\gamma}}^{\delta} = {R_{\alpha\beta\gamma}}^{\delta}(g)$ then takes the form

$${R_{\alpha\beta\gamma}}^{\delta} = \Gamma^{\delta}_{\alpha\gamma,\beta} - \Gamma^{\delta}_{\beta\gamma,\alpha} + \left(\Gamma^{\mu}_{\alpha\gamma} \Gamma^{\delta}_{\beta\mu} - \Gamma^{\mu}_{\beta\gamma} \Gamma^{\delta}_{\alpha\mu} \right) . \tag{2.5}$$

The above-defined Christoffel symbols essentially gauge the curvature the spacetime metric introduces on the manifold. They measure the deviation from flat space as all described curvature quantities in this section can be expressed in terms of these symbols. If all Christoffel symbols vanish, the Riemann tensor and, hence, the Ricci tensor and scalar also disappear, and the manifold is said to be flat or *Minkowskian*[6]. The other way around is, in general, not accurate. There are Ricci flat spacetimes with nonvanishing Riemann tensor, e.g., the Schwarzschild[7] or Kerr[8] spacetime. Note also that despite having indices as the other tensor quantities in this chapter, Christoffel symbols are no tensors as they transform quite differently under a change of coordinates. We will have multiple objects in this work with indices despite being no tensors, so it is worth paying attention to the nature of the quantities we manipulate.

This section's quantities are written in local coordinates $(x^0, \ldots, x^3)$. It is, however, possible to define all curvature terms in a more geometric mindset

[5]Elwin Bruno Christoffel, 10 November 1829 – 15 March 1900, German mathematician.
[6]Hermann Minkowski, June 22, 1864 – January 12, 1909, German mathematician.
[7]Karl Schwarzschild, October 9, 1873 – May 11, 1916, German physicist.
[8]Roy Patrick Kerr, born May 16, 1934, New Zealand mathematician.

and not rely on coordinates at all. In that way, one has certain gauge freedom established as a feature of the theory itself, which opens the door for much exciting mathematics and physics. However, the following work is better formulated in coordinates, hence our choice of definitions here.

2.1.2 The Field Equations and their Uniqueness

To obtain the simplest form *Einstein Field Equations* (EFE), one has to vary the action in Eq. (2.1) with respect to the metric tensor $g_{\alpha\beta}$ to obtain

$$G_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = 0 \,. \tag{2.6}$$

The newly defined quantity $G_{\alpha\beta}$ is known as *Einstein tensor*. This is the simplest form due to the lack of matter terms and no cosmological constant introduced. Hence this is typically known as the *Einstein Vacuum Equations* and can be more easily written as $R_{\alpha\beta} = 0$ by tracing out both sides. A more complex action would look like

$$S = \frac{c^4}{16\pi G} \int_M (R - 2\Lambda) \sqrt{-g}\, \mathrm{d}^4 x + S_{\mathrm{matt}} \,, \tag{2.7}$$

where we have again the speed of light in vacuum c, the gravitational constant G and Ricci scalar R. The cosmological constant Λ and the collected matter action denoted by S_{matt} are also newly introduced here. Varying this action yields the complete Einstein Field Equations

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta} \,. \tag{2.8}$$

These field equations here generalize the vacuum equations above in two ways. First, the right-hand side is now given by an *energy momentum tensor* $T_{\alpha\beta}$. It appears as the result of varying the matter action S_{matt} and incorporates all physical quantities put into the system. This might be a perfect fluid, isolated particles, or electromagnetic quantities, among others. The energy-momentum tensor obeys a conservation law formulated via a *covariant derivative*, an adapted version of a standard derivative for manifolds equipped with metric tensors:

$$0 = \nabla_\beta T^{\alpha\beta} := T^{\alpha\beta}{}_{;\beta} := T^{\alpha\beta}{}_{,\beta} + \Gamma^{\alpha}_{\mu\beta} T^{\mu\beta} + \Gamma^{\beta}_{\mu\beta} T^{\alpha\mu} \,, \tag{2.9}$$

where we have introduced the notion of a semi-colon to refer to said covariant derivatives. The exact calculation, of course, changes with the rank of the tensor being differentiated. The other newly introduced concept is the cosmological constant Λ appearing as a scaling of the metric in Eq. (2.8). This parameter allows us to explain certain large-scale phenomena of the universe and introduces the concept of expansion to the theory. It has been the center point of many discussions over the past century. See [115] for a modern approach.

In theory, in Eq. (2.8), we have a system of ten (due to the symmetry) partial differential equations for the ten independent metric components $g_{\alpha\beta}$. In principle, one inserts all the physical components one is interested in inside the energy-momentum tensor $T_{\alpha\beta}$ on the left-hand side of Eq. (2.8) and then solves for the metric components on the right-hand side. This is how the famous Wheeler[9] quote „Spacetime tells matter how to move; matter tells spacetime how to curve“, comes to life. Generally, two problems arise while trying to solve the Einstein equations:

[9]John Archibald Wheeler, July 9, 1911 – April 13, 2008, American physicist.

1. Nonlinearity: In contrast to other physical theories such as Maxwell's[10] description of electromagnetism or the Schrödinger[11] Equation in Quantum Mechanics, the Eqs. (2.8) (and Eqs. (2.6) for that matter) are non linear in their solutions $g_{\alpha\beta}$. This means that, in general, two different solutions cannot be summed together to generate a new third solution.

2. No (apparent) PDE character: Naively inspected, the EFE in our form here does not fall in any clear category like hyperbolic, parabolic, or elliptic, meaning that no powerful mathematical theory can be utilized to guarantee existence or uniqueness of solutions (there is, however, other methods of creating such characteristics like the famous 3+1 decomposition [112, 116, 117]).

The majority of this thesis is to explore one relatively recent framework to solve the EFE iteratively for a particular use case which, to some extent, involves linearization of the problem to circumvent problem 1. explained above. This will be detailed thoroughly in Chapter 3.

The action and the resulting field equations discussed here are far from arbitrary choices, as the following uniqueness theorem elegantly demonstrates.

Theorem 1 *(Lovelock [4, 118, 119])*
In four spacetime dimensions, the only divergence-free symmetric second rank tensor constructed solely from the metric $g_{\alpha\beta}$ and its derivatives up to second differential order, and preserving diffeomorphism invariance, is the Einstein tensor (Eq. (2.6)) plus a cosmological term.

[10]James Clerk Maxwell, June 13, 1831 – November 5, 1879, Scottish mathematician.
[11]Erwin Rudolf Josef Alexander Schrödinger, August 12, 1887 – January 4, 1961, Austrian physicist.

This theorem builds up on work established by Vermeil[12] [120] and Cartan[13] [121] and reveals the mathematical gem that the EFE, as in Eq. (2.8), is everything one might hope for in a theory of tensorial nature. But one more useful conclusion one might draw from Lovelock's Theorem is that every modified or generalized theory of gravity must necessarily brake one of the assumptions made here. Hence, a road map in alternative theories can be categorized by exactly these assumptions. In Section 2.3, we will explain what braking specific assumption yields and discuss the viability of the emerging gravitational models.

2.2 Testing General Relativity

As with any physical theory born out of theoretical assumption, the testing of said theory against reproducible empirical data is paramount. The same is necessarily true for General Relativity. Einstein suggested three observational tests [3], now known as the classical tests of GR. These tests have been instrumental in shaping our understanding of gravity and include:

1. Mercury's perihelion precession: The orbit of Mercury was observed to deviate from the theoretical predictions of Newtonian[14] gravity. This was already found out in 1859 by Urbain Le Verrier [122, 123], and many attempts to solve this issue were discussed but ultimately failed. Hence, this phenomenon was a great test for any upcoming theory of gravity. Using perturbation methods, Einstein himself could show that GR predicted the measured orbit and correct perihelion advance of Mercury already in 1915 [124].

[12] Hermann Vermeil, 1889 – 1959, German mathematician.
[13] Élie Joseph Cartan, April 9, 1869 – May 6, 1951, French mathematician.
[14] Sir Isaac Newton, December 25, 1642 – March 20, 1726/27, English scientist.

2. Gravitational redshift: GR forecasts a loss of energy of electromagnetic waves or photons moving away from matter. This has been measured as early as 1925 [125] and, with much more precision, reproduced multiple times over the years [126].

3. Deflection of light: Newtonian gravity already predicts the bending of light around compact objects. GR, however, forecasts a deflection of twice the amount calculated in Newtonian Gravity [127–129]. During an eclipse in 1919, an experiment was able to confirm Einstein's prediction [130]. Many tests, later on, were also able to confirm that result as this first measurement was of poor accuracy [129, 131].

Over the years, many more tests emerged, and GR still stands the test of time remarkably well today. GR is well tested and the gold standard, especially in the *weak field regime*, that is, tests performed with relatively slow velocities and noncompact objects. However, the *strong field regime* is still not thoroughly explored, and alternatives may still be viable. Modern tests of this regime include studying double pulsars [132] and the measurement of *gravitational waves* (GWs) [5], which is the subject of the next section.

2.2.1 Gravitational Waves

Gravitational waves are a prediction of General Relativity, which states that gravitation itself is not instant and instead behaves wave-like in spacetime. When compact objects move or accelerate, they produce changes in the curvature of spacetime that propagate at the speed of light as gravitational waves. This contrasts the Newtonian gravity model and is a much-discussed feature of GR for the past century [133].

Measuring these gravitational waves is one of the biggest challenges in modern physics. The waves are weak and can only be detected by sensitive instruments. After much effort, in 2015, scientists succeeded for the first time in measuring gravitational waves directly. The experimental setup used laser interferometers such as Laser Interferometer Gravitational Wave Observatory (LIGO) and Virgo [1]. After that initial milestone, which has been honored with a Nobel Prize in 2017 to Rainer Weiss, Barry C. Barish, and Kip S. Thorne „for decisive contributions to the LIGO detector and the observation of gravitational waves“, more events have been observed [1, 134–141].

These instruments can measure minor changes in spacetime geometry caused by gravitational waves. To date, we can observe multiple binary mergers consisting of binary black holes (BBH), binary neutron stars (BNS), and mixed binaries. Our detectors are not sensitive enough to measure continuous gravitational waves emitted by nonmerging systems or single compact objects [142, 143]. Hence, we will focus on the inspiral, merging, and ring-down phases of the end life of a binary system.

Figure 2.1 depicts the general form of a gravitational wave we are interested in. All events measured to date follow this schematic and differ, e.g., in the length of observation before the merger occurred. We will go deeper into the mathematics of GWs in Chapter 3, but for now, it is sufficient to imagine them as illustrated here. Obtaining the theoretical waveform always involves solving the Einstein equations (2.8) in one way or another. In the previous section, we briefly explained why this task is difficult. Hence, different ideas to calculate the waveform for the multiple stages of the binaries lifespan emerged over time. We are most concerned about the early to late inspiral phase. Fully relativistic, nonlinear effects do not significantly impact the binary's behavior at first. This is

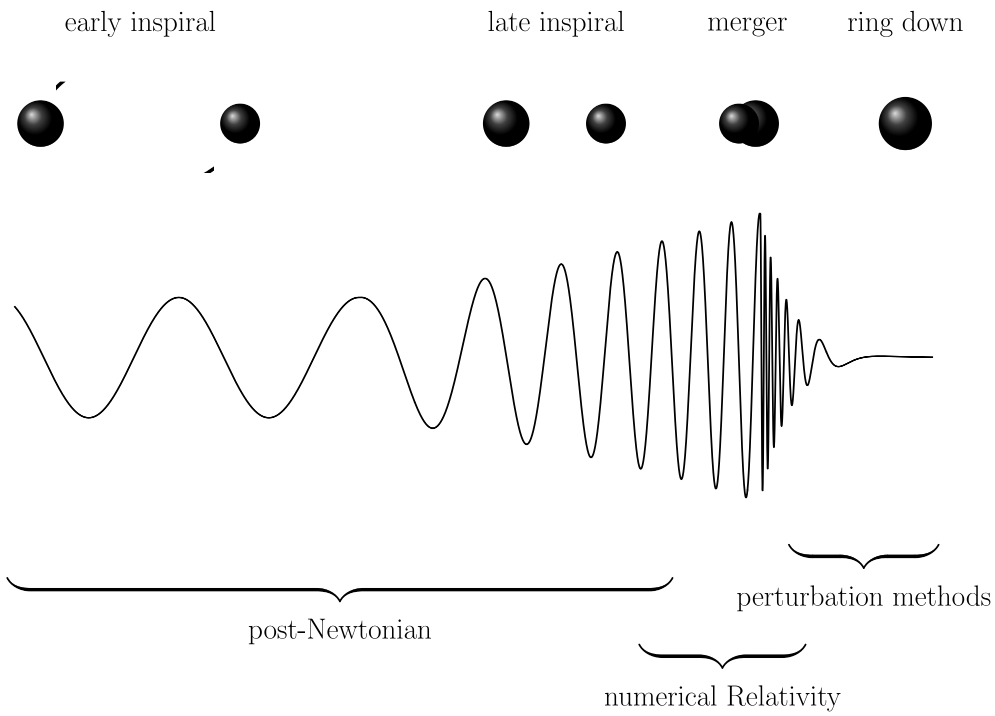

Figure 2.1: Qualitative description of a gravitational wave stemming from a binary merger. The two inspiral phases are sufficiently well modeled via post-Newtonian analysis. Then, there is a smooth transition to numerical methods for the merger since the nonlinear effects at play can no longer be captured by the PN expansion. In the end, neutron star or black hole perturbation methods model the ring down depending on which compact object emerged.

why a post-Newtonian expansion works very well here; the leading contribution is the Newtonian potential followed by relativistic corrections. Pure relativistic effects mainly govern the last inspiral and merger stages. Hence, a fully nonlinear evolution code from numerical Relativity is best suited to study this stage [5, 144]. Immediately after the merger, the ring down of the newly created compact objects occurs. The neutron star or black hole is still very excited, which is best analyzed via well-established perturbation methods [145, 146].

Let us comment further on why gravitational wave detection helps test GR further in the strong field regime. The previous section listed the classical weak field tests, which are already well-established. All viable alternatives need to be pretty close to GR in these potentials. This leaves deviations for the strong field regime, precisely the physics of very compact and heavy objects in a binary. It is not as straightforward as comparing measurements to theoretical GR predictions to see if it fits. Since the signal of binary mergers is still very weak and buried under noise, the way to detect incoming waves is via a method called *matched filtering*. One essentially filters out and identifies gravitational wave signals from ambient noise by having templates beforehand and checking continuously for correlation. If only GR templates are available, all you will ever discover is GR-related gravitational waves [147, 148]. Hence, a vast database is needed to support the matched filtering process in all directions concerned with fundamental physics. A few of those paths will be highlighted in the next section.

2.3 Road map towards Alternatives

Lovelock's Theorem (Thm. 1) explains the unique standing point of the Einstein field equations (2.8). As mentioned above, this allows us to classify the most viable alternatives by sorting them in what way they break the theory's assumptions. Notable exceptions are theories that may emerge without any action to motivate them or models that do not use a tensorial description of gravity at all. However, since those are the exception, we will not discuss them further and concentrate on the generalized version of Einstein's General Relativity.

We discuss the topic utilizing Figure 2.2. The main breaking of assumptions of Lovelock's Theorem (Thm. 1) can be collected as

1. diffeomorphism invariance violations,
2. higher than second-order derivatives,
3. higher dimensions,
4. nonvanishing divergence of the field equations.
5. more fields in addition to the metric tensor.

We will briefly discuss those violations and refer to literature for more details (e.g., [4, 5, 149]).

1. **Diffeomorphism invariance violations.** According to many, the gauge freedom and the geometric nature is one of the key features of GR. This makes the theory especially interesting for mathematicians and is the basis of much work in mathematical Relativity. The most common occurrence of breaking this invariance is in the particular form of Lorentz invariance via extra scalar fields that we discuss separately later. Other than that, these theories are hard to justify given modern measurements of Lorentz symmetry [150–152].

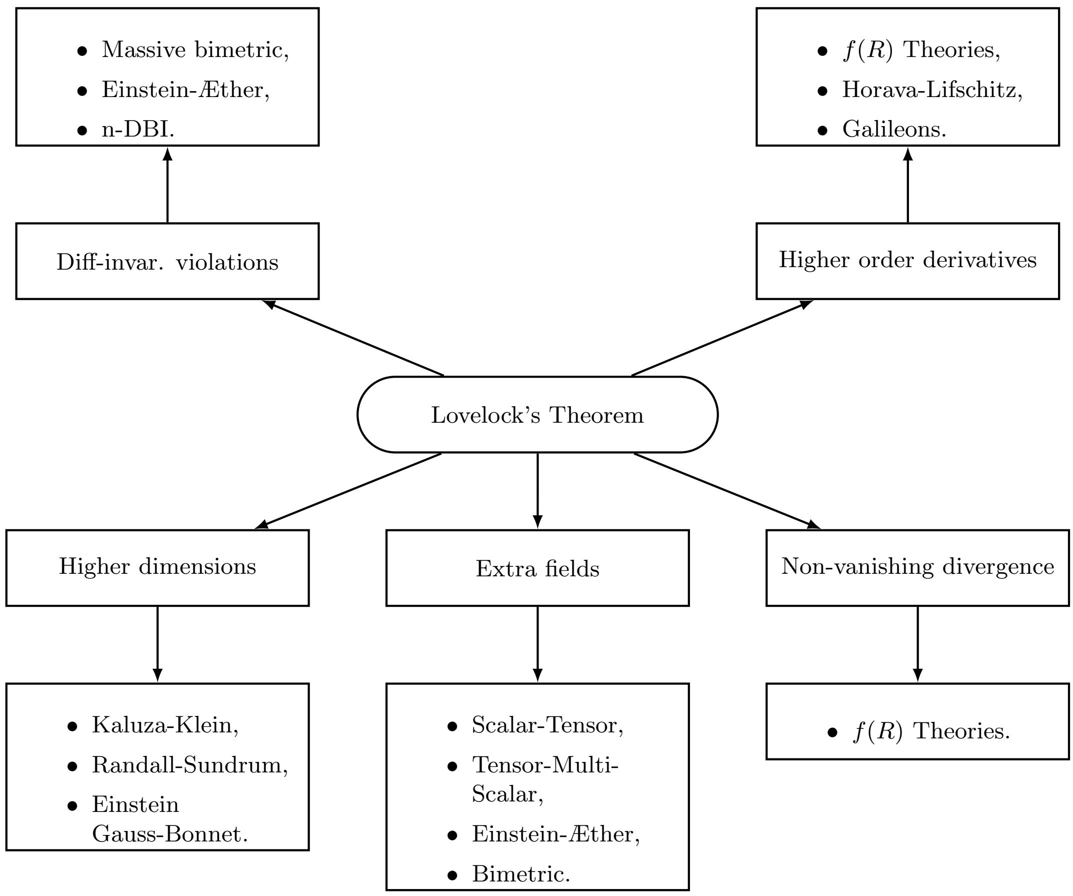

Figure 2.2: Road map towards alternative theories using the assumptions of Lovelock's Theorem (Thm. 1) [118]. This is not an extensive list and can be appended quite a bit [4, 149]. We focused ourselves here on the most common families of alternatives. However, One must realize that many of the here listed theories are vast in their rights and may differ substantially in their formalism. Hence, some theories are listed to break multiple assumptions. This only means that there exists one way to formulate a version of this theory in such a way that said specific assumption breaks. There might be other, different versions that still satisfy this assumption.

2. **Higher order derivatives.** Differential equations in classical or quantum physics are rarely given by any higher order than two. Higher-order derivatives can help regarding UV divergence and renormalization efforts in GR. Some of this is studied in forms of $f(R)$ theories [153–155] or, in terms of purely higher spatial derivatives, the more recent Horava-Lifschitz theories as studied in [150, 156–158].

3. **Higher dimensions.** No specific theory stands out in more than four spacetime dimensions since every model motivated by an action can, in principle, be generalized to higher (or lower) dimensions. It is widespread to analyze important theorems related to GR in more than $3 + 1$ dimensions in the mathematical Relativity community, e.g., the Positive Mass Theorem [159]. Beyond mathematical approaches, Quantum theories of gravity are commonly formulated in higher dimensions. Even multiple time-like dimensions have been studied [160]. Despite that, the Kazula-Klein (4+1 including electrodynamics) [161, 162], and Einstein Gauss-Bonnet families are common higher dimensional formulations of gravity [163].

4. **Nonvanishing divergence.** Imposing a nonvanishing divergence on the left-hand side of the field equations necessarily leads to a divergence of the energy-momentum tensor, and the conservation law (2.9) is no longer satisfied. This is linked to the weak equivalence principle being satisfied, an assumption generally very well established and tested [164]. Some versions of $f(R)$ theories study these phenomena in more detail [165, 166].

5. **Extra fields.** We regard this as the most common and natural extension of GR. There are a variety of methods extra fields can be appended to the

spacetime metric depending on choosing massless versus massive fields and the way they couple to the actual matter. Also, the extra fields can be just a single scalar, vector, or even higher rank tensor added to the metric. Suppose the extra fields are dynamic, which they are in most cases. In that case, the strong equivalence principle is violated, as the outcome of any experiment can change depending on the value the fields take around that region in spacetime. We are not too concerned with that and study a general class of Tensor Multi-Scalar Theories in the upcoming chapters.

At the end of this section, we present three tables (Tab. 2.1, Tab. 2.2, and Tab. 2.3) taken from the topical review headed by Berti et al. [4]. These tables give a fantastic insight into the work done in alternative theories up to 2015. Of course, the field has changed somewhat eight years later, mainly due to astonishing results in gravitational wave observations. Each question mark in the following tables deserves a close analysis and can be, as the authors describe it, a great Ph.D. project. Our work here aims to shine some light on the Tensor Multi-Scalar Theories row, as they are littered with question marks. This is part of the motivation for this work.

Theory	Field content	Strong EP	Massless graviton	Lorentz symmetry	Linear $T_{\mu\nu}$	Weak EP	Well-posed?	Weak-field constraints
Extra scalar field								
Scalar-tensor	S	✗	✓	✓	✓	✓	✓[167]	[103, 168, 169]
Multiscalar	S	✗	✓	✓	✓	✓	✓[170]	[21]
Metric $f(R)$	S	✗	✓	✓	✓	✓	✓[171, 172]	[173]
Quadratic gravity								
Gauss-Bonnet	S	✗	✓	✓	✓	✓	✓?	[174]
Chern-Simons	P	✗	✓	✓	✓	✓	✗✓? [175]	[176]
Generic	S/P	✗	✓	✓	✓	✓	?	
Horndeski	S	✗	✓	✓	✓	✓	✓?	
Lorentz-violating								
Æ-gravity	SV	✗	✓	✗	✓	✓	✓?	[177–180]
Khronometric/ Hořava-Lifshitz	S	✗	✓	✗	✓	✓	✓?	[179–182]
n-DBI	S	✗	✓	✗	✓	✓	?	none ([183])
Massive gravity								
dRGT/Bimetric	SVT	✗	✗	✓	✓	✓	?	[152]
Galileon	S	✗	✓	✓	✓	✓	✓?	[152, 184]
Nondynamical fields								
Palatini $f(R)$	–	✓	✓	✓	✗	✓	✓	none
Eddington-Born-Infeld	–	✓	✓	✓	✗	✓	?	none
Others, not covered here								
TeVeS	SVT	✗	✓	✓	✓	✓	?	[169]
$f(R)\mathcal{L}_m$	?	✗	✓	✓	✓	✗	?	
$f(T)$	?	✗	✓	✗	✓	✓	?	[185]

Table 2.1: Taken from [4], 2015. Catalog of several theories of gravity and their relation with the assumptions of Lovelock's theorem. Each theory violates at least one assumption (see also Figure 2.2), and can be seen as a proxy for testing a specific principle underlying GR. See text for details of the entries. Key to abbreviations: S: scalar; P: pseudoscalar; V: vector; T: tensor; ?: unknown; ✓?: not explored in detail or not rigorously proven, but there exist arguments to expect ✓. The occurrence of ✗✓? means that there exist arguments in favor of well-posedness within the EFT formulation and against well-posedness for the full theory. Weak-field constraints (as opposed to strong-field constraints) refer to Solar System and binary pulsar tests.

Theory	Solutions	Stability	Geodesics	Quadrupole
Extra scalar field				
Scalar-tensor	$\equiv$GR [186–191]	[192–198]	–	–
Multiscalar/Complex scalar	$\supset$GR [187, 199, 200]	?	?	[199, 200]
Metric $f(R)$	$\supset$GR [189, 190]	[201, 202]	?	?
Quadratic gravity				
Gauss-Bonnet	NR [203–205]; SR [206, 207]; FR [208]	[209, 210]	SR [206, 211, 212]; FR [208]	[207, 213]
Chern-Simons	SR [214–216]; FR [217]	NR [218–221]; SR [210]	[205, 222]	[216]
Generic	SR [211]	?	[211]	Qu. Moment
Horndeski	[223–225]	? [226, 227]	?	?
Lorentz-violating				
Æ-gravity	NR [228–230]	?	[229, 230]	?
Khronometric/				
Hořava-Lifshitz	NR, SR [229–232]	? [233]	[229, 230]	?
n-DBI	NR[234, 235]	?	?	?
Massive gravity				
dRGT/Bimetric	$\supset$GR, NR [236–239]	[240–243]	?	?
Galileon	[244]	?	?	?
Nondynamical fields				
Palatini $f(R)$	$\equiv$GR	–	–	–
Eddington-Born-Infeld	$\equiv$GR	–	–	–

Table 2.2: Taken from [4], 2015. Catalog of BH properties in several theories of gravity. The column "Solutions" refers to asymptotically-flat, regular solutions. Legend: ST="Scalar-Tensor," $\equiv$GR="Same solutions as in GR," $\supset$GR="GR solutions are also solutions of the theory," NR="Non rotating," SR="Slowly rotating," FR="Fast rotating/Generic rotation," ?=unknown or uncertain.

Theory	Structure			Collapse	Sensitivities	Stability	Geodesics
	NR	SR	FR				
Extra scalar field							
Scalar-Tensor	[40, 245–249]	[247, 250, 251]	[252–254]	[255–262]	[263]	[264–274]	[253, 275]
Multiscalar	?	?	?	?	?	?	?
Metric $f(R)$	[276–288]	[289]	[290]	[291, 292]	?	[293, 294]	?
Quadratic gravity							
Gauss-Bonnet	[295]	[295]	[213]	?	?	?	?
Chern-Simons	$\equiv$ GR	[176, 296–299]	?	?	[298]	?	?
Horndeski	?	?	?	?	?	?	?
Lorentz-violating							
Æ-gravity	[300, 301]	?	?	[302]	[179, 180]	[293]	?
Khronometric/							
Hořava-Lifshitz	[303]	?	?	?	[179, 180]	?	?
n-DBI	?	?	?	?	?	?	?
Massive gravity							
dRGT/Bimetric	[304, 305]	?	?	?	?	?	?
Galileon	[306]	[306]	?	[307, 308]	?	?	?
Nondynamical fields							
Palatini $f(R)$	[309–313]	?	?	?	–	?	?
Eddington-Born-Infeld	[314–320]	[314, 315]	?	[315]	–	[321, 322]	?

Table 2.3: Taken from [4], 2015. Catalog of NS properties in several theories of gravity. Symbols and abbreviations are the same as in Table 2.2.

2.4 Tensor Mulsti-Scalar Theories of Gravity

It is now time to introduce the leading actor of the original work done in our work here, *Tensor Multi-Scalar Theories* (TMSTs) of gravitation. As discussed in the previous section and equivalent to Scalar Tensor Theories (STTs), this family breaks the assumption of Lovelock's theorem (Thm. 1) of only having the metric tensor describing the nature of gravity. It is, however, the most general form in breaking this assumption, generalizing single scalar field models to their fullest potential.

The general form of the action in TMST has the following form [21]

$$S := \frac{1}{16\pi G_\star}\int \Big(F(\varphi)\widetilde{R} - 2\widetilde{\nabla}_\alpha\varphi^a\,\widetilde{\nabla}_\beta\varphi^b\,\widetilde{g}^{\alpha\beta}\gamma_{ab}(\varphi) + V(\varphi)\Big)\sqrt{-\widetilde{g}}\,\mathrm{d}^4x \\ + S_{\mathrm{matt}}\left[\widetilde{g}_{\alpha\beta},\Psi\right] . \tag{2.10}$$

Here we used the spacetime metric $\widetilde{g}_{\alpha\beta}$ in the physical Jordan frame (JF) to which the collective matter fields Ψ are coupled. $G_\star$ is the bare gravitational constant, $\widetilde{R}$ is the Ricci scalar curvature and $\widetilde{\nabla}$ to covariant derivative with respect to the Jordan frame metric $\widetilde{g}$. The scalar curvature is coupled to the scalar fields $\varphi = (\varphi^1, \dots, \varphi^n)$ via the field $F(\varphi)$. $V(\varphi) \geq 0$ is the potential of the scalar fields φ. The JF volume element includes $\widetilde{g} = \det(\widetilde{g}_{\alpha\beta})$. The scalar fields are contracted with a tensor field γ_{ab}, a Riemannian metric on the n-dimensional target space manifold T^n. The importance and meaning of (T^n, γ_{ab}) will be discussed thoroughly later.

First note, while working in the physical Jordan frame has the advantage that every calculated field is directly coupled to matter and measured in the real world, the drawbacks are mathematical. Converting to a conformal frame allows us to isolate the Einstein-Hilbert (EH) contribution of the action in any theory introducing new fields [323]. This means, by introducing a new metric $A^2(\varphi)g_{\alpha\beta} := \tilde{g}_{\alpha\beta}$ we can drop the coupling field $F(\varphi)$ in front of the Ricci scalar (precisely what we mean by isolating the EH contribution of the action from the scalar fields) for the price of introducing a new conformal factor $A(\varphi)$. This was the approach followed by [21, 27], where the post-Newtonian formalism in TMST was explored for the first time. The single scalar field theory analysis up to 2.5 PN order was performed in the Jordan frame [15]. That is why even though we follow the DIRE formalism of [15], some critical differences related to using a different frame will be discussed below. The Einstein frame is very natural for the definition of TMST [21, 27] because of multiple scalar fields. The freedom to choose a conformal factor $A(\varphi)$ that might depend on them in a nontrivial way and the relation of this factor to the nonminimal coupling between the scalar field and the Ricci scalar in the Jordan frame can lead to a significantly higher degree of complexity of the Jordan frame field equations compared to the single scalar field case. Hence, we adopt the conformal Einstein frame (EF) throughout the calculations for our analysis.

The Einstein frame form of the action (2.10) in TMST is then given as [21]

$$\begin{aligned} S := \frac{1}{16\pi G_\star} \int \Big(R - 2\nabla_\alpha \varphi^a \, \nabla_\beta \varphi^b \, g^{\alpha\beta} \gamma_{ab}(\varphi) - 4V(\varphi) \Big) \sqrt{-g} \, \mathrm{d}^4 x \\ + S_{\mathrm{matt}} \left[A^2(\varphi) g_{\alpha\beta}, \Psi \right] . \end{aligned} \tag{2.11}$$

The matter model Ψ still couples to the physical JF metric $A^2(\varphi)g_{\alpha\beta}$, while the Ricci scalar R and covariant derivative ∇ are now with respect to the EF metric

$g_{\alpha\beta}$. We realize immediately that the conformal action can be decomposed as

$$S = S_{\mathrm{EH}}\left[g_{\alpha\beta}\right] + S_{\varphi}\left[g_{\alpha\beta}, \varphi\right] + S_{\mathrm{matt}}\left[A^2(\varphi) g_{\alpha\beta}, \Psi\right] \tag{2.12}$$

which is the basis of the mathematical advantages the Einstein frame yield since this results in separated second-order derivatives of the gravitational variables $(g_{\alpha\beta}, \varphi^a)$.

To make sense of the indices crowded notation intrinsic to our subject, we use the convention of Greek letters $\{\alpha, \beta, \gamma, \mu, \nu, \dots\}$ for fields concerning the Lorentzian spacetime metric $g_{\alpha\beta}$ and the Latin letters $\{i, j, k, l, \dots\}$ for purely spatial components of said metric. The indices for the target space fields φ^a, that is, with respect to the Riemannian target space metric γ_{ab}, are labeled via the different Latin letters $\{a, b, c, d, \dots\}$. By slightly abusing notation, these last indices might be added to the left of the fields when certain functions become too crowded with labels. To make it easier to compare to commonly cited articles with similar setups, we include Table 2.4 for quick conversion.

	EF metric	JF metric	extra field(s)	con. factor
Our work	$g_{\alpha\beta}$	$\widetilde{g}_{\alpha\beta}$	$\varphi^1, \dots, \varphi^n$	$A^2(\varphi)$
DEF [21]	$g^*_{\alpha\beta}$	$\widetilde{g}_{\alpha\beta}$	$\varphi^1, \dots, \varphi^n$	$A^2(\varphi)$
MW [15]	$\widetilde{g}_{\alpha\beta}$	$g_{\alpha\beta}$	ϕ	$\varphi = \phi/\phi_0$

Table 2.4: A quick comparison of the notation used in relevant previous work by Damour and Esposito-Farèse (DEF) [21] and Mirshekari and Will (MW) [15] compared to our approach. We changed the main convention from MW to a more appealing look in the Einstein frame (EF) since this is how the bulk of our work is formulated. MW, however, did their analysis in the physical Jordan frame (JF).

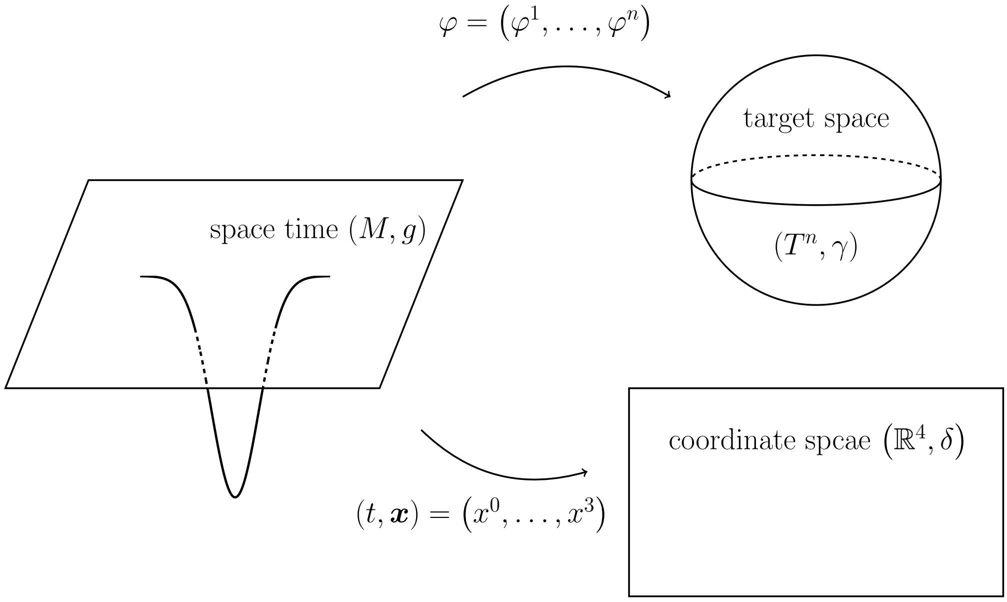

Figure 2.3: Presented is a qualitative depiction of the interaction between the spacetime (M, g), target space (T^n, γ), and canonical coordinate space $(\mathbb{R}^4, \delta)$. The spacetime is pictured as an asymptotically flat manifold with a central contribution of matter (and, hence, curvature), as this is the scenario we are most interested in later on. The n-dimensional target space (T^n, γ) is illustrated as a spherically symmetric object as some work has been done using this simplification [36, 39]. In this work, however, we will not impose any conditions on the target curvature but instead keep it in a general form.

As indicated in the volume element of integration in (2.11), we work in a set of coordinates $(t, \boldsymbol{x}) = (x^0, \dots, x^3)$ for the spacetime manifold (M, g). The multiple scalar fields are each function of spacetime events themselves, so one may collect all n fields to an n-dimensional vector $\varphi = (\varphi^1, \dots, \varphi^n)$ that acts as some sort of generalized coordinates $\varphi\colon (M, g) \to (T^n, \gamma)$. We impose the Riemannian nature on the target space (T^n, γ_{ab}) to avoid ghosts that may occur in a semi-Riemannian setting. The curvature and topology of the target space are essential to the theory as they directly govern the predictions of the theory itself. We will analyze this in quite some depth in Chapter 4.

To obtain the field equations of TMST out of the conformal action (2.11), we vary it with respect to the Einstein frame metric $g_{\alpha\beta}$ and the scalar fields φ^a (instead of only the metric as in pure GR) and obtain

$$
\begin{aligned}
R_{\alpha\beta} &= 2\gamma_{ab}(\varphi)\nabla_\alpha\varphi^a\,\nabla_\beta\varphi^b + 2V(\varphi)g_{\alpha\beta} \\
&\quad +8\pi G_\star\left(T_{\alpha\beta} - \frac{1}{2}Tg_{\alpha\beta}\right), && (2.13) \\
g^{\mu\nu}\nabla_\mu\,\nabla_\nu\varphi^a &= -\gamma^a_{\ bc}(\varphi)g^{\alpha\beta}\,\nabla_\alpha\varphi^b\,\nabla_\beta\varphi^c + \gamma^{ab}(\varphi)\frac{\partial V(\varphi)}{\partial\varphi^b} \\
&\quad -4\pi G_\star\gamma^{ab}\alpha_b(\varphi)T\,, && (2.14)
\end{aligned}
$$

where $\gamma^a_{\ bc}$ are the Christoffel symbols with respect to the target space metric γ_{ab}. Note that Eq. (2.13) is not the standard form containing the Einstein tensor $G_{\alpha\beta}$ from Eq. (2.6) but rather written in terms of the conformal energy-momentum tensor

$$T_{\alpha\beta} := -\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matt}}\left[A^2(\varphi)g_{\alpha\beta},\Psi\right]}{\delta g^{\alpha\beta}} \tag{2.15}$$

and its trace $T := g^{\alpha\beta}T_{\alpha\beta}$. Equivalently, to replace Eq. (2.13), we may write

$$G_{\alpha\beta} = 2\gamma_{ab}(\varphi)\nabla_\alpha\varphi^a\,\nabla_\beta\varphi^b + 2V(\varphi)g_{\alpha\beta} + 8\pi G_\star T_{\alpha\beta}\,. \tag{2.16}$$

The reason for choosing the form (2.13) instead of the more common (2.16) is to adapt already existing results from [15] later on.

The matter contribution to the scalar field equations is given in the last term of Eq. (2.14) and in the form of the energy-momentum trace T coupled with the new quantity $\alpha_a(\varphi)$. Making the standard assumption that the matter fields are independent of the scalar fields, one can show that (see, e.g., [21, 27, 28, 33, 34])

$$\alpha_a(\varphi) := \frac{\partial \log\left(A(\varphi)\right)}{\partial \varphi^a} = A^{-1}(\varphi)\frac{\partial A(\varphi)}{\partial \varphi^a}\,. \tag{2.17}$$

This coupling function is of great interest to us later in our analysis, especially regarding self-gravitating phenomena in TMST.

The first part of the field equations, Eq. (2.13), is a system of ten second-order, nonlinear PDEs for the metric components $g_{\alpha\beta}$ while the second part, Eq. (2.14), is set of n nonlinear, curved wave equations for the scalars φ^a. Assuming some matter model $T_{\alpha\beta}$ and keeping the free parts $(A(\varphi), V(\varphi), \gamma_{ab})$ mostly general, the bulk of this work is to solve for these $10 + n$ gravitational fields $(g_{\alpha\beta}, \varphi)$ in a particular scenario explained in Chapters 3 and 4.

The transformation of the energy-momentum tensor (2.15) into the physical Jordan frame is then given by

$$\widetilde{T}_{\alpha\beta} = A^{-2}(\varphi)T_{\alpha\beta} \tag{2.18}$$

$$\widetilde{T}^{\alpha\beta} = A^{-6}(\varphi)T^{\alpha\beta}\,. \tag{2.19}$$

The energy-momentum conservation of the physical matter model

$$\widetilde{\nabla}_\beta \widetilde{T}^{\alpha\beta} = 0 \tag{2.20}$$

translates to

$$\nabla_\beta T^{\alpha\beta} = \alpha_a(\varphi) T \nabla^\alpha \varphi^a\,, \tag{2.21}$$

with $\alpha_a(\varphi)$ defined as in (2.17). This can be seen from the conservation identity

$$\nabla^\alpha \left(T_{\alpha\beta} + T^\varphi_{\alpha\beta}\right) = 0\,, \tag{2.22}$$

where $T^\varphi_{\alpha\beta}$ is defined as the variational derivative of the scalar action $S_\varphi[g_{\alpha\beta}, \varphi]$ (Eq. (2.12))

$$\begin{aligned} T^\varphi_{\alpha\beta} :=& -\frac{2}{\sqrt{g}}\frac{\delta S_\varphi\,[g_{\alpha\beta}, \varphi]}{\delta g^{\alpha\beta}} \\ =& 2\gamma_{ab}(\varphi)\nabla_\alpha\varphi^a\,\nabla_\beta\varphi^b + 2V(\varphi)g_{\alpha\beta}\,. \end{aligned} \tag{2.23}$$

In the post-Newtonian formalism we adopt, as part of the skeletonization procedure, one can assume, however, that the masses of the individual self-gravitating objects depend on the scalar fields as well [15, 21, 27], which makes the resulting energy-momentum tensor also φ-dependent. Hence, derivatives of the trace of the energy-momentum tensor with respect to the scalar field have to be included in the field equation (2.14). It was demonstrated in [21, 27] that these derivatives can be introduced through a redefinition of $\alpha_a(\varphi)$. Here we will follow the approach of [21], that is to keep the expression for $\alpha_a(\varphi)$ in its general form in the first part of the analysis and only later present its explicit form when we discuss the matter fields and skeletonization.

Let us remark on some general properties of TMST. The action (2.10) is invariant under spacetime and target space diffeomorphism. This keeps the covariant, geometrical nature of GR alive and allows for scalar field redefinitions. TMSTs not only generalize GR and single scalar field extensions, including all Brans-Dicke theories [324] but also a large subset of $f(R)$ theories [325]. More well-studied models also include single scalar field theories with a complex scalar field [21, 28, 35, 39] These are equivalent to TMSTs with two scalar fields and, hence, analysis of, e.g., no hair theorems of complex scalar field models translate to specific TMSTs [200]. TMSTs were also investigated under the 3+1 formalism crucial for numerical Relativity [28]. This allows for a much deeper analysis of especially late inspiral and merger phases of binaries. To our knowledge, this has not been done yet and is, in our mind, a perfect project for future work. More recently, TMSTs were also investigated in a cosmological context [326, 327], revealing promising features for future, testable measurements.

Chapter 3

Post-Newtonian Analysis via DIRE

Parts of our work in this chapter are based on the publication „Tensor-multiscalar gravity: Equations of motion to 2.5 post-Newtonian order“, in Phys. Rev. D, 105.064034[1] *[19] by O. Schön, and D. D. Doneva. Please refer to our Contribution Statement at the beginning of this dissertation for more information.*

It is time to introduce the central toolkit we use to conduct our analysis, the *direct integration of the relaxed field equations* (DIRE) [6, 8–14, 73]. We chose this framework because it is a well-studied model that is theory agnostic in its approach. Generally, every theory motivated via an action, and hence, field equations, can be studied with this mathematical setup. This is quite powerful for multiple reasons, such as it allows one to promptly compare with other theories that have been studied using DIRE [15, 16, 108, 328]. It is also possible to generalize previous results even further using this framework.

DIRE is a tool for post-Newtonian analysis which, in turn, is a tool for

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calculating the inspiral phase gravitational waves stemming from binaries. It has been developed by multiple people, including Epstein and Wagoner [6], Wiseman [7] and made more rigorous by a series of Will and Pati [8–12]. This chapter will use all these resources, with a topical book by Poisson and Will [44] and our previous work [19].

This chapter introduces DIRE vie pure GR, whereas Chapter 4 starts with adapting to TMSTs.

3.1 Landau-Lifshitz Formulation of Gravity

Post-Newtonian and the more general post-Minkowskian expansion start with a framework developed by Landau and Lifshitz [74], which is fittingly named Landau-Lifshitz formulation of GR. We again start with a four-dimensional spacetime $(M, g_{\alpha\beta})$ satisfying the Einstein field equations (2.8) for some given matter model $T_{\alpha\beta}$ as the energy-momentum tensor. We still use the signature $(-+++)$ and a coordinate system $(t, \boldsymbol{x}) = (x^0, \ldots, x^3)$. The goal is, as per usual, to calculate the spacetime metric for a given matter model. Knowing the metric gives you access to all information you desire from spacetime in GR. The Landau-Lifshitz formulation, however, starts with another object, the *gothic metric density*

$$\mathfrak{g}^{\alpha\beta} := \sqrt{-g}\, g^{\alpha\beta}\,, \tag{3.1}$$

where $g = \det(g_{\alpha\beta})$ is again the determinant of the matrix representation of the spacetime metric. Please observe that the gothic metric itself is not a tensor anymore but rather a tensor density due to the scaling with the determinant. The gothic metric will be the most important quantity in this chapter as we calculate various fields. Knowing all components of the gothic metric allows us to reconstruct the full spacetime metric $g_{\alpha\beta}$ via Eq. (3.1) and the easily verifiable fact that $\det\left(\mathfrak{g}^{\alpha\beta}\right) = g$. Next, we introduce the tensor density

$$H^{\alpha\mu\beta\nu} := \mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\mu\nu} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\beta\mu} \,. \tag{3.2}$$

While also not a tensor, this object possesses the same symmetries as the rank four Riemann tensor (2.5). More specifically, it is skew-symmetric in the first, as well as in the last two indices (i.e., $0 = H^{(\alpha\mu)\beta\nu} = H^{\alpha\mu(\beta\nu)}$). It also has the interchange symmetry $H^{\alpha\mu\beta\nu} = H^{\beta\nu\alpha\mu}$. Furthermore, the density, or rather the second derivatives of it, interestingly reproduces the Einstein tensor (2.6) plus some other terms that we collect in the *Landau-Lifshitz pseudotensor* $t^{\alpha\beta}_{LL}$, such that

$$\partial_{\mu\nu}H^{\alpha\mu\beta\nu} = 16\pi G(-g)\left(T^{\alpha\beta} + t^{\alpha\beta}_{LL}\right) \tag{3.3}$$

with

$$\begin{aligned} 16\pi G(-g)t^{\alpha\beta}_{LL} \;:=\;& \partial_\lambda\mathfrak{g}^{\alpha\beta}\partial_\mu\mathfrak{g}^{\lambda\mu} - \partial_\lambda\mathfrak{g}^{\alpha\lambda}\partial_\mu\mathfrak{g}^{\beta\mu} + \frac{1}{2}g^{\alpha\beta}g_{\lambda\mu}\partial_\rho\mathfrak{g}^{\lambda\nu}\partial_\nu\mathfrak{g}^{\mu\rho} \\ & -g^{\alpha\lambda}g_{\mu\nu}\partial_\rho\mathfrak{g}^{\beta\nu}\partial_\lambda\mathfrak{g}^{\mu\rho} - g^{\beta\lambda}g_{\mu\nu}\partial_\rho\mathfrak{g}^{\alpha\nu}\partial_\lambda\mathfrak{g}^{\mu\rho} + g^{\nu\rho}g_{\lambda\mu}\partial_\nu\mathfrak{g}^{\alpha\lambda}\partial_\rho\mathfrak{g}^{\beta\mu} \\ & +\frac{1}{8}\left(2g^{\alpha\lambda}g^{\beta\mu} - g^{\alpha\beta}g^{\lambda\mu}\right)\left(2g_{\nu\rho}g_{\sigma\tau} - g_{\rho\sigma}g_{\nu\tau}\right)\partial_\lambda\mathfrak{g}^{\nu\tau}\partial_\mu\mathfrak{g}^{\rho\sigma} \,, \end{aligned} \tag{3.4}$$

where $\partial_{\mu\nu} = \partial_\mu\partial_\nu$.

Equation (3.3) is the starting point of a post-Minkowskian theory which, in turn, is the foundation of the post-Newtonian theory used in this paper. These field equations are as usual of second order: The left-hand side has second derivatives of the tensor density $H^{\alpha\mu\beta\nu}$, which implies second derivatives of the gothic metric and the spacetime metric. Notice that all these are pure geometric quantities, precisely as the left-hand side of the standard Einstein field equations (2.8). The Landau-Lifshitz pseudotensor $t_{LL}^{\alpha\beta}$ itself does not contain second-order derivatives anymore but is rather proportional to quadratics of first-order derivatives of the gothic metric, $t_{LL}^{\alpha\beta} \sim \partial\mathfrak{g} \cdot \partial\mathfrak{g}$. Further, Eq. (3.3) is still mathematically equivalent to the full Einstein field equations (2.8) in the sense that no approximations have been introduced yet. We did lose, however, the covariant nature of GR by writing everything in Lorentzian coordinates because not all quantities are pure tensors anymore. This loss is not too devastating, though, since in almost all practical use cases in physics relating to GR, one has to choose a coordinate system at one point. Here, we already used one on the hierarchy of the field equations. Note here, as with any attempt to solve Einstein's field equations, the manipulations introduced here do not give away their usefulness. Similar to the 3+1 formulation or conformal rewriting, this is a preliminary step with some advantages later.

Moving on, the antisymmetry of $H^{\alpha\mu\beta\nu}$ in the last pair of indices yields a resemblance of a conservation law

$$\partial_\beta \partial \mu\nu H^{\alpha\mu\beta\nu} = \partial_\beta \left[(-g)\left(T^{\alpha\beta} + t_{LL}^{\alpha\beta}\right)\right] = 0\,. \tag{3.5}$$

This, together with the Einstein equations (2.8), is equivalent to the energy-momentum conservation $\nabla_\beta T^{\alpha\beta} = 0$. Using this, we can formulate global conservation properties (here only for the energy, but similar integral identities can be defined straight from Eq. (3.5) for angular momentum, linear momentum, and center of mass)

$$E \;:=\; \int_V (-g)\left(T^{00} + t_{LL}^{00}\right) \mathrm{d}^3x \tag{3.6}$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \oint_S (-g)t^{0j}_{LL}\,\mathrm{d}^2 S_j\,, \tag{3.7}$$

where the former integral is assumed to be over a three-dimensional domain V with nonvanishing matter contribution, and the latter equation is a surface integral over a two-sphere S containing all support of the energy-momentum tensor $T^{\alpha\beta}$. The involvement of the Landau-Lifshitz pseudotensor in the energy rate of change (3.7) hints at its involvement in the energy loss of a system due to gravitational radiation. This contrasts Newtonian gravity, where energy was a strictly conserved quantity. GR allows energy to be reduced in forms of radiation away from the source towards spatial infinity.

3.2 The Relaxed Einstein equations

To proceed, we impose *harmonic coordinate (gauge) conditions* $\partial_\beta \mathfrak{g}^{\alpha\beta} = 0$ and define the potentials

$$h^{\alpha\beta} := \eta^{\alpha\beta} - \mathfrak{g}^{\alpha\beta}\,, \tag{3.8}$$

with inverse Minkowski metric $\eta^{\alpha\beta}$ in Lorentzian coordinates $(t := x^0, x^j)$. It is easily verified that such coordinates always exist as our formulation inherits the coordinate freedom from the usual GR formulation. The fields $h^{\alpha\beta}$ are, next to the scalar fields, the main focus of the rest of this thesis. Together, they are the unknowns in the wave equations we will derive below. First, note that knowing all $h^{\alpha\beta}$ is sufficient to recreate the spacetime metric via the definition (3.8) and Eq. (3.1). Next, it is easy to imagine that any ansatz to solve for $h^{\alpha\beta}$ works best if the spacetime geometry is close to flat Minkowski space since then Eq. (3.8) implies that the $h^{\alpha\beta}$ are small and, in some sense, a perturbation to a flat background.

Later on, we will realize that, while not suitable for physics directly at the event horizon, this ansatz produces strong results even for the space between compact objects assuming certain slow-motion conditions. Hence, this toolkit is very usable even for calculating orbits of slow-moving (compared to the speed of light) binary black holes.

The above introduced harmonic gauge translates nicely to these fields as $\partial_\beta h^{\alpha\beta} = 0$. Using all this, the left-hand side of (3.3) becomes

$$\partial_{\mu\nu} H^{\alpha\mu\beta\nu} = -\Box h^{\alpha\beta} - 16\pi G_\star (-g) t_H^{\alpha\beta} , \tag{3.9}$$

for the usual harmonic-gauge pseudotensor

$$16\pi G_\star (-g) t_H^{\alpha\beta} := \partial_\mu h^{\alpha\nu} \partial_\nu h^{\beta\mu} - h^{\mu\nu} \partial_\mu \partial_\nu h^{\alpha\beta} \tag{3.10}$$

and for the Minkowskian wave operator $\Box := \eta^{\mu\nu}\partial_\mu\partial_\nu$. The right-hand side of (3.3) also simplifies as some terms in the Landau-Lifshitz pseudotensor (3.4) vanish due to our chosen gauge. Now, substituting the identity (3.3) into Eq. (3.9) and isolating the box operator to the left-hand side, we arrive at the flat wave equation

$$\Box h^{\alpha\beta} = -16\pi G_\star \tau^{\alpha\beta} , \tag{3.11}$$

where the source

$$\tau^{\alpha\beta} := (-g) \left(T^{\alpha\beta} + t_{LL}^{\alpha\beta} + t_H^{\alpha\beta} \right) \tag{3.12}$$

plays the role of an effective energy-momentum pseudotensor and satisfies the conservation law

$$\partial_\beta \tau^{\alpha\beta} = 0 \, . \tag{3.13}$$

Note that with Eq. (3.8) in mind, we hint at the idea that we will assume the spacetime to be close to the Minkowski solution to expand in terms of the gravitational potentials $h^{\alpha\beta}$. Until now, however, we did not impose anything else on pure GR but a preferred coordinate system. Everything here is a direct reformulation of the Einstein field equations (2.8) and mathematically equivalent to a change of coordinates. Further on, approximations will be introduced to be helpful for our analysis, and at that point, this equivalence will break. The rewriting was to obtain the flat wave equation (3.11). While highly nonlinear and complicated, it is nonetheless a wave equation; we can utilize decades of existing research to solve this problem.

The reformulating of the field equations in the previous section lead us, in principle, to the following set of equations

$$\Box h^{\alpha\beta} = -16\pi G_\star \tau^{\alpha\beta}\,, \tag{3.14}$$

$$\partial_\beta \tau^{\alpha\beta} = 0\,. \tag{3.15}$$

The idea is to solve for the gravitational potentials $h^{\alpha\beta}$ in terms of the matter variables included in the source $\tau^{\alpha\beta}$ via Eq. (3.12) and then make sure the divergence condition of the second equation is satisfied. In some sense, this is similar to the famous Wheeler quote cited in Section 2.1.2 in that in Eq. (3.14), matter dictates the fields $h^{\alpha\beta}$ and, in turn, the metric tensor which tells spacetime how to curve. On the other hand, Eq. (3.15) is essentially a form of an equation of motion, meaning spacetime tells matter how to move.

As the title of the section indicates, we will occupy ourselves with the *relaxed field equations*, that is, the wave equation (3.14) independent from the conservation (3.15). This is where the basic framework of DIRE starts, and the integration techniques are explained in the next section.

3.3 The Direct Integration of the Relaxed Field Equations

Luckily for us, there already exists a formal solution of wave equations in the form of Eq. (3.14) in

$$h^{\alpha\beta}(x) = 4G\int G(x,x')\,\tau^{\alpha\beta}(x')\,\mathrm{d}^4x'\,, \tag{3.16}$$

for an event $x = (t, \boldsymbol{x})$ with the *retarded Green's function* $G(x,x')$ satisfying the Minkowski wave equation

$$\Box G(x,x') = -4\pi\delta(x-x')\ , \tag{3.17}$$

for the standard δ distribution. Note that we call Eq. (3.16) a formal solution due to the dependence of $h^{\alpha\beta}$ in the source $\tau^{\alpha\beta}$ via the Landau-Lifshitz, $t^{\alpha\beta}_{LL}$, and harmonic gauge, $t^{\alpha\beta}_{H}$, pseudetensors in Eq. (3.12). The retarded Green's function as a solution of Eq. (3.17) is a function only of $x - x'$ and can be written down explicitly as

$$G(x,x') = \frac{\delta(t - t' - |\boldsymbol{x} - \boldsymbol{x}'|)}{|\boldsymbol{x} - \boldsymbol{x}'|}\,, \tag{3.18}$$

for the standard Euclidean norm $|\cdot|$ where we used the fact that

$$\delta(x-x') = \delta(t-t')\,\delta(\boldsymbol{x} - \boldsymbol{x}')\ . \tag{3.19}$$

Substituting the Green's function (3.18) back in the formal solution (3.16) then yields

$$h^{\alpha\beta}(x) = 4G \int \frac{\tau^{\alpha\beta}(x')\,\delta\left(t - t' - |\boldsymbol{x} - \boldsymbol{x}'|\right)}{|\boldsymbol{x} - \boldsymbol{x}'|}\,\mathrm{d}^4x'\,. \tag{3.20}$$

This can be simplified immediately further by integrating out the time component to obtain (as demonstrated in Fig. 3.1)

$$h^{\alpha\beta}(t, \boldsymbol{x}) = 4G \int \frac{\tau^{\alpha\beta}\left(t - |\boldsymbol{x} - \boldsymbol{x}'|, \boldsymbol{x}'\right)}{|\boldsymbol{x} - \boldsymbol{x}'|}\,\mathrm{d}^3x'\,, \tag{3.21}$$

where this integral is meant to be evaluated over the past light cone $\mathcal{C}(t, \boldsymbol{x})$ of the field event $(t, \boldsymbol{x})$. We still observe a retardation in time in the source $\tau^{\alpha\beta}\left(t - |\boldsymbol{x} - \boldsymbol{x}'|, \boldsymbol{x}'\right)$ via $t - |\boldsymbol{x} - \boldsymbol{x}'|$. This is, again, in contrast to Newtonian gravity, where no retardation would occur due to gravity being an immediate effect. Here, relativistic effects such as noninstant gravitation are incorporated. Approximating this retardation will be one of the key components in our analysis later on.

The key to unlocking solutions from the right-hand side of Eq. (3.21) is via an iterative process of the form

$$\begin{aligned} \Box h^{\alpha\beta}_{N+1} &= -16\pi G_\star \tau^{\alpha\beta}\left(h^{\alpha\beta}_N\right)\,, & (3.22)\\ h^{\alpha\beta}_{N+1} &= 4G \int \frac{\tau^{\alpha\beta}\left(h^{\alpha\beta}_N\right)\left(t - |\boldsymbol{x} - \boldsymbol{x}'|, \boldsymbol{x}'\right)}{|\boldsymbol{x} - \boldsymbol{x}'|}\,\mathrm{d}^3x'\,. & (3.23) \end{aligned}$$

So we start with the initial data $h^{\alpha\beta}_0 = 0$, pluck this in the effective energy-momentum pseudotensor $\tau^{\alpha\beta}$ and then evaluate the integral (3.23) to obtain $h^{\alpha\beta}_1$. This process can be iterated, and due to the involvement of the gravitational constant G in the above's equations, we end up with an expansion in G of the

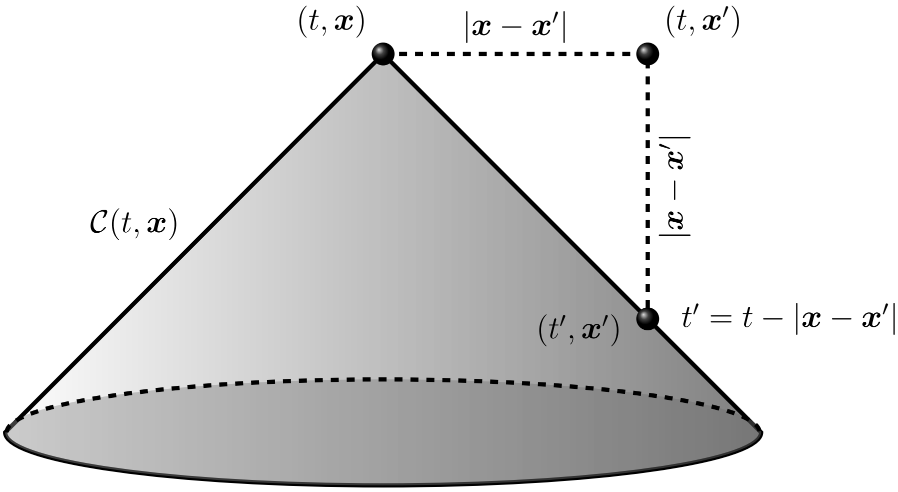

Figure 3.1: Past light cone $\mathcal{C}(t, \boldsymbol{x})$ of an event $(t, \boldsymbol{x})$. Any other event $(t, \boldsymbol{x}')$ happening at the same time t is moved to the the past light cone $\mathcal{C}(t, \boldsymbol{x})$ via the time retardation $t' = t - |\boldsymbol{x} - \boldsymbol{x}'|$.

form

$$h^{\alpha\beta} = \underbrace{\underbrace{\underbrace{\underbrace{G k_1^{\alpha\beta}}_{h_1^{\alpha\beta}} + G^2 k_2^{\alpha\beta}}_{h_2^{\alpha\beta}} + G^3 k_3^{\alpha\beta}}_{h_3^{\alpha\beta}} + G^4 k_4^{\alpha\beta}}_{h_4^{\alpha\beta}} + \ldots \,, \tag{3.24}$$

with the iterative defined auxiliary quantities

$$h_N^{\alpha\beta} =: \sum_{j=1}^{N} G^j k_j^{\alpha\beta} \,. \tag{3.25}$$

A power series in the gravitational constant G like here is known as a *post-Minkowskian expansion.* Using a quantity with dimension as an expansion parameter might seem unusual. In involving units, there is no way of making sense of „small“ typically needed for the parameter of a series like that. Later on, the physical expansion parameter will be of the form

$$\frac{Gm_c}{c^2 r_c} \tag{3.26}$$

for the characteristic mass m_c and radius r_c of the physical system of interest. To not overcrowd the analysis until the matter matters, it is established to keep the parameter as G for now. This also means that there is not exactly convergence expected from Eq. (3.22) but rather hope that the truncated series produces good enough results for the region we are interested in due to the nature of the framework itself.

3.3.1 Domain Separation

Evaluating the integral (3.21) is not as straightforward as one might hope. There are, however, methods to approximate the integral in specific domains (similar to problems encountered in electrodynamics). To make full use of these methods, we split the past light cone $\mathcal{C}(t, \boldsymbol{x})$ of any event $(t, \boldsymbol{x})$ into a *near-zone* $\mathcal{N}(t, \boldsymbol{x})$ and a *wave-zone* $\mathcal{W}(t, \boldsymbol{x})$ (sometimes also called the *far-zone* or *radiation-zone*). These zones are defined via a three-dimensional sphere of radius $\mathcal{R}$ around the matter source of the physical system we are modeling (see Fig. 3.2). In the most common use case of a binary system, the radius $\mathcal{R}$ roughly equals the characteristic wavelength the system radiates in the forms of gravitational waves. This ensures that all the matter lies within the sphere and that the methods below yield the best results.

More precisely, the near-zone $\mathcal{N}(t, \boldsymbol{x})$ then is the intersection of a world tube

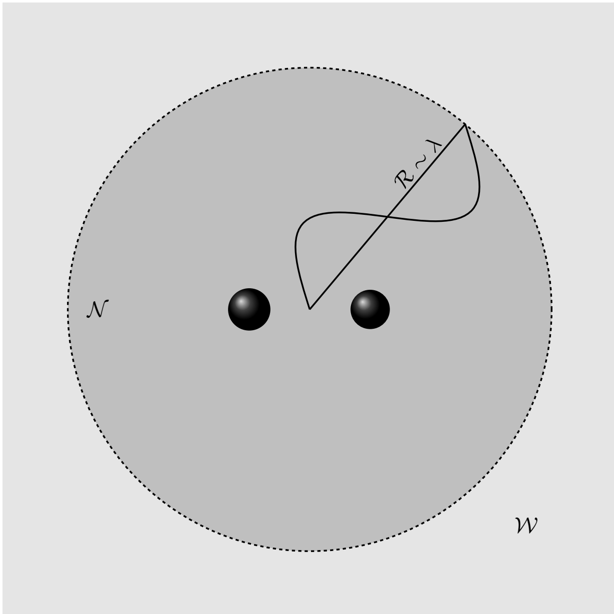

Figure 3.2: Binary system separated in near- and far zone via a sphere of radius $\mathcal{R}$. The size of the sphere is proportional to the wavelength λ of the system-specific characteristic radiation, $\mathcal{R} \sim \lambda$.

$\mathcal{D}$, traced by our sphere of radius $\mathcal{R}$

$$\mathcal{D} := \left\{ (t, \boldsymbol{x}) \,\middle|\, |\boldsymbol{x} - \boldsymbol{x}_{CM}| < \mathcal{R} \text{ for the center of mass } (t, \boldsymbol{x}_{CM}) \text{ at time } t \right\} \tag{3.27}$$

and the past light cone $\mathcal{C}(t, \boldsymbol{x})$ of any event $(t, \boldsymbol{x})$ (see Fig. 3.3), so

$$\mathcal{N}(t, \boldsymbol{x}) := \mathcal{D} \cap \mathcal{C}(t, \boldsymbol{x}) \subset \mathcal{C}(t, \boldsymbol{x}) \,, \tag{3.28}$$

and the wave-zone is hence given as the set difference

$$\mathcal{W}(t, \boldsymbol{x}) := \mathcal{C}(t, \boldsymbol{x}) \setminus \mathcal{N}(t, \boldsymbol{x}) \subset \mathcal{C}(t, \boldsymbol{x}) \,. \tag{3.29}$$

As mentioned earlier, depending on whether the event $(t, \boldsymbol{x})$ is located in

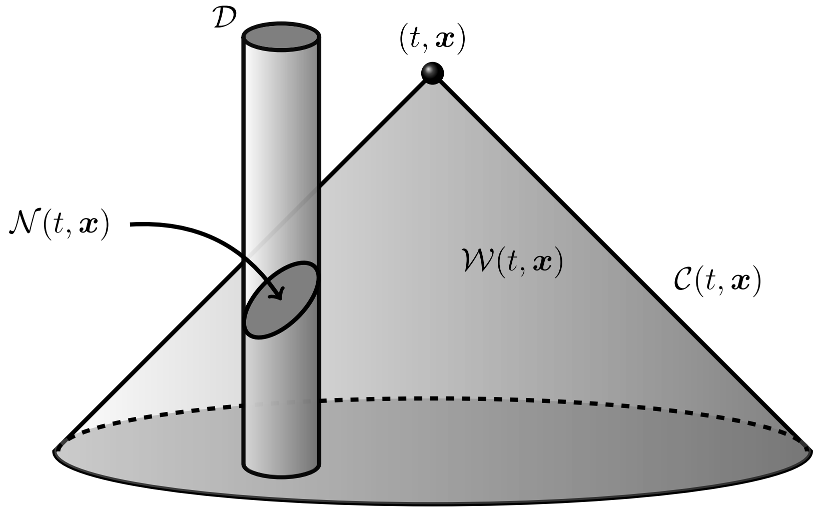

Figure 3.3: Past light cone $\mathcal{C}(t, \boldsymbol{x})$ of an event $(t, \boldsymbol{x})$. The domain $\mathcal{D}$ is a world tube traced by the matter sphere of radius $\mathcal{R}$ where the support of the energy-momentum tensor is located. The near-zone $\mathcal{N}(t, \boldsymbol{x})$ is depicted as the intersection of the light cone with $\mathcal{D}$. Here, we arranged the field point $(t, \boldsymbol{x})$ in the wave-zone $\mathcal{W}(t, \boldsymbol{x})$.

the near-zone $\mathcal{N}(t, \boldsymbol{x})$ or wave-zone $\mathcal{W}(t, \boldsymbol{x})$, different techniques can be used to approximately evaluate the integral (3.21) and, hence, calculate $h^{\alpha\beta}(t, \boldsymbol{x})$ at this specific event. Be aware that in both incidents, there is a near- and wave-zone contribution to the gravitational potential $h^{\alpha\beta}$. Hence, to obtain the entire spacetime geometry, one has to calculate the four individual components

$$h^{\alpha\beta}(t, \boldsymbol{x}) = \begin{cases} h_{\mathcal{N}}^{\alpha\beta}(t, \boldsymbol{x}) \; + \; h_{\mathcal{W}}^{\alpha\beta}(t, \boldsymbol{x}) & \text{for } (t, \boldsymbol{x}) \in \mathcal{N}(t, \boldsymbol{x}) \,, \\ h_{\mathcal{N}}^{\alpha\beta}(t, \boldsymbol{x}) \; + \; h_{\mathcal{W}}^{\alpha\beta}(t, \boldsymbol{x}) & \text{for } (t, \boldsymbol{x}) \in \mathcal{W}(t, \boldsymbol{x}) \,. \end{cases} \tag{3.30}$$

The near- and wave-zone potentials might contain terms dependent on the cut-off radius $\mathcal{R}$ due to its involvement in the integration boundary. This parameter, however, has to necessarily cancel out by adding up the two solutions. Hence, in further calculations, we can drop $\mathcal{R}$-depended terms from our analysis.

Of course, one might expect that the radiation zone solutions will not contribute much to the full potential for a field point in the near-zone as the near-zone dynamics will dominate the system due to the effects of the retardation being minimal. Basically, inside the near-zone, everything happens almost instantaneously. But the nonlinear nature of our theory will necessarily force such contribution at higher orders as the value of nonmatter fields (such as $t^{\alpha\beta}_{LL}$, Eq. (3.4), and $t^{\alpha\beta}_{H}$, Eq. (3.10)) might decrease away from the near-zone, the integration domain grows more extensive as well. Hence the magnitude of the contribution is not apparent right away.

This contrasts the previously mentioned similarity to electrodynamics, where you only ever integrate over the source of your system. Our source term, $\tau^{\alpha\beta}$ from Eq, (3.12), contains the gravitational potential $h^{\alpha\beta}$ itself and, hence, the source has to be integrated over the whole past light cone. In this sense, gravity itself can create gravity.

It is crucial to keep in mind that when we talk about the near-zone dynamics, when do not mean just the near-zone part $h^{\alpha\beta}_{\mathcal{N}}$ of the gravitational potentials $h^{\alpha\beta}$. Instead, we talk about the behavior of $h^{\alpha\beta}$ at a near-zone event $(t, \boldsymbol{x})$, which includes radiation-zone integrals as depicted in Eq. (3.30).

3.4 Near-Zone Field Point Evaluation

As our primary goal is to derive the equation of motion in Tensor Multi-Scalar Theories of gravitation, we are mainly interested in near-zone dynamics. This means we must thoroughly evaluate the first case of Eq. (3.30) to a high enough order for our analysis. As briefly mentioned above, the main contribution is the near-zone integration which will be analyzed in the following. After, we will briefly discuss the wave-zone contribution to the evaluation of $h^{\alpha\beta}$ at a near-zone event.

3.4.1 Near-Zone Integration of Near-Zone Event

The near-zone $\mathcal{N}(t,\boldsymbol{x})$ of any event $(t,\boldsymbol{x})$ is a nontrivial geometrical region. As an intersection of the past light cone, it has nonvanishing curvature making it a complicated integration domain. Examining the formal solution of Eq. (3.21) more closely, we observe that for a near-zone event $(t,\boldsymbol{x})$ the near-zone integration

$$h^{\alpha\beta}_{\mathcal{N}}\left(t,\boldsymbol{x}\right) = 4G\int_{\mathcal{N}} \frac{\tau^{\alpha\beta}\left(t-|\boldsymbol{x}-\boldsymbol{x}'|,\boldsymbol{x}'\right)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x'\,, \tag{3.31}$$

depends on the distance $|\boldsymbol{x}-\boldsymbol{x}'|$ in two separate ways. Once via time retardation in the wave equation source $\tau^{\alpha\beta}$ and the Newtonian-like denominator. Since here we assume $(t,\boldsymbol{x}) \in \mathcal{N}(t,\boldsymbol{x})$, we now that

$$|\boldsymbol{x}-\boldsymbol{x}'| < \mathcal{R} \tag{3.32}$$

for the near-zone integration over $\boldsymbol{x}'$. This allows us to interpret $|\boldsymbol{x}-\boldsymbol{x}'|$ as a small quantity compared to the time and validates a Taylor expansion of the source as

$$\tau^{\alpha\beta}\left(t-|\boldsymbol{x}-\boldsymbol{x}'|,\boldsymbol{x}'\right) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\partial^k}{\partial t^k}\tau^{\alpha\beta}\left(t,\boldsymbol{x}'\right)|\boldsymbol{x}-\boldsymbol{x}'|^k\,. \tag{3.33}$$

We immediately note that the pseudo tensor $\tau^{\alpha\beta}$ is now evaluated at nonretarded time t. Hence, integrating over the near-zone now involves the new domain $\mathcal{M}(t,\boldsymbol{x})$

$$\mathcal{M}(t,\boldsymbol{x}) := \mathcal{D}\cap\Sigma_t \;\subset\; \Sigma_t\,, \tag{3.34}$$

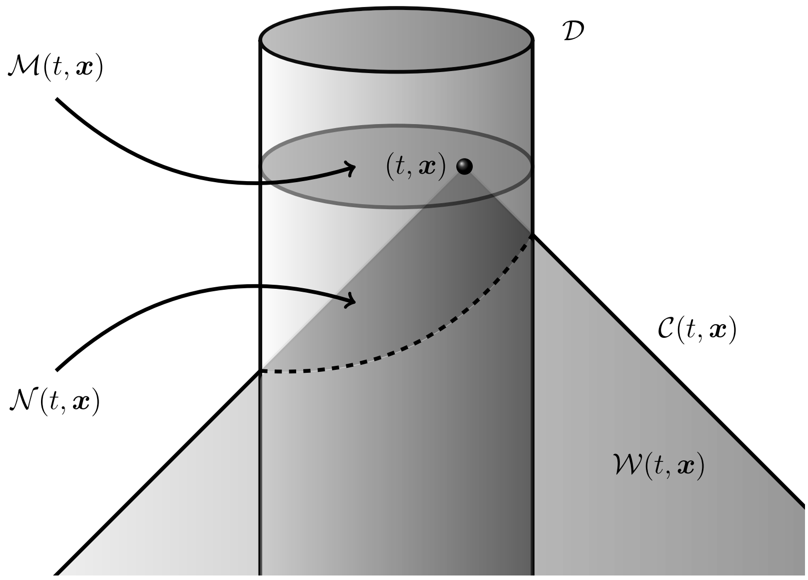

Figure 3.4: Field point $(t, \boldsymbol{x})$ in the near-zone $\mathcal{N}(t, \boldsymbol{x})$. We approximate the near-zone with the intersection $\mathcal{M}(t, \boldsymbol{x})$ of our world tube $\mathcal{D}$ and a time slice Σ_t of constant time t (Cauchy surface).

where Σ_t is a Cauchy surface of constant time t (as depicted in Fig. 3.4). This, again, mimics the behavior one would study in electrodynamics as we have instantaneous interaction of gravity via this approximation. In the near-zone, this is true enough for our purposes.

Putting it all together to calculate $h_{\mathcal{N}}^{\alpha\beta}$, and keeping in mind that the source is weighted by $1/|\boldsymbol{x} - \boldsymbol{x}'|$, we obtain

$$h_{\mathcal{N}}^{\alpha\beta}(t, \boldsymbol{x}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\partial^k}{\partial t^k} \int_{\mathcal{M}} \tau^{\alpha\beta}(t, \boldsymbol{x}') \, |\boldsymbol{x} - \boldsymbol{x}'|^{k-1} \, \mathrm{d}^3 x' \, . \tag{3.35}$$

Calculating the sum above to a high order is the bulk of analyzing near-zone behavior as we intend to do. The calculations are more involved as the wave-zone integration does not contribute early. This post-Newtonian expansion right here is often depicted as a power series in terms of the reciprocal of the speed of light,

$1/c$, since in nongeometric units, the retardation could read $t - |\boldsymbol{x} - \boldsymbol{x}'|/c$. We omitted it here since our original analysis, later on, will be in said units, as stated before.

As a sanity check, we may briefly investigate the series in Eq. (3.35). The first term, and hence lowest order contribution, has the form

$$\int_{\mathcal{M}} \frac{\tau^{\alpha\beta}\left(t, \boldsymbol{x}'\right)}{|\boldsymbol{x} - \boldsymbol{x}'|} \, \mathrm{d}^3 x' \,. \tag{3.36}$$

This is, just as expected, the Newtonian potential itself. All additional terms are counted as relativistic corrections in powers of $|\boldsymbol{x} - \boldsymbol{x}'|$.

3.4.2 Wave-Zone Integration of Near-Zone Event

We will not spend too much time and effort on this part since the wave-zone contribution to the near-zone analysis only plays a vital role in an order beyond our goal of accurately constructing the equation of motion thru 2.5 post-Newtonian order. This is also what one would expect from a physical point of view, as the masses in the near-zone should dominate behavior related to gravity quite a bit. However, it has to be carefully checked every time one calculates contribution if we discover new terms beyond gravity. As mentioned, this is a brief discussion, and for details, please consult [10, 15, 16, 44].

By adapting the spatial integration variables via the retardation

$$\tau' = t' - R' \tag{3.37}$$

the wave-zone $\mathcal{W}$ integration yields

$$\begin{aligned} h^{\alpha\beta}_{\mathcal{W}}(t, \boldsymbol{x}) &= 4\int_{\tau-2\mathcal{R}}^{\tau-2\mathcal{R}+2R} \mathrm{d}\tau' \int_0^{2\pi} \mathrm{d}\phi' \int_{1-\xi}^{1} \frac{\tau^{\alpha\beta}\left(\tau'+R', \boldsymbol{x}'\right)}{t-\tau'-\boldsymbol{N}'\cdot\boldsymbol{x}} \left[R'\left(\tau',\Omega'\right)\right] \mathrm{d}\cos\left(\theta'\right) \\ &\quad +4\int_{-\infty}^{\tau-2\mathcal{R}} \mathrm{d}\tau' \oint \frac{\tau^{\alpha\beta}\left(\tau'+R', \boldsymbol{x}'\right)}{t-\tau'-\boldsymbol{N}'\cdot\boldsymbol{x}} \left[R'\left(\tau',\Omega'\right)\right]^2 \mathrm{d}^2\Omega' \qquad (3.38) \end{aligned}$$

with

$$\xi := \frac{\tau-\tau'}{2R\mathcal{R}}\left(2R-2\mathcal{R}+\tau-\tau'\right) \qquad (3.39)$$

$$R'\left(\tau',\Omega'\right) = \frac{\left(t-\tau'\right)^2-R^2}{2\left(t-\tau'-\boldsymbol{N}'\cdot\boldsymbol{x}\right)}, \qquad (3.40)$$

and the unit normal $\boldsymbol{N}' = \boldsymbol{x}'/R'$ for $R' = |\boldsymbol{x}'|$.

3.5 Post-Newtonian Spacetime Metric for Near-Zone Events

Let us now take the time to explain how exactly we convert between the gravitational potentials $h^{\alpha\beta}$ as defined in Eq. (3.8) and the full spacetime metric $g_{\alpha\beta}$. More precisely, we need to ask ourselves how accurately we need to calculate the fields $h^{\alpha\beta}$ via the post-Minkowskian approximation explained above to have the (conformal) spacetime metric sufficiently correct for the physical problem we want to analyze. Combining Eq. (3.8) and the definition of the goth metric Eq. (3.1), we obtain

$$g^{\alpha\beta} = \frac{1}{\sqrt{-g}}\left(\eta^{\alpha\beta}-h^{\alpha\beta}\right), \qquad (3.41)$$

with $g = \det(g_{\alpha\beta})$ again.

The factor $1/\sqrt{-g}$ in the above's equation is important to understand to calculate $g^{\alpha\beta}$ in any meaningful way. Its expansion will guide the expansion of all other spacetime fields later on. First, we recall

$$(-g) \;=\; -\det\left(g_{\alpha\beta}\right) = -\det\left(\mathfrak{g}^{\alpha\beta}\right) = -\det\left(\eta^{\alpha\beta} - h^{\alpha\beta}\right) . \tag{3.42}$$

Hence, we need to approximate the determinant somehow in terms of the gravitational potentials $h^{\alpha\beta}$. We note the formal expansion in a parameter $\lambda \in \mathbb{R}$

$$\begin{aligned} \det\left(\mathrm{Id} + \lambda M\right) &= \exp\left(\mathrm{tr}\left(\log\left(\mathrm{Id} - \lambda M\right)\right)\right) \\ &= \exp\left(\lambda \mathrm{tr}(M) - \frac{1}{2}\lambda^2 \mathrm{tr}\left(M^2\right) + \frac{1}{3}\lambda^3 \mathrm{tr}\left(M^3\right) + \mathcal{O}\left(M^4\right)\right) \\ &= 1 + \lambda \mathrm{tr}(M) + \frac{1}{2}\lambda^2\left(\mathrm{tr}^2\left(M\right) - \mathrm{tr}\left(M^2\right)\right) \\ &\quad + \frac{1}{6}\lambda^3\left(\mathrm{tr}^3\left(M\right) - 3\mathrm{tr}\left(M\right)\mathrm{tr}\left(M^2\right) + 2\mathrm{tr}\left(M^3\right)\right) \\ &\quad + \mathcal{O}\left(M^4\right) . \end{aligned} \tag{3.43}$$

Adapted to our scenario, meaning $\mathrm{Id} = \eta^{\alpha\beta}$, $M = h^{\alpha\beta}$, and $\lambda = -1$, we obtain

$$\begin{aligned} (-g) &= -\det\left(g_{\alpha\beta}\right) = -\det\left(\mathfrak{g}^{\alpha\beta}\right) = -\det\left(\eta^{\alpha\beta} - h^{\alpha\beta}\right) \\ &= 1 - h + \frac{1}{2}h^2 - \frac{1}{2}h^{\alpha\beta}h_{\alpha\beta} - \frac{1}{6}h^3 + \frac{1}{2}h\,h^{\alpha\beta}h_{\alpha\beta} - \frac{1}{3}h_{\alpha\beta}h^{\beta}{}_{\gamma}h^{\alpha\gamma} \\ &\quad + \mathcal{O}\left(G^4\right) , \end{aligned} \tag{3.44}$$

since $h^{\alpha\beta} \propto G$ and for $h := \mathrm{tr}_\eta(h) = \eta_{\alpha\beta}h^{\alpha\beta}$. It is generally understood that the indices of the gravitational potentials $h^{\alpha\beta}$ are raised and lowered with the Minkowski metric $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$.

Using Eq. (3.44), we can expand the square root needed for Eq. (3.41) as

$$\sqrt{-g} \;=\; 1 - \frac{1}{2}h + \frac{1}{8}h^2 - \frac{1}{4}h^{\alpha\beta}h_{\alpha\beta} - \frac{1}{48}h^3 + \frac{1}{8}h\,h^{\alpha\beta}h_{\alpha\beta} - \frac{1}{6}h_{\alpha\beta}h^{\beta}{}_{\gamma}h^{\alpha\gamma}$$

$$+\mathcal{O}\left(G^4\right) . \tag{3.45}$$

From this we can easily expand $1/\sqrt{-g}$ and substitute the solution in Eq. (3.41) to obtain the inverse spacetime metric

$$\begin{aligned} g^{\alpha\beta} &= \frac{1}{\sqrt{-g}}\left(\eta^{\alpha\beta} - h^{\alpha\beta}\right) \\ &= \eta^{\alpha\beta} - h^{\alpha\beta} + \frac{1}{2}h\eta^{\alpha\beta} - \frac{1}{2}hh^{\alpha\beta} + \left(\frac{1}{8}h^2 + \frac{1}{4}h^{\alpha\beta}h_{\alpha\beta}\right)\eta^{\alpha\beta} \\ &\quad -\frac{1}{8}h^2h^{\alpha\beta} - \frac{1}{4}h^{\alpha\beta}h_{\mu\nu}h^{\mu\nu} + \frac{1}{48}h^3\eta^{\alpha\beta} + \frac{1}{8}hh_{\mu\nu}h^{\mu\nu}\eta^{\alpha\beta} + \frac{1}{6}h_{\mu\nu}h^{\nu}{}_{\gamma}h^{\mu\gamma}\eta^{\alpha\beta} \\ &\quad +\mathcal{O}\left(G^4\right) . \end{aligned} \tag{3.46}$$

And after inverting the full spacetime metric has the form

$$\begin{aligned} g_{\alpha\beta} &= \eta_{\alpha\beta} + h_{\alpha\beta} - \frac{1}{2}h\eta^{\alpha\beta} - \frac{1}{2}hh_{\alpha\beta} + h_{\alpha\mu}h^{\mu}{}_{\beta} + \left(\frac{1}{8}h^2 - \frac{1}{4}h^{\alpha\beta}h_{\alpha\beta}\right)\eta_{\alpha\beta} \\ &\quad +\mathcal{O}\left(G^3\right) . \end{aligned} \tag{3.47}$$

Let us analyze the (inverse) spacetime metrics Eqs. (3.46) an (3.47) more closely. The good news is that we do not need to calculate all ten spacetime potentials in the same order. Depending on the physical problem one wants to study, the component g_{00} might be needed to higher order accuracy than some spatial component g_{ij}. In this application, we switch from the formal post-Minkowskian expansion in G to the physical post-Newtonian expansion parameter already indicated in Eq. (3.26)

$$\varepsilon \sim \frac{Gm_c}{c^2r_c} . \tag{3.48}$$

This also shows why PN approximations are sometimes given in orders of $1/c$.

However, we stick to ε since we will work in geometrized units later on and be consistent with previous literature. A crucial assumption needed to further the investigation is called *slow-motion condition.* This entails that the physical objects move slowly compared to the speed of light in a way such that $\varepsilon \sim v_c^2$ for the characteristic velocity v_c. Keeping this in mind, the factor

$$\left(-g_{00} - 2g_{0i}v^i - g_{ij}v^iv^j\right)^{1/2} \tag{3.49}$$

the standard Lagrangian for the equation of motion reveals the order in which the metric components need to be evaluated. To obtain the equation of motion to our desired 2.5 PN order, we need g_{00} to $\varepsilon^{7/2}$ but g_{0i} only to ε^3 and g_{ij} to $\varepsilon^{5/2}$ due to the multiplication with the velocities $v^i \sim \sqrt{\varepsilon}$ and $v^iv^j \sim \varepsilon$, respectively. So, in total,

$$g_{00} \sim \mathcal{O}\left(\varepsilon^{7/2}\right) , \tag{3.50}$$

$$g_{0j} \sim \mathcal{O}\left(\varepsilon^{3}\right) , \tag{3.51}$$

$$g_{ij} \sim \mathcal{O}\left(\varepsilon^{5/2}\right) . \tag{3.52}$$

Combining the fact above with the expanded spacetime metric Eq. (3.47), we obtain

$$\begin{aligned} g_{00} &= -\left(1 - \frac{1}{2}h^{00} + \frac{3}{8}\left(h^{00}\right)^2 - \frac{5}{16}\left(h^{00}\right)^3\right) + \frac{1}{2}h^k{}_k\left(1 - \frac{1}{2}h^{00}\right) + \frac{1}{2}h^{0k}h_{0k} \\ &\quad + \mathcal{O}\left(\varepsilon^4\right) , &(3.53\text{a}) \\ g_{0i} &= -h^{0i}\left(1 - \frac{1}{2}h^{00}\right) + \mathcal{O}\left(\varepsilon^{7/2}\right) , &(3.53\text{b}) \\ g_{ij} &= \delta^{ij}\left(1 + \frac{1}{2}h^{00} - \frac{1}{8}\left(h^{00}\right)^2\right) + h^{ij} - \frac{1}{2}h^k{}_k\delta^{ij} + \mathcal{O}\left(\varepsilon^3\right) , &(3.53\text{c}) \\ (-g) &= 1 + h^{00} - h^k{}_k + \mathcal{O}\left(\varepsilon^3\right) . &(3.53\text{d}) \end{aligned}$$

We can analyze how and where $h^{\alpha\beta}$ components enter the spacetime metric (3.53). We find that the lapse h^{00} enters all metric fields at first post-Newtonian order despite g_{0i} where it enters at $\varepsilon^{5/2}$. Next, despite the obvious contribution to g_{0i}, the field h^{0j} also enters at ε^3 in g_{00}. Hence, for the equations of motion to 1.5 PN order, h^{0j} does not contribute to the spacetime metric field g_{00}. Last, the spatial part h^{ij} enters twice only as a trace, namely in g_{00} and g_{ij}, while the full spatial part also appears in the latter. Note that the spatial part does not enter g_{ij} for analysis to 1.5 PN order making it a second-order correction term.

Chapter 4

Equations of Motion to 2.5 Post-Newtonian Order

Parts of our work in this chapter are based on the publication „Tensor-multiscalar gravity: Equations of motion to 2.5 post-Newtonian order“, in Phys. Rev. D, 105.064034[1] *[19] by O. Schön, and D. D. Doneva. Please refer to our Contribution Statement at the beginning of this dissertation for more information.*

In this chapter, we combine all the ingredients discussed in the previous chapter to calculate the main focus of our analysis: The motion of compact bodies in Tensor Multi-Scalar Theories of gravity. More precisely, we take the field equations (2.13)-(2.14) and bring them into the relaxed form introduced in Sec 3.2. Then, we iterate via the DIRE process to obtain a spacetime metric which we will use to calculate the Christoffel symbols accurately enough to obtain the equations of motion through 2.5 PN order finally.

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4.1 Adapting DIRE to TMST

At first, we collect all scalar field-related terms in (2.13) via

$$\frac{16\pi G_\star}{(-g)} t^{\varphi}_{\alpha\beta} := -2 g_{\alpha\beta}\gamma_{ab}(\varphi) g^{\mu\nu}\nabla_\mu\varphi^a\,\nabla_\nu\varphi^b + 4\gamma_{ab}(\varphi)\nabla_\alpha\varphi^a\,\nabla_\beta\varphi^b - 4V(\varphi)g_{\alpha\beta}\,, \tag{4.1}$$

and we rewrite (2.13) as

$$G_{\alpha\beta} = 8\pi G_\star T_{\alpha\beta} + \frac{8\pi G_\star}{(-g)} t^{\varphi}_{\alpha\beta}\,. \tag{4.2}$$

The equation above visualizes the difference to pure GR quite directly as the additional terms are all encoded in $t^{\varphi}_{\alpha\beta}$. Thanks to this specific form, we can insert this Eq. (4.2) into our newly defined field equation (3.3) and obtain

$$\partial_\mu\partial_\nu H^{\alpha\mu\beta\nu} = 16\pi G_\star(-g)\left(T^{\alpha\beta} + t^{\alpha\beta}_{LL} + t^{\alpha\beta}_{\varphi}\right)\,. \tag{4.3}$$

Let us take a closer look at the right-hand side. The physical quantities are, as usual, incorporated in the energy-momentum tensor $T^{\alpha\beta}$. As before, the difference to GR is the inclusion of the field $t^{\alpha\beta}_{\varphi}$. Note that while Eq. (4.3) looks algebraically identical to the single scalar field case in [15], the difference is hidden in the definition of $t^{\alpha\beta}_{\varphi}$ as all the multiple scalar fields are contracted in there.

Until now, all manipulations done in this section can be seen as equivalent to the theory defined by the field equation (2.13), and we have yet to focus our attention on the extra field equations (2.14). The following steps explicitly show the advantage of this Landau-Lifshitz formulation of gravity. The goal is to isolate the spacetime metric potentials and the multiple scalar fields in a flat wave equation. Once we succeed, we can fully use PDE theory and solve for those desired fields.

To proceed, we again impose the harmonic coordinate (gauge) conditions $\partial_\beta \mathfrak{g}^{\alpha\beta} = 0$ and define the potentials

$$h^{\alpha\beta} := \eta^{\alpha\beta} - \mathfrak{g}^{\alpha\beta} \tag{4.4}$$

with inverse Minkowski metric $\eta^{\alpha\beta}$ in Lorentzian coordinates $(t := x^0, x^j)$ as we did in Eq. (3.8). The fields $h^{\alpha\beta}$ are, next to the scalar fields, the main focus of the rest of this chapter. Together, they are the unknowns in the wave equations we will derive below.

Using all this, the left-hand side of (4.3) becomes

$$\partial_\mu \partial_\nu H^{\alpha\mu\beta\nu} = -\Box h^{\alpha\beta} - 16\pi G_\star (-g) t_H^{\alpha\beta} \,, \tag{4.5}$$

for the usual harmonic-gauge pseudotensor

$$16\pi G_\star (-g) t_H^{\alpha\beta} := \partial_\mu h^{\alpha\nu} \partial_\nu h^{\beta\mu} - h^{\mu\nu} \partial_\mu \partial_\nu h^{\alpha\beta} \tag{4.6}$$

and for the Minkowskian wave operator $\Box := \eta^{\mu\nu}\partial_\mu\partial_\nu$. The right-hand side of (4.3) also simplifies as some terms in the Landau-Lifshitz pseudotensor (3.4) vanish due to our chosen gauge. Now, substituting the identity (4.3) into Eq. (3.9) and isolating the box operator to the left-hand side, we arrive at the flat wave equation

$$\Box h^{\alpha\beta} = -16\pi G_\star \tau^{\alpha\beta} \,, \tag{4.7}$$

where the source

$$\tau^{\alpha\beta} := (-g)\left(T^{\alpha\beta} + t_{LL}^{\alpha\beta} + t_H^{\alpha\beta} + t_\varphi^{\alpha\beta}\right) \tag{4.8}$$

plays the role of an effective energy-momentum pseudotensor. Again, compared to GR, this source has the additional term $t_\varphi^{\alpha\beta}$ encoding the multiple scalar field contributions. In addition, similar to the conservation of energy-momentum $\widetilde{\nabla}_\beta \widetilde{T}^{\alpha\beta} = 0$ in the physical Jordan frame, our conformal effective source necessarily obeys an equivalent conservation law in

$$\partial_\beta \tau^{\alpha\beta} = 0\,. \tag{4.9}$$

The crucial conceptual difference in those two described conservation laws is that the first one is fundamental in the sense that this should be true in any viable theory. The latter is a consequence of assuming our field equations to be fulfilled.

The second part of the field equations, Eq. (2.14), already has the form of a wave equation with respect to a curved metric. We can, however, transform it into a flat wave equation via

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi^a = \frac{1}{\sqrt{-g}}\partial_\mu\left(\mathfrak{g}^{\mu\nu}\partial_\nu\varphi^a\right) = \frac{1}{\sqrt{-g}}\left(\Box\varphi^a - h^{\alpha\beta}\partial_\alpha\partial_\beta\varphi^a\right)\,. \tag{4.10}$$

Hence, Eq. (2.14) can be brought into the form

$$\Box\varphi^a = -8\pi G_\star \tau_\varphi^a\,, \tag{4.11}$$

where the source is given as

$$\begin{aligned}\tau_\varphi^a \;:=\; & -\frac{\sqrt{-g}}{8\pi G_\star}\left[-\gamma^a_{bc}(\varphi) g^{\alpha\beta}\,\nabla_\alpha\varphi^b\,\nabla_\beta\varphi^c + \gamma^{ab}(\varphi)\frac{\partial V(\varphi)}{\partial\varphi^b} - 4\pi G_\star\gamma^{ab}\alpha_b(\varphi)T\right] \\ & -\frac{1}{8\pi G_\star}h^{\alpha\beta}\partial_\alpha\partial_\beta\varphi^a\,. \end{aligned} \tag{4.12}$$

The difference between TMST to STT becomes more evident in the source term

(4.12). For each scalar field φ^a, the target space metric γ^{ab} and its associated Christoffel symbols γ^a_{bc} contribute differently to the source. Hence, the curvature of the n-dimensional Riemannian target manifold (T^n, γ_{ab}) directly decides the source term difference. Also, one may introduce symmetry conditions to the metric γ_{ab} to reduce certain degrees of freedom.

In total, we have a system of wave equations

$$
\begin{aligned}
\Box h^{\alpha\beta} &= -16\pi G_\star \tau^{\alpha\beta}\,, &(4.13)\\
\Box \varphi^a &= -8\pi G_\star \tau^a_\varphi\,. &(4.14)
\end{aligned}
$$

These equations, in the absence of any coordinate conditions, are referred to as the relaxed Einstein field equations, or, more accurately, in our case, relaxed Tensor Multi-Scalar Theory field equations. This paper aims to solve these two entangled partial differential equations in the near-zone. The formal solutions are given by the standard retarded Green functions

$$
\begin{aligned}
h^{\alpha\beta}(t,\boldsymbol{x}) &= 4\int \frac{\tau^{\alpha\beta}\left(t-|\boldsymbol{x}-\boldsymbol{x}'|,\boldsymbol{x}'\right)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x'\,, &(4.15)\\
\varphi^a(t,\boldsymbol{x}) &= 2\int \frac{\tau^a_\varphi\left(t-|\boldsymbol{x}-\boldsymbol{x}'|,\boldsymbol{x}'\right)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x'\,. &(4.16)
\end{aligned}
$$

These will be calculated iteratively with the framework DIRE as explained in [11, 44]. That is, the integrals will be expanded, incorporating a slow-motion and weak-field assumption in terms of a small parameter $\varepsilon \sim v^2 \sim G_\star m/r$ for the characteristic mass m, size r, and velocity v of the physical objects we are interested in.

Now that we have defined the main equations we want to study further and solve in (4.13) and (4.14), it is worthwhile to think about the whole setting we find ourselves in. Various fields govern Tensor Multi-Scalar Theory in the Einstein frame. Besides the natural spacetime metric $g_{\alpha\beta}$ and the scalar fields φ^a, we have the conformal factor $A^2(\varphi)$, the target space metric $\gamma_{ab}(\varphi)$, the scalar field potential $V(\varphi)$, and, of course, the energy-momentum tensor $T_{\alpha\beta}$. The unknowns of our system (4.13) and (4.14) are the tuple $(g_{\alpha\beta}, \varphi^a)$, so for a four-dimensional spacetime with n scalar fields that is $10 + n$ total independent fields. The initial data we need to provide therefore $[A^2(\varphi), \gamma_{ab}(\varphi), V(\varphi), T_{\alpha\beta}]$ consisting of $1 + n(n+1)/2 + 1 + 10 = n(n+1)/2 + 12$ independent fields fully defining our approach.

To close this section, we set ourselves up to utilize previous results. Following [11, 15], we define the quantities

$$\Lambda^{\alpha\beta} := 16\pi G_\star(-g)\left(t^{\alpha\beta}_{LL} + t^{\alpha\beta}_{H}\right), \tag{4.17}$$

$$\Lambda^{\alpha\beta}_{\varphi} := 16\pi G_\star(-g)t^{\alpha\beta}_{\varphi}, \tag{4.18}$$

to rewrite the metric-potential source in (4.13) and obtain

$$16\pi G_\star \tau^{\alpha\beta} = 16\pi G_\star(-g)T^{\alpha\beta} + \Lambda^{\alpha\beta} + \Lambda^{\alpha\beta}_{\varphi}. \tag{4.19}$$

This equation mimics the formulas in [11] and [15]. Hence, the fields defined in (4.17) and (4.18) have the same algebraic structure as in GR and single STT and can be used in our analysis.

For the remaining work of this dissertation, we restrict ourselves to a class of TMST with vanishing potential of the scalar fields $V(\varphi) = 0$. Furthermore, we adapt geometric coordinate units to set the bare gravitational constant $G_\star = 1$.

4.2 Formal structure of the near-zone fields

At this stage, we are ready to calculate the general form of the near-zone metric. This forms the basis of the post-Newtonian theory analyzed in the presented work.

Out of convenience, we assign less indices-crowded notation (as in [10, 15])

$$N := h^{00}\,, \qquad K^j := h^{0j}\,, \qquad B^{ij} := h^{ij}\,, \qquad B^i{}_i := B\,. \tag{4.20}$$

These fields inherit their leading order in ε from the energy-momentum tensor in Eq. (4.13). The exact form of the matter model we are using will be discussed in more detail in Section 4.4, but for now, it is sufficient to know that there exists a hierarchy of the form $T^{0i}/T^{00} \sim \sqrt{\varepsilon}\, T^{ij}/T^{00}$. Using this fact together with Eq. (4.13) yields $h^{0i}/h^{00} \sim \sqrt{\varepsilon}\, h^{ij}/h^{00}$. These physically meaningful relationships can be written more handily via the shortcuts

$$N \sim \mathcal{O}(\varepsilon)\,, \qquad K^j \sim \mathcal{O}\left(\varepsilon^{3/2}\right)\,, \qquad B^{ij} \sim \mathcal{O}\left(\varepsilon^2\right)\,, \qquad B \sim \mathcal{O}\left(\varepsilon^2\right)\,. \tag{4.21}$$

We can now calculate the expansion of our Einstein frame metric in terms of the potentials (4.20) using the leading orders in (4.21). Utilizing (3.8) to get an expansion for the gothic inverse metric $\mathfrak{g}^{\alpha\beta}$ allows us to calculate the inverse spacetime metric $g^{\alpha\beta}$ via (3.1). Finally, inverting this metric perturbatively yields

$$\begin{aligned} g_{00} &= -\left(1 - \frac{1}{2}N + \frac{3}{8}N^2 - \frac{5}{16}N^3\right) + \frac{1}{2}B\left(1 - \frac{1}{2}N\right) + \frac{1}{2}K^jK^j \\ &\quad + \mathcal{O}\left(\varepsilon^4\right)\,, && \text{(4.22a)} \\ g_{0i} &= -K^i\left(1 - \frac{1}{2}N\right) + \mathcal{O}\left(\varepsilon^{7/2}\right)\,, && \text{(4.22b)} \end{aligned}$$

$$g_{ij} = \delta^{ij}\left(1+\frac{1}{2}N-\frac{1}{8}N^2\right)+B^{ij}-\frac{1}{2}B\delta^{ij}+\mathcal{O}\left(\varepsilon^3\right)\,, \tag{4.22c}$$

$$(-g) = 1+N-B+\mathcal{O}\left(\varepsilon^3\right)\,, \tag{4.22d}$$

which is the same as calculated in pure GR in Eqs. (3.53).

The fields' covariant and contravariant components, such as K^i or B^{ij}, are naturally interchangeable since the spatial metric to raise and lower those indices is δ^{ij}. Note that these potentials do not depend explicitly on the extra scalar fields (instead via the $h^{\alpha\beta}$ fields) as in the single scalar field case in [15] since we stay in the conformal Einstein frame. In contrast, the cited paper converts to the physical Jordan frame. If one collapses all equations down to one scalar field, we can yield the relevant fields in the Einstein frame, which is an auxiliary new result of this dissertation.

Following the convention in [21] and the work built on top of it, we define underneath quantities from the energy-momentum tensor $T^{\alpha\beta}$

$$\sigma := T^{00}+T^{ii}\,, \tag{4.23a}$$

$$\sigma^i := T^{0i}\,, \tag{4.23b}$$

$$\sigma^{ij} := T^{ij}\,, \tag{4.23c}$$

$$\sigma^a_\varphi := \alpha^a(\varphi)\,T\,. \tag{4.23d}$$

These densities will aid in expanding the sources of the wave equations (4.13) and (4.14). In contrast to GR and similar to STT, the field σ^a_φ is added. It is coupled to the matter model via $\alpha^a(\varphi)$ defined in (2.17). Hence, the conformal factor $A^2(\varphi)$ contributes here.

As mentioned earlier, the source fields (4.17) and (4.18) are algebraic equivalent with GR; hence, those fields to the required order are given in [10] as

$$\begin{aligned}
\Lambda^{00} &= -\frac{7}{8}(\nabla N)^2 + \left\{\frac{5}{8}\dot{N}^2 - \ddot{N}N - 2\dot{N}^{,k}K^k + \frac{1}{2}K^{i,j}\left(3K^{j,i} + K^{i,j}\right)\right. \\
&\quad \left. + \dot{K}^j N^{,j} - B^{ij}N^{,ij} + \frac{1}{4}\nabla N\cdot\nabla B + \frac{7}{8}N(\nabla N)^2\right\} + \mathcal{O}(\rho\varepsilon^3)\,, \quad (4.24a) \\
\Lambda^{0i} &= \left\{N^{,k}(K^{k,i} - K^{i,k}) + \frac{3}{4}\dot{N}N^{,i}\right\} + \mathcal{O}\left(\rho\varepsilon^{5/2}\right)\,, \quad (4.24b) \\
\Lambda^{ij} &= \frac{1}{4}\left\{N^{,i}N^{,j} - \frac{1}{2}\delta^{ij}(\nabla N)^2\right\} + \left\{2K^{k,(i}K^{j),k} - K^{k,i}K^{k,j} - K^{i,k}K^{j,k}\right. \\
&\quad + 2N^{,(i}\dot{K}^{j)} + \frac{1}{2}N^{,(i}B^{,j)} - \frac{1}{2}N\left(N^{,i}N^{,j} - \frac{1}{2}\delta^{ij}(\nabla N)^2\right) \\
&\quad \left. -\delta^{ij}\left(K^{l,k}K^{[k,l]} + N^{,k}\dot{K}^k + \frac{3}{8}\dot{N}^2 + \frac{1}{4}\nabla N\cdot\nabla B\right)\right\} + \mathcal{O}(\rho\varepsilon^3)\,, \quad (4.24c) \\
\Lambda^{ii} &= -\frac{1}{8}(\nabla N)^2 + \left\{K^{l,k}K^{[k,l]} - N^{,k}\dot{K}^k - \frac{1}{4}\nabla N\cdot\nabla B\right. \\
&\quad \left. -\frac{9}{8}\dot{N}^2 + \frac{1}{4}N(\nabla N)^2\right\} + \mathcal{O}(\rho\varepsilon^3)\,. \quad (4.24d)
\end{aligned}$$

To ease the reading, we employ the following notation: Parentheses denote the symmetrization of a tensor with respect to those indices. In contrast, square brackets denote the antisymmetrization of a tensor with respect to those indices. A comma represents a partial derivative with respect to the spatial coordinate, while a dot indicates a time derivative. Time derivatives of order three or higher will be denoted as a number in parentheses over the field.

To expand the extra source terms for the multiple scalar fields, we rely on an asymptotic expansion in terms of φ^a around φ^a_∞, the cosmological values of the multiple scalar fields. For the target space metric $\gamma_{ab} = \gamma_{ab}(\varphi)$, this means

$$\gamma_{ab}(\varphi) = \gamma_{ab}(\varphi_\infty) + \left.\frac{\partial\gamma_{ab}(\varphi)}{\partial\varphi^c}\right|_{\varphi_\infty}(\varphi^c - \varphi^c_\infty) + \mathcal{O}(\varphi^2)\,. \quad (4.25)$$

Now, in order not to overcrowd our notations, we understand every occurrence of

γ_{ab} as asymptotically evaluated, such that

$$\gamma_{ab} \equiv \gamma_{ab}(\varphi_\infty)\,, \qquad \gamma_{ab,c} \equiv \left.\frac{\partial \gamma_{ab}(\varphi)}{\partial \varphi^c}\right|_{\varphi_\infty}\,. \tag{4.26}$$

Using the same techniques for the Christoffel symbols $\gamma^a_{bc} = \gamma^a_{bc}(\varphi)$, we obtain

$$\gamma^a_{bc}(\varphi) = \gamma^a_{bc}(\varphi_\infty) + \left.\frac{\partial \gamma^a_{bc}(\varphi)}{\partial \varphi^d}\right|_{\varphi_\infty} \left(\varphi^d - \varphi^d_\infty\right) + \mathcal{O}(\varphi^2)\,, \tag{4.27}$$

where it is from now on again understood that

$$\gamma^a_{bc} \equiv \gamma^a_{bc}(\varphi_\infty)\,, \qquad \gamma^a_{bc,d} \equiv \left.\frac{\partial \gamma^a_{bc}(\varphi)}{\partial \varphi^d}\right|_{\varphi_\infty}\,. \tag{4.28}$$

Without loss of generality in what follows, we can assume that the cosmological value of the scalar field is zero similar to [21], i.e., $\varphi^a_\infty = 0$. We will, however, in contrast to [21], not make any further simplifications by choosing specific coordinates for the target space (T^n, γ_{ab}). In their analysis, field coordinates were selected to be asymptotically geodesic; that is, the cosmological value φ_∞ let the Christoffel symbols vanish, i.e., $\gamma^a_{bc}(\varphi_\infty) \equiv 0$. By keeping the coordinates general ourselves, we can identify specific spots where the geometry of the target space contributes via these Christoffel symbols. The goal is to gain some insight into the physical meaning of the target space and its form.

Keeping in mind that $\partial_t \sim \sqrt{\varepsilon}\,\nabla$, we can calculate the expanded scalar field source terms to be

$$\begin{aligned}\Lambda^{00}_\varphi \;=\;& \left\{2\gamma_{ab}\delta^{ij}\varphi^{a,i}\varphi^{b,j}\right\} + \left\{4\gamma_{ab}N\delta^{ij}\varphi^{a,i}\varphi^{b,j} + 2\gamma_{ab}\dot{\varphi}^a\dot{\varphi}^b + 2\gamma_{ab,c}\delta^{ij}\varphi^{a,i}\varphi^{b,j}\varphi^c\right\}\\ &+\mathcal{O}\left(\rho\varepsilon^3\right)\,,\end{aligned} \tag{4.29a}$$

$$\Lambda^{0i}_{\varphi} = -4\gamma_{ab}\dot{\varphi}^a\varphi^{b,i} + \mathcal{O}\left(\rho\varepsilon^{5/2}\right) , \tag{4.29b}$$

$$\begin{aligned}\Lambda^{ij}_{\varphi} = &\; 2\left\{4\gamma_{ab}\varphi^{a,i}\varphi^{b,j} + 2\gamma_{ab}\delta^{ij}\delta^{kl}\varphi^{a,k}\varphi^{b,l}\right\} + 2\left\{2\gamma_{ab}N\varphi^{a,i}\varphi^{b,j}\right.\\ &+\gamma_{ab}\delta^{ij}\left(N\delta^{kl}\varphi^{a,k}\varphi^{b,l} + \dot{\varphi}^a\dot{\varphi}^b\right) 2\gamma_{ab,c}\,\varphi^{a,i}\varphi^{b,j}\varphi^c\\ &\left.+\gamma_{ab,c}\,\delta^{ij}\delta^{kl}\varphi^{a,k}\varphi^{b,l}\varphi^c\right\} + \mathcal{O}\left(\rho\varepsilon^3\right) ,\end{aligned} \tag{4.29c}$$

$$\begin{aligned}\Lambda^{ii}_{\varphi} = &\; \left\{10\gamma_{ab}\delta^{ij}\varphi^{a,i}\varphi^{b,j}\right\} + 10\left\{\gamma_{ab}N\delta^{ij}\varphi^{a,i}\varphi^{b,j} + \gamma_{ab}\dot{\varphi}^a\dot{\varphi}^b + \gamma_{ab,c}\,\delta^{ij}\varphi^{a,i}\varphi^{b,j}\varphi^c\right\}\\ &+\mathcal{O}\left(\rho\varepsilon^3\right) .\end{aligned} \tag{4.29d}$$

The source of the scalar fields wave equations (4.12) then can be expanded as

$$\begin{aligned}\tau^a_{\varphi} = &\; \frac{1}{2}\sigma^a_{\varphi} + \left\{-\frac{1}{4}N\sigma^a_{\varphi} + \frac{1}{8\pi}\gamma^a_{bc}\delta^{ij}\varphi^{b,i}\varphi^{c,j}\right\}\\ &+\left\{\frac{1}{8}N^2\sigma^a_{\varphi} + \frac{1}{8\pi}\left(-N\ddot{\varphi}^a - 2\dot{\varphi}^{a,k}K^k - \varphi^{a,ij}\,B^{ij} - \gamma^a_{bc}\dot{\varphi}^b\dot{\varphi}^c\right.\right.\\ &\left.\left.+\gamma^a_{bc,d}\,\delta^{ij}\varphi^{b,i}\varphi^{c,j}\varphi^d\right)\right\} + \mathcal{O}\left(\rho\varepsilon^3\right) .\end{aligned} \tag{4.30}$$

Again, we emphasize the difference to STT by examining Eqs. (4.29) and (4.30) more closely. The target space metric is directly involved in contracting the scalar field indices; hence, the target space's geometry is directly involved here. Besides the obvious contribution of the Christoffel symbols γ^a_{bc} in the expanded source (4.30), one can also notice the direct contribution of the conformal factor $A(\varphi)$ in σ^a_{φ}. As the derivative of $A(\varphi)$ in direction φ^a is contributing, and the Einstein frame in TMST gives the freedom to choose a conformal factor $A(\varphi)$, one can see that different scalar fields might behave vastly different, depending on their dependency in $A(\varphi)$.

We continue to follow the process of DIRE and give the formal near-zone expansions of the retarded Green functions. The integration domain is a bounded time-slice: a spatial hypersurface $\mathcal{M}$ at a constant time t bounded by a world-tube of radius $\mathcal{R}$. This radius naturally bounds the near-zone from the far-zone. We can disregard all near-zone potentials that depend on this auxiliary cut-off parameter $\mathcal{R}$ since they need to cancel out with their counterparts from the far-zone. Furthermore, the equations of the formal near-zone expansions of the retarded Green functions are again algebraically the same as in [15] due to the inheritance of the conservation law shown in Eq. (4.9) adapted for TMST. Knowing this, the expansions to the order we need are given as

$$\begin{aligned}
N_{\mathcal{N}} &= 4\varepsilon \int_{\mathcal{M}} \frac{\tau^{00}(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x' + 2\varepsilon^2\partial_t^2 \int_{\mathcal{M}} \tau^{00}(t,\boldsymbol{x}')|\boldsymbol{x}-\boldsymbol{x}'|\,\mathrm{d}^3x' - \frac{2}{3}\varepsilon^{5/2}\,\overset{(3)}{\mathcal{I}}{}^{kk}(t) \\
&\quad + \frac{1}{6}\varepsilon^3\partial_t^4 \int_{\mathcal{M}} \tau^{00}(t,\boldsymbol{x}')|\boldsymbol{x}-\boldsymbol{x}'|^3\,\mathrm{d}^3x' - \frac{1}{30}\varepsilon^{7/2}\left\{\left((4x^{kl}+2r^2\delta^{kl}\right)\overset{(5)}{\mathcal{I}}{}^{kl}(t)\right. \\
&\quad \left. -4x^k\,\overset{(5)}{\mathcal{I}}{}^{kll}(t) + \overset{(5)}{\mathcal{I}}{}^{kkll}(t)\right\} + N_{\partial\mathcal{M}} + \mathcal{O}(\varepsilon^4)\,, \qquad (4.31a)
\end{aligned}$$

$$\begin{aligned}
K^i_{\mathcal{N}} &= 4\varepsilon^{3/2} \int_{\mathcal{M}} \frac{\tau^{0i}(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x' + 2\varepsilon^{5/2}\partial_t^2 \int_{\mathcal{M}} \tau^{0i}(t,\boldsymbol{x}')|\boldsymbol{x}-\boldsymbol{x}'|\,\mathrm{d}^3x' \\
&\quad + \frac{2}{9}\varepsilon^3\left\{3x^k\,\overset{(4)}{\mathcal{I}}{}^{ik}(t) - \overset{(4)}{\mathcal{I}}{}^{ikk}(t) + 2\varepsilon^{mik}\,\overset{(3)}{\mathcal{J}}{}^{mk}(t)\right\} \\
&\quad + K^i_{\partial\mathcal{M}} + \mathcal{O}\left(\varepsilon^{7/2}\right)\,, \qquad (4.31b)
\end{aligned}$$

$$\begin{aligned}
B^{ij}_{\mathcal{N}} &= 4\varepsilon^2 \int_{\mathcal{M}} \frac{\tau^{ij}(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x' - 2\varepsilon^{5/2}\,\overset{(3)}{\mathcal{I}}{}^{ij}(t) + 2\varepsilon^3\partial_t^2 \int_{\mathcal{M}} \tau^{ij}(t,\boldsymbol{x}')|\boldsymbol{x}-\boldsymbol{x}'|\,\mathrm{d}^3x' \\
&\quad - \frac{1}{9}\varepsilon^{7/2}\left\{3r^2\,\overset{(5)}{\mathcal{I}}{}^{ij}(t) - 2x^k\,\overset{(5)}{\mathcal{I}}{}^{ijk}(t) - 8x^k\varepsilon^{mki}\,\overset{(4)}{\mathcal{J}}{}^{m|j}(t) + 6\,\overset{(3)}{M}{}^{ijkk}(t)\right\} \\
&\quad + B^{ij}_{\partial\mathcal{M}} + \mathcal{O}(\varepsilon^4)\,, \qquad (4.31c)
\end{aligned}$$

$$\begin{aligned}
\varphi^a_{\mathcal{N}} &= 2\varepsilon \int_{\mathcal{M}} \frac{\tau^a_\varphi(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x' - 2\varepsilon^{3/2}\dot{M}^a_\varphi + \varepsilon^2\partial_t^2 \int_{\mathcal{M}} \tau^a_\varphi(t,\boldsymbol{x}')|\boldsymbol{x}-\boldsymbol{x}'|\,\mathrm{d}^3x' \\
&\quad - \frac{1}{3}\varepsilon^{5/2}\left\{r^2\,\overset{(3)}{M}{}^a_\varphi(t) - 2x^j\,{}^a\overset{(3)}{\mathcal{I}}{}^j_\varphi(t) + {}^a\overset{(3)}{\mathcal{I}}{}^{kk}_\varphi(t)\right\}
\end{aligned}$$

$$+\frac{1}{12}\varepsilon^3\partial_t^4\int_{\mathcal{M}}\tau_\varphi^a(t,\boldsymbol{x}')|\boldsymbol{x}-\boldsymbol{x}'|^3\,\mathrm{d}^3x' - \frac{1}{60}\varepsilon^{7/2}\Bigg\{r^4\ \overset{(5)}{M_\varphi^a}(t) - 4r^2x^j\ \overset{(5)}{{}^a\mathcal{I}_\varphi^j}(t)$$
$$+\left(4x^{kl}+2r^2\delta^{kl}\right)\overset{(5)}{{}^a\mathcal{I}_\varphi^{kl}}(t) - 4x^k\ \overset{(5)}{{}^a\mathcal{I}_\varphi^{kll}}(t) + \overset{(5)}{{}^a\mathcal{I}_\varphi^{kkll}}(t)\Bigg\}$$
$$+\mathcal{O}(\varepsilon^4)\,. \tag{4.31d}$$

The key difference with STT in [15] here is the adaption of the expansion to multiple scalar fields φ^a in the last equation. To present this expansion in a readable manner, we made use of the momentum already employed in [15] and adapted it to our needs:

$$\mathcal{I}^Q := \int_{\mathcal{M}}\tau^{00}x^Q\,\mathrm{d}^3x\,, \tag{4.32a}$$
$$\mathcal{J}^{iQ} := \varepsilon^{ikl}\int_{\mathcal{M}}\tau^{0l}x^{kQ}\,\mathrm{d}^3x\,, \tag{4.32b}$$
$$M^{ijQ} := \int_{\mathcal{M}}\tau^{ij}x^Q\,\mathrm{d}^3x\,, \tag{4.32c}$$
$${}^a\mathcal{I}_\varphi^Q := \int_{\mathcal{M}}\tau_\varphi^a x^Q\,\mathrm{d}^3x\,, \tag{4.32d}$$
$$M_\varphi^a := \int_{\mathcal{M}}\tau_\varphi^a\,\mathrm{d}^3x\,. \tag{4.32e}$$

Here, Q is understood as a multi-index in the following sense: Take, as an example, the scalar dipole moments ${}^a\mathcal{I}_\varphi^j(t)$. For those we have $Q=j$ and

$${}^a\mathcal{I}_\varphi^Q(t) = {}^a\mathcal{I}_\varphi^j(t) = \int_{\mathcal{M}}\tau_\varphi^a x^j\,\mathrm{d}^3x\,. \tag{4.33}$$

As in the single scalar field scenario, the boundary terms $N_{\partial M}$, $K^i_{\partial\mathcal{M}}$ and $B^{ij}_{\partial\mathcal{M}}$ have no effect for the order we are interested in. They are, however, given in Appendix C in [10] and will have the same algebraic form for TMST.

The near-zone expansions in (4.31) are essentially an expansion in terms of time derivatives and powers of $|\boldsymbol{x}-\boldsymbol{x}'|$ where we integrate out $\boldsymbol{x}'$ over the previously explained set $\mathcal{M}$. This expansion is suitable since for any event $(t, \boldsymbol{x})$ in the near-zone, the difference $|\boldsymbol{x} - \boldsymbol{x}'|$ is small or, more precisely, $|\boldsymbol{x} - \boldsymbol{x}'| < 2\mathcal{R}$ for the above introduced cut-off radius $\mathcal{R}$. Examining these equations a bit more, we notice that first-time derivatives are missing in, for example, the near-zone expansion of $N_{\mathcal{N}} = h^{00}_{\mathcal{N}}$. This is a direct consequence of the conservation law (4.9) together with Gauss's theorem. However, no such conservation law exists for the extra multiple scalar fields φ^a; hence, we have a 1.5 PN contribution term in the expansion (4.31d). In the next section, we deal with that fact more closely in calculating the potential.

The potentials resulting from integrating the source terms via Eq. (4.31) will be Poisson-like. We follow the notation of [10] further and generalize to multiscalar potentials when appropriate. We then get for any source f the Poisson potential

$$P(f) := \frac{1}{4\pi} \int_{\mathcal{M}} \frac{f(t, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \, \mathrm{d}^3x' \,, \quad \nabla^2 P(f) = -f \,. \tag{4.34}$$

The fields stemming from the energy-matter distribution and hence the source of the wave-equations σ, σ^i, σ^{ij}, and σ^a_φ inherit potentials such as

$$\begin{aligned}
\Sigma(f) &:= \int_{\mathcal{M}} \frac{\sigma(t, \boldsymbol{x}') f(t, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \, \mathrm{d}^3x' = P(4\pi\sigma f) \,, && \text{(4.35a)} \\
\Sigma^i(f) &:= \int_{\mathcal{M}} \frac{\sigma^i(t, \boldsymbol{x}') f(t, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \, \mathrm{d}^3x' = P(4\pi\sigma^i f) \,, && \text{(4.35b)} \\
\Sigma^{ij}(f) &:= \int_{\mathcal{M}} \frac{\sigma^{ij}(t, \boldsymbol{x}') f(t, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \, \mathrm{d}^3x' = P(4\pi\sigma^{ij} f) \,, && \text{(4.35c)} \\
\Sigma^a_\varphi(f) &:= \int_{\mathcal{M}} \frac{\sigma^a_\varphi(t, \boldsymbol{x}') f(t, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \, \mathrm{d}^3x' = P(4\pi\sigma^a_\varphi f) \,, && \text{(4.35d)}
\end{aligned}$$

where we added the theory specific Σ^a_φ stemming from the source σ^a_φ. Integrating a source against higher powers of $|\boldsymbol{x} - \boldsymbol{x}'|^{-1}$, i.e. $|\boldsymbol{x} - \boldsymbol{x}'|$, $|\boldsymbol{x} - \boldsymbol{x}'|^3$, are commonly referred to as superpotentials [15]. For these, we introduce the notation

$$
\begin{aligned}
X(f) &:= \int_{\mathcal{M}} \sigma(t,\boldsymbol{x}')f(t,\boldsymbol{x}')|\boldsymbol{x}-\boldsymbol{x}'|\,\mathrm{d}^3x'\,, && (4.36\text{a})\\
Y(f) &:= \int_{\mathcal{M}} \sigma(t,\boldsymbol{x}')f(t,\boldsymbol{x}')|\boldsymbol{x}-\boldsymbol{x}'|^3\,\mathrm{d}^3x'\,, && (4.36\text{b})
\end{aligned}
$$

and likewise their counterparts and generalizations X^i, X^a_φ and analogs for Y.

To improve readability and reduce long expressions we introduce similar definitions as in [10] and again adapt them to our generalized formulation. The most often used potentials are the *Newtonian-like* constructions

$$
\begin{aligned}
U &:= \int_{\mathcal{M}} \frac{\sigma(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x' = P(4\pi\sigma) = \Sigma(1)\,, && (4.37\text{a})\\
U^a_\varphi &:= \int_{\mathcal{M}} \frac{\sigma^a_\varphi(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x' = P\left(4\pi\sigma^a_\varphi\right) = \Sigma^a_\varphi(1)\,. && (4.37\text{b})
\end{aligned}
$$

We use the GR potentials to PN order of [10, 15]:

$$
V^i := \Sigma^i(1)\,,\quad \Phi^{ij}_1 := \Sigma^{ij}(1)\,,\quad \Phi_1 := \Sigma^{ii}(1)\,,\quad \Phi_2 := \Sigma(U)\,,\quad X := X(1)\,,
$$

and the 2 PN potentials

$$
\begin{aligned}
V^i_2 &:= \Sigma^i(U)\,, & \Phi^i_2 &:= \Sigma(V^i)\,,\\
Y &:= Y(1)\,, & X^i &:= X^i(1)\,,\\
X_1 &:= X^{ii}(1)\,, & X_2 &:= X(U)\,,\\
P^{ij}_2 &:= P(U^{,i}U^{,j})\,, & P_2 &:= P^{ii}_2 = \Phi_2 - \frac{1}{2}U^2\,,\\
G_1 &:= P(\dot{U}^2)\,, & G_2 &:= P(U\ddot{U})\,,\\
G_3 &:= -P(\dot{U}^{,k}V^k)\,, & G_4 &:= P(V^{i,j}V^{j,i})\,,
\end{aligned}
$$

$$G_5 := -P(\dot{V}^k U^{,k}) \,, \qquad G_6 := P(U^{,ij}\Phi_1^{ij}) \,,$$
$$G_7^i := P(U^{,k}V^{k,i}) + \frac{3}{4}P(U^{,i}\dot{U}) \,, \qquad H := P(U^{,ij}P_2^{ij}) \,.$$

To avoid confusion with too many indices, we keep the abbreviations for potentials, including target space indices, to a minimum. The ones used are listed as

$$X_\varphi^a := X_\varphi^a(1) \,, \qquad Y_\varphi^a := Y_\varphi^a(1) \,. \tag{4.38}$$

4.3 Expansion of near-zone fields to 2.5 PN order

We follow the convention in [15] and [11] to split the metric fields in terms of their PN contributions via

$$N = \varepsilon \left(N_0 + \varepsilon N_1 + \varepsilon^{3/2} N_{1.5} + \varepsilon^2 N_2 + \varepsilon^{5/2} N_{2.5}\right) + \mathcal{O}\left(\varepsilon^4\right) \,, \tag{4.39a}$$
$$K^i = \varepsilon^{3/2} \left(K_1^i + \varepsilon K_2^i + \varepsilon^{3/2} K_{2.5}^i\right) + \mathcal{O}\left(\varepsilon^{7/2}\right) \,, \tag{4.39b}$$
$$B = \varepsilon^2 \left(B_1 + \varepsilon^{1/2} B_{1.5} + \varepsilon B_2 + \varepsilon^{3/2} B_{2.5}\right) + \mathcal{O}\left(\varepsilon^4\right) \,, \tag{4.39c}$$
$$B^{ij} = \varepsilon^2 \left(B_2^{ij} + \varepsilon^{1/2} B_{2.5}^{ij}\right) + \mathcal{O}\left(\varepsilon^3\right) \,, \tag{4.39d}$$
$$\varphi^a = \varepsilon \left(\varphi_0^a + \varepsilon^{1/2}\varphi_{0.5}^a + \varepsilon\varphi_1^a + \varepsilon^{3/2}\varphi_{1.5}^a + \varepsilon^2\varphi_2^a + \varepsilon^{5/2}\varphi_{2.5}^a\right) + \mathcal{O}\left(\varepsilon^4\right) , \tag{4.39e}$$

where the subscript number on each metric field denotes the leading order contribution to the equations of motion. Writing the equations this way helps to visualize where and how much any field of interest contributes. As expected, the lapse $N = h^{00}$ and the scalar fields φ^a are most involved as they start to contribute already at first post-Newtonian order. A complete map of the iterative process to calculate all the above fields and our succeeding analysis is given in Figure 4.1.

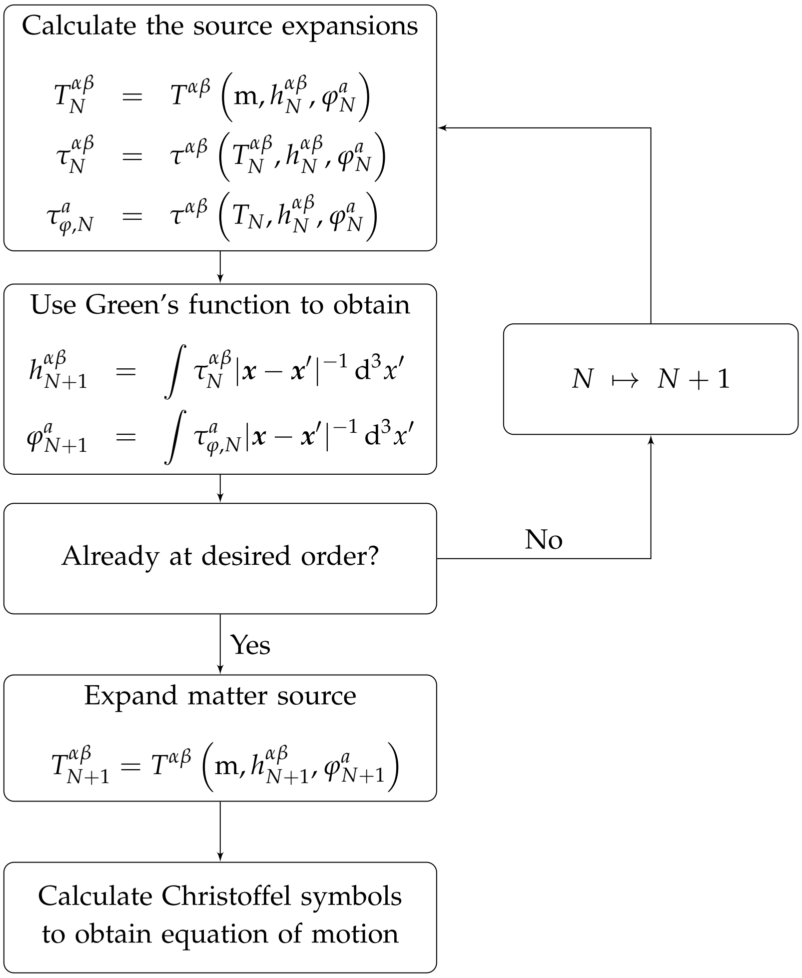

Figure 4.1: Flowchart of the general scheme of our calculations. This is similar to the one found in [10] but adapted to our TMST case here. The iterative process starts with setting $0 \equiv \varphi^a \equiv h^{\alpha\beta}$ and uses that to calculate the wave equation sources in (4.13)–(4.14) to lowest order. These sources are then inserted in the retarded Green's function, which will be evaluated via the expansions detailed in Eqs. (4.31). This yields the first set of the metric and scalar potentials of Eqs. (4.39). Now, depending on the problem of interest, one can iterate this process as long as needed, reinserting these fields in the wave equation sources and calculating those one order more accurately. Once the desired order is reached, we expand the actual matter model assumed in our work utilizing the prior metric and scalar field potentials calculated. At last, we can calculate the Christoffel symbols from our expanded metric, which, in turn, yields the equation of motion.

4.3.1 Calculation of Newtonian, 1 PN, and 1.5 PN Fields

The lowest order in our PN expansion relies only on

$$\tau^{00} = (-g)T^{00} + \mathcal{O}(\rho\varepsilon) = \sigma + \mathcal{O}(\rho\varepsilon) \tag{4.40}$$

since there are no other contributions in the source and $\sigma^{ii} \sim \varepsilon\sigma$. This gives us

$$N_0 = 4U\,. \tag{4.41}$$

This result is expected since it resembles the Newtonian potential itself. The source to the Newtonian order of the scalar fields is given by

$$\tau_\varphi^a = \frac{1}{2}\sigma_\varphi^a + \mathcal{O}(\rho\varepsilon) \tag{4.42}$$

which returns

$$\varphi_0^a = U_\varphi^a\,. \tag{4.43}$$

To the next PN order, we substitute the field to the prior order in the source and obtain

$$\begin{aligned}
\tau^{00} &= \sigma - \sigma^{ii} + 4\sigma U - \frac{7}{8\pi}(\nabla U)^2 + \frac{1}{8\pi}\gamma_{ab}\delta^{ij}U_\varphi^{a,i}U_\varphi^{b,j} + \mathcal{O}(\rho\varepsilon^2)\,, && (4.44)\\
\tau^{0i} &= \sigma^i + \mathcal{O}(\rho\varepsilon^{3/2})\,, && (4.45)\\
\tau^{ii} &= \sigma^{ii} - \frac{1}{8\pi}(\nabla U)^2 + \frac{5}{8\pi}\gamma_{ab}\delta^{kl}U_\varphi^{a,k}U_\varphi^{b,l} + \mathcal{O}(\rho\varepsilon^2)\,, && (4.46)\\
\tau^{ij} &= \mathcal{O}(\rho\varepsilon)\,, && (4.47)\\
\tau_\varphi^a &= \frac{1}{2}\sigma_\varphi^a - \sigma_\varphi^a U + \frac{1}{8\pi}\gamma_{bc}^a\delta^{ij}U_\varphi^{b,i}U_\varphi^{c,j} + \mathcal{O}(\rho\varepsilon^2) && (4.48)
\end{aligned}$$

At this point, comparing these equations to those in [15] is worthwhile. The terms presented here are natural generalizations to their single scalar field counterparts. There is, however, one notable difference: In (4.44), the counterpart for τ^{00} (Eq. (4.9a) in [15]) has a term σU_s. We do not have this term in our analysis. The reason for that is the difference in the underlying frame used.

Substituting all sources above into Eqs. (4.31), we obtain

$$
\begin{aligned}
N_1 &= 7U^2 - 4\Phi_1 + 2\Phi_2 + 2\ddot{X} - \gamma_{ab}\left(U_\varphi^a U_\varphi^b + 2\Sigma_\varphi^a\left(U_\varphi^b\right)\right), && (4.49)\\
K_1^i &= 4V^i\,, && (4.50)\\
B_1 &= U^2 + 4\Phi_1 - 2\Phi_2 - 5\gamma_{ab}\left(U_\varphi^a U_\varphi^b + 2\Sigma_\varphi^a\left(U_\varphi^b\right)\right), && (4.51)\\
\varphi_1^a &= -\gamma_{bc}^a\left(U_\varphi^b U_\varphi^c + 2\Sigma_\varphi^b\left(U_\varphi^c\right)\right) - \Sigma_\varphi^a(U) + \frac{1}{2}\ddot{X}_\varphi^a\,, && (4.52)\\
N_{1.5} &= -\frac{2}{3}\overset{(3)}{\mathcal{I}^{kk}}(t)\,, && (4.53)\\
B_{1.5} &= -2\overset{(3)}{\mathcal{I}^{kk}}(t)\,, && (4.54)\\
\varphi_{1.5}^a &= -2\dot{M}_\varphi^a(t) + \frac{2}{3}x^j\,{}^a\overset{(3)}{\mathcal{I}_\varphi^j}(t) - \frac{1}{3}\,{}^a\overset{(3)}{\mathcal{I}_\varphi^{kk}}(t)\;. && (4.55)
\end{aligned}
$$

Like the single scalar field case [15], $M_\varphi^a(t)$ is constant to the lowest PN order. This can be verified assuming that our compact bodies have a stationary internal structure and the conservation of the baryon number in our system. Hence the term $M_\varphi^a(t)$ does not contribute to $\varphi_{0.5}^a$ as shown in Eq. (4.31d) but rather to 1.5 PN order as shown above. Hence, $\varphi_{0.5}^a$ vanishes here.

To calculate the 1 PN and 1.5 PN metric and scalar field contributions listed in Eqs. (4.49)–(4.55) we made use of some identities. First note that for any sufficiently regular function f the Poisson potential of $\nabla^2 f$, as defined in Eq. (4.34), is given as

$$P\left(\nabla^2 f\right) = -f + \mathcal{B}_P(f)\,, \qquad (4.56)$$

for the boundary term given in the surface integral

$$\mathcal{B}_P(f) := \frac{1}{4\pi}\oint_{\partial\mathcal{M}}\left[\frac{f(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\partial'_r \log\left(f(t,\boldsymbol{x}')|\boldsymbol{x}-\boldsymbol{x}'|\right)\right]_{r'=\mathcal{R}}\mathcal{R}^2\,\mathrm{d}\Omega'\,. \tag{4.57}$$

For 1 PN and 1.5 PN correction terms, this term will not produce any $\mathcal{R}$-independent terms and can be neglected. At higher order, however, essential contributions might arise. Hence, it is necessary to carefully evaluate this integral at every step the formula above is used.

To apply this formula, we present useful in-between results of integrating the sources $\tau^{\alpha\beta}$, Eqs. (4.44)–(4.48). For example, note that

$$\begin{aligned}\delta^{ij}U^{a,i}_{\varphi}U^{b,j}_{\varphi} &= \nabla U^a_{\varphi}\cdot\nabla U^b_{\varphi}\\ &= \frac{1}{2}\nabla^2\left(U^a_{\varphi}U^b_{\varphi}\right) - U^a_{\varphi}\nabla^2 U^b_{\varphi} - U^b_{\varphi}\nabla^2 U^a_{\varphi}\,,\end{aligned} \tag{4.58}$$

leading to

$$P\left(\nabla U^a_{\varphi}\cdot\nabla U^b_{\varphi}\right) = -\frac{1}{2}\left(U^a_{\varphi}U^b_{\varphi} - P\left(U^a_{\varphi}\nabla^2 U^b_{\varphi}\right) - P\left(U^b_{\varphi}\nabla^2 U^a_{\varphi}\right)\right)\,. \tag{4.59}$$

The last step, invoking the fact that γ_{ab} is symmetric together with Poisson's equation

$$\nabla^2 U^a_{\varphi} = 4\pi\sigma^a_{\varphi}\,, \tag{4.60}$$

and the potentials in Eqs. (4.35), yields

$$P\left(\gamma_{ab}\delta^{ij}U^{a,i}_{\varphi}U^{b,j}_{\varphi}\right) = -\frac{1}{2}\gamma_{ab}U^a_{\varphi}U^b_{\varphi} + \gamma_{ab}\Sigma^a_{\varphi}\left(U^b_{\varphi}\right)\,. \tag{4.61}$$

Since the Christoffel symbols are symmetric in the lower indices, i.e., $\gamma^a_{bc} = \gamma^a_{cb}$, integrating the term

$$\gamma^a_{bc}\delta^{ij}U^{b,i}_\varphi U^{c,j}_\varphi \tag{4.62}$$

from the scalar-field source τ^a_φ, Eq. (4.48), works similar as

$$P\left(\gamma^a_{bc}\delta^{ij}U^{b,i}_\varphi U^{c,j}_\varphi\right) = -\frac{1}{2}\gamma^a_{bc}U^b_\varphi U^c_\varphi + \gamma^a_{bc}\Sigma^b_\varphi\left(U^c_\varphi\right) . \tag{4.63}$$

All of the above together explains the terms beyond GR in Eqs. (4.49)–(4.55) stemming from Einstein frame TMSTs. The GR counterpart to Eq. (4.61) is then given as [10]

$$P(\nabla U\cdot\nabla U) = -\frac{1}{2}U^2 + \Phi_2 . \tag{4.64}$$

With similar calculations, we are also able to provide some valuable formulae for other GR and TMST potentials, such as

$$\begin{aligned} P(U) &= -\frac{1}{2}X , && (4.65)\\ P(U^a_\varphi) &= -\frac{1}{2}X^a_\varphi , && (4.66)\\ & && (4.67) \end{aligned}$$

utilizing the identities

$$\begin{aligned} \nabla^2 X &= 2U , && (4.68)\\ \nabla^2 X^a_\varphi &= 2U^a_\varphi . && (4.69) \end{aligned}$$

4.3.2 Spacetime Metric and Scalar Fields to 1.5 PN Order

To better see the full picture of our results until here, we can put the 1 PN and 1.5 PN order contributions of Eqs. (4.49)–(4.55) in the context of the spacetime metric (4.22). This allows us to analyze the interactions between the various contributions of the expansions (4.39) in the actual setting of the gravitational fields $g_{\alpha\beta}$. So, substituting all previous results in the 1.5 PN expansion of the metric (4.22) yields

$$\begin{aligned}
g_{00} &= -1 + 2U - 2U^2 + \ddot{X} - 3\gamma_{ab}\left(U^a_\varphi U^b_\varphi + 2\Sigma^a_\varphi\left(U^b_\varphi\right)\right) \\
&\quad -\frac{4}{3}\overset{(3)}{\mathcal{I}^{kk}}(t) + \mathcal{O}\left(\varepsilon^3\right), && \text{(4.70a)} \\
g_{0i} &= -4V^i + \mathcal{O}\left(\varepsilon^{5/2}\right), && \text{(4.70b)} \\
g_{ij} &= \delta^{ij}\left(1 + 2U\right) + \mathcal{O}\left(\varepsilon^2\right), && \text{(4.70c)} \\
(-g) &= 1 + 4U + \mathcal{O}\left(\varepsilon^2\right), && \text{(4.70d)} \\
\varphi^a &= U^a_\varphi - \gamma^a_{bc}\left(U^b_\varphi U^c_\varphi + 2\Sigma^b_\varphi\left(U^c_\varphi\right)\right) - \Sigma^a_\varphi(U) + \frac{1}{2}\ddot{X}^a_\varphi \\
&\quad -2\dot{M}^a_\varphi(t) + \frac{2}{3}x^j\, {}^a\overset{(3)}{\mathcal{I}^j_\varphi}(t) - \frac{1}{3}\, {}^a\overset{(3)}{\mathcal{I}^{kk}_\varphi}(t) + \mathcal{O}\left(\varepsilon^3\right). && \text{(4.70e)}
\end{aligned}$$

Taking a step back and examining these potentials more closely is worthwhile. This result agrees with [21] and can be compared to the one scalar field case given in [15]. In there, the notation is as follows: The extra scalar field given in the field equations is denoted by ϕ with cosmological value ϕ_0 and the rescaling $\phi/\phi_0 =: 1 - \Psi$. This allows us to collapse our equations to one scalar field, $\varphi^a \equiv \phi$, and choose our free parameters as

$$A^2(\phi) := \frac{\phi_0}{\phi}\,; \qquad \gamma_{ab}(\phi) \equiv \gamma_{00}(\phi) := \frac{2\omega(\phi) + 3}{4\phi^2}\,. \tag{4.71}$$

Now, calculating the physical metric $\widetilde{g}_{\alpha\beta}$ via the conformal relation for single STTs, $\widetilde{g}_{\alpha\beta} = \phi_0/\phi\, g_{\alpha\beta}$, we obtain the same result as in [15], Eqs. (4.11).

Next, let us further analyze the physical Jordan frame metric $\widetilde{g}_{\alpha\beta} = A^2(\varphi)g_{\alpha\beta}$

in our Tensor Multi-Scalar setting. Notice that, to the here discussed 1.5 PN order, the only scalar field contribution in the tensorial part of the Einstein frame metric $g_{\alpha\beta}$ is in the $\mathrm{d}t^2$ component g_{00}, Eq. (4.70a). This is not true for the physical metric, as one can see with an asymptotic expansion of the conformal factor

$$A^2(\varphi) = A^2(\varphi_\infty) + 2\frac{\partial A(\varphi)}{\partial \varphi^a}\bigg|_{\varphi_\infty} (\varphi^a - \varphi^a_\infty) + \mathcal{O}\left(\varphi^2\right) , \tag{4.72}$$

or, simply written (keeping $\varphi_\infty \equiv 0$ from earlier in mind), $A^2(\varphi) = A_0^2 + 2A_{0,a}\varphi^a + \mathcal{O}(\varphi^2)$. Now, substituting in the lowest order contribution of φ^a via Eq. (4.43), the nontrivial contribution to the first PN order of the physical Jordan frame metric is given as

$$\widetilde{g}_{00} = -A_0^2 + 2A_0^2 U - 2A_{0,a}U^a_\varphi + \mathcal{O}\left(\varepsilon^2\right) . \tag{4.73}$$

The form of the physical metric here makes sense as it is a linear combination of the Newtonian-like gravitational potentials U and U^a_φ. At the same time, the coefficient in front can be interpreted as rescaled effective gravitational coupling constants. The fact that there is now a combination of $1+n$ contributing potentials as a post-Newtonian addition to gravity is, of course, expected as a result of coupling n scalar fields to the gravitational potentials as is done in Tensor Multi-Scalar Theory studied here.

4.3.3 Calculation of 2 PN and 2.5 PN Fields

At 2 PN and 2.5 PN orders, we obtain

$$\tau^{ij} \;=\; \sigma^{ij} + \frac{1}{4\pi}\left(U^{,i}U^{,j} - \frac{1}{2}\delta^{ij}(\nabla U)^2\right)$$

$$+\frac{1}{8\pi}\gamma_{ab}\left(2U_{\varphi}^{a,i}U_{\varphi}^{b,j}+\delta^{ij}\delta^{kl}U_{\varphi}^{a,k}U_{\varphi}^{b,l}\right)+\mathcal{O}(\rho\varepsilon^2)\,, \tag{4.74}$$

$$\tau^{0i} = \sigma^i+4\sigma^iU+\frac{2}{\pi}U^{,j}V^{[j,i]}+\frac{3}{4\pi}\dot{U}U^{,i}-\frac{1}{4\pi}\gamma_{ab}\dot{U}_{\varphi}^{a}U_{\varphi}^{b,i}+\mathcal{O}\left(\rho\varepsilon^{5/2}\right). \tag{4.75}$$

Using Eqs. (4.31b) and (4.31c), the integrals yield

$$\begin{aligned} B_2^{ij} &= 4\varphi_1^{ij}+4P_2^{ij}-\delta^{ij}(2\Phi_2-U^2) \\ &\quad +4\gamma_{ab}\,{}^{ab}P_{2\varphi}^{ij}-\gamma_{ab}\delta^{ij}\left(U_{\varphi}^{a}U_{\varphi}^{b}+2\Sigma_{\varphi}^{a}\left(U_{\varphi}^{b}\right)\right), \end{aligned} \tag{4.76}$$

$$K_2^i = 8V_2^i-8\Phi_2^i+8UV^i+16G_7^i+2\ddot{X}^i-4\gamma_{ab}\delta^{ij}P\left(\dot{U}_{\varphi}^{a}U_{\varphi}^{b,j}\right), \tag{4.77}$$

$$B_{2.5}^{ij} = -2\overset{(3)}{\mathcal{I}^{ij}}(t)\,, \tag{4.78}$$

$$K_{2.5}^i = \frac{2}{3}x^k\overset{(4)}{\mathcal{I}^{ik}}(t)-\frac{2}{9}\overset{(4)}{\mathcal{I}^{ikk}}(t)+\frac{4}{9}\varepsilon^{mik}\overset{(3)}{\mathcal{J}^{mk}}(t)\;. \tag{4.79}$$

To calculate the source terms of our wave equations to the final order needed, we substitute all prior results of this section in Eqs. (4.19) and (4.30) to obtain

$$\begin{aligned} \tau^{00} &= \sigma-\sigma^{ii}+4\sigma U-\frac{7}{8\pi}(\nabla U)^2+\frac{1}{8\pi}\gamma_{ab}\delta^{ij}U_{\varphi}^{a,i}U_{\varphi}^{b,j} \\ &\quad +\sigma\left(7U^2-8\Phi_1+2\Phi_2+2\ddot{X}-5\gamma_{ab}U_{\varphi}^{a}U_{\varphi}^{b}-10\gamma_{ab}\Sigma_{\varphi}^{a}\left(U_{\varphi}^{b}\right)\right)-4\sigma^{ii}U \\ &\quad +\frac{1}{4\pi}\left\{\frac{5}{2}\dot{U}^2-4U\ddot{U}-8\dot{U}^{,k}V^k+2V^{i,j}\left(3V^{j,i}+V^{i,j}\right)+4\dot{V}^jU^{,j}\right. \\ &\quad -4U^{,ij}\Phi_1^{ij}+8\nabla U\cdot\nabla\Phi_1-4\nabla U\cdot\nabla\Phi_2-\frac{7}{2}\nabla U\cdot\nabla\ddot{X}-10U(\nabla U)^2 \\ &\quad -4U^{,ij}\left(P_2^{ij}-\gamma_{ab}P\left(U_{\varphi}^{a,i}U_{\varphi}^{b,j}\right)\right)-6\gamma_{ab}U_{\varphi}^{a}\nabla U\cdot\nabla U_{\varphi}^{b}-6\gamma_{ab}\nabla U \\ &\quad \times\nabla\Sigma_{\varphi}^{a}\left(U_{\varphi}^{b}\right)+4\gamma_{ab}U\nabla U_{\varphi}^{a}\cdot\nabla U_{\varphi}^{b}+\frac{1}{2}\gamma_{ab}\dot{U}_{\varphi}^{a}\dot{U}_{\varphi}^{b}-\gamma_{ab}\gamma_{cd}^{b}\nabla U_{\varphi}^{a}\cdot\nabla\left(U_{\varphi}^{c}U_{\varphi}^{d}\right) \\ &\quad -2\gamma_{ab}\gamma_{cd}^{b}\nabla U_{\varphi}^{a}\cdot\nabla\Sigma_{\varphi}^{c}\left(U_{\varphi}^{d}\right)-\gamma_{ab}\nabla U_{\varphi}^{a}\cdot\nabla\Sigma_{\varphi}^{b}(U)+\frac{1}{2}\gamma_{ab}\nabla U_{\varphi}^{a}\cdot\nabla\ddot{X}_{\varphi}^{b} \\ &\quad \left.+\frac{1}{2}\gamma_{ab,c}U_{\varphi}^{c}\nabla U_{\varphi}^{a}\cdot\nabla U_{\varphi}^{b}\right\}+\frac{4}{3}\sigma\overset{(3)}{\mathcal{I}^{kk}}(t)+\frac{1}{2\pi}U^{,ij}\overset{(3)}{\mathcal{I}^{ij}}(t)+\mathcal{O}(\rho\varepsilon^3)\,, \end{aligned} \tag{4.80}$$

$$\begin{aligned} \tau^{ii} &= \sigma^{ii}-\frac{1}{8\pi}(\nabla U)^2+\frac{5}{8\pi}\gamma_{ab}\delta^{ij}U_{\varphi}^{a,i}U_{\varphi}^{b,j}+4\sigma^{ii}U \\ &\quad -\frac{1}{4\pi}\left\{\frac{9}{2}\dot{U}^2+4V^{i,j}V^{[i,j]}+4\dot{V}^jU^{,j}+\frac{1}{2}\nabla U\cdot\nabla\ddot{X}-\frac{1}{2}\gamma_{ab}U_{\varphi}^{a}\nabla U\cdot\nabla U_{\varphi}^{b}\right. \end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{2}\gamma_{ab}\nabla U\cdot\nabla\Sigma^a_\varphi\left(U^b_\varphi\right)-\frac{5}{2}\gamma_{ab}\dot{U}^a_\varphi\dot{U}^b_\varphi-\frac{5}{2}\gamma_{ab}\nabla U^a_\varphi\cdot\nabla\ddot{X}^b_\varphi \\
& +5\gamma_{ab}\gamma^b_{cd}\nabla U^a_\varphi\cdot\nabla\left(U^c_\varphi U^d_\varphi\right)+10\gamma_{ab}\gamma^b_{cd}\nabla U^a_\varphi\cdot\nabla\Sigma^c_\varphi\left(U^d_\varphi\right)+5\gamma_{ab}\nabla U^a_\varphi \\
& \times\nabla\Sigma^b_\varphi(U)-10\gamma_{ab}U\nabla U^a_\varphi\cdot\nabla U^b_\varphi-\frac{5}{2}\gamma_{ab,c}U^c_\varphi\nabla U^a_\varphi\cdot\nabla U^b_\varphi\Big\} \\
& +\mathcal{O}(\rho\varepsilon^3)\,, \qquad (4.81)
\end{aligned}
$$

$$
\begin{aligned}
\tau^a_\varphi \;=\; & \frac{1}{2}\sigma^a_\varphi-\sigma^a_\varphi U+\frac{1}{8\pi}\gamma^a_{bc}\delta^{ij}U^{b,i}_\varphi U^{c,j}_\varphi \\
& +\sigma^a_\varphi\left(\frac{21}{4}U^2+\Phi_1-\frac{1}{2}\Phi_2-\frac{1}{2}\ddot{X}+\frac{1}{4}\gamma_{bc}\left(U^b_\varphi U^c_\varphi+2\Sigma^b\left(U^c_\varphi\right)\right)\right) \\
& +\frac{1}{4\pi}\gamma^a_{bc}\left\{-2\gamma^c_{de}U^d_\varphi\nabla U^b_\varphi\cdot\nabla U^e_\varphi-2\gamma^c_{de}\nabla U^b_\varphi\cdot\nabla\Sigma^d_\varphi\left(U^e_\varphi\right)\right. \\
& \left.-\nabla U^b_\varphi\cdot\nabla\Sigma^c_\varphi(U)+\frac{1}{2}\nabla U^b_\varphi\cdot\nabla\ddot{X}^c_\varphi\right\} \\
& +\frac{1}{8\pi}\left\{-\gamma^a_{bc}\dot{U}^b_\varphi\dot{U}^c_\varphi-4U\ddot{U}^a_\varphi+8V^k\dot{U}^{a,k}_\varphi+U^{a,ij}_\varphi B^{ij}_2+\gamma^a_{bc,d}\delta^{ij}U^{b,i}_\varphi U^{c,j}_\varphi U^d_\varphi\right\} \\
& +\frac{1}{6}\sigma^a_\varphi\overset{(3)}{\mathcal{I}^{kk}}(t)-\frac{1}{4\pi}U^{a,ij}_\varphi\overset{(3)}{\mathcal{I}^{ij}}(t)+\mathcal{O}(\rho\varepsilon^3)\,. \qquad (4.82)
\end{aligned}
$$

Substituting this into Eqs. (4.31a), (4.31c), and (4.31d), yields

$$
\begin{aligned}
N_2 \;=\; & -16U\Phi_1+8U\Phi_2+7U\ddot{X}+\frac{20}{3}U^3-4V^iV^i-16\Sigma(\Phi_1)+\Sigma(\ddot{X}) \\
& +8\Sigma^i(V^i)-2\ddot{X}_1+\ddot{X}_2+\frac{1}{6}\overset{(4)}{Y}-4G_1-16G_2+32G_3+24G_4-16G_5 \\
& -16G_6-16H+2\gamma_{ab}\ddot{X}^a_\varphi\left(U^b_\varphi\right)+2\gamma_{ab}P\left(\ddot{U}^a_\varphi U^b_\varphi\right)+6\gamma_{ab}P\left(\dot{U}^a_\varphi\dot{U}^b_\varphi\right) \\
& -32\gamma_{ab}\Sigma\left(U^a_\varphi U^b_\varphi\right)-52\gamma_{ab}\Sigma\left(\Sigma^a_\varphi\left(U^b_\varphi\right)\right)+16P\left(U^{,ij}\gamma_{ab}\,{}^{ab}P^{ij}_{2\varphi}\right) \\
& +12\gamma_{ab}U\Sigma^a_\varphi\left(U^b_\varphi\right)-38\gamma_{ab}\Sigma^a_\varphi\left(UU^b_\varphi\right)+4\gamma_{ab}\gamma^b_{cd}U^a_\varphi\Sigma^c_\varphi\left(U^d_\varphi\right) \\
& -4\gamma_{ab}\gamma^b_{cd}\Sigma^c_\varphi\left(U^a_\varphi U^d_\varphi\right)-4\gamma_{ab}\gamma^b_{cd}\Sigma^a_\varphi\left(\Sigma^c_\varphi\left(U^d_\varphi\right)\right)+2\gamma_{ab}U^a_\varphi\Sigma^b_\varphi(U) \\
& -2\gamma_{ab}\Sigma^a_\varphi\left(\Sigma^b_\varphi(U)\right)-\gamma_{ab}U^a_\varphi\ddot{X}^b_\varphi+\gamma_{ab}\Sigma^a_\varphi\left(\ddot{X}^b_\varphi\right)+4\gamma_{ab}\gamma^b_{cd}U^a_\varphi U^c_\varphi U^d_\varphi \\
& +4\gamma_{ab}\gamma^b_{cd}P\left(U^a_\varphi\nabla U^c_\varphi\cdot\nabla U^d_\varphi\right)-8\gamma_{ab}\gamma^b_{cd}\Sigma^d_\varphi\left(U^a_\varphi U^c_\varphi\right)-4\gamma_{ab}\gamma^b_{cd}\Sigma^a_\varphi\left(U^c_\varphi U^d_\varphi\right) \\
& -\gamma_{ab,c}U^a_\varphi U^b_\varphi U^c_\varphi-\gamma_{ab,c}P\left(U^a_\varphi\nabla U^b_\varphi\cdot\nabla U^c_\varphi\right)+2\gamma_{ab,c}\Sigma^c_\varphi\left(U^a_\varphi U^b_\varphi\right) \\
& +\gamma_{ab,c}\Sigma^a_\varphi\left(U^b_\varphi U^c_\varphi\right)+12\gamma_{ab}UU^a_\varphi U^b_\varphi+28\gamma_{ab}P\left(U\nabla U^a_\varphi\cdot\nabla U^b_\varphi\right)\,, \qquad (4.83)
\end{aligned}
$$

$$
\begin{aligned}
B_2 &= U\ddot{X} + 4V^iV^i - \Sigma(\ddot{X}) - 8\Sigma^i(V^i) + 16\Sigma^{ii}(U) + 2\ddot{X}_1 - \ddot{X}_2 - 20G_1 \\
&+8G_4 + 16G_5 + 10\gamma_{ab}\ddot{X}^a_\varphi\left(U^b_\varphi\right) + 10\gamma_{ab}P\left(\ddot{U}^a_\varphi U^b_\varphi\right) + 30\gamma_{ab}P\left(\dot{U}^a_\varphi \dot{U}^b_\varphi\right) \\
&+\gamma_{ab}U\Sigma^a_\varphi\left(U^b_\varphi\right) - \gamma_{ab}\Sigma^a_\varphi\left(UU^b_\varphi\right) - \gamma_{ab}\Sigma\left(\Sigma^a_\varphi\left(U^b_\varphi\right)\right) - \gamma_{ab}UU^a_\varphi U^b_\varphi \\
&+39\gamma_{ab}P\left(U\nabla U^a_\varphi \cdot U^b_\varphi\right) + 2\gamma_{ab}\Sigma^a_\varphi\left(UU^b_\varphi\right) + \gamma_{ab}\Sigma\left(U^a_\varphi U^b_\varphi\right) - 5\gamma_{ab}U^a_\varphi\ddot{X}^b_\varphi \\
&+5\gamma_{ab}\Sigma^a_\varphi\left(\ddot{X}^b_\varphi\right) + 20\gamma_{ab}\gamma^b_{cd}U^a_\varphi U^c_\varphi U^d_\varphi + 20\gamma_{ab}\gamma^b_{cd}P\left(U^a_\varphi\nabla U^c_\varphi \cdot \nabla U^d_\varphi\right) \\
&-60\gamma_{ab}\gamma^b_{cd}\Sigma^d_\varphi\left(U^a_\varphi U^c_\varphi\right) - 20\gamma_{ab}\gamma^b_{cd}\Sigma^a_\varphi\left(U^c_\varphi U^d_\varphi\right) + 20\gamma_{ab}\gamma^b_{cd}U^a_\varphi\Sigma^c_\varphi\left(U^d_\varphi\right) \\
&-20\gamma_{ab}\gamma^b_{cd}\Sigma^a_\varphi\left(\Sigma^c_\varphi\left(U^d_\varphi\right)\right) + 10\gamma_{ab}U^a_\varphi\Sigma^b_\varphi(U) - 10\gamma_{ab}\Sigma^a_\varphi\left(UU^b_\varphi\right) \\
&-10\gamma_{ab}\Sigma^a_\varphi\left(\Sigma^b_\varphi(U)\right) - 5\gamma_{ab,c}U^a_\varphi U^b_\varphi U^c_\varphi - 5\gamma_{ab,c}P\left(U^a_\varphi\nabla U^b_\varphi \cdot \nabla U^c_\varphi\right) \\
&+10\gamma_{ab,c}\Sigma^c_\varphi\left(U^a_\varphi U^b_\varphi\right) + 5\gamma_{ab,c}\Sigma^a_\varphi\left(U^b_\varphi U^c_\varphi\right)\,, \qquad (4.84) \\
\varphi^a_2 &= \frac{1}{24}\overset{(4)}{Y^a_\varphi} - \ddot{X}^a_\varphi(U) + \gamma^a_{bc}\ddot{X}^b_\varphi\left(U^c_\varphi\right) + \frac{3}{2}\gamma^a_{bc}P\left(\ddot{U}^b_\varphi U^c_\varphi\right) + \gamma^a_{bc}P\left(\dot{U}^b_\varphi\dot{U}^c_\varphi\right) \\
&+\frac{21}{2}\Sigma^a_\varphi\left(U^2\right) + 2\Sigma^a_\varphi\left(\Phi_1\right) - \Sigma^a_\varphi\left(\Phi_2\right) - \Sigma^a_\varphi\left(\ddot{X}\right) + \frac{1}{2}\gamma_{bc}\Sigma^a_\varphi\left(U^b_\varphi U^c_\varphi\right) \\
&+\gamma_{bc}\Sigma^a_\varphi\left(\Sigma^b_\varphi\left(U^c_\varphi\right)\right) + \gamma^a_{bc}\gamma^c_{de}U^b_\varphi U^d_\varphi U^e_\varphi + \gamma^a_{bc}\gamma^c_{de}P\left(U^b_\varphi\nabla U^d_\varphi \cdot \nabla U^e_\varphi\right) \\
&-2\gamma^a_{bc}\gamma^c_{de}\Sigma^e_\varphi\left(U^b_\varphi U^d_\varphi\right) - \gamma^a_{bc}\gamma^c_{de}\Sigma^b_\varphi\left(U^d_\varphi U^e_\varphi\right) + 2\gamma^a_{bc}\gamma^c_{de}U^b_\varphi\Sigma^d_\varphi\left(U^e_\varphi\right) \\
&-2\gamma^a_{bc}\gamma^c_{de}\Sigma^d_\varphi\left(U^b_\varphi U^e_\varphi\right) - 2\gamma^a_{bc}\gamma^c_{de}\Sigma^b_\varphi\left(\Sigma^d_\varphi\left(U^e_\varphi\right)\right) + \gamma^a_{bc}U^b_\varphi\Sigma^c_\varphi(U) \\
&-\gamma^a_{bc}\Sigma^c_\varphi\left(UU^b_\varphi\right) - \gamma^a_{bc}\Sigma^b_\varphi\left(\Sigma^c_\varphi(U)\right) - \frac{1}{4}\gamma^a_{bc}U^b_\varphi\ddot{X}^c_\varphi + \frac{1}{4}\gamma^a_{bc}\Sigma^b_\varphi\left(\ddot{X}^c_\varphi\right) \\
&-4P\left(U\ddot{U}^a_\varphi\right) + 8P\left(V^k\dot{U}^{a,k}_\varphi\right) + P\left(U^{a,ij}_\varphi B^{ij}_2\right) - \frac{1}{2}\gamma^a_{bc,d}U^b_\varphi U^c_\varphi U^d_\varphi \\
&-\frac{1}{2}\gamma^a_{bc,d}P\left(U^d_\varphi\nabla U^b_\varphi \cdot \nabla U^c_\varphi\right) + \gamma^a_{bc,d}\Sigma^b_\varphi\left(U^c_\varphi U^d_\varphi\right) + \frac{1}{2}\gamma^a_{bc,d}\Sigma^d_\varphi\left(U^b_\varphi U^c_\varphi\right) \qquad (4.85) \\
N_{2.5} &= -\frac{1}{15}(2x^{kl} + r^2\delta^{kl})\overset{(5)}{\mathcal{I}^{kl}}(t) + \frac{2}{15}x^k\overset{(5)}{\mathcal{I}^{kll}}(t) - \frac{1}{30}\overset{(5)}{\mathcal{I}^{kkll}}(t) \\
&+\frac{16}{3}U\overset{(3)}{\mathcal{I}^{kk}}(t) - 4X^{,kl}\overset{(3)}{\mathcal{I}^{kl}}(t)\,, \qquad (4.86) \\
B_{2.5} &= -\frac{1}{3}r^2\overset{(5)}{\mathcal{I}^{ii}}(t) + \frac{2}{9}x^k\overset{(5)}{\mathcal{I}^{iik}}(t) + \frac{8}{9}x^k\varepsilon^{mki}\overset{(4)}{\mathcal{J}^{mi}}(t) - \frac{2}{3}\overset{(3)}{M^{iikk}}(t)\,, \qquad (4.87) \\
\varphi^a_{2.5} &= -\frac{1}{3}r^2\overset{(3)}{M^a_\varphi}(t) - 4r^2x^j\,{}^a\overset{(5)}{\mathcal{I}^j_\varphi}(t) + (4x^{kl} + 2r^2\delta^{kl})\,{}^a\overset{(5)}{\mathcal{I}^{kl}_\varphi}(t) - 4x^k\,{}^a\overset{(5)}{\mathcal{I}^{kll}_\varphi}(t)
\end{aligned}
$$

$$+ \overset{(5)}{{}^{a}\mathcal{I}_{\varphi}^{kkll}}(t) + \frac{1}{3} U_{\varphi}^{a} \overset{(3)}{\mathcal{I}^{kk}}(t) - X_{\varphi}^{a,kl} \overset{(3)}{\mathcal{I}^{kl}}(t) \ . \tag{4.88}$$

We again present the most important mathematical identities used to arrive at these equations. Of course, the metric potentials to 2 PN and 2.5 PN orders are quite involved, so we will try to explain the methods more generally and in a way that can be adapted to multiple scenarios. Of course, the identities explained at the end of Section 4.3.1, Eqs. (4.56)–(4.63), are still valid and useful here. For general and sufficiently regular fields f, g, we have

$$P\left(\nabla f \cdot \nabla g\right) = -\frac{1}{2}\left(fg + P\left(f\nabla^2 g\right) + P\left(g\nabla^2 f\right) - \mathcal{B}_P(fg)\right) \tag{4.89}$$

where the surface integral $\mathcal{B}_P(fg)$ is given as in Eq. (4.57). This formula becomes especially useful if either f or g is a Newtonian-like potential U, Eq. (4.37a), or U_{φ}^{a}, Eq. (4.37b), since then we can utilize the Poisson's equation

$$\nabla^2 U = 4\pi\sigma \,, \tag{4.90}$$

$$\nabla^2 U_{\varphi}^{a} = 4\pi\sigma_{\varphi}^{a} \,. \tag{4.91}$$

Other things we encounter while integrating the 2 PN sources Eqs. (4.81)–(4.82) are potentials including three factors such as

$$f\nabla U \cdot \nabla U \,, \qquad f\nabla U_{\varphi}^{a} \cdot \nabla U \,, \qquad f\nabla U_{\varphi}^{a} \cdot \nabla U_{\varphi}^{b} \,. \tag{4.92}$$

These kinds of potentials are, in general, difficult to integrate. One instance is given as

$$\begin{aligned} P\left(f\nabla U\cdot\nabla U\right) &= -\frac{1}{2}\left(fU^2+P\left(U^2\nabla^2 f\right)-2\Sigma(fU)+4P\left(U\nabla U\cdot\nabla f\right)\right.\\ &\left.-\mathcal{B}_P\left(fU^2\right)\right)\,. \end{aligned} \tag{4.93}$$

We already used Poisson's equation here, and for $f=U^a_\varphi$ this would simplify further as

$$P\left(U^2\nabla^2 U^a_\varphi\right)=\Sigma^a_\varphi\left(U^2\right)\,. \tag{4.94}$$

Generally, again for sufficiently regular fields f,g,h, we calculate

$$\begin{aligned} P\left(f\nabla g\cdot\nabla h\right) &= -\frac{1}{2}\Big[fgh+2P(g\nabla f\cdot\nabla h)+2P(h\nabla f\cdot\nabla g)+P(fg\nabla^2 h)\\ &+P(fh\nabla^2 g)+P(hg\nabla^2 f)-\mathcal{B}_P(fgh)\Big]\,. \end{aligned} \tag{4.95}$$

This identity tells us that for different functions f,g, and h, we cannot simplify those potentials much further. This is not much of an issue in GR or STTs, at least not to this order here, but in TMST, we do have source terms of the form

$$U\nabla U^a_\varphi\cdot\nabla U^b_\varphi\,. \tag{4.96}$$

In GR, no other Newtonian-like potential despite U appears where in STTs it is at most one, namely U_s [15]. TMST allows for $n+1$ different versions, hence the difficulties arising here.

Other valuable in-between calculations include of 2 PN sources include, using again the identities Eqs. (4.69) and (4.69),

$$P(X) = -\frac{1}{12}Y\,, \tag{4.97}$$

$$\begin{aligned} P(X_\varphi^a) &= -\frac{1}{12}Y_\varphi^a\,, && (4.98)\\ P\left(\nabla U\cdot\nabla\ddot{X}\right) &= -\frac{1}{2}U\ddot{X}+\frac{1}{2}\Sigma(\ddot{X})-P(U\ddot{U})+\mathcal{O}(\varepsilon^4)\,, && (4.99)\\ P\left(\nabla U_\varphi^a\cdot\nabla\ddot{X}_\varphi^b\right) &= -\frac{1}{2}U_\varphi^a\ddot{X}_\varphi^b+\frac{1}{2}\Sigma_\varphi^a(\ddot{X}_\varphi^b)-P(U_\varphi^a\ddot{U}_\varphi^a)+\mathcal{O}(\varepsilon^4)\,. && (4.100)\end{aligned}$$

Another essential part of integrating the 2 PN sources Eqs. (4.81)–(4.82) are the Poisson superpotentials Eq. (4.36), generally given as

$$S(f) := \int_{\mathcal{M}} f(t,\boldsymbol{x}')|\boldsymbol{x}-\boldsymbol{x}'|\,\mathrm{d}^3x'\,, \tag{4.101}$$

with the property

$$S\left(\nabla^2 f\right) = 2P(f)+\mathcal{B}_S(f)\,, \tag{4.102}$$

where the boundary this time as calculated as

$$\mathcal{B}_S(f) := \frac{1}{4\pi}\oint_{\mathcal{M}}\left[f(t,\boldsymbol{x}')|\boldsymbol{x}-\boldsymbol{x}'|\partial_r'\log\left(\frac{f(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)\right]_{r'=\mathcal{R}}\mathcal{R}^2\,\mathrm{d}\Omega'\,. \tag{4.103}$$

Applying this to the TMST source

$$\nabla U_\varphi^a\cdot\nabla U_\varphi^b = \frac{1}{2}\nabla^2\left(U_\varphi^a U_\varphi^b\right)-U_\varphi^a\nabla^2U_\varphi^b-U_\varphi^b\nabla^2U_\varphi^a\,, \tag{4.104}$$

yields

$$\begin{aligned} S\left(\nabla U_\varphi^a\cdot\nabla U_\varphi^b\right) &= \frac{1}{2}S\left(\nabla^2\left(U_\varphi^aU_\varphi^b\right)\right)-S\left(U_\varphi^a\nabla^2U_\varphi^b\right)-S\left(U_\varphi^b\nabla^2U_\varphi^a\right)+\mathcal{O}(\varepsilon^4)\\ &= P\left(U_\varphi^aU_\varphi^b\right)-X_\varphi^b\left(U_\varphi^a\right)-X_\varphi^a\left(U_\varphi^b\right)+\mathcal{O}(\varepsilon^4)\,. && (4.105)\end{aligned}$$

Since the superpotential in the expansions Eq. (4.31) appear as second order time derivatives, we calculate further

$$\begin{aligned}
\ddot{S}\left(\nabla U^a_\varphi \cdot \nabla U^b_\varphi\right) &= P\left(\partial_t^2\left(U^a_\varphi U^b_\varphi\right)\right) - \ddot{X}^b_\varphi\left(U^a_\varphi\right) - \ddot{X}^a_\varphi\left(U^b_\varphi\right) + \mathcal{O}(\varepsilon^4) \\
&= P\left(U^a_\varphi \ddot{U}^b_\varphi\right) + P\left(U^b_\varphi \ddot{U}^a_\varphi\right) + 2P\left(\dot{U}^a_\varphi \dot{U}^b_\varphi\right) \\
&\quad -\ddot{X}^b_\varphi\left(U^a_\varphi\right) - \ddot{X}^a_\varphi\left(U^b_\varphi\right) + \mathcal{O}(\varepsilon^4)\,. \qquad (4.106)
\end{aligned}$$

In GR, we simply get

$$\ddot{S}\left(\nabla^2 U^2\right) = 2\ddot{P}\left(U^2\right) + \mathcal{O}(\varepsilon^4) = 4G_1 + 4G_2 + \mathcal{O}(\varepsilon^4)\,, \qquad (4.107)$$

for G_1 and G_2 as defined after Eq. (4.37).

4.4 Energy-Momentum Tensor and its Expansion

4.4.1 Expansion of Mass Distribution

As a matter model, we use the idea of modeling compact bodies as skeletonized point masses. This approach has already been used in [15, 21] and is based on the work in [329, 330]. The model involves using δ-functions for encoding boundary conditions derived from the effects of scalar gravitational fields. The Einstein frame matter action then takes the form

$$S_{\mathrm{matt}} = -\sum_A \int m_A\left(\varphi\left(z^\mu_A\right)\right)\sqrt{-g_{\alpha\beta}\left(z^\mu_A\right)\mathrm{d}z^\alpha_A\,\mathrm{d}z^\beta_A}\,, \qquad (4.108)$$

where we sum over the various bodies A (not to be conflicted with the conformal factor $A(\varphi)$), and $m_A = m_A(\varphi)$ denotes the Einstein frame masses of the objects corresponding to the worldlines z^μ_A.

In the matter action (4.108), it is taken into account that the mass of self-gravitating objects m_A can depend explicitly on the scalar fields, i.e., to have compact objects such as neutron star or a black hole endowed with scalar hair. This will effectively bring an additional contribution to the scalar fields equation (2.14) connected with the derivative of the energy-momentum tensor with respect to the scalar fields that can now be nonzero [15, 329]. It was demonstrated in [21] that this additional contribution can be encoded in an elegant way in the expression of $\alpha_a(\varphi)$ appearing in Eq. (2.14). More precisely, instead of employing Eq. (2.17) that is valid for non-self-gravitating objects, we can generalize the expression for $\alpha_a(\varphi)$ in the following way

$$\alpha^A_a(\varphi) := \frac{\partial \log\left(m_A(\varphi)\right)}{\partial \varphi^a} = m_A^{-1}(\varphi)\frac{\partial m_A(\varphi)}{\partial \varphi^a}\,. \tag{4.109}$$

We see that $\alpha^A_a(\varphi)$ practically acts as an effective coupling function between compact object A and the contribution of the scalar fields. Following [21] one can show that

$$\alpha^A_a(\varphi) = \alpha_a(\varphi) + \frac{\partial \log\left(\widetilde{m}_A(\varphi)\right)}{\partial \varphi^a}\,, \tag{4.110}$$

where $\alpha_a(\varphi)$ is defined in Eq. (2.17) and $\widetilde{m}_A$ is the Jordan frame mass of the objects connected to the Einstein frame one via the conformal factor $m_A = A(\varphi)\widetilde{m}_A$. Clearly, for non-self-gravitating objects, $\widetilde{m}_A$ is independent of the scalar field, and the second term in the above equation is zero.

For convenience, one can define

$$M_A(z_A) := m_A(\varphi)\frac{1}{\sqrt{-g(z_A)}}\frac{1}{\sqrt{-g_{\alpha\beta}(z_A)\,v_A^\alpha v_A^\beta}}\,, \tag{4.111}$$

with the 4-velocities of the Ath compact object

$$u_A^\alpha = \frac{\mathrm{d}z_A^\alpha}{\mathrm{d}z_A^0} = \left(1, \frac{\mathrm{d}\boldsymbol{z}_A}{\mathrm{d}t}\right)\,. \tag{4.112}$$

Put together, varying the action in Eq. (4.108) and inserting the quantities above yields the distributional Einstein frame energy-momentum tensor

$$T^{\alpha\beta}\left(t,\boldsymbol{x}\right) = \sum_A M_A(t) u_A^\alpha u_A^\beta \delta^3\left(\boldsymbol{x} - \boldsymbol{z}_A(t)\right)\,. \tag{4.113}$$

To get the matter quantity to desired order, we need to expand the φ dependent masses and hence the coupling function (4.109) around the asymptotic values of the extra scalar fields φ_∞^a. Remember that without loss of generality, we have assumed that these are zero, similar to [21]. Formally, this yields

$$\begin{aligned} m_A(\varphi) \quad = \quad & m_{A0}\Big[1 + \alpha_a^{A0}\,\varphi^a + \frac{1}{2}\left(\alpha_a^{A0}\alpha_a^{A0} + \beta_{ab}^{A0}\right)\varphi^a\varphi^b \\ & + \frac{1}{6}\left(\alpha_a^{A0}\alpha_a^{A0}\alpha_a^{A0} + \beta_{ab}^{A0}\alpha_a^{A0} + \alpha_a^{A0}\,\beta_{ac}^{A0} + \alpha_a^{A0}\,\beta_{bc}^{A0} + \beta_{abc}^{A0}\right)\varphi^a\varphi^b\varphi^c\Big] \\ & + \mathcal{O}\left(\varphi^4\right)\,. \end{aligned} \tag{4.114}$$

Here, as in [21, 27], we introduced the notation $m_{A0} = m_A(\varphi_\infty)$ and collected the covariant derivatives D_a of the target space metric $\gamma_{ab}(\varphi)$ in the symmetric quantity

$$\beta_{ab}^A := D_a\,D_b\,\log(m_A(\varphi)) = D_{a\,b}\alpha_A\,, \tag{4.115}$$

and $\beta^A_{abc} := D_a \beta^A_{bc}$. The superscript $A0$ denotes an evaluation of the derivative at the background value φ_∞.

Now, following [15], we introduce the shorthand $m_A(\varphi) =: m_{A0}[1 + \mathcal{S}(\alpha, \varphi)] + \mathcal{O}(\varepsilon^4)$, where α collects all α_a fields. To expand the energy tensor completely, we make use of the fact that in GR (see, e.g., [11]), we have

$$T^{\alpha\beta} = \frac{\rho^*}{\sqrt{-g}} u^\alpha u^\beta (u^0)^{-1}\,, \tag{4.116}$$

where the newly introduced quantity ρ^* satisfies the continuity equation

$$\partial\rho^*/\partial t + \nabla\cdot(\rho^* \boldsymbol{v}) = 0\,. \tag{4.117}$$

As in the single scalar field case in [15], we can identify baryonic mass in the density ρ^* as point masses via the delta distribution to get

$$\rho^* = \sum_A m_{A0}\, \delta^3\left(\boldsymbol{x} - \boldsymbol{z}_A\right)\,. \tag{4.118}$$

Substituting this in Eq. (4.116) then yields

$$T^{\alpha\beta} = \frac{\rho^*}{\sqrt{-g}} v^\alpha v^\beta u^0 [1 + \mathcal{S}(\alpha, \varphi)]\,, \tag{4.119}$$

with the ordinary velocities $u^\alpha = u^0 v^\alpha$ and $v^\alpha := \mathrm{d}x^\alpha/\mathrm{d}t = (1, \boldsymbol{v})$. The task for the rest of this section is to express all σ-densities related to the energy-momentum tensor (Eqs. (4.23)) via ρ^*:

$$\sigma = T^{00} + T^{ii} = \frac{\rho^*}{\sqrt{-g}} u^0 \left(1 + v^2\right) [1 + \mathcal{S}(\alpha, \varphi)]\,, \tag{4.120a}$$

$$\sigma^i = T^{0i} = \frac{\rho^*}{\sqrt{-g}} u^0 v^i [1 + \mathcal{S}(\alpha, \varphi)]\,, \tag{4.120b}$$

$$\sigma^{ij} \quad = \quad T^{ij} = \frac{\rho^*}{\sqrt{-g}} u^0 v^i v^j \left[1 + \mathcal{S}(\alpha, \varphi)\right] . \tag{4.120c}$$

Note that these equations look algebraically similar to the single scalar field theory. The difference in our frame choice, the conformal Einstein frame, compared to the physical Jordan frame used in [15] is hidden in the velocities, and the contribution of the multiple scalar fields is encrypted in $[1 + \mathcal{S}(\alpha, \varphi)]$.

The updated density of the scalar fields can be calculated as

$$\sigma^a_\varphi = -\frac{\rho^*}{u^0\sqrt{-g}} \left[\alpha^a_A + \alpha^a_A \mathcal{S}(\alpha, \varphi)\right] . \tag{4.121}$$

Before continuing to expand all those densities to the desired order, note that we can calculate u^0 via

$$\begin{aligned} u^0 \quad &= \quad \frac{1}{\sqrt{-g_{00} - 2g_{0i}v^i - g_{ij}v^i v^j}} \\ &= \quad 1 + \varepsilon\left(\frac{1}{4}N_0 + \frac{1}{2}v^2\right) + \varepsilon^2\left(-\frac{3}{32}N_0^2 + \frac{1}{4}N_1 + \frac{1}{4}B_1 - v^i K^i_1\right. \\ &\quad \left. -\frac{1}{8}N_0 v^2 + \frac{3}{8}v^4\right) + \varepsilon^{5/2}\left(\frac{1}{4}N_{1.5} + \frac{1}{4}B_{1.5}\right) + \mathcal{O}(\varepsilon^3) , \end{aligned} \tag{4.122}$$

and remember that

$$\frac{1}{\sqrt{-g}} = 1 - \varepsilon\frac{1}{2}N_0 + \varepsilon^2\frac{1}{2}\left(-N_1 + \frac{3}{4}N_0^2 + B_1\right) + \varepsilon^{5/2}\frac{1}{2}\left(-N_{1.5} + B_{1.5}\right) + \mathcal{O}(\varepsilon^3) . \tag{4.123}$$

To expand all of the above σ-densities we need to insert the metric (4.22) and the expansion (4.39) to get

$$\sigma \quad = \quad \rho^*\Bigg[1 + \varepsilon\left(\frac{3}{2}v^2 - U_\sigma + \alpha^{A0}_a U^a_{\varphi\sigma}\right) + \varepsilon^2\left(\frac{7}{8}v^4 + v^2 U_\sigma - 4v^j V^j_\sigma - \frac{1}{4}N_1\right.$$

$$
\begin{aligned}
&+\frac{3}{4}B_1+\frac{5}{2}U_\sigma^2+\alpha_a^{A0}\varphi_1^a+\frac{1}{2}\left(\alpha_a^{A0}\alpha_b^{A0}+\beta_{ab}^{A0}\right)U_{\varphi\sigma}^a U_{\varphi\sigma}^b+\alpha_a^{A0}U_{\varphi\sigma}^a U_\sigma \\
&+\frac{3}{2}\alpha_a^{A0}U_\varphi^a v^2\Big)+\varepsilon^{5/2}\left(2N_{1.5}+\alpha_a^{A0}\varphi_{1.5}^a\right)+\mathcal{O}(\varepsilon^3)\Big], \qquad (4.124a)
\end{aligned}
$$

$$
\sigma^i = \rho^* v^i\left[1+\varepsilon\left(v^2-U_\sigma+\alpha_a^{A0}U_{\varphi\sigma}^a\right)+\mathcal{O}(\varepsilon^2)\right], \qquad (4.124b)
$$

$$
\sigma^{ij} = \rho^* v^i v^j\left[1+\mathcal{O}(\varepsilon)\right], \qquad (4.124c)
$$

$$
\sigma^{ii} = \rho^* v^2\left[1+\varepsilon\left(\frac{1}{2}v^2-U_\sigma+\alpha_a^{A0}U_{\varphi\sigma}^a\right)+\mathcal{O}(\varepsilon^2)\right], \qquad (4.124d)
$$

where the subscript σ in U_σ, $U_{\varphi\sigma}^a$, and V_σ^j indicates definition via the σ-potentials. Similarly, the σ-densities stemming from all extra scalar fields are then given as

$$
\begin{aligned}
\sigma_\varphi^a = \rho^*\Big[&\alpha_A^a-\varepsilon\left(\frac{1}{2}\alpha_A^a v^2-3\alpha_A^a U_\sigma+\alpha_A^a\alpha_b^{A0}U_{\varphi\sigma}^b\right)+\varepsilon^2\Big(-\frac{1}{8}\alpha_A^a v^4+\frac{1}{2}\alpha_A^a U_\sigma v^2 \\
&+2\alpha_A^a U_\sigma^2+4\alpha_A^a V_\sigma^i v^i-\frac{3}{4}\alpha_A^a N_1+\frac{1}{4}\alpha_A^a B_1+\alpha_A^a\alpha_b^{A0}\varphi_1^b \\
&+\frac{1}{2}\alpha_A^a\left(\alpha_b^{A0}\alpha_c^{A0}+\beta_{bc}^{A0}\right)U_{\varphi\sigma}^b U_{\varphi\sigma}^c+\alpha_A^a\alpha_b^{A0}U_{\varphi\sigma}^b U_\sigma-\frac{1}{2}\alpha_A^a\alpha_b^{A0}U_{\varphi\sigma}^b v^2\Big) \\
&+\varepsilon^{5/2}\alpha_A^a\alpha_b^{A0}\varphi_{1.5}^b+\mathcal{O}(\varepsilon^3)\Big]. \qquad (4.125)
\end{aligned}
$$

With those new densities, one can express all other fields stemming from the potentials (4.35) and (4.36) in terms of the redefined sources (4.124) and (4.125). Similar to [15], to avoid overcrowding the notation, we will use the same notation as before and redefine

$$
U := \int_{\mathcal{M}}\frac{\rho^*(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x', \qquad (4.126a)
$$

$$
U_\varphi^a := \int_{\mathcal{M}}\frac{\alpha_A^a(t,\boldsymbol{x}')\rho^*(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x', \qquad (4.126b)
$$

with the analogous rewritten potentials and superpotentials Σ, X, and Y as in [15]. Calulating these potentials, we obtain

$$
\begin{aligned}
U_\sigma &= U + \varepsilon\left\{\frac{3}{2}\Phi_1 - \Phi_2 + \Sigma^\varphi_a\left(U^a_\varphi\right)\right\} + \varepsilon^2\left\{\frac{7}{8}\Sigma(v^4) + \frac{1}{2}\Sigma\left(v^2U\right) - 4\Sigma\left(v^jV^j\right)\right.\\
&+\frac{5}{2}\Sigma(\Phi_1) - \Sigma(\Phi_2) + \frac{3}{2}\Sigma\left(U^2\right) - \frac{1}{2}\Sigma\left(\ddot{X}\right) - \alpha^{A0}_a\gamma^a_{bc}\Sigma\left(U^b_\varphi U^c_\varphi\right)\\
&-2\alpha^{A0}_a\gamma^a_{bc}\Sigma\left(\Sigma^b_\varphi\left(U^c_\varphi\right)\right) + 2\alpha^{A0}_a\Sigma\left(\Sigma^a_\varphi(U)\right) - \frac{1}{2}\alpha^{A0}_a\Sigma\left(\ddot{X}^a_\varphi\right)\\
&\left.+\frac{3}{2}\alpha^{A0}_a\Sigma\left(U^a_\varphi v^2\right) + \frac{1}{2}\Sigma\left(\left(\alpha^{A0}_a\alpha^{A0}_b + \beta^{A0}_{ab}\right)U^a_\varphi U^b_\varphi\right) + \alpha^{A0}_a\Sigma\left(U^a_\varphi U\right)\right\}\\
&+\varepsilon^{5/2}\left\{-\frac{4}{3}U\overset{(3)}{\mathcal{I}^{jj}}(t) - 2U^\varphi_a\dot{M}^a_\varphi(t) - \frac{1}{3}U^\varphi_a\,{}^a\overset{(3)}{\mathcal{I}^{kk}_\varphi}(t)\right.\\
&\left.+\frac{2}{3}\,{}^a\overset{(3)}{\mathcal{I}^{j}_\varphi}(t)\left(x^jU^\varphi_a - X^{\varphi,j}_a\right)\right\} + \mathcal{O}(\varepsilon^3)\,, \qquad (4.127)\\
U^a_{\varphi\sigma} &= U^a_\varphi + \varepsilon\left\{-\frac{1}{2}\Sigma^a_\varphi(v^2) + 3\Sigma^a_\varphi(U) - \Sigma^a_\varphi\left(\alpha^{A0}_bU^b_\varphi\right)\right\}\\
&+\varepsilon^2\left\{-\frac{1}{8}\Sigma^a_\varphi(v^4) + \frac{1}{2}\Sigma^a_\varphi\left(Uv^2\right) - 3\Sigma^a_\varphi\left(U^2\right) + 4\Sigma^a_\varphi\left(V^iv^i\right)\right.\\
&+\frac{1}{2}\Sigma^a_\varphi\left(\left(\alpha^{A0}_b\alpha^{A0}_c + \beta^{A0}_{bc}\right)U^b_\varphi U^c_\varphi\right) + \Sigma^a_\varphi\left(\alpha^{A0}_bU^b_\varphi U\right) - \frac{1}{2}\Sigma^a_\varphi\left(\alpha^{A0}_bU^b_\varphi v^2\right)\\
&+\Sigma^a_\varphi\left(\alpha^{A0}_b\varphi^b_1\right) + 4\Sigma^a_\varphi\left(\Phi_1\right) - 2\Sigma^a_\varphi\left(\Phi_2\right) - \frac{3}{2}\Sigma^a_\varphi\left(\ddot{X}\right)\\
&\left.-\frac{1}{2}\Sigma^a_\varphi\left(\gamma_{bc}\left(U^b_\varphi U^c_\varphi + 2\Sigma^b_\varphi\left(U^c_\varphi\right)\right)\right)\right\} + \varepsilon^{5/2}\left\{-2\Sigma^a_\varphi\left(\alpha^{A0}_b\right)\dot{M}^b_\varphi(t)\right.\\
&\left.-\frac{1}{3}\Sigma^a_\varphi\left(\alpha^{A0}_b\right){}^b\overset{(3)}{\mathcal{I}^{kk}_\varphi}(t) + \frac{2}{3}\,{}^b\overset{(3)}{\mathcal{I}^{j}_\varphi}(t)\left(x^j\Sigma^a_\varphi\left(\alpha^{A0}_b\right) - X^{a,j}_\varphi\left(\alpha^{A0}_b\right)\right)\right\}\\
&+\mathcal{O}(\varepsilon^3)\,. \qquad (4.128)
\end{aligned}
$$

To arrive at these conversions between the σ- and ρ^*-densities we relied on the identities

$$\Sigma(\alpha_a f) = \Sigma^\varphi_a(f)\,, \qquad (4.129)$$

$$\Sigma(\alpha_a f^a) = \Sigma^\varphi_a(f^a)\,, \qquad (4.130)$$

$$\Sigma(x^k f) = x^k\Sigma(f) - X^{,k}(f)\,, \qquad (4.131)$$

$$\Sigma(x^k \alpha_a f^a) \quad = \quad x^k \Sigma_a^\varphi(f^a) - X_a^{\varphi,k}(f^a)\,, \tag{4.132}$$

valid for any (vector-) fields f, f^a. The first two follow directly from the definitions whereas the ladder require partial integration to obtain the identity.

Using these to the ρ^*-density converted potentials, we can rewrite also the pseudo energy-momentum tensor $\tau^{\alpha\beta}$, Eq. (4.12) in terms of these new fields. The pseudo energy-momentum tensor written in this form is actually the starting point of analyzing the near-zone contribution to $h^{\alpha\beta}(t, \boldsymbol{x})$ evaluated for radiation-zone events $(t, \boldsymbol{x}) \in \mathcal{W}$. We will expand on that in more detail how this source term is used in the Outlook Section of Chapter 6 and the Appendix A.

4.4.2 Christoffel Symbols and their Expansion

In order to calculate the equation of motion, we first need to calculate the expansions of the Christoffel symbols to our desired order. Due to working in the Einstein frame, our metric expansion (4.22) is algebraically the same as in pure GR in [11]. Hence, calculating the Christoffel symbols via the standard identity

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\lambda} \left(g_{\lambda\beta,\gamma} + g_{\lambda\gamma,\beta} - g_{\beta\gamma,\lambda} \right) \tag{4.133}$$

yields

$$\begin{aligned}
\Gamma^0_{00} \quad &= \quad -\varepsilon \dot{U} - \varepsilon^2 \left(\frac{1}{4}(\dot{N}_1 + \dot{B}_1) - 4U\dot{U} - 4V^i U^{,i} \right) - \varepsilon^{5/2} \dot{N}_{1.5} \\
&\quad + \mathcal{O}(\varepsilon^3)\,, && \text{(4.134a)} \\
\Gamma^0_{0i} \quad &= \quad -\varepsilon^{1/2} U^{,i} - \varepsilon^{3/2} \left(\frac{1}{4}(N_1^{;i} + B_1^{;i}) - 4UU^{,i} \right) + \mathcal{O}\left(\varepsilon^{5/2}\right)\,, && \text{(4.134b)} \\
\Gamma^0_{ij} \quad &= \quad \varepsilon \left(4V^{(i,j)} + \dot{U}\delta^{ij} \right) + \mathcal{O}(\varepsilon^2)\,, && \text{(4.134c)}
\end{aligned}$$

$$
\begin{aligned}
\Gamma^i_{00} &= -\varepsilon^{1/2} U^{,i} - \varepsilon^{3/2}\left(\frac{1}{4}(N_1^{;i} + B_1^{;i}) + 4\dot{V}^i - 8UU^{,i}\right) \\
&\quad -\varepsilon^{5/2}\left(\frac{1}{4}(N_2^{;i} + B_2^{;i}) + \dot{K}_2^i - 2N_1U^{,i} - 2UN_1^{;i} - UB_1^{;i} - B_2^{ij}U^{,j}\right. \\
&\quad \left. -4V^i\dot{U} - 16U\dot{V}^i + 8V^jV^{j,i} + 48U^2U^{,i}\right) \\
&\quad -\varepsilon^3\left(\frac{1}{4}\left(N_{2.5}^{;i} + B_{2.5}^{;i}\right) + \dot{K}_{2.5}^i - 2N_{1.5}U^{,i} - B_{2.5}^{ij}U^{,j}\right) \\
&\quad +\mathcal{O}\left(\varepsilon^{7/2}\right) , && (4.134d) \\
\Gamma^i_{0j} &= \varepsilon\left(\dot{U}\delta^{ij} - 4V^{[i,j]}\right) + \varepsilon^2\left(\frac{1}{4}(\dot{N}_1 - \dot{B}_1)\delta^{ij} - K_2^{[i,j]} + \frac{1}{2}\dot{B}_2^{ij} - 4U\dot{U}\delta^{ij}\right. \\
&\quad \left. -4V^jU^{,i} + 16UV^{[i,j]}\right) - \varepsilon^{5/2}\left(\frac{1}{2}\dot{N}_{1.5}\delta^{ij} + K_{2.5}^{[i,j]} - \frac{1}{2}\dot{B}_{2.5}^{ij}\right) \\
&\quad +\mathcal{O}(\varepsilon^3) , && (4.134e) \\
\Gamma^i_{jk} &= \varepsilon^{1/2}\left(U^{,k}\delta^{ij} + U^{,j}\delta^{ik} - U^{,i}\delta^{jk}\right) \\
&\quad +\varepsilon^{3/2}\left(\frac{1}{4}\left(N_1^{;k}\delta^{ij} + N_1^{;j}\delta^{ik} - N_1^{;i}\delta^{jk}\right) - \frac{1}{4}\left(B_1^{;k}\delta^{ij} + B_1^{;j}\delta^{ik} - B_1^{;i}\delta^{jk}\right)\right. \\
&\quad \left. -4U\left(U^{,k}\delta^{ij} + U^{,j}\delta^{ik} - U^{,i}\delta^{jk}\right) + \frac{1}{2}\left(B_2^{ij,k} + B_2^{ik,j} - B_2^{jk,i}\right)\right) \\
&\quad +\mathcal{O}(\varepsilon^2) . && (4.134f)
\end{aligned}
$$

Examining these Christoffel symbols more closely, we immediately recognize that the Newtonian-like potential U, Eq. (4.126a), contributes to each symbol at its lowest order. This potential contributes either as its time derivative $\dot{U}$ or as the Newtonian acceleration field $U^{,j}$. This suggests that the equation of motion calculated in the following section will have this acceleration potential as the lowest term and then add post-Newtonian corrections.

4.4.3 Equation of Motion to 2.5 PN Order

Derivation of Equation of Motion

From the general contracted Bianchi identity applied to the field equation (2.13), we obtain

$$\nabla_\nu T^{\mu\nu} = \alpha_a(\varphi) T \, \nabla^\mu \varphi^a \, . \tag{4.135}$$

This TMST version of the conservation law valid for the Einstein frame energy-momentum tensor naturally differs from its physical Jordan frame counterpart, where the right-hand side vanishes. In our case here, the right-hand side incorporates self-gravitating effects in terms of the $\alpha_a(\varphi)$ coupled to the matter trace T as explained below Eq. (4.109) in the previous section. Now, using the energy-momentum tensor as given in (4.113),

$$T^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{1}{u^0} m_A(\varphi) u^\mu u^\nu \delta^3 \left(\boldsymbol{x} - \boldsymbol{z}_A\right) \, , \tag{4.136}$$

we can project both sides of Eq. (4.135) via the operator $P_\mu{}^\beta = \delta_\mu{}^\beta + u_\mu \, u^\beta$. This projection yields a modified geodesic identity for each compact body of the form

$$u^\nu \nabla_\nu u^\beta = - \, \alpha_a(\varphi) \left[\nabla^\beta \varphi^a + u^\beta u_\mu \nabla^\mu \varphi^a \right] \, . \tag{4.137}$$

In this form, we already see a derivative of a velocity on the left-hand side, and we realize that the right-hand side depends on the scalar fields in two ways. Both involve only the derivatives due to the equation stemming from a contracted Bianchi identity. Still, they differ because the latter contribution is a directional derivative along the velocity u^μ. Now, replacing $\alpha_a(\varphi)$ with the appropriate mass dependent version of Eq. (4.109)

$$\alpha_a^A(\varphi) = \frac{\partial \log(m_A(\varphi))}{\partial \varphi^a} = \frac{1}{m_A(\varphi)} \frac{\partial m_A(\varphi)}{\partial \varphi^a}\,, \tag{4.138}$$

and rewriting the covariant derivatives in terms of the Christoffel symbols (4.134), we obtain via a $3+1$ decomposition of the form

$$\frac{dv^j}{dt} + \Gamma^j_{\alpha\beta} v^\alpha v^\beta - \Gamma^0_{\alpha\beta} v^\alpha v^\beta v^j = -\frac{1}{m_A(\varphi)(u^0)^2} \frac{\partial m_A(\varphi)}{\partial \varphi^a} \left(\varphi^{a,j} - \dot{\varphi}^a v^j \right) . \tag{4.139}$$

Equation of Motion in terms of Metric Potentials

It is time to calculate the Equation of Motion to our desired order in terms of the 3-velocities $v^j \sim \sqrt{\varepsilon}$ and $v^0 = v_0 \sim \mathcal{O}(1)$. We collect all previously calculated terms and sort them according to their post-Newtonian contribution via the expansion

$$\frac{dv^j}{dt} = a^j_N + \varepsilon a^j_{PN} + \varepsilon^{3/2} a^j_{1.5PN} + \varepsilon^2 a^j_{2PN} + \varepsilon^{5/2} a^j_{2.5PN} + \mathcal{O}\left(\varepsilon^3\right) . \tag{4.140}$$

Now, substituting all relevant fields in (4.139), we obtain our final Newtonian order coefficients as

$$a^j_N = -\alpha_a^{A0} U_\varphi^{a,j} + v_0^2\, U^{,j} \,. \tag{4.141}$$

The first post-Newtonian correction is given as

$$\begin{aligned} a^j_{PN} \;=\;& 4v_0^2 \dot{V}^j - v_0^2 \dot{U} v^j + \alpha_a^{A0} \dot{U}_\varphi^a v^j - 2v_0 v^i v^j U^{,i} \\ & -\delta^{jk} v^i v^k U^{,i} - 4v_0 v^i V^{j,i} + \frac{1}{4} v_0^2 B_1^{,j} + \frac{1}{4} v_0^2 N_1^{,j} + 2v^2 \alpha_a^{A0} U_\varphi^{a,j} \\ & +2\alpha_a^{A0} U U_\varphi^{a,j} - \beta_{ab}^{A0} U_\varphi^b U_\varphi^{a,j} - \alpha_a^{A0} \varphi_1^{a,j} - 8v_0^2 U U^{,j} + \delta^{ik} v^i v^k U^{,j} \end{aligned}$$

$$+4v_0 v^i V^{i,j} - \delta^{ij} v^i \left(2v_0 \dot{U} + v^k U^{,k}\right) . \tag{4.142}$$

Similar to single STTs [15], but unlike GR [44], we have 1.5 PN correction to the motion as

$$a^j_{1.5PN} = -\alpha^{A0}_a \varphi^{a,j}_{1.5} . \tag{4.143}$$

Finally, we present the 2 and 2.5 PN coefficients

$$\begin{aligned}
a^j_{2PN} \quad = \quad & v_0^2 \dot{K}_2^{,j} - 16 v_0^2 \dot{V}^{,j} U - v_0 \dot{B}_2^{ij} v^i - \frac{1}{2} v_0 \dot{B}_1 \delta^{ij} v^i - \frac{1}{2} v_0 \dot{N}_1 \delta^{ij} v^i + 8 v_0 \dot{U} \delta^{ij} U v^i \\
& - \frac{1}{4} v_0^2 \dot{B}_1 v^j - \frac{1}{4} v_0^2 \dot{N}_1 v^j - 4\alpha^{A0}_a \left(\frac{1}{2} v^2 + U\right) \dot{U}^a_\varphi v^j + \alpha^{A0}_a \dot{\varphi}^a_1 v^j \\
& - \alpha^{A0}_a \alpha^{A0}_b \dot{U}^a_\varphi U^b_\varphi v^j + \left(\alpha^{A0}_a \alpha^{A0}_b + \beta^{A0}_{ab}\right) \dot{U}^a_\varphi U^b_\varphi v^j + 4 v_0^2 \dot{U} U v^j + \dot{U} \delta^{ik} v^i v^j v^k \\
& - 4 v_0^2 \dot{U} V^j - \frac{1}{2} v_0 v^i v^j B_1^{,i} + \frac{1}{4} \delta^{jk} v^i v^k B_1^{,i} - \frac{1}{2} v^i v^k B_2^{jk,i} - v_0 v^i K_2^{j,i} \\
& - \frac{1}{2} v_0 v^i v^j N_1^{,i} - \frac{1}{4} \delta^{jk} v^i v^k N_1^{,i} + 8 v_0 U v^i v^j U^{,i} + 4 \delta^{jk} U v^i v^k U^{,i} \\
& + 16 v_0 U v^i V^{j,i} + 2 v^i v^j v^k V^{k,i} - v_0^2 U B_1^{,j} - \frac{1}{4} \delta^{ik} v^i v^k B_1^{,j} + \frac{1}{4} v_0^2 B_2^{,j} \\
& + \frac{1}{2} v^i v^k B_2^{ik,j} + v_0 v^i K_2^{i,j} - 2 v_0^2 U N_1^{,j} + \frac{1}{4} \delta^{ik} v^i v^k N_1^{,j} + \frac{1}{4} v_0^2 N_2^{,j} \\
& - \alpha^{A0}_a \left(\frac{1}{2} v^2 + U\right)^2 U^{a,j}_\varphi + \alpha^{A0}_a \alpha^{A0}_b \varphi^a_1 U^{b,j}_\varphi - 2\alpha^{A0}_a \alpha^{A0}_b \left(\frac{1}{2} v^2 + U\right) U^a_\varphi U^{b,j}_\varphi \\
& - \alpha^{A0}_a \alpha^{A0}_b \alpha^{A0}_c U^a_\varphi U^b_\varphi U^{c,j}_\varphi + 2\left(\alpha^{A0}_a \alpha^{A0}_b + \beta^{A0}_{ab}\right) \left(\frac{1}{2} v^2 + U\right) U^b_\varphi U^{a,j}_\varphi \\
& + \frac{3}{2} \alpha^{A0}_c \left(\alpha^{A0}_a \alpha^{A0}_b + \beta^{A0}_{ab}\right) U^a_\varphi U^b_\varphi U^{c,j}_\varphi \\
& - \frac{1}{2} \left(\alpha^{A0}_a \alpha^{A0}_b \alpha^{A0}_c + \alpha^{A0}_c \beta^{A0}_{ab} + \alpha^{A0}_b \beta^{A0}_{ac} + \beta^{A0}_{abc}\right) U^b_\varphi U^c_\varphi U^{a,j}_\varphi \\
& + 2\alpha^{A0}_a \left(\frac{5}{8} v^4 + \frac{1}{4} B_1 + v^2 U - \frac{1}{2} U^2 + \frac{1}{4} N_1 - K_1^i v^i\right) U^{a,j}_\varphi \\
& - \left(\alpha^{A0}_a \alpha^{A0}_a + \beta^{A0}_{ab}\right) \varphi^b_1 U^{a,j}_\varphi + 2\alpha^{A0}_a \left(\frac{1}{2} v^2 + U\right) \varphi^{a,j}_1 + \alpha^{A0}_a \alpha^{A0}_b U^a_\varphi \varphi^{b,j}_1 \\
& + \left(-\alpha^{A0}_a \alpha^{A0}_b - \beta^{A0}_{ab}\right) U^b_\varphi \varphi^{a,j}_1 - \alpha^{A0}_a \varphi^{a,j}_2 - 2 v_0^2 N_1 U^{,j} + 48 v_0^2 U^2 U^{,j}
\end{aligned}$$

$$
\begin{aligned}
&-4\delta^{ik}Uv^iv^kU^{,j}+8v_0v^iV^iU^{,j}-16v_0Uv^iV^{i,j}+8v_0^2V^lV^{l,j}+\frac{1}{4}\delta^{ij}v^iv^kB_1^{,k}\\
&-\frac{1}{2}v^iv^kB_2^{ij,k}-\frac{1}{4}\delta^{ij}v^iv^kN_1^{,k}+4\delta^{ij}Uv^iv^kU^{,k}+2v^iv^jv^kV^{i,k}\\
&-v_0^2B_2^{jl}U^{,l}\,, \qquad (4.144)
\end{aligned}
$$

$$
\begin{aligned}
a^j_{2.5PN} \;=\;& v_0^2\dot{K}^j_{2.5}-v_0\dot{B}^{ij}_{2.5}v^i+v_0\dot{N}_{1.5}\delta^{ij}v^i-\dot{N}_{1.5}v_0^2v^j+\alpha_a^{A0}\dot{\varphi}^a_{1.5}v^j-v_0v^iK^{j,i}_{2.5}\\
&+\frac{1}{4}v_0^2B^{,j}_{2.5}+v_0v^iK^{i,j}_{2.5}+\frac{1}{4}v_0^2N^{,j}_{2.5}+2\alpha_a^{A0}N_{1.5}U_\varphi^{a,j}+\alpha_a^{A0}\alpha_b^{A0}\varphi^a_{1.5}U_\varphi^{b,j}\\
&-\left(\alpha_a^{A0}\alpha_b^{A0}+\beta_{ab}^{A0}\right)\varphi^b_{1.5}U_\varphi^{a,j}+2\alpha_a^{A0}\left(\frac{1}{2}v^2+U\right)\varphi^{a,j}_{1.5}+\alpha_a^{A0}\alpha_b^{A0}U_\varphi^a\varphi^{b,j}_{1.5}\\
&-\left(\alpha_a^{A0}\alpha_b^{A0}+\beta_{ab}^{A0}\right)U_\varphi^b\varphi^{a,j}_{1.5}-2v_0^2N_{1.5}U^{,j}-v_0^2B^{jl}_{2.5}U^{,l}\,. \qquad (4.145)
\end{aligned}
$$

Chapter 5

Self-Gravitating Bodies and Compact Binaries in TMST

Parts of our work in this chapter are based on the publication „Tensor-multiscalar gravity: Equations of motion to 2.5 post-Newtonian order", in Phys. Rev. D, 105.064034[1] *[19] by O. Schön, and D. D. Doneva. Please refer to our Contribution Statement at the beginning of this dissertation for more information.*

In the present dissertation, we have derived a ready-to-use version of the equation of motion to 2.5 post-Newtonian order in a general class of Tensor Multi-Scalar Theories (TMST) as introduced in [21]. To achieve this, we adapted the direct integration of the relaxed field equations approach [9–14] beyond general relativity and the single scalar field case. Due to the specifics of the TMST and the significant simplification of the field equations, we have performed our calculations in the conformal Einstein frame similar to [21, 27] and in contrast to previous PN studies in the single scalar field case that employs the physical Jordan frame [15, 16, 94–96, 101]. Thus, as a complementary result of our studies, the Einstein frame

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2.5 PN single scalar field equation of motion follows our results when multiple scalar fields are contracted to one.

We have consistently performed PN expansion of the metric and the scalar field up to 2.5 PN order. Using a skeletonization procedure to describe matter and compact objects in general, we have derived the generalized Binachi identity and the equation of motion in TMST. We have considered the possibility that a compact object's mass can depend on the scalar field for self-gravitating objects. In all these calculations, we have kept a general form of TMST admitting an arbitrary number of scalar fields without imposing restrictions on the target space metric.

Below we will summarize some of the main differences compared with previous studies in the single scalar field case and GR. We will also emphasize the physical interpretation of our result, especially concerning inspiraling binary compact objects.

5.1 Target Space Involvement

Among the most important differences between Tensor Multi-Scalar Theories to other alternative theories is the addition of the target space (T^n, γ_{ab}). Remember that this n-dimensional Riemannian manifold allows us to interpret the n extra scalar fields $\varphi = (\varphi^1, \ldots, \varphi^n)$ of our analyzed theory of gravity as generalized coordinates of this target space φ: *spacetime* $\rightarrow$ *target space* such that

$$\mathrm{d}\sigma^2 = \gamma_{ab}(\varphi)\,\mathrm{d}\varphi^a\,\mathrm{d}\varphi^b \tag{5.1}$$

is the line element of said target space. The addition of this construct naturally gives rise to the physical interpretation of said space. In particular, it is interesting how this manifold's curvature contributes to the TMST analysis. We try to hint at some answers here. First, note that we kept our work as general as possible concerning the target space, i.e., we did not choose any coordinates or make any topology/symmetry assumptions. Those two methods are generally the first steps to simplify equations involving the manifold (T^n, γ_{ab}). For example, choosing specific coordinates would allow the Christoffel symbols in Eqs. (4.27)-(4.28) to vanish at lowest order. However, since derivatives of these Christoffel symbols also enter our equations and as any choice of coordinates cannot guarantee globally both the symbols and their derivatives to vanish, some curvature terms will inevitably contribute to our post-Newtonian analysis here.

First, as explained in Section 4.2, the target space Christoffel symbols are contracted with the scalar fields and their derivatives in the source (4.30). This source then is integrated with the scalar field expansion (4.31d). Due to the coupling with the scalar fields, these curvature terms do not contribute to Newtonian order but rather start at first PN order in φ_1^a as evident from Eq. (4.52). As seen in the 1.5 PN spacetime metric (4.70), φ_1^a and hence the Christoffel symbols do not enter the gravitational fields $g_{\alpha\beta}$. This means that the tensorial waveform calculated in future work will be unaffected by these symbols to 1.5 PN order making the explicit curvature a 2 PN order effect. After that, we see multiple contributions from the Christoffel symbols in N_2 and B_2 given by Eqs. (4.83)-(4.84), and, of course, φ_2^a in Eq. (4.85). These contributions are always linked to some form of the Newtonian-like potentials U and U_φ^a through Eqs. (4.126a) and (4.126b). In φ_2^a, we even have the occurrence of products of Christoffel symbols as seen in Eq. (4.85), making the contribution of the target space curvature even more prominent.

From all the contributions mentioned above, it is clear that the geometry of the target space has a physical relevance in the post-Newtonian motion of compact objects. While these contributions are small in the sense that they occur at a higher post-Newtonian order than, for example, the contribution of the scalar fields themselves, the number of terms containing target space curvature fields is quite numerous in N_2 and B_2 (see Eqs. (4.83)-(4.84)). Hence, we expect their noticeable contributions to the tensorial waveform and detectable differences from general relativity and single scalar-tensor theories.

5.2 Self-Gravitating Bodies

Binaries consisting of strongly self-gravitating bodies play an important role in studying generalized theories of gravity. In the series [329–331], Eardley and Will showed that in a wide class of Brans-Dicke theories, such binaries are governed by dipole-radiation term. This term promised some new physics as no GR counterpart exists; hence, it is quite useful to distinguish GR from various versions of STTs. As explained in Section 4.4.1, the method to measure self-gravitating effects is in generalizing the coupling coefficients $\alpha_a(\varphi) = \partial \log(A(\varphi))/\partial\varphi^a$ given in Eq. (2.17), to

$$\alpha_a^A(\varphi) = \frac{\partial \log(m_A(\varphi))}{\partial\varphi^a} = \alpha_a(\varphi) + \frac{\partial \log(\widetilde{m}_A(\varphi))}{\partial\varphi^a}\,, \tag{5.2}$$

for the physical Jordan frame masses $\widetilde{m}_A(\varphi) = A^{-1}(\varphi)\, m_A(\varphi)$ and each self-gravitating compact object A. Hence, the magnitude of the coefficients $\alpha_a^A(\varphi)$ captures the coupling strength of the self-gravitating forces of a compact object A to the multiple scalar fields. This is manifested in the wave equation (4.14), as to lowest order it yields

$$\Box\varphi^a = -4\pi G_\star \sum_A \alpha^a_A T_A + \mathcal{O}(\varepsilon)\,, \tag{5.3}$$

for the localized version of T at body A.

We want to point out that we follow the approach of [21, 27] and work with $\alpha^A_a(\varphi)$ that is different from the standard sensitivities s_A defined in the Jordan frame formulation of the single scalar field PN approach (see, e.g., [15]) and is connected to the scalar charge of the body. The exact relation between $\alpha^A_a(\varphi)$ and s_A is extensively discussed in [15]. Here we will point out only that $\alpha^A_a(\varphi)$ is proportional to $1 - 2s_A$ in the case of a single scalar field which means that the standard value of $s_A = 1/2$ for a GR black hole translates to $\alpha^A_a(\varphi) = 0$.

Let us now discuss these self-gravitating effects in the context of our post-Newtonian analysis here. The explicit contributions of the sensitivities α^A_a to the expanded equation of motion potentials (4.141)–(4.145) is due to the direct dependency of the Einstein frame masses $m_A(\varphi)$ and its derivative on the right-hand side of the TMST equation of motion (4.139). All contributions are products and covariant derivatives with respect to the target space connection of the above given $\alpha^A_a(\varphi)$, namely

$$\beta^A_{ab} = D_a\,\alpha^A_b = \partial_a\alpha^A_b - \gamma^c_{ab}\,\alpha^A_c\,, \tag{5.4a}$$

$$\beta^A_{abc} = D_a D_b\,\alpha^A_c\,. \tag{5.4b}$$

Due to the way we have formulated the expanded equation of motion (4.141)–(4.145), we see that all explicit scalar field contributions get contracted with combinations of the here listed coupling fields. Of course, implicit scalar field contributions in the potentials $N_1, B_1, B_2^{ij}, K_2^i, N_2$, and B_2. Still, those fields' free target space indices get contracted with the Riemannian metric γ_{ab} and its Christoffel symbols. Hence, we can easily distinguish the scalar field contributions related to self-gravitating effects as the explicit appearances in (4.141)–(4.145). In the case of non-self-gravitating bodies, the physical Jordan frame masses $\widetilde{m}_A(\varphi)$ are then independent of the multiple scalar fields φ^a and the body-dependent fields (5.2) and (5.4) will reduce to their natural body-independent counterpart. The exact influence of these self-gravitating effects, especially about the dipole-radiation phenomena of TMST, is beyond the scope of this work and will be analyzed much more profoundly in future work when we tackle gravitational waveforms and scalar flux.

5.3 Binary Compact Objects

Studying the dynamics of binary compact objects and the observed waveforms its full complexity requires one to derive, on the one hand, the integrals of motion, as well as the equation of motion in the center-of-mass frame, and on the other to derive the expansion of the fields in the radiation zone that is prone to future work. In Appendix A, we already lay the groundwork for this analysis. Here we will discuss some conclusions that can be drawn from the equation of motion presented in this paper.

5.3.1 Binary Black Holes

It is well-known that black holes in single scalar field theories obey no-scalar-hair theorems that cover many possibilities (see, e.g., [332] and references therein). Their PN dynamics are also indistinguishable from GR at least up to 3 PN orders.

The nonlinear numerical simulations of binary black hole mergers confirm this also for regimes beyond the validity of the PN approach [333]. If we consider nonrotating black holes, similar conclusions will also be valid in TMST [334]. Using the results in the present dissertation, we can study whether the dynamics of binary black hole systems will also converge to GR if the conditions of this no-hair theorem in [334] are satisfied. If we assume that the scalar field is a constant (or zero) and the black hole mass is independent of the scalar field, then $\alpha_a^A(\varphi)$ and its derivatives are zero. If one examines closely the different terms entering the equation of motion (4.141)-(4.145), it is clear that the scalar field contribution will be held in σ_φ^a, similar to the single scalar field case [15], that are the sources of the field equations of the multiple scalar fields. The explicit form of these sources written in terms of $\alpha_a^A(\varphi)$ and its derivatives is given in (4.125). Clearly, zero $\alpha_a^A(\varphi)$ would lead to vanishing σ_φ^a. Therefore, the motion of bald black holes in TMST will coincide with GR at least up to the 2.5 PN order.

The extension to multiple scalar fields brings a higher degree of complexity to the equation of motion and offers possibilities for new phenomenology. Namely, it is possible to violate the no-scalar hair theorem and produce rotating black holes in TMST with nonzero scalar field [39] that is not allowed in the single scalar field case. The scalar field should have a nonzero scalar field potential, though, that is beyond the studies in the present paper and is a topic of future work (we refer the reader to [103, 105] for calculations in the single scalar field case performed though to a lower PN order).

5.3.2 Binary Neutron Stars

Neutron stars, unlike black holes, can easily develop scalar hair in modified gravity since the matter has a nonzero trace of the energy-momentum tensor and thus acts as a scalar field source. Again this is encoded in the PN formalist through the quantities $\alpha_a^A(\varphi)$ and its derivatives. What is interesting in TMST is that we can have a set of $\alpha_a^A(\varphi)$ associated with every scalar field that can be significantly different depending on the target space manifold and its metric. This will lead to exciting possibilities. For example, in TMST, topological and scalarized neutron star solutions have nonzero scalar hair with a vanishing scalar charge [34, 36]. It will be interesting to see how the resulting waveforms differ from GR, which can be done once we develop the PN formalist in TMST in the radiation zone.

5.3.3 Black Hole – Neutron Star Dynamics

First, let us limit ourselves to the case of nonrotating black holes with zero $\alpha_{a\,\mathrm{BH}}^A(\varphi)$ while we allow for a nonvanishing $\alpha_{a\,\mathrm{NS}}^A(\varphi)$ for the neutron star. In the single scalar field case, it was argued that the equation of motion at least up to 3 PN order depends on only a single combination of parameters involving the $\alpha_{a\,\mathrm{NS}}^A(\varphi)$ [15, 16, 94]. This dependence appears so that it is impossible to distinguish Brans-Dicke theory from other single scalar field theories based on mixed black hole-neutron star binary observations up to this PN order. Such a statement is not valid for the general case of TMST because we have an additional structure that is the target space equipped with a nontrivial metric γ_{ab}. As extensively discussed above, this metric and its first and second derivatives enter the PN expansion nontrivially, one of the main qualitative differences between the TMST and the single scalar field. The detailed analysis of the two body equation of motion and the related conserved quantities will be the topic of the second publication of this series. The basis of the calculations performed in the present paper is that one can conclude that the dynamics of a black hole-neutron star system will depend on the particular TMST under consideration, at least for a proper nontrivial choice of the target space metric. Thus the GW observations of such systems can help us discriminate between different subclasses of TMST.

We should, of course, always keep in mind that if one allows for rotating black holes in TMST, the scalar field and thus $\alpha^{A}_{a\,\mathrm{BH}}(\varphi)$ can be nonzero leading to a much more complicated and rich dynamics compared to the single scalar field case.

Chapter 6

Conclusion and Outlook

6.1 Conclusion

We adapted established mathematical machinery known as *direct integration of the relaxed field equations* (DIRE) [6–14] to a vast class of *Tensor Multi-Scalar Theories* (TMSTs) [21], a generalization of single *Scalar Field Theories* (SFTs) and, in turn, General Relativity (GR) itself. TMSTs are interesting for several reasons, including that they are a vast class of theories when adding fields to the Einstein-Hilbert action. Hence, it is a logical next step to test GR to calculate gravitational wave templates obtained from said theories. Our work here starts this process by analyzing the near-zone dynamics of compact objects in TMSTs. We found out that some unique characteristics of TMSTs, such as the geometry of the target space, a n-dimensional Riemannian Manifold (T^n, γ_{ab}) equipped with Riemannian metric γ_{ab} acting as the image of the scalar fields as

$$(\varphi^1, \dots, \varphi^n) = \varphi\colon\ \textit{space time} \ \rightarrow\ \textit{target space}. \tag{6.1}$$

It is very much an open problem how to interpret the target space and its geometry in a physically meaningful way. We established in our work here that the curvature of (T^n, γ_{ab}) explicitly impacts the equation of motion in TMSTs, hence the physical dynamics of compact objects. Work in TMSTs often relies on making symmetry assumptions to the target space to simplify the equations considerably [33, 35, 36, 39]. We did not limit ourselves to these cases and kept our equations in a general form to try to understand how exactly the target space involves itself.

The DIRE machinery allowed us to accurately calculate the equation of motion in TMST to 2.5 PN order. We iterated the relaxed field equations and the extra scalar fields wave equations in a post-Minkowskian setting to achieve this. We then incorporated slow-motion conditions to end up in a proper post-Newtonian analysis. Our study agrees with previous results obtained in GR and STTs [10, 15] when collapsed to no or one single scalar field, respectively. We found that compact binaries behave as expected from previous results with the interesting new dynamics from the target space. Another main difference in this work compared to GR is the existence of a 1.5 PN contribution to the equation of motion. This is not surprising due to its presence in STTs already. In our case, however, the 1.5 PN term is a sum over n Newtonian-like potentials allowing for further deviations to GR than STTs would allow.

Our analysis also includes the auxiliary result of studying STTs in the Einstein frame to an accuracy of 2.5 PN order by collapsing the equations to one single scalar field as demonstrated in Section 4.3.2. This allows for a more intuitive comparison of TMST and, hence, STT, to GR as it represents a closer relation in the action.

6.2 Outlook

This work can be considered a starting point of a series of papers to obtain a large set of gravitational wave templates in the vast space of Tensor Multi-Scalar Theories of Gravitation. This is necessary to further test General Relativity against viable alternatives and generalizations to gain further impact on a fundamental theoretical level.

This dissertation is concerned with generalizing previous fundamental results obtained in General Relativity and single Scalar Field Theories to a much broader class of Tensor Multi-Scalar Theories. This previous research contains the inception and implementation of the direct integration of the relaxed field equations (DIRE) in Relativity itself [6–14] as well as the more recent calculations in theories appending a scalar field [15–18]. As explained in Chapter 3, discussing any action motivated theory of a spacetime (M, g) with DIRE involves calculating the gravitational potentials $h^{\alpha\beta}$ via

$$\begin{aligned} &\eta^{\alpha\beta}(t,\boldsymbol{x}) - \sqrt{-g}\, g^{\alpha\beta}(t,\boldsymbol{x}) \\ =:\quad &h^{\alpha\beta}(t,\boldsymbol{x}) = \begin{cases} h^{\alpha\beta}_{\mathcal{N}}(t,\boldsymbol{x}) \;+\; h^{\alpha\beta}_{\mathcal{W}}(t,\boldsymbol{x}) & \text{for } (t,\boldsymbol{x}) \in \mathcal{N}(t,\boldsymbol{x})\,, \\ h^{\alpha\beta}_{\mathcal{N}}(t,\boldsymbol{x}) \;+\; h^{\alpha\beta}_{\mathcal{W}}(t,\boldsymbol{x}) & \text{for } (t,\boldsymbol{x}) \in \mathcal{W}(t,\boldsymbol{x})\,, \end{cases} \end{aligned} \tag{6.2}$$

where $\eta^{\alpha\beta}$ is the flat Minkowski metric, in the form of a post-Newtonian expansion in terms of a parameter $\varepsilon \sim Gm_c/c^2 r_c$ for the gravitational constant G, the speed of light c and the characteristic mass m_c and radius r_c of the physical system we are modeling separated in the near zone $\mathcal{N}(t,\boldsymbol{x})$ and wave zone $\mathcal{W}(t,\boldsymbol{x})$. In cases of added fields to the action, as in our work here, a similar equation for the added fields needs to be solved. We emphasize only the concept of the calculations needed here.

Hence, adapting the DIRE machinery, the first region to analyze is the near zone, as we have done to an accuracy of 2.5 PN order in the presented dissertation. For SFTs, the same work was done in [15]. Calculating the wave, or far away, zone dynamics is the second step, as results from the near zone can be adapted to lessen the number of calculations needed. This entails the second part of the cases presented in Eq. (6.2). Again, for single SFTs, this has been done already in [16–18]. Thus, the logical next step is to generalize these results to obtain the complete picture of TMSTs in a post-Newtonian setting via DIRE. The bulk of the work needed to calculate $h^{\alpha\beta}\,(t,\boldsymbol{x})$ for a wave zone event $(t,\boldsymbol{x})$ is again in obtaining the near zone contribution via calculating fields known as multi-index Epstein-Wagoner moments [6]. More precisely, it can be shown that the spatial near-zone dynamics $h^{ij}_{\mathcal{N}}(t,\boldsymbol{x})$ for a wave-zone field point $(t,\boldsymbol{x}) \in \mathcal{W}$ can be calculated as

$$h^{ij}_{\mathcal{N}}(t,\boldsymbol{x}) = \frac{2}{R}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\sum_{m=0}^{\infty}\hat{N}^{k_1}\dots\hat{N}^{k_m}I^{ijk_1\dots k_m}_{\mathrm{EW}}(\tau)\,, \tag{6.3}$$

where $\hat{\boldsymbol{N}} = \boldsymbol{x}/R$ for $R \gg \mathcal{R}$, that is, in a regime far away from the matter world tube with radius $\mathcal{R}$ where a gravitational wave detector may operate. This is obtained by expanding the gravitational potentials $h^{\alpha\beta}_{\mathcal{N}}$ in powers of $1/R$ and only keeping the lowest order term, most relevant for detectors satisfying $R \gg \mathcal{R}$, in addition to the conservation law related to the Lorenz gauge condition [16]

$$\tau^{\alpha\beta}{}_{,\beta} = 0\,. \tag{6.4}$$

The Epstein-Wagoner moments, as well as more details and possible challenges of this novel approach to TMST, are given in Appendix A

It is worth noticing that the results as mentioned above [16–18] have been checked via a different framework than DIRE, namely a symmetric trace-free approach [101]. The authors did find inconsistencies with [17] in the scalar and energy flux calculations and went into great detail on how these might arise. Hence, continuing the work in multiple scalar fields might shed some light on the discrepancies published in the paper mentioned above.

Appendix A

Epstein-Wagoner Moments in TMST

Let us briefly discuss how this dissertation might be continued in the future and which problems might arise by generalizing previous work to TMST. First, the formal solutions obtained using the retarded Green's functions, Eqs. (4.16) and (4.16), can be expanded in the near-zone in powers of $|\boldsymbol{x}'|/R$ for a radiation-zone event $(t, \boldsymbol{x})$ and retarded time $\tau = t - R$ to get [8, 16]

$$h_{\mathcal{N}}^{\alpha\beta}(t, \boldsymbol{x}) = 4 \sum_{q=0}^{\infty} \frac{(-1)^q}{q!} \left(\frac{1}{R} M^{\alpha\beta k_1 \cdots k_q} \right)_{,k_1 \cdots k_q} , \tag{A.1}$$

$$\varphi_{\mathcal{N}}^{a}(t, \boldsymbol{x}) = 2 \sum_{q=0}^{\infty} \frac{(-1)^q}{q!} \left(\frac{1}{R} {}^{a}M^{k_1 \cdots k_q} \right)_{,k_1 \cdots k_q} , \tag{A.2}$$

with the multipoles

$$M^{\alpha\beta k_1 \cdots k_q}(\tau) := \int_{\mathcal{M}} \tau^{\alpha\beta}(\tau, \boldsymbol{x}') x'^{k_1} \cdots x'^{k_q} \, \mathrm{d}^3 x' , \tag{A.3}$$

$${}^{a}M^{k_1 \cdots k_q}(\tau) := \int_{\mathcal{M}} \tau^{a}(\tau, \boldsymbol{x}') x'^{k_1} \cdots x'^{k_q} \, \mathrm{d}^3 x' , \tag{A.4}$$

where we have used

$$|\boldsymbol{x}-\boldsymbol{x}'|^q = \sum_{m=0}^{\infty} (-x')^{i_1\ldots i_m} R^q{}_{,i_1\ldots i_m} \,. \tag{A.5}$$

A critical difference between the integration here and the near-zone contribution as given in Eq. (3.35) is that the manifold $\mathcal{M}$ is now a hypersurface with respect to the constant retarded time $\tau = t - R$. First, remember that for any gravitational wave detector, only the spatial part of the gravitational potential, $h^{ij}_{\mathcal{N}}(t,\boldsymbol{x})$, matter. After that, we impose a far-away condition $R \gg \mathcal{R}$, reasonable for a detector on earth and far away from, e.g., compact binaries. This allows us only to keep the leading $1/R$ term to obtain

$$\begin{aligned}
h^{ij}_{\mathcal{N}}(t,\boldsymbol{x}) &= \frac{4}{R}\sum_{m=0}^{\infty}\frac{1}{m!}\frac{\partial^m}{\partial t^m}\int_{\mathcal{M}} \tau^{ij}(\tau,\boldsymbol{x}')\left(\hat{\boldsymbol{N}}\cdot\boldsymbol{x}'\right)^m \mathrm{d}^3x' + \mathcal{O}\left(R^{-2}\right)\,, && \text{(A.6)}\\
\varphi^{a}_{\mathcal{N}}(t,\boldsymbol{x}) &= \frac{2}{R}\sum_{m=0}^{\infty}\frac{1}{m!}\frac{\partial^m}{\partial t^m}\int_{\mathcal{M}} \tau^{a}(\tau,\boldsymbol{x}')\left(\hat{\boldsymbol{N}}\cdot\boldsymbol{x}'\right)^m \mathrm{d}^3x' + \mathcal{O}\left(R^{-2}\right)\,, && \text{(A.7)}
\end{aligned}$$

utilizing

$$\tau^{,i} = -\hat{N}^i := \frac{x^i}{R} \tag{A.8}$$

for the detector direction $\hat{\boldsymbol{N}}$.

Now, to obtain the tensorial gravitational waveform, we can further simplify $h^{ij}_{\mathcal{N}}(t,\boldsymbol{x})$ using the already in Chapter 6 mentioned conservation law Eq. (4.9), in its spatial form

$$\begin{aligned}
\tau^{ij} &= \frac{1}{2}\left(\tau^{00}x^{ij}\right)_{,00} + 2\left(\tau^{l(i}x^{j)}\right)_{,l} - \frac{1}{2}\left(\tau^{kl}x^{ij}\right)_{kl}\,, && \text{(A.9)}\\
\tau^{ij}x^k &= \frac{1}{2}\left(2\tau^{0(i}x^{j)}x^k - \tau^{0k}x^{ij}\right)_{,0} + \frac{1}{2}\left(2\tau^{l(i}x^{j)}x^k - \tau^{kl}x^{ij}\right)_{,l}\,. && \text{(A.10)}
\end{aligned}$$

Substituting this in Eq. (A.6) yields the final form

$$h^{ij}_{\mathcal{N}}(t,\boldsymbol{x}) = \frac{2}{R}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\sum_{m=0}^{\infty}\hat{N}^{k_1}\dots\hat{N}^{k_m}I^{ijk_1\dots k_m}_{\mathrm{EW}}(\tau) + \mathcal{O}\left(R^{-2}\right)\,, \tag{A.11}$$

where the Epstein-Wagoner moments [6] are explicitly given as

$$\begin{aligned}
I^{ij}_{\mathrm{EW}} &:= \int_{\mathcal{M}} \tau^{00}x^{ij}\,\mathrm{d}^3x \\
&\quad + \frac{\mathrm{d}^{-2}}{\mathrm{d}t^{-2}}\oint_{\partial\mathcal{M}}\left[4\tau^{l(i}x^{j)} - \left(\tau^{kl}x^{ij}\right)_{,k}\right]\mathcal{R}^2\hat{n}^l\,\mathrm{d}^2\Omega\,, && \text{(A.12)}\\
I^{ijk}_{\mathrm{EW}} &:= \int_{\mathcal{M}}\left(2\tau^{0(i}x^{j)k} - \tau^{0k}x^{ij}\right)\mathrm{d}^3x \\
&\quad + \frac{\mathrm{d}^{-1}}{\mathrm{d}t^{-1}}\oint_{\partial\mathcal{M}}\left[2\tau^{l(i}x^{j)k} - \tau^{kl}x^{ij}\right]\mathcal{R}^2\hat{n}^l\,\mathrm{d}^2\Omega\,, && \text{(A.13)}\\
I^{ijk_1\dots k_m}_{\mathrm{EW}} &:= \frac{2}{m!}\frac{\mathrm{d}^{m-2}}{\mathrm{d}t^{m-2}}\int_{\mathcal{M}}\tau^{ij}x^{k_1\dots k_m}\,\mathrm{d}^3x\,, && \text{(A.14)}
\end{aligned}$$

for the boundary $\partial\mathcal{M}$ of the hypersurface $\mathcal{M}$ with outward pointing unit normal $\hat{n}^l$. Writing the gravitational potentials in terms of these Epstein-Wagoner moments comes with the advantage that the energy-momentum pseudotensor in the integrand is always evaluated at the same retarded time $\tau = t - R$. We would also expect multiple $\mathcal{R}$-dependent terms arising in the expansion. As before, we disregard these terms as they necessarily need to cancel out with radiation-zone counterparts. During the calculation, there is another way to simplify the process. Only the transverse-traceless (TT) part of the gravitational potential tensor is needed for a gravitational wave. The TT contribution of any tensor can be obtained via a projection

$$h^{ij}_{\mathrm{TT}} = \left(P^{ip}P^{jq} - \frac{1}{2}P^{ij}P^{pq}\right)h^{ij}\,, \tag{A.15}$$

where $P^{pq} = \delta^{pq} - \hat{N}^p\hat{N}^q$ is the transverse projection operator [16]. Hence, potentials that cannot contain a TT contribution might be dropped immediately.

Calculating the Epstein-Wagoner moments is the bulk of the work in [16]. It should also be noted that no such conservation law as described above exists for the scalar-field source τ^a, making the calculations of the scalar energy flux more complicated [17]. There is also a debate about the potentials given in [17] as they do not coincide with results obtained using a symmetric trace-free approach in [101].

To evaluate these Epstein-Wagoner moments, we first need to transform the energy-momentum pseudotensor $\tau^{\alpha\beta}$, Eq. (4.12), in terms of the ρ^*-density, Eq. (4.118), introduced in Section 4.4.1. Using the ρ^*-density converted potentials, Eqs. (4.128) and (4.128), and substituting them, for instance in the expanded τ^{00}, Eq. (4.81), we calculate

$$\begin{aligned}
\tau^{00} \;=\; & \rho^* \Big\{ 1 + \frac{1}{2}v^2 + 3U + \alpha_a U_\varphi^a + \frac{1}{2}\Phi_1 - 3\Phi_2 - 14\Sigma_a^\varphi \left(U_\varphi^a\right) + \frac{3}{2}\ddot{X} \\
& + \frac{9}{2}U^2 - \frac{17}{2}\gamma_{ab}U_\varphi^a U_\varphi^b + 2v^2U + \frac{5}{2}v^2\alpha_a U_\varphi^a + 5\alpha_a U_\varphi^a U + \frac{3}{8}v^4 \\
& - 4v^jV^j + \frac{1}{2}\left(\alpha_a\alpha_b + \beta_{ab}\right)U_\varphi^a U_\varphi^b - \frac{1}{2}\alpha_a\Sigma_\varphi^a(v^2) + 2\alpha_a\Sigma_\varphi^a\left(U\right) \\
& - \alpha_a\Sigma_\varphi^a\left(\alpha_b U_\varphi^b\right) + \frac{1}{2}\alpha_a\ddot{X}_\varphi^a - \alpha_a\gamma_{bc}^a\left(U_\varphi^b U_\varphi^c + 2\Sigma_\varphi^b\left(U_\varphi^c\right)\right)\Big\} \\
& + \frac{1}{4\pi}\Big\{ -\frac{7}{2}(\nabla U)^2 + \frac{5}{2}\dot{U}^2 - 4U\ddot{U} - 8\dot{U}^{,k}V^k + 2V^{i,j}\left(3V^{j,i} + V^{i,j}\right) \\
& + \frac{1}{2}\gamma_{ab}\delta^{ij}U_\varphi^{a,i}U_\varphi^{a,j} - 4U^{,ij}\Phi_1^{ij} + 8\nabla U\cdot\nabla\Phi_1 - 4\nabla U\cdot\nabla\Phi_2 - \frac{7}{2}\nabla U\cdot\nabla\ddot{X} \\
& - 10U(\nabla U)^2 - 4U^{,ij}\left(P_2^{ij} - \gamma_{ab}P\left(U_\varphi^{a,i}U_\varphi^{b,j}\right)\right) - 6\gamma_{ab}U_\varphi^a\nabla U\cdot\nabla U_\varphi^b \\
& + 4\dot{V}^jU^{,j} - 6\gamma_{ab}\nabla U\nabla\Sigma_\varphi^a\left(U_\varphi^b\right) + 4\gamma_{ab}U\nabla U_\varphi^a\cdot\nabla U_\varphi^b + \frac{1}{2}\gamma_{ab}\dot{U}_\varphi^a\dot{U}_\varphi^b \\
& - \gamma_{ab}\gamma_{cd}^b\nabla U_\varphi^a\cdot\nabla\left(U_\varphi^c U_\varphi^d\right) - 2\gamma_{ab}\gamma_{cd}^b\nabla U_\varphi^a\cdot\nabla\Sigma_\varphi^c\left(U_\varphi^d\right) - \gamma_{ab}\nabla U_\varphi^a \\
& \times\nabla\Sigma_\varphi^b(U) + \frac{1}{2}\gamma_{ab}\nabla U_\varphi^a\cdot\nabla\ddot{X}_\varphi^b + \frac{1}{2}\gamma_{ab,c}U_\varphi^c\nabla U_\varphi^a\cdot\nabla U_\varphi^b\Big\} \\
& + \mathcal{O}\left(\rho\varepsilon^{5/2}\right)\,. \qquad \text{(A.16)}
\end{aligned}$$

To arrive at this form of τ^{00}, we must carefully evaluate all possible contributions. The most potentials in Eq. (A.16) stem from the σ-density counterpart Eq. (4.81). We also have to count in the contribution of σ an σ^{ii}, Eq. (4.124), inside the source τ^{00}, Eq. (4.81). Furthermore, inside the potentials σ an σ^{ii}, Eq. (4.124), we have contributions of the 1 PN potentials N_1, B_1, and φ_1^a, Eqs. (4.49), (4.51), and (4.52), which also need to be included to the correct order in Eq. (A.16). These origins might also include the Newtonian-like potentials U and U_φ^a, which must be translated to ρ^*-density as in Eqs. (4.128) and (4.128), before calculating them in here.

Let us look even closer at the first, the two-index, Epstein-Wagoner moment Eq. (A.13). We start with the common decomposition of the form

$$\begin{aligned} I^{ij}_{\mathrm{EW}} &= \int_{\mathcal{M}} \tau^{00} x^i x^j \, \mathrm{d}^3x + I^{ij}_{\mathrm{EW(surf)}} \\ &=: I^{ij}_{\mathrm{C}} + I^{ij}_{\mathrm{F}} + I^{ij}_{\mathrm{S}} \end{aligned} \tag{A.17}$$

including the moment I^{ij}_{C} only containing the compact support part of (A.16); the moment including the field part I^{ij}_{F}; and the surface integral part of Eq. (A.13)

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} I^{ij}_{\mathrm{EW(surf)}} := \frac{\mathrm{d}^2}{\mathrm{d}t^2} I^{ij}_{\mathrm{S}} := \oint_{\partial\mathcal{M}} \left[4\tau^{l(i}x^{j)} - \left(\tau^{kl}x^{ij}\right)_{,k} \right] \mathcal{R}^2 \hat{n}^l \, \mathrm{d}^2\Omega \,. \tag{A.18}$$

Hence, the first thing on the to-do list is integrating the compact support of τ^{00}, Eq. (A.16), weighted with the coordinates $x^{ij} = x^i x^j$ over the spatial hypersurface $\mathcal{M}$ of constant retarded time $\tau = t - R$. We have accommodated this calculation by collecting all of the terms we have to analyze in the first curly bracket in Eq. (A.16) above. All potentials naturally have the density ρ^* as a factor. From Eq. (4.118), we know

$$\rho^* = \sum_A m_A \, \delta^3 \left(\boldsymbol{x} - \boldsymbol{x}_A\right) . \tag{A.19}$$

Next, we will inspect the relevant terms of τ^{00}, Eq. (A.16), to the relevant order to prepare for the integration. Using Eq. (4.124), we find

$$\begin{aligned}
U(\tau, \boldsymbol{x}) &= \sum_A \frac{m_A}{|\boldsymbol{x} - \boldsymbol{x}_A|} \left(1 + \frac{3}{2} v_A^2 + \sum_{B \neq A} \frac{m_B}{|\boldsymbol{x}_A - \boldsymbol{x}_B|} \left(-1 + \alpha_a^A \alpha_B^a\right)\right) \\
&\quad + \mathcal{O}(\varepsilon^3) && \text{(A.20)} \\
U_\varphi^a(\tau, \boldsymbol{x}) &= \sum_A \frac{\alpha_A^a m_A}{|\boldsymbol{x} - \boldsymbol{x}_A|} \left(1 - \frac{1}{2} v_A^2 + \sum_{B \neq A} \frac{m_B}{|\boldsymbol{x}_A - \boldsymbol{x}_B|} \left(3 - \alpha_b^A \alpha_B^b\right)\right) \\
&\quad + \mathcal{O}(\varepsilon^3) && \text{(A.21)} \\
X(\tau, \boldsymbol{x}) &= \sum_A m_A |\boldsymbol{x} - \boldsymbol{x}_A| \left(1 + \mathcal{O}(\varepsilon)\right) && \text{(A.22)} \\
X_\varphi^a(\tau, \boldsymbol{x}) &= \sum_A \alpha_A^a m_A |\boldsymbol{x} - \boldsymbol{x}_A| \left(1 + \mathcal{O}(\varepsilon)\right) && \text{(A.23)} \\
V^k(\tau, \boldsymbol{x}) &= \sum_A \frac{m_A v_A^k}{|\boldsymbol{x} - \boldsymbol{x}_A|} + \mathcal{O}(\varepsilon^{5/2}) . && \text{(A.24)}
\end{aligned}$$

The superpotentials X and X_φ^a appear as second time-derivatives. Hence, we calculate

$$\partial_t^2 X = \sum_A \frac{m_A}{|\boldsymbol{x} - \boldsymbol{x}_A|} \left[v_A^2 - \left(\frac{\boldsymbol{x} - \boldsymbol{x}_A}{|\boldsymbol{x} - \boldsymbol{x}_A|} \cdot \boldsymbol{v}_A \right)^2 \right] - \sum_A m_A \boldsymbol{a}_A \frac{\boldsymbol{x} - \boldsymbol{x}_A}{|\boldsymbol{x} - \boldsymbol{x}_A|} , \tag{A.25}$$

for the acceleration $\boldsymbol{a}_A$. To the Newtonian order, this acceleration is given as

$$- \sum_{B \neq A} \frac{m_B}{|\boldsymbol{x}_A - \boldsymbol{x}_B|^3} \left(\boldsymbol{x}_A - \boldsymbol{x}_B\right) . \tag{A.26}$$

The calculation can exactly be adopted to obtain the counterpart result for $\ddot{X}^a_\varphi$ in

$$\partial_t^2 X^a_\varphi = \sum_A \frac{\alpha^a_A m_A}{|\boldsymbol{x}-\boldsymbol{x}_A|}\left[v_A^2 - \left(\frac{\boldsymbol{x}-\boldsymbol{x}_A}{|\boldsymbol{x}-\boldsymbol{x}_A|}\cdot \boldsymbol{v}_A\right)^2\right] - \sum_A \alpha^a_A m_A \boldsymbol{a}_A \frac{\boldsymbol{x}-\boldsymbol{x}_A}{|\boldsymbol{x}-\boldsymbol{x}_A|}\,. \quad \text{(A.27)}$$

Naively trying to carry out the integration runs into immediate problems not often discussed in the literature. Take, for instance, the potentials $\rho^* U$ and $\rho^* U^a_\varphi$ directly from τ^{00}, Eq. (A.16). Those terms each include two sums over the bodies. Naturally, at one point, the sum reaches the same body, and fractions like

$$\sum_A m_A^2 \frac{\delta^3\left(\boldsymbol{x}-\boldsymbol{x}_A\right)}{|\boldsymbol{x}-\boldsymbol{x}_A|}\,, \qquad \text{and} \qquad \sum_A \alpha^a_A m_A^2 \frac{\delta^3\left(\boldsymbol{x}-\boldsymbol{x}_A\right)}{|\boldsymbol{x}-\boldsymbol{x}_A|}\,, \qquad \text{(A.28)}$$

occur. This expression is not a well-defined distribution, as we encounter a "1/0" situation here. This reflects the problems of describing an extended, compact body as a point particle in GR, a highly nonlinear theory. To avoid ambiguity in the evaluation of the radiative quadrupole moment, we can, however, impose a regularization prescription of the form [44]

$$\frac{\delta^3\left(\boldsymbol{x}-\boldsymbol{x}_A\right)}{|\boldsymbol{x}-\boldsymbol{x}_A|} := 0\,, \qquad \text{(A.29)}$$

making the integral well-defined. This is generally known as Hadamard regularization [335, 336]. This regularization is implied in all calculations from this point moving forward.

Finally, inserting Eqs. (A.19)–(A.27) together with the other compact support potentials in τ^{00}, Eq. (A.16), into I^{ij}_C, always keeping the regularization (A.29) in mind, we find

$$
\begin{aligned}
I_C^{ij} \;=\; & \sum_A m_A x_A^{ij} \left\{ 1 + \frac{1}{2} v_A^2 + \sum_{B\neq A} \frac{m_B}{|\boldsymbol{x}_A - \boldsymbol{x}_B|} \left(3 - \gamma_{ab}\alpha_A^a \alpha_B^b\right) \right\} + \frac{3}{8} \sum_A m_A x_A^{ij} v_A^4 \\
& + \sum_A \sum_{B\neq A} \frac{m_A m_B x_A^{ij}}{|\boldsymbol{x}_A - \boldsymbol{x}_B|} \Bigg\{ v_A^2 \left(2 + \frac{5}{2}\gamma_{ab}\alpha_A^a \alpha_B^b\right) + 2v_B^2 - 4\boldsymbol{v}_A \cdot \boldsymbol{v}_B \\
& - \left(\boldsymbol{a}_B \cdot (\boldsymbol{x_A} - \boldsymbol{x}_B) + \left(\boldsymbol{v}_B \cdot \frac{\boldsymbol{x_A} - \boldsymbol{x}_B}{|\boldsymbol{x_A} - \boldsymbol{x}_B|} \right)^2 \right) \left(\frac{3}{2} + \frac{1}{2}\gamma_{ab}\alpha_A^a \alpha_B^b \right) \\
& + \sum_{C\neq A} \frac{m_C}{|\boldsymbol{x}_A - \boldsymbol{x}_C|} \left[\frac{9}{2} + 5\gamma_{ab}\alpha_A^a \alpha_C^b + \frac{1}{2}\left(\beta_{ab}^A + \alpha_a^A \alpha_b^A\right) \alpha_B^a \alpha_C^b \right. \\
& \left. - \alpha_a^A \gamma_{bc}^a \alpha_B^b \alpha_C^c \right] + \sum_{C\neq B} \frac{m_C}{|\boldsymbol{x}_B - \boldsymbol{x}_C|} \left[-3 - 14\gamma_{ab}\alpha_A^a \alpha_B^b - \gamma_{ab}\alpha_B^a \alpha_C^b \right. \\
& \left. \left. - 2\alpha_a^A \gamma_{bc}^a \alpha_B^b \alpha_C^c \right] \right\} .
\end{aligned} \tag{A.30}
$$

It gets more complicated for the field moment I_F^{ij} and the surface moment I_S^{ij}. Exactly as in [16], we have 24 potentials to integrate in τ^{00}, Eq. (A.16). For GR as well as single STTs, this is less complicated than in TMST here. For instance, in STT, the treatment of potentials of the form $(\nabla U)^2$ and $(\nabla U_s)^2$ is essentially indistinguishable, whereas, in TMST, the ladder term would read $\nabla U_\varphi^a \cdot \nabla U_\varphi^b$. This makes integration much more difficult, as the same mathematical machinery can not be used for different gradients. Furthermore, also potentials involving three factors become more difficult as all three Newtonian-like potentials can differ in terms like

$$
U_\varphi^c \nabla U_\varphi^a \cdot \nabla U_\varphi^b , \tag{A.31}
$$

making a big difference to the single STT version $U_s(\nabla U_s)^2$. Other three factor potentials include

$$
U_\varphi^a \nabla U \cdot \nabla U_\varphi^b , \qquad \text{and} \qquad U \nabla U_\varphi^a \cdot \nabla U_\varphi^b . \tag{A.32}
$$

Again, in pure GR these terms do, of course, not exist and in their STT counterparts they read

$$U_s \nabla U \cdot \nabla U_s \,, \qquad \text{and} \qquad U \left(\nabla U_s\right)^2 \,. \tag{A.33}$$

Having only two different potentials makes a big difference in the methods used to evaluate. As we advance, we need to develop new techniques to capture these differences and overcome these challenges.

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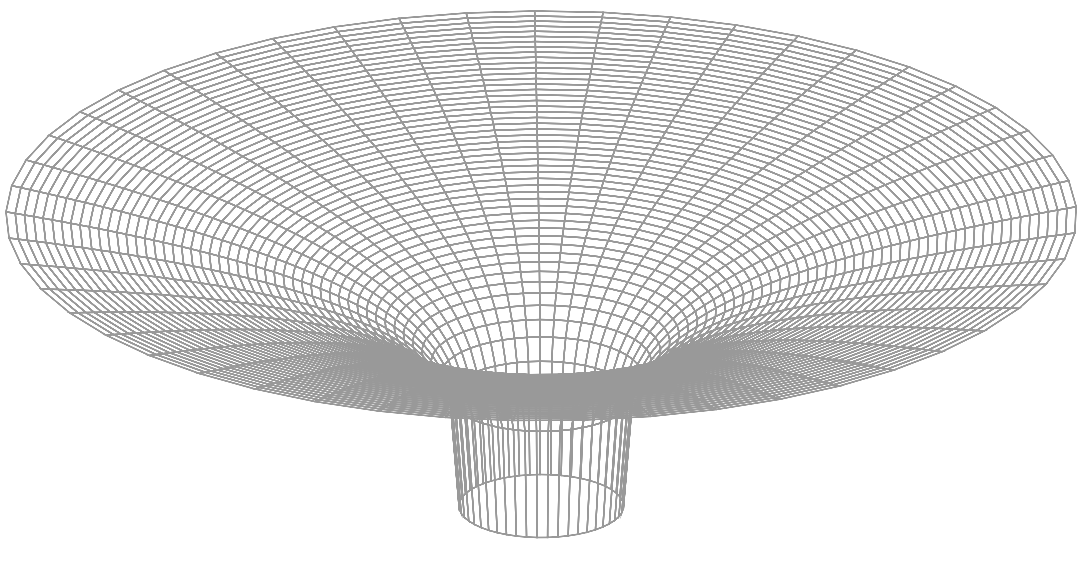
The End



Exterior of a Schwarzschild black hole. The end is somewhere in the middle.