Enumerative aspects of Tropical Geometry: Curves with cross-ratio constraints and Mirror Symmetry

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Christoph Jan Goldner
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14H52	Elliptic curves
14H81	Relationships between algebraic curves and physics
14J27	Elliptic surfaces, elliptic or Calabi-Yau fibrations
14J81	Relationships with physics
81T18	Feynman diagrams

Contents

Chapter 1

Introduction

This thesis is about answering enumerative questions using tropical geometry.

Tropical geometry, a toolbox for enumerative questions. *Tropical geometry* is a rather young field of mathematics that is intimately connected to algebraic geometry, non-Archimedean analytic geometry and combinatorics. It can be viewed as a degeneration of classical algebraic geometry. An important property of this *tropicalization* is that it usually is compatible with combinatorial structures. Which means that combinatorial structures of algebraic varieties are not only preserved when this degeneration is applied, but they are rather carved out and thus become visible in the tropical world.

In the past, tropical geometry turned out to provide powerful methods to answer enumerative questions. To apply methods from tropical geometry to an enumerative problem, a so-called *correspondence theorem* is required. A correspondence theorem states that an enumerative number equals its tropical counterpart, where in tropical geometry we have to count each tropical object with a suitable multiplicity reflecting the number of classical objects in our counting problem that tropicalize to the given tropical object. Thus tropical geometry hands us a new approach to enumerative problems:

(1) Find a suitable correspondence theorem, then

(2) use combinatorics to enumerate the tropical objects in question.

In many situations such correspondence theorems are known, which enables us to work on the tropical side for the largest parts of this thesis.

Mikhalkin [Mik05] pioneered the use of tropical methods in enumerative geometry by proving a correspondence theorem for counts of curves in toric surfaces satisfying point conditions. Later, other correspondence theorems were proven.

The tools: Tropical moduli spaces and intersection theory. *(Tropical) moduli spaces* and *(tropical) intersection theory* therein play – as in enumerative algebraic geometry – an important role in enumerative tropical geometry. Often, an enumerative problem can be expressed as an intersection product on the moduli space parametrizing the objects to be counted. Gathmann and Markwig [GM08] started to use these tropical moduli space techniques to answer an enumerative question.

Moduli spaces of abstract rational tropical curves were studied in [Mik07]. They also show up in the study of the tropical Grassmannian as the space of trees [AK06, SS06]. It turns out that these tropical moduli spaces are tropicalizations of the corresponding moduli spaces in algebraic geometry in a suitable embedding [Tev07, GM10]. Tropicalizations of moduli spaces of curves of higher genus (in a toroidal and non-Archimedean setting) were studied by Abramovich, Caporaso and Payne [ACP15]. The theory of rational tropical stable maps was introduced by Gathmann, Kerber and Markwig in [GKM09]. Ranganathan [Ran17] tropicalized the moduli space of rational stable maps to toric surfaces using logarithmic and non-Archimedean geometry.

A coherent tropical intersection theory was initiated by Mikhalkin [Mik06] and established by Allermann and Rau [AR10, Rau16]. Katz [Kat12] related tropical intersection theory to intersection theory on toric varieties studied by Fulton and Sturmfels in [FS97]. For matroidal fans (i.e. tropicalizations of linear spaces) Shaw offers in [Sha13] a framework of tropical intersection theory.

1.1 Topics

The present thesis consists of two main parts, which are to a great extend independent.

Part I is about counting rational tropical curves in $\mathbb{R}^2$ and $\mathbb{R}^3$ with so-called *cross-ratio* conditions. It is based on the published papers [Gol20a, Gol20c] and the preprint [Gol20b], which are written by the author of this thesis.

Part II is about counting tropical covers and counting tropical curves on a cylinder surface, using and extending tropical mirror symmetry techniques. It is based on the published paper [BGM20] and the preprint [BGM18], which are joint work with Janko Böhm and Hannah Markwig.

Figure 1.1 outlines the general structure of the thesis. On the left side, it shows Part I, which breaks down into three parts, that are different approaches to (almost) the same counting problem. Each of these approaches yields deeper insights into the original counting problem. As Figure 1.1 indicates, the combinatorial approach generalizes to rational tropical curves in $\mathbb{R}^3$. On the right side, the two subparts of Part II, their relation and the dimensions of their counting problems are shown.

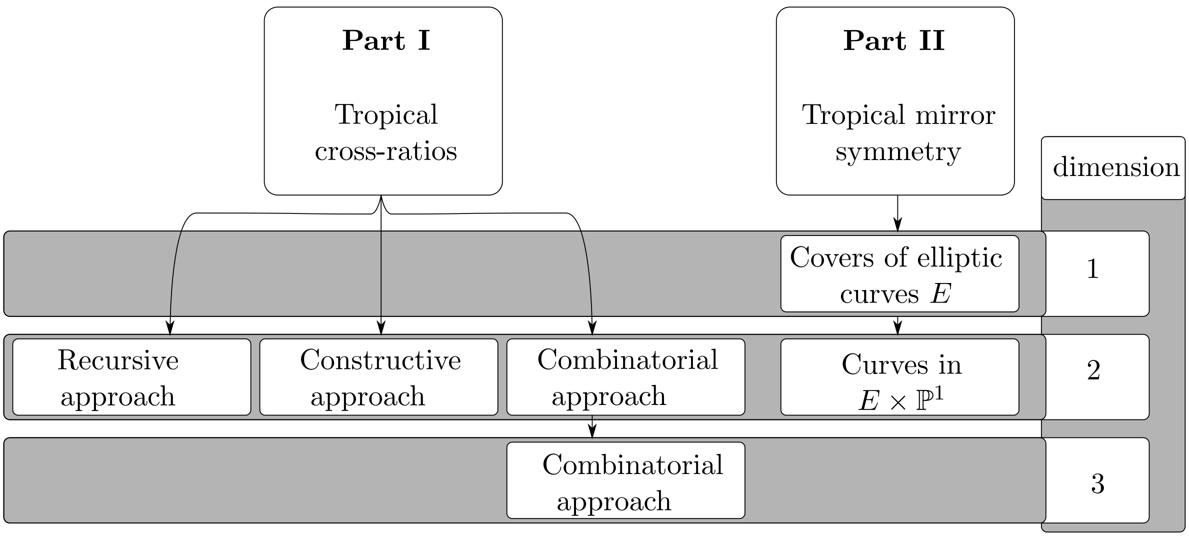

Figure 1.1: The overall structure of the present thesis.

1.1.1 Part I: Tropical cross-ratios

Consider the following enumerative problem: Determine the number N_d of rational degree d curves in $\mathbb{P}^2_{\mathbb{C}}$ passing through $3d-1$ general positioned points. For small d, this question can be answered using methods from classical algebraic geometry. It took until '94 when Kontsevich, inspired from developments in physics, presented a recursive formula to calculate the numbers N_d for all degrees. This recursion is known as *Kontsevich's formula.*

Theorem (Kontsevich's formula, [KM94])**.** *The numbers N_d are determined by the recursion*

$$N_d = \sum_{\substack{d_1+d_2=d \\ d_1,d_2>0}} \left(d_1^2 d_2^2 \cdot \binom{3d-4}{3d_1-2} - d_1^3 d_2 \cdot \binom{3d-4}{3d_1-1} \right) N_{d_1} N_{d_2}$$

with initial value $N_1 = 1$.

Tropical geometry, which is efficient when it comes to enumerative problems, provides a tropical proof of Kontsevich's formula. Recall that tropical geometry is applied in two steps:

(1) Find a suitable correspondence theorem. Mikhalkin's correspondence theorem [Mik05] states that the numbers N_d equal their tropical counterparts, i.e. they can be obtained from the weighted[1] count of rational tropical degree d curves in $\mathbb{R}^2$ passing through $3d-1$ general positioned points. Hence Kontsevich's formula translates into a recursion on the tropical side called *tropical Kontsevich's formula* and vice versa.

(2) Use combinatorics to enumerate the tropical objects in question. Gathmann and Markwig demonstrated the efficiency of tropical methods by giving a purely tropical proof of tropical Kontsevich's formula [GM08]. Applying Mikhalkin's correspondence theorem then yields Kontsevich's formula.

In the tropical proof — similar to the classical one — rational tropical degree d curves that satisfy point conditions, two line conditions and one *tropical cross-ratio* condition are considered. In fact, a byproduct of the proof of (tropical) Kontsevich's formula is a recursive formula for the number of rational (tropical) plane degree d curves that satisfy point conditions, two line conditions and one (tropical) cross-ratio condition in general position.

(Tropical) cross-ratios. A *cross-ratio* is the element

$$\frac{(q_3-q_1)(q_4-q_2)}{(q_3-q_2)(q_4-q_1)}$$

associated to four ordered points q_1, q_2, q_3, q_4 on a line. It encodes the relative position of these four points to each other. It is well-known that it is invariant under projective transformations and is therefore well-defined when four points on the projective line are considered. If a cross-ratio is given, then the positions of the points subject to that cross-ratio are restricted. Hence a cross-ratio can be seen as a condition that can be imposed on elements of the moduli space of n-pointed rational stable maps to a toric variety.

The tropical counterpart of a cross-ratio, a *tropical cross-ratio*, was first introduced by Mikhalkin in [Mik07] under the name "tropical double ratio" and can be thought of as paths of fixed lengths in a tropical curve. More precisely: A plane rational tropical curve is a 1-dimensional polyhedral complex in $\mathbb{R}^2$ whose unbounded polyhedra are uniquely labeled. Let (q_1, q_2, q_3, q_4) be a quadruple of labels of unbounded polyhedra. Mikhalkin defined the tropical double ratio as signed length of the intersection of the geodesic paths $\overline{q_1q_2}$ and $\overline{q_3q_4}$, where the sign is positive if and only if the orientations of those paths are compatible.

[1] Tropical curves are always counted with multiplicities.

Correspondence theorem. Tyomkin [Tyo17] provides a correspondence theorem that involves tropical cross-ratios. It states that the number of rational algebraic curves in a toric variety over an algebraically closed field of characteristic zero that satisfy general positioned point and cross-ratio conditions equals its tropical counterpart.

Counting problems. The idea of the proof of Kontsevich's formula raises the following enumerative question, which serves as a starting point of a whole list of enumerative questions.

(Q1) How many rational plane degree d curves are there that satisfy any appropriate[2]number of general positioned point and cross-ratio conditions?

Inspired from Kontsevich's recursive formula, we specialize (Q1) to the following question:

(Q2) Is there a general version of Kontsevich's formula that recursively calculates the number from Question (Q1)?

Further questions we may ask are:

(Q3) Can the rational (tropical) plane degree d curves from Question (Q1) be constructed?

(Q4) What about rational curves in other toric surfaces? What is the number of rational curves of a given degree in a toric surface that satisfy conditions as in Question (Q1)?

(Q5) What about rational curves in higher dimensional spaces? What is, for example, the number of rational degree d space curves that satisfy conditions as in Question (Q1)?

In order to find answers to questions (Q1)–(Q5), we use Tyomkin's correspondence theorem [Tyo17] for cross-ratios such that tropical geometry can be applied. It turns out that questions (Q2)–(Q5) require different approaches within tropical geometry. Each of the three approaches, which we call *recursive*, *constructive* and *combinatorial* approach, sheds light on different aspects of tropical curves that satisfy tropical cross-ratio conditions.

Recursive approach. In Chapter 4, Question (Q2) is answered by proving a *general tropical Kontsevich's formula* (Theorem 4.3.4) that recursively calculates the weighted number of rational plane tropical curves of degree d that satisfy point conditions, curve conditions and tropical cross-ratio conditions. In order to obtain a classical general Kontsevich's formula (Corollary 4.3.5), Tyomkin's correspondence theorem [Tyo17] is applied.

Notice that Tyomkin's correspondence theorem only holds for point and cross-ratio conditions. There is no correspondence theorem that relates the tropical numbers that also involve curve conditions to their classical counterparts, yet. That is, the general tropical Kontsevich's formula we obtain is capable of not only computing the algebro-geometric numbers we are looking for in Question (Q1), but also of computing further tropical numbers for which there is no correspondence theorem, yet.

The general Kontsevich's formula we derive allows us to recover Kontsevich's formula, see Corollary 4.3.7. The initial values of the general Kontsevich's formula are the numbers provided by the original Kontsevich's formula and so-called *cross-ratio multiplicities*, which are purely combinatorial (Definition 3.2.16).

[2] "Appropriate" means that there are enough conditions such that the numbers obtained from counting are finite.

We remark that Kontsevich's formula was generalized in different ways before, e.g. Ernström and Kennedy took tangency conditions into account [EK98, EK99] and Di Francesco and Itzykson [DFI95] generalized it among others to $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$. We are not aware of any generalization that involves multiple cross-ratios.

Constructive approach. There is a well-known combinatorial tool tropical geometry provides to explicitly construct tropical curves, the so-called *lattice path algorithm*. In Chapter 5, the tropical version of Question (Q3) is answered, i.e. all rational tropical plane degree d curves that satisfy an appropriate number of general positioned point conditions and tropical cross-ratio conditions are constructed. For that, the lattice path algorithm is extended such that it involves tropical cross-ratios. The resulting generalized lattice path algorithm is called *cross-ratio lattice path algorithm*. It is capable of explicitly constructing the tropical curves we want and thus gives an alternative answer to Question (Q1) using Tyomkin's correspondence theorem [Tyo17]. Moreover, our algorithm also works in arbitrary compact toric surfaces and thus answers Question (Q4).

Lattice paths were used in [Mik03, Mik05] to construct tropical curves that satisfy point conditions. Besides our generalization to tropical cross-ratios, there are other generalizations (in particular [MR09]) of lattice paths that inspired our approach. The lattice path algorithm can also be extended to determine invariants connected to counts of real curves as well, see [Shu06].

Combinatorial approach. In Chapter 6, the concept of so-called *floor diagrams* is generalized in order to answer question (Q5). Floor diagrams are graphs that arise from so-called floor-decomposed tropical curves by forgetting some information. Floor diagrams were introduced by Mikhalkin and Brugallé in [BM07, BM08] to give a combinatorial description of Gromov-Witten invariants of Hirzebruch surfaces. Floor diagrams have also been used to establish polynomiality of the node polynomials [FM10] and to give an algorithm to compute these polynomials in special cases – see [Blo11]. Moreover, floor diagrams have been generalized, for example in case of Psi-conditions, see [BGM12], or for counts of curves relative to a conic [Bru15].

We extend the concept of floor diagrams to involve tropical cross-ratios, and call the resulting combinatorial objects *cross-ratio floor diagrams*. Each of these cross-ratio floor diagrams is counted with a suitable multiplicity which reflects how many tropical curves degenerate to that cross-ratio floor diagram. Hence a weighted count of cross-ratio floor diagrams offers an alternative way of answering Question (Q1) using Tyomkin's correspondence theorem [Tyo17].

However, the main benefit of the combinatorial cross-ratio floor diagram approach is that it generalizes to higher dimension. Floor diagrams (resp. floor-diagram-like approaches) have been used in higher dimensions before, see [BM] (resp. [Tor14]). Using cross-ratio floor diagrams in higher dimension, we are able to provide an answer for the second part of Question (Q5), i.e. in case of space curves.

1.1.2 Part II: Tropical mirror symmetry

Mirror symmetry. Mirror symmetry is a duality relation that originated from string theory, where it relates 3-dimensional Calabi-Yau manifolds. Beyond that, mirror symmetry is at the base of many interesting developments in mathematics. For example, it appears in enumerative contexts which makes it particularly interesting for us. More precisely, enumerative results relate Gromov-Witten invariants to so-called *Feynman integrals*, where a Feynman integral is a formal expression that can be calculated using a computer algebra system.

Hurwitz numbers. Hurwitz numbers are counts of simply ramified covers with fixed genus and degree. It is known from Okounkov-Pandharipande's Gromov-Witten/Hurwitz correspondence [OP06] that Hurwitz numbers are a special case of descendant Gromov-Witten invariants. So some results of enumerative mirror symmetry are stated for Hurwitz numbers, like Dijkgraaf's famous mirror symmetry theorem for elliptic curves. For that, let $N_{d,g}$ denote a Hurwitz number of an elliptic curve, i.e. a count of simply ramified genus g covers of degree d of an elliptic curve with $2g-2$ fixed branch points.

Theorem (Dijkgraaf's mirror symmetry theorem for elliptic curves, [Dij95])**.** *For fixed $g \geq 2$, the equation*

$$\sum_d N_{d,g} q^d = \sum_\Gamma \frac{1}{|\operatorname{Aut}(\Gamma)|} I_\Gamma(q)$$

of formal power series holds, where the sum on the right goes over all 3*-valent connected graphs of genus g and $I_\Gamma(q)$ denotes a Feynman integral.*

Quasimodularity. Dijkgraaf's mirror symmetry theorem was used in [Dij95, KZ95] to prove that the generating series of Hurwitz numbers is a quasimodular form of weight $6g-6$. Quasimodularity behavior is desirable because it controls the asymptotic of the generating series.

Tropical mirror symmetry. Batyrev [Bat94] and Batyrev-Borisov [BB96] used combinatorics and toric geometry to study mirror symmetry. Their constructions influenced the well-known Gross-Siebert program for mirror symmetry, which aims at constructing new mirror pairs and providing an algebraic framework for SYZ-mirror symmetry [SYZ96, GS03, GS06, GS10]. The Gross-Siebert program established tropical geometry as a tool to prove such mirror symmetry relations [Gro11]. The philosophy how tropical geometry can be exploited is illustrated in the following triangle[3]:

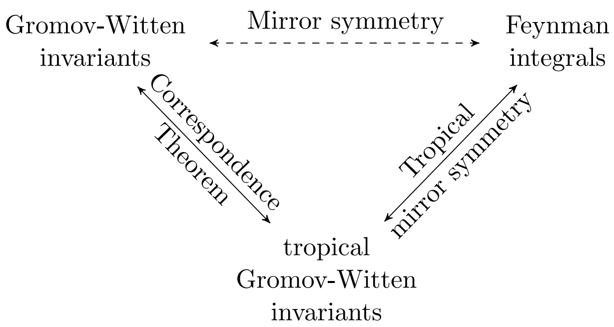

Figure 1.2: Tropical geometry and mirror symmetry.

In many situations, correspondence theorems relating Gromov-Witten invariants resp. enumerative invariants to their tropical counterparts are known [Mik05, NS06, CJM10, BBM11]. If we can relate the generating series of tropical invariants to Feynman integrals, we obtain a proof of the desired mirror symmetry relation using a detour via tropical geometry [Gro10, Ove15].

[3] A similar version of this triangle can be found in [Gro11].

Tropical mirror symmetry for elliptic curves

The tropical mirror symmetry part of the current thesis contains two subparts, see Figure 1.1. The first of these subparts (Chapter 8) is about tropical mirror symmetry for elliptic curves.

Dijkgraaf's mirror symmetry theorem via tropical geometry. Böhm, Bringmann, Buchholz and Markwig [BBBM17] investigated the triangle of Figure 1.2 for the case of Hurwitz numbers of an elliptic curve and Feynman integrals. They proved a tropical mirror symmetry relation, i.e. they provided the right arrow of the triangle of Figure 1.2 in this situation. Correspondence theorems for Hurwitz numbers existed already [CJM10, BBM11], where tropical Hurwitz numbers essentially count certain decorated graphs. Thus Dijkgraaf's mirror symmetry theorem follows as a corollary from [BBBM17] and a suitable correspondence theorem. Interestingly, the tropical approach of [BBBM17] revealed that the tropical mirror symmetry relation holds on an even finer level. Tropically, Feynman integrals and generating series of (labeled) tropical covers can be related graph by graph and order by order. As a consequence, one obtains interesting new quasimodularity statements for graph generating series [GM20].

The tropical mirror symmetry theorem for Hurwitz numbers of an elliptic curve can be viewed as a support for the strategy of the Gross-Siebert program, or more generally for the philosophy of using tropical geometry as a tool in mirror symmetry.

Fock space. The traditional approach to mirror symmetry of an elliptic curve involves operators on Fock spaces. There are two Fock spaces, a fermionic and a bosonic Fock space, and an isomorphism between them called the *Boson-Fermion correspondence.* The latter is usually viewed as the essence of mirror symmetry for elliptic curves. The generating series of Gromov-Witten invariants can be interpreted on the fermionic side. The Boson-Fermion correspondence expresses them in terms of matrix elements on the bosonic Fock space, which can be related to Feynman integrals [KR87, OP06, Li11a, Li11b]. Thus mathematical physics provides a proof of Dijkgraaf's theorem that involves operators on Fock spaces.

Generalizing tropical mirror symmetry of elliptic curves. The previous results about tropical mirror symmetry of elliptic curves yield the following question (we continue the numbering of the questions from above):

(Q6) Can the tropical mirror symmetry relation of tropical Hurwitz numbers of an elliptic curve be extended to tropical descendant Gromov-Witten invariants of an elliptic curve?

As it turns out in Chapter 8, the answer to Question (Q6) is positive. Therefore previous results involving the Fock space suggest the next question:

(Q7) Is there a relation between tropical descendant Gromov-Witten invariants of an elliptic curve and the Fock space?

It is also shown in Chapter 8 that the answer to Question (Q7) is positive as well. Again, we want to emphasize that Chapter 8 is based on [BGM18], which is joint work of the author of the present thesis with Janko Böhm and Hannah Markwig.

Methodology. The most important new tool of [BBBM17] in the study of mirror symmetry for elliptic curves was a bijection between tropical covers (i.e. decorated graphs) satisfying fixed discrete data and sets of monomials contributing to a coefficient in a Feynman integral (Theorem 2.30, [BBBM17]). To answer Question (Q6), it is generalized in two directions, both involving the source curves of the tropical covers in question:

(a) we need to allow vertices of valence different from 3, and

(b) we need to allow genus at vertices.

The task in (a) is a major extension of the bijection. The task in (b) involves the multiplicities with which covers are counted. In case of tropical Hurwitz numbers the multiplicity of a tropical cover is simply the product of expansion factors of its edges. In the case of descendant Gromov-Witten invariants, we also obtain local vertex contributions which are 1-point relative descendant Gromov-Witten invariants. In dimension one, the generating series of 1-point relative descendant Gromov-Witten invariants has a nice form and can be given in terms of sinus hyperbolicus [OP09]. This nice form enables us to single out vertex contributions in Feynman integrals and to prove a tropical version of Dijkgraaf's mirror symmetry theorem that involves descendant Gromov-Witten invariants.

To answer Question (Q7), we restrict to the case of tropical Hurwitz numbers since the general case of tropical descendant Gromov-Witten invariants could be treated similarly but the amount of notation required would increase largely. The main ingredient to link tropical Hurwitz numbers to the Fock space is a version of Wick's Theorem [Wic50] which encodes matrix elements in a bosonic Fock space as weighted sums of graphs, which can then directly be related to tropical Hurwitz covers.

Tropical mirror symmetry for $E \times \mathbb{P}^1$

The second subpart of tropical mirror symmetry (see Figure 1.1) is Chapter 9 in which the tropical mirror symmetry relation between tropical Gromov-Witten invariants of an elliptic curve E and Feynman integrals (Figure 1.2) is utilized to study generating series of Gromov-Witten invariants of $E \times \mathbb{P}^1$.

Relating Gromov-Witten invariants of $E \times \mathbb{P}^1$ to Feynman integrals. Gromov-Witten invariants of $E \times \mathbb{P}^1$ can be viewed as counts of curves in $E \times \mathbb{P}^1$ of fixed bidegree (d_1, d_2) and genus g that satisfy generic point conditions. Such a number is denoted by $N_{(d_1,d_2,g)}$. A main result of Chapter 9 states that, for fixed d_2 and g, the generating series $\sum_{d_1} N_{(d_1,d_2,g)} q^{d_1}$ equals a sum of Feynman integrals (see Corollary 9.3.5):

$$\sum_{d_1} N_{(d_1,d_2,g)} q^{d_1} = \sum_{\mathcal{P}} \frac{1}{|\operatorname{Aut}(\mathcal{P})|} I_{\mathcal{P}}(q), \tag{1.1}$$

where the sum on the right goes over particular graphs $\mathcal{P}$ which we call *pearl chains* (see Definition 9.2.1) of type (d_2, g). For fixed d_2 and g, there is a finite list of pearl chains of type (d_2, g). Equation (1.1) and, more generally, Chapter 9 is based on [BGM20], which is joint work of the author of the present thesis with Janko Böhm and Hannah Markwig.

Obviously, Dijkgraaf's mirror symmetry theorem for elliptic curves inspired equation (1.1). It is interesting that our generating series of Gromov-Witten invariants of $E \times \mathbb{P}^1$ can be expressed as a sum over the same kind of Feynman integrals that appear in Dijkgraaf's mirror symmetry theorem. Only the graphs over which we sum change when comparing Dijkgraaf's mirror symmetry theorem which involves Hurwitz numbers of an elliptic curve to Equation (1.1) which involves Gromov-Witten invariants of $E \times \mathbb{P}^1$.

Floor diagrams and curled pearl chains. In order to obtain (1.1), tropical geometry is used. To do so, we prove a correspondence theorem that states the equality of the Gromov-Witten invariant $N_{(d_1,d_2,g)}$ to its tropical counterpart. As in Part I (see "combinatorial approach" there), floor diagram techniques are then utilized. They allow us to reduce the counting problem of the tropical curves in tropical $E \times \mathbb{P}^1$ to certain tropical covers of a tropical elliptic curve. More precisely, floor diagrams are used to relate counts of tropical curves to counts of *curled pearl chains*, which can essentially be viewed as (combinatorial types of) tropical covers of a tropical elliptic curve with a particular source graph, namely a pearl chain. Thus results of Chapter 8 can be applied to deduce (1.1).

Further generalization and limitations. For our study of generating series of Gromov-Witten invariants of $E \times \mathbb{P}^1$, we restrict to invariants evaluating point conditions. This puts us within the scope of current techniques for correspondence theorems. It is also possible to evaluate points and insert Psi-conditions, i.e. study stationary descendant Gromov-Witten invariants. Preliminary work on correspondences exists for this case [MR20, CJMR21]. However, it is more complicated to relate the generating series of descendant Gromov-Witten invariants to Feynman integrals. The difficulty that arises can be expressed in terms of the multiplicity with which tropical objects are counted. In case of descendant Gromov-Witten invariants, local vertex contributions appear, which are one-point relative descendant Gromov-Witten invariants. Recall (b): in dimension one, the generating series of one-point relative descendant Gromov-Witten invariants has a nice form and can be given in terms of sinus hyperbolicus [OP09]. To count descendant Gromov-Witten invariants of $E \times \mathbb{P}^1$ tropically, we need one-point relative descendant Gromov-Witten invariants of $\mathbb{P}^1 \times \mathbb{P}^1$. We are not aware of a nice form of their generating series. This momentarily limits our possibilities to generalize Equation (1.1) to descendant Gromov-Witten invariants.

1.2 Overview of the results

Each of the parts I and II contains a preliminary chapter, namely chapters 2 and 7. The preliminary chapters contain no new results of this thesis. The results of Part I can be found in chapters 3 to 6, whereas the results of Part II can be found in chapters 8 and 9.

1.2.1 Part I

The results of Part I involve a description of tropical cross-ratios via tropical intersection theory on tropical moduli spaces, a generalized Kontsevich's formula, an algorithm to construct rational tropical curves that satisfy cross-ratio conditions and a combinatorial tool that enables us to determine the number of rational tropical curves in $\mathbb{R}^3$ that satisfy point and cross-ratio conditions.

Results of Chapter 3: Tropical cross-ratios and their degenerations

Most of the results of this chapter can be found in the paper [Gol20a] of the author that is published in Mathematische Zeitschrift.

A novel approach to tropical cross-ratios that uses tropical intersection theory on tropical moduli spaces is introduced (Definition 3.1.1, Remark 3.1.2). It is shown that the arising zero-dimensional cycles tropical intersection theory produces are enumerative. That means, on the one hand, that it is shown that the tropical curves that contribute to such zero-dimensional cycles are precisely the ones we expect according to the definition of tropical cross-ratios of Mikhalkin [Mik07] (Corollary 3.1.17). On the other hand, it means that it is shown that the multiplicities of tropical curves that the tropical intersection theoretic framework provides coincides with the multiplicities used in Tyomkin's correspondence theorem [Tyo17] (Proposition 3.1.19). Therefore our novel approach to tropical cross-ratios goes well with existing results.

In particular, it provides tools to determine the numbers of algebraic curves that satisfy point and cross-ratio conditions (Corollary 3.1.20).

Tropical intersection theory allows us to define degenerated tropical cross-ratios (Definition 3.2.1). We show that zero-dimensional cycles that are associated to such degenerated tropical cross-ratios are also enumerative (in the sense that they yield the desired numbers of Tyomkin's correspondence theorem), see Corollary 3.2.10. Compared to non-degenerated tropical cross-ratios, degenerated ones allow a simple combinatorial description of tropical curves that satisfy them (Corollary 3.2.12). This advantage of degenerated tropical cross-ratios comes with a minor trade-off: To determine the multiplicity of a tropical curve that satisfies degenerated tropical cross-ratios, an additional factor, called cross-ratio multiplicity, is required (Definition 3.2.16). It is shown that cross-ratio multiplicities are well-behaved: they are local and purely combinatorial (Corollary 3.2.22). Moreover, the well-known evaluation-multiplicity of tropical curves in $\mathbb{R}^2$ is generalized in order to describe multiplicities of tropical curves in $\mathbb{R}^2$ that satisfy degenerated tropical cross-ratios (Lemma 3.2.32).

Results of Chapter 4: General Kontsevich's formula

The results of this chapter can be found in the paper [Gol20c] of the author that was accepted for publication in the Electronic Journal of Combinatorics.

A general tropical Kontsevich's formula is established (Theorem 4.3.4) which is an answer to leading questions (Q1), (Q2), see Corollary 4.3.5. Moreover, it is shown that it implies the well-known original Kontsevich's formula (Corollary 4.3.7). We remark, that the general tropical Kontsevich's formula (Theorem 4.3.4) is not only capable of computing the numbers of rational algebraic curves that satisfy point and cross-ratio conditions we are interested in, but it also computes tropical numbers that involve tropical curve conditions for which there is no correspondence theorem, yet.

To prove the general tropical Kontsevich's formula, new methods in tropical geometry are developed. One of the new concepts arises in the context of movements of tropical curves. More precisely, if there are not enough conditions given, then rational tropical curves satisfying them may be moved and thus give rise to families of rational tropical curves. Such movements are described in special cases using the concept of *chains* inside the so-called movable component of a rational tropical curve, see the proof of Proposition 4.1.1. Another new method is used to describe multiplicities of rational tropical curves which have a so-called contracted bounded edge and thus split into two "smaller" rational tropical curves. Our approach is based on considering additional "artificial" degenerated tropical line conditions to the smaller rational tropical curves (Notation 4.2.1, Proposition 4.2.3).

Results of Chapter 5: Constructing tropical curves algorithmically

The results of this chapter can be found in the paper [Gol20a] of the author that is published in Mathematische Zeitschrift.

An algorithm, called cross-ratio lattice path algorithm, is presented. It calculates the numbers from leading question (Q1) by explicitly constructing all rational tropical curves that contribute to these numbers for a specific configuration of the point conditions (Theorem 5.2.4, Corollary 5.2.6). The cross-ratio lattice path algorithm therefore answers the leading question (Q3). Moreover, the cross-ratio lattice path algorithm works in arbitrary compact toric surfaces, i.e. it provides an answer to question (Q4).

Results of Chapter 6: Counting curves via cross-ratio floor diagrams

The results of this chapter can be found in the preprint [Gol20b] of the author.

Cross-ratio floor diagrams in $\mathbb{R}^2$ and $\mathbb{R}^3$ are introduced. We show that they are useful combinatorial tools to determine the numbers we are interested in. Cross-ratio floor diagrams in $\mathbb{R}^2$ provide an alternative answer to Question (Q1) when the tropical cross-ratios satisfy a simple additional condition, see Theorem 6.2.13. To define cross-ratio floor diagrams in $\mathbb{R}^3$, the concept of so-called *condition flows* on rational tropical curves is introduced[4] (Subsection 6.3.1). It is shown that cross-ratio floor diagrams in $\mathbb{R}^3$ are capable of determining the numbers of rational tropical curves in $\mathbb{R}^3$ that satisfy point conditions and tropical cross-ratio conditions if the tropical cross-ratios satisfy a simple additional condition (Theorem 6.3.34). As a result, the second part of Question (Q5) is answered (Corollary 6.3.37). Interestingly, the "artificial" degenerated tropical line conditions introduced in Chapter 4 also play an important role in the proof of Theorem 6.3.34.

1.2.2 Part II

The results of Part II involve a tropical mirror symmetry relation for generating series of tropical descendant Gromov-Witten invariants of an elliptic curve E, its interpretation in terms of operators on the Fock space, a mirror symmetry theorem for $E \times \mathbb{P}^1$ and results about quasimodularity of certain generating series of invariants of E and $E \times \mathbb{P}^1$.

Results of Chapter 8: Tropical mirror symmetry for elliptic curves

The results of this chapter can be found in the preprint [BGM18] which is joint work of the author with Janko Böhm and Hannah Markwig.

Question (Q6) is answered, i.e. the tropical mirror symmetry relation of tropical Hurwitz numbers of tropical elliptic curves is extended to tropical descendant Gromov-Witten invariants, see Theorem 8.1.9. It is remarkable that tropical mirror symmetry naturally holds on a finer level (Theorem 8.1.9, Theorem 8.1.14). Using a correspondence theorem, we show that the tropical mirror symmetry relation implies the mirror symmetry relation for descendant Gromov-Witten invariants of elliptic curves (Theorem 8.1.4). Moreover, the finer level on which the tropical mirror symmetry relation holds allows us to deduce quasimodularity results for generating functions of tropical covers (resp. for sums contributing to Feynman integrals), see Corollary 8.1.20.

We also show that in case of Hurwitz numbers there is a connection between the Fock space approach to mirror symmetry of elliptic curves and the tropical approach (Theorem 8.2.10). More precisely, we shown that tropical Hurwitz numbers of elliptic curves can directly be linked to matrix elements on the bosonic Fock space. Figure 1.3 illustrates the philosophy of how tropical geometry acts as a shortcut to the bosonic Fock space, which supports the slogan "tropicalization is bosonification" from [CJMR18], and the intuition underlying the Gross-Siebert program that tropical geometry is a natural language in the context of mirror symmetry.

[4]We want to remark that a similar construction was independently introduced by Mandel and Ruddat [MR19] to study multiplicities of tropical curves.

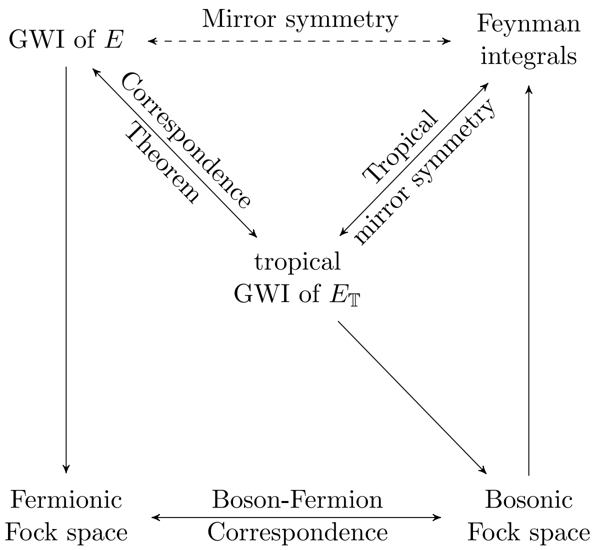

Figure 1.3: Boson-Fermion correspondence and tropical geometry as a shortcut.

Results of Chapter 9: Tropical mirror symmetry for $E \times \mathbb{P}^1$

The results of this chapter can be found in the paper [BGM20] which will appear in Annales de l'Institut Henri Poincaré D: Combinatorics Physics and their Interactions. It is joint work of the author with Janko Böhm and Hannah Markwig.

The main results of Chapter 9 are Theorem 9.3.1 and its refined version which is Theorem 9.3.11. Both theorems are stated using so-called *pearl chains* which we introduce in Definition 9.2.1. We show that pearl chains are an effective tool to count tropical curves since counting them leads to the same numbers as counting tropical curves in tropical $E \times \mathbb{P}^1$ (Theorem 9.2.8).

Theorem 9.3.1 yields a tropical mirror symmetry statement which relates generating series of tropical curves in tropical $E \times \mathbb{P}^1$ to sums of Feynman integrals (Corollary 9.3.4). Using the correspondence theorem 9.1.16, a mirror symmetry statement for $E \times \mathbb{P}^1$ is established, see Corollary 9.3.5. It relates generating series of Gromov-Witten invariants of $E \times \mathbb{P}^1$ to sums of Feynman integrals.

Moreover, Theorem 9.3.1 as well as Theorem 9.3.11 is stated for tropical curves of so-called *leaky degree*. This is done with a view towards further generalizations involving tangency conditions. Most natural, and most important for our main application, which is Corollary 9.3.5, is the case where the leaky degree is zero. Equation (1.1) of Corollary 9.3.5 and Theorem 7.3.7 is the equality of the very left side with the very right side of Figure 1.4. For the intermediate equalities, we prove more general versions involving tropical curve counts of leaky degree, curled pearl chains with leaking and Feynman integrals which are (non-constant) coefficients of power series (see theorems 9.3.1, 9.3.11 and Corollary 9.3.12). These generalizations can be obtained essentially with the same effort and have potentially further applications in the theory of tropical curve counts.

As in the case of an elliptic curve, the relation of Gromov-Witten invariants of $E \times \mathbb{P}^1$ and Feynman integrals can be used to obtain new quasimodularity statements: the quasimodularity of a summand $I_{\mathcal{P}}(q)$ of Equation 1.1 for a fixed graph $\mathcal{P}$ can be deduced from [OP18], see Theorem 9.4.5.

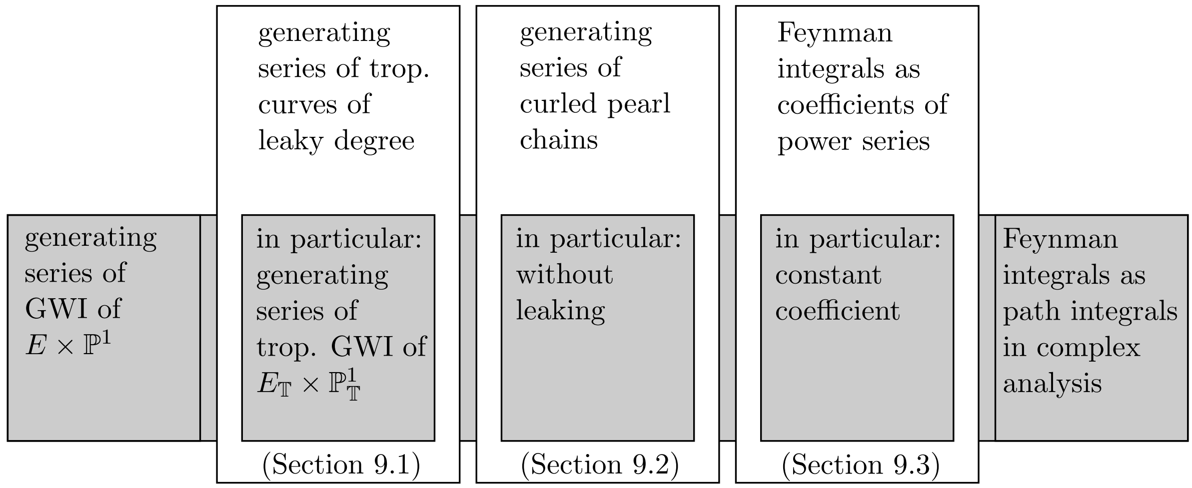

Figure 1.4: A chart partially summing up results of Chapter 9.

1.3 Outlook and open questions

The approaches and results of the current thesis raise further questions. We briefly gather them in this section and make suggestions for further research.

Tropical curves and tropical cross-ratios.

(1) The multiplicity of a rational tropical curve that satisfies degenerated tropical cross-ratios depends on a cross-ratio multiplicity, which is a combinatorial factor that can be calculated locally at vertices (Definition 3.2.16). Although this cross-ratio multiplicity can be calculated by distinguishing all finitely many cases, it raises the question whether there is a more efficient way of calculating it, see 3.2.27.

(2) The floor decomposition technique that is used in Chapter 6 generalizes to higher dimension (Proposition 6.1.6). In order to use this technique for counting rational tropical curves that satisfy general positioned point conditions and degenerated tropical cross-ratio conditions, a sufficiently local description of the multiplicity of a floor decomposed rational tropical curve is required, see sections 3.2.2, 6.3.2. Currently, no such description is available for floor decomposed rational tropical curves in $\mathbb{R}^n$ for $n > 3$. This prevents us from extending our combinatorial cross-ratio floor diagram approach to higher dimensions.

(3) Tropical geometry can be used to deduce results for *real* enumerative problems like Welschinger invariants [IKS03, IKS09]. It would be interesting to see whether tropical cross-ratios could be incorporated into such real enumerative problems.

Tropical mirror symmetry.

(4) Our approach to tropical mirror symmetry for $E \times \mathbb{P}^1$ involves point conditions only, whereas our results for tropical mirror symmetry for E involve point conditions as well as Psi-conditions. Thus the question is raised whether Psi-conditions can be incorporated into our tropical mirror symmetry theorem 9.3.1 for $E \times \mathbb{P}^1$. Again, multiplicities of tropical curves prevent us from doing so: When considering Psi-conditions (i.e. in case of

descendant Gromov-Witten invariants) local vertex contributions appear, which are one-point relative descendant Gromov-Witten invariants. In case of tropical mirror symmetry for E the generating series of one-point relative descendant Gromov-Witten invariants has a nice form and can be given in terms of sinus hyperbolicus [OP09]. To count descendant Gromov-Witten invariants of $E \times \mathbb{P}^1$ tropically, we need one-point relative descendant Gromov-Witten invariants of $\mathbb{P}^1 \times \mathbb{P}^1$. We are not aware of a nice form of their generating series. This momentarily prevents us from incorporating Psi-conditions into our results for tropical mirror symmetry for $E \times \mathbb{P}^1$.

(5) There is a proof of tropical mirror symmetry for E in case of Hurwitz numbers via the bosonic Fock space (Section 8.2). Thus the question whether there is a Fock space approach to tropical mirror symmetry for $E \times \mathbb{P}^1$ comes up. Intuitively, we expect that there is such an approach since tropical geometry usually acts as a shortcut to the bosonic Fock space, see Figure 1.3. Indeed, the only difficulties we anticipate are of technical nature and arise due to the amount of notation. More precisely, we expect that one can define *black* and *white labeled cut-join* operators similar to the labeled cut-joint operators used for proving tropical mirror symmetry for E (Definition 8.2.9). Once suitable operators are found, proofs of Section 8.2 can be imitated to yield the desired bosonic Fock space approach to tropical mirror symmetry for $E \times \mathbb{P}^1$.

1.4 Acknowledgements

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... her tireless and friendly way when answering my mathematical questions,

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Last, I want to thank anonymous persons: the referees, that provided useful comments and helped to improve the quality of my papers, and the various people that answer LaTeX related questions online.

Part I

Enumeration of tropical curves with tropical cross-ratios

Chapter 2

Preliminaries

In the preliminary section, we first give a brief overview of the necessary tropical intersection theory. After that, standard notations and definitions from tropical geometry (e.g. moduli spaces of rational tropical stable maps) are recalled, see [Mik07, GM08, GKM09]. It is pointed out that tropical intersection theory can be applied to the moduli spaces of interest. Besides this, we try to make notations used as clear as possible by introducing them in separate blocks to which we refer later.

Notation 2.0.1. We write $[m] := \{1, \ldots, m\}$ if $0 \neq m \in \mathbb{N}$, and if $m = 0$, then define $[m] := \emptyset$. Underlined symbols indicate a set of symbols, e.g. $\underline{n} \subset [m]$ is a subset of $\{1, \ldots, m\}$. We may also use sets S of symbols as an index, e.g. p_S, to refer to the set of all symbols p with indices taken from S, i.e. $p_S := \{p_i \mid i \in S\}$. The #-symbol is used to indicate the number of elements in a set, for example $\#[m] = m$.

2.1 Tropical intersection theory

This section collects intersection theoretic background. For more details about tropical intersection theory, see [FS97, Rau09, All10, AR10, Kat12, Sha13, AHR16, Rau16].

2.1.1 Affine tropical cycles

Definition 2.1.1 (Normal vectors and balanced fans)**.** Let $V := \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ be the real vector space associated to a given lattice Γ and let X be a fan in V. The lattice generated by $\operatorname{span}(\kappa) \cap \Gamma$, where κ is a cone of X, is denoted by Γ_κ. Let σ be a cone of X and τ be a face of σ of dimension $\dim(\tau) = \dim(\sigma) - 1$ (we write $\tau < \sigma$). A vector $u_\sigma \in \Gamma_\sigma$ that generates $\Gamma_\sigma / \Gamma_\tau$ such that $u_\sigma + \tau \subset \sigma$ defines a class $u_{\sigma/\tau} := [u_\sigma] \in \Gamma_\sigma / \Gamma_\tau$ that does not depend on the choice of u_σ. This class is called *normal vector of σ relative to τ.*

X is a *weighted fan of dimension k* if X is of pure dimension k and there are weights on its facets (i.e. its k-dimensional faces), that is there is a map $\omega_X : X^{(k)} \to \mathbb{Z}$. The number $\omega_X(\sigma)$ is called *weight* of the facet σ of X. To simplify notation, we write $\omega(\sigma)$ if X is clear. Moreover, a weighted fan (X, ω_X) of dimension k is called a *balanced* fan of dimension k if

$$\sum_{\sigma \in X^{(k)}, \tau < \sigma} \omega(\sigma) \cdot u_{\sigma/\tau} = 0$$

holds in $V / \langle \tau \rangle_{\mathbb{R}}$ for all faces τ of dimension $\dim(\tau) = \dim(\sigma) - 1$.

Definition 2.1.2 (Affine cycles)**.** Let $V := \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ be the real vector space associated to a given lattice Γ. A *tropical fan X (of dimension k)* is a balanced fan of dimension k in V and $[(X, \omega_X)]$

denotes the refinement class of X with weights ω_X (see Definition 2.8 and Construction 2.10 of [AR10]). Such a class is also called an *affine (tropical) k-cycle* in V. Denote the set of all affine k-cycles in V by $Z_k^{\mathrm{aff}}(V)$. For a fan X in V, we may also define an *affine k-cycle* in X as an element $[(Y,\omega_Y)]$ of $Z_k^{\mathrm{aff}}(V)$ such that the support of Y with nonzero weights lies in the support of X (see Definition 2.15 of [AR10]). Define $|[(X,\omega_X)]|$ as the support of X with nonzero weights.

The set $Z_k^{\mathrm{aff}}(V)$ (resp. $Z_k^{\mathrm{aff}}([(X,\omega_X)])$) can be turned into an abelian group by taking unions while refining appropriately.

Example 2.1.3. Consider the vector space $\mathbb{R}^2$ with its standard lattice $\mathbb{Z}^2$. Let e_1, e_2 be the vectors of the standard basis of $\mathbb{R}^2$. Let X be the 1-dimensional fan whose rays are given by $-e_1, -e_2, e_1+e_2$. Define all weights of X to be 1. Hence (X,ω_X) is a balanced fan and its refinement class $[(X,\omega_X)]$ is an affine (tropical) 1-cycle.

Definition 2.1.4 (Rational functions)**.** Let $[(X,\omega_X)]$ be an affine k-cycle. A *(nonzero) rational function on* $[(X,\omega_X)]$ is a continuous piecewise linear function

$$\varphi : |[(X,\omega_X)]| \to \mathbb{R},$$

i.e. there exists a representative (X,ω_X) of $[(X,\omega_X)]$ such that on each cone $\sigma \in X$ the map φ is the restriction of an integer affine linear function. The set of (nonzero) rational functions of $[(X,\omega_X)]$ is denoted by $\mathcal{K}^*([(X,\omega_X)])$.

Define

$$\mathcal{K}\left([(X,\omega_X)]\right) := \mathcal{K}^*([(X,\omega_X)]) \cup \{-\infty\}$$

such that $(\mathcal{K}([(X,\omega_X)]), \max, +)$ is a semifield, where the constant function $-\infty$ is the "zero" function.

Definition 2.1.5 (Divisor associated to a rational function)**.** Let $[(X,\omega_X)]$ be an affine k-cycle in $V := \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ and $\varphi \in \mathcal{K}^*([(X,\omega_X)])$ a rational function on $[(X,\omega_X)]$. Let (X,ω) be a representative of $[(X,\omega_X)]$ on whose cones φ is affine linear and denote these linear pieces by φ_σ. We denote by $X^{(i)}$ the set of all i-dimensional cones of X. We define

$$\operatorname{div}(\varphi) := \varphi \cdot [(X,\omega_X)] := [(\bigcup_{i=0}^{k-1} X^{(i)}, \omega_\varphi)] \in Z_{k-1}^{\mathrm{aff}}([(X,\omega_X)]),$$

where

$$\begin{aligned} \omega_\varphi : X^{(k-1)} &\to \mathbb{Z} \\ \tau &\mapsto \sum_{\sigma \in X^{(k)}, \tau < \sigma} \varphi_\sigma(\omega(\sigma) v_{\sigma/\tau}) - \varphi_\tau \left(\sum_{\sigma \in X^{(k)}, \tau < \sigma} \omega(\sigma) v_{\sigma/\tau} \right) \end{aligned}$$

and the $v_{\sigma/\tau}$ are arbitrary representatives of the normal vectors $u_{\sigma/\tau}$. If $[(Y,\omega_Y)]$ is an affine k-cycle in $[(X,\omega_X)]$, we define $\varphi \cdot [(Y,\omega_Y)] := \varphi\mid_{|[(Y,\omega_Y)]|} \cdot [(Y,\omega_Y)]$.

Example 2.1.6. Let $[(X,\omega_X)]$ be the affine 1-cycle with representative (X,ω_X) whose weights are all 1 and whose 1-dimensional rays are given by $-e_1, -e_2, e_1+e_2$ as in Example 2.1.3. Then

$$\begin{aligned} \varphi : X &\to \mathbb{R} \\ (x,y) &\mapsto \max(x,y,0) \end{aligned}$$

is a rational function on $[(X,\omega_X)]$ and (X,ω_X) is a representative such that φ is integer linear affine on each cone. The divisor associated to φ, namely $\varphi \cdot X$, is given by the codimension-1-skeleton of X which is just one point (namely $0 \in \mathbb{R}^2$) and that point has weight 1. We calculate this weight as an example: Let $\tau = 0 \in \mathbb{R}^2$, $\sigma_1 = \operatorname{cone}(-e_1), \sigma_2 = \operatorname{cone}(-e_2)$ and $\sigma_3 = \operatorname{cone}(e_1+e_2)$ be cones of X. Applying Definition 2.1.5 yields

$$\begin{aligned}
\omega_\varphi(\tau) &= \varphi_{\sigma_1}\left(\omega(\sigma_1)v_{\sigma_1/\tau}\right) + \varphi_{\sigma_2}\left(\omega(\sigma_2)v_{\sigma_2/\tau}\right) + \varphi_{\sigma_3}\left(\omega(\sigma_3)v_{\sigma_3/\tau}\right) \\
&\quad - \varphi_\tau\left(\omega(\sigma_1)v_{\sigma_1/\tau} + \omega(\sigma_2)v_{\sigma_2/\tau} + \omega(\sigma_3)v_{\sigma_3/\tau}\right) \\
&= \varphi_{\sigma_3}\left(\omega(\sigma_3)v_{\sigma_3/\tau}\right) \\
&= \varphi_{\sigma_3}\left(1(e_1+e_2)\right) = 1
\end{aligned}$$

because $\varphi_{\sigma_1}, \varphi_{\sigma_2}, \varphi_\tau \equiv 0$ and $\varphi_{\sigma_3}(e_1+e_2) = \max(1,1,0)$.

Definition 2.1.7 (Affine intersection product)**.** Let $[(X,\omega_X)]$ be an affine k-cycle. The subgroup of globally linear functions in $\mathcal{K}^*([(X,\omega_X)])$ with respect to $+$ is denoted by $\mathcal{O}^*([(X,\omega_X)])$. We define the *group of affine divisors of* $[(X,\omega_X)]$ to be the quotient group

$$\operatorname{Div}([(X,\omega_X)]) := \mathcal{K}^*([(X,\omega_X)])/\mathcal{O}^*([(X,\omega_X)]).$$

Let $[\varphi] \in \operatorname{Div}([(X,\omega_X)])$ be a divisor. The divisor associated to this function is denoted by $\operatorname{div}([\varphi]) := \operatorname{div}(\varphi)$ and is well-defined. The following bilinear map is called *affine intersection product*

$$\begin{aligned}
\cdot : \operatorname{Div}([(X,\omega_X)]) \times Z_k^{\mathrm{aff}}([(X,\omega_X)]) &\to Z_{k-1}^{\mathrm{aff}}([(X,\omega_X)]) \\
([\varphi],[(Y,\omega_Y)]) &\mapsto [\varphi]\cdot[(Y,\omega_Y)] := \varphi\cdot[(Y,\omega_Y)].
\end{aligned}$$

Definition 2.1.8 (Morphisms of affine cycles)**.** Let X be a fan in $V = \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ and Y a fan in $V' = \Gamma' \otimes_{\mathbb{Z}} \mathbb{R}$. A *morphism* $f : X \to Y$ is a $\mathbb{Z}$-linear map from $|X| \subseteq V$ to $|Y| \subseteq V'$ induced by a $\mathbb{Z}$-linear map on the lattices. A *morphism of weighted fans* is a morphism of fans. A *morphism of affine cycles* $f : [(X,\omega_X)] \to [(Y,\omega_Y)]$ is a morphism of weighted fans $f : |[(X,\omega_X)]| \to |[(Y,\omega_Y)]|$ that is independent of the choice of representatives, where $|[(X,\omega_X)]|$ (resp. $|[(Y,\omega_Y)]|$) denotes the support of X (resp. Y) with nonzero weights.

Example 2.1.9. Let e_1, e_2 be the vectors of the standard basis of $\mathbb{R}^2$. Let $[(X,\omega_X)]$ be the affine 1-cycle from Example 2.1.3. Let $[(Y,\omega_Y)]$ be another affine 1-cycle in $\mathbb{R}^2$ with representative (Y,ω_Y) whose weights are all 1 and whose two 1-dimensional rays are given by $-e_1-e_2, e_1+e_2$. Then

$$\begin{aligned}
f : |[(X,\omega_X)]| &\to |[(Y,\omega_Y)]| \\
-e_1 &\mapsto -e_1 - e_2 \\
-e_2 &\mapsto -e_1 - e_2
\end{aligned}$$

is a morphism of affine cycles which is illustrated in Figure 2.1.

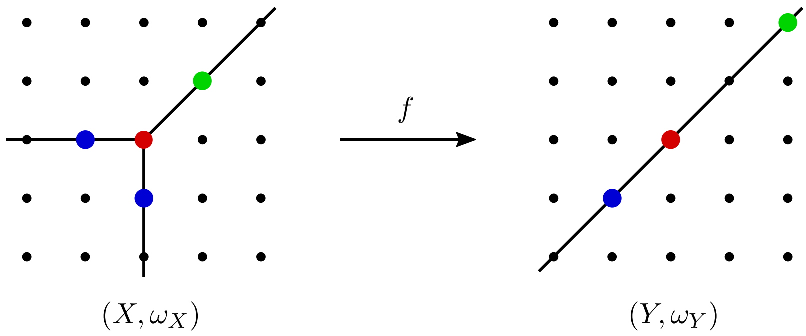


Figure 2.1: A morphism of affine cycles. The colors indicate where lattice points of $|X|$ are mapped to.

Definition 2.1.10 (Push-forward of affine cycles)**.** Let $V = \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ and $V' = \Gamma' \otimes_{\mathbb{Z}} \mathbb{R}$. Let $[(X, \omega_X)] \in Z_m^{\text{aff}}(V)$ and $[(Y, \omega_Y)] \in Z_n^{\text{aff}}(V')$ be cycles with representatives (X, ω_X) and (Y, ω_Y). Let $f : X \to Y$ be a morphism. Choosing a refinement of (X, ω_X), the set of cones

$$f_*X := \{f(\sigma) \mid \sigma \in X \text{ contained in a maximal cone of } X \text{ on which } f \text{ is injective}\}$$

is a tropical fan in V' of dimension m with weights

$$\omega_{f_*X}(\sigma') := \sum_{\sigma \in X^{(m)} \colon f(\sigma) = \sigma'} \omega_X(\sigma) \cdot |\Gamma'_{\sigma'}/f(\Gamma_\sigma)|$$

for all $\sigma' \in f_*X^{(m)}$. The equivalence class of (f_*X, ω_{f_*X}) is uniquely determined by the equivalence class of (X, ω_X). For $[(Z, \omega_Z)] \in Z_k^{\text{aff}}([(X, \omega_X)])$, we define

$$f_*[(Z, \omega_Z)] := [(f_*(|[(Z, \omega_Z)]|), \omega_{f_*(|[(Z,\omega_Z)]|)})] \in Z_k^{\text{aff}}([(Y, \omega_Y)])$$

The map

$$Z_k^{\text{aff}}([(X, \omega_X)]) \to Z_k^{\text{aff}}([(Y, \omega_Y)]),\ [(Z, \omega_Z)] \mapsto f_*[(Z, \omega_Z)]$$

is well-defined, $\mathbb{Z}$-linear and $f_*[(Z, \omega_Z)]$ is called *push-forward of* $[(Z, \omega_Z)]$ *along* f.

Example 2.1.11. Let f be the morphism of Example 2.1.9. Notice that no ray of X is mapped to a point under f. Therefore $f_*X = Y$. Let σ'_- (resp. σ'_+) be the ray of (Y, ω_Y) that is generated by $-e_1 - e_2$ (resp. $e_1 + e_2$). There are two rays of X that are mapped onto σ'_- under f (indicated by blue dots in Figure 2.1) such that primitive direction vectors are mapped to a primitive direction vector again. Hence

$$\omega_{f_*X}(\sigma'_-) = 2.$$

On the other hand, there is exactly one ray of (X, ω_X) that is mapped to σ'_+. As indicated by the green dots in Figure 2.1, a primitive direction vector is stretched to twice its length. Thus

$$\omega_{f_*X}(\sigma'_+) = 2.$$

So in total

$$f_*[(X, \omega_X)] = [(Y, 2 \cdot \omega_Y)].$$

Definition 2.1.12 (Pull-back of divisors)**.** Let $[(X,\omega_X)] \in Z_m^{\text{aff}}(V)$ and $[(Y,\omega_Y)] \in Z_n^{\text{aff}}(V')$ be cycles in $V = \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ and $V' = \Gamma' \otimes_{\mathbb{Z}} \mathbb{R}$. Let $f : [(X,\omega_X)] \to [(Y,\omega_Y)]$ be a morphism. The map

$$\begin{aligned} \operatorname{Div}([(Y,\omega_Y)]) &\to \operatorname{Div}([(X,\omega_X)]) \\ [h] &\mapsto f^*[h] := [h \circ f] \end{aligned}$$

is well-defined, $\mathbb{Z}$-linear and $f^*[h]$ is called *pull-back of* $[h]$ *along* f.

Proposition 2.1.13 (Projection formula)**.** *Let* $[(X,\omega_X)] \in Z_m^{\text{aff}}(V)$ *and* $[(Y,\omega_Y)] \in Z_n^{\text{aff}}(V')$ *be cycles in* $V = \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ *and* $V' = \Gamma' \otimes_{\mathbb{Z}} \mathbb{R}$. *Let* $f : [(X,\omega_X)] \to [(Y,\omega_Y)]$ *be a morphism. Let* $[(Z,\omega_Z)] \in Z_k^{\text{aff}}([(X,\omega_X)])$ *be a cycle and let* $\varphi \in \operatorname{Div}([(Y,\omega_Y)])$. *Then the equality*

$$\varphi \cdot (f_*[(Z,\omega_Z)]) = f_*\left(f^*\varphi \cdot [(Z,\omega_Z)]\right) \in Z_{k-1}^{\text{aff}}([(Y,\omega_Y)])$$

holds.

2.1.2 Abstract tropical cycles

So far, we introducted affine cycles only. Affine cycles are building blocks of *abstract* cycles. Since the whole “affine-to-abstract”-procedure is quite technical, we omit it here and refer to section 5 of [AR10] instead. We want to remark that concepts like pull-backs and push-forwards carry over to abstract cycles. In particular, the projection formula (Proposition 2.1.13) also holds for abstract cycles. For our purposes, the following definition of abstract cycles is sufficient:

Definition 2.1.14 (Abstract cycles)**.** An *abstract* k*-cycle* C is a class under a refinement relation of a balanced polyhedral complex of pure dimension k which is locally isomorphic to tropical fans.

Remark 2.1.15 (Rational functions on abstract cycles)**.** In the same way rational functions on affine cycles led to an affine intersection product, one can also consider *rational functions on abstract cycles* to obtain an intersection product. Again, we want to omit technicalities and refer to Definition 6.1 of [AR10] instead. The main point of considering rational functions on abstract cycles is that they are no longer piecewise linear but piecewise affine linear.

As we see below, it happens that we start with an affine cycle $[(X,\omega_X)]$ and want to intersect it with a rational function f that is piecewise affine linear. In order to do so, we need to refine $[(X,\omega_X)]$ in such a way that f is a rational function on the affine cycle locally around each codimension one face. Hence $[(X,\omega_X)]$ becomes a polyhedral complex which is a representative of an abstract cycle C. Then we can intersect f with C.

In the following we want to restrict to tropical intersection theory on $\mathbb{R}^n$.

Remark 2.1.16 (Rational equivalence of abstract cycles)**.** There is a concept of *rational equivalence* of abstract cycles (section 8 of [AR10]). When we consider abstract cycles, we usually consider them up to this equivalence relation.

Definition 2.1.17 (Degree map)**.** Let $A_0(\mathbb{R}^n)$ denote the set of abstract 0-cycles in $\mathbb{R}^n$ up to rational equivalence. The map

$$\begin{aligned} \deg : A_0(\mathbb{R}^n) &\to \mathbb{Z} \\ [\omega_1 P_1 + \cdots + \omega_r P_r] &\mapsto \sum_{i=1}^r \omega_i \end{aligned}$$

is a well-defined morphism and for $D \in A_0(\mathbb{R}^n)$ the number $\deg(D)$ is called the *degree of* D.

2.2 Tropical curves and their moduli spaces

In this section moduli spaces of tropical abstract curves and moduli spaces of tropical stable maps are recalled. For more background on these moduli spaces see [Mik07, GM08, GKM09].

Definition 2.2.1 (Moduli space of abstract rational tropical curves)**.** Notation 2.0.1 is used. An *abstract rational tropical curve* is a metric tree Γ with unbounded edges called *ends* and with $\mathrm{val}(v) \geq 3$ for all vertices $v \in \Gamma$. It is called *M-marked abstract tropical curve* $(\Gamma, x_{[M]})$ if Γ has exactly M ends that are labeled with pairwise different $x_1, \dots, x_M \in \mathbb{N}$. Two M-marked tropical curves $(\Gamma, x_{[M]})$ and $(\tilde{\Gamma}, \tilde{x}_{[M]})$ are isomorphic if there is a homeomorphism $\Gamma \to \tilde{\Gamma}$ mapping x_i to $\tilde{x}_i$ for $i \in [M]$ and each edge of Γ is mapped onto an edge of $\tilde{\Gamma}$ by an affine linear map of slope ± 1. The set $\mathcal{M}_{0,M}$ of all M-marked tropical curves up to isomorphism is called *moduli space of M-marked abstract tropical curves*. Forgetting all lengths of an M-marked abstract tropical curve gives us its *combinatorial type*.

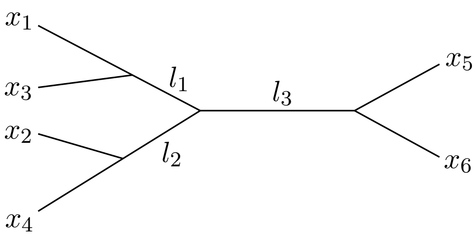


Figure 2.2: An example of a 6-marked abstract tropical curve $(\Gamma, x_{[6]})$. The lengths of the bounded edges of Γ are denoted by $l_{[3]}$.

Theorem 2.2.2 ($\mathcal{M}_{0,M}$ is a tropical fan, [SS06, Mik07, GKM09, GM10])**.** *The moduli space $\mathcal{M}_{0,M}$ can explicitly be embedded into a $\mathbb{R}^t$ such that $\mathcal{M}_{0,M}$ is a fan of pure dimension $M-3$ with its fan structure given by combinatorial types. Equip $\mathbb{R}^t$ with a lattice which arises from considering integer edge lengths of abstract tropical curves in $\mathcal{M}_{0,M}$ and let all weights of $\mathcal{M}_{0,M}$ be one. Then $\mathcal{M}_{0,M} \subset \mathbb{R}^t$ is a tropical fan, i.e. $\mathcal{M}_{0,M}$ represents an affine cycle in $\mathbb{R}^t$. This allows us to use tropical intersection theory on $\mathcal{M}_{0,M}$. For an example, see Figure 2.3.*

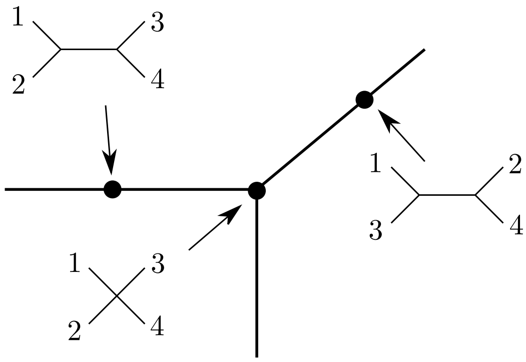


Figure 2.3: One way of embedding the moduli space $\mathcal{M}_{0,4}$ into $\mathbb{R}^2$ centered at the origin of $\mathbb{R}^2$. The length of a bounded edge of an abstract tropical curve depicted above is given by the distance of the point in $\mathcal{M}_{0,4}$ corresponding to this curve from the origin of $\mathbb{R}^2$. The ends of $\mathcal{M}_{0,4}$ correspond to different distributions of labels on ends of abstract tropical curves with four ends. We refer to these ends as $(12|34), (13|24), (14|23)$.

Definition 2.2.3 (Degree)**.** A tuple (Δ, l) is called a *degree* in $\mathbb{R}^m$ if Δ is a nonempty finite multiset, $l : \Delta \hookrightarrow \mathbb{N}$ is an injective map and each entry v of Δ is a nonzero element of $\mathbb{Z}^m$ such that

$$\sum_{v\in\Delta} v = 0 \quad \text{and} \quad \langle v \mid v \in \Delta\rangle = \mathbb{R}^m.$$

Thus each entry of Δ is equipped with a unique natural number called *label*. Let $\Delta = \Delta_1\dot\cup \dots \dot\cup\Delta_r$ be a decomposition of Δ into multisets such that each Δ_i consists of copies of a $v_i \in \Delta$ and $\#\Delta_i$ is maximal for $i \in [r]$. The number of bijective maps $\bar{l} : \Delta \to [\#\Delta]$ with $\bar{l}\mid_{\Delta_i}: \Delta_i \to [\#\Delta_i+\sum_{t=1}^{i-1}\#\Delta_t]$ is called the *number of ways to label* Δ. Let $v = (v_1, \dots, v_m) \in \Delta$, then $\gcd(v_1, \dots, v_m)$ is called *weight* of v. Most of the time the map l is suppressed in the notation, i.e. we usually write Δ and assume that elements of Δ are labeled. If we refer to a degree to be *partially labeled*, then we mean that l is restricted to a subset of Δ, i.e. some entries of Δ are labeled and some are not. In particular, a degree is called *unlabeled* if all entries of Δ are not labeled.

Notation 2.2.4 (Standard directions and certain degrees)**.** Let $e_i := (\delta_{ti})_{t\in[m]}$ for $i \in [m]$ denote the vectors of the standard basis of $\mathbb{R}^m$, and define $e_0 := \sum_{i=1}^m e_i$. We call $e_0, -e_1, \dots, -e_m$ *standard directions* of $\mathbb{R}^m$. The following degrees are used often:

(a) For $m \in \mathbb{N}_{>0}$ and $d \in \mathbb{N}$, we define the degree Δ_d^m to be the multiset consisting of d copies of e_0 and d copies of each $-e_i$ for $i \in [m]$.

(b) Let $\alpha := (\alpha_i)_{i\in\mathbb{N}}$ and $\beta := (\beta_i)_{i\in\mathbb{N}}$ be two sequences with $\alpha_i, \beta_i \in \mathbb{N}$ such that

$$|\alpha| := \sum_{i\in\mathbb{N}} \alpha_i \quad \text{and} \quad |\beta| := \sum_{i\in\mathbb{N}} \beta_i$$

are finite. Let $d \in \mathbb{N}$ such that $d - \sum_{i\in\mathbb{N}} i \cdot \alpha_i + \sum_{i\in\mathbb{N}} i \cdot \beta_i = 0$ and define

$$\Delta_d^m(\alpha, \beta) := \Delta_d^m \setminus \underbrace{\{-e_m, \dots, -e_m\}}_{d \text{ many}} \cup \bigcup_{i\in\mathbb{N}} \underbrace{\{i(-e_m), \dots, i(-e_m)\}}_{\alpha_i \text{ many}} \cup \bigcup_{i\in\mathbb{N}} \underbrace{\{i \cdot e_m, \dots, i \cdot e_m\}}_{\beta_i \text{ many}},$$

where unions are actually unions of multisets. We write α^{lab} (resp. β^{lab}) to refer to the set of labels associated to ends in α (resp. β). Notice that by definition

$$\Delta_d^m = \Delta_d^m((0, \dots), (0, \dots)).$$

If a degree Δ is given and we refer to α^{lab} (resp. β^{lab}), we mean that $\alpha^{\text{lab}} = \beta^{\text{lab}} = \emptyset$ if $\Delta \neq \Delta_d^m(\alpha, \beta)$ for suitable d, m, α, β.

Remark 2.2.5 (Degree associated to a polytope)**.** Notice that each degree of Notation 2.2.4 arises as outer normal fan of a lattice polytope (e.g. each degree $\Delta_d^2(\alpha, \beta)$ is associated to a polytope that defines a Hirzebruch surface). Whenever a given degree appears as such a normal fan, then we denote its associated lattice polytope by $\Sigma(\Delta)$. Note also that each degree in $\mathbb{R}^2$ corresponds to a lattice polytope.

Example 2.2.6. Let $\Delta_1^2((1, 1, 0, \dots), (0, 1, 0, \dots))$ be a degree as in Notation 2.2.4. Then

$$\Delta_1^2((1, 1, 0, \dots), (0, 1, 0, \dots)) = \{e_0, -e_1, -e_2, 2 \cdot (-e_2), 2 \cdot e_2\}$$

holds by definition, where the equality is an equality of multisets. Notice that

$$\{e_0, -e_1, -e_2, 2 \cdot (-e_2), 2 \cdot e_2\} = \{e_0\} \dot\cup \{-e_1\} \dot\cup \{-e_2\} \dot\cup \{-2e_2\} \dot\cup \{2e_2\}$$

is a decomposition as in Definition 2.2.3. Hence there is one way to label, namely

vector	e_0	$-e_1$	$-e_2$	$-2e_2$	$2e_2$
label	1	2	3	4	5

.

Figure 2.4 shows the polytope associated to $\Delta_1^2\left((1,1,0,\dots),(0,1,0,\dots)\right)$, see Remark 2.2.5.

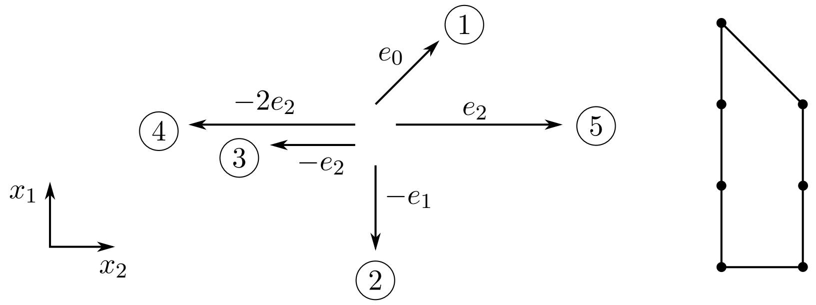


Figure 2.4: On the left, the degree $\Delta_1^2\left((1,1,0,\dots),(0,1,0,\dots)\right)$ of Example 2.2.6 is shown. The circled numbers indicate the labels. On the right, its associated lattice polytope $\Sigma\left(\Delta_1^2\left((1,1,0,\dots),(0,1,0,\dots)\right)\right)$ that defines a Hirzebruch surface is shown.

Definition 2.2.7 (Moduli space of rational tropical stable maps to $\mathbb{R}^m$)**.** Let (Δ, l) be a degree in $\mathbb{R}^m$ as in Definition 2.2.3 and let $N \in \mathbb{N}$. A *rational tropical stable map of degree* (Δ, l) *to* $\mathbb{R}^m$ *with* N *contracted ends* is a tuple $(\Gamma, x_{[M]}, h)$, where $(\Gamma, x_{[M]})$ is an M-marked abstract tropical curve with $M = N + \#\Delta$, $x_{[N]} \notin l(\Delta)$, and a map $h : \Gamma \to \mathbb{R}^m$ that satisfies the following:

(a) Let $e \in \Gamma$ be an edge with length $|e| \in [0, \infty]$, identify e with $[0, |e|]$ and denote the vertex of e that is identified with $0 \in [0, |e|]$ by V. The map h is integer affine linear, i.e. $h\mid_e\colon t \mapsto tv + a$ with $a \in \mathbb{R}^m$ and $v(e, V) := v \in \mathbb{Z}^m$, where $v(e, V)$ is called *direction vector of* e *at* V and the *weight* of an edge (denoted by $\omega(e)$) is the gcd of the entries of $v(e, V)$ if $v(e, V) \neq 0$ and zero otherwise. The vector $\frac{1}{\omega(e)} \cdot v(e, V)$ is called the *primitive* direction vector of e at V. If $e = x_i \in \Gamma$ is an end, then $v(x_i)$ denotes the direction vector of x_i pointing away from its one vertex it is adjacent to.

(b) The direction vector $v(x_i)$ of an end labeled with x_i is $0 \in \mathbb{R}^m$ if $i \in [N]$. Otherwise, $v(x_i)$ equals the unique $v \in \Delta$ with $l(v) = x_i \in \mathbb{N}$. Ends with direction vector zero are called *contracted ends*.

(c) The *balancing condition*

$$\sum_{\substack{e \in \Gamma \text{ an edge,} \\ V \text{ vertex of } e}} v(e, V) = 0$$

holds for every vertex $V \in \Gamma$.

The *combinatorial type* of a rational tropical stable map $(\Gamma, x_{[M]}, h)$ is the combinatorial type of its underlying M-marked abstract tropical curve $(\Gamma, x_{[M]})$. Two rational tropical stable maps of degree Δ (the map l of (Δ, l) is usually suppressed in the notation) with N contracted ends, namely $(\Gamma, x_{[M]}, h)$ and $(\Gamma', x'_{[M]}, h')$, are isomorphic if there is an isomorphism φ of their underlying M-marked tropical curves such that $h' \circ \varphi = h$. The set $\mathcal{M}_{0,N}\left(\mathbb{R}^m, \Delta\right)$ of all rational tropical stable maps of degree Δ to $\mathbb{R}^m$ with N contracted ends up to isomorphism is called *moduli space of rational tropical stable maps of degree* Δ *to* $\mathbb{R}^m$ *(with* N *contracted ends).*

Example 2.2.8. Figure 2.5 provides an example of a rational tropical stable map $(\Gamma, x_{[6]}, h)$ of degree $\Delta_1^2((0,1,0,\dots),(1,0,\dots))$ to $\mathbb{R}^2$. The weights and the directions of ends of Γ in $\mathbb{R}^2$ are prescribed by its degree. The lengths of bounded edges are given by the lengths $l_{[3]}$ of Γ. The directions of the bounded edges of Γ in $\mathbb{R}^2$ are determined by the balancing condition.

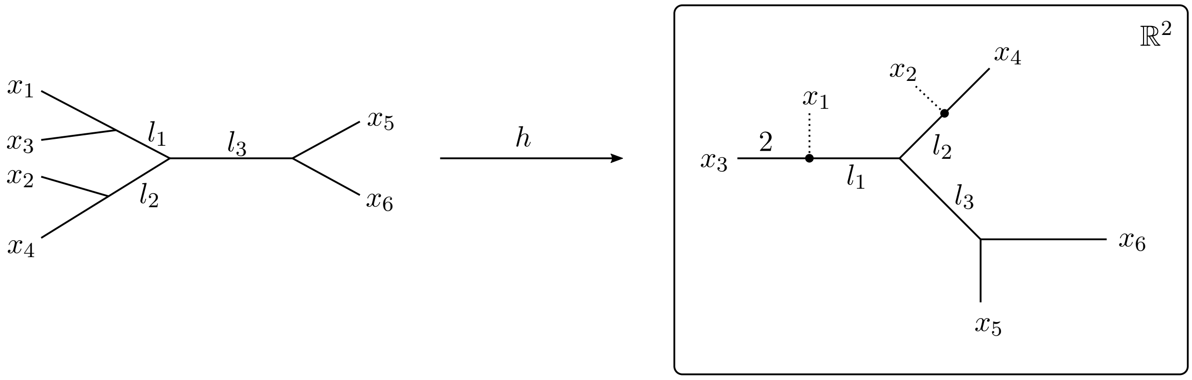


Figure 2.5: Left: The 6-marked abstract tropical curve from Figure 2.2. Right: The image of Γ under h, where the ends x_1, x_2 are contracted to points which is indicated by drawing them dotted.

Notation 2.2.9. See Notation 2.2.4 for the following: The projection

$$\pi : \mathbb{R}^m \to \mathbb{R}^{m-1}, (x_1, \dots, x_m) \mapsto (x_1, \dots, x_{m-1})$$

induces a map

$$\tilde{\pi} : \mathcal{M}_{0,1}\left(\mathbb{R}^m, \Delta_d^m(\alpha,\beta)\right) \to \mathcal{M}_{0,1+|\alpha|+|\beta|}\left(\mathbb{R}^{m-1}, \Delta_d^{m-1}\right),$$

where ends in $\Delta_d^m(\alpha,\beta)\backslash\left(\Delta_d^m\backslash\{-e_m,\dots,-e_m\}\right)$ are contracted. So $\tilde{\pi}$ induces labels on contracted ends by contracting labeled ends of direction parallel to $\pm e_m$. To emphasize how non-contracted ends are labeled, we write $\mathcal{M}_{0,1+|\alpha|+|\beta|}\left(\mathbb{R}^{m-1}, \pi\left(\Delta_d^m(\alpha,\beta)\right)\right)$ instead of writing $\mathcal{M}_{0,1+|\alpha|+|\beta|}\left(\mathbb{R}^{m-1}, \Delta_d^{m-1}\right)$.

Theorem 2.2.10 ($\mathcal{M}_{0,N}(\mathbb{R}^m, \Delta)$ is a fan, [GKM09])**.** *The map*

$$\begin{aligned} \mathcal{M}_{0,N}(\mathbb{R}^m, \Delta) &\to \mathcal{M}_{0,M} \times \mathbb{R}^m \\ (\Gamma, x_{[M]}, h) &\mapsto \left(\left(\Gamma, x_{[M]}\right), h(x_1)\right) \end{aligned}$$

with $M = N + \#\Delta$ is bijective and $\mathcal{M}_{0,N}(\mathbb{R}^m, \Delta)$ is a tropical fan of dimension $\#\Delta + N - 3 + m$ (notice that $h(x_1)$ is an arbitrary choice of a base point). Hence $\mathcal{M}_{0,N}(\mathbb{R}^m, \Delta)$ represents an affine cycle in a $\mathbb{R}^t$. This allows us to use tropical intersection theory on $\mathcal{M}_{0,N}(\mathbb{R}^m, \Delta)$.

Definition 2.2.11 (Local coordinates on $\mathcal{M}_{0,M}$ and $\mathcal{M}_{0,N}(\mathbb{R}^m, \Delta)$)**.** The bounded edges' lengths of M-marked abstract tropical curves parametrized by $\mathcal{M}_{0,M}$ give rise to local coordinates on $\mathcal{M}_{0,M}$. Consequently, the identification of Theorem 2.2.10 yields local coordinates on $\mathcal{M}_{0,N}(\mathbb{R}^m, \Delta)$ as well. They are given by the bounded edges' lengths and the position of a vertex in $\mathbb{R}^m$ to which we refer as *base point.*

Definition 2.2.12 (Combinatorial types of cells)**.** Since $\mathcal{M}_{0,M}$ (resp. $\mathcal{M}_{0,N}(\mathbb{R}^m, \Delta)$) are in particular polyhedral complexes (see Theorem 2.2.2, Theorem 2.2.10) we often refer to their cones as cells. Let σ be a cell of $\mathcal{M}_{0,M}$ (resp. of $\mathcal{M}_{0,N}(\mathbb{R}^m, \Delta)$). Let $(\Gamma, x_{[M]})$ be an abstract rational tropical curve (resp. let $(\Gamma, x_{[M]}, h)$ be a rational tropical stable map) in σ

such that $(\Gamma, x_{[M]})$ (resp. $(\Gamma, x_{[M]}, h)$) is in the interior of σ if σ is not 0-dimensional. Then the cell σ is determined by the combinatorial type of $(\Gamma, x_{[M]})$ (resp. $(\Gamma, x_{[M]}, h)$) because combinatorial types are constant on the interior of σ if σ is not zero-dimensional. Thus defining the *combinatorial type* $\mathfrak{c}(\sigma)$ of σ to be equal to the combinatorial type of $(\Gamma, x_{[M]})$ (resp. $(\Gamma, x_{[M]}, h)$) makes sense.

Definition 2.2.13 (Evaluation maps I)**.** Define the map

$$\begin{aligned} \mathrm{ev}_i : \mathcal{M}_{0,N}\left(\mathbb{R}^m, \Delta\right) &\to \mathbb{R}^m \\ (\Gamma, x_{[M]}, h) &\mapsto h(x_i) \end{aligned}$$

for $i \in [M]$. Typically, ev_i is used for contracted ends. That is, if $i \in [N]$, then the map ev_i is called *i-th evaluation map*. Under the identification from Theorem 2.2.10, the i-th evaluation map is a morphism of fans $\mathrm{ev}_i : \mathcal{M}_{0,M} \times \mathbb{R}^m \to \mathbb{R}^m$. Let $\varphi_{[r]} \in \mathrm{Div}\left([(\mathbb{R}^m, \omega_{\mathbb{R}^m})]\right)$ for $r \in \mathbb{N}_{>0}$. Consider the cycle $C := \varphi_1 \cdots \varphi_r \cdot [(\mathbb{R}^m, \omega_{\mathbb{R}^m})]$ and define the *pull-back of C along the i-th evaluation map* ev_i as

$$\mathrm{ev}_i^*(C) := \mathrm{ev}_i^*(\varphi_1) \cdots \mathrm{ev}_i^*(\varphi_r) \cdot \mathcal{M}_{0,N}\left(\mathbb{R}^m, \Delta\right).$$

This is well-defined because of Proposition 1.12 of [Rau16] and allows us to pull-back cycles like C via the i-th evaluation map.

If follows from [Rau16] that the support of $\mathrm{ev}_i^*(C)$ for a cycle C is the set of all rational tropical stable maps in $\mathcal{M}_{0,N}\left(\mathbb{R}^m, \Delta\right)$ whose contracted end x_i is mapped to $C \subset \mathbb{R}^m$. We may therefore refer to $\mathrm{ev}_i^*(C)$ as rational tropical stable maps on which a condition is imposed. If C is a point, then the imposed condition is called *point condition*.

In the following, we shorten notation by writing $\mathbb{R}^m$ instead of $[(\mathbb{R}^m, \omega_{\mathbb{R}^m})]$ if all weights are one.

Example 2.2.14 (Pull-back of a point)**.** A point $p = (p_1, \dots, p_m) \in \mathbb{R}^m$ is an intersection product of m rational functions, e.g.

$$p = \prod_{j=1}^{m} \max\{p_j, x_j\} \cdot \mathbb{R}^m,$$

where $x_{[m]}$ are the standard coordinates of $\mathbb{R}^m$. The pull-back of the point p along the i-th evaluation map ev_i is

$$\begin{aligned} \mathrm{ev}_i^*(p) &= \prod_{j=1}^{m} \mathrm{ev}_i^*(\max\{p_j, x_j\}) \cdot \mathcal{M}_{0,N}\left(\mathbb{R}^m, \Delta\right) \\ &= \prod_{j=1}^{m} \max\{p_j, (\mathrm{ev}_i(\star))_j\} \cdot \mathcal{M}_{0,N}\left(\mathbb{R}^m, \Delta\right) \end{aligned}$$

according to Definitions 2.2.13 and 2.1.12.

Example 2.2.15. Let $(\Gamma, x_{[6]}, h)$ be the rational tropical stable map of Example 2.2.8. Denote the vertex of Γ that is adjacent to x_i by v_i for $i = 1, 2$. Let $p_1, p_2 \in \mathbb{R}^2$ be two non-collinear points. Then $(\Gamma, x_{[6]}, h) \in \mathrm{ev}_1^*(p_1) \cdot \mathrm{ev}_2^*(p_2) \cdot \mathcal{M}_{0,2}\left(\mathbb{R}^2, \Delta_1^2\left((0,1,0,\dots),(1,0,\dots)\right)\right)$ if $h(v_i) = p_i$ for $i = 1, 2$.

Definition 2.2.16 (Evaluation maps II)**.** Let $\Delta_d^m(\alpha, \beta)$ be a degree in $\mathbb{R}^m$ as in Notation 2.2.4. Let π be the projection from Notation 2.2.9 that forgets the last canonical coordinate of $\mathbb{R}^m$. For $k \in \alpha^{\text{lab}} \cup \beta^{\text{lab}}$ the map

$$\begin{aligned} \partial \operatorname{ev}_k : \mathcal{M}_{0,N}\left(\mathbb{R}^m, \Delta_d^m(\alpha,\beta)\right) &\to \mathbb{R}^{m-1} \\ (\Gamma, x_{[M]}, h) &\mapsto \left((h(x_k))_i\right)_{i=1,\dots,m-1} \end{aligned}$$

is by abuse of notation also called *k-th evaluation map* since it evaluates the position of certain non-contracted ends under the projection π. Equivalently, we may write

$$\partial \operatorname{ev}_k = \pi \circ \operatorname{ev}_k .$$

To see that pull-backs of cycles along $\partial \operatorname{ev}_k$, which are defined analogously to pull-backs of cycles along ev_i, make sense, consider the map

$$\mathcal{M}_{0,N}\left(\mathbb{R}^m, \Delta_d^m(\alpha,\beta)\right) \to \mathbb{R}^{m-1} \times \mathcal{M}_{0,M} \times \mathbb{R}$$

which is similar to the one of Theorem 2.2.10, where the vertex v_k adjacent to the end labeled with k is the base point, i.e. the first $m-1$ coordinates of v_k in $\mathbb{R}^m$ are in the $\mathbb{R}^{m-1}$-factor and the last coordinate of v_k is in the $\mathbb{R}$-factor. Thus Proposition 1.12 of [Rau16] can be applied and yields that pull-backs of cycles like C in Definition 2.2.13 along $\partial \operatorname{ev}_k$ are well-defined.

Similar to pull-backs of cycles via evaluation maps, the support of pull-backs along $\partial \operatorname{ev}_k^*(C)$ for a cycle C is the set of all rational tropical stable maps in $\mathcal{M}_{0,N}\left(\mathbb{R}^m, \Delta_d^m(\alpha,\beta)\right)$ whose end x_k has its first $m-1$ coordinates in C. We may refer to $\partial \operatorname{ev}_k^*(C)$ as rational tropical stable maps on which a *tangency condition* is imposed.

Definition 2.2.17 (Forgetful maps)**.** For $M \geq 4$ the map

$$\begin{aligned} \operatorname{ft}_{x_{[M-1]}} : \mathcal{M}_{0,M} &\to \mathcal{M}_{0,M-1} \\ (\Gamma, x_{[M]}) &\mapsto (\Gamma', x_{[M-1]}), \end{aligned}$$

where Γ' is the stabilization (straighten 2-valent vertices) of Γ after removing its end marked by x_M is called the M-th *forgetful map.* Applied recursively, it can be used to forget several ends with markings in $I^C \subset x_{[M]}$, denoted by ft_I, where I^C is the complement of $I \subset x_{[M]}$. With the identification from Theorem 2.2.10, and additionally forgetting the map h to the plane, we can also consider

$$\begin{aligned} \operatorname{ft}_I : \mathcal{M}_{0,N}\left(\mathbb{R}^m, \Delta\right) &\to \mathcal{M}_{0,|I|} \\ (\Gamma, x_{[M]}, h) &\mapsto \operatorname{ft}_I(\Gamma, x_i \mid i \in I). \end{aligned}$$

Any forgetful map is a morphism of fans. This allows us to pull-back cycles via the forgetful map.

Before proceeding with the next section, we want to briefly recall facts about rational equivalence that are then frequently used in the following chapters.

Remark 2.2.18 (Rational equivalence)**.** Throughout Part I of this thesis, we consider intersection products of the form $\varphi_1^*(Z_1) \cdots \varphi_r^*(Z_r) \cdot \mathcal{M}_{0,N}\left(\mathbb{R}^m, \Delta\right)$, where φ_i is either an evaluation map ev_i (resp. $\partial \operatorname{ev}_k$) from Definition 2.2.13 (resp. 2.2.16) or a forgetful map ft_I to $\mathcal{M}_{0,4}$ from Definition 2.2.17, and Z_i is a cycle we want to pull-back via φ_i for $i \in [r]$. When considering such cycles Z_i that are conditions we impose on tropical stable maps, then we usually want to ensure that a 0-dimensional cycle $\varphi_1^*(Z_1) \cdots \varphi_r^*(Z_r) \cdot \mathcal{M}_{0,N}\left(\mathbb{R}^m, \Delta\right)$ is independent of the exact positions of the conditions Z_i for $i \in [r]$. This is where rational equivalence comes into play. We usually consider cycles like Z_i up to a rational equivalence relation (cf. Remark 2.1.16). The most important facts about this relation are the following:

(a) Two cycles Z, Z' in $\mathbb{R}^m$ that only differ by a translation are rationally equivalent.

(b) Pull-backs $\varphi^*(Z), \varphi^*(Z')$ of rationally equivalent cycles Z, Z' are rationally equivalent.

(c) The *degree* of a 0-dimensional cycle (see Definition 2.1.17) is compatible with rational equivalence, i.e. if two 0-dimensional cycles are rationally equivalent, then their degrees are the same.

Notice that (a)-(c) allows us to "move" all conditions we consider slightly without affecting counts of tropical stable maps we are interested in.

Definition 2.2.19 (Tropical curves and multi-lines)**.** A *tropical curve C of degree Δ* is the abstract 1-dimensional cycle a rational tropical stable map of degree Δ gives rise to, i.e. C is a weighted embedded 1-dimensional polyhedral complex in $\mathbb{R}^m$. A *(tropical) multi-line L* is a rational tropical curve in $\mathbb{R}^m$ with $m+1$ ends such that the primitive direction of each of this ends is one of the standard directions of $\mathbb{R}^m$, see Notation 2.2.4. The weight with which an end of L appears is denoted by $\omega(L)$.

Another well-known fact about rational equivalence is the following:

Theorem 2.2.20 (Recession fan, [AHR16])**.** *Notation 2.2.4 is used. Each tropical curve C of degree Δ_d^m is rationally equivalent to a multi-line L_C with weights $\omega(L_C) = d$. Hence pull-backs of C and L_C along the evaluation maps are rationally equivalent. The multi-line L_C is also called recession fan of C.*

It was shown in Example 2.1.6 that a multi-line L which is centered at the origin of $\mathbb{R}^2$ and that has weight $\omega(L) = 1$ is cut out by the rational function $\max_{(x,y)\in\mathbb{R}^2}(x, y, 0)$. Analogously, one can show that a multi-line centered at $0 \in \mathbb{R}^2$ of weight ω is cut out by the rational function $\max_{(x,y)\in\mathbb{R}^2}(\omega \cdot x, \omega \cdot y, 0)$. This gives rise to the following definition.

Definition 2.2.21 (Degenerated multi-lines in $\mathbb{R}^2$)**.** Let $\omega \in \mathbb{N}_{>0}$. The following tropical intersections $\max_{(x,y)\in\mathbb{R}^2}(\omega \cdot x, 0) \cdot \mathbb{R}^2$, $\max_{(x,y)\in\mathbb{R}^2}(\omega \cdot y, 0) \cdot \mathbb{R}^2$ and $\max_{(x,y)\in\mathbb{R}^2}(\omega \cdot (x - y), 0) \cdot \mathbb{R}^2$ and any translations thereof are called *degenerated (tropical) multi-lines* of weight ω. In particular, degenerated multi-lines with $\omega = 1$ are called *degenerated lines*. They (and translations of them) are denoted by $L_{10} := \max_{(x,y)\in\mathbb{R}^2}(x, 0) \cdot \mathbb{R}^2$, $L_{01} := \max_{(x,y)\in\mathbb{R}^2}(y, 0) \cdot \mathbb{R}^2$ and $L_{1\text{-}1} := \max_{(x,y)\in\mathbb{R}^2}(x, -y) \cdot \mathbb{R}^2$, see Figure 2.6.

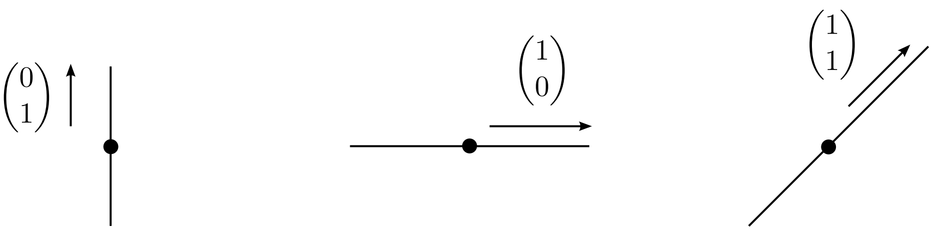


Figure 2.6: Degenerated tropical lines (from left to right) L_{10}, L_{01} and $L_{1\text{-}1}$ in $\mathbb{R}^2$ with ends of weight one.

2.3 Tyomkin's correspondence theorem

The tropical counterpart to classical cross-ratios was first introduced by Mikhalkin under the name *tropical double ratio* in [Mik07]. Tyomkin's correspondence theorem [Tyo17] states that the number of classical curves satisfying point and cross-ratio conditions and the number of tropical curves satisfying point and tropical double ratio conditions are equal. Since different classical curves may tropicalize to the same tropical curve, each tropical curve has to be counted with a *multiplicity*. We recall the definition of these multiplicities. For that, we stick to the notation used in [Tyo17]. For more details, see (4.1) of [Tyo17].

Definition 2.3.1 (Tropical double ratios defined by [Mik07, Tyo17])**.** Let $(\Gamma, x_{[M]}) \in \mathcal{M}_{0,M}$. Let $\{\beta_{i_1}, \beta_{i_3}\}$ and $\{\beta_{i_2}, \beta_{i_4}\}$ be two sets of labels of ends of Γ such that $\beta_{i_1}, \dots, \beta_{i_4}$ are pairwise different. A bounded edge γ of Γ *separates* β_{i_1}, β_{i_2} from β_{i_3}, β_{i_4} if β_{i_1}, β_{i_2} belong to one of the two connected components of $\Gamma \backslash \{\gamma\}$ and β_{i_3}, β_{i_4} to another.

The *tropical double ratio* λ_i^{DR} of $\{\beta_{i_1}, \beta_{i_2}\}$ and $\{\beta_{i_3}, \beta_{i_4}\}$ is given by

$$\lambda_i^{\mathrm{DR}} := \sum_{\gamma} \epsilon(\gamma, i)|\gamma|,$$

where the sum goes over all bounded edges of Γ and $|\gamma|$ is the length of a bounded edge and

$$\epsilon(\gamma, i) := \begin{cases} 1, & \text{if } \gamma \text{ separates the ends } \beta_{i_1}, \beta_{i_2} \text{ from } \beta_{i_3}, \beta_{i_4}, \\ -1, & \text{if } \gamma \text{ separates the ends } \beta_{i_1}, \beta_{i_4} \text{ from } \beta_{i_2}, \beta_{i_3}, \\ 0, & \text{otherwise.} \end{cases}$$

Notice that by abuse of notation the β_i's are not incorporated into the notation of a tropical double ratio λ_i^{DR}. Moreover, the definition of tropical double ratios also applies to tropical stable maps $(\Gamma, x_{[M]}, h) \in \mathcal{M}_{0,N}(\mathbb{R}^m, \Delta)$.

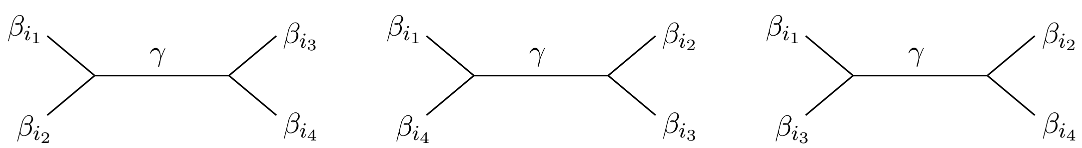


Figure 2.7: Schematic picture of the three cases of $\epsilon(\gamma, i)$ for λ_i' as in Definition 2.3.1 for a 4-marked abstract rational tropical curve. From left to right: $\epsilon(\gamma, i) = 1, -1, 0$.

Remark 2.3.2 (Tropical double ratios and tropicalizations)**.** Tropical double ratios are indeed tropicalizations of classical cross-ratios (see Lemma 3.1 of [Tyo17]), i.e. given an algebraic curve over a valued field that satisfies a classical cross-ratio, then its tropicalization satisfies a tropical double ratio which is given by applying the valuation map of the ground field to the classical cross-ratio.

Example 2.3.3. Let $p_1, p_2 \in \mathbb{R}^2$ be two non-collinear points. Let λ_1^{DR} be a tropical double ratio of $\{x_1, x_2\}$ and $\{x_5, x_6\}$ with length $l > 0$. The rational tropical stable map $(\Gamma, x_{[6]}, h)$ of Example 2.2.8 satisfies the point conditions and the tropical double ratio condition λ_1^{DR} if its lengths $l_{[3]}$ are chosen appropriately. In particular, we need to set $l_3 = l$. Notice that $(\Gamma, x_{[6]}, h)$ is fixed by these conditions, i.e. the position of $(\Gamma, x_{[6]}, h)$ in $\mathbb{R}^2$ and the lengths of its bounded edges are completely determined by the conditions $p_1, p_2, \lambda_1^{\mathrm{DR}}$.

Definition 2.3.4 (Correspondence theorem's multiplicities)**.** Let $C = (\Gamma, x_{[M]}, h)$ be a tropical stable map in $\in \mathcal{M}_{0,N}(\mathbb{R}^m, \Delta)$ that satisfies given point conditions $p_{[n]}$ (i.e. prescribed contracted ends are mapped to prescribed points) and that satisfies tropical double ratios $\lambda_{[l]}^{\mathrm{DR}}$.

Let x_1 be the end of Γ that is contracted to p_1 under h. We refer to the vertex adjacent to x_1 in Γ as *root vertex* and orient all edges of Γ away from the root vertex. The *head* of a bounded edge γ is denoted by $\mathfrak{h}(\gamma)$ and its *tail* by $\mathfrak{t}(\gamma)$. Let $V(\Gamma)$ be the set of vertices of Γ and let $E^b(\Gamma)$ be the set of bounded edges of Γ. We refer to a vertex of Γ as v and to a bounded edge of Γ as γ for now. The vertices adjacent to ends $x_{[M]}$ are denoted by $v_{[M]}$ and do not need to be different. Define the complex

$$\theta : \underbrace{\bigoplus_{v\in V(\Gamma)} \mathbb{Z}^m \oplus \bigoplus_{\gamma\in E^b(\Gamma)} \mathbb{Z}}_{M_1} \xrightarrow{B} \underbrace{\bigoplus_{\gamma\in E^b(\Gamma)} \mathbb{Z}^m \oplus \bigoplus_{i=1}^{n} \mathbb{Z}^m \oplus \bigoplus_{j=1}^{l} \mathbb{Z}}_{M_2} \tag{2.1}$$

given by the maps (that are defined copywise)

$$1_\gamma \mapsto n_\gamma + \sum_{i=1}^{l} \epsilon(\gamma, i) \quad \text{and} \quad a_v \mapsto \sum_{\gamma} \tilde{\epsilon}(\gamma, v) a_v + \sum_{i=1}^{n} \delta(v, v_i) a_v,$$

where a_v is the coordinate vector of $h(v)$ and where (see Definition 2.2.7 for the notation $v(\gamma, \mathfrak{t}(\gamma))$)

$$\tilde{\epsilon}(\gamma, v) := \begin{cases} 1, & \text{if } v = \mathfrak{t}(\gamma) \\ -1, & \text{if } v = \mathfrak{h}(\gamma) \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad n_\gamma := v(\gamma, \mathfrak{t}(\gamma))$$

and

$$\delta(v, v_i) := \begin{cases} 1, & \text{if } v = v_i \\ 0, & \text{otherwise.} \end{cases}$$

Let $\theta_{\mathbb{Z}}$ be the map from above in the complex (2.1) $\otimes_{\mathbb{Z}} \mathbb{Z}$. Finally, we can define the *multiplicity of C* by

$$m_{\mathbb{C}}(\Gamma, h) := \# \operatorname{coker} \theta_{\mathbb{Z}},$$

which is equal to $|\det(B)|$.

Example 2.3.5. Let $(\Gamma, x_{[6]}, h)$ be the rational tropical stable map of Example 2.3.3 that satisfies point conditions p_1, p_2 and a tropical double ratio λ_1^{DR}. Let γ_i denote the bounded edge of Γ which is of length l_i for $i \in [3]$ (see Figure 2.5). The matrix M_B associated to the map B of (2.1) is given by

$$M_B = \begin{array}{cc} & \begin{array}{ccccccccccc} v_1 & & v_2 & & v_3 & & v_4 & & \gamma_1 & \gamma_2 & \gamma_3 \end{array} \\ \begin{array}{c} \gamma_1 \\ \\ \gamma_2 \\ \\ \gamma_3 \\ \\ p_1 \\ \\ p_2 \\ \\ \lambda_1^{\mathrm{DR}} \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Laplace expansion yields

$$m_{\mathbb{C}}(\Gamma, h) = |\det(M_B)| = 2.$$

Theorem 2.3.6 (Tyomkin's correspondence theorem — Theorem 5.1 of [Tyo17])**.** *Notation 2.0.1 is used. Let Δ be a degree as in Definition 2.2.3 that arises from a lattice polytope $\Sigma(\Delta)$ as in Remark 2.2.5. Let $X_{\Sigma(\Delta)}$ be the toric variety associated to $\Sigma(\Delta)$. Let $q_{\underline{n}}$ be points in $X_{\Sigma(\Delta)}$ and let $\mu_{[l]}$ be classical cross-ratios. If the conditions $q_{\underline{n}}, \mu_{[l]}$ are in general position, i.e. there is only a finite number of rational algebraic curves in $X_{\Sigma(\Delta)}$ over an algebraically closed field of characteristic zero that fulfill them, then denote this number by $N_{\Delta}^{\mathrm{alg}}\left(q_{\underline{n}}, \mu_{[l]}\right)$. Let $p_{\underline{n}}, \lambda_{[l]}^{\mathrm{DR}}$ be the tropicalizations (see Remark 2.3.2) of the conditions above. Then*

$$N_{\Delta}^{\mathrm{alg}}\left(q_{\underline{n}}, \mu_{[l]}\right) = N_{\Delta}\left(p_{\underline{n}}, \lambda_{[l]}^{\mathrm{DR}}\right)$$

holds, where $N_{\Delta}\left(p_{\underline{n}}, \lambda_{[l]}^{\mathrm{DR}}\right)$ is the weighted number of rational tropical stable maps of degree Δ that satisfy the point conditions $p_{\underline{n}}$ and the tropical double ratio conditions $\lambda_{[l]}^{\mathrm{DR}}$. The multiplicity with which each rational tropical stable map is counted is the one of Definition 2.3.4.

The numbers $N_{\Delta}^{\mathrm{alg}}\left(q_{\underline{n}}, \mu_{[l]}\right)$ (resp. $N_{\Delta}\left(p_{\underline{n}}, \lambda_{[l]}^{\mathrm{DR}}\right)$) are independent of the exact positions of the points $q_{\underline{n}}$ (resp. $p_{\underline{n}}$) as long as all conditions are in general position (see Corollary 3.1.20 and Remark 3.1.10). Incorporating the point conditions explicitly in the notation allows us to directly access them which simplifies notation later.

Chapter 3

Tropical cross-ratios and their degenerations

Tyomkin's correspondence theorem 2.3.6 shows that enumerative problems that involve cross-ratios can be formulated tropically. An approach to enumerative questions that proved to be fruitful in the past is using intersection theory on moduli spaces. Thus a tropical intersection theoretic description of tropical stable maps that satisfy tropical double ratios is required. In this chapter, *tropical cross-ratios* are introduced in such a way that they provide a tropical intersection theoretic description we are looking for. The main advantage of tropical cross-ratios over tropical double ratios is that they allow a degeneration that yields *degenerated tropical cross-ratios.* This degenerated version of tropical cross-ratios yields a local description of so-called *cross-ratio multiplicities* from which we profit from in chapters 4, 5, 6.

3.1 Tropical cross-ratios

In this section, tropical cross-ratios are introduced. The tropical intersection theoretic framework provides multiplicities for tropical stable maps satisfying tropical cross-ratios. It turns out that these multiplicities have an enumerative meaning.

3.1.1 Tropical cross-ratios via intersection products

Definition 3.1.1 (Tropical cross-ratios)**.** A *tropical cross-ratio* λ' is an unordered pair of pairs of unordered numbers $(\beta_1\beta_2|\beta_3\beta_4)$ together with an element in $\mathbb{R}_{>0}$ denoted by $|\lambda'|$, where $\beta_1,\dots,\beta_4$ are pairwise distinct labels of ends of an abstract tropical curve in $\mathcal{M}_{0,M}$ (resp. of a tropical stable map in $\mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$). We say that $C\in\mathcal{M}_{0,M}$ (resp. $C\in\mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$) *satisfies* the tropical cross-ratio condition λ' if $C\in \mathrm{ft}^*_{\lambda'}(|\lambda'|)\cdot\mathcal{M}_{0,M}$ (resp. $C\in \mathrm{ft}^*_{\lambda'}(|\lambda'|)\cdot\mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$), where $|\lambda'|$ is the canonical local coordinate of the end $(\beta_1\beta_2|\beta_3\beta_4)$ of $\mathcal{M}_{0,4}$, see Figure 2.3.

Remark 3.1.2. Definition 3.1.1 expresses the definition of tropical double ratios 2.3.1 used by Mikhalkin and Tyomkin in terms of tropical intersection theory. This can be seen by applying a suitable projection $\pi_{\lambda'}:\mathcal{M}_{0,4}\to\mathbb{R}$ shrinking one end to zero, sending another one to $\mathbb{R}_{>0}$ and the last one to $\mathbb{R}_{<0}$ such that $\pi_{\lambda'}\circ \mathrm{ft}_{\lambda'}$ coincides with Definition 2.3.1.

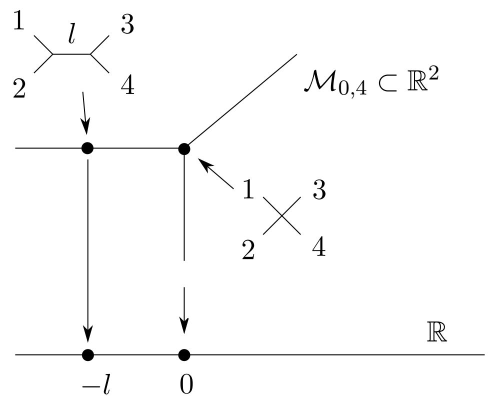


Figure 3.1: As in Theorem 2.2.2, one way of embedding $\mathcal{M}_{0,4}$ into $\mathbb{R}^2$ is shown. A projection that maps the shown rational tropical stable map with an edge of length l to $-l \in \mathbb{R}$ is indicated.

Definition 3.1.3 (General position I)**.** Notation 2.0.1 is used. Let $m, N \in \mathbb{N}_{>0}$. Let $\underline{n}, \underline{\kappa}$ be disjoint subsets of the set $[N]$. Let $p_{\underline{n}}$ be points in $\mathbb{R}^m$ and let $L_{\underline{\kappa}}$ be tropical multi-lines in $\mathbb{R}^m$. Let Δ be a degree as in Definition 2.2.3 and let $\lambda'_{[l]}$ be tropical cross-ratios for some $l \in \mathbb{N}$. Let $\underline{\eta}^\gamma \subset \gamma^{\text{lab}}$ and $\underline{\kappa}^\gamma \subset \gamma^{\text{lab}}$ for $\gamma = \alpha, \beta$ be pairwise disjoint sets of labels as in Notation 2.2.4. Let $P_{\underline{\eta}^\gamma}$ be points in $\mathbb{R}^{m-1}$ for $\gamma = \alpha, \beta$. Let $L_{\underline{\kappa}^\gamma}$ be tropical multi-lines in $\mathbb{R}^{m-1}$ for $\gamma = \alpha, \beta$ such that in total

$$\#\Delta - 3 + N + m = m \cdot \#\underline{n} + (m-1) \cdot \# \left(\underline{\kappa} \cup \underline{\eta}^\alpha \cup \underline{\eta}^\beta\right) + (m-2) \cdot \# \left(\underline{\kappa}^\alpha \cup \underline{\kappa}^\beta\right) + l \tag{3.1}$$

holds. The conditions $p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda'_{[l]}$ are in *general position* if

$$Z_\Delta \left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda'_{[l]}\right) :=$$
$$\prod_{k \in \underline{\kappa}^\alpha \cup \underline{\kappa}^\beta} \partial \operatorname{ev}_k^*(L_k) \cdot \prod_{f \in \underline{\eta}^\alpha \cup \underline{\eta}^\beta} \partial \operatorname{ev}_f^*(P_f) \cdot \prod_{h \in \underline{\kappa}} \operatorname{ev}_h^*(L_h) \cdot \prod_{i \in \underline{n}} \operatorname{ev}_i^*(p_i) \cdot \prod_{j \in [l]} \operatorname{ft}_{\lambda'_j}^* \left(|\lambda'_j|\right) \cdot \mathcal{M}_{0,N}(\mathbb{R}^m, \Delta) \tag{3.2}$$

is a zero-dimensional that is not rationally equivalent to a zero cycle and it lies inside top-dimensional cells of $\mathcal{M}_{0,N}(\mathbb{R}^m, \Delta)$.

Usually, we refer to $p_{\underline{n}}$ as *point conditions* and to $L_{\underline{\kappa}}$ as *multi-line conditions*. Moreover, we refer to $P_{\underline{\eta}^\alpha \cup \underline{\eta}^\beta}$ and $L_{\underline{\kappa}^\alpha \cup \underline{\kappa}^\beta}$ as *tangency conditions*, where, in particular, $P_{\underline{\eta}^\alpha \cup \underline{\eta}^\beta}$ are called *codimension one tangency conditions* and $L_{\underline{\kappa}^\alpha \cup \underline{\kappa}^\beta}$ are called *codimension two tangency conditions.*

Remark 3.1.4. Notice that if there are tangency conditions in a set of general positioned conditions, then $\Delta = \Delta_d^m(\alpha, \beta)$ for some m, d, α, β, see Notation 2.2.4.

Notation 3.1.5. A convention is used in Definition 3.1.3 to which we stick from now on: Given a degree Δ and general positioned conditions, we know which conditions we expect to be satisfied by which labeled ends as we use the same index for conditions and ev (resp. $\partial \operatorname{ev}$) maps. In particular, we may e.g. consider a submultiset of $\Delta_d^m(\alpha, \beta)$ which contains all ends satisfying the tangency conditions $L_{\underline{\kappa}^\alpha} \cup L_{\underline{\kappa}^\beta}$.

Remark 3.1.6. Given an intersection product as in Definition 3.1.3, where L_k is a rational tropical curve in $\mathbb{R}^m$ of degree Δ_d^m (resp. a rational tropical curve in $\mathbb{R}^{m-1}$ of degree Δ_d^{m-1}), we can pass to its recession fan and obtain an intersection product that is rationally equivalent to the one we started with, see Remark 2.2.18 and Theorem 2.2.20. Therefore we can always assume that L_k is in fact a tropical multi-line in $\mathbb{R}^m$ (resp. in $\mathbb{R}^{m-1}$).

Remark 3.1.7. We assume in the following that all given sets of conditions are in general position. If we refer to a set of conditions to be in general condition although it has not enough elements, then we mean that there are some conditions that we can add to this set such that all together these conditions are in general position.

Definition 3.1.8. Remark 3.1.4 is used. Let $Z_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda'_{[l]}\right)$ be the zero-dimensional cycle associated to general positioned conditions $p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda'_{[l]}$ as in Definition 3.1.3. Define the number

$$N_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda'_{[l]}\right) := \deg\left(Z_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda'_{[l]}\right)\right),$$

where deg is the degree map of Definition 2.1.17. If we write $N_\Delta\left(p_{\underline{n}}, \lambda'_{[l]}\right)$, we mean that there are no multi-line and tangency conditions in the set of given conditions.

Remark 3.1.9. Recall Remark 2.2.18: for $t \in \mathbb{N}$ translated cycles in $\mathbb{R}^t$ are rationally equivalent. Hence embedding $\mathcal{M}_{0,4}$ into $\mathbb{R}^2$ as in Theorem 2.2.2 implies that all points of $\mathcal{M}_{0,4}$ are rationally equivalent. Thus pulling-back different points of $\mathcal{M}_{0,4}$ yields rationally equivalent cycles, see Remark 2.2.18. Therefore the numbers of Definition 3.1.8 are independent of the exact lengths $|\lambda'_j|$ of the tropical cross-ratios λ'_j for $j \in [l]$. Additionally, the numbers do not depend on the partition of the four entries of each tropical cross-ratio into pairs. Moreover, the numbers of Definition 3.1.8 are also independent of the exact positions of the point, multi-line and tangency conditions involved.

Remark 3.1.10. Given conditions as in Definition 3.1.8, then the set of positions where these conditions can be moved as in Remark 3.1.9 while remaining in general position is open and dense in the space of all possible positions of conditions. To see this, consider the map

$$\varphi := \bigtimes_{k\in\underline{\kappa}^\alpha\cup\underline{\kappa}^\beta} \partial\operatorname{ev}_k \times \bigtimes_{f\in\underline{\eta}^\alpha\cup\underline{\eta}^\beta} \partial\operatorname{ev}_f \times \bigtimes_{h\in\underline{\kappa}} \operatorname{ev}_h \times \bigtimes_{i\in\underline{n}} \operatorname{ev}_i \times \bigtimes_{j\in[l]} \operatorname{ft}_{\lambda'_j} :$$
$$\mathcal{M}_{0,N}\left(\mathbb{R}^m, \Delta\right) \to \mathbb{R}^t \times \left(\mathcal{M}_{0,4}\right)^l,$$

where t is a suitable natural number. The map φ is a morphism of fans. The first part of Corollary 3.1.17 does not require general positioned conditions and thus yields that pull-backs along φ lie in the intersections of the preimages of the evaluation and forgetful maps involved in φ. Hence the set where general positioned conditions can be moved while remaining in general position is the complement of a codimension one skeleton of a pure dimensional fan and thus it is open and dense.

3.1.2 Multiplicities of rational tropical stable maps.

The intersection theoretic definition of tropical cross-ratios automatically assigns a multiplicity to each tropical stable map that satisfies given point, multi-lines, tangency and tropical cross-ratio conditions. We now recall how to calculate such multiplicities.

Definition 3.1.11 (ev-ft-matrix)**.** Let $p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda'_{[l]}$ be general positioned conditions in the sense of Remark 3.1.7, i.e. the cycle $Z_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda'_{[l]}\right)$ associated to these conditions as in Definition 3.1.8 is not necessary zero-dimensional. Let C be a rational tropical stable map in a top-dimensional cell σ of $Z_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda'_{[l]}\right)$ such that C lies in the interior of σ if σ is not zero-dimensional. Let φ_i for $i \in [r]$ for a suitable $r \in \mathbb{N}$ denote the pull-backs that appear in (3.2), i.e.

$$Z_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda'_{[l]}\right) = \varphi_1 \cdots \varphi_r \cdot \mathcal{M}_{0,N}\left(\mathbb{R}^m, \Delta\right).$$

Locally (around C) each pull-back φ_i is of the form $\max\{h_i, 0\}$ for an integer linear map h_i. Thus the map

$$\begin{aligned} H : \mathcal{M}_{0,N}\left(\mathbb{R}^m, \Delta\right) &\to \mathbb{R}^d \times \mathbb{R}^l \\ x &\mapsto (h_1(x), \dots, h_r(x)) \end{aligned}$$

with $d := m \cdot \#\underline{n} + (m-1) \cdot \#(\underline{\kappa} \cup \underline{\eta}^\alpha \cup \underline{\eta}^\beta) + (m-2) \cdot \#(\underline{\kappa}^\alpha \cup \underline{\kappa}^\beta)$ is linear locally around C as well. It gives rise to a matrix with respect to the local coordinates of Definition 2.2.11. This matrix is called ev-ft-*matrix* and is denoted by $M(C)$. If there are no tropical cross-ratio condition, then $M(C)$ is also called ev-*matrix*. Notice that the entries of $M(C)$ are in $\mathbb{Z}$ and define

$$\mathrm{mult}_{\mathrm{ev,ft}}(C) := |\,\mathrm{ind}\,(M(C))\,|,$$

where ind is the index of $M(C)$, i.e. the product of elementary divisors that appear in its Smith normal form over $\mathbb{Z}$. If there are no tropical cross-ratio conditions, then define $\mathrm{mult}_{\mathrm{ev}(C)}$ analogously for the ev-matrix $M(C)$ of C.

Lemma 1.2.9 of [Rau09] states that $\mathrm{mult}_{\mathrm{ev,ft}}(C)$ (resp. $\mathrm{mult}_{\mathrm{ev}}(C)$) equals the weight $\omega(\sigma)$ of the top-dimensional cell σ of $Z_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda'_{[l]}\right)$ (resp. of the top-dimensional cell σ of $Z_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}\right)$).

Remark 3.1.12. If the cycle $Z_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda'_{[l]}\right)$ appearing in Definition 3.1.11 is zero-dimensional, then $\mathrm{mult}_{\mathrm{ev,ft}}(C)$ equals the absolute value of the determinant of the ev-ft-matrix $M(C)$. In particular, the multiplicity with which a rational tropical stable map C in the cycle $Z_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda'_{[l]}\right)$ contributes to $N_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda'_{[l]}\right)$ is precisely $|\det(M(C))|$.

Remark 3.1.13. The absolute value of the index of an ev-ft-matrix (resp. ev-matrix) equals a weight of a top-dimensional cell in an intersection product. Thus it does not depend on the base point of the local coordinates, see Definition 2.2.11.

Example 3.1.14. Consider the rational tropical stable maps C whose image in $\mathbb{R}^3$ is shown in Figure 3.2. The ends of C are labeled by $1, \dots, 6$. The labels are indicated with circled numbers in Figure 3.2. The direction vectors of edges and ends of C are shown in Figure 3.2. Moreover, the lengths of the three bounded edges of C are denoted by l_1, l_2, l_3. The end labeled with 1 which is drawn dotted indicates a contracted end. The degree of C is $\Delta_1^3(\alpha, \beta)$, where $\alpha = (0, 1, 0, \dots)$ and $\beta = (1, 0, \dots)$ (see Notation 2.2.4), i.e. C has one end of primitive direction $-e_3$ whose weight is 2 and C has one end of primitive direction e_3 whose weight is 1.

The rational tropical stable map C satisfies the following conditions by which it is fixed: p_1 is a point condition to which the end labeled with 1 is contracted to. The end labeled with 3 satisfies a codimension two tangency condition L_3, where L_3 is a multi-line with ends of weight 1 which is indicated by a dashed line in Figure 3.2. Moreover, the end labeled with 6 satisfies a codimension one tangency condition P_6. Notice that Notation 3.1.5 was used.

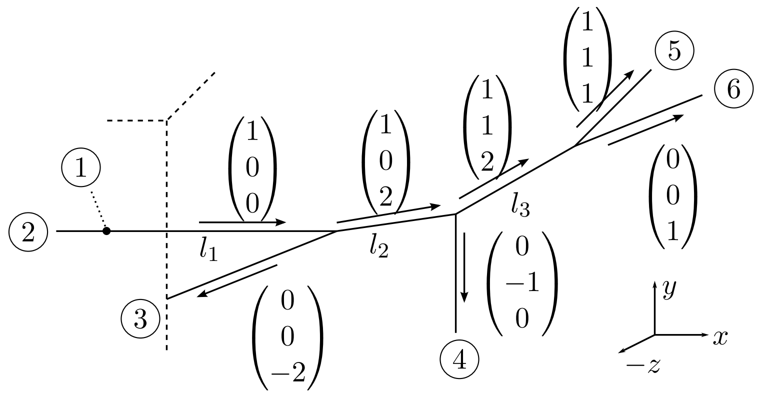


Figure 3.2: The tropical stable map C that is fixed by a point p_1 and two tangency conditions L_3, P_6. The arrows indicate the direction vectors of the edges.

Then the ev-matrix $M(C)$ with respect to the base point p_1 of C reads as

$$M(C) = \begin{array}{rc} & \begin{array}{cccccc} \multicolumn{3}{c}{\text{Base } p_1} & l_1 & l_2 & l_3 \end{array} \\ \begin{array}{r} \mathrm{ev}_1 \\ \\ \\ \partial\,\mathrm{ev}_3 \\ \partial\,\mathrm{ev}_6 \\ \\ \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

The first 3 rows describe the position of p_1. The fourth row describes the position of L_3 and the last two rows describe the position of P_6 using the coordinates l_1, l_2, l_3.

Example 3.1.15. Let C be the rational tropical stable map whose image in $\mathbb{R}^2$ is shown in Figure 3.3. The ends of C are labeled by $1, \dots, 6$. The labels are indicated with circled numbers in Figure 3.3. The lengths of the three bounded edges of C are denoted by l_1, l_2, l_3. The rational tropical stable map C satisfies two point conditions p_1, p_2 with its contracted ends $1, 2$ and a tropical cross-ratio $\lambda'_1 = (12|56)$ of length $|\lambda'_1| = l_3$.

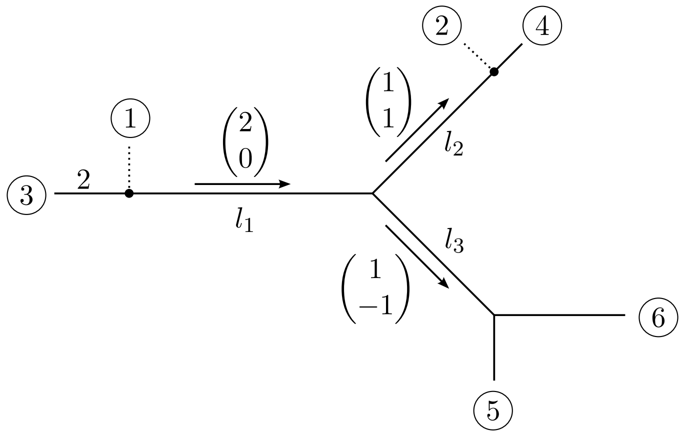


Figure 3.3: The tropical stable map C that is fixed the points p_1, p_2 and a tropical cross-ratio λ'_1. The arrows indicate the direction vectors of the edges.

The ev-ft-matrix $M(C)$ of C with respect to the base point p_1 is

$$M(C) = \begin{array}{c} \\ \mathrm{ev}_1 \\ \\ \mathrm{ev}_2 \\ \\ \mathrm{ft}_{\lambda'_1} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Base } p_1 & & l_1 & l_2 & l_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Therefore

$$\mathrm{mult}_{\mathrm{ev,ft}}(C) = |\det(M(C))| = 2.$$

The rational tropical stable map C is actually the one of Example 2.3.5. There, its multiplicity used in Tyomkin's correspondence theorem was computed. Notice that both multiplicities coincide. This observation holds in general since Proposition 3.1.19 states that these two multiplicities are indeed always the same.

3.1.3 Enumerative properties of tropical cross-ratios

Often, tropical intersection theory yields multiplicities needed for correspondence theorems, which enables us to count tropical stable maps by means of tropical intersection theory on tropical moduli spaces. The same holds true for the counts of tropical stable maps satisfying tropical cross-ratio conditions we consider here. More precisely, our next aim is to show that the set of rational tropical stable maps that appears in Tyomkin's correspondence theorem 2.3.6 equals the set of rational tropical stable maps contributing to suitable numbers from Definition 3.1.8. Each rational tropical stable map in such a set has two multiplicities, namely the one used in Tyomkin's correspondence theorem 2.3.6 and the one assigned to it via tropical intersection theory (Remark 3.1.12). Showing that these multiplicities coincide yields that the numbers $N_\Delta\left(p_{\underline{n}}, \lambda'_{[l]}\right)$ of Definition 3.1.8 are indeed enumerative in the sense that Tyomkin's correspondence theorem can be applied.

Lemma 3.1.16. *Let Δ be a degree in $\mathbb{R}^m$ and let $\lambda'_{[l]}$ be tropical cross-ratios which are not necessary in general position. Then*

$$|\prod_{j\in[l]} \mathrm{ft}^*_{\lambda'_j}\left(|\lambda'_j|\right)\cdot \mathcal{M}_{0,N}\left(\mathbb{R}^m,\Delta\right)| \subset \bigcap_{j\in[l]} \mathrm{ft}^{-1}_{\lambda'_j}\left(|\lambda'_j|\right) \tag{3.3}$$

holds, where $|\star|$ denotes the support of $\prod_{j\in[l]} \mathrm{ft}^*_{\lambda'_j}\left(|\lambda'_j|\right)\cdot\mathcal{M}_{0,N}\left(\mathbb{R}^m,\Delta\right)$ with nonzero weights.

Moreover, if the tropical cross-ratios $\lambda'_{[l]}$ are in general position (in the sense of Remark 3.1.7), then $\prod_{j\in[l]} \mathrm{ft}^*_{\lambda'_j}\left(|\lambda'_j|\right)\cdot\mathcal{M}_{0,N}\left(\mathbb{R}^m,\Delta\right)$ equals the polyhedral set $\bigcap_{j\in[l]} \mathrm{ft}^{-1}_{\lambda'_j}\left(|\lambda'_j|\right)$ with the additional data of weights such that some of them might be zero.

Proof. To show (3.3), induction on the number l of tropical cross-ratios is used. Let $l = 1$ and denote $\lambda'_1 = (\beta_1\beta_2|\beta_3\beta_4)$. The map $\mathrm{ft}_{\lambda'_1}$ takes rational tropical stable maps to $\mathcal{M}_{0,4}$, which can without loss of generality be embedded into $\mathbb{R}^2$ such that the direction vector of its end $(\beta_1\beta_2|\beta_3\beta_4)$ is e_0, see Theorem 2.2.2, Figure 2.3 and Notation 2.2.4. Thus $P := (|\lambda'_1|, |\lambda'_1|) \in \mathbb{R}^2$ is the point which is pulled-back along $\mathrm{ft}_{\lambda'_1}$. Therefore P is cut out by

$$\begin{aligned} \max\{\star,\star,|\lambda'_1|\} : \mathcal{M}_{0,4} &\to \mathbb{R} \\ (x,y) &\mapsto \max\{x,y,|\lambda'_1|\}. \end{aligned}$$

Hence

$$\mathrm{ft}^*_{\lambda'_1}\left(|\lambda'_1|\right) = \max\{\left(\mathrm{ft}_{\lambda'_1}(\star)\right)_x, \left(\mathrm{ft}_{\lambda'_1}(\star)\right)_y, |\lambda'_1|\} \tag{3.4}$$

is affine linear on each connected component of the complement of $\mathrm{ft}^{-1}_{\lambda'_1}\left(|\lambda'_1|\right)$. Hence the left-hand side of (3.3) is a subset of the right-hand side of (3.3) for $l=1$ according to Definition 2.1.5.

For $l>1$, observe again that

$$\mathrm{ft}^*_{\lambda'_l}\left(|\lambda'_l|\right) = \max\{\left(\mathrm{ft}_{\lambda'_l}(\star)\right)_x, \left(\mathrm{ft}_{\lambda'_l}(\star)\right)_y, |\lambda'_l|\}$$

can only be locally nonlinear at $\mathrm{ft}^{-1}_{\lambda'_l}\left(|\lambda'_l|\right)$ and use the induction hypothesis.

To show the second part of Lemma 3.1.16, recall that $\prod_{j\in[l]}\mathrm{ft}^*_{\lambda'_j}\left(|\lambda'_j|\right)\cdot\mathcal{M}_{0,N}\left(\mathbb{R}^m,\Delta\right)$ is a polyhedral complex of pure codimension l. We just proved that its nonzero part is contained in $\bigcap_{j\in[l]}\mathrm{ft}^{-1}_{\lambda'_j}\left(|\lambda'_j|\right)$. Thus the second part of Lemma 3.1.16 holds if the claim that $\bigcap_{j\in[l]}\mathrm{ft}^{-1}_{\lambda'_j}\left(|\lambda'_j|\right)$ is of codimension l is true. For each map $\mathrm{ft}_{\lambda'_j}$ with $j\in[l]$, consider a suitable projection $\pi_{\lambda'_j}$ as in Remark 3.1.2 such that $\pi_{\lambda'_j}\circ\mathrm{ft}_{\lambda'_j}$ maps $\mathrm{ft}_{\lambda'_j}\left(|\lambda'_j|\right)$ to $|\lambda'_j|\in\mathbb{R}$. Notice that for $j\in[l]$

$$\mathrm{ft}^*_{\lambda'_j}\left(|\lambda'_j|\right) = \left(\pi_{\lambda'_j}\circ\mathrm{ft}_{\lambda'_j}\right)^*\left(|\lambda'_j|\right)$$

holds, where by abuse of notation $|\lambda'_j|$ denotes a point in $\mathcal{M}_{0,4}$ on the left-hand side and it also denotes the corresponding point in $\mathbb{R}$ on the right-hand side. Thus

$$\prod_{j\in[l]}\mathrm{ft}^*_{\lambda'_j}\left(|\lambda'_j|\right)\cdot\mathcal{M}_{0,N}\left(\mathbb{R}^m,\Delta\right) = \prod_{j\in[l]}\left(\pi_{\lambda'_j}\circ\mathrm{ft}_{\lambda'_j}\right)^*\left(|\lambda'_j|\right)\cdot\mathcal{M}_{0,N}\left(\mathbb{R}^m,\Delta\right)$$

holds. If we take Remark 3.1.10 into account, then Proposition 1.15 of [Rau16] can be applied to prove the claim and hence finishes the proof. □

The analogous statement of Lemma 3.1.16 that involves evaluation maps instead of forgetful maps is well-known, see for example Remark 1.16 of [Rau16]. Thus combining Lemma 3.1.16 with its ev-analogue immediately yields the following corollary.

Corollary 3.1.17. *Let Δ be a degree in $\mathbb{R}^m$. Let $p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda'_{[l]}$ be conditions with the usual notation from Definition 3.1.3 that are not necessary in general position. Then*

$$\begin{aligned}&|Z_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda'_{[l]}\right)| \subset \\ &\bigcap_{k\in\underline{\kappa}^\alpha\cup\underline{\kappa}^\beta}\partial\,\mathrm{ev}_k^{-1}(L_k)\cap\bigcap_{f\in\underline{\eta}^\alpha\cup\underline{\eta}^\beta}\partial\,\mathrm{ev}_f^{-1}(P_f)\cap\bigcap_{h\in\underline{\kappa}}\mathrm{ev}_h^{-1}\left(L_h\right)\cap\bigcap_{i\in\underline{n}}\mathrm{ev}_i^{-1}\left(p_i\right)\cap\bigcap_{j\in[l]}\mathrm{ft}^{-1}_{\lambda'_j}\left(|\lambda'_j|\right)\end{aligned} \tag{3.5}$$

holds, where $|\star|$ denotes the support of $Z_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda'_{[l]}\right)$ with nonzero weights.

Moreover, if all conditions are in general positioned (in the sense of Remark 3.1.7), then $Z_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda'_{[l]}\right)$ equals the polyhedral set on the right-hand side of (3.5) *with the additional data of weights such that some of them might be zero.*

Remark 3.1.18. Let $N_\Delta\left(p_{\underline{n}}, \lambda_{[l]}^{\mathrm{DR}}\right)$ be a tropical number as in Theorem 2.3.6. Let λ'_j be the tropical cross-ratio that is associated to the tropical double ratio λ_j^{DR} for $j \in [l]$ by Remark 3.1.2. Notice that $N_\Delta\left(p_{\underline{n}}, \lambda_{[l]}^{\mathrm{DR}}\right)$ is by definition the weighted number of rational tropical stable maps C with $C \in \bigcap_{i\in\underline{n}} \mathrm{ev}_i^{-1}(p_i) \cap \bigcap_{j\in[l]} \mathrm{ft}_{\lambda'_j}^{-1}\left(|\lambda'_j|\right)$, where the multiplicity $m_{\mathbb{C}}(\Gamma, h)$ of C is the one of Definition 2.3.4 which appears in Tyomkin's correspondence theorem 2.3.6. On the other hand, Corollary 3.1.17 shows that $N_\Delta\left(p_{\underline{n}}, \lambda'_{[l]}\right)$ is the weighted number of rational tropical stable maps C with $C \in \bigcap_{i\in\underline{n}} \mathrm{ev}_i^{-1}(p_i) \cap \bigcap_{j\in[l]} \mathrm{ft}_{\lambda'_j}^{-1}\left(|\lambda'_j|\right)$, where the multiplicity with which C is counted is given by $|\det(M(C))|$ as in Remark 3.1.12. The two multiplicities associated to C are compared in the next proposition which we prove using methods well-known to the experts in the area.

Proposition 3.1.19. *Let Δ be a degree. Let $p_{[n]}, \lambda'_{[l]}$ be general positioned condition as in Definition 3.1.3. Let $C \in \bigcap_{i\in[n]} \mathrm{ev}_i^{-1}(p_i) \cap \bigcap_{j\in[l]} \mathrm{ft}_{\lambda'_j}^{-1}\left(|\lambda'_j|\right)$ be a rational tropical stable map. Then the intersection theoretic multiplicity $|\det(M(C))|$ of C coincides with $m_{\mathbb{C}}(\Gamma, h)$ used in Tyomkin's correspondence theorem 2.3.6.*

Proof. Let $(\Gamma, x_{[M]}, h)$ be a rational tropical stable map in $\bigcap_{i\in\underline{n}} \mathrm{ev}_i^{-1}(p_i) \cap \bigcap_{j\in[l]} \mathrm{ft}_{\lambda'_j}^{-1}\left(|\lambda'_j|\right)$. In terms of tropical intersection theory the multiplicity of $(\Gamma, x_{[M]}, h)$ is given by $|\det(A)|$, where A is the ev-ft-matrix of $(\Gamma, x_{[M]}, h)$, see Definition 3.1.11. We want to sketch how to prove that $|\det(A)|$ and $|\det(B)|$ (from Definition 2.3.4) are equal. For that, we start with the following complex

$$\underbrace{\mathbb{Z}^m \oplus \bigoplus_{\gamma\in E^b(\Gamma)} \mathbb{Z}}_{N_1} \xrightarrow{A} \underbrace{\bigoplus_{i=1}^{n} \mathbb{Z}^m \oplus \bigoplus_{j=1}^{l} \mathbb{Z}}_{N_2},$$

where the first summand on the left belongs to the root vertex defined in 2.3.4. There are maps between the complex above and the complex (2.1) in the following way: Let $\alpha_2 : N_2 \to M_2$ be the canonical embedding and let

$$\alpha_1 : N_1 \to M_1,\ (a, \underline{e}) \mapsto (a, a + \sum \pm e_i u_{e_i}, \underline{e})$$

be a map where a is the coordinate of the root vertex, e_i is the length of the edge γ_i and u_{e_i} is the primitive direction vector of γ_i. Moreover, we choose $a + \sum \pm e_i u_{e_i}$ in such a way that it is the shortest path between the root vertex and the vertex associated to the j-th contracted end depending on which entry of the vector in the image we are considering (the choice of $\pm$ should be consistent with the orientation on Γ). Note that α_1, α_2 are both injective and that the diagram given by the maps A, B, α_1, α_2 commutes. This commutative diagram extends to the commutative diagram shown below. By definition

$$\operatorname{coker}\alpha_1 \cong (\mathbb{Z}^m)^{\#V(\Gamma)-1} \quad \text{and} \quad \operatorname{coker}\alpha_2 = (\mathbb{Z}^m)^{\#E^b(\Gamma)}.$$

Considering the definitions of B, ζ_2, we can see that $\zeta_2 \circ B$ is surjective. Hence C is surjective. Since C is a surjective morphism of free modules of the same rank it is an isomorphism. Therefore $\operatorname{coker}\alpha_3$ vanishes which guarantees that α_3 is surjective. The map ∂ which we obtain from applying the snake lemma yields that G vanishes. Therefore α_3 is an isomorphism. Thus

$$|\det(A)| = |\det(B)|$$

follows.

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & G \arrow[d] \arrow[dddll, rounded corners, "\partial"] & \\
0 \arrow[r] & N_1 \arrow[r, "A"] \arrow[d, "\alpha_1"] & N_2 \arrow[r] \arrow[d, "\alpha_2"] & \operatorname{coker} A \arrow[r] \arrow[d, "\alpha_3"] & 0 \\
0 \arrow[r] & M_1 \arrow[r, "B"] \arrow[d, "\zeta_1"] & M_2 \arrow[r] \arrow[d, "\zeta_2"] & \operatorname{coker} B \arrow[r] \arrow[d] & 0 \\
 & \operatorname{coker} \alpha_1 \arrow[r, "C"] & \operatorname{coker} \alpha_2 \arrow[r] & \operatorname{coker} \alpha_3 &
\end{tikzcd}$$

□

The following corollary is an immediate consequence of Remark 3.1.18 and Proposition 3.1.19. It gives an enumerative meaning to the numbers $N_\Delta\left(p_{\underline{n}}, \lambda'_{[l]}\right)$ in the sense that Tyomkin's correspondence theorem 2.3.6 can be applied to them.

Corollary 3.1.20. *Let Δ be a degree. Let $p_{\underline{n}}, \lambda^{\mathrm{DR}}_{[l]}$ be general positioned conditions as in Theorem 2.3.6. Let λ'_j be the tropical cross-ratio that is associated to the tropical double ratio λ^{DR}_j for $j \in [l]$ by Remark 3.1.2. Each rational tropical stable map C that contributes to $N_\Delta\left(p_{\underline{n}}, \lambda^{\mathrm{DR}}_{[l]}\right)$ with multiplicity* $\operatorname{mult}(C)$ *also contributes to $N_\Delta\left(p_{\underline{n}}, \lambda'_{[l]}\right)$ with multiplicity* $\operatorname{mult}(C)$ *and vice versa. In particular, Tyomkin's correspondence theorem 2.3.6 can be applied to $N_\Delta\left(p_{\underline{n}}, \lambda'_{[l]}\right)$. Hence the numbers $N_\Delta\left(p_{\underline{n}}, \lambda'_{[l]}\right)$ we compute via tropical intersection theory are equal to the corresponding algebro-geometric counts of curves that satisfy point and cross-ratio conditions.*

3.2 Degenerated tropical cross-ratios

An advantage of our intersection theoretic definition of tropical cross-ratios is that it allows us to *degenerate* tropical cross-ratios easily. It was indicated in Remark 3.1.9 that — up to rational equivalence — it does not matter which points of $\mathcal{M}_{0,4}$ are pulled-back to describe the cycle $Z_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda'_{[l]}\right)$ (resp. the numbers $N_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda'_{[l]}\right)$). In particular, the 3-valent vertex identified with $0 \in \mathbb{R}^2$ of $\mathcal{M}_{0,4}$ might be pulled-back. In other words, we could allow the lengths of tropical cross-ratios to become zero, which gives rise to the notion of *degenerated* tropical cross-ratios.

Definition 3.2.1 (Degenerated tropical cross-ratios)**.** A *degenerated (tropical) cross-ratio* λ is a set $\{\beta_1, \dots, \beta_4\}$, where $\beta_1, \dots, \beta_4$ are pairwise distinct labels of ends of an abstract tropical curve in $\mathcal{M}_{0,M}$ (resp. of a tropical stable map in $\mathcal{M}_{0,N}(\mathbb{R}^m, \Delta)$). We say that $C \in \mathcal{M}_{0,M}$ (resp. $C \in \mathcal{M}_{0,N}(\mathbb{R}^m, \Delta)$) *satisfies* the degenerated tropical cross-ratio condition λ if $C \in \mathrm{ft}_\lambda^*(0) \cdot \mathcal{M}_{0,M}$ (resp. $C \in \mathrm{ft}_\lambda^*(0) \cdot \mathcal{M}_{0,N}(\mathbb{R}^m, \Delta)$).

Another way of thinking about a degenerated cross-ratio λ is that λ is the *degeneration* of a tropical cross-ratio λ' when $|\lambda'|$ becomes zero in the process of taking a limit.

Notation 3.2.2. In the following we stick to the convention to denote a non-degenerated tropical cross-ratio by λ' and to denote its degeneration by λ.

Definition 3.2.3 (Tropical cross-ratios on contracted ends)**.** A tropical cross-ratio λ' given by $\lambda' = (\beta_1\beta_2|\beta_3\beta_4)$ (resp. a degenerated tropical cross-ratio $\lambda = \{\beta_1, \dots, \beta_4\}$) is called (degenerated) tropical cross-ratio *on contracted ends* if β_i is the label of a contracted end for $i \in [4]$. Let $\tilde{\lambda}'_{[l']}$ be tropical cross-ratios and let $\lambda_{[l]}$ be degenerated tropical cross-ratios. The set $\tilde{\lambda}'_{[l']}, \lambda_{[l]}$ is called *on contracted ends* if each (degenerated) tropical cross-ratio therein is a (degenerated) tropical cross-ratio on contracted ends.

The notation used in the following definition is consistent with the one of Definition 3.1.3. In particular, Remark 3.1.4 also holds for Definition 3.2.4.

Definition 3.2.4 (General position II)**.** Notation 2.0.1 is used. Let $m, N \in \mathbb{N}_{>0}$. Let $\underline{n}, \underline{\kappa}$ be disjoint subsets of the set $[N]$. Let $p_{\underline{n}}$ be points in $\mathbb{R}^m$ and let $L_{\underline{\kappa}}$ be tropical multi-lines in $\mathbb{R}^m$. Let Δ be a degree as in Definition 2.2.3. Let $\tilde{\lambda}'_{[l']}$ be tropical cross-ratios for some $l' \in \mathbb{N}$ and let $\lambda_{[l]}$ be degenerated cross-ratios for some $l \in \mathbb{N}$. Let $\underline{\eta}^\gamma \subset \gamma^{\mathrm{lab}}$ and $\underline{\kappa}^\gamma \subset \gamma^{\mathrm{lab}}$ for $\gamma = \alpha, \beta$ be pairwise disjoint sets of labels as in Notation 2.2.4. Let $P_{\underline{\eta}^\gamma}$ be points in $\mathbb{R}^{m-1}$ for $\gamma = \alpha, \beta$. Let $L_{\underline{\kappa}^\gamma}$ be tropical multi-lines in $\mathbb{R}^{m-1}$ for $\gamma = \alpha, \beta$ such that in total

$$\#\Delta - 3 + N + m = m \cdot \#\underline{n} + (m-1) \cdot \# \left(\underline{\kappa} \cup \underline{\eta}^\alpha \cup \underline{\eta}^\beta\right) + (m-2) \cdot \# \left(\underline{\kappa}^\alpha \cup \underline{\kappa}^\beta\right) + l + l' \tag{3.6}$$

holds. The conditions $p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \tilde{\lambda}'_{[l']}, \lambda_{[l]}$ are in *general position* if

$$\begin{aligned} &Z_\Delta \left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \tilde{\lambda}'_{[l']}, \lambda_{[l]}\right) := \\ &\prod_{k \in \underline{\kappa}^\alpha \cup \underline{\kappa}^\beta} \partial \operatorname{ev}_k^*(L_k) \cdot \prod_{f \in \underline{\eta}^\alpha \cup \underline{\eta}^\beta} \partial \operatorname{ev}_f^*(P_f) \cdot \prod_{h \in \underline{\kappa}} \operatorname{ev}_h^*(L_h) \cdot \prod_{i \in \underline{n}} \operatorname{ev}_i^*(p_i) \cdot \prod_{\tilde{j} \in [l']} \mathrm{ft}^*_{\tilde{\lambda}'_{\tilde{j}}} \left(|\tilde{\lambda}'_{\tilde{j}}|\right) \cdot \prod_{j \in [l]} \mathrm{ft}^*_{\lambda_j}(0) \\ &\cdot \mathcal{M}_{0,N}(\mathbb{R}^m, \Delta) \end{aligned} \tag{3.7}$$

is a zero-dimensional cycle that is not rationally equivalent to a zero cycle and it lies inside top-dimensional cells of

$$\mathcal{X}_\Delta\left(\lambda_{[l]}\right) := \prod_{j \in [l]} \mathrm{ft}^*_{\lambda_j}(0) \cdot \mathcal{M}_{0,N}(\mathbb{R}^m, \Delta).$$

Remark 3.1.7 also applies to 3.2.4, i.e. if we refer to a set of conditions to be in general position according to Definition 3.2.4, then we mean that some conditions can be added to this set such that all together these conditions are in general position.

Definition 3.2.5. Remark 3.1.4 is used. Let $Z_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \tilde{\lambda}'_{[l']}, \lambda_{[l]}\right)$ be the zero-dimensional cycle associated to general positioned conditions as in Definition 3.2.4. Define the number

$$N_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \tilde{\lambda}'_{[l']}, \lambda_{[l]}\right) := \deg\left(Z_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \tilde{\lambda}'_{[l']}, \lambda_{[l]}\right)\right),$$

where deg is the degree map of Definition 2.1.17. If we write $N_\Delta\left(p_{\underline{n}}, \lambda_{[l]}\right)$, we mean that there are no multi-line and tangency conditions in the set of given conditions.

Remark 3.2.6. Given conditions as in Definition 3.2.5, then the set of positions where these conditions can be moved as in Remark 3.1.9 while remaining in general position is open and dense in the space of all possible positions of conditions. This follows analogously to Remark 3.1.10 with the morphism

$$\bigtimes_{k\in\underline{\kappa}^\alpha\cup\underline{\kappa}^\beta} \partial\,\mathrm{ev}_k \times \bigtimes_{f\in\underline{\eta}^\alpha\cup\underline{\eta}^\beta} \partial\,\mathrm{ev}_f \times \bigtimes_{h\in\underline{\kappa}} \mathrm{ev}_h \times \bigtimes_{i\in\underline{n}} \mathrm{ev}_i \times \bigtimes_{j\in[l']} \mathrm{ft}_{\tilde{\lambda}'_j} : \mathcal{X}_\Delta\left(\lambda_{[l]}\right) \to \mathbb{R}^t \times \left(\mathcal{M}_{0,4}\right)^{l'}$$

for a suitable $t \in \mathbb{N}$.

The following proposition ensures that the numbers $N_\Delta\left(p_{\underline{n}}, \lambda_{[l]}\right)$ are enumerative in the sense that Tyomkin's correspondence theorem can be applied.

Proposition 3.2.7. *Let $p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \tilde{\lambda}'_{[l']}, \lambda'_{[l]}$ be general positioned conditions as in Definition 3.1.8. Let λ_j denote the degeneration of λ'_j for $j \in [l]$. Then there is an open dense subset in the set of all positions where $p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \tilde{\lambda}'_{[l']}, \lambda'_{[l]}$ can be moved (see Remark 3.1.10) such that the conditions $p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \tilde{\lambda}'_{[l']}, \lambda'_{[l]}$ and the conditions $p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \tilde{\lambda}'_{[l']}, \lambda_{[l]}$ are simultaneously in general position.*

In particular, if the cycle $Z_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \tilde{\lambda}'_{[l']}, \lambda'_{[l]}\right)$ of Definition 3.1.3 and the cycle $Z_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \tilde{\lambda}'_{[l']}, \lambda_{[l]}\right)$ of Definition 3.2.4 are zero-dimensional, then

$$N_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \tilde{\lambda}'_{[l']}, \lambda'_{[l]}\right) = N_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \tilde{\lambda}'_{[l']}, \lambda_{[l]}\right)$$

holds on an open dense subset in the set of all positions of conditions. In particular, Tyomkin's correspondence theorem 2.3.6 can be applied to $N_\Delta\left(p_{\underline{n}}, \lambda_{[l]}\right)$.

Proof. By Remark 3.1.9 the cycles $Z_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \tilde{\lambda}'_{[l']}, \lambda'_{[l]}\right)$ of Definition 3.1.3 and $Z_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \tilde{\lambda}'_{[l']}, \lambda_{[l]}\right)$ of Definition 3.2.4 are rationally equivalent. Hence none of them is rationally equivalent to a zero cycle if either

$$S' := \{p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \tilde{\lambda}'_{[l']}, \lambda'_{[l]}\} \text{ or } S := \{p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \tilde{\lambda}'_{[l']}, \lambda_{[l]}\}$$

are in general position. Moreover, there are open dense sets U, U' associated to S, S' by remarks 3.1.10, 3.2.6. Since $U \cap U'$ is again open dense, it is the desired open dense set. Thus

$$N_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \tilde{\lambda}'_{[l']}, \lambda'_{[l]}\right) = N_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \tilde{\lambda}'_{[l']}, \lambda_{[l]}\right). \tag{3.8}$$

Therefore, in particular,

$$N_\Delta\left(p_{\underline{n}}, \lambda'_{[l]}\right) = N_\Delta\left(p_{\underline{n}}, \lambda_{[l]}\right) \tag{3.9}$$

in the special case of only point and (degenerated) tropical cross-ratio conditions. Notice that the equalities (3.8), (3.9) hold on an open dense subset of positions of conditions as we defined the numbers $N_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \tilde{\lambda}'_{[l']}, \lambda'_{[l]}\right)$, $N_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \tilde{\lambda}'_{[l']}, \lambda_{[l]}\right)$ (resp. the numbers $N_\Delta\left(p_{\underline{n}}, \lambda'_{[l]}\right)$, $N_\Delta\left(p_{\underline{n}}, \lambda_{[l]}\right)$) for general positioned conditions only. Tyomkin's correspondence theorem can be applied due to Corollary 3.1.20. □

Remark 3.2.8. One could define the numbers $N_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \tilde{\lambda}'_{[l']}, \lambda'_{[l]}\right)$ (resp. the numbers $N_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \tilde{\lambda}'_{[l']}, \lambda_{[l]}\right)$) as degree of a zero-dimensional cycle where the conditions are not necessarily in general position. Then (3.8) would still hold using Remark 2.2.18. However, it is easier to restrict to general positioned conditions since tropical curves satisfying them are combinatorially more intuitive (corollaries 3.1.17, 3.2.10) and their weights can be calculated via indices of matrices, see Definition 3.1.11.

Lemma 3.2.9. *Let Δ be a degree in $\mathbb{R}^m$. Let $\tilde{\lambda}'_{[l']}$ be tropical cross-ratios and let $\lambda_{[l]}$ be degenerated tropical cross-ratios such that $\tilde{\lambda}'_{[l']}, \lambda_{[l]}$ are in general position. Then $\prod_{j\in[l]} \mathrm{ft}^*_{\lambda_j}(0) \cdot \mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$ is contained in the codimension-l skeleton of $\mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$. Additionally, the cycle $\prod_{\tilde{j}\in[l']} \mathrm{ft}^*_{\tilde{\lambda}'_{\tilde{j}}}\left(|\tilde{\lambda}'_{\tilde{j}}|\right)\cdot\prod_{j\in[l]} \mathrm{ft}^*_{\lambda_j}(0)\cdot\mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$ equals the polyhedral set $\bigcap_{\tilde{j}\in[l']} \mathrm{ft}^{-1}_{\tilde{\lambda}'_{\tilde{j}}}\left(|\tilde{\lambda}'_{\tilde{j}}|\right)\cap \bigcap_{j\in[l]} \mathrm{ft}^{-1}_{\lambda_j}(0)$ with the additional data of weights such that some of them might be zero.*

Proof. Notice that analogously to the proof of Lemma 3.1.16

$$\mathrm{ft}^*_{\lambda_j}(0) = \max\{\left(\mathrm{ft}_{\lambda_j}(\star)\right)_x, \left(\mathrm{ft}_{\lambda_j}(\star)\right)_y, 0\} \tag{3.10}$$

holds for all $j\in[l]$. To see that $\prod_{j\in[l]} \mathrm{ft}^*_{\lambda_j}(0)\cdot\mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$ is contained in the codimension-l skeleton of $\mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$, induction on l is used. If $l=1$, then $\mathrm{ft}^*_{\lambda_1}(0)$ is because of (3.10) affine linear on each top-dimensional cell of $\mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$. Thus no refinement of $\mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$ is necessary when intersecting with $\mathrm{ft}^*_{\lambda_1}(0)$. Therefore $\mathrm{ft}^*_{\lambda_1}(0)\cdot\mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$ is contained in the codimension-1 skeleton of $\mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$. If $l>1$, then $\prod_{j\in[l-1]} \mathrm{ft}^*_{\lambda_j}(0)\cdot\mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$ is contained in the codimension-$(l-1)$ skeleton of $\mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$ by the induction hypothesis. Equation (3.10) implies that $\mathrm{ft}^*_{\lambda_l}(0)$ is affine linear on each top-dimensional cell of $\prod_{j\in[l-1]} \mathrm{ft}^*_{\lambda_j}(0)\cdot\mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$. Therefore $\prod_{j\in[l]} \mathrm{ft}^*_{\lambda_j}(0)\cdot\mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$ is contained in the codimension-l skeleton of $\mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$.

In the same way (3.10) was used above, we can use it (and (3.4)) to inductively deduce — like in Lemma 3.1.16 — that

$$|\prod_{\tilde{j}\in[l']} \mathrm{ft}^*_{\tilde{\lambda}'_{\tilde{j}}}\left(|\tilde{\lambda}'_{\tilde{j}}|\right)\cdot\prod_{j\in[l]} \mathrm{ft}^*_{\lambda_j}(0)\cdot\mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)| \subset \bigcap_{\tilde{j}\in[l']} \mathrm{ft}^{-1}_{\tilde{\lambda}'_{\tilde{j}}}\left(|\tilde{\lambda}'_{\tilde{j}}|\right)\cap\bigcap_{j\in[l]} \mathrm{ft}^{-1}_{\lambda_j}(0) \tag{3.11}$$

holds, where $|\star|$ denotes the support of $\prod_{j\in[l]} \mathrm{ft}^*_{\lambda_j}(0)\cdot\mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$ with nonzero weights.

Before showing the general statement, we want to see that $\prod_{j\in[l]} \mathrm{ft}^*_{\lambda_j}(0)\cdot\mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$ equals the polyhedral set $\bigcap_{j\in[l]} \mathrm{ft}^{-1}_{\lambda_j}(0)$ with the additional data of weights. For that, induction on l is again utilized. If $l=1$, then $\mathrm{ft}^*_{\lambda_1}(0)$ is of codimension-1 in $\mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$ such that the statement follows from (3.11) since $\mathrm{ft}^*_{\lambda_1}(0)\cdot\mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$ is of pure codimension-1. Let $l>1$. There are two cases. First, the dimensions of $\prod_{j\in[l]} \mathrm{ft}^*_{\lambda_j}(0)\cdot\mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$ and $\bigcap_{j\in[l]} \mathrm{ft}^{-1}_{\lambda_j}(0)$ coincide. Thus induction and (3.11) yield the desired statement repeating the arguments from the initial step $l=1$. In the second case,

$$\dim\left(\bigcap_{j\in[l-1]} \mathrm{ft}^{-1}_{\lambda_j}(0)\right) = \dim\left(\bigcap_{j\in[l]} \mathrm{ft}^{-1}_{\lambda_j}(0)\right) \tag{3.12}$$

holds by induction and (3.11). Hence, by (3.11) and (3.12), $\prod_{j\in[l]} \mathrm{ft}^*_{\lambda_j}(0)\cdot \mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$ is a codimension-1 complex in $\bigcap_{j\in[l]} \mathrm{ft}^{-1}_{\lambda_j}(0)$ with respect to a suitable refinement. Since $\prod_{j\in[l]} \mathrm{ft}^*_{\lambda_j}(0)\cdot \mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$ is contained in the codimension-l skeleton of $\mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$ there is no need to refine $\bigcap_{j\in[l]} \mathrm{ft}^{-1}_{\lambda_j}(0)$ further. Notice that $\mathrm{ft}^*_{\lambda_l}(0)$ is locally linear around the codimension-1 skeleton of $\bigcap_{j\in[l]} \mathrm{ft}^{-1}_{\lambda_j}(0)$. Hence $\prod_{j\in[l]} \mathrm{ft}^*_{\lambda_j}(0)\cdot \mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$ is a zero cycle which contradicts that $\lambda_{[l]}$ are in general position.

To show that the cycle $\prod_{\tilde{j}\in[l']} \mathrm{ft}^*_{\tilde{\lambda}'_{\tilde{j}}}\left(|\tilde{\lambda}'_{\tilde{j}}|\right)\cdot\prod_{j\in[l]} \mathrm{ft}^*_{\lambda_j}(0)\cdot \mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$ equals the polyhedral set $\bigcap_{\tilde{j}\in[l']} \mathrm{ft}^{-1}_{\tilde{\lambda}'_{\tilde{j}}}\left(|\tilde{\lambda}'_{\tilde{j}}|\right)\cap\bigcap_{j\in[l]} \mathrm{ft}^{-1}_{\lambda_j}(0)$ with the additional data of weights, apply Proposition 1.15 of [Rau16] as in the proof of Lemma 3.1.16 to $\prod_{j\in[l]} \mathrm{ft}^*_{\lambda_j}(0)\cdot \mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$ instead of $\mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$. □

Analogously to Corollary 3.1.17, which follows from Lemma 3.1.16, the following corollary is a direct consequence of Lemma 3.2.9.

Corollary 3.2.10. *Let Δ be a degree in $\mathbb{R}^m$. Let $p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \tilde{\lambda}'_{[l']}, \lambda_{[l]}$ be conditions with the usual notation from Definition 3.2.4 that are in general position. Then the cycle $Z_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \tilde{\lambda}'_{[l']}, \lambda_{[l]}\right)$ equals the polyhedral set*

$$\bigcap_{k\in\underline{\kappa}^\alpha\cup\underline{\kappa}^\beta} \partial\,\mathrm{ev}_k^{-1}(L_k)\cap\bigcap_{f\in\underline{\eta}^\alpha\cup\underline{\eta}^\beta} \partial\,\mathrm{ev}_f^{-1}(P_f)\cap\bigcap_{h\in\underline{\kappa}} \mathrm{ev}_h^{-1}(L_h)$$
$$\cap\bigcap_{i\in\underline{n}} \mathrm{ev}_i^{-1}(p_i)\cap\bigcap_{\tilde{j}\in[l']} \mathrm{ft}^{-1}_{\tilde{\lambda}'_{\tilde{j}}}\left(|\tilde{\lambda}'_{\tilde{j}}|\right)\cap\bigcap_{j\in[l]} \mathrm{ft}^{-1}_{\lambda_j}(0)$$

with the additional data of weights such that some of them might be zero.

Given a tropical stable map C that satisfies a tropical cross-ratio condition λ', we can think of this condition as a path of fixed length inside this stable map, see Lemma 3.1.16. Given a degenerated tropical cross-ratio condition λ instead, we it can be thought of as a path of length zero inside a tropical tropical stable map, see Lemma 3.2.9. Hence there is a vertex of valence > 3 in a tropical stable map satisfying a degenerated tropical cross-ratio.

Definition 3.2.11. Let λ be a degenerated tropical cross-ratio. Let C be a tropical stable map. If there is a vertex $v\in C$ such that the image of v under ft_λ is 4-valent, then we say that λ is *satisfied at v*. We define the set λ_v of tropical cross-ratios associated to a vertex v that consists of all given degenerated tropical cross-ratios whose images of v using the forgetful maps associated to the degenerated cross-ratios are 4-valent.

A direct consequence of Lemma 3.2.9 is the following criterion to decide whether a tropical stable map satisfies a degenerated tropical cross-ratio.

Corollary 3.2.12 (Path criterion)**.** *Let C be a rational tropical stable map and let $\lambda=\{\beta_1,\dots,\beta_4\}$ be a degenerated tropical cross-ratio. Then C satisfies λ if and only if there is a vertex v of C that satisfies λ. To check whether λ is satisfied at v the so-called path criterion can be used: A pair (β_i,β_j) with different $i,j\in[4]$ induces a unique path in C. If the paths associated to $(\beta_{i_1},\beta_{i_2})$ and $(\beta_{i_3},\beta_{i_4})$ intersect in exactly one vertex v of C for all pairwise different choices of $i_1,\dots,i_4$ such that $\{i_1,\dots,i_4\}=[4]$, then and only then the tropical cross-ratio λ is satisfied at v.*

Remark 3.2.13. Note that “for all choices” in Corollary 3.2.12 above is equivalent to “for one choice”.

3.2.1 Cross-ratio multiplicities

Compared to non-degenerated tropical cross-ratios, degenerated ones allow a simple combinatorial description of the rational tropical stable maps satisfying them, see Corollary 3.2.12. This advantage of degenerated tropical cross-ratios comes with a trade-off. To determine the multiplicity of a rational tropical stable map C that satisfies given degenerated tropical cross-ratio conditions $\lambda_{[l]}$, a combinatorial factor, the so-called *cross-ratio multiplicity*, needs to be considered. Let $\lambda'_{[l]}$ be non-degenerated tropical cross-ratios that degenerate to $\lambda_{[l]}$. Then the cross-ratio multiplicity reflects how many rational tropical stable maps that satisfy $\lambda'_{[l]}$ degenerate to C when the tropical cross-ratios $\lambda'_{[l]}$ degenerate to $\lambda_{[l]}$. The advantage we take from this situation is that the cross-ratio multiplicities can be described locally on the level of vertices of rational tropical stable maps.

Definition 3.2.14 (Resolutions of vertices)**.** Let v be a vertex of an abstract rational tropical curve and let $\mathrm{val}(v)$ denote the valance of v. Let $\lambda_j \in \lambda_v$ be a degenerated tropical cross-ratio that is satisfied at v as in Definition 3.2.11. Let λ'_j denote a tropical cross-ratio that degenerates to λ_j. We say that v is *resolved according to* λ'_j if the following conditions are satisfied:

(1) The following equality holds:

$$\mathrm{val}(v) = 3 + \#\lambda_v.$$

(2) The vertex v is replaced by two vertices v_1, v_2 that are connected by a new edge such that λ'_j is satisfied. In particular, the length $|e|$ of e equals $|\lambda'_j|$.

(3) The following equality holds for $k = 1, 2$:

$$\mathrm{val}(v_k) = 3 + \#\lambda_{v_k}.$$

(4) The set λ_v decomposes according to the vertices v_1, v_2, i.e.

$$\lambda_v = \{\lambda_j\} \dot\cup \lambda_{v_1} \dot\cup \lambda_{v_2}$$

is a union of pairwise disjoint sets.

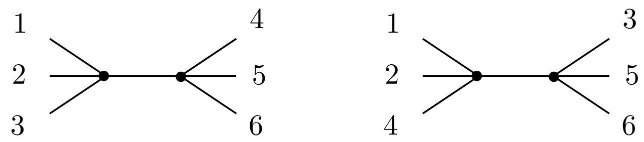
Figure 3.4: Let $\lambda_1 := \{1,2,3,4\}$ and $\lambda_2 := \{1,2,3,5\}$ be two degenerated tropical cross-ratios. On the left there is a 5-valent vertex v with $\lambda_v = \{\lambda_1, \lambda_2\}$. On the right v is resolved according to $\lambda'_1 := (12|34)$. Notice that the resolution is unique in this case.

Example 3.2.15. Resolving a vertex according to a tropical cross-ratio is not unique. It is neither unique in the sense (A) that the edges adjacent to v_1, v_2 are uniquely determined nor in the (weaker) sense (B) that the sets λ_{v_k} are uniquely determined. Let Γ be an abstract rational curve consisting of a single vertex v to which all six ends are adjacent to. Define the following degenerated tropical cross-ratios:

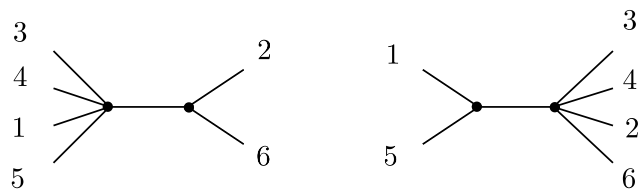
$$\lambda_1 = \{1,2,3,4\},\ \lambda_2 = \{3,4,5,6\},\ \lambda_3 = \{1,2,5,6\}$$

such that $\lambda_v = \{\lambda_1, \lambda_2, \lambda_3\}$.

(A) If v is resolved according to $\lambda_3' = (12|56)$, then there are at least two ways of doing so which are shown in the Figure below.



(B) If another non-degenerated tropical cross-ratio $\lambda_3' = (15|26)$ is chosen, then there are at least two resolutions shown in the Figure below.



Definition 3.2.16 (Cross-ratio multiplicity)**.** Let v be a vertex of an abstract rational tropical curve with

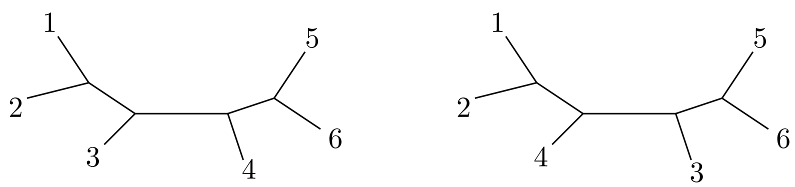
$$\lambda_v = \{\lambda_{j_1}, \dots, \lambda_{j_r}\} \quad \text{and} \quad \mathrm{val}(v) = 3 + r.$$

Let λ'_{j_t} be tropical cross-ratios that degenerate to λ_{j_t} for $t \in [r]$ such that $|\lambda'_{j_1}| > \cdots > |\lambda'_{j_r}|$ is a total order. A *total resolution* of v is a 3-valent labeled rational abstract tropical curve on $r+1$ vertices that arises from v by resolving v according to the following recursion. First, resolve v according to λ'_{j_1}. The two new vertices are denoted by v_1, v_2. Choose v_k with $\lambda_{j_2} \in \lambda_{v_k}$ and resolve it according to λ'_{j_2} (this may not be unique, pick one resolution). Now we have 3 vertices v_1, v_2, v_3 from which we pick the one with $\lambda_{j_3} \in \lambda_{v_{\tilde{k}}}$, resolve it and so on. We define the *cross-ratio multiplicity* $\mathrm{mult}_{\mathrm{cr}}(v)$ of v to be the number of total resolution of v. Notice that in the special case of $\#\lambda_v = 0$ the cross-ratio multiplicity of v equals one. The cross-ratio multiplicity of a vertex v of a rational tropical stable map is defined to be the cross-ratio multiplicity of v in its underlying abstract tropical curve. For an abstract rational tropical curve C (resp. a rational tropical stable map) that satisfies given degenerated cross-ratio conditions, the *cross-ratio multiplicity* $\mathrm{mult}_{\mathrm{cr}}(C)$ of C is defined as

$$\mathrm{mult}_{\mathrm{cr}}(C) := \prod_{v \in C} \mathrm{mult}_{\mathrm{cr}}(v),$$

where the product goes over all vertices of C.

Example 3.2.17. Let v be a 6-valent vertex such that $\lambda_v = \{\lambda_1, \lambda_2, \lambda_3\}$ and the non-degenerated tropical cross-ratios are given by $\lambda_1' := (12|56), \lambda_2' := (34|56), \lambda_3' = (12|34)$. The following two 3-valent trees schematically show all total resolutions of v with respect to $|\lambda_1'| > |\lambda_2'| > |\lambda_3'|$.



Construction 3.2.18. Let $\lambda_{[l]}$ be general positioned degenerated tropical cross-ratios. To each λ_j with $j \in [l]$, we want to associate a non-degenerated tropical cross-ratio λ'_j. For that, recall that λ'_j consists of two information, namely a split of λ'_j into a pair of pairs and a lengths denoted by $|\lambda'_j|$. Let λ'_j (by abuse of notation) be some split of λ_j into a pair of pairs for all $j \in [l]$. If $\mathcal{M}_{0,4}$ is given such that the four appearing labels of ends of rational abstract tropical curves it parametrizes are in λ_j, then let $E_{\lambda'_j}$ denote the end of $\mathcal{M}_{0,4}$ which is associated to the split λ'_j for $j \in [l]$. A length $|\lambda'_j|$ is associated to λ'_j via the following recursive procedure.

Let $i \in [l]$ and assume that the lengths $|\lambda'_j|$ for $j \in [i-1]$ are already defined. Consider the map

$$\varphi_i := \mathrm{ft}_{\lambda_i} : \prod_{j=1}^{i-1} \mathrm{ft}^*_{\lambda'_j}\left(|\lambda'_j|\right) \cdot \prod_{t=i+1}^{l} \mathrm{ft}^*_{\lambda_t}(0) \cdot \mathcal{M}_{0,N}(\mathbb{R}^m, \Delta) \to \mathcal{M}_{0,4}. \tag{3.13}$$

It induces a subdivision S_i on $\mathcal{M}_{0,4}$ on the right-hand side of (3.13). Let $\tau_{\lambda'_i}$ be the top-dimensional cell of $S_i\,|_{E_{\lambda'_i}}$ that is adjacent to the vertex $0 \in \mathcal{M}_{0,4}$ (see Figure 2.3). Choose a point $|\lambda'_i|$ (expressed in the local coordinates of $E_{\lambda'_i}$ as in Definition 2.2.11) in the interior of $\tau_{\lambda'_i}$ such that $|\lambda'_{i-1}| > |\lambda'_i|$.

Remark 3.2.19. It can be assumed that the tropical cross-ratios $\lambda'_{[l]}$ of Construction 3.2.18 are in general position, see Proposition 3.2.7.

Construction 3.2.18 associates tropical cross-ratios $\lambda'_{[l]}$ to degenerated tropical cross-ratios $\lambda_{[l]}$. The lengths of the tropical cross-ratios $\lambda'_{[l]}$ are chosen in such a way that $\lambda'_1, \dots, \lambda'_l$ can be degenerated one by one while tropical stable maps that satisfy them are in a particular nice form. More precisely, if λ'_1 is degenerated to λ_1, then all tropical stable maps that satisfy $\lambda'_{[l]}$ have exactly one edge e that contributes to λ'_1, see Lemma 3.2.9 and Construction 3.2.18. Thus degenerating λ'_1 means to shrink exactly the edge e in each of those tropical stable maps. We then may proceed with $\lambda'_2, \dots, \lambda'_l$ in the same way. Such degeneration arguments are useful to prove the following lemma which describes the weights of the cycle $\mathcal{X}_\Delta\left(\lambda_{[l]}\right)$ of Definition 3.2.4.

Lemma 3.2.20 (Weights of $\mathcal{X}_\Delta\left(\lambda_{[l]}\right)$)**.** *Notation of Definition 2.2.12 is used. Let Δ be a degree. Let $\lambda_{[l]}$ degenerated tropical cross-ratios in general position. Let λ'_j be tropical cross-ratios that degenerate to λ_j for $j \in [l]$ such that $|\lambda'_1| > \cdots > |\lambda'_l|$. Let*

$$\mathcal{X}_\Delta\left(\lambda_{[l]}\right) = \prod_{j \in [l]} \mathrm{ft}^*_{\lambda_j}(0) \cdot \mathcal{M}_{0,N}(\mathbb{R}^m, \Delta)$$

be the intersection product of Definition 3.2.4. Let τ be a top-dimensional cell of $\mathcal{X}_\Delta\left(\lambda_{[l]}\right)$ and let $\mathfrak{c}(\tau)$ denote the combinatorial type of τ. Then τ equals a top-dimensional cell of the codimension-l skeleton of $\mathcal{M}_{0,N}(\mathbb{R}^m, \Delta)$ such that for all vertices v of $\mathfrak{c}(\tau)$

$$\mathrm{val}(v) = 3 + \#\lambda_v$$

holds. Let $\tilde{v} \in \mathfrak{c}(\tau)$ be a vertex such that $\lambda_1 \in \lambda_{\tilde{v}}$. The weight of τ is recursively given by

$$\omega(\tau) = \begin{cases} 1 & \text{, if } l = 1 \\ \sum_\sigma \omega(\sigma) & \text{, otherwise} \end{cases}$$

where the sum goes over all top-dimensional cells of $\prod_{j=2}^{l} \mathrm{ft}^*_{\lambda_j}(0) \cdot \mathcal{M}_{0,N}(\mathbb{R}^m, \Delta)$ such that their combinatorial types $\mathfrak{c}(\sigma)$ are given by resolving the vertex $\tilde{v} \in \mathfrak{c}(\tau)$ according to λ'_1. In particular, all weights of $\mathcal{X}_\Delta\left(\lambda_{[l]}\right)$ are non-negative.

Remark 3.2.21. The intersection product $\mathcal{X}_\Delta\left(\lambda_{[l]}\right)$ in Lemma 3.2.20 does not depend on the non-degenerated tropical cross-ratios $\lambda'_{[l]}$. In particular, the weights of $\mathcal{X}_\Delta\left(\lambda_{[l]}\right)$ are independent of the exact choice of the $\lambda'_{[l]}$ in Lemma 3.2.20.

Proof of Lemma 3.2.20. According to Remark 3.2.21, we may assume that the non-degenerated tropical cross-ratios associated to $\lambda_{[l]}$ are given by Construction 3.2.18.

To prove the lemma, identify

$$\mathcal{M}_{0,N}\left(\mathbb{R}^m,\Delta\right)\cong\mathcal{M}_{0,M}\times\mathbb{R}^m \tag{3.14}$$

as in Theorem 2.2.10 such that it is sufficient to prove the statement for $\mathcal{M}_{0,M}$ because tropical cross-ratio conditions do not interact with the $\mathbb{R}^m$-factor in (3.14). Let $M\in\mathbb{N}_{>3}$. Induction on the number l of degenerated tropical cross-ratios is used.

Let $l=1$ and let $\lambda_1=\{\beta_1,\dots,\beta_4\}$ be the degenerated cross-ratio. A top-dimensional cell τ of $\operatorname{ft}^*_{\lambda_1}(0)\cdot\mathcal{M}_{0,M}$ is a top-dimensional cell of the codimension-1 skeleton of $\mathcal{M}_{0,M}$ such that there is exactly one 4-valent vertex v in the combinatorial type $\mathfrak{c}(\tau)$ of τ such that

$$\operatorname{val}(v)=3+\#\lambda_v$$

holds, see Lemma 3.2.9 and Corollary 3.2.12. The codimension-1 cell τ is adjacent to three top-dimensional cells of $\mathcal{M}_{0,M}$ that arise form resolving v according to $(\beta_1\beta_2|\beta_3\beta_4),(\beta_1\beta_3|\beta_2\beta_4)$ or $(\beta_1\beta_4|\beta_2\beta_3)$. Denote these top-dimensional cells by $\sigma_{(\beta_1\beta_2|\beta_3\beta_4)},\sigma_{(\beta_1\beta_3|\beta_2\beta_4)},\sigma_{(\beta_1\beta_4|\beta_2\beta_3)}$, that is $\sigma_{(\beta_{i_1}\beta_{i_2}|\beta_{i_3}\beta_{i_4})}$ denotes the cell where in $\sigma_{(\beta_{i_1}\beta_{i_2}|\beta_{i_3}\beta_{i_4})}$ a newly inserted edge separates $\{\beta_{i_1}\beta_{i_2}\}$ from $\{\beta_{i_3}\beta_{i_4}\}$. Recall (3.10), that is $\operatorname{ft}^*_{\lambda_1}(0)$ equals $\max\{(\operatorname{ft}_{\lambda_1}(\star))_x,(\operatorname{ft}_{\lambda_1}(\star))_y,0\}$. Observe that on two of the cells $\sigma_{(\beta_1\beta_2|\beta_3\beta_4)},\sigma_{(\beta_1\beta_3|\beta_2\beta_4)},\sigma_{(\beta_1\beta_4|\beta_2\beta_3)}$ the map $\max\{(\operatorname{ft}_{\lambda_1}(\star))_x,(\operatorname{ft}_{\lambda_1}(\star))_y,0\}$ is the zero function and on one of them it maps tropical stable maps to the length of the edge that was obtained from resolving the vertex v. Which of the cells $\sigma_{(\beta_1\beta_2|\beta_3\beta_4)},\sigma_{(\beta_1\beta_3|\beta_2\beta_4)}$ or $\sigma_{(\beta_1\beta_4|\beta_2\beta_3)}$ are mapped to zero depends on the choice of how to embed $\mathcal{M}_{0,4}$ into $\mathbb{R}^2$.

Let $v_{(\beta_1\beta_2|\beta_3\beta_4)}$ denote the direction vector in $\mathcal{M}_{0,M}$ associated to an abstract tropical curve that has only one bounded edge of length one that separates the ends β_1,β_2 from β_3,β_4 (see the following Figure) and define $v_{(\beta_1\beta_3|\beta_2\beta_4)},v_{(\beta_1\beta_4|\beta_2\beta_3)}$ analogously.

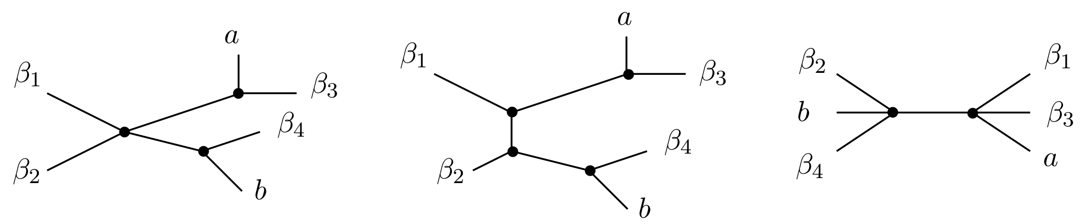


Figure 3.5: From left to right: an arbitrary τ with its $\sigma_{(\beta_1\beta_3|\beta_2\beta_4)}$ and the curve associated to $v_{(\beta_1\beta_3|\beta_2\beta_4)}$.

We assume without loss of generality that $\sigma_{(\beta_1\beta_2|\beta_3\beta_4)}$ is not mapped to zero under the map $\max\{(\operatorname{ft}_{\lambda_1}(\star))_x,(\operatorname{ft}_{\lambda_1}(\star))_y,0\}$. Therefore $v_{(\beta_1\beta_2|\beta_3\beta_4)}$ is mapped to 1 and $v_{(\beta_1\beta_3|\beta_2\beta_4)},v_{(\beta_1\beta_4|\beta_2\beta_3)}$ are mapped to zero. To shorten notation, write $\varphi:=\max\{(\operatorname{ft}_{\lambda_1}(\star))_x,(\operatorname{ft}_{\lambda_1}(\star))_y,0\}$. The weight $\omega_\varphi(\tau)$ is (see Definition 2.1.5)

$$\omega_\varphi(\tau)=\sum_{\substack{\sigma=\sigma_{(\beta_1\beta_2|\beta_3\beta_4)},\\ \sigma_{(\beta_1\beta_3|\beta_2\beta_4)},\sigma_{(\beta_1\beta_4|\beta_2\beta_3)}}}\varphi_\sigma\left(\omega(\sigma)\cdot v_{\sigma/\tau}\right)-\varphi_\tau\left(\sum_{\substack{\sigma=\sigma_{(\beta_1\beta_2|\beta_3\beta_4)},\\ \sigma_{(\beta_1\beta_3|\beta_2\beta_4)},\sigma_{(\beta_1\beta_4|\beta_2\beta_3)}}}\omega(\sigma)\cdot v_{\sigma/\tau}\right), \tag{3.15}$$

where $\varphi_\sigma, \varphi_\tau$ denote the linear parts of φ on σ, τ, and $\omega(\sigma) = 1$ denotes the weight of a top-dimensional cell σ in $\mathcal{M}_{0,M}$. Moreover, $v_{\sigma/\tau}$ denotes an arbitrary representative of the normal vector $u_{\sigma/\tau}$. Therefore set $v_{\sigma_{(\beta_1\beta_2|\beta_3\beta_4)}/\tau} = v_{(\beta_1\beta_2|\beta_3\beta_4)}$ and $v_{\sigma_{(\beta_1\beta_3|\beta_2\beta_4)}/\tau}$, $v_{\sigma_{(\beta_1\beta_4|\beta_2\beta_3)}/\tau}$, respectively. Note that the second sum of the right-hand side of (3.15) is in τ as $\mathcal{M}_{0,M}$ is balanced. Hence it vanishes under φ_τ since $\tau \subset \mathrm{ft}_{\lambda_1}^{-1}(0)$. As discussed above, only one summand of the first sum of (3.15) is nonzero, namely the one with $\sigma = \sigma_{(\beta_1\beta_2|\beta_3\beta_4)}$. Hence $\omega_\varphi(\tau) = 1$.

Next, the induction step from $l-1$ to l is performed. Denote the elements of λ_1 as above, that is $\lambda_1 = \{\beta_1, \dots, \beta_4\}$ with $|\lambda_1| = 0$. We use that

$$\prod_{j=1}^{l} \mathrm{ft}_{\lambda_j}^{*}(0) \cdot \mathcal{M}_{0,M} = \mathrm{ft}_{\lambda_1}^{*}(0) \cdot \left(\prod_{j=2}^{l} \mathrm{ft}_{\lambda_j}^{*}(0) \cdot \mathcal{M}_{0,M} \right)$$

and then apply the induction hypothesis to $\prod_{j=2}^{l} \mathrm{ft}_{\lambda_j}^{*}(0) \cdot \mathcal{M}_{0,M}$. A top-dimensional polyhedron τ of $\mathrm{ft}_{\lambda_1}^{*}(0) \cdot \left(\prod_{j=2}^{l} \mathrm{ft}_{\lambda_j}^{*}(0) \cdot \mathcal{M}_{0,M} \right)$ is a top-dimensional polyhedron of the codimension-l skeleton of $\mathcal{M}_{0,M}$ such that there is a vertex v of $\mathfrak{c}(\tau)$ with $\lambda_1 \in \lambda_v$, see Lemma 3.2.9 and Corollary 3.2.12. Since the interior of τ is in the codimension-1-boundary of $\prod_{j=2}^{l} \mathrm{ft}_{\lambda_j}^{*}(0) \cdot \mathcal{M}_{0,M}$ and the lengths of the tropical cross-ratios are given by Construction 3.2.18, the vertex v is obtained by shrinking an edge connecting two vertices v_1, v_2 in the combinatorial type neighboring top-dimensional cells of $\prod_{j=2}^{l} \mathrm{ft}_{\lambda_j}^{*}(0) \cdot \mathcal{M}_{0,M}$ such that

$$\begin{aligned} \mathrm{val}(v) &= 3 + \#\lambda_{v_1} + 3 + \#\lambda_{v_2} - 2 \\ &= 4 + \#\left(\lambda_{v_1} \cup \lambda_{v_2}\right) \\ &= 3 + \#\left(\lambda_{v_1} \cup \lambda_{v_2} \cup \{\lambda_1\}\right) \\ &= 3 + \#\lambda_v, \end{aligned}$$

where the first equality holds by induction. Again, there are three types of resolutions of v concerning λ_1 and $\mathcal{M}_{0,4} \subset \mathbb{R}^2$ is embedded in such a way that the top-dimensional cells of $\prod_{j=2}^{l} \mathrm{ft}_{\lambda_j}^{*}(0) \cdot \mathcal{M}_{0,M}$ given by resolving the vertex v according to the pairs of unordered numbers of λ_1' are not mapped to zero. Analogously to the start of the induction, the weight $\omega_\varphi(\tau)$ is

$$\omega_\varphi(\tau) = \sum_\sigma \varphi_\sigma\left(\omega(\sigma) \cdot v_{\sigma/\tau}\right) - \varphi_\tau\left(\sum_\sigma \omega(\sigma) \cdot v_{\sigma/\tau}\right), \tag{3.16}$$

where the sums goes over all top-dimensional cells of $\prod_{j=2}^{l} \mathrm{ft}_{\lambda_j}^{*}(0) \cdot \mathcal{M}_{0,M}$ that have τ in their boundaries. Since $\prod_{j=2}^{l} \mathrm{ft}_{\lambda_j}^{*}(0) \cdot \mathcal{M}_{0,M}$ is balanced, the second sum of (3.16) is in τ and vanishes. Moreover, the arguments from the start of the induction yield that $\varphi_\sigma\left(v_{\sigma/\tau}\right)$ is zero if and only if v is not resolved according to λ_1'. By definition $\varphi_\sigma\left(v_{\sigma/\tau}\right) = 1$ otherwise. □

An immediate consequence of Lemma 3.2.20 is the following corollary.

Corollary 3.2.22 (Weighs of $\mathcal{X}_\Delta\left(\lambda_{[l]}\right)$ via resolutions)**.** *Make the same assumption as in Lemma 3.2.20. In particular, let τ be a top-dimensional cell of $\mathcal{X}_\Delta\left(\lambda_{[l]}\right)$. Let C be a rational tropical stable map in τ such that C lies in the interior of τ if τ is not zero-dimensional. Then the weight $\omega(\tau)$ of τ is given by*

$$\omega(\tau) = \prod_{v \in C} \mathrm{mult}_{\mathrm{cr}}(v), \tag{3.17}$$

where the product goes over all vertices of C and $\mathrm{mult}_{\mathrm{cr}}(v)$ *is the cross-ratio multiplicity of v, see Definition 3.2.16.*

Remark 3.2.23. Let $\mathrm{mult}_{\mathrm{cr}}(v)$ be a cross-ratio multiplicity of a vertex v. By Definition 3.2.16, $\mathrm{mult}_{\mathrm{cr}}(v)$ depends on a choice of non-degenerated cross-ratios $\lambda'_{j_1},\dots,\lambda'_{j_r}$ whose degenerations are $\{\lambda_{j_1},\dots,\lambda_{j_r}\}=\lambda_v$. Moreover, it also depends on an order of lengths $|\lambda'_{j_1}|>\dots>|\lambda'_{j_r}|$. The left-hand side of (3.17) is independent of such choices. Hence $\mathrm{mult}_{\mathrm{cr}}(v)$ does in fact not depend on the choice of non-degenerated cross-ratios $\lambda'_{j_1},\dots,\lambda'_{j_r}$, in particular, it does not depend on their order $|\lambda'_{j_1}|>\dots>|\lambda'_{j_r}|$.

Corollary 3.2.22 allows us to deduce the following nice property which is extensively used later.

Corollary 3.2.24. *Let $\mathcal{X}_\Delta\left(\lambda_{[l]}\right)$ be the cycle from Definition 3.2.4. Let τ be a top-dimensional cell of $\mathcal{X}_\Delta\left(\lambda_{[l]}\right)$ such that its weight $\omega(\tau)$ is nonzero. Let C be a rational tropical stable map in τ such that C lies in the interior of τ if τ is not zero-dimensional. Let $v\in C$ be a vertex of C with* $\mathrm{val}(v)>3$. *Then for every edge e adjacent to v in C there is a β_i in some $\lambda_j\in\lambda_v$ such that e is in the shortest path from v to the end labeled with β_i.*

Proof. Assume that there is a vertex v of C with $\mathrm{val}(v)>3$ and that there is an edge e of v such that e does not appear in some shortest path to some β_i in some $\lambda_j\in\lambda_v$. Then a total resolution of v cannot have 3-valent vertices only since each 3-valent vertex that arises from resolving v according to a tropical cross-ratio cannot be adjacent to e. Thus Corollary 3.2.22 implies that the cell τ in which C lives has weight zero which is a contradiction. □

Proposition 3.2.25. *Let Δ be a degree in $\mathbb{R}^m$. Let $p_{\underline{n}},L_{\underline{\kappa}},L_{\underline{\kappa}^\alpha},L_{\underline{\kappa}^\beta},P_{\underline{\eta}^\alpha},P_{\underline{\eta}^\beta},\tilde{\lambda}'_{[l']},\lambda_{[l]}$ be conditions with the usual notation from Definition 3.2.4 that are in general position. Let σ be a top-dimensional cell of the cycle $Z_\Delta\left(p_{\underline{n}},L_{\underline{\kappa}},L_{\underline{\kappa}^\alpha},L_{\underline{\kappa}^\beta},P_{\underline{\eta}^\alpha},P_{\underline{\eta}^\beta},\tilde{\lambda}'_{[l']},\lambda_{[l]}\right)$ which is cut out by the given conditions. Let C be a rational tropical stable map in σ such that C lies in the interior of σ if σ is not zero-dimensional. Associate an* ev-ft-*matrix $M(C)$ to C as in Definition 3.1.11 (i.e. replace $\mathcal{M}_{0,N}(\mathbb{R}^m,\Delta)$ by $\mathcal{X}_\Delta\left(\lambda_{[l]}\right)$). Then the weight $\omega(\sigma)$ of σ equals the multiplicity* $\mathrm{mult}(C)$ *of C which is defined by*

$$\mathrm{mult}(C):=\mathrm{mult}_{\mathrm{ev,ft}}(C)\cdot\mathrm{mult}_{\mathrm{cr}}(C),$$

where $\mathrm{mult}_{\mathrm{cr}}(C)$ *is the product of the cross-ratio multiplicities over all vertices of C (see Definition 3.2.16), and* $\mathrm{mult}_{\mathrm{ev,ft}}(C)$ *equals the absolute value of the index of the* ev-ft-*matrix $M(C)$ associated to C.*

Proof. The proposition follows from Corollary 3.2.22 and Lemma 1.2.9 of [Rau09]. □

Remark 3.2.26. If the cycle $Z_\Delta\left(p_{\underline{n}},L_{\underline{\kappa}},L_{\underline{\kappa}^\alpha},L_{\underline{\kappa}^\beta},P_{\underline{\eta}^\alpha},P_{\underline{\eta}^\beta},\tilde{\lambda}'_{[l']},\lambda_{[l]}\right)$ appearing in Proposition 3.2.25 is zero-dimensional, then $\mathrm{mult}_{\mathrm{ev,ft}}(C)$ equals the absolute value of the determinant of the ev-ft-matrix $M(C)$. In particular, a rational tropical stable map C in the zero-dimensional cycle $Z_\Delta\left(p_{\underline{n}},L_{\underline{\kappa}},L_{\underline{\kappa}^\alpha},L_{\underline{\kappa}^\beta},P_{\underline{\eta}^\alpha},P_{\underline{\eta}^\beta},\tilde{\lambda}'_{[l']},\lambda_{[l]}\right)$ contributes with multiplicity $|\det(M(C))|\cdot\mathrm{mult}_{\mathrm{cr}}(C)$ to $N_\Delta\left(p_{\underline{n}},L_{\underline{\kappa}},L_{\underline{\kappa}^\alpha},L_{\underline{\kappa}^\beta},P_{\underline{\eta}^\alpha},P_{\underline{\eta}^\beta},\tilde{\lambda}'_{[l']},\lambda_{[l]}\right)$.

Future research 3.2.27. The numbers $\mathrm{mult}_{\mathrm{cr}}(v)$, which are the local multiplicities of degenerated tropical cross-ratios, are not well understood. Of course, one can calculate them by considering all trees with an appropriate number of labeled ends and pick the ones that are total resolutions of v with respect to the given tropical cross-ratios. This approach is neither fast nor pleasing. So a question for future research naturally comes up: is there another, more efficient way to calculate the cross-ratio multiplicity $\mathrm{mult}_{\mathrm{cr}}(v)$ of a vertex v satisfying degenerated tropical cross-ratios?

3.2.2 Special case: Multiplicities of rational tropical stable maps to $\mathbb{R}^2$.

Proposition 3.2.25 implies that the multiplicity of a rational tropical stable map C that contributes to the number $N_\Delta\left(p_{\underline{n}}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}\right)$ equals a product $\mathrm{mult}_{\mathrm{ev}}(C)\cdot\mathrm{mult}_{\mathrm{cr}}(C)$ of two multiplicities (see Definition 3.1.11 for the definition of $\mathrm{mult}_{\mathrm{ev}}(C)$). The cross-ratio factor $\mathrm{mult}_{\mathrm{cr}}(C)$ of this product is local, that is, it can be calculated as a product over cross-ratio multiplicities of each vertex of C. The evaluation factor $\mathrm{mult}_{\mathrm{ev}}(C)$ of the product $\mathrm{mult}_{\mathrm{ev}}(C)\cdot\mathrm{mult}_{\mathrm{cr}}(C)$ does not need to be local in the sense that it cannot be calculated via a product of multiplicities on the level of vertices of C.

If C is a rational tropical stable map to $\mathbb{R}^2$ with 3-valent vertices only, then it is well-known [Mik05, GM08] that multiplicities of evaluation maps are local, i.e. they can be calculated on the level of vertices of C. Degenerated tropical cross-ratios lead to vertices of valence > 3. It turns out that the well-known local evaluation-multiplicities can be generalized to this situation. Thus multiplicities of rational tropical stable maps to $\mathbb{R}^2$ that satisfy (degenerated) tropical cross-ratio conditions are completely local.

Definition 3.2.28 (Free and fixed components)**.** For conditions, the usual notation from Definition 3.2.4 is used. Let Δ be a degree in $\mathbb{R}^2$. Let C be a rational tropical stable map (possibly with vertices of higher valence) of degree Δ that is fixed by general positioned conditions $p_{\underline{n}}, \lambda_{[l]}$. Let v be an r-valent vertex of C. Denote adjacent bounded edges of v by $e_{[r]}$. Fix $i \in [r]$, cut the edge e_i and stretch it to infinity. Now there are two rational tropical stable maps, namely one that contains v and one that does not. The rational tropical stable map C_i that does not contain v is called a *component of* v. A component of v is called a *fixed component of* v if it is fixed by the conditions on it. If this component is only a line, then it is considered fixed if there is a vertex on it that is adjacent to a contracted end that satisfies a point condition. Otherwise, a component is called a *free component of* v.

Note that if a vertex v as in Definition 3.2.28 is not adjacent to a contracted end that satisfies a point condition, then it has exactly two fixed components: It is clear that v has at least two fixed components, otherwise its position in $\mathbb{R}^2$ would not be fixed. On the other hand, general positioned conditions $p_{\underline{n}}, \lambda_{[l]}$ do not allow the number of fixed components to be greater than two (move the point conditions slightly, see Remark 3.2.6). Hence the following multiplicities that generalize the well-known local ev-multiplicities for 3-valent vertices are well-defined.

Definition 3.2.29 (Local ev-multiplicities)**.** The usual notation from Definition 3.2.4 is used for given conditions. Let Δ be a degree in $\mathbb{R}^2$. Let C be a rational tropical stable map (possibly with vertices of higher valence) of degree Δ that is fixed by general positioned conditions $p_{\underline{n}}, \lambda_{[l]}$. Let v be a vertex of C. Two cases are distinguished:

(a) If v is not adjacent to a contracted end that satisfies a point condition, then there are exactly two fixed components C_1, C_2 of v associated to different edges e_1, e_2 that are adjacent to v. Let $v(e_i, v)$ denote the direction vectors of e_i at v for $i = 1, 2$, see Definition 2.2.7. Define the *multiplicity of* v by

$$\mathrm{mult}_{\mathrm{ev}}(v) := |\det\left(v(e_1, v), v(e_2, v)\right)|.$$

(b) If v is adjacent to a contracted end that satisfies a point condition, then define the *multiplicity of* v by

$$\mathrm{mult}_{\mathrm{ev}}(v) := 1.$$

Remark 3.2.30. If v of part (a) of Definition 3.2.29 is 3-valent, i.e. there are exactly three edges e_1, e_2, e_3 adjacent to v, then, by the balancing conditions, $|\det(v(e_i, v), v(e_j, v))|$ does not depend on the choice of $i, j \in [3]$ such that $i \neq j$. Thus Definition 3.2.29 generalizes the well-known local ev-multiplicities from [Mik05, GM08].

Remark 3.2.31. Another way to think about the local ev-multiplicity of a higher-valent vertex is to add up edges of free components, to be more precise, consider the following example:

On the left there is a 4-valent vertex whose black edges belong to fixed components and its red edges belong to free components. The multiplicity of this vertex is completely determined by its black edges. If we "add" these red edges (add their direction vectors), we obtain the 3-valent vertex on the right whose multiplicity is again completely determined by its black edges. Notice that the multiplicity of the right vertex equals the well-known local ev-multiplicity.

Lemma 3.2.32. *Let Δ be a degree in $\mathbb{R}^2$. Let $p_{\underline{n}}, \lambda_{[l]}$ be general positioned conditions. Let C be a rational tropical stable map in the zero-dimensional cycle $Z_\Delta\left(p_{\underline{n}}, \lambda_{[l]}\right)$ of Definition 3.2.4. Then*

$$\operatorname{mult}_{\mathrm{ev}}(C) = \prod_{v | v \text{ vertex of } C} \operatorname{mult}_{\mathrm{ev}}(v)$$

hold, where $\operatorname{mult}_{\mathrm{ev}}(C)$ equals the absolute value of the determinant of the ev*-matrix of C as in Remark 3.2.26.*

Proof. The following prove uses ideas from Proposition 3.8 of [GM08]. Let v_1, v_2 be two vertices of C that are connected by an edge e. Cut e into e_1, e_2, stretch the loose edges e_1, e_2 to infinity to obtain two rational tropical stable maps C_1, C_2 with $v_i \in C_i$ such that v_i is adjacent to the end e_i for $i = 1, 2$. Let Δ_i be the degree of C_i. Moreover, splitting C into C_1, C_2 also splits $p_{\underline{n}}$ into $p_{\underline{n_i}}$ and $[l]$ into $\underline{l_i}$ for $i = 1, 2$. Additionally, the total number N of contracted ends of C equals the sum of the numbers N_i of contracted ends of C_i for $i = 1, 2$. Since the conditions $p_{\underline{n}}, \lambda_{[l]}$ are in general position, (3.6) yields

$$\underbrace{(\#\Delta_1 + \#\Delta_2 - 2)}_{=\#\Delta} - 3 + \underbrace{(N_1 + N_2)}_{=N} + 2 = 2 \cdot \underbrace{\left(\#\underline{n_1} + \#\underline{n_2}\right)}_{=\#\underline{n}} + \underbrace{\left(\#\underline{l_1} + \#\underline{l_2}\right)}_{=l}. \tag{3.18}$$

Notice that $p_{\underline{n_i}}, \lambda_{\underline{l_i}}$ are in general position as well. Hence

$$\#\Delta_1 - 3 + N_1 + 2 \geq 2 \cdot \#\underline{n_1} + \#\underline{l_1} \tag{3.19}$$
$$\#\Delta_2 - 3 + N_2 + 2 \geq 2 \cdot \#\underline{n_2} + \#\underline{l_2} \tag{3.20}$$

hold. Adding (3.19) and (3.20) leads to

$$(\#\Delta_1 + \#\Delta_2 - 2) - 3 + (N_1 + N_2) + 2 + 1 \geq 2 \cdot \left(\#\underline{n_1} + \#\underline{n_2}\right) + (\#\underline{l_1} + \#\underline{l_2}). \tag{3.21}$$

Compare (3.21) to (3.18) to deduce that not both of the inequalities (3.19), (3.20) can be strict. Thus assume without loss of generality that (3.19) is an equality. Therefore C_1 is a fixed component of v_2. Hence the ev-matrix $M(C)$ of C is of the following form if v_1 is chosen as base point, see Remark 3.1.13:

$$M(C) = \begin{array}{rc} & \begin{array}{ccc}\text{Base } v_1 & \text{lengths in } C_1 & \text{lengths in } C_2\end{array} \\ \begin{array}{r}\text{conditions in } C_1 \\[2em] \text{conditions in } C_2\end{array} & \left(\begin{array}{c|c} A & 0 \\ \hline * & B \end{array}\right) \end{array}.$$

Since we assumed that (3.19) is an equality

$$\underbrace{\#\Delta_1 - 3 + N_1}_{\#\text{ of lengths in } C_1} + \underbrace{2}_{\substack{\#\text{ of columns}\\ \text{of the base point}}} = \underbrace{2 \cdot \#\underline{n_1} + \#\underline{l_1}}_{\substack{\#\text{ of rows associated}\\ \text{to conditions in } C_1}}$$

holds. Thus the upper left matrix A in $M(C)$ is square. Moreover, A equals $M(C_1)$. Therefore

$$\begin{aligned} \operatorname{mult}_{\mathrm{ev}}(C) &= |\det(M(C))| \\ &= |\det(M(C_1))| \cdot |\det(A)| \\ &= \operatorname{mult}_{\mathrm{ev}}(C_1) \cdot |\det(A)|. \end{aligned}$$

Add a new 3-valent vertex v' to C_2 such that v' is adjacent to the end e_2, the vertex v_2 and a new contracted end that satisfies a new point condition. Denote this new rational tropical stable map by C_2'. The ev-matrix $M(C_2')$ of C_2' with respect to the base point v' fulfills

$$M(C_2') = \left(\begin{array}{cc|c} 1 & 0 & \multirow{2}{*}{0} \\ 0 & 1 & \\ \hline \multicolumn{2}{c|}{*} & A \end{array}\right).$$

Hence

$$\operatorname{mult}_{\mathrm{ev}}(C) = \operatorname{mult}_{\mathrm{ev}}(C_1) \cdot \operatorname{mult}_{\mathrm{ev}}(C_2').$$

Applying this procedure recursively yields two final cases:

(1) A free component gives rise to C' that consists of exactly one vertex v' that is not adjacent to a contracted end which satisfies a point conditions plus n other vertices that are 3-valent and each of them is adjacent to a contracted end that satisfies a point condition. Let N' be the number of contracted ends adjacent to v'. Since C' is fixed by the conditions it satisfies, equation (3.6) yields that

$$(\operatorname{val}(v') - N') - 3 + (n + N') + 2 = 2n + \#\lambda_{v'}.$$

Lemma 3.2.20 implies that $\#\lambda_{v'} = \operatorname{val}(v') - 3$, hence

$$(\operatorname{val}(v') - N') - 3 + (n + N') + 2 = 2n + \operatorname{val}(v') - 3.$$

Thus $n = 2$, i.e. there are two vertices v_1', v_2' that are adjacent to v' via e_1', e_2' and that are adjacent to contracted bounded edges such that each of these contracted bounded edges satisfies a point condition. Therefore

$$M(C') = \left(v(e_1', v_1'), v(e_2', v_2')\right),$$

where $v(e_i, v)$ denotes the direction vector of e_i' at v_i' for $i = 1, 2$, see Definition 2.2.7. Thus

$$\text{mult}_{\text{ev}}(C') = \text{mult}_{\text{ev}}(v').$$

(2) A fixed component C'' consists of a single vertex v'' adjacent to a contracted end that satisfies a point condition. Thus

$$\text{mult}_{\text{ev}}(C'') = 1 = \text{mult}_{\text{ev}}(v'')$$

since $M(C'')$ equals the 2×2 identity matrix.

□

Construction 3.2.33. Let $p_{\underline{n}}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}$ be general positioned conditions such that the cycle $Z_\Delta\left(p_{\underline{n}}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}\right)$ is zero-dimensional. For each rational tropical stable map C in the cycle $Z_\Delta\left(p_{\underline{n}}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}\right)$ let $e_f(C)$ for $f \in \underline{\eta} := \underline{\eta}^\alpha \cup \underline{\eta}^\beta$ denote its end that satisfies P_f. Let $\mathring{e}_f(C)$ denote the interior of $e_f(C)$. Let $p_f \in \mathbb{R}^2$ be a point such that

$$p_f \in \bigcap_C \mathring{e}_f(C)$$

for $f \in \underline{\eta}$, where the intersection goes over all C in $Z_\Delta\left(p_{\underline{n}}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}\right)$. We can assume that $p_{\underline{n}}, p_{\underline{\eta}}, \lambda_{[l]}$ are in general position with respect to Δ, see Remark 3.2.6. To each rational tropical stable map C in $Z_\Delta\left(p_{\underline{n}}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}\right)$, a rational tropical stable map C_p is associated the following way: Given C as above, subdivide its end $e_f \subset \mathbb{R}^2$ by adding a new 3-valent vertex to it that is adjacent to a new contracted end such that this contracted end satisfies p_f. Doing this for all $f \in \underline{\eta}$ gives us C_p, see Figure 3.6. Notice that C_p is in $Z_\Delta\left(p_{\underline{n}}, p_{\underline{\eta}}, \lambda_{[l]}\right)$ by Corollary 3.2.10.

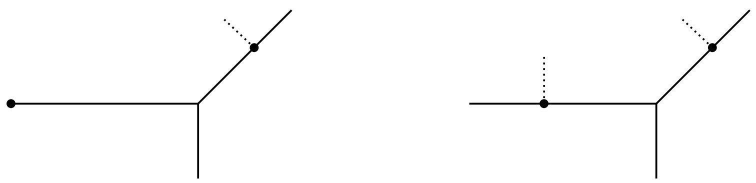


Figure 3.6: Left: A rational tropical stable map C to $\mathbb{R}^2$ of degree Δ_1^2 with one contracted end (dotted) that satisfies a point condition and C satisfies one codimension one tangency condition (left black dot). Right: The rational tropical stable map C_p of Construction 3.2.33 that satisfies two point conditions.

Proposition 3.2.34. *Notation from Definition 3.2.4 is used. Let Δ be a degree in $\mathbb{R}^2$. Let $p_{\underline{n}}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}$ be general positioned conditions. Let C be a rational tropical stable map in the zero-dimensional cycle $Z_\Delta\left(p_{\underline{n}}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}\right)$ of Definition 3.2.4. Let C_p denote the rational tropical stable map that is associated to C by Construction 3.2.33. Then*

$$\text{mult}_{\text{ev}}(C) = \prod_{v|v \text{ vertex of } C_p} \text{mult}_{\text{ev}}(v) \cdot \prod_{f \in \underline{\eta}^\alpha \cup \underline{\eta}^\beta} \frac{1}{\omega(e_f)}$$

holds, where $\text{mult}_{\text{ev}}(C)$ *equals the absolute value of the determinant of the* ev*-matrix of C as in Remark 3.2.26, and where $\omega(e_f)$ denotes the weight of the end labeled by e_f.*

Proof. Let v be a vertex of C. Notice that v is also a vertex of C_p by Construction 3.2.33. Consider the ev-matrix $M(C_p)$ of C_p with respect to the base point v. Let l_f denote the column of $M(C_p)$ that is associated to the length which appeared in C_p due to subdividing the end e_f of C as in construction 3.2.33. Notice that there is exactly one nonzero entry, namely $\pm\omega(e_f)$, in each l_f-column of $M(C_p)$ for $f \in \underline{\eta}^\alpha \cup \underline{\eta}^\beta$. Laplace expanding each l_f-column yields $\operatorname{mult}_{\text{ev}}(C) \cdot \prod_{f \in \underline{\eta}^\alpha \cup \underline{\eta}^\beta} \omega(e_f)$. Thus

$$\operatorname{mult}_{\text{ev}}(C_p) = \operatorname{mult}_{\text{ev}}(C) \cdot \prod_{f \in \underline{\eta}^\alpha \cup \underline{\eta}^\beta} \omega(e_f)$$

holds. Applying Lemma 3.2.32 proves the proposition. □

Corollary 3.2.35. *Let C be a rational tropical stable map as in Proposition 3.2.34. Then the* ev*-multiplicity* $\operatorname{mult}_{\text{ev}}(C)$ *of C can be calculated locally at vertices.*

Proof. Let C_p be the rational tropical stable map associated to C by Construction 3.2.33. Let v_f denote the vertex of C_p that satisfies p_f for $f \in \underline{\eta}^\alpha \cup \underline{\eta}^\beta$ as in Construction 3.2.33. Then, by Definition 3.2.29,

$$\operatorname{mult}_{\text{ev}}(v_f) = 1$$

for each $f \in \underline{\eta}^\alpha \cup \underline{\eta}^\beta$. Therefore

$$\prod_{v \mid v \text{ vertex of } C_p} \operatorname{mult}_{\text{ev}}(v) = \prod_{\substack{v \mid v \text{ vertex of } C_p \\ \text{and vertex of } C}} \operatorname{mult}_{\text{ev}}(v)$$

since the remaining vertices of C_p are precisely the vertices of C by construction. This finishes the proof since the factor $\prod_{f \in \underline{\eta}^\alpha \cup \underline{\eta}^\beta} \frac{1}{\omega(e_f)}$ of Proposition 3.2.34 is already known considering Δ and the labels $\underline{\eta}^\alpha \cup \underline{\eta}^\beta$. □

Chapter 4

Recursive approach: General Kontsevich's formula

By the end of this chapter, a general Kontsevich's formula 4.3.4 is established. It answers the leading question (Q2), which asks whether there is a recursion that calculates rational plane degree d curves that satisfy general positioned point and cross-ratio conditions. Our approach to a general Kontsevich's formula is inspired by the one of Gathmann and Markwig in [GM08]. Let us sum up the (for our purposes) most relevant ideas and techniques used in [GM08]:

1 Splitting rational tropical stable maps
An important observation is that a count of rational tropical stable maps that satisfy a tropical cross-ratio condition λ' is independent of the length λ'. In particular, one can choose a large length for λ'. An even more important observation, which, at the end of the day, gives rise to a recursion is the following: If the length of λ' is large enough, then all rational tropical stable maps that satisfy λ' have a contracted bounded edge. Hence they can be split into two rational tropical stable maps.

2 Splitting multiplicities
Rational tropical stable maps are counted with multiplicities. So splitting them using a large length for a tropical cross-ratio only yields a recursion if their multiplicities split accordingly.

3 Using rational equivalence
A tropical cross-ratio condition appears as a pull-back of a point of $\mathcal{M}_{0,4}$ and pull-backs of different point of $\mathcal{M}_{0,4}$ are rationally equivalent (Remark 2.2.18). Hence the number of rational tropical stable maps that satisfy a tropical cross-ratio $\lambda' = (\beta_1\beta_2|\beta_3\beta_4)$ does not depend on how the labels $\beta_1, \dots, \beta_4$ are grouped together — we could also consider the cross-ratio $\tilde{\lambda}' = (\beta_1\beta_3|\beta_2\beta_4)$ and obtain the same number. This yields an equation.

As a result, we obtain a general tropical Kontsevich's formula (Theorem 4.3.4) that recursively calculates the weighted number of rational tropical stable maps to $\mathbb{R}^2$ of degree Δ_d^2 that satisfy point conditions, curve conditions and tropical cross-ratio conditions which are on contracted ends, see Definition 3.2.3. In order to obtain a classical general Kontsevich's formula (Corollary 4.3.5), Tyomkin's correspondence theorem 2.3.6 is applied (cf. Corollary 3.1.20). Notice that Tyomkin's correspondence theorem only holds for point and tropical cross-ratio conditions, i.e. for $N_\Delta\left(p_{\underline{n}}, \lambda'_{[l]}\right)$. There is no correspondence theorem that relates the tropical numbers $N_\Delta\left(p_{\underline{n}}, L_{\underline{\kappa}}, \lambda'_{[l]}\right)$ that also involve multi-line conditions to their classical counterparts, yet.

The general Kontsevich's formula allows us to recover Kontsevich's formula, see Corollary 4.3.7. The initial values of the general Kontsevich's formula are the numbers provided by the original Kontsevich's formula and the cross-ratio multiplicities, which are purely combinatorial (Definition 3.2.16).

To deduce the general Kontsevich's formula, it is necessary to assume throughout the current chapter that all degenerated and non-degenerated tropical cross-ratios are on contracted ends only, i.e. that no entry of a tropical cross-ratio is a non-contracted end, see Definition 3.2.3.

4.1 Splitting rational tropical stable maps

A first step towards a recursion that calculates the numbers $N_{\Delta_d^2}\left(p_{\underline{n}}, L_{\underline{\kappa}}, \lambda'_{[l]}\right)$ of Definition 3.1.8 (resp. $N_{\Delta_d^2}\left(p_{\underline{n}}, L_{\underline{\kappa}}, \lambda_{[l]}\right)$ of Definition 3.2.5) is to split rational tropical stable maps into "smaller" ones. One way of doing so is to force the existence of a contracted bounded edge which can be cut to obtain two rational tropical stable maps (Subsection 4.1.1). In order to glue two such rational tropical stable maps back together, a description of the movement of the new loose ends is required, see Subsection 4.1.2.

4.1.1 Existence of contracted bounded edges

The aim of this subsection is to prove Propositions 4.1.1, 4.1.25, which are crucial for the recursion we aim for. They guarantee that the tropical stable maps we are dealing with have a contracted bounded edge at which we can split them. Proposition 4.1.1 covers the case when there is at least one point condition. Proposition 4.1.25 covers the case of no point conditions.

Proposition 4.1.1. *Notation from Definition 3.2.4 is used. Let C be a rational tropical stable map that contributes to $N_{\Delta_d^2}\left(p_{\underline{n}}, L_{\underline{\kappa}}, \lambda_{[l-1]}, \lambda'_l\right)$, where λ'_l is a non-degenerated tropical cross-ratio (Notation 3.2.2) and $\lambda_{[l-1]}, \lambda'_l$ are on contracted ends (Definition 3.2.3). If $\#\underline{n} \geq 1$ and $|\lambda'_l|$ is large, then there is exactly one contracted bounded edge in C.*

To keep track of the overall structure of the proof of Proposition 4.1.1, important steps are briefly outlined:

- Definition 4.1.2: Forget λ'_l, to obtain a 1-dimensional cycle $Y_{\Delta_d^2}\left(p_{\underline{n}}, L_{\underline{\kappa}}, \lambda_{[l-1]}\right)$.
- Definition 4.1.2, Remark 4.1.4, Example 4.1.5: Consider the ends of $Y_{\Delta_d^2}\left(p_{\underline{n}}, L_{\underline{\kappa}}, \lambda_{[l-1]}\right)$. They correspond to rational tropical stable maps D that satisfy $p_{\underline{n}}, L_{\underline{\kappa}}, \lambda_{[l-1]}$ such that D admits a movement which gives rise to an unbounded 1-dimensional family of rational tropical stable maps of the same combinatorial type as D. Hence we should study rational tropical stable maps D that have a *movable component* (i.e. a subgraph) B which can be moved unboundedly without changing the combinatorial type of D.
- Definition 4.1.9, Corollary 4.1.22: Show that B contains a single vertex. For this, *chains* of vertices in B are defined and it is shown that no chain has more than one element.
- Proof of Proposition 4.1.1: Conclude that there must be a contracted bounded edge.

Definition 4.1.2 (Movable component)**.** Notation from Definition 3.2.4 is used and it is assumed that all degenerated tropical cross-ratios are on contracted ends only. Let C be a rational tropical stable map that has no contracted bounded edge and that lies in the 1-dimensional cycle

$$Y_{\Delta_d^2}\left(p_{\underline{n}}, L_{\underline{\kappa}}, \lambda_{[l-1]}\right) := \prod_{k\in\underline{\kappa}} \mathrm{ev}_k^*(L_k) \cdot \prod_{i\in\underline{n}} \mathrm{ev}_i^*\left(p_i\right) \cdot \prod_{j\in[l-1]} \mathrm{ft}_{\lambda_j}^*\left(0\right) \cdot \mathcal{M}_{0,N}\left(\mathbb{R}^2, \Delta_d^2\right)$$

such that C gives rise to a 1-dimensional family of rational tropical stable maps by moving some of its vertices. Since the family obtained by moving vertices of C is 1-dimensional, no vertex can be moved freely, i.e. in each possible direction. Hence each vertex of C is either *fixed*, i.e. it can not be moved at all, or *movable* in a direction $b(v)$ given by a vector in $\mathbb{R}^2$ which is called *direction of movement* of v. Directions of movements of vertices are indicated in Figure 4.1 of Example 4.1.5. Since each movable vertex v cannot move freely, its movement is restricted by a condition imposed to it via an edge adjacent to v. More precisely, v either needs to be adjacent to a fixed vertex or to a contracted end which satisfies a multi-line condition. The connected component of C which consists of all movable vertices of C (and edges connecting movable vertices) is called the *movable component* $B(C)$ of C. Notice that there is exactly one movable component since C gives rise to a 1-dimensional family only. A connected component of $C\backslash B(C)$ is called *fixed component*. We say that a movable component allows an *unbounded movement*, if the movement of the movable component gives rise to a family of rational tropical stable maps of the same combinatorial type as C that is unbounded.

Remark 4.1.3. Showing that the movable component $B(C)$ contains a single vertex is non-trivial. However, the difficulties arise primarily due to the degenerated tropical cross-ratios. If there are no degenerated tropical cross-ratios, then every vertex in a rational tropical stable map in question is 3-valent and the $B(C)$ boils down to a *string* as introduced in [GM08], which can be thought of as a single chain.

Remark 4.1.4. Consider a 1-dimensional family of rational tropical stable maps of the same combinatorial type that is unbounded, and consider a movable component $B(C)$ within a rational tropical stable map C of this family that allows an unbounded movement. Notice that the direction of movement $b(v)$ of a vertex v in $B(C)$ might change as moving the component $B(C)$ generates the family. Since v is either adjacent to a fixed vertex or adjacent to an end satisfying a multi-line condition, $b(v)$ can only change, when v is adjacent to an end that satisfies a multi-line condition L. Thus $b(v)$ can only change if v passes over the vertex of L, see Example 4.1.5. Hence the direction of movement of a vertex in the movable component cannot change if we already moved the movable component enough. In the following, we focus on movable components that allow an unbounded movement and that already have been moved sufficiently such that it can assumed that the direction of movement of each vertex therein does not change anymore when moving. In particular, we may assume that the direction of movement of a vertex satisfying a multi-line condition is parallel to $(-1,0),(0,-1)$ or $(1,1)$.

Example 4.1.5. Figure 4.1 provides an example of a rational tropical stable maps C to $\mathbb{R}^2$ whose contracted ends labeled with $1,2,4,5$ satisfy point conditions and the contracted end labeled with 3 satisfies a multi-line condition (the dashed line). The vertex v adjacent to the end labeled with 3 is in the movable component of C and the direction of movement $b(v)$ (indicated by an arrow) of v might changes as v is moved. The movement shown in Figure 4.1 is bounded.

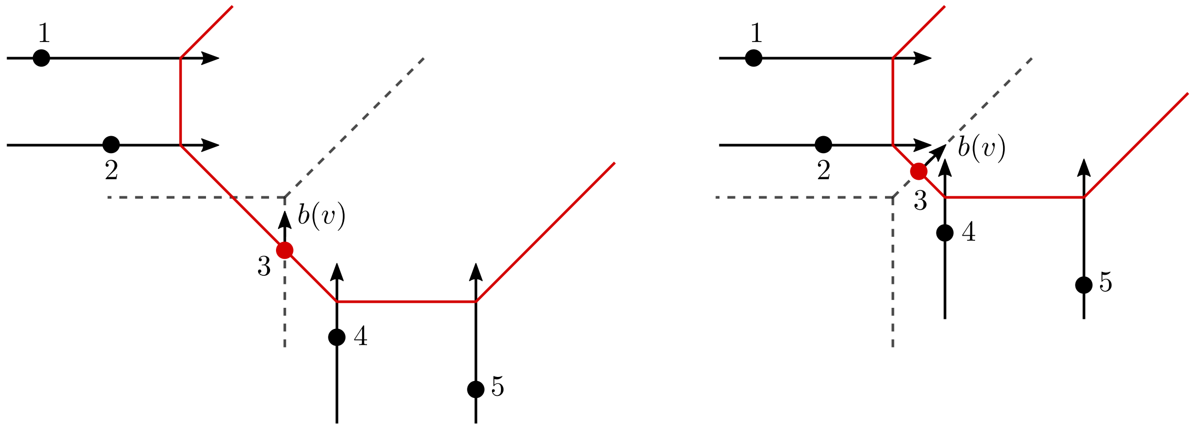


Figure 4.1: A rational tropical stable maps satisfying point conditions and one multi-line condition. The movable component is drawn in red. The arrows indicate the directions of movement. The movement shown is bounded.

Classification 4.1.6 (Types of movable vertices)**.** Let C be a rational tropical stable map as in Definition 4.1.2. If there is a vertex v in the movable component of C that is adjacent to a fixed component and all of its adjacent edges and ends which are non-contracted are parallel, then the movable component of C has exactly one vertex, namely v. Otherwise C would not give rise to a 1-dimensional family only.

Hence the following classification is complete if we assume that the movable component of C has more than 1 vertex (if it has exactly 1 vertex, then we can directly jump to the proof of Proposition 4.1.1): We distinguish 4 types of vertices in the movable component.

- **Type (I)** vertices are adjacent to a fixed component and not all adjacent edges and non-contracted ends are parallel.
- **Type (II)** vertices are not 3-valent and adjacent to a contracted end which satisfies a multi-line condition.
- **Type (IIIa)** vertices are 3-valent, adjacent to two bounded edges and adjacent to a contracted end which satisfies a multi-line condition
- **Type (IIIb)** vertices are 3-valent, adjacent to one bounded edge, a contracted end which satisfies a multi-line condition and an end in standard direction.

Throughout this section the assumption that the movable component of C has more than 1 vertex is used whenever we refer to this classification of vertices.

Construction 4.1.7. In the following we often forget the vertices of type (IIIa) and type (IIIb) in C by gluing the non-contracted edges adjacent to a vertex of type (IIIa) (resp. type (IIIb)) together and obtain a rational tropical stable map denoted by $\tilde{C}$. *This notation of $\tilde{C}$ is fixed throughout this section.*

If $\tilde{C}$ allows no 1-dimensional movement, then the only vertices in the movable component $B(C)$ of C are of type (IIIa) or (IIIb). Hence there is no type (I) vertex in $B(C)$. Thus C has no fixed component. In particular, $p_{\underline{n}} = \emptyset$, but this case is treated separately in Lemma 4.1.23, Lemma 4.1.24 and Proposition 4.1.25. Therefore it can be assumed that $\tilde{C}$ allows an unbounded 1-dimensional movement.

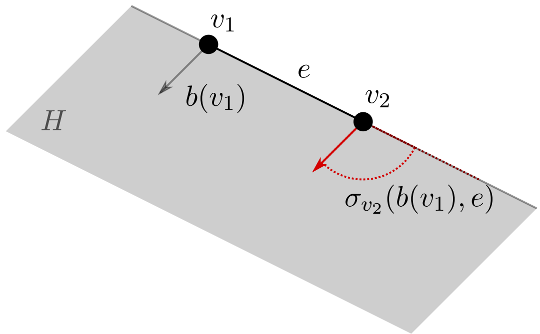


Figure 4.2: The cone $\sigma_{v_2}(b(v_1), e)$ in which the direction of movement of v_2 lies. The slope of the edge connecting v_1, v_2 is fixed during the movement. Hence the translation $b(v_2) + v_2$ of the direction of movement $b(v_2)$ of v_2 is contained in the open half-plane H whose boundary is $\langle e\rangle + v_2$ and whose interior contains $b(v_1) + v_1$.

Lemma 4.1.8 (Angle Lemma)**.** *Let $\tilde{C}$ be a rational tropical stable map to $\mathbb{R}^2$ as in Construction 4.1.7 that allows an unbounded 1-dimensional movement. Let v_1, v_2 be adjacent vertices in the movable component $B(\tilde{C})$. Let $b(v_1) \neq 0$ be the direction of movement of v_1 and let $v(e, v_1) \neq b(v_1)$ be the direction vector at v_1 of the edge e that connects v_1 and v_2. Then the direction of movement $b(v_2)$ of v_2 lies in the half-open cone*

$$\sigma_{v_2}(b(v_1), e) := \{x \in \mathbb{R}^2 \mid x = v_2 + \lambda_1 v(e, v_1) + \lambda_2 b(v_1),\ \lambda_1 \in \mathbb{R}_{\geq 0},\ \lambda_2 \in \mathbb{R}_{>0}\}$$

centered at v_2 that is spanned by $b(v_1)$ and $v(e, v_1)$, where half-open means that the boundary of $\sigma_{v_2}(b(v_1), e)$ that is generated by $b(v_1)$ is part of the cone and the boundary that is generated by $v(e, v_1)$ is not part of the cone, while v_2 itself is also not part of the cone.

Proof. This is true since the length of the edge e' that connects v_1 and v_2 cannot shrink when moving v_1 and v_2, otherwise the movement would be bounded. Therefore the (affine) lines $\langle b(v_1)\rangle + v_1$ and $\langle b(v_2)\rangle + v_2$ must either be parallel or their point of intersection does not lie in H. □

Definition 4.1.9 (Partial order)**.** Notation from Construction 4.1.7 is used. Let $\tilde{C}$ be a rational tropical stable map to $\mathbb{R}^2$ that allows an unbounded 1-dimensional movement and let H be an open half-plane. If we translate H to a vertex $v \in \tilde{C}$, i.e. v is contained in the boundary of H, then the translated half-plane is denoted by H_v. Let M be the set of all vertices of the movable component of $\tilde{C}$, i.e. M consists of all type (I) and type (II) vertices of the movable component of C. The half-plane H induces a partial order $\Omega(H)$ on M as follows: For $v_1, v_2 \in M$ define

$$v_1 \geq v_2 :\Longleftrightarrow \begin{cases} v_1 = v_2, \text{ or} \\ v_2 \text{ is adjacent to } v_1 \text{ and } v_2 \in H_{v_1}. \end{cases}$$

Here, we only use open half-planes H such that $b(v_1) + v_1 \in H_{v_1}$. Therefore if $v_1 \geq \dots \geq v_r$ is a maximal chain and $b(v_i)$ is the direction of movement of v_i for $i \in [r]$, then $b_i + v_i \in H_{v_i}$ for $i \in [r]$ by inductively applying Lemma 4.1.8.

Notation 4.1.10. Given a chain $v_1 \geq \dots \geq v_r$ in the movable component of $\tilde{C}$, the direction of movement of v_i is denoted by $b(v_i)$ for $i \in [r]$ throughout this section. If such a chain is maximal, then an edge connecting v_i and v_{i+1} is usually denoted by e_i for $i \in [r-1]$. by abuse of notation, we often write e_i instead of the direction vector $v(e_i, v_i)$ at v_i from Definition 2.2.7.

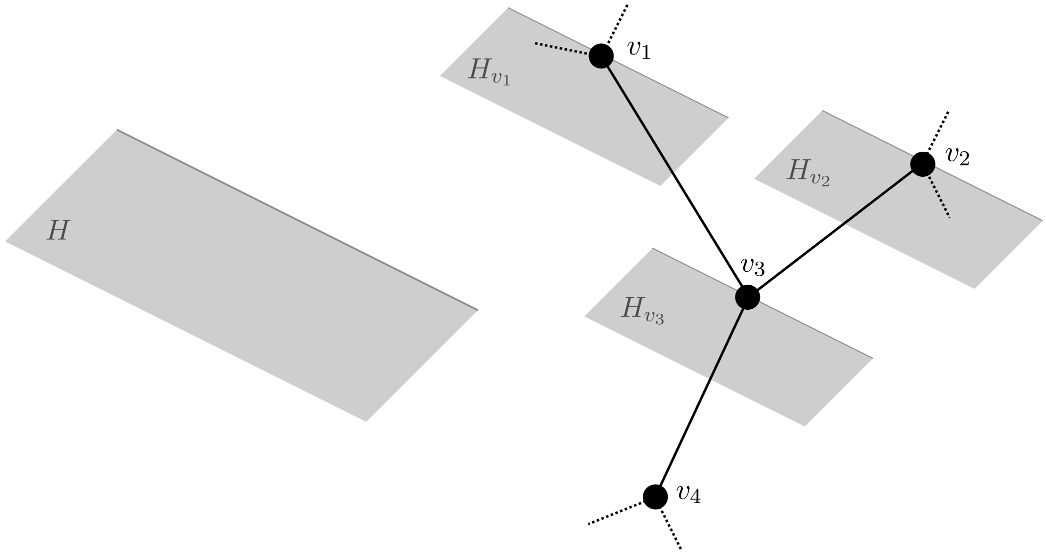


Figure 4.3: This is an example of the partial order $\Omega(H)$ for $H \subset \mathbb{R}^2$ which is an open half-plane as shown on the left (the boundary of the half-plane is darkened). On the right there is a sketch of a tropical curve in $\mathbb{R}^2$ such that $v_1 \geq v_3 \geq v_4$ and $v_2 \geq v_3 \geq v_4$ with respect to the order $\Omega(H)$.

Lemma 4.1.11 (Maximal chains)**.** *Notation 4.1.10 is used. Let $\tilde{C}$ be a rational tropical stable map to $\mathbb{R}^2$ as in Construction 4.1.7 that allows an unbounded 1-dimensional movement. Let $v_1 \geq \cdots \geq v_r$ be a maximal chain with $r > 1$ and $b(v_1) + v_1 \in H_{v_1}$ in $\tilde{C}$ with respect to $\Omega(H)$ as in Definition 4.1.9. Then there is no vertex $v_{r+1} \in \tilde{C}$ adjacent to v_r such that $v_{r+1} \in H_{v_r}$.*

Proof. Notation 4.1.10 is used. By definition, $v_n, b(v_{r-1}) + v_{n-1} \in H_{v_{r-1}}$ and there is an edge e_{r-1} connecting v_{r-1} to v_r. If $\langle b(v_{r-1})\rangle = \langle e_{r-1}\rangle$, then $b(v_{r-1})$ and $b(v_r)$ are parallel. Thus we have a 2-dimensional movement which yields a contradiction since we just allow a 1-dimensional movement. In total, the requirements of Lemma 4.1.8 are fulfilled such that $b(v_r) + v_r \in H_{v_r}$ follows. Since there is an edge e_r that connects v_r to v_{r+1} and $v_{r+1} \in H_{v_r}$, Definition 4.1.9 yields $v_r \geq v_{r+1}$ with respect to $\Omega(H)$. This contradicts our maximality assumption. $\square$

Definition 4.1.12 (Special half-planes)**.** Let $e \in \mathbb{R}^2$ be a vector of standard direction, see Notation 2.2.4. An open half-plane is called *special half-plane* if the affine subspace $\langle e\rangle + v \subset \mathbb{R}^2$ for some $v \in \mathbb{R}^2$ that is generated by e is the boundary of H. There are six special half-planes up to translation, see Figure 4.4.

Figure 4.4: All six special half-planes up to translation. The boundary of each is darkened.

Definition 4.1.13. An open half-plane H is called 1-ray (resp. 2-ray) half-plane if it contains exactly one (resp. two) rays of standard direction. Notice that special half-planes are 1-ray half-planes.

Lemma 4.1.14. *Let $\tilde{C}$ be a rational tropical stable map to $\mathbb{R}^2$ as in Construction 4.1.7 that allows an unbounded 1-dimensional movement. Let v_1 be a vertex of the movable component of $\tilde{C}$. Let H be a 1-ray half-plane that contains a ray of standard direction D. If $v_1 \geq \dots \geq v_r$ is a maximal chain starting at v_1 with respect to $\Omega(H)$ such that $r > 1$ and $b(v_1) + v_1 \in H_{v_1}$, then there is an end e of $\tilde{C}$ adjacent to v_r which is parallel to D.*

Proof. Notation 4.1.10 is used. Notice that $v_{r-1} \geq v_r$. Hence $e_{r-1} + v_r \notin \overline{H}_{v_r}$, where $\overline{H}_{v_r}$ denotes the closure of H_{v_r}. Thus by balancing, there is an edge $e \in \tilde{C}$ adjacent to v_r such that $e \in H_{v_r}$. If e connects v_r to a fixed component, then $b(v_r) + v_r \notin \overline{H}_{v_r}$ because the movement of v_r should be unbounded, i.e. $b(v_r)$ moves v_r away from that fixed component while $\langle e \rangle + v_r = \langle b(v_r) \rangle + v_r$, which contradicts that $b(v_r) + v_r \in H_{v_r}$ by Lemma 4.1.8. Hence e is an end of $\tilde{C}$ by Lemma 4.1.11. Since H_{v_r} is a 1-ray half-plane containing exactly 1 ray of standard direction D, the direction of e is D. □

Lemma 4.1.15 (About maximal chains, weak version)**.** *Let $\tilde{C}$ be a rational tropical stable map to $\mathbb{R}^2$ as in Construction 4.1.7 that allows an unbounded 1-dimensional movement. Let v_1 be a vertex of the movable component of $\tilde{C}$. If there is a 1-ray half-plane H and $v_1 \geq \dots \geq v_r$ is a maximal chain starting at v_1 with respect to $\Omega(H)$ such that $r > 1$ and $b(v_1) + v_1 \in H_{v_1}$, then v_r is a 3-valent type (I) vertex.*

Proof. Notation 4.1.10 is used. By Lemma 4.1.14, there is an end e of $\tilde{C}$ adjacent to v_r. Moreover, since H is a 1-ray half-plane containing exactly 1 ray of standard direction D, the direction of e is D. Assume that the valence of v_r is greater than 3, i.e. there is a degenerated tropical cross-ratio in λ_{v_r}. Since we assumed in Classification 4.1.6 that all degenerated tropical cross-ratios are on contracted ends only (see Definition 3.2.3), Corollary 3.2.24 can be applied. Therefore there is a vertex $v \in C$ connected to v_r via e such that v is of type (IIIa) or type (IIIb) such that v satisfies a multi-line condition. Since the movement of v is unbounded, its direction of movement, denoted by $b(v)$, is parallel to e (cf. Remark 4.1.4). Therefore the movable component of C allows a 2-dimensional movement, which is a contradiction.

In total, v_r can only be a 3-valent type (I) vertex since the other cases were ruled out. □

Corollary 4.1.16. *If we make the same assumptions as in Lemma 4.1.15 and additionally require that H is a special half-plane (see Definition 4.1.12), then there exists no chain $v_1 \geq \dots \geq v_r$ with respect to $\Omega(H)$ such that $r > 1$ and $b(v_1) + v_1 \in H_{v_1}$.*

Proof. Notation 4.1.10 is used. It is sufficient to show the statement for maximal chains $v_1 \geq \dots \geq v_r$ starting at v_1. So we assume that our chain is maximal. The vertex v_r is 3-valent of type (I) by Lemma 4.1.15. Let D denote the ray of standard direction that is contained in H. By Lemma 4.1.14, there is an end e adjacent to v_r of standard direction D. Denote the edge that connects v_r to a fixed component by f, and because $\langle f \rangle + v_r = \langle b(v_r) \rangle + v_r$, we know that $f + v_r \notin \overline{H}_{v_r}$. Since all non-contracted ends determined by Δ_d^2 are of weight 1, the end e is also of weight 1. Using balancing and the definition of special half-planes, we conclude that the edge e_{r-1} that connects v_{r-1} to v_r lies in the boundary of H_{v_r}, which contradicts $v_{r-1} \geq v_r$. □

Observation 4.1.17. Let v_1, v_2 be two vertices of the movable component of $\tilde{C}$. Let e be an edge that connects v_1 and v_2 and let $b(v_1)$ be the direction of movement of v_1. Corollary 4.1.16 shows that there cannot be an open half-plane H such that $b(v_1) + v_1, e + v_1 \in H_{v_1}$, and such that H_{v_1} is a special half-plane. Note that $\langle b(v_1) \rangle \neq \langle e \rangle$, otherwise our movable component would move in a 2-dimensional way. Therefore, for each pair of directions of $b(v_1)$ and e, there are open half-planes that contain $b(v_1)$ and e. But each of these open half-planes is not a special half-plane. This observation gives rise to the following classification.

Classification 4.1.18 (Dependence of $b(v_1)$ and e)**.** Let $\tilde{C}$ be as in Construction 4.1.7. In particular, we assume that $\tilde{C}$ has more than one vertex. Use the notation of Observation 4.1.17, i.e. let $v_1 \in \tilde{C}$ be a vertex with direction of movement $b(v_1)$. If $b(v_1)+v_1$ is in one of the dashed red cones of Figure 4.5, then $e+v_1$ has to lie in the opposite cone. Otherwise, there would be a special half-plane H such that $b(v_1)+v_1, e+v_1 \in H_{v_1}$, which contradicts Observation 4.1.17. The 3 cases depicted in Figure 4.5 are distinguished: If $b(v_1)+v_1$ and $e+v_1$ lie in the red cones depicted on the left, then v_1 is said to be of type F_1. The other two cases can be seen in Figure 4.5.

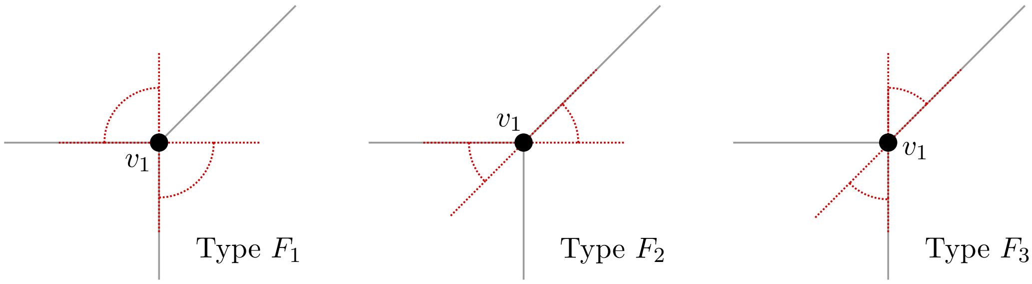


Figure 4.5: A vertex v_1 with its cones in which $b(v_1)+v_1$ and $e+v_1$ can lie. From left to right: A vertex v_1 of type F_1, F_2 and F_3.

The other way round, given a vertex $v_1 \in \tilde{C}$ and its type F_i, the positions of $b(v_1)+v_1$ and $e+v_1$ can be estimated. See Figure 4.5 for the following: If v_1 is of type F_i, then $b(v_1)+v_1$ and $e+v_1$ need to lie in the red cones depicted in Figure 4.5 in such a way that $b(v_1)+v_1$ and $e+v_1$ lie in opposite cones.

Remark 4.1.19. If there is a maximal chain $v_1 \geq \cdots \geq v_r$ in $\tilde{C}$ with respect to $\Omega(H)$ such that $b(v_1)+v_1 \in H_{v_1}$ and v_1 is of type F_i, then v_j is also of type F_i for $j = 2, \ldots, r$.

Proof. Notation 4.1.10 is used. By induction, is is sufficient to show the statement for $v_1 \geq v_2$. Let e_1 be the edge adjacent to v_1, v_2. Let F_i be the type of v_1 such that $\sigma_{e_1}+v_1$ and $\sigma_{b(v_1)}+v_1$ are its two opposing cones, where $e_1+v_1 \in \sigma_{e_1}+v_1$ and $b(v_1)+v_1 \in \sigma_{b(v_1)}+v_1$. Hence $-e_1+v_2 \in \sigma_{b(v_1)}+v_2$. By Observation 4.1.17, we obtain $b(v_2)+v_2 \in \sigma_{e_1}+v_2$. □

Lemma 4.1.20. *Notation 4.1.10 is used. Let $\tilde{C}$ be a rational tropical stable map to $\mathbb{R}^2$ as in Construction 4.1.7 that allows an unbounded 1-dimensional movement. Let v_1 be a vertex of the movable component of $\tilde{C}$. Let H be an open half-plane. Let $v_1 \geq \cdots \geq v_r$ be a maximal chain with respect to $\Omega(H)$ such that $r > 1$ and $b(v_1)+v_1 \in H_{v_1}$. If b_r is of non-standard direction, then v_r is adjacent to two ends of $\tilde{C}$ of different standard directions. If b_r is of standard direction, then v_r is adjacent to one end of $\tilde{C}$ of standard direction parallel to b_r.*

Proof. Assume that v_r is of type F_i for an $i \in [3]$ and that b_r is of non-standard direction. Thus, by Classification 4.1.18, $b(v_r)+v_r$ lies in the interior of one of the dashed red cones of Figure 4.5 and all bounded edges adjacent to v_r lie in the opposite cone. Therefore, by the balancing condition, v_r needs to be adjacent to at least two ends of different standard directions.

Next, assume that b_r is of standard direction. Hence $b(v_r)+v_r$ appears in the boundary of two of the red cones σ_1, σ_2 of Classification 4.1.18. Therefore all edges which are no ends adjacent to $v_r \in \tilde{C}$ are in the union $\sigma'_1 \cup \sigma'_2$ of the opposing cones σ'_j of σ_j for $j = 1, 2$. Therefore balancing guarantees that there is an end adjacent to $v_r \in \tilde{C}$ which is parallel to $b(v_r)$. □

The following lemma generalizes Lemma 4.1.15 from 1-ray half-planes to arbitrary half-planes.

Lemma 4.1.21 (About maximal chains, strong version)**.** *Let $\tilde{C}$ be a rational tropical stable map to $\mathbb{R}^2$ as in Construction 4.1.7 that allows an unbounded 1-dimensional movement. Let v_1 be a vertex of the movable component of $\tilde{C}$. If there is an open half-plane H such that $v_1 \geq \cdots \geq v_r$ is a maximal chain starting at v_1 with respect to $\Omega(H)$ such that $r > 1$ and $b(v_1) + v_1 \in H_{v_1}$, then v_r is a 3-valent type (I) vertex.*

Proof. We use Notation 4.1.10, assume that $\operatorname{val}(v_r) > 3$, that v_r is of type F_i for an $i \in [3]$ and that b_r is of non-standard direction. By Lemma 4.1.20, v_r needs to be adjacent to at least two ends E_1, E_2 of different standard directions. By Corollary 3.2.24, a vertex of type (IIIb) can be reached via each of the edges E_1, E_2 in C. The direction of movement of such a type (IIIb) vertex cannot be parallel to the end of standard direction it is connected to, otherwise we would have a 2-dimensional movement. Recall that type (IIIb) vertices can only move in standard direction since their contracted ends satisfy multi-line conditions. See Figure 4.6 for the following: If $i = 1$, i.e. v_r is of type F_1, we consider the cone in which $b(v_r) + v_r$ lies and go through all different directions of movements of the type (IIIb) vertices. In each case a contradiction to the unbounded movement occurs.

We still get a contradiction if $b(v_r) + v_r$ would lie in the other red cone of Figure 4.6. More generally, the same arguments and conclusion of the case $i = 1$ are true for $i = 2, 3$ and lead to contradictions as well.

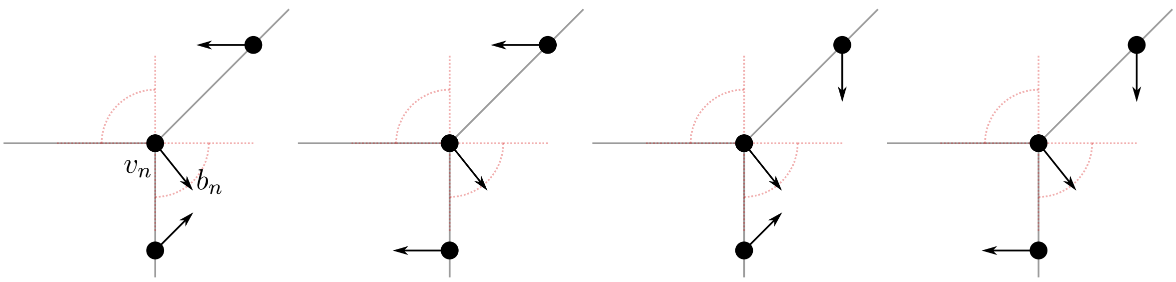


Figure 4.6: A vertex v_r of type F_1 connected to two type (IIIb) vertices which move along the directions of the arrows.

Next, we assume that $b(v_r)$ is of standard direction. By Lemma 4.1.20, there is an end E_1 adjacent to $v_r \in \tilde{C}$ which is parallel to $b(v_r)$. Since we assumed that $\operatorname{val}(v_r) > 3$, there must, again, be a type (IIIb) vertex adjacent to v_r via E_1. Notice that this vertex can only move unboundedly in the direction of $b(v_r)$, which is a contradiction because our movement is only 1-dimensional.

In total, v_r can only be a type (I) vertex that is 3-valent. □

Corollary 4.1.22. *Let $v_1, b(v_1)$ and H be as in Lemma 4.1.21. Then there is no chain $v_1 \geq \cdots \geq v_r$ with $r > 1$ and $b(v_1) + v_1 \in H_{v_1}$ in the movable component of $\tilde{C}$.*

Proof. We use Notation 4.1.10 and assume that there is a maximal chain $v_1 \geq \cdots \geq v_r$ starting at v_1. Hence v_r must be a 3-valent type (I) vertex by Lemma 4.1.21. By Lemma 4.1.20, there is an end E of $\tilde{C}$ adjacent to v_r. Moreover, denote the direction vector at v_r of the edge that connects v_r to a fixed component by f. Therefore the direction of movement of v_r, denoted by $b(v_r)$, is given by $-f$ since v_r moves unboundedly, i.e. it moves away from the fixed component it is adjacent to. All cases of Classification 4.1.18 are distinguished for v_r: Let the type of the vertex v_r be F_i for an $i \in [3]$ (see Figure 4.5). Since $b(v_r) = -f$, the edges e_{r-1} and f adjacent to v_r lie in the same cone. Then there exists no end E such that v_r is balanced (for each possible end E we find a half-plane P such that $E + v_r, f + v_r, -e_{r-1} + v_r \in P_{v_r}$) which is a contradiction. □

The following proof builds on ideas of Proposition 5.1 in [GM08].

Proof of Proposition 4.1.1. Consider the 1-dimensional cycle

$$Y_{\Delta_d^2}\left(p_{\underline{n}}, L_{\underline{\kappa}}, \lambda_{[l-1]}\right) = \prod_{k\in\underline{\kappa}} \mathrm{ev}_k^*(L_k) \cdot \prod_{i\in\underline{n}} \mathrm{ev}_i^*\left(p_i\right) \cdot \prod_{j\in[l-1]} \mathrm{ft}_{\lambda_j}^*\left(0\right) \cdot \mathcal{M}_{0,N}\left(\mathbb{R}^2, \Delta_d^2\right)$$

from Definition 4.1.2. We need to show that

$$\{\mathrm{ft}_{\lambda_l'}(C) \mid C \in Y_{\Delta_d^2}\left(p_{\underline{n}}, L_{\underline{\kappa}}, \lambda_{[l-1]}\right) \text{ has no contracted bounded edge}\}$$

is bounded in $\mathcal{M}_{0,4}$. If it is unbounded, then there is a rational tropical stable map C in the cycle $Y_{\Delta_d^2}\left(p_{\underline{n}}, L_{\underline{\kappa}}, \lambda_{[l-1]}\right)$ without a contracted bounded edge which allows an unbounded movement. Hence the movable component of C has exactly one vertex v by Corollary 4.1.22 which is not of type (IIIa) or (IIIb) as in Classification 4.1.6. Notice that C has at least one fixed component as well since we assume that there is at least one point condition that C satisfies.

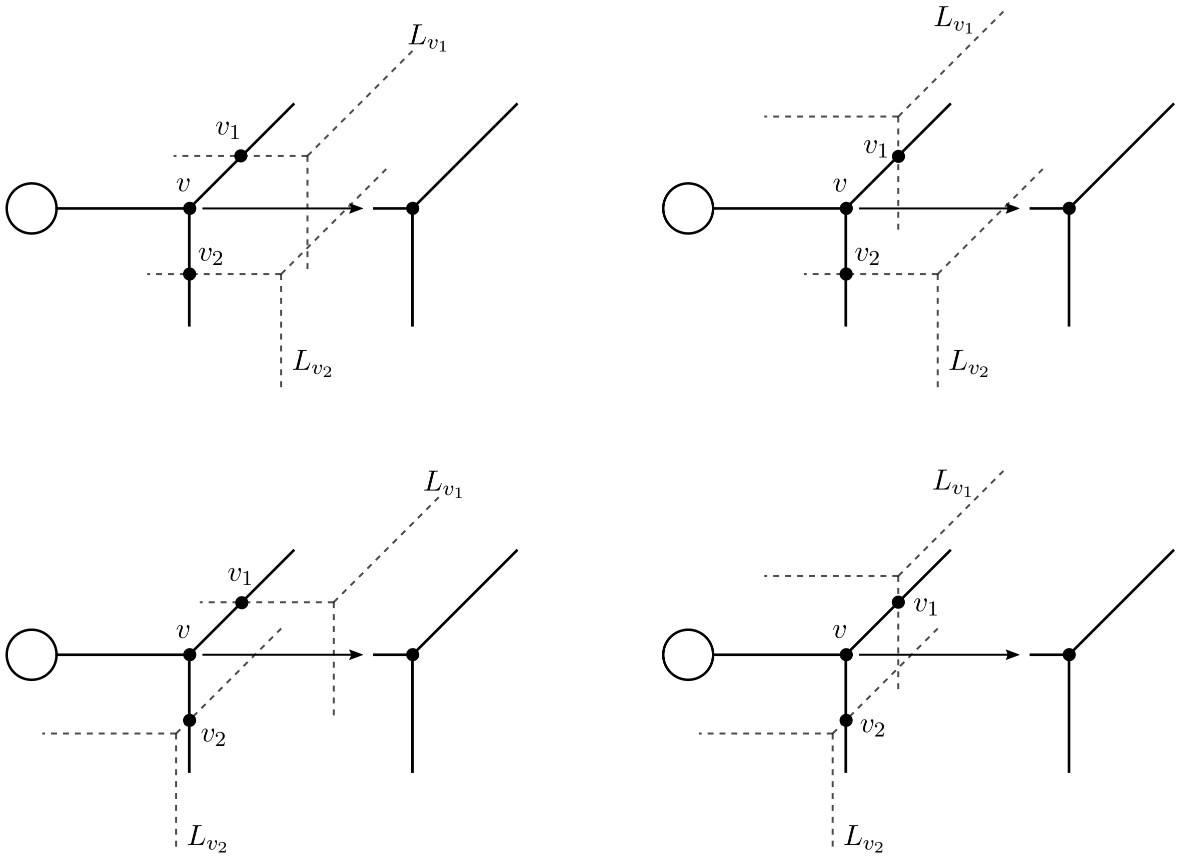


Figure 4.7: The movable vertex v and its movement away from the fixed component.

We distinguish different cases for v.

(1) Assume that $\mathrm{val}(v) = 3$ and that v is adjacent to two edges E_1, E_2 which are parallel to two ends of different direction. The edges E_1, E_2 lead to other vertices in the movable component since moving v varies $\mathrm{ft}_{\lambda_l'}(C)$. There are 3 cases (choose 2 different directions for E_1, E_2 from the 3 standard directions) that we need to distinguish. Moving v unboundedly, we obtain an end adjacent to v. More precisely, Figure 4.7 shows one of

the 3 case where the directions are $(1,1)$ and $(0,-1)$ (the other two cases are analogous). Hence moving v further in its direction of movement eventually produces a combinatorial type that does not allow $\mathrm{ft}_{\lambda'_l}(C)$ to become larger as v is moved.

(2) Assume that $\mathrm{val}(v)=3$ and that all edges adjacent to v are parallel. Since all ends of C are of weight 1 (the degree of C is Δ^2_d), the two edges E_1, E_2 adjacent to v, which lead to other vertices in the movable component, are on the same side of v. Therefore moving v as before (analogous to Figure 4.7 but with v_1, v_2 lying on parallel ends) does not make the coordinate $\mathrm{ft}_{\lambda'_l}(C)$ larger.

(3) Assume that $\mathrm{val}(v)>3$, then there are edges E_1, E_2 adjacent to v (by Corollary 3.2.24) which connect v to vertices v_1, v_2 of the movable component that satisfy line conditions L_{v_1}, L_{v_2}. The same movement as in the case of $\mathrm{val}(v)=3$ yields a combinatorial type where there is an end adjacent to v which contradicts Corollary 3.2.24 since $\mathrm{val}(v)>3$, see again Figure 4.7.

In total, choosing a large value for $|\lambda'_l|$ implies that only rational tropical stable maps with a contracted bounded edge can contribute to $N_{\Delta^2_d}\left(p_{\underline{n}}, L_{\underline{\kappa}}, \lambda_{[l-1]}, \lambda'_l\right)$. Moreover, there is exactly one contracted bounded edge. Otherwise, a rational tropical stable map C contributing to the number $N_{\Delta^2_d}\left(p_{\underline{n}}, L_{\underline{\kappa}}, \lambda_{[l-1]}, \lambda'_l\right)$ would give rise to a 1-dimensional family of stable maps contributing to $N_{\Delta^2_d}\left(p_{\underline{n}}, L_{\underline{\kappa}}, \lambda_{[l-1]}, \lambda'_l\right)$ which is a contradiction. □

Notice that it was assumed in Proposition 4.1.1 that $\#\underline{n}\geq 1$, i.e. that there is at least one point condition. However, even without point conditions we can still assume that there is a contracted bounded edge, see Proposition 4.1.25.

Lemma 4.1.23. *Let C be a rational tropical stable map that contributes to $N_{\Delta^2_d}\left(L_{\underline{\kappa}}, \lambda_{[l]}\right)$, where all degenerated tropical cross-ratios are on contracted ends only. Then there is a vertex v of C which is adjacent to two contracted ends e_1, e_2 such that e_1 satisfies a multi-line condition L_a and e_2 satisfies a multi-line condition L_b with $a, b\in\underline{\kappa}$.*

Proof. Assume that each vertex of C is at most adjacent to one contracted end that satisfies a multi-line condition. Hence each vertex of the rational tropical stable map associated to C allows a 1-dimensional movement since its movement is only restricted by at most one multi-line condition (there are no point conditions). Thus C give rise to a 1-dimensional family which is a contradiction. □

Lemma 4.1.24. *Let v be the vertex adjacent to e_1, e_2 from Lemma 4.1.23. Then* $\mathrm{val}(v)>3$ *and there is a degenerated tropical cross-ratio $\lambda\in\lambda_{[l]}$ such that $\lambda=\{e_1, e_2, \beta_3, \beta_4\}$.*

Proof. Notation from Lemma 4.1.23 is used. If $\mathrm{val}(v)=3$, then, by Lemma 4.1.23, there is a contracted bounded edge adjacent to v. Hence C cannot be fixed by the set of given conditions which is a contradiction. Thus $\mathrm{val}(v)>3$.

By Corollary 3.2.24, there is a degenerated tropical cross-ratio λ as desired or there are degenerated tropical cross-ratios $\lambda_1=\{e_1,\dots\}$ and $\lambda_2=\{e_2,\dots\}$ such that $e_2\notin\lambda_1$ and $e_1\notin\lambda_2$. Assume that there is no degenerated tropical cross-ratio λ as desired. Then v can be resolved by adding a contracted bounded edge e to C that is adjacent to v and a new 3-valent vertex v' which is adjacent to e_1, e_2. Notice that this resolution of v is compatible with λ_1, λ_2 but gives rise to a 1-dimensional family of rational tropical stable maps satisfying $L_{\underline{\kappa}}, \lambda_{[l]}$ which is a contradiction. □

Proposition 4.1.25. *We use notation from Lemma 4.1.23 and Lemma 4.1.24 and assume without loss of generality that e_1, e_2 are entries of the degenerated tropical cross-ratio λ_l. Let λ_l' be a non-degenerated cross-ratio that degenerates to λ_l, where e_1, e_2 are grouped together. Then every rational tropical stable map C' that contributes to $N_{\Delta_d^2}\left(L_{\underline{\kappa}}, \lambda_{[l-1]}, \lambda_l'\right)$ (where all (degenerated) tropical cross-ratios are on contracted ends only) arises from a rational tropical stable map C that contributes to $N_{\Delta_d^2}\left(L_{\underline{\kappa}}, \lambda_{[l]}\right)$ by adding a contracted bounded edge e to C that is adjacent to v and a new vertex v' which is in turn adjacent to e_1, e_2.*

Proof. Let C be a rational tropical stable map that contributes to $N_{\Delta_d^2}\left(L_{\underline{\kappa}}, \lambda_{[l]}\right)$ and let v be the vertex from Lemma 4.1.23 at which λ_l is satisfied. Assume that the edge e' we add by resolving v according to λ_l' is not contracted. Denote the rational tropical stable map obtained this way by C''. Denote the vertex adjacent to e' and e_1, e_2 by $\tilde{v}$. Consider C'' as a point in the cycle that arises from dropping the tropical cross-ratio condition λ_l' (cf. Definition 4.1.2). Then C'' is in the boundary of a 2-dimensional cell of the same cycle that arises from C'' by adding a contracted bounded edge e to C'' that separates $\tilde{v}$ from e_1, e_2. Hence there is a 2-dimensional cell inside a 1-dimensional cycle, which is a contradiction.

Each rational tropical stable map C that contributes to $N_{\Delta_d^2}\left(L_{\underline{\kappa}}, \lambda_{[l]}\right)$ yields a contribution to $N_{\Delta_d^2}\left(L_{\underline{\kappa}}, \lambda_{[l-1]}, \lambda_l'\right)$ if the vertex v at which λ_l is satisfied is resolved according to λ_l' and each resolution of v according to λ_l' produces a contracted bounded edge e. Hence Proposition 3.2.7 and the description of $\operatorname{mult}(C)$ via resolutions of vertices (see Proposition 3.2.25) guarantees that there cannot be more stable maps C' contributing to $N_{\Delta_d^2}\left(L_{\underline{\kappa}}, \lambda_{[l-1]}, \lambda_l'\right)$ than the ones obtained from adding a contracted bounded edge e to tropical stable maps C. $\square$

4.1.2 Behavior of cut contracted bounded edges

After a contracted bounded edge e was identified in Propositions 4.1.1, 4.1.25, we can cut this edge which yields a split of the original rational tropical stable map into two new ones. The aim of this subsection is to prove Corollary 4.1.31, in which the behavior of the two new ends that arise from cutting e is described.

Construction 4.1.26 (Cutting the contracted bounded edge)**.** Let C be a rational tropical stable map that contributes to $N_{\Delta_d^2}\left(p_{\underline{n}}, L_{\underline{\kappa}}, \lambda_{[l-1]}, \lambda_l'\right)$, where λ_l' is a non-degenerated tropical cross-ratio such that $|\lambda_l'|$ is large and all (degenerated) tropical cross-ratios are on contracted ends. Assume that C has a contracted bounded edge e.

If e is cut, we obtain two rational tropical stable maps C_1 and C_2 with contracted ends e_1 and e_2 that come from e. By abuse of notation, the label of e_i is also e_i for $i = 1, 2$. We usually denote the vertices adjacent to the ends e_1, e_2 by v_1, v_2. Notice that C_i is of degree $\Delta_{d_i}^2$ for $i = 1, 2$ such that $\Delta_{d_1}^2 \cup \Delta_{d_2}^2 = \Delta_d^2$ as a union of multisets since C is balanced and of degree Δ_d^2. Recall that we assume that all degrees are labeled (see Definition 2.2.3). Thus the notation $\Delta_{d_1}^2 \cup \Delta_{d_2}^2 = \Delta_d^2$ implies that the labels l of $\left(\Delta_d^2, l\right)$ are distributed among $\left(\Delta_{d_1}^2, l_1\right)$ and $\left(\Delta_{d_2}^2, l_2\right)$, i.e. $\operatorname{Im}(l) = \operatorname{Im}(l_1) \cup \operatorname{Im}(l_2)$ by abuse of notation.

If a contracted bounded edge e is cut, the degenerated tropical cross-ratios can be *adapted* the following way: If λ_j is a degenerated tropical cross-ratio that is satisfied at some vertex $v \in C_i$ for an $i \in [2]$, then, by the path criterion (Corollary 3.2.12), either all entries of λ_j are labels of contracted ends of C_i or 3 entries of λ_j are labels of contracted ends of C_i and one entry β is a label of a contracted end of C_t for $t \neq i$. In the first case, we do not change λ_j and in the latter case, we replace the entry β of λ_j by e_i. A degenerated tropical cross-ratio that is adapted to e_i is denoted by $\lambda_j^{\to e_i}$.

Let $\underline{f}$ denote the set of labels of contracted ends of C that satisfy no point or line condition, i.e. $(\underline{\kappa}, \underline{n}, \underline{f})$ is a partition of $[N]$ using the usual notation (see Definition 3.2.4). Each C_i of degree $\Delta^2_{d_i}$ for $i = 1, 2$ satisfies point conditions $p_{\underline{n_i}}$, multi-line conditions $L_{\underline{\kappa_i}}$ and degenerated tropical cross-ratio conditions $\lambda^{\to e_i}_{\underline{l_i}}$ such that $\underline{n_1} \dot\cup \underline{n_2} = \underline{n}$, $\underline{\kappa_1} \dot\cup \underline{\kappa_2} = \underline{\kappa}$, $\underline{f_1} \dot\cup \underline{f_2} = \underline{f}$ and $\underline{l_1} \dot\cup \underline{l_2} = [l-1]$ hold, where all degenerated tropical cross-ratios were adapted to the cut edge e.

We say that C *splits* into the two rational tropical stable maps C_1 and C_2 and the *splitting type* of C is defined as $(\Delta^2_{d_1}, \underline{n_1}, \underline{\kappa_1}, \underline{l_1}, \underline{f_1} \mid \Delta^2_{d_2}, \underline{n_2}, \underline{\kappa_2}, \underline{l_2}, \underline{f_2})$.

Definition 4.1.27 (1/1 and 2/0 splits)**.** Let Δ^2_d be a degree, let $p_{\underline{n}}, L_{\underline{\kappa}}, \lambda_{[l-1]}$ be given conditions and let $\underline{f}$ be labels of contracted ends that satisfy no conditions as in Construction 4.1.26. We refer to

$$(\Delta^2_{d_1}, \underline{n_1}, \underline{\kappa_1}, \underline{l_1}, \underline{f_1} \mid \Delta^2_{d_2}, \underline{n_2}, \underline{\kappa_2}, \underline{l_2}, \underline{f_2})$$

as a *split* (of conditions) if $\Delta^2_{d_1} \cup \Delta^2_{d_2} = \Delta^2_d$ as a union of multisets, $\underline{n_1} \dot\cup \underline{n_2} = \underline{n}$, $\underline{\kappa_1} \dot\cup \underline{\kappa_2} = \underline{\kappa}$, $\underline{l_1} \dot\cup \underline{l_2} = [l-1]$, $\underline{f_1} \dot\cup \underline{f_2} = \underline{f}$ hold and each degenerated tropical cross-ratio in $\lambda_{\underline{l_i}}$ has at least 3 of its entries in $\underline{n_i} \cup \underline{\kappa_i} \cup \underline{f_i}$. If we write $\lambda^{\to e_i}_{\underline{l_i}}$, we mean that each entry of each cross-ratio in $\lambda_{\underline{l_i}}$ that is not in $\underline{n_i} \cup \underline{\kappa_i} \cup \underline{f_i}$ is replaced by the label e_i. Such a split is called a 1/1 *split* if

$$3d_i = \#\underline{n_i} + \#\underline{l_i} - \#\underline{f_i} + 1 \tag{4.1}$$

holds for $i = 1, 2$. If

$$3d_i = \#\underline{n_i} + \#\underline{l_i} - \#\underline{f_i} \text{ and } 3d_t = \#\underline{n_t} + \#\underline{l_t} - \#\underline{f_t} + 2 \tag{4.2}$$

holds for $i = 1, 2$ with $t \neq i$ for some choice of $i, t \in \{1, 2\}$, then we refer to the split $(\Delta^2_{d_1}, \underline{n_1}, \underline{\kappa_1}, \underline{l_1}, \underline{f_1} \mid \Delta^2_{d_2}, \underline{n_2}, \underline{\kappa_2}, \underline{l_2}, \underline{f_2})$ as a 2/0 *split.*

Definition 4.1.28 (1/1 and 2/0 edges)**.** Let $(\Delta^2_{d_1}, \underline{n_1}, \underline{\kappa_1}, \underline{l_1}, \underline{f_1} \mid \Delta^2_{d_2}, \underline{n_2}, \underline{\kappa_2}, \underline{l_2}, \underline{f_2})$ be a split of conditions as in Definition 4.1.27. Define for the (adapted) conditions $p_{\underline{n_i}}, L_{\underline{\kappa_i}}, \lambda^{\to e_i}_{\underline{l_i}}$ and for $i = 1, 2$ the cycles

$$Y_i := \operatorname{ev}_{e_i,*}\left(\prod_{k\underline{\kappa_i}} \operatorname{ev}_k^*(L_k) \cdot \prod_{t \in \underline{n_i}} \operatorname{ev}_t^*(p_t) \cdot \prod_{j \in \underline{l_i}} \operatorname{ft}^*_{\lambda_j^{\to e_i}}(0) \cdot \mathcal{M}_{0,N_i}\left(\mathbb{R}^2, \Delta^2_{d_i}\right)\right) \subset \mathbb{R}^2,$$

where $N_i := \#\underline{n_i} + \#\underline{\kappa_i} + \#\underline{f_i} + 1$. Notice that $(\Delta^2_{d_1}, \underline{n_1}, \underline{\kappa_1}, \underline{l_1}, \underline{f_1} \mid \Delta^2_{d_2}, \underline{n_2}, \underline{\kappa_2}, \underline{l_2}, \underline{f_2})$ is a 1/1 split if and only if Y_1, Y_2 are 1-dimensional. It is a 2/0 split if and only if Y_i is 0-dimensional and Y_t is 2-dimensional (see (4.2) in Definition 4.1.27).

Let C be a rational tropical stable map with a contracted bounded edge e such that C is of splitting type $(\Delta^2_{d_1}, \underline{n_1}, \underline{\kappa_1}, \underline{l_1}, \underline{f_1} \mid \Delta^2_{d_2}, \underline{n_2}, \underline{\kappa_2}, \underline{l_2}, \underline{f_2})$. Then N_i is the number of contracted ends of C_i and the cycle Y_i is the condition C_i imposes on $C_{\bar{i}}$ for $\bar{i} \neq i$ via e. For example, if Y_1 is 0-dimensional, then the position of v_2 (for notation, see Construction 4.1.26) is completely determined by Y_1 since v_2 is connected to v_1 via e in C and C is fixed by the given conditions $p_{\underline{n}}, L_{\underline{\kappa}}, \lambda_{[l-1]}, \lambda_l$. Since all given conditions are in general position, the dimension of Y_2 is 2 in this case, i.e. v_2 cannot impose a condition via e to v_1. In general, there are two cases for C:

(1) One of the cycles Y_i is 0-dimensional and the other one is 2-dimensional. We then refer to e as a 2/0 *edge.*

(2) Both of the cycles Y_i are 1-dimensional. We then refer to e as a 1/1 *edge.*

Which case occurs depends only on $d_i, \#\underline{n_i}, \#\underline{\kappa_i}, \#\underline{l_i}, \#\underline{f_i}$ for $i = 1, 2$.

Example 4.1.29. An example of a 1/1 split is provided below, see Example 4.2.2. An example of a 2/0 split is the following: Let C be a degree Δ_2^2 rational tropical stable map that satisfies point conditions $p_{[2]}$, multi-line conditions $L_{[6]\backslash[2]}$, degenerated tropical cross-ratios $\lambda_1 = \{p_1, L_3, L_4, L_5\}$, $\lambda_2 = \{p_1, p_2, L_3, L_4\}$ and a non-degenerated tropical cross-ratio $\lambda_3' = (p_1 L_3 | p_2 L_6)$ whose length is large enough such that C has a contracted bounded edge e. Construction 4.1.26 yields a split of C into C_1 and C_2, where the vertices adjacent to the split edge e are denoted by $v_i \in C_i$ for $i = 1, 2$. Figure 4.8 shows C_1 and C_2, where we shifted C_2 in order to get a better picture (in fact v_1 and v_2 are the same point in $\mathbb{R}^2$).

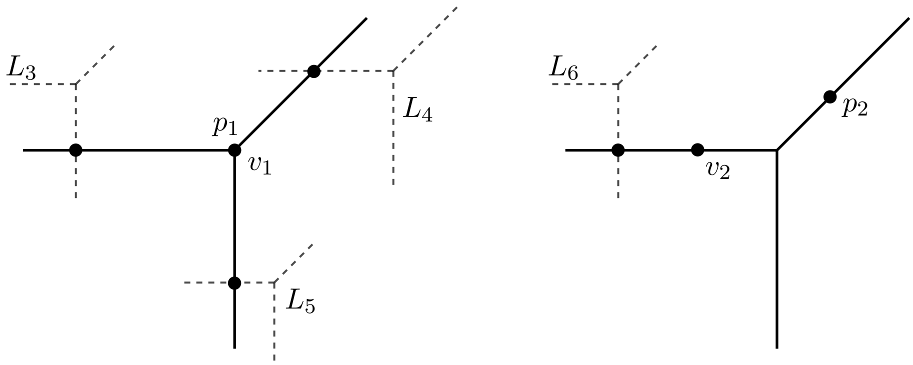


Figure 4.8: The rational tropical stable map C_1 satisfying $p_1, L_{[5]\backslash[2]}, \lambda_{[2]}$ is shown on the left, the rational tropical stable map C_2 satisfying p_2, L_6 is shown on the right. Notice that the length of e in C is given by λ_3', i.e. C is fixed by the given conditions.

Remark 4.1.30. Fix a degree Δ_d^2, point conditions $p_{\underline{n}}$, multi-line conditions $L_{\underline{\kappa}}$ and degenerated tropical cross-ratio conditions $\lambda_{[l-1]}$. Let $(\Delta_{d_1}^2, \underline{n_1}, \underline{\kappa_1}, \underline{l_1}, \underline{f_1} \mid \Delta_{d_2}^2, \underline{n_2}, \underline{\kappa_2}, \underline{l_2}, \underline{f_2})$ be a split of these conditions. Consider degree $\Delta_{d_i}^2$ rational tropical stable maps C_i for $i = 1, 2$ with $\#\underline{n_i} + \#\underline{\kappa_i} + \#\underline{f_i} + 1$ contracted ends that satisfy the point conditions $p_{\underline{n_i}}$, the multi-line conditions $L_{\underline{\kappa_i}}$ and the degenerated tropical cross-ratio conditions $\lambda_{\underline{l_i}}^{\to e_i}$. The cycles Y_i for $i = 1, 2$ of Definition 4.1.28 tell us how to glue the end e_1 of C_1 to the end e_2 of C_2 to form a contracted bounded edge e such that the new rational tropical stable map C satisfies all given conditions.

If Y_1 is 0-dimensional and p_{e_2} is a point in Y_1, then considering rational tropical stable maps C_2 that satisfy $p_{\underline{n_2}}, L_{\underline{\kappa_2}}, \lambda_{\underline{l_2}}^{\to e_2}$ and that satisfy p_{e_2} with the end e_2 allows us to glue C_1 to C_2, where the contracted bounded edge is contracted to $p_{e_2} \in \mathbb{R}^2$.

If both Y_i are 1-dimensional, then we can consider tropical stable maps C_2 that satisfy the conditions $p_{\underline{n_2}}, L_{\underline{\kappa_2}}, \lambda_{\underline{l_2}}^{\to e_2}$ and Y_1. Since $\mathrm{ev}_{e_2}(C_2) \in Y_2$, i.e. C_2 satisfies Y_2, the position of the contracted end e_2 of C_2 in $\mathbb{R}^2$ is a point p contributing to the 0-dimensional cycle $Y_1 \cdot Y_2$. On the other hand, there is a rational tropical stable map C_1 that satisfies $p_{\underline{n_2}}, L_{\underline{\kappa_2}}, \lambda_{\underline{l_2}}^{\to e_1}$ and Y_2 such that its end e_1 is contracted to p. Thus e_1 of C_1 and e_2 of C_2 can be glued to form a bounded edge e that is contracted to p.

Corollary 4.1.31 (of Proposition 4.1.1)**.** *If C is a rational tropical stable map as in Proposition 4.1.1 whose contracted bounded edge is a 1/1 edge, then the 1-dimensional cycles Y_i from Definition 4.1.28 have ends of primitive directions $(1,1), (-1,0)$ and $(0,-1) \in \mathbb{R}^2$ only. In other words, the 1-dimensional conditions that a contracted bounded 1/1 edge passes from one vertex to the other has ends of standard directions.*

Proof. Proposition 4.1.25 implies that each contracted bounded edge that appears in the no-point-conditions case is a 2/0 edge. Hence we may assume that at least one point condition is given.

Let C be a rational tropical stable map in Y_i whose movement is unbounded, i.e. that gives rise to an end of Y_i. Corollary 4.1.22 yields that the movable component of C consists of exactly one vertex v_i of type (I) or (II). Thus v_i is of type (I) since we assumed that there is at least one point condition. If there is a cross-ratio $\lambda_j \in \lambda_{[l-1]}$ such that $\lambda_j^{\to e_i}$ is satisfied at v_i, i.e. $\lambda_j^{\to e_i} \in \lambda_v$, then Corollary 3.2.24 guarantees that v_i is not adjacent to unbounded edges. This yields a contradiction when v_i moves unboundedly as the proof of Proposition 4.1.1 shows. Hence v_i is a 3-valent type (I) vertex which is adjacent to e_i and an end E of C. Therefore, v_i moves parallel to E. □

Corollary 4.1.32. *Notation from Construction 4.1.26 is used, i.e. denote the vertex adjacent to the end e_i of C_i by v_i. Under the same assumptions as in Corollary 4.1.31, it follows that v_i is 3-valent and adjacent to an end of C_i for $i = 1, 2$.*

Proof. This follows immediately from the proof of Corollary 4.1.31. □

4.2 Multiplicities of splits

In order to obtain a recursion, splitting a rational tropical stable map C into C_1, C_2 as in Construction 4.1.26 is not enough. It is also necessary that the multiplicity of C splits as well. However, it is observed in this subsection that the multiplicity of C does not have to be equal to $\operatorname{mult}(C_1) \cdot \operatorname{mult}(C_2)$. We have to deal with this problem later.

Notation 4.2.1 (Replacing 1/1 edge conditions)**.** Let C be a rational tropical stable map that contributes to $N_{\Delta_d^2}\left(p_{\underline{n}}, L_{\underline{\kappa}}, \lambda_{[l-1]}, \lambda'_l\right)$ such that C has a contracted bounded edge e that is a 1/1 edge (see Definition 4.1.28). Split e as in Construction 4.1.26 to obtain C_1, C_2 and let $Y_{\bar{i}}$ denote the 1-dimensional condition C_i satisfies for $i \neq \bar{i}$ as in Definition 4.1.28. Let v_i be the vertex of C_i that is adjacent to e_i (e_i is the contracted end of C_i that came from cutting e) which satisfies $Y_{\bar{i}}$. Let $st \in \{01, 10, 1\text{-}1\}$ and let L_{st} be a degenerated line as in Definition 2.2.21 such that its vertex is translated to v_i. Let $C_{i,st}$ denote the rational tropical stable map that equals C_i, but where we replaced the $Y_{\bar{i}}$ condition with L_{st}, i.e. $C_{i,st}$ satisfies L_{st} instead of $Y_{\bar{i}}$.

Notice that only the multiplicities of C_i and $C_{i,st}$ may differ. In particular, the multiplicity of $C_{i,st}$ may be zero, whereas the multiplicity of C_i can be nonzero.

We fix the convention that if the ev-matrix $M(C_{i,st})$ is considered, then the replaced condition appears in the first row of $M(C_{i,st})$.

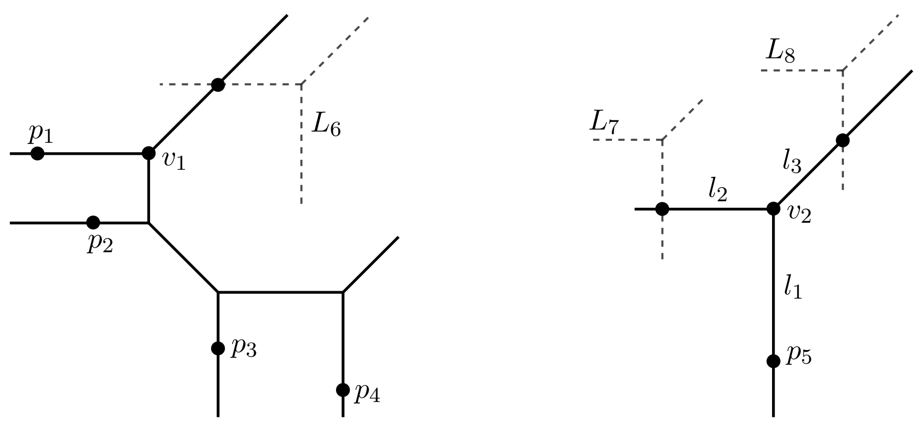


Figure 4.9: The rational tropical stable map C_1 of Example 4.2.2 satisfying $p_{[4]}, L_6, \lambda_1$ is shown on the left, the rational tropical stable map C_2 satisfying p_5, L_7, L_8, λ_2 is shown on the right. Notice that the length of e in C is given by λ'_3, i.e. C is fixed by the given conditions.

Example 4.2.2. Let C be a rational tropical stable map of degree Δ_3^2 that satisfies point conditions $p_{[5]}$, multi-line conditions $L_{[8]\backslash[5]}$, degenerated tropical cross-ratio conditions $\lambda_1 = \{p_1, p_2, p_5, L_6\}$, $\lambda_2 = \{p_1, p_5, L_7, L_8\}$ and a non-degenerated tropical cross-ratio condition $\lambda_3' = (p_1 p_2 | L_7 L_8)$ whose length is large enough such that C has a contracted bounded edge e. Construction 4.1.26 yields a split of C into C_1 and C_2, where the vertices adjacent to the split edge e are denoted by $v_i \in C_i$ for $i = 1, 2$. Figure 4.9 shows C_1 and C_2, where we shifted C_2 in order to get a better picture (in fact v_1 and v_2 are the same point in $\mathbb{R}^2$ as in Example 4.1.29).

Notice that e is a 1/1 edge, so we use Notation 4.2.1 to replace conditions. For example, $C_{2,10}$ equals C_2, where the end e_2 adjacent to v_2 satisfies the degenerated line condition L_{10}. Figure 4.9 shows that $C_{2,10}$ is not fixed by its conditions, i.e. $\operatorname{mult}(C_{2,10}) = 0$. If we consider $C_{2,01}$ instead, then its multiplicity is 1 since it is the absolute value of the determinant the matrix

$$M(C_{2,01}) = \begin{array}{c} \\ L_{01} \\ \\ \\ L_7 \\ L_8 \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Base } p_5 & l_1 & l_2 & l_3 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

of Remark 3.2.26, where p_5 is chosen as base point and the first row is associated to L_{01} satisfied by e_2.

Proposition 4.2.3. *Let C be a rational tropical stable map such that it contributes to the number $N_{\Delta_d^2}\left(p_{\underline{n}}, L_{\underline{\kappa}}, \lambda_{[l-1]}, \lambda_l'\right)$ and such that it has a contracted bounded edge e. The components arising from cutting e as in Construction 4.1.26 are denoted by C_1, C_2.*

(a) If e is a 2/0 edge, then

$$\operatorname{mult}(C) = \operatorname{mult}(C_1) \cdot \operatorname{mult}(C_2).$$

(b) If e is a 1/1 edge, then

$$\operatorname{mult}(C) = |\det(M(C_{1,10})) \cdot \det(M(C_{2,01})) - \det(M(C_{1,01})) \cdot \det(M(C_{2,10}))|,$$

where $C_{i,st}$ is defined in Notation 4.2.1. In particular, if $\det(M(C_{1,01}))$ or $\det(M(C_{2,10}))$ vanishes, then

$$\operatorname{mult}(C) = \operatorname{mult}(C_{1,10}) \cdot \operatorname{mult}(C_{2,01}).$$

Proof. It is sufficient to prove (a), (b) for ev-multiplicities only since the cross-ratio multiplicities can be expressed locally at vertices, see Proposition 3.2.25. Thus contributions from vertices to cross-ratio multiplicities do not depend on cutting edges.

(a) If e is a 2/0 edge, then the situation is similar to Lemma 3.2.32 as the following proof shows. Denote the vertices adjacent to e by v_1, v_2 such that $v_1 \in C_1$ and $v_2 \in C_2$ and assume without loss of generality that Y_1 (notation from Definition 4.1.28) is 0-dimensional. Consider the ev-matrix $M(C)$ of C of Definition 3.1.11 with base point v_1, i.e.

$$M(C) = \begin{array}{r} \\ \text{conditions in } C_1 \\ \\ \text{conditions in } C_2 \end{array} \begin{array}{c} \begin{array}{ccc} \text{Base } v_1 & \text{lengths in } C_1 & \text{lengths in } C_2 \end{array} \\ \left(\begin{array}{c|c|c} * & * & 0 \\ \hline * & 0 & * \end{array}\right) \end{array}$$

Let y_1 be the number of rows that belong to the conditions in C_1, let x_1 be the number of columns belonging to the base point and the lengths in C_1. Using notation from Definition 4.1.28, we obtain

$$\begin{aligned} x_1 &= 2 + 3d_1 - 3 + \#\underline{n_1} + \#\underline{\kappa_1} - \#\underline{l_1} + \#\underline{f_1} + 1, \\ y_1 &= 2 \cdot \#\underline{n_1} + \#\underline{\kappa_1}. \end{aligned}$$

On the other hand, C_1 is fixed by its set of conditions since Y_1 is 0-dimensional, i.e. we can apply (3.6) for $N = \#\underline{n_1} + \#\underline{\kappa_1} + (\#\underline{f_1} + 1)$ to obtain $x_1 = y_1$. Thus the bold red lines in $M(C)$ above divide $M(C)$ into squares, hence

$$|\det(M(C))| = \operatorname{mult}(C_1) \cdot |\det(M)|,$$

where M is the square matrix on the bottom right. Define the matrix

$$M(C_{2,v_2}) := \begin{array}{c} \text{Base } v_2 \qquad \\ \left(\begin{array}{cc|c} 1 & 0 & \multirow{2}{*}{0} \\ 0 & 1 & \\ \hline \multicolumn{2}{c|}{*} & M \end{array}\right) \end{array}$$

where the first two columns are chosen in such a way that $M(C_{2,v_2})$ is the ev-matrix of C_2 with respect to the base point v_2. Notice that

$$|\det(M)| = |\det(M(C_{2,v_2}))|$$

and

$$|\det(M(C_{2,v_2}))| = \operatorname{mult}(C_2)$$

hold, where C_2 satisfies the additional point condition imposed on e_2 by Y_1.

(b) We assume that the weight of each multi-line $\omega(L_k)$ (see Definition 2.2.19) for $k \in \kappa$ equals 1 since we can pull out the factor $\omega(L_k)$ from each row of the *ev*-matrix, apply all the following arguments and multiply with $\omega(L_k)$ later.

Denote the vertex of C_1 adjacent to the cut edge e by v_1 and the other vertex adjacent to e by v_2. The ev-matrix $M(C)$ of C with respect to the base point v_1 is given by

$$M(C) = \begin{array}{r} \\ \text{conditions in } C_1 \\ \\ \text{conditions in } C_2 \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} \text{Base } v_1 & \text{lengths in } C_1 & & \text{lengths in } C_2 \end{array} \\ \left(\begin{array}{c|c|c|c} * & * & \begin{matrix} * \\ \vdots \\ * \end{matrix} & 0 \\ \hline * & 0 & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & * \end{array}\right) \end{array}$$

The bold red lines divide $M(C)$ into square pieces at the upper left and the lower right. This follows from similar arguments as used in the proof of part (a). Let M be the matrix consisting of the lower right block of $M(C)$ whose entries (see above) are indicated by $*$ and its columns are associated to lengths in C_2. Let $A = (a_{ij})_{ij}$ be the submatrix of $M(C)$ given by the rows that belong to conditions of C_1 and by the base point's columns and the columns that are associated to lengths in C_1, i.e. A consists of all the $*$-entries above the bold red line in $M(C)$.

Consider the Laplace expansion of the rightmost column of A. Recursively, use Laplace expansion on every column that belongs to the lengths in C_1 starting with the rightmost column. Eventually, we end up with a sum in which each summand contains a factor $\det(M_{a_{r1}a_{r2}})$ for a matrix $M_{a_{r1}a_{r2}}$, which is one of the following three matrices, namely

$$M_{a_{r1}a_{r2}} := \begin{array}{c} \text{lenghts in } C_2 \\ \left(\begin{array}{cc|ccc} a_{r1} & a_{r2} & 0 & \dots & 0 \\ \hline & & & & \\ & * & & M & \\ & & & & \end{array}\right) \end{array},$$

where $(a_{r1}, a_{r2}) = (1, 0)$, $(a_{r1}, a_{r2}) = (0, 1)$ or $(a_{r1}, a_{r2}) = (1, -1)$ are the remaining entries of A in its r-th row after the recursive procedure. Notice that in each of the three cases the entries of the first two columns are of such a form that M_{st} for $st = 10, 01, \text{1-1}$ is the ev-matrix of $C_{2,st}$ (see Notation 4.2.1) with base point v_2. We can group the summands according to the values a_{r1}, a_{r2} and obtain in total

$$|\det(M(C))| = |F_{10} \cdot \det(M_{10}) + F_{01} \cdot \det(M_{01}) + F_{\text{1-1}} \cdot \det(M_{\text{1-1}})|, \tag{4.3}$$

where $F_{st} \in \mathbb{R}$ for $st = 10, 01, \text{1-1}$ are factors occurring due to the recursive Laplace expansion. More precisely, let b be the number of bounded edges in C_1, i.e. the number of Laplace expansions we applied. Then

$$F_{st} = \sum_{r:(a_{r1},a_{r2})=(s,t)} \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{j=3}^{3+b} a_{\sigma(j)j}, \tag{4.4}$$

where the second sum goes over all bijections $\sigma : \{3, \dots, 3+b\} \to \{1, \dots, r-1, r+1, \dots, b+1\}$, i.e. it goes over all possibilities of choosing for each column Laplace expansion was used on an entry in a row of A which is not the r-th row.

Let $A_{10}, A_{01}, A_{\text{1-1}}$ be the square matrices obtained from A by adding the new first row $(1, 0, 0, \dots, 0)$, $(0, 1, 0 \dots, 0)$ or $(1, -1, 0, \dots, 0)$ to A. Again, notice that A_{st} for $st = 10, 01, \text{1-1}$ is the ev-matrix of $C_{1,st}$ (see Notation 4.2.1, Definition 3.1.11) with base point v_1. We claim that

$$\det(A_{10}) = F_{01} - F_{\text{1-1}} \tag{4.5}$$

holds. Let N be the number of columns and rows of A_{st}. Denote the entries of the ev-matrix $M(C)$ by $m(C)_{ij}$. Define

$$S_{st} := \{r \in [N-1] \mid m(C)_{r1} = s,\ m(C)_{r2} = t\}$$

for $(s, t) = (1, 0), (0, 1), (1, -1)$ and notice that $\#S_{10} + \#S_{01} + \#S_{\text{1-1}} = N - 1$. Denote the entries of A_{10} by $a_{ij}^{(10)}$ and apply Leibniz' determinant formula to obtain

$$\begin{aligned} \det(A_{10}) &= \sum_{\sigma \in \mathbb{S}_N} \operatorname{sgn}(\sigma) \prod_{j=1}^{N} a_{\sigma(j)j}^{(10)} \\ &= \sum_{\substack{\sigma \in \mathbb{S}_N \\ \sigma(2) \in S_{01}}} \operatorname{sgn}(\sigma) \prod_{j=1}^{N} a_{\sigma(j)j}^{(10)} + \sum_{\substack{\sigma \in \mathbb{S}_N \\ \sigma(2) \in S_{\text{1-1}}}} \operatorname{sgn}(\sigma) \prod_{j=1}^{N} a_{\sigma(j)j}^{(10)} = F_{01} - F_{\text{1-1}}, \end{aligned}$$

where the second equality holds by definition of S_{st} and the third equality holds by considering how contributions of F_{01} and $F_{1\text{-}1}$ arise as choices of entries of A, see (4.4). The minus sign comes from the factor $a^{(10)}_{\sigma(2),2} = -1$ in each product in the last sum. Thus (4.5) holds.

It can be shown in a similar way that

$$\det(A_{01}) = -(F_{10} + F_{1\text{-}1}) = -F_{10} - F_{1\text{-}1}, \tag{4.6}$$

$$\det(A_{1\text{-}1}) = F_{10} + F_{1\text{-}1} + F_{01} - F_{1\text{-}1} = F_{10} + F_{01} \tag{4.7}$$

hold. Solving the system of linear equations (4.5), (4.6), (4.7) for $F_{10}, F_{01}, F_{1\text{-}1}$ yields

$$\begin{pmatrix} F_{10} \\ F_{01} \\ F_{1\text{-}1} \end{pmatrix} \in \begin{pmatrix} -\det(A_{01}) \\ \det(A_{10}) \\ 0 \end{pmatrix} + \langle \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \rangle, \tag{4.8}$$

where the 1-dimensional part appears because of the relation

$$-\det(M_{10}) + \det(M_{01}) + \det(M_{1\text{-}1}) = 0.$$

Combining (4.3) with (4.8) proves part (b), where $A_{st} = C_{1,st}$ and $M_{st} = C_{2,st}$. In particular,

$$\begin{aligned} \operatorname{mult}(C) &= |\det(M(C_{1,10})) \cdot \det(M(C_{2,01})) - \det(M(C_{1,01})) \cdot \det(M(C_{2,10}))| \\ &= |\det(M(C_{1,10})) \cdot \det(M(C_{2,01}))| \\ &= |\det(M(C_{1,10}))| \cdot |\det(M(C_{2,01}))| \\ &= \operatorname{mult}(C_{1,10}) \cdot \operatorname{mult}(C_{2,01}) \end{aligned}$$

holds if $\det(M(C_{1,01}))$ or $\det(M(C_{2,10}))$ vanishes.

□

4.3 General Kontsevich's formula

Results from Section 4.1 and Section 4.2 are now combined. In particular, we deduce that it can be assumed that one summand in part (b) of Proposition 4.2.3 always vanishes. Hence multiplicities split in a way which is desirable when aiming for a recursion. Indeed, a general tropical Kontsevich's formula is established by the end of this section. Applying Tyomkin's correspondence theorem 2.3.6 then yields a general (classical) Kontsevich's formula. It is also shown that the original Kontsevich's formula [KM94] is a consequence of our general version.

Definition 4.3.1. Given a split $(\Delta^2_{d_1}, \underline{n_1}, \underline{\kappa_1}, \underline{l_1}, \underline{f_1} \mid \Delta^2_{d_2}, \underline{n_2}, \underline{\kappa_2}, \underline{l_2}, \underline{f_2})$ and a tropical non-degenerated cross-ratio $\lambda'_l = (\beta_1\beta_2|\beta_3\beta_4)$ with entries in $\underline{n_1} \cup \underline{\kappa_1} \cup \underline{f_1} \cup \underline{n_2} \cup \underline{\kappa_2} \cup \underline{f_2}$ and $\beta_1 = \min_{i=1}^4(\beta_i)$ (the labels of ends of abstract tropical curves are natural numbers), we say that

$$(\Delta^2_{d_1}, \underline{n_1}, \underline{\kappa_1}, \underline{l_1}, \underline{f_1} \mid \Delta^2_{d_2}, \underline{n_2}, \underline{\kappa_2}, \underline{l_2}, \underline{f_2})$$

is a split *respecting* λ'_l if $\beta_1, \beta_2 \in \underline{n_1} \cup \underline{\kappa_1} \cup \underline{f_1}$ and $\beta_3, \beta_4 \in \underline{n_2} \cup \underline{\kappa_2} \cup \underline{f_2}$. Using the minimum here prevents a factor of $\frac{1}{2}$ later, which would come from renaming C_1 to C_2 and vice versa.

Lemma 4.3.2. *Notation of Construction 4.1.26 is used. Fix a 2/0 split of general positioned conditions as in Remark 4.1.30 and Definition 4.1.27* $(\Delta^2_{d_1}, \underline{n_1}, \underline{\kappa_1}, \underline{l_1}, \underline{f_1} \mid \Delta^2_{d_2}, \underline{n_2}, \underline{\kappa_2}, \underline{l_2}, \underline{f_2})$ *that respects* λ'_l *such that additionally*

$$3d_1 = \#\underline{n_1} + \#\underline{l_1} - \#\underline{f_1}$$

holds. Then

$$\sum_{\substack{C: \\ (\Delta^2_{d_1}, \underline{n_1}, \underline{\kappa_1}, \underline{l_1}, \underline{f_1} | \Delta^2_{d_2}, \underline{n_2}, \underline{\kappa_2}, \underline{l_2}, \underline{f_2})}} \operatorname{mult}(C) = N_{\Delta^2_{d_1}}\left(p_{\underline{n_1}}, L_{\underline{\kappa_1}}, \lambda_{\underline{l_1}}^{\to e_1}\right) \cdot N_{\Delta^2_{d_2}}\left(p_{\underline{n_2}}, p_{e_2}, L_{\underline{\kappa_2}}, \lambda_{\underline{l_2}}^{\to e_2}\right) \tag{4.9}$$

holds, where the sum goes over all rational tropical stable maps C with a contracted bounded edge e such that C contributes to $N_{\Delta^2_d}\left(p_{\underline{n}}, L_{\underline{\kappa}}, \lambda_{[l-1]}, \lambda'_l\right)$*, where λ'_l is the large non-degenerated cross-ratio C satisfies such that C has a contracted bounded edge, and C is of splitting type* $(\Delta^2_{d_1}, \underline{n_1}, \underline{\kappa_1}, \underline{l_1}, \underline{f_1} \mid \Delta^2_{d_2}, \underline{n_2}, \underline{\kappa_2}, \underline{l_2}, \underline{f_2})$*, and p_{e_2} is a point condition imposed on e_2.*

Proof. Each rational tropical stable map C on the left-hand side of (4.9) can be cut at its contracted bounded edge as in Construction 4.1.26 to obtain a rational tropical stable map C_1 that contributes to $N_{\Delta^2_{d_1}}\left(p_{\underline{n_1}}, L_{\underline{\kappa_1}}, \lambda_{\underline{l_1}}^{\to e_1}\right)$ and a rational tropical stable map C_2 that contributes to $N_{\Delta^2_{d_2}}\left(p_{\underline{n_2}}, p_{e_2}, L_{\underline{\kappa_2}}, \lambda_{\underline{l_2}}^{\to e_2}\right)$.

The other way around, each pair of rational tropical stable maps C_1, C_2 such that C_1 contributes to $N_{\Delta^2_{d_1}}\left(p_{\underline{n_1}}, L_{\underline{\kappa_1}}, \lambda_{\underline{l_1}}^{\to e_1}\right)$ and C_2 contributes to $N_{\Delta^2_{d_1}}\left(p_{\underline{n_2}}, p_{e_2}, L_{\underline{\kappa_2}}, \lambda_{\underline{l_2}}^{\to e_2}\right)$ can be glued to a rational tropical stable map C using Remark 4.1.30.

Proposition 4.2.3 states that

$$\operatorname{mult}(C) = \operatorname{mult}(C_1) \cdot \operatorname{mult}(C_2)$$

and thus proves the lemma. □

Lemma 4.3.3. *Notation of Construction 4.1.26 is used. Fix a 1/1 split of general positioned conditions as in Remark 4.1.30 and Definition 4.1.27* $(\Delta^2_{d_1}, \underline{n_1}, \underline{\kappa_1}, \underline{l_1}, \underline{f_1} \mid \Delta^2_{d_2}, \underline{n_2}, \underline{\kappa_2}, \underline{l_2}, \underline{f_2})$ *that respects λ'_l. Then*

$$\sum_{\substack{C: \\ (\Delta^2_{d_1}, \underline{n_1}, \underline{\kappa_1}, \underline{l_1}, \underline{f_1} | \Delta^2_{d_2}, \underline{n_2}, \underline{\kappa_2}, \underline{l_2}, \underline{f_2})}} \operatorname{mult}(C) = N_{\Delta^2_{d_1}}\left(p_{\underline{n_1}}, L_{\underline{\kappa_1}}, L_{e_1}, \lambda_{\underline{l_1}}^{\to e_1}\right) \cdot N_{\Delta^2_{d_2}}\left(p_{\underline{n_2}}, L_{\underline{\kappa_2}}, L_{e_2}, \lambda_{\underline{l_2}}^{\to e_2}\right) \tag{4.10}$$

holds, where the sum goes over all rational tropical stable maps C with a contracted bounded edge e such that C contributes to $N_{\Delta^2_d}\left(p_{\underline{n}}, L_{\underline{\kappa}}, \lambda_{[l-1]}, \lambda'_l\right)$*, where λ'_l is the large non-degenerated tropical cross-ratio C satisfies such that C has a contracted bounded edge and all (degenerated) tropical cross-ratios are on contracted ends only. Additionally, C is of splitting type* $(\Delta^2_{d_1}, \underline{n_1}, \underline{\kappa_1}, \underline{l_1}, \underline{f_1} \mid \Delta^2_{d_2}, \underline{n_2}, \underline{\kappa_2}, \underline{l_2}, \underline{f_2})$*, and L_{e_i} for $i = 1, 2$ is a multi-line condition with ends of weight one that is imposed on e_i, see Definition 2.2.19.*

Proof. The ends of Y_1 and Y_2 (see Definition 4.1.28) are of standard directions, i.e. of direction $(1,1), (-1,0)$ and $(0,-1)$ by Corollary 4.1.31. The position of Y_1 and Y_2 in $\mathbb{R}^2$ depends only on the position of the given conditions. In particular, moving the given conditions (while keeping the property of being in general position, see Remark 3.2.6) moves Y_1 and Y_2 as well.

Assume that the given conditions are positioned in such a way that Y_1 and Y_2 intersect only in their ends as shown in Figure 4.10. Define the multi-line conditions L_{e_1} and L_{e_2} with weights one as in Figure 4.10 and consider a rational tropical stable map C_1 that contributes to $N_{\Delta^2_{d_1}}\left(p_{\underline{n_1}}, L_{\underline{\kappa_1}}, L_{e_1}, \lambda_{\underline{l_1}}^{\to e_1}\right)$ and a rational tropical stable map C_2 that contributes to the number $N_{\Delta^2_{d_2}}\left(p_{\underline{n_2}}, L_{\underline{\kappa_2}}, L_{e_2}, \lambda_{\underline{l_2}}^{\to e_2}\right)$. The contracted end of C_i for $i = 1, 2$ that satisfies L_{e_i} is e_i. Let v_i denote the vertex adjacent to e_i for $i = 1, 2$. Notice that $\mathrm{ev}_{e_i}(C_i) \in Y_i$, i.e. C_i satisfies Y_i by definition. Hence v_i is a point in $Y_i \cdot L_{e_i}$ for $i = 1, 2$. Each pair of points (v_1, v_2) is uniquely associated to a point p in $Y_1 \cdot Y_2$, see Figure 4.10. By Corollary 4.1.32 each of the vertices v_i is 3-valent and adjacent to an end of C_i for $i = 1, 2$. Hence (by moving v_1, v_2 along those ends) each pair of rational tropical stable maps (C_1, C_2) as above can be glued to a rational tropical stable map C as in Remark 4.1.30 such that the ends e_1, e_2 are glued to form a bounded edge that is contracted to p. On the other hand, each rational tropical stable map C on the left-hand side of (4.10) can be split into a pair (C_1, C_2) of rational tropical stable maps as above using Construction 4.1.26. Moreover,

$$\mathrm{mult}(C) = \mathrm{mult}(C_1) \cdot \mathrm{mult}(C_2)$$

holds by Proposition 4.2.3 since $\det(M(C_{1,01}))$ and $\det(M(C_{2,10}))$ both vanish by our choice of positions of Y_1 and Y_2.

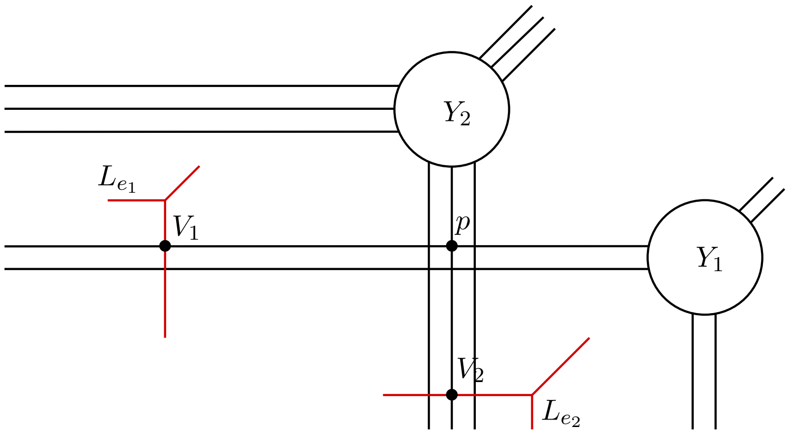


Figure 4.10: The 1-dimensional conditions Y_1 and Y_2 after movement, together with the (multi) line conditions L_{e_1} and L_{e_2}, where $p \in Y_1 \cdot Y_2$ is the point associated to $V_1 \in Y_1 \cdot L_{e_1}$ and $V_2 \in Y_2 \cdot L_{e_2}$.

To finish the proof, we need to see that we can always assume that Y_1 and Y_2 intersect as shown in Figure 4.10, i.e. we want to show that the left-hand side of (4.10) does not depend on the position of Y_1 and Y_2. Let C be a rational tropical stable map contributing to $N_{\Delta^2_{d_1+d_2}}\left(p_{\underline{n_1}}, p_{\underline{n_2}}, L_{\underline{\kappa_1}}, L_{\underline{\kappa_2}}, \lambda_{\underline{l_1}}, \lambda_{\underline{l_2}}, \lambda'_l\right)$ as in Proposition 4.1.1. Notice that $n \geq 1$ by Proposition 4.1.25 since we have a 1/1 edge. The tropical cross-ratio's length $|\lambda'_l|$ is so large such that there is a contracted bounded edge e in C, and C is of splitting type $(\Delta^2_{d_1}, \underline{n_1}, \underline{\kappa_1}, \underline{l_1}, \underline{f_1} \mid \Delta^2_{d_2}, \underline{n_2}, \underline{\kappa_2}, \underline{l_2}, \underline{f_2})$. Consider the cycle Z_i that arises from forgetting the point conditions $p_{\underline{n_i}}$ and the line conditions $L_{\underline{\kappa_i}}$ for $i = 1, 2$ imposed on C. Hence C gives rise to a top-dimensional cell of Z_i, where points in that cell correspond to C together with some movement of the conditions $p_{\underline{n_i}}, L_{\underline{\kappa_i}}$. The proof of Proposition 4.1.1 implies that if $|\lambda'_l|$ is large enough, then the given conditions can be moved in a bounded area B (say $B \subset \mathbb{R}^2$ is a rectangular box) and all rational tropical stable maps that satisfy this moved conditions still have a contracted bounded edge. Moreover, the splitting type of those rational tropical stable maps cannot change since that would require two contracted bounded edges which would contradict that our given conditions are in general position. Since Z_1, Z_2 are balanced, we might choose different positions for our point and line conditions for every splitting type without effecting the overall count. Let $B_1, B_2 \subset B$ be disjoint small rectangular boxes such that B_1 lies in the lower right corner of B and B_2 lies in the upper left corner of B. Move the conditions $p_{\underline{n_1}}, L_{\underline{\kappa_1}}, \lambda_{\underline{l_1}}$ into B_1 and the conditions $p_{\underline{n_2}}, L_{\underline{\kappa_2}}, \lambda_{\underline{l_2}}$ into B_2 while maintaining their property of being in general position, see Remark 3.2.6. By choosing B_1 and B_2 small enough, we can bring Y_1 and Y_2 in the desired position from Figure 4.10. Therefore (4.10) follows. $\square$

Theorem 4.3.4 (General tropical Kontsevich's formula)**.** *Notation from Notation 2.0.1, Definition 4.1.28, 4.3.1, 3.2.3 and Remark 4.1.30 is used. Fix a degree Δ_d^2, point conditions $p_{\underline{n}}$, multi-line conditions $L_{\underline{\kappa}}$ and degenerated tropical cross-ratios $\lambda_{[l]}$ on contracted ends only such that these conditions are in general position. Let λ_l' denote a cross-ratio that degenerates to λ_l.*

(a) If there is at least one point condition, i.e. $p_{\underline{n}} \neq \emptyset$, then the equation

$$\begin{aligned}
&N_{\Delta_d^2}\left(p_{\underline{n}}, L_{\underline{\kappa}}, \lambda_{[l]}\right) = \\
&\sum_{\substack{(\Delta_{d_1}^2,\underline{n_1},\underline{\kappa_1},\underline{l_1},\underline{f_1}|\Delta_{d_2}^2,\underline{n_2},\underline{\kappa_2},\underline{l_2},\underline{f_2})\\ \text{is a } 1/1 \text{ split respecting } \lambda_l'}} N_{\Delta_{d_1}^2}\left(p_{\underline{n_1}}, L_{\underline{\kappa_1}}, L_{e_1}, \lambda_{\underline{l_1}}^{\to e_1}\right) \cdot N_{\Delta_{d_2}^2}\left(p_{\underline{n_2}}, L_{\underline{\kappa_2}}, L_{e_2}, \lambda_{\underline{l_2}}^{\to e_2}\right) \\
&+ \sum_{\substack{(\Delta_{d_1}^2,\underline{n_1},\underline{\kappa_1},\underline{l_1},\underline{f_1}|\Delta_{d_2}^2,\underline{n_2},\underline{\kappa_2},\underline{l_2},\underline{f_2})\\ \text{is a } 2/0 \text{ split respecting } \lambda_l' \text{ and}\\ 3d_1=\#\underline{n_1}+\#\underline{l_1}-\#\underline{f_1}}} N_{\Delta_{d_1}^2}\left(p_{\underline{n_1}}, L_{\underline{\kappa_1}}, \lambda_{\underline{l_1}}^{\to e_1}\right) \cdot N_{\Delta_{d_2}^2}\left(p_{\underline{n_2}}, p_{e_2}, L_{\underline{\kappa_2}}, \lambda_{\underline{l_2}}^{\to e_2}\right) \\
&+ \sum_{\substack{(\Delta_{d_1}^2,\underline{n_1},\underline{\kappa_1},\underline{l_1},\underline{f_1}|\Delta_{d_2}^2,\underline{n_2},\underline{\kappa_2},\underline{l_2},\underline{f_2})\\ \text{is a } 2/0 \text{ split respecting } \lambda_l' \text{ and}\\ 3d_2=\#\underline{n_2}+\#\underline{l_2}-\#\underline{f_2}}} N_{\Delta_{d_1}^2}\left(p_{\underline{n_1}}, p_{e_1}, L_{\underline{\kappa_1}}, \lambda_{\underline{l_1}}^{\to e_1}\right) \cdot N_{\Delta_{d_2}^2}\left(p_{\underline{n_2}}, L_{\underline{\kappa_2}}, \lambda_{\underline{l_2}}^{\to e_2}\right)
\end{aligned} \tag{4.11}$$

holds.

(b) If there are no point conditions, i.e. $p_{\underline{n}} = \emptyset$, then the equation

$$N_{\Delta_d^2}\left(L_{\underline{\kappa}}, \lambda_{[l]}\right) = \sum_{\substack{(\underline{l_1},\underline{f_1}|\underline{l_2},\underline{f_2})\\ \text{is a } 2/0 \text{ split respecting } \lambda_l'}} N_{\Delta_0^2}\left(L_a, L_b, \lambda_{\underline{l_1}}^{\to e}\right) \cdot N_{\Delta_d^2}\left(p, L_{\underline{\kappa}}\backslash\{L_a, L_b\}, \lambda_{\underline{l_2}}^{\to e}\right) \tag{4.12}$$

holds, where the line conditions L_a, L_b are the ones of Lemma 4.1.23.

Moreover, (4.11) *and* (4.12) *give rise to a recursion with two types of initial values:*

(1) The numbers $N_{\Delta_d^2}\left(p_{\underline{n}}\right)$ which tropical Kontsevich's formula (Corollary 4.3.7) provides.

(2) The numbers $N_{\Delta_0^2}\left(L_a, L_b, \lambda_{\underline{l_1}}^{\to e}\right)$ which satisfy

$$N_{\Delta_0^2}\left(L_a, L_b, \lambda_{\underline{l_1}}^{\to e}\right) = \omega(L_a) \cdot \omega(L_b) \cdot \mathrm{mult}_{\mathrm{cr}}(v'), \tag{4.13}$$

where v' denotes the only vertex of the only rational tropical stable map contributing to $N_{\Delta_0^2}\left(L_a, L_b, \lambda_{\underline{l_1}}^{\to e}\right)$ and $\operatorname{mult}_{\mathrm{cr}}(v')$ is its cross-ratio multiplicity, see Definition 3.2.16. Notice that in the special case of $\lambda_{\underline{l_1}}^{\to e} = \emptyset$ we have

$$N_{\Delta_0^2}(L_a, L_b) = \omega(L_a) \cdot \omega(L_b). \tag{4.14}$$

Using Tyomkin's correspondence theorem 2.3.6 (more precisely, Corollary 3.1.20 and Proposition 3.2.7), Theorem 4.3.4 immediately yields the following corollary.

Corollary 4.3.5 (Algebro-geometric general Kontsevich's formula)**.** *Let $N_{\Delta_d^2}^{\mathrm{alg}}\left(p_{\underline{n}}, \mu_{[l]}\right)$ denote the number of plane rational degree d curves over an algebraically closed field of characteristic zero that satisfy point conditions and classical cross-ratios $\mu_{[l]}$ as in Theorem 2.3.6 such that all conditions are in general position. If the tropicalizations of $\mu_{[l]}$ are on contracted ends only, then Theorem 4.3.4 provides a recursive formula to calculate these numbers with initial values as in Theorem 4.3.4.*

Example 4.3.6. We want to give an example of how to compute numbers we are looking for using our general tropical Kontsevich's formula. Say we want to compute $N_{\Delta_2^2}\left(p_{[3]}, L_4, L_5, \lambda_{[2]}\right)$. For degenerated tropical cross-ratios

$$\lambda_1 := \{1,2,3,4\} \quad \text{and} \quad \lambda_2 := \{1,2,3,5\}.$$

Notice that (3.6) is satisfied so your input data makes sense. Recall the conventions we used for labeling ends: in this example, we want to count rational tropical stable maps C of degree Δ_2^2 in $\mathbb{R}^2$ that have 5 contracted ends. A contracted end labeled with i satisfies the point condition p_i for $i = 1,2,3$ and satisfies the multi-line condition L_i for $i = 4,5$. There is no non-contracted end which satisfies no condition. To use Theorem 4.3.4, we need to fix a tropical cross-ratio λ_2' that degenerates to λ_2. We choose

$$\lambda_2' := (12|35).$$

If C splits into C_1, C_2, then by Definition 4.3.1 ends $1,2$ are contracted ends of C_1, i.e. p_1, p_2 are satisfied in C_1, and $3,5$ are contracted ends of C_2, i.e. p_3, L_5 are satisfied in C_2. Therefore λ_1 is satisfied in C_1 such that 4 is a contracted end of C_1 that satisfies L_4. Going through the three cases of different types of splits using (4.1) and (4.2), we see that the only possible splits are the 2/0 splits

$$(\Delta_1^2, p_1, p_2, L_4, \lambda_1 \mid \Delta_1^2, p_3, L_5).$$

Since they only differ in the distribution of the labels l of $\left(\Delta_2^2, l\right)$ among the two degrees $\left(\Delta_1^2, l_1\right)$ and $\left(\Delta_1^2, l_2\right)$ there are $\binom{2}{1}^3 = 8$ of them (for notation, see Construction 4.1.26). Hence part (a) of Theorem 4.3.4 yields

$$N_{\Delta_2^2}\left(p_{[3]}, L_4, L_5, \lambda_{[2]}\right) = 8 \cdot N_{\Delta_1^2}\left(p_1, p_2, L_4, \lambda_1^{\to e_1}\right) \cdot N_{\Delta_1^2}\left(p_3, p_{e_2}, L_5\right),$$

where the rightmost factor can be written as

$$N_{\Delta_1^2}\left(p_3, p_{e_2}, L_5\right) = \omega(L_5) \cdot \underbrace{N_{\Delta_1^2}\left(p_3, p_{e_2}\right)}_{=1}$$

by tropical Bézout's Theorem [AR10].

So it remains to calculate $N_{\Delta_1^2}(p_1,p_2,L_4,\lambda_1^{\to e_1})$. For that, we want to use Theorem 4.3.4 again. A rational tropical stable map C contributing to $N_{\Delta_1^2}(p_1,p_2,L_4,\lambda_1^{\to e_1})$ has 4 contracted ends. A contracted end labeled with i satisfies p_i for $i=1,2$ and L_i for $i=4$. The remaining contracted end is labeled with e_1 and satisfies no point condition. To stick to our convention of labeling ends with natural numbers, we relabel e_1 by 6. Again, fix a tropical cross-ratio $\lambda_1'^{\to e_1}$ that degenerates to $\lambda_1^{\to e_1}=\{1,2,6,4\}$. We choose

$$\lambda_1'^{\to e_1} := (12|46).$$

If C splits into C_1,C_2 then $1,2$ are contracted ends of C_1, i.e. p_1,p_2 are satisfied in C_1, and $4,6$ are contracted ends of C_2, i.e. L_4 is satisfied by C_2 and there is one contracted end, labeled 6, in C_2 that satisfies no condition. As before, we can go through all splits and notice that

$$(\Delta_1^2,p_1,p_2\mid\Delta_0^2,L_4,6)$$

is the only possible split. Hence part (a) of Theorem 4.3.4 yields

$$N_{\Delta_1^2}(p_1,p_2,L_4,\lambda_1^{\to e_1}) = N_{\Delta_1^2}\left(p_1,p_2,L_{e_1'}\right)\cdot N_{\Delta_0^2}\left(L_4,L_{e_2'}\right),$$

where

$$N_{\Delta_1^2}\left(p_1,p_2,L_{e_1'}\right) = \underbrace{\omega(L_{e_1'})}_{=1}\cdot\underbrace{N_{\Delta_1^2}(p_1,p_2)}_{=1}$$

holds by tropical Bézout's Theorem and by definition of $L_{e_1'}$. Moreover,

$$N_{\Delta_0^2}\left(L_4,L_{e_2'}\right)=\omega(L_4)$$

by Theorem 4.3.4.

In total, we calculated

$$N_{\Delta_2^2}\left(p_{[3]},L_4,L_5,\lambda_{[2]}\right) = 8\cdot\omega(L_4)\cdot\omega(L_5)$$

for λ_1,λ_2 defined as above.

We now prove Theorem 4.3.4, discuss the initial values of the recursion Theorem 4.3.4 provides and then proceed with tropical Kontsevich's formula which is a corollary of part (a) of Theorem 4.3.4.

Proof of part (a) of Theorem 4.3.4. Proposition 3.2.7 yields that

$$N_{\Delta_d^2}\left(p_{\underline{n}},L_{\underline{\kappa}},\lambda_{[l]}\right) = N_{\Delta_d^2}\left(p_{\underline{n}},L_{\underline{\kappa}},\lambda_{[l-1]},\lambda_l'\right)$$

for a tropical cross-ratio λ_l' that degenerates to λ_l. Since $N_{\Delta_d^2}\left(p_{\underline{n}},L_{\underline{\kappa}},\lambda_{[l-1]},\lambda_l'\right)$ does not depend on $|\lambda_l'|$ (see Remark 3.1.9), choose it to be large as in Proposition 4.1.1. Hence each rational tropical stable map contributing to $N_{\Delta_d^2}\left(p_{\underline{n}},L_{\underline{\kappa}},\lambda_{[l-1]},\lambda_l'\right)$ has a contracted bounded edge e which can be cut using Construction 4.1.26 and thus gives rise to some splitting type $(\Delta_{d_1}^2,\underline{n_1},\underline{\kappa_1},\underline{l_1},\underline{f_1}\mid\Delta_{d_2}^2,\underline{n_2},\underline{\kappa_2},\underline{l_2},\underline{f_2})$ that respects λ_l'. Therefore

$$N_{\Delta_d^2}\left(p_{\underline{n}},L_{\underline{\kappa}},\lambda_{[l-1]},\lambda_l'\right) = \sum_{\substack{(\Delta_{d_1}^2,\underline{n_1},\underline{\kappa_1},\underline{l_1},\underline{f_1}\mid\Delta_{d_2}^2,\underline{n_2},\underline{\kappa_2},\underline{l_2},\underline{f_2})\\ \text{is a split respecting }\lambda_l'}}\ \sum_C \operatorname{mult}(C), \tag{4.15}$$

where the second sum goes over all rational tropical stable maps C that give rise to the split $(\Delta_{d_1}^2,\underline{n_1},\underline{\kappa_1},\underline{l_1},\underline{f_1}\mid\Delta_{d_2}^2,\underline{n_2},\underline{\kappa_2},\underline{l_2},\underline{f_2})$. Reordering the first sum of (4.15) as in (4.11) and applying Lemmas 4.3.2, 4.3.3 proves part (a) of Theorem 4.3.4. $\square$

Proof of part (b) of Theorem 4.3.4. We use notation from lemmas 4.1.23, 4.1.24 and Proposition 4.1.25. Proposition 3.2.7 yields that

$$N_{\Delta_d^2}\left(L_{\underline{\kappa}}, \lambda_{[l]}\right) = N_{\Delta_d^2}\left(L_{\underline{\kappa}}, \lambda_{[l-1]}, \lambda_l'\right), \tag{4.16}$$

holds. Conclude with Proposition 4.1.25 that each rational tropical stable map contributing to the right-hand side of (4.16) has a contracted bounded edge e which is adjacent to a vertex v' which is in turn adjacent to e_1, e_2. Notice that cutting e yields a 2/0 split. Thus Lemma 4.3.2 gives us equation (4.13). □

Proof of the initial values part of Theorem 4.3.4. Notice that equations (4.11) and (4.12) of Theorem 4.3.4 allow us to successively reduce the number of point, multi-line or degenerated tropical cross-ratio conditions. There are three cases:

(1) We run out of degenerated tropical cross-ratio conditions. Then, if there are point conditions left, tropical Bézout's Theorem [AR10] can be applied to reduce the initial value problem to the numbers $N_{\Delta_d^2}\left(p_{\underline{n}}\right)$ which tropical Kontsevich's formula (Corollary 4.3.7) provides. If there are no point conditions left, then

$$N_{\Delta_d^2}\left(L_{\underline{\kappa}}\right) = 0$$

for all $d \neq 0$ applies. Otherwise $d = 0$, $\#\underline{\kappa} = \#\{a, b\} = 2$ and $\#\underline{f} = 1$ (for notation, see Construction 4.1.26) such that

$$N_{\Delta_0^2}\left(L_{\kappa}\right) = \omega(L_a) \cdot \omega(L_b)$$

holds.

(2) We run out of point conditions. Then (4.12) reduces the initial value problem to calculating $N_{\Delta_0^2}\left(L_a, L_b, \lambda_{\underline{l_1}}^{\to e}\right)$. This can be done via (4.13).

For equation (4.13), notice that each edge of a rational tropical stable map of degree Δ_0^2 must be contracted. Thus there cannot be a bounded edge since all tropical cross-ratios are degenerated. Hence there is exactly one vertex v' in such a rational tropical stable map whose position is determined by the unique point of intersection of L_a and L_b. Therefore there is exactly one rational tropical stable map contributing to $N_{\Delta_0^2}\left(L_a, L_b, \lambda_{\underline{l_1}}^{\to e}\right)$ whose multiplicity is $\omega(L_a) \cdot \omega(L_b) \cdot \mathrm{mult}_{\mathrm{cr}}(v')$ by Proposition 3.2.25.

(3) We run out of multi-line conditions. Then (4.11) can still be applied, so cases (1) and (2) apply.

□

Gathmann and Markwig proved the following tropical Kontsevich's formula [GM08] in order to give a tropical proof of Kontsevich's original formula via Mikhalkin's correspondence theorem [Mik05]. Here, it is a consequence of certain special cases of our general Kontsevich's formula 4.3.4.

Corollary 4.3.7 (Tropical Kontsevich's formula, [GM08])**.** *For $\#\underline{n} = 3d - 1 > 0$ general positioned point conditions the equality*

$$\frac{N_{\Delta_d^2}\left(p_{\underline{n}}\right)}{(d!)^3} = \sum_{\substack{d_1+d_2=d \\ d_1,d_2>0}} \left(d_1^2 d_2^2 \cdot \binom{3d-4}{3d_1-2} - d_1^3 d_2 \cdot \binom{3d-4}{3d_1-1}\right) \frac{N_{\Delta_{d_1}^2}\left(p_{\underline{n_1}}\right)}{(d_1!)^3} \cdot \frac{N_{\Delta_{d_2}^2}\left(p_{\underline{n_2}}\right)}{(d_2!)^3}$$

holds and provides a recursion to calculate $N_{\Delta_d^2}\left(p_{\underline{n}}\right)$ from the initial value $N_{\Delta_1^2}\left(p_1, p_2\right) = 1$.

Remark 4.3.8. The factors $\frac{1}{(d!)^3}, \frac{1}{(d_1!)^3}, \frac{1}{(d_2!)^3}$ in Corollary 4.3.7 appear since ends of our rational tropical stable maps are labeled. Let $\Delta_d^{2,\text{unlab}}$ denote the unlabeled degree associated to Δ_d^2. For each rational tropical stable map to $\mathbb{R}^2$ of degree Δ_d^2 there are $(d!)^3$ ways to label its non-contracted ends. Thus $\frac{N_{\Delta_d^2}(p_{\underline{n}})}{(d!)^3}$ equals the number of rational tropical stable maps to $\mathbb{R}^2$ of degree $\Delta_d^{2,\text{unlab}}$ that satisfy the point conditions $p_{\underline{n}}$. Often tropical Kontsevich's formula is stated for rational tropical stable maps of unlabeled degrees such that the factors $\frac{1}{(d!)^3}, \frac{1}{(d_1!)^3}, \frac{1}{(d_2!)^3}$ disappear.

Notice that the following proof of tropical Kontsevich's formula that we present here is inspired by the one given by Gathmann and Markwig in [GM08].

Proof of Corollary 4.3.7. Let $p_{\underline{n}}$ be point conditions, let L_a, L_b be line conditions, i.e. multi-lines with weights $\omega(L_a) = \omega(L_b) = 1$ and let $\lambda = \{L_a, L_b, p_c, p_d\}$ be a degenerated tropical cross-ratio, where $p_c, p_d \in p_{\underline{n}}$ are points and the labels are chosen in such a way that $a < b < c < d$.

Consider the tropical cross-ratio $\lambda' := (L_a p_c | L_b p_d)$ that degenerates to λ. We claim that (4.11) reduces to

$$N_{\Delta_d^2}\left(p_{\underline{n}}, L_a, L_b, \lambda\right) = \sum_{\substack{(\Delta_{d_1}^2, \underline{n_1} | \Delta_{d_2}^2, \underline{n_2}) \\ \text{is a 1/1 split respecting } \lambda'}} N_{\Delta_{d_1}^2}\left(p_{\underline{n_1}}, L_a, L_{e_1}\right) \cdot N_{\Delta_{d_2}^2}\left(p_{\underline{n_2}}, L_b, L_{e_2}\right) \tag{4.17}$$

in our case. Since we only have two line conditions and no contracted ends without point or line conditions, each split we deal with can be written as $(\Delta_{d_1}^2, \underline{n_1} \mid \Delta_{d_2}^2, \underline{n_2})$ since λ' determines the distribution of L_a and L_b in each possible split respecting λ'. To show the claim, it remains to show that the last two sums of (4.11) vanish. For that, it is because of symmetry sufficient to show that the second sum vanishes. Let $N_{\Delta_{d_1}^2}\left(p_{\underline{n_1}}, L_a\right) \cdot N_{\Delta_{d_2}^2}\left(p_{\underline{n_2}}, p_{e_2}, L_b\right)$ be a factor of the second sum. Let C_1 be a rational tropical stable map that contributes to $N_{\Delta_{d_1}^2}\left(p_{\underline{n_1}}, L_a\right)$ and let C_2 be a rational tropical stable map that contributes to $N_{\Delta_{d_2}^2}\left(p_{\underline{n_2}}, p_{e_2}, L_b\right)$. Using Remark 4.1.30, C_1 and C_2 can be glued to form a rational tropical stable map C which has a contracted bounded edge e. Since our split was a 2/0 split, the 3-valent vertex v_1 of C that is adjacent to e is fixed. Hence there is a contracted end satisfying a point condition that is adjacent to v_1. Thus there is another contracted end adjacent to v_1 which needs to satisfy either a point or a line condition which is a contradiction since all conditions are in general position.

Now consider the tropical cross-ratio $\tilde{\lambda}' := (L_a L_b | p_c p_d)$ that degenerates to λ. We claim that (4.11) reduces to

$$\begin{aligned} N_{\Delta_d^2}\left(p_{\underline{n}}, L_a, L_b, \lambda\right) = & \sum_{\substack{(\Delta_{d_1}^2, \underline{n_1} | \Delta_{d_2}^2, \underline{n_2}) \\ \text{is a 1/1 split respecting } \tilde{\lambda}'}} N_{\Delta_{d_1}^2}\left(p_{\underline{n_1}}, L_a, L_b, L_{e_1}\right) \cdot N_{\Delta_{d_2}^2}\left(p_{\underline{n_2}}, L_{e_2}\right) \\ & + N_{\Delta_0^2}\left(L_a, L_b\right) \cdot N_{\Delta_d^2}\left(p_{\underline{n}}, p_{e_2}\right) \end{aligned} \tag{4.18}$$

in this case. As before, splits can be written as $(\Delta_{d_1}^2, \underline{n_1} \mid \Delta_{d_2}^2, \underline{n_2})$. The last sum of (4.11) vanishes with the same arguments from before. It remains to see that the second sum of (4.11) equals the product $N_{\Delta_0^2}\left(L_a, L_b\right) \cdot N_{\Delta_d^2}\left(p_{\underline{n}}, p_{e_2}\right)$. If $d_1 > 0$ and we consider a product contributing to the last sum, then the same arguments from before show that this product vanishes. Hence the only remaining contribution from the second sum that is possible is $N_{\Delta_0^2}\left(L_a, L_b\right) \cdot N_{\Delta_d^2}\left(p_{\underline{n}}, p_{e_2}\right)$.

Notice that there are no degenerated tropical cross-ratios on the right-hand sides of (4.17) and (4.18) such that tropical Bézout's Theorem [AR10] yields

$$\begin{aligned}
&\sum_{\substack{(\Delta^2_{d_1},\underline{n_1}|\Delta^2_{d_2},\underline{n_2})\\ \text{is a 1/1 split respecting } \lambda'}} d_1^2 N_{\Delta^2_{d_1}}\left(p_{\underline{n_1}}\right)\cdot d_2^2 N_{\Delta^2_{d_2}}\left(p_{\underline{n_2}}\right)\\
=&\sum_{\substack{(\Delta^2_{d_1},\underline{n_1}|\Delta^2_{d_2},\underline{n_2})\\ \text{is a 1/1 split respecting } \tilde{\lambda}'}} d_1^3 N_{\Delta^2_{d_1}}\left(p_{\underline{n_1}}\right)\cdot d_2 N_{\Delta^2_{d_2}}\left(p_{\underline{n_2}}\right) + N_{\Delta^2_0}\left(L_a,L_b\right)\cdot N_{\Delta^2_d}\left(p_{\underline{n}},p_{e_2}\right)
\end{aligned}$$

since $\omega(L_a)=\omega(L_b)=\omega(L_{e_1})=\omega(L_{e_2})=1$. Using $N_{\Delta^2_0}(L_a,L_b)=\omega(L_a)\omega(L_b)=1$, we obtain

$$\begin{aligned}
N_{\Delta^2_d}\left(p_{\underline{n}},p_{e_2}\right)=&\\
&\sum_{\substack{(\Delta^2_{d_1},\underline{n_1}|\Delta^2_{d_2},\underline{n_2})\\ \text{is a 1/1 split respecting } \lambda'}} d_1^2 d_2^2 N_{\Delta^2_{d_1}}\left(p_{\underline{n_1}}\right) N_{\Delta^2_{d_2}}\left(p_{\underline{n_2}}\right)\\
&-\sum_{\substack{(\Delta^2_{d_1},\underline{n_1}|\Delta^2_{d_2},\underline{n_2})\\ \text{is a 1/1 split respecting } \tilde{\lambda}'}} d_1^3 d_2 N_{\Delta^2_{d_1}}\left(p_{\underline{n_1}}\right) N_{\Delta^2_{d_2}}\left(p_{\underline{n_2}}\right).
\end{aligned}$$

Since all conditions we started with are in general position

$$3d=\#\underline{n}+1+1$$

holds, i.e. each choice of $\underline{n_1},\underline{n_2}$ in a split for fixed d_1,d_2 is a choice of distributing the remaining $3d-4$ points. There are $\binom{3d-4}{3d_1-2}$ choices if $p_c\in\underline{n_1}$ and $\binom{3d-4}{3d_1-1}$ choices if $p_c,p_d\in\underline{n_2}$. Using $3d_i=\#\underline{n_i}+1$ provides the following index for the sum (notation from Construction 4.1.26 is used), namely

$$N_{\Delta^2_d}\left(p_{\underline{n}}\right)=\sum_{\substack{(\Delta^2_{d_1}|\Delta^2_{d_2}):\\ \Delta^2_d=\Delta^2_{d_1}\cup\Delta^2_{d_2},\\ d_1,d_2>0}}\left(d_1^2d_2^2\cdot\binom{3d-4}{3d_1-2}-d_1^3d_2\cdot\binom{3d-4}{3d_1-1}\right)N_{\Delta^2_{d_1}}\left(p_{\underline{n_1}}\right)N_{\Delta^2_{d_2}}\left(p_{\underline{n_2}}\right).$$

For each choice of $d_1,d_2>0$ there are

$$\binom{d}{d_1}^3=\frac{d!^3}{d_1!^3(d-d_1)!^3}=\frac{d!^3}{d_1!^3\cdot d_2!^3}$$

summands $(\Delta^2_{d_1}\mid\Delta^2_{d_2})$ with $\Delta^2_d=\Delta^2_{d_1}\cup\Delta^2_{d_2}$ associated to it. Hence

$$N_{\Delta^2_d}\left(p_{\underline{n}}\right)=\sum_{\substack{d_1+d_2=d\\ d_1,d_2>0}}\left(d_1^2d_2^2\cdot\binom{3d-4}{3d_1-2}-d_1^3d_2\cdot\binom{3d-4}{3d_1-1}\right)\cdot\frac{d!^3}{d_1!^3\cdot d_2!^3}\cdot N_{\Delta^2_{d_1}}\left(p_{\underline{n_1}}\right)N_{\Delta^2_{d_2}}\left(p_{\underline{n_2}}\right)$$

holds, which yields the desired formula. □

Further generalizations The same methods Gathmann and Markwig used to prove tropical Kontsevich's formula [GM08] also yield a recursive formula for counting rational tropical stable maps of bidegree (d_1,d_2) (i.e. with ends of directions $(\pm1,0),(0,\pm1)$) to $\mathbb{R}^2$ that satisfy point conditions, see [FM11]. Analogously, the methods developed here yield a recursive formula for rational tropical stable maps to $\mathbb{R}^2$ of bidegree (d_1,d_2) that satisfy point conditions, *degenerated* multi-line conditions and degenerated tropical cross-ratio conditions.

Chapter 5

Constructive approach: Constructing tropical curves algorithmically

In this chapter, a *cross-ratio lattice path algorithm* is presented. It calculates $N_\Delta\left(p_{\underline{n}}, \lambda_{[l]}\right)$ which is the number of rational tropical stable maps that satisfy point conditions and degenerated tropical cross-ratio conditions, when Δ is a degree in $\mathbb{R}^2$. Notice that these numbers can also be calculated recursively using the general tropical Kontsevich's formula 4.3.4 if all degenerated tropical cross-ratios are on contracted ends only. The benefits that the cross-ratio lattice path algorithm offers are that it takes degenerated tropical cross-ratios into account that are not on contracted ends only and that it is constructive. That is, it allows us to explicitly construct all rational tropical stable maps to $\mathbb{R}^2$ that contribute to $N_\Delta\left(p_{\underline{n}}, \lambda_{[l]}\right)$ for a specific configuration of the points $p_{\underline{n}}$. The cross-ratio lattice path algorithm therefore answers the leading question (Q3) which asks whether contributions to $N_\Delta\left(p_{\underline{n}}, \lambda_{[l]}\right)$ can be constructed. Moreover, the cross-ratio lattice path algorithm works in arbitrary compact toric surfaces, i.e. it provides an answer to question (Q4).

Dual subdivision. Let C be a rational tropical stable map of degree Δ to $\mathbb{R}^2$. Let $\Sigma(\Delta)$ denote the lattice polytope associated to Δ (see Remark 2.2.5). If C is *simple* (which we can always assume, see Remark 5.2.2), then C induces a subdivision of $\Sigma(\Delta)$ into lattice polytopes the following way: edges locally around a vertex v_i of C give rise to a local degree Δ_i which corresponds to a lattice polytope $\Sigma(\Delta_i)$. Gluing these polytopes together with respect to the structure of C, which is a polyhedral complex, yields a a subdivision of $\Sigma(\Delta)$. It is called *dual subdivision* of C and is denoted by $\mathcal{S}_C$. See Figure 5.1 for an example of a dual subdivision. For more details about dual subdivision, we refer to [Mik03, Mik05].

Original lattice path algorithm. Before diving into technical details, we shortly recall the original lattice path algorithm introduced by Mikhalkin in [Mik03, Mik05]. The original lattice path algorithm determines the number $N_\Delta\left(p_{[n]}\right)$ of rational tropical stable maps of a degree Δ in $\mathbb{R}^2$ that satisfy general positioned point conditions $p_{[n]}$. It does so by explicitly constructing the images of these rational tropical stable maps in $\mathbb{R}^2$ for a specific configuration of points. To obtain a suitable point configuration, pick points $p_{[n]}$ in general position linearly ordered on a line θ with a small negative slope such that distances of consecutive points grow, i.e.

$$|p_i - p_{i-1}| << |p_{i+1} - p_i|.$$

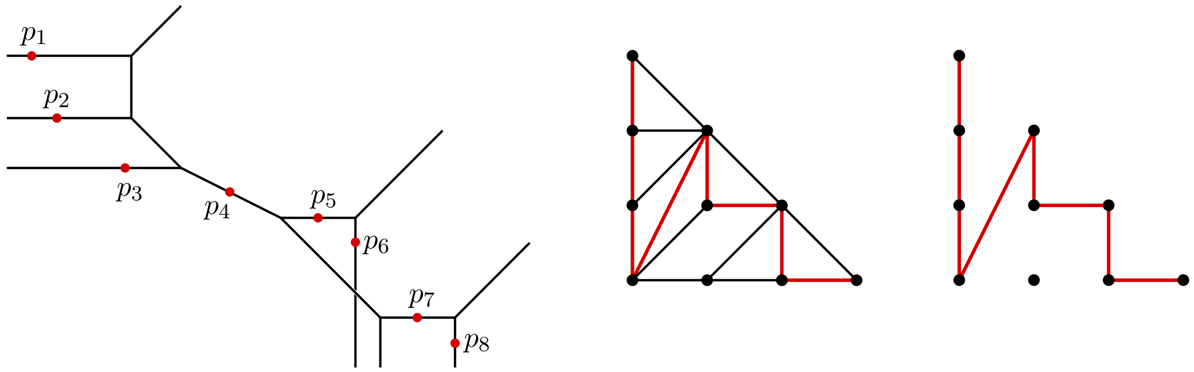


Figure 5.1: From left to right: A rational tropical stable map of degree Δ_3^2 satisfying the point conditions $p_{[8]}$, its associated dual subdivision and its associated lattice path in bold red. Although $p_{[8]}$ are not lying on a line with small negative slope, they can be moved into this position without effecting the combinatorial type (resp. the dual subdivision) of the rational tropical stable map drawn. We just draw the points this way to get a better picture.

Let C be a rational tropical stable map of degree Δ to $\mathbb{R}^2$ that satisfies these point conditions and let $\mathcal{S}_C$ be its dual subdivision of the polytope $\Sigma(\Delta)$ (see Remark 2.2.5). Using that the point conditions are in general position, it can be achieved that $\mathcal{S}_C$ consists only of triangles and parallelograms since C is 3-valent (see Remark 5.2.2). Hence each contracted end x_i of C that satisfies a point condition p_i is dual to an edge a_i of $\mathcal{S}_C$. A crucial observation is that the set $\mathcal{A} := \{a_i \mid i \in [n]\}$ forms a path in $\mathcal{S}_C$, a so-called *lattice path* with respect to the chosen line θ on which $p_{[n]}$ lie. Figure 5.1 provides an example of a rational tropical stable map of degree Δ_3^2 that satisfies eight point conditions, its dual subdivision and its associated lattice path.

The idea of the original lattice path algorithm is to go the other way round: start with a lattice path $\mathcal{A}$ inside $\Sigma(\Delta)$ and reconstruct all rational tropical stable maps C that satisfy $p_{[n]}$ and yield the given lattice path $\mathcal{A}$. To do so, construct all possible dual subdivisions of $\Sigma(\Delta)$ by starting with the lattice path $\mathcal{A}$ and then recursively fill in missing polytopes. The original lattice path algorithm provides the necessary set of rules which govern how triangles and parallelograms can be filled in. For more details about the original lattice path algorithm, we refer to [Mik03, Mik05].

5.1 Cross-ratio lattice path algorithm

We now want to generalize the original lattice path algorithm to rational tropical stable maps to $\mathbb{R}^2$ that satisfy point conditions and degenerated tropical cross-ratio conditions. Notice that degenerated tropical cross-ratios lead to vertices of valence > 3, which means that nor there are only triangles and parallelograms in our subdivisions, neither $\mathcal{A}$ needs to be a path (i.e. a collection of connected edges). We overcome these technical problems by considering a wider range of polytopes and carefully adapting the rules of filling them in when completing a path-like $\mathcal{A}$ to an appropriate subdivision. Moreover, a consequence of having vertices of valence > 3 is that edges of a rational tropical stable map may be mapped onto vertices in $\mathbb{R}^2$. To deal with this problem, our polytopes are equipped with additional discrete data, see Figure 5.4.

5.1.1 Building blocks

The building blocks which form subdivisions later on are now introduced. Notice that all additional data polytopes are equipped with is discrete. Notice that the following definition of *edge* is only used in the current chapter and is different from the definition of an edge of a rational tropical stable map.

Definition 5.1.1 (Segments and labeled polytopes)**.** Notation 2.0.1 is used.

- An *edge* E is a 1-dimensional lattice polytope in $\mathbb{R}^2$ consisting of one 1-dimensional face and two 0-dimensional faces. A *labeled edge* is a tuple $\left(E, \tau^E\right)$, where τ^E is a multiset of $m > 0$ elements denoted by $\tau_1^E, \dots, \tau_m^E$ in $\mathbb{N}_{>0}$ such that $\sum_i \tau_i^E = |E|$, where $|E|$ denotes the lattice length of E. We refer to τ^E as *labeling* of E and to $\tau_{[m]}^E$ as *labels* of E.
- In particular, a labeled edge $\left(E, \tau^E\right)$ with $\tau^E = \{n\}$ for some $n \in \mathbb{N}_{>0}$ is called a *segment.*
- Let P be a lattice polytope in $\mathbb{R}^2$ where each of its e facets is a labeled edge. Denote the labeling of an edge E^j of P by τ^j. Then (P, τ) with $\tau = \left(\tau^1, \dots, \tau^e\right)$ is called a *labeled polytope.*

Definition 5.1.2 (Minkowski labeled polytopes)**.** Notation 2.0.1 is used. Let P be the Minkowski sum of a labeled polytope $\tilde{P} \subset \mathbb{R}^2$ that is either 0-dimensional or 2-dimensional and segments $S_1, \dots, S_r$ such that each segment is parallel to an edge of $\tilde{P}$ and P is 2-dimensional. Note that if $\tilde{P}$ is a point, then every segment is by definition parallel to it. Moreover, we require that if $\tilde{P}$ is 0-dimensional, then there are two segments $S_{i_1}, S_{i_2} \in S_{[r]}$ such that all other Minkowski summands of P are parallel to S_{i_1} or S_{i_2}. Let E be an edge of P and denote by $F_{[k]}$ edges of the Minkowski summands $\tilde{P}, S_{[r]}$ that contribute to E. If τ^{F_i} is the labeling of F_i, then we define τ^E to be the multiset

$$\tau^E := \tau^{F_1} \dot{\cup} \dots \dot{\cup} \tau^{F_k}.$$

A pair (P, τ) of such a polytope P with e edges $E^{[e]}$ and a tuple of multisets $\tau = \left(\tau^{E^1}, \dots, \tau^{E^e}\right)$, where τ^{E^i} is defined above, together with maps that *match* labels to the summands they come from

$$f_P \mid_E : \tau^E \to \{\tilde{P}, S_1, \dots, S_r\}$$

such that if $f_P \mid_E (t) = A \in \{\tilde{P}, S_1, \dots, S_r\}$, then $t \in \tau^{F_i}$ for $F_i \subset A$, is called a *Minkowski labeled polytope.*

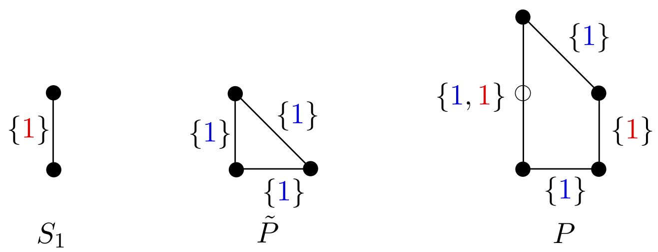


Figure 5.2: From left to right: A Segment S_1 and a 2-dimensional labeled polytope $\tilde{P}$ whose Minkowski sum forms the labeled polytope P on the right. The colors indicate the matching of labelings of P_1, P_2 to their Minkowski summands.

Notation 5.1.3. The non-segment Minkowski summand of a Minkowski labeled polytope P is always denoted by $\tilde{P}$ as in Definition 5.1.2.

Definition 5.1.4 (Valid Polytopes and pointed segments)**.** Notation 5.1.3 is used.

- A Minkowski labeled polytope P is called *k-marked* if $\tilde{P}$ has e edges E^j with labelings τ^j such that $\sum_{j=1}^{e} \#\tau^j = 3 + k$ holds, where $\#\tau^j \in \mathbb{N}_{>0}$ is the number of entries of τ^j. If $k = 0$ or $\tilde{P}$ is 0-dimensional, then P is called *unmarked*.
- A Minkowski labeled polytope is called *valid polytope* if it is either unmarked or k-marked. Two valid polytopes that share an edge E are *compatible* if their labelings of E coincide.
- Let $\tilde{P}$ be a 1-dimensional polytope where each side of its edge E is equipped with a labeling. The Minkowski sum of $\tilde{P}$ with segments $S_1, \ldots, S_r$ parallel to it, where each summand contributes a label to the two labelings of E as in Definition 5.1.2 is called a *pointed segment*. If $\tilde{P}$ is 0-dimensional, then it is called a non-pointed segment (all S_i are then parallel). The notion of compatibility extends to (non-)pointed segments as well: If a valid polytope and a (non-)pointed segment share an edge, then they are compatible if their labelings on this (side of the) edge coincide. We can refer to a (non-)pointed segment as k-marked as above.

5.1.2 Coloring

The building blocks introduced in the last section are now enhanced with colors. Later, they help us decide whether a rational tropical stable map which we constructed is fixed by the given conditions or not.

Definition 5.1.5 (Coloring)**.** Notation 5.1.3 is used. A *coloring* of a labeled polytope P is a 2-coloring of all of its labels on each of its edges. The two colors are called *fixed* and *free*. A colored polytope is called *free* (or *fixed*) if it is monochromatic of the color free (or fixed). Given a colored Minkowski labeled polytope P, we say that exactly $\tilde{P}$ is fixed if all labels associated to $\tilde{P}$ are colored fixed and the rest is colored free.

Algorithm 5.1.6 (Adjusting colors of two compatible polytopes.)**.** Notation 5.1.3 is used. Let P_1, P_2 be two colored polytopes that are compatible and denote their shared edge by E with labelings $\tau^E_{P_1}, \tau^E_{P_2}$. Let $f_{P_1}|_E, f_{P_2}|_E$ be maps as in Definition 5.1.2 and let $g : \tau^E_{P_1} \to \tau^E_{P_2}$ be a bijective map such that $g(t) = t$ for all $t \in \tau^E \cap \mathbb{N}_{>0}$. Let $t \in \tau^E_{P_1}$ be a colored label of E in P_1 and let $g(t)$ be its image under g in $\tau^E_{P_2}$. When comparing and *adjusting* the colors of t and $g(t)$, we follow the slogan "fixed wins":

(1) If t is colored fixed and $g(t)$ is colored fixed, we leave the colors the way they are.

(2) If t is colored fixed and $g(t)$ is colored free, we change $g(t)$ to fixed. When changing $g(t)$ to fixed, we check whether all other labels coming from $f_{P_2}|_E\,(g(t))$ are fixed. If this is not the case, then change them to fixed if $f_{P_2}|_E\,(g(t))$ is a segment. If $f_{P_2}|_E$ associates $g(t)$ to $\tilde{P}_2$, then change the labels associated to $\tilde{P}_2$ to fixed if exactly two of the labels associated to $\tilde{P}_2$ are fixed (where $g(t)$ is one of them).

(3) If t is colored free and $g(t)$ is colored fixed, then do the same as in (2) but with the roles of $t, g(t)$ and P_1, P_2 exchanged.

(4) If t is colored free and so is $g(t)$, then do nothing.

We repeat this procedure using different labels in $\tau_{P_1}^E$ until no color of labels of P_1, P_2 can be changed according to the rules above. Note that this algorithm terminates since colors can only be changed from free to fixed.

Algorithm 5.1.7 (Adjusting colors of a set of polytopes)**.** Let $P_{[z]}$ be a finite set of colored polytopes, where two polytopes are compatible if they share an edge. Go through all pairs of compatible polytopes of $P_{[z]}$ and adjust their colors according to Algorithm 5.1.6. Repeat this procedure until no colors can be changed. This algorithm terminates because we only allow changing a color from free to fixed, following the slogan that fixed wins.

Note that the notion of coloring and adjusting colors extends to (non-)pointed segments.

5.1.3 Cross-ratio lattice paths and subdivisions

We are now ready to define cross-ratio lattice paths. The following two definitions are similar to definitions of lattice paths in [Mik03] and [MR09]. Having these lattice paths, Construction 5.1.11 can then be used to construct subdivisions from lattice paths. It turns out, however, that this gives us too many subdivisions such that some of them have to be sorted out.

Definition 5.1.8 (Lattice path)**.** Fix θ to be a linear map of the form

$$\theta : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto x - \epsilon y,$$

where ϵ is a small irrational number. A path $\gamma : [0, n] \to \mathbb{R}^2$ is called a *lattice path* if $\gamma\ |_{[j-1,j]}$ for $j \in [n]$ is an affine-linear map and $\gamma(j) \in \mathbb{Z}^2$ for all $j = 0, \dots, n$. For $j \in [n]$, we call $\gamma\ |_{[j-1,j]}\ ([j-1, j])$ a *step* (the j-th step) of the lattice path γ. A lattice path is called *θ-increasing* if $\theta \circ \gamma$ is strictly increasing. If every step in a lattice path is a labeled edge, the lattice path is called *labeled lattice path.*

Definition 5.1.9 (Cross-ratio lattice path)**.** Notation 5.1.3 is used. Let Δ be a degree in $\mathbb{R}^2$ and let $\Sigma(\Delta)$ be its associated polytope (Remark 2.2.5). Let $n \in \mathbb{N}_{>0}$. Let $\mathcal{A}$ be an ordered set $P_{[n+z]}$ of colored polytopes in $\Sigma(\Delta)$ such that there are polytopes $\{P_{i_1}, \dots, P_{i_n}\} \subset \mathcal{A}$ such that P_{i_j} is a pointed segment or a valid polytope such that $\tilde{P}_{i_j}$ is fixed and not 0-dimensional for $j \in [n]$. The other polytopes in $\mathcal{A} \backslash \{P_{i_1}, \dots, P_{i_n}\}$ are non-pointed segments that are colored free. The set $\mathcal{A}$ is called a *cross-ratio lattice path* if the following conditions are satisfied:

(1) Two polytopes $P_i, P_j \in \mathcal{A}$ intersect in at most one point.

(2) If an edge E of a polytope $P_i \in \mathcal{A}$ lies in the boundary $\partial\Sigma(\Delta)$ of $\Sigma(\Delta)$, then it is labeled by $\tau^E = (1, \dots, 1)$.

(3) Let π_x denote the projection of $\mathbb{R}^2$ onto the x-axis. There are sets γ_+, γ_- of edges of $P_1, \dots, P_{n+z}$ such that γ_+, γ_- form θ-increasing labeled lattice paths, $\gamma_+ \cup \gamma_-$ is the set of all edges of $P_1, \dots, P_{n+z}$ and for all $x \in \pi_x(\Sigma(\Delta))$ and all $E_+ \in \gamma_+, E_- \in \gamma_-$ such that there are points $(x, y_+) \in E_+ \subset \mathbb{R}^2, (x, y_-) \in E_- \subset \mathbb{R}^2$ the inequality $y_+ \geq y_-$ holds (see Figure 5.3).

(4) The order of the polytopes $P_{[n+z]}$ agrees with the obvious order given by γ_+ and γ_-, respectively.

(5) Let p and q be the points in $\Sigma(\Delta)$ where $\theta\ |_{\Sigma(\Delta)}$ reaches its minimum (resp. maximum), then $p = \gamma_+(0) = \gamma_-(0)$ and $q = \gamma_+(n_+) = \gamma_-(n_-)$, where $\gamma_+ : [0, n_+] \to \mathbb{R}^2$ and $\gamma_- : [0, n_-] \to \mathbb{R}^2$ are defined as above.

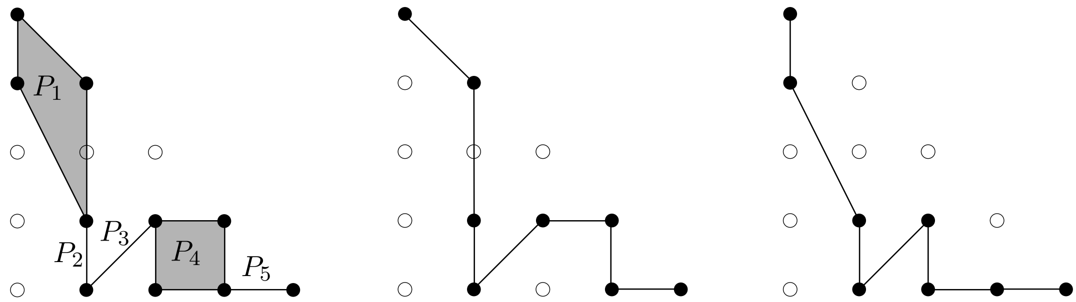


Figure 5.3: Consider the polytope $\Sigma\left(\Delta_4^2\right)$ which is a 2-simplex stretched to side length 4. From left to right: $\mathcal{A} = \{P_1, \ldots, P_5\}, \gamma_+, \gamma_-$ in $\Sigma\left(\Delta_4^2\right)$.

Convention 5.1.10. Throughout the rest of this section, we fix a degree Δ in $\mathbb{R}^2$ such that all vectors in Δ are of weight one, see Definition 2.2.3. Moreover, fix point conditions $p_{[n]}$ and degenerated tropical cross-ratio conditions $\lambda_{[l]}$ in general position such that

$$\#\Delta - 1 = n + l$$

holds, see Definition 3.2.4. That is, each contracted end satisfies a point condition.

Construction 5.1.11 (Constructing subdivisions of $\Sigma(\Delta)$ from $\mathcal{A}$)**.** Let $\mathcal{A}$ be a cross-ratio lattice path in the polytope $\Sigma(\Delta)$ with $\#\mathcal{A} = n+z$ for some $z \in \mathbb{N}$ such that $z \leq \#\left(\Sigma(\Delta) \cap \mathbb{Z}^2\right)$. Let γ_+ be the associated labeled lattice path from Definition 5.1.9.

Let $\gamma_+(j)$ and $\gamma_+(j+1)$ be the j-th and the $(j+1)$-th labeled edge of γ_+ that form the first left turn. Fill up this left turn with a valid polytope $P \subset \Sigma(\Delta)$ that is colored free, whose edges that equal $\gamma_+(j)$ and $\gamma_+(j+1)$ are compatible with $\gamma_+(j)$ and $\gamma_+(j+1)$ and if P shares other edges with our polytopes, it should there be compatible, too. Whenever two compatible labeled edges with labelings τ^E come together, we choose a bijective map $g : \tau^E \to \tau^E$ such that $g(t) = t$ for all $t \in \tau^E \cap \mathbb{N}_{>0}$. Moreover, we use Algorithm 5.1.7 to adjust the colors of the set of polytopes we have so far. If P shares an edge E with the boundary $\partial\Sigma(\Delta)$, then we require $\tau^E = (1, \ldots, 1)$ and we choose a bijective map $g' : \tau^E \to M$, where M is a submultiset of the labels of the degree Δ that are associated to vectors orthogonal (and pointing away from $\Sigma(\Delta)$) to E. When another polytope P' shares an edge with $\partial\Sigma(\Delta)$, then we choose M' in the set of labels of Δ minus M. In the same way the right turns of γ_- can be filled up.

Repeating these steps (i.e. fill up the first left (resp. right) turn), a subdivisions of $\Sigma(\Delta)$ is obtained if and only if

$$\Sigma(\Delta) = \mathcal{A} \cup \bigcup_{P} \{P\},$$

where the union runs over all valid polytopes P used to fill up turns during the process described above. The cells of such a subdivision are valid polytopes which are compatible and connected via maps called g above. Such a subdivision is called a *lattice path subdivision of* $\mathcal{A}$ if all polytopes are colored fixed. The set of all lattice path subdivisions of $\mathcal{A}$ is denoted by $\mathcal{S}_0(\mathcal{A})$.

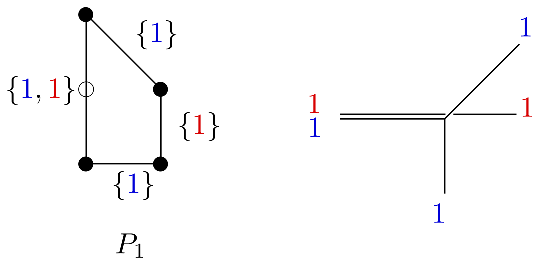


Figure 5.4: On the left is the Minkowski labeled polytope P introduced in Figure 5.2 and on the right is its dual rational tropical stable map.

Construction 5.1.12 (Dual tropical curve)**.** Notation 5.1.3 is used. Let $\mathcal{S} \in \mathcal{S}_0(\mathcal{A})$ be a lattice path subdivision. We want to construct the dual tropical stable map $C_{\mathcal{S}}$ to $\mathcal{S}$. For that, draw a k-valent vertex v for every k-marked ($k > 0$) polytope P in $\mathcal{S}$ and an edge passing through this vertex for every segment of P (the edge is orthogonal to the segment). An edge e adjacent to v is dual (and orthogonal) to an edge E of $\tilde{P}$, that is, the weight of e is given by an entry of the labeling τ^E of E. The weight of an edge passing through v is given by the label of its associated segment that is dual to this edge. If two polytopes $P, Q \in \mathcal{S}_0(\mathcal{A})$ share an edge E with labeling τ^E, we connect the edge associated to τ_i^E in P with the edge associated to $g\left(\tau_i^E\right)$ in Q for all i, where g is a map as in Construction 5.1.11. Moreover, if $P \in \mathcal{A}$ and P is neither a pointed segment nor a non-pointed segment, then add a contracted end (which has to satisfy a point condition) to the vertex dual to $\tilde{P}$. If $P \in \mathcal{A}$ and P is a pointed segment, then the edges dual to the labelings associated to $\tilde{P}$ meet in one vertex which is in addition adjacent to a contracted end (which has to satisfy a point condition). In this way, a combinatorial type $C_{\mathcal{S}}$ is obtained. The general construction of tropical stable maps dual to lattice paths (see [Mik05]) and the fact that all polytopes are fixed implies that for given points $p_{[n]}$ in general position linearly ordered on a line with a small negative slope such that distances grow ($|p_i - p_{i-1}| << |p_{i+1} - p_i|$) there is exactly one tropical stable map whose combinatorial type equals $C_{\mathcal{S}}$. Moreover, this tropical stable map satisfies the point conditions $p_{[n]}$. Hence $C_{\mathcal{S}}$ can uniquely be turned into a tropical stable map.

Since we are only interested in rational tropical stable maps, we need to remove subdivisions whose dual tropical stable maps are reducible. The set of lattice path subdivisions for a given cross-ratio lattice path $\mathcal{A}$ which are dual to irreducible tropical stable maps is denoted by $\mathcal{S}_1(\mathcal{A})$.

Definition 5.1.13. Notation 5.1.3 is used. Let Λ denote the union of all given degenerated tropical cross-ratio conditions $\lambda_{[l]}$, i.e.

$$\Lambda := \bigcup_{j \in [l]} \lambda_j.$$

Let $\mathcal{S}$ be a lattice path subdivision in $\mathcal{S}_1(\mathcal{A})$ and let P be a valid polytope or a pointed segment in $\mathcal{S}$. Consider the summand $\tilde{P}$ of P and define for all entries $\tau_1, \ldots, \tau_m$ of labelings of edges of P associated to $\tilde{P}$ the sets $\Lambda(P, i) \subset \Lambda$ of points and ends appearing in the tropical cross-ratios $\lambda_{[l]}$ that can be reached from P via τ_i. That is, the elements of $\Lambda(P, i)$ are obtained with the following procedure:

- If the edge E of P where τ_i appears is contained in the boundary $\partial\Sigma(\Delta)$, then its dual edge is a labeled end determined by $g(\tau_i)$ (Construction 5.1.11), and we add it to $\Lambda(P, i)$.

- If there is a non-pointed segment S in $\mathcal{A}$ such that P and S share an edge E such that τ_i appears in τ^E, then there is exactly one τ_i' on the other side of the edge of S that is mapped to the same segment of S as τ_i. Continue with τ_i'.
- Else, there is a valid polytope (or a pointed segment) Q in $\mathcal{S}$ such that $Q \neq P$ and P, Q share an edge E such that τ_i appears in τ^E. Then either:
  - τ_i is mapped to $\tilde{Q}$ (via the map $f_Q \mid_E$ from Definition 5.1.2) and $Q \notin \mathcal{A}$, then continue with all other labels mapped to $\tilde{Q}$ instead of τ_i.
  - τ_i is mapped to $\tilde{Q}$ and $Q = P_{t_j} \in \{P_{t_1}, \dots, P_{t_n}\} \subset \mathcal{A}$ (where $\{P_{t_1}, \dots, P_{t_n}\}$ are all valid polytopes and pointed segments in $\mathcal{A}$), then add the label of the contracted end x_{t_j} to $\Lambda(P, i)$ and continue with all other labels mapped to $\tilde{Q}$ instead of τ_i
  - τ_i is mapped to a segment of Q, then there is exactly one τ_i' in another edge E' of Q that is mapped to the same segment. We continue with τ_i'.

  In each case, we follow all appearing edges until either ends associated to $\partial\Sigma(\Delta)$ are reached (for which we add the labels of such an end to $\Lambda(P, i)$) or contracted ends are reached.

Let $\{P_{t_1}, \dots, P_{t_n}\}$ denote the set of all valid polytopes and pointed segments in $\mathcal{A}$. If P equals $P_{t_j} \in \{P_{t_1}, \dots, P_{t_n}\}$, then set

$$\Lambda(P, 0) := \{x_j\},$$

where x_j is the label of the j-th contracted end. Otherwise, set

$$\Lambda(P, 0) := \emptyset.$$

Moreover, define

$$\Lambda(P) := \{\lambda_j = \{\beta_{j1}, \dots, \beta_{j4}\} \in \lambda_{[l]} \mid \beta_{ji} \in \Lambda(P, k_i) \text{ for } i \in [4] \text{ and } k_i \neq k_{i'} \text{ if } i \neq i'\}.$$

We say that the lattice path subdivision $\mathcal{S}$ *fits* the degenerated tropical cross-ratios $\lambda_{[l]}$ if

$$\sum_P \#\Lambda(P) = l,$$

where the sum goes over all valid polytopes and pointed segments in $\mathcal{S}$ and

$$\#\Lambda(P) = \begin{cases} k & \text{, if } P \text{ is } k\text{-marked} \\ 0 & \text{, otherwise.} \end{cases}$$

For a cross-ratio lattice path $\mathcal{A}$, the subset of $\mathcal{S}_1(\mathcal{A})$ of subdivisions which fit the given degenerated tropical cross-ratios is denoted by $\mathcal{S}_2(\mathcal{A})$.

5.1.4 Multiplicities of subdivisions

The algorithm of the last section which yields a set of subdivisions $\mathcal{S}_2(\mathcal{A})$ is now enriched such that is also yields a multiplicity with which each subdivision in $\mathcal{S}_2(\mathcal{A})$ can be counted. The overall algorithm which then constructs subdivisions and their multiplicities is called *cross-ratio lattice path algorithm.*

Definition 5.1.14 (Multiplicity of a subdivision)**.** Notation 5.1.3 is used. In order to associate a multiplicity to a lattice path subdivision $\mathcal{S} \in \mathcal{S}_2(\mathcal{A})$, define the ev-*multiplicity* of $\mathcal{S}$ by

$$\mathrm{mult}_{\mathrm{ev}}(\mathcal{S}) := \prod_{P} \mathrm{mult}_{\mathrm{ev}}(P),$$

where the product goes over all valid polytopes and pointed segments in $\mathcal{S}$, and $\mathrm{mult}(P)$ is defined as follows: If $\tilde{P}$ is 0-dimensional or $P \in \mathcal{A}$, then $\mathrm{mult}(P) := 1$. Otherwise, let $\tau_1, \ldots, \tau_m$ denote the entries of labelings of edges of P associated to $\tilde{P}$, let $\mathcal{E}_i$ be the number of ends that can be reached from P via τ_i and let $\mathcal{C}_i$ be the number of constraints that can be reached from P via τ_i (using the procedure from Definition 5.1.13), that is

$$\mathcal{C}_i := \mathcal{C}_i^{(\text{points})} + \mathcal{C}_i^{(\text{cross-ratios})},$$
$$\mathcal{C}_i^{(\text{cross-ratios})} := \sum_{P'} \#\Lambda(P'),$$

where the sum goes over all valid polytopes and pointed segments in $\mathcal{S}$ that can be reached from P via τ_i, $\Lambda(P')$ is defined in 5.1.13 and $\mathcal{C}_i^{(\text{points})}$ is the number of contracted ends that can be reached from P via τ_i. We either have

$$\mathcal{E}_i - 1 = \mathcal{C}_i \quad \text{or} \quad \mathcal{E}_i - 2 = \mathcal{C}_i.$$

In the first case, the edge dual to τ_i in the associated rational tropical stable map $C_{\mathcal{S}}$ (Construction 5.1.12) leads to a fixed component, in the second case it leads to a free component (see Definition 3.2.28). Every vertex of the dual rational tropical stable map has exactly two fixed components, we use the indices i_0 and i_1 for those labels corresponding to edges $C_{\mathcal{S}}$ that lead to a fixed component. Then we set

$$\mathrm{mult}_{\mathrm{ev}}(P) := |\det(\tau_{i_0} \cdot v_0, \tau_{i_1} \cdot v_1)|,$$

where v_0 (resp. v_1) is the primitive vector of the edge E_0 (resp. E_1) of P that belongs to τ_{i_0} (resp. τ_{i_1}).

Let $P \notin \mathcal{A}$ be a valid polytope or a pointed segment in $\mathcal{S}$. Again, let $\tau_1, \ldots, \tau_m$ denote the entries of labelings of edges of P associated to $\tilde{P}$. Let $\lambda_j \in \Lambda(P)$ with $= \{\beta_{j_1}, \ldots, \beta_{j_4}\}$, then (by Definition 5.1.13) there are pairwise different k_i for $i \in [4]$ such that $\beta_{j_i} \in \Lambda(P, k_i)$. Define the degenerated tropical cross-ratio that is *adapted* to P by

$$\lambda^{\to P} := \{\tau_{k_1}, \ldots, \tau_{k_4}\}$$

and define

$$\Lambda^{\to P}(P) := \{\lambda^{\to P} \mid \lambda \in \Lambda(P)\}.$$

Consider a vertex v of valence m that is adjacent to only ends which are labeled by $\tau_1, \ldots, \tau_m$. Define

$$\mathrm{mult}_{\mathrm{cr}}(P) := \mathrm{mult}_{\mathrm{cr}}(v),$$

where $\mathrm{mult}_{\mathrm{cr}}(v)$ is the cross-ratio multiplicity of v with respect to $\Lambda^{\to P}(P)$. If P is the t-th of the n pointed segments or valid polytopes in $\mathcal{A}$, then define $\mathrm{mult}_{\mathrm{cr}}(P)$ analogously to above with

$$\lambda^{\to P} := \{\tau_{k_1}, \ldots, \tau_{k_4}\} \cup \{x_t\},$$

where x_t is the label to which the t-th point condition $p_t \in p_{[n]}$ is associated to. Define the *cross-ratio multiplicity* of $\mathcal{S}$ by

$$\operatorname{mult}_{\mathrm{cr}}(\mathcal{S}) := \prod_P \operatorname{mult}_{\mathrm{cr}}(P),$$

where the product goes over all valid polytopes and pointed segments in $\mathcal{S}$. The *multiplicity* $\operatorname{mult}(\mathcal{S})$ of the subdivision $\mathcal{S}$ is defined as

$$\operatorname{mult}(\mathcal{S}) := \operatorname{mult}_{\mathrm{ev}}(\mathcal{S}) \cdot \operatorname{mult}_{\mathrm{cr}}(\mathcal{S}).$$

Definition 5.1.15 (Output cross-ratio lattice path algorithm)**.** Let Δ be a degree in $\mathbb{R}^2$ and let $p_{[n]}, \lambda_{[l]}$ be general positioned conditions as in Convention 5.1.10. Define

$$N_\Delta^{\mathrm{lpa}}\left(p_{[n]}, \lambda_{[l]}\right) := \sum_{\mathcal{S}} \operatorname{mult}(\mathcal{S}),$$

where the sum goes over all $\mathcal{S} \in \mathcal{S}_2(\mathcal{A})$ for all cross-ratio lattice paths $\mathcal{A}$ with respect to Δ and with $n+z$ steps for all z.

Remark 5.1.16 (Arbitrary degree)**.** In Convention 5.1.10 we restricted to degrees Δ such that all vectors in Δ are of weight one. This was done to keep notation simple. Indeed, the whole cross-ratio lattice path algorithm can be extended to arbitrary degrees in $\mathbb{R}^2$.

Example 5.1.17. We want to give an example of the cross-ratio lattice path algorithm. Fix the degree Δ_3^2, see Notation 2.2.4. We choose points $p_{[7]}$ and a degenerated tropical cross-ratio $\lambda = \{1, 2, 14, 15\}$. It turns out that all cross-ratio lattice paths we need to consider have 7 steps. The top row of Figure 5.5 shows these cross-ratio lattice paths. There are no labels on polytopes and colors in Figure 5.5 because all labels are 1 and all labels are colored fixed. The column under each of these cross-ratio lattice paths shows the subdivisions arising from these cross-ratio lattice paths. The maps that glue together the polytopes in a subdivision (maps like g from Construction 5.1.11) are not mentioned in Figure 5.5 since they are the obvious ones. However, the gluing maps that connect the polytopes in the subdivsion to the boundary of Δ_3 are not unique since we labeled ends of rational tropical stable maps (we come back to this later). The grey polytopes are 1-marked, that is λ leads to a vertex of valence four that is dual to these polytopes. Note that all subdivisions fit the degenerated tropical cross-ratio λ for an appropriate choice of gluing the polytopes to the boundary.

The numbers in the rightmost column correspond to subdivisions shown on the left. Each of these numbers is a product, where the first factor is the multiplicity $\operatorname{mult}(\mathcal{S})$ of its associated subdivision $\mathcal{S}$. Note that $\operatorname{mult}_{\mathrm{cr}}(\mathcal{S}) = 1$ for all subdivisions since there is only one way of resolving the 4-valent vertex dual to each 1-marked polytope according to some λ' degenerating to λ. The second factor comes from different gluings of polytopes to the boundary of Δ_3^2 and can easily be seen from an example, see Figure 5.6.

The total sum of the numbers in the right column is 40, which is the number of rational tropical stable maps of an partially labeled degree such that they satisfy the given point conditions and the degenerated tropical cross-ratio condition. Since the second factor of each product in the rightmost column equals the number of ways to label ends with direction vector $(1,1) \in \mathbb{R}^2$, we obtain the number of rational tropical stable maps of a labeled degree by multiplying 40 with $(3!)^2$, which is 1440.

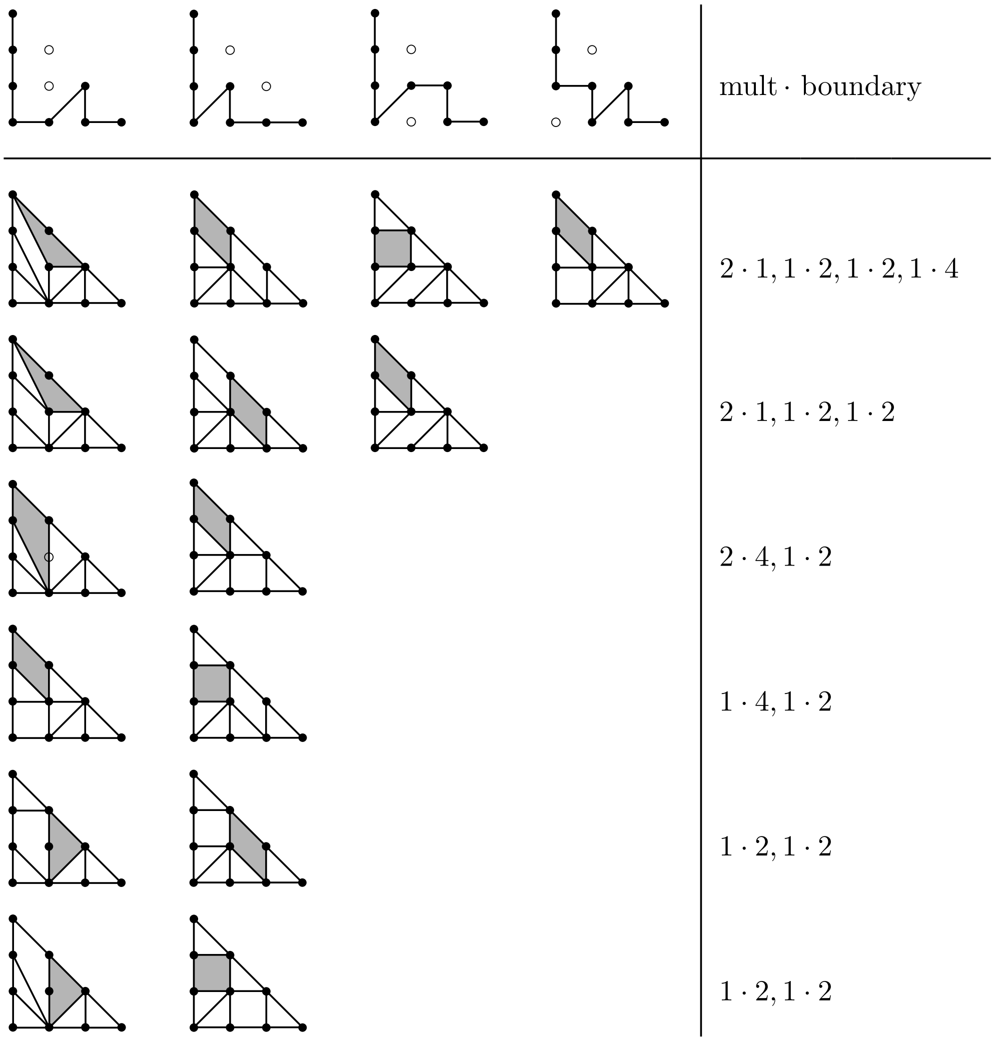


Figure 5.5: A complete example of cross-ratio lattice paths, subdivisions and their multiplicities.

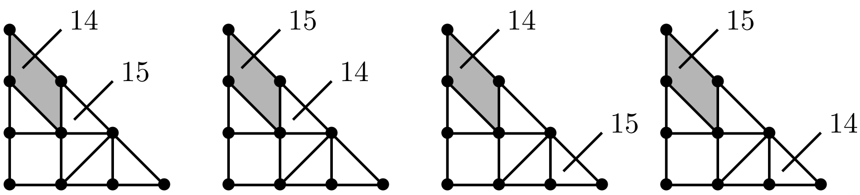


Figure 5.6: The subdivision in the right top corner of Figure 5.5 and the four different choices of labels of ends that appear in λ such that the subdivision still fits λ.

5.2 Duality: Tropical curves and subdivisions

In this section, Theorem 5.2.4 is proved. It relates the numbers obtained from the cross-ratio lattice path algorithm to the numbers $N_\Delta\left(p_{\underline{n}}, \lambda_{[l]}\right)$ of rational tropical stable maps we are interested in (where Δ is a degree in $\mathbb{R}^2$).

The general tropical Kontsevich's formula 4.3.4 allows us to determine some of these numbers $N_\Delta\left(p_{\underline{n}}, \lambda_{[l]}\right)$ already. Notice, however, that it does not allow to compute the numbers

$N_\Delta\left(p_{\underline{n}}, \lambda_{[l]}\right)$ if Δ is an arbitrary degree in $\mathbb{R}^2$ and if not all degenerated tropical cross-ratios are on contracted ends only. The cross-ratio lattice path algorithm determines these numbers for arbitrary degrees in $\mathbb{R}^2$ and arbitrary degenerated tropical cross-ratios instead. Moreover, it is constructive allowing us to explicitly construct all rational tropical stable maps that contribute to such a number $N_\Delta\left(p_{\underline{n}}, \lambda_{[l]}\right)$.

As a consequence, the corresponding algebraic numbers (via Tyomkin's correspondence theorem 2.3.6) become computable too.

Definition 5.2.1 (Simple rational tropical stable maps)**.** A rational tropical stable map $\left(\Gamma, x_{[M]}, h\right)$ in $\mathcal{M}_{0,N}\left(\mathbb{R}^2, \Delta\right)$ is called *simple* if is satisfies:

- The map h that embeds Γ in $\mathbb{R}^2$ is injective on vertices of Γ.
- If $h(v) \in h(e)$ for a vertex v and an edge e, then there is an edge e' adjacent to v such that $h(e)$ and $h(e')$ intersect in infinitely many points. Additionally, there is a finite sequence $(e_i)_{i\in[r]}$ of edges of Γ with $e_1 := e'$ and $e_r := e'$ such that $h(e_i)$ is contained in the affine span of $h(e)$ for all $i \in [r]$ and such that two consecutive elements of the sequence are incident in Γ.
- If there is a point $p \in \mathbb{R}^2$ through which more than two edges pass, then divide these edges into equivalence classes depending on the slope of the affine line they are mapped to. Then there are at most two equivalence classes.

Remark 5.2.2. Consider point conditions $p_{\underline{n}}$ and degenerated tropical cross-ratio conditions $\lambda_{[l]}$ in general position and a rational tropical stable map C that satisfies them such that C is not simple. Since the position of every vertex of C depends on the positions of the point conditions only, it follows that moving the point conditions slightly (see Remark 3.2.6) deforms C into a simple rational tropical stable map C' that satisfies the moved conditions. By definition, the multiplicity $\operatorname{mult}(C')$ equals $\operatorname{mult}(C)$. Moreover, the set of positions where the conditions can be moved such that all rational tropical stable maps that satisfy them are simple is open and dense in the space of all possible positions of conditions.

Remark 5.2.3. Given a rational tropical stable map $(\Gamma, x_{[M]}, h)$ to $\mathbb{R}^2$, two graph structures can be associated to $h(\Gamma)$ in $\mathbb{R}^2$. The first graph structure is the one coming from Γ. The second graph structure is the one of $h(\Gamma) \subset \mathbb{R}^2$, i.e. whenever edges of $h(\Gamma)$ in $\mathbb{R}^2$ intersect, this intersection is considered a vertex, see Figure 5.7. Notice that Minkowski labeled polytopes as in Figure 5.4 help to keep track of the graph structures.

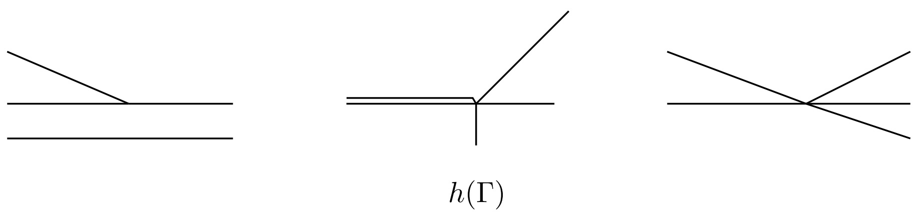


Figure 5.7: Left: A local picture of Γ and its graph structure. Middle: A local picture of a vertex of $h(\Gamma)$, where the two edges on the left are mapped on top of each other, we shifted them slightly to get a better picture. Right: The graph structure of $h(\Gamma)$ induced from intersections of edges.

Theorem 5.2.4. *For notation, see Notation 2.0.1, Definition 3.2.5 and Definition 5.1.15. Let Δ be a degree in $\mathbb{R}^2$. Let $p_{\underline{n}}$ be point conditions and let $\lambda_{[l]}$ be degenerated tropical cross-ratio conditions such that these conditions are in general position with respect to λ such that $\#\Delta - 1 = \#\underline{n} + l$. Then the equality*

$$N_\Delta\left(p_{[n]}, \lambda_{[l]}\right) = N_\Delta^{\mathrm{lpa}}\left(p_{[n]}, \lambda_{[l]}\right)$$

holds. Moreover, the lattice path algorithm explicitly constructs all rational tropical stable maps that contribute to $N_\Delta\left(p_{[n]}, \lambda_{[l]}\right)$ if the position of the conditions is chosen appropriately.

Proof. Remark 3.2.6 and Remark 5.2.2 allow us to assume that the point conditions $p_{\underline{n}}$ are linearly ordered on a line with a small negative slope such that distances grow, i.e.

$$|p_i - p_{i-1}| << |p_{i+1} - p_i|,$$

and that all rational tropical stable maps that contribute to $N_\Delta\left(p_{\underline{n}}, \lambda_{[l]}\right)$ are simple.

Let $\mathcal{S}_\Delta\left(p_{\underline{n}}, \lambda_{[l]}\right)$ denote the set of elements that contribute to $N_\Delta^{\mathrm{lpa}}\left(p_{\underline{n}}, \lambda_{[l]}\right)$. Denote the set of rational tropical stable maps that contribute to $N_\Delta\left(p_{\underline{n}}, \lambda_{[l]}\right)$ by $\mathcal{R}_\Delta\left(p_{\underline{n}}, \lambda_{[l]}\right)$. Consider the map

$$\begin{aligned} \phi : \mathcal{S}_\Delta\left(p_{\underline{n}}, \lambda_{[l]}\right) &\to \mathcal{R}_\Delta\left(p_{\underline{n}}, \lambda_{[l]}\right) \\ \mathcal{S} &\mapsto C_{\mathcal{S}} \end{aligned}$$

that maps a lattice path subdivision $\mathcal{S}$ to its dual rational tropical stable map $C_{\mathcal{S}}$ given by Construction 5.1.12. This map is obviously well-defined because in all subdivisions all polytopes are colored fixed, i.e. each rational tropical stable map $C_{\mathcal{S}}$ is fixed by the given conditions. Moreover, the map is injective because rational tropical stable maps of different combinatorial types are different. To see that ϕ is surjective, we need to construct a preimage for a given rational tropical stable map $C := (\Gamma, x_{[N]}, h)$ in $\mathcal{R}_\Delta\left(p_{\underline{n}}, \lambda_{[l]}\right)$. For that, the two graph structures of C of Remark 5.2.3 are used: if we refer to a vertex in $h(\Gamma)$, we mean the graph structure induced by h and if we refer to a vertex in Γ, we mean the graph structure of Γ.

First of all, associate a valid polytope (resp. a pointed segment) to every vertex $v \in h(\Gamma)$: Let v be a vertex of $h(\Gamma)$ and consider its dual polytope P_v. The polytope P_v can be turned into a labeled polytope (resp. a pointed segment) if we label its edges E_i with weights of its dual edges $e_{i_1}, \dots, e_{i_m} \in \Gamma$. Moreover, denote by $\tilde{P}_v$ the dual polytope of $v \in \Gamma$ and label its edges as before. Note that P_v is a Minkowski sum of $\tilde{P}_v$ and segments $S_1, \dots, S_r$ that correspond to edges of $v \in h(\Gamma)$ that are no edges of $v \in \Gamma$. Since C is simple it follows that edges of $v \in h(\Gamma)$ that are no edges of $v \in \Gamma$ can only be parallel to edges of $v \in \Gamma$. Furthermore, if $\tilde{P}_v$ is 0-dimensional, then there are two segments $S_{i_1}, S_{i_2} \in S_{[r]}$ such that all other Minkowski summands of P_v are parallel to S_{i_1} or S_{i_2}. Note also that there are mappings of entries of labeled edges of P_v to its Minkowski summands. In addition, P_v is unique because permuting parallel edges of $v \in h(\Gamma)$ leads to the same dual polytope. In this way, a valid polytope (resp. pointed segment) is associated to every vertex $v \in h(\Gamma)$.

The second step is to associate a subdivision $\mathcal{S}_C \in \mathcal{S}_\Delta\left(p_{\underline{n}}, \lambda_{[l]}\right)$ to C: The rational tropical stable map C determines how to glue the polytopes P_v (via maps called g in Construction 5.1.11) for all vertices $v \in h(\Gamma)$ together. Notice that if two vertices $v, v' \in h(\Gamma)$ are adjacent, then their dual valid polytopes $P_v, P_{v'}$ are compatible. Denote the subdivision obtained this way by $\mathcal{S}_C$. The dual polytopes resp. segments associated to the vertices and edges of $h(\Gamma)$ meeting the points $p_{\underline{n}}$ and non-pointed segment (we associate in the obvious way to the edges of C that intersect the line the points $p_{\underline{n}}$ lie on) form a cross-ratio lattice path $\mathcal{A}$. Hence $\mathcal{S}_C$ is a lattice path subdivision whose dual tropical stable map is C, the genus of C is zero, all polytopes of $\mathcal{S}_C$ are fixed and $\mathcal{S}_C$ fits to the given degenerated tropical cross-ratios by the path criterion (Corollary 3.2.12). Therefore $\mathcal{S}_C \in \mathcal{S}_2(\mathcal{A})$ for a cross-ratio lattice path $\mathcal{A}$. Thus ϕ is bijective.

Comparing Definition 5.1.14 with Lemma 3.2.32 yields that

$$\mathrm{mult}(\mathcal{S}) = \mathrm{mult}(\phi(\mathcal{S}))$$

holds for all $\mathcal{S} \in \mathcal{S}_\Delta\left(p_{\underline{n}}, \lambda_{[l]}\right)$. Hence Theorem 5.2.4 follows. $\square$

Example 5.2.5. To illustrate the proof of Theorem 5.2.4, consider the subdivision of Example 5.1.17 given be the entry $(4,2)$ of Figure 5.5, where matrix notation is used. Figure 5.8 shows this entry on the left and its associated rational tropical stable map on the right. The degenerated tropical cross-ratio λ of Example 5.1.17 is satisfied at the vertex v which corresponds to the gray square in the subdivision.

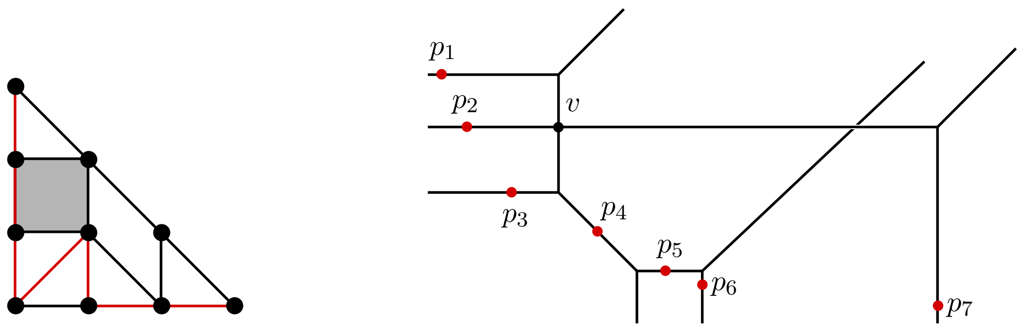


Figure 5.8: Left: Entry $(4,2)$ of Figure 5.5 with its lattice path colored red. Right: Its associated rational tropical stable map that satisfies the point conditions $p_{[7]}$. Moreover, it satisfies a degenerated tropical cross-ratio λ at the vertex v. Labels of ends are not shown here.

Using Tyomkin's correspondence theorem 2.3.6 (more precisely, Corollary 3.1.20 and Proposition 3.2.7), Theorem 5.2.4 immediately yields the following corollary.

Corollary 5.2.6 (Algebro-geometric count via cross-ratio lattice path algorithm)**.** *Notation 2.0.1 is used. Let Δ be a degree in $\mathbb{R}^2$ as in Definition 2.2.3 and let $\Sigma(\Delta)$ be its associated lattice polytope, see Remark 2.2.5. Let $X_{\Sigma(\Delta)}$ be the toric variety associated to $\Sigma(\Delta)$. Then the number of rational algebraic curves in $X_{\Sigma(\Delta)}$ over an algebraically closed field of characteristic zero that satisfy point conditions and classical cross-ratio conditions in general position can be calculated via the cross-ratio lattice path algorithm.*

Chapter 6

Combinatorial approach: Counting curves via cross-ratio floor diagrams

In this chapter, *cross-ratio floor diagrams* are presented. They are combinatorial objects that arise as degenerations of rational tropical stable maps. Each such cross-ratio floor diagram is equipped with a multiplicity that reflects how many rational tropical stable maps degenerate to it. Counting cross-ratio floor diagrams with their multiplicities then yields numbers of interest, namely the counts of rational tropical stable maps that involve degenerated tropical cross-ratio conditions. In case of rational tropical stable maps to $\mathbb{R}^2$ these numbers are known thanks to the lattice path algorithm from Chapter 5 and general tropical Kontsevich's formula 4.3.4 of Chapter 4. Here, cross-ratio floor diagrams provide another — combinatorial — way to calculate those numbers. The benefit of this combinatorial approach is that it can be generalized to rational tropical stable maps of degree $\Delta_d^3(\alpha,\beta)$ (Notation 2.2.4) to $\mathbb{R}^3$ which is done in Theorem 6.3.34. Thus cross-ratio floor diagrams provide (partial) answers to leading questions (Q1) and (Q5). We remark that almost all techniques presented in this chapter extend to even higher dimensions. What prevents us from extending the cross-ratio floor diagram approach to rational tropical stable maps to $\mathbb{R}^m$ with $m > 3$ is the currently unknown behavior of multiplicities of so-called *floor-decomposed* rational tropical stable maps to $\mathbb{R}^m$ if degenerated tropical cross-ratios are involved.

Original floor diagrams. The original concept of floor diagrams was introduced by Brugallé and Mikhalkin in [BM07, BM08] to describe, among other things, the numbers $N_{\Delta_d^2}\left(p_{[n]}\right)$ (Definition 3.1.8). More floor diagram techniques were then elaborated in [FM10]. Recall that the numbers $N_{\Delta_d^2}\left(p_{[n]}\right)$ are independent of the exact positions of the general positioned point conditions $p_{[n]}$. Moreover, by Remark 3.1.10, the set of all positions of point conditions that are in general position is open and dense in the set of all possible positions for the point conditions. Hence it can be assumed that the given point conditions are in a so-called *stretched configuration*, which means that they lie in a thin stripe $(-\varepsilon,\varepsilon)\times\mathbb{R}\subset\mathbb{R}^2$ with large distances between them. A crucial observation is that all rational tropical stable maps that contribute to $N_{\Delta_d^2}\left(p_{[n]}\right)$ (where the point conditions $p_{[n]}$ are in a stretched configuration) are of a particularly nice form. This form is referred to as *floor-decomposed*.

Figure 6.1 provides an example of a floor-decomposed rational tropical stable map C of degree Δ_3^2 satisfying eight point conditions $p_{[8]}$. Notice that the distances between the given point conditions are not large. However, stretching the point conditions further apart yields a rational tropical stable map of the same combinatorial type as C. Notice that the floor-decomposed rational tropical stable map C can easily be degenerated to a graph. In Figure 6.1 it is indicated with dotted lines which parts of C are contracted to vertices of the graph

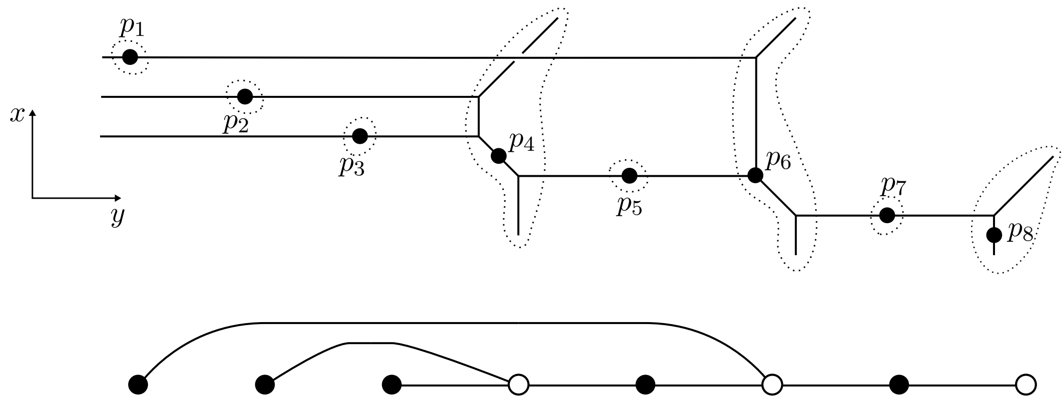


Figure 6.1: Top: A floor-decomposed rational tropical stable map C of degree Δ_3^2 that satisfies eight point conditions $p_{[8]}$. Its so-called *floors* are indicated by dotted lines. Bottom: Its associated floor diagram, where larger floors are shrunk to white vertices.

downstairs. Such a graph is called *floor diagram.*

Each such floor diagram is then counted with a suitable multiplicity that reflects how many rational tropical stable maps degenerate to it such that the overall count equals $N_{\Delta_d^2}\left(p_{[n]}\right)$. Thus determining the numbers $N_{\Delta_d^2}\left(p_{[n]}\right)$ boils down to a purely combinatorial counting problem.

6.1 Floor decomposition

To incorporate degenerated tropical cross-ratios into the floor diagram approach, it is essential to show that rational tropical stable maps that satisfy given degenerated tropical cross-ratio conditions are floor-decomposed (see also Figure 6.1). For that, it is necessary that we restrict to (degenerated) tropical cross-ratios whose entries are contracted ends or ends of certain directions only.

Definition 6.1.1 (Stretched configuration)**.** Let $\Delta_d^m(\alpha,\beta)$ be a degree as in Notation 2.2.4. Let $p_{[n]}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}$ be general positioned conditions with respect to $\Delta_d^m(\alpha,\beta)$ as in Definition 3.2.4. Let $\pi : \mathbb{R}^m \to \mathbb{R}^{m-1}$ be the projection that forgets the x_m-coordinate as in Notation 2.2.9. Let $\varepsilon > 0$ be a real number, and define for the open interval $(-\varepsilon,\varepsilon) \subset \mathbb{R}$ the open box $B_\varepsilon^t := (-\varepsilon,\varepsilon)^t \subset \mathbb{R}^t$. The conditions $p_{[n]}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}$ are said to be in a *stretched configuration* if the following properties are satisfied.

(1) $\pi\left(P_{\underline{\eta}^\gamma}\right) \subset B_\varepsilon^{m-1}$ for $\gamma = \alpha, \beta$.

(2) $L_k^{(0)} \in B_\varepsilon^{m-1}$ for $k \in \underline{\kappa}^\alpha \cup \underline{\kappa}^\beta$, where $L_k^{(0)}$ denotes the 0-skeleton of L_k.

(3) $\pi(p_{[n]}) \subset B_\varepsilon^{m-1}$ and the distances of the x_m-coordinates p_{i,x_m} of the points p_i for $i \in [n]$ are large compared to the size of the box B_ε^{m-1}, i.e. $|p_{i,x_m} - p_{j,x_m}| >> \varepsilon$ for all $i \neq j \in [n]$.

(4) Each entry of each degenerated tropical cross-ratio $\lambda_j \in \lambda_{[l]}$ is either the label of a contracted end or the label of and end of primitive direction $\pm e_m$ (see Notation 2.2.4).

Remark 6.1.2. Stretched configurations exist, because the set of all positions of general positioned conditions is open and dense in the set of all possible positions of conditions (Remark 3.2.6), i.e. the property of being in general position can be preserved when stretching the points $p_{[n]}$ in x_m-direction.

Definition 6.1.3 (Floor-decomposed)**.** Notation 2.2.4 is used. An *elevator* of a rational tropical stable map C of degree $\Delta_d^m(\alpha,\beta)$ is an edge whose primitive direction is $\pm e_m$. A connected component C_i of C that remains if the interiors of the elevators are removed is called *floor* of the rational tropical stable map C. The number $s_i \in \mathbb{N}$ of ends of C_i that are of direction e_0 is called the *size* of the floor C_i. A rational tropical stable map that is fixed by general positioned conditions as in Definition 3.2.4 is called *floor-decomposed* if each of the points $p_{[n]}$ lies on its own floor. Notice that floors can be of size zero, i.e. a floor can have exactly one vertex.

Later, we equip floors with additional ends by cutting elevators (Construction 6.3.16) and stretching them to infinity. By abuse of notation we refer to these rational tropical stable maps as floors as well when no confusion can occur.

Example 6.1.4. The top part of Figure 6.1 shows a floor-decomposed rational tropical stable map to $\mathbb{R}^2$. Its horizontal edges are the elevators and its floors are indicated by dotted lines.

Example 6.1.5. Figure 6.2 shows a floor-decomposed rational tropical stable map C. The labels of some of its ends are indicated with circled numbers. The ends labeled with 1 and 2 are drawn dotted which indicates that these ends are contracted. The other labeled ends are of primitive direction $\pm e_3 \in \mathbb{R}^3$ using Notation 2.2.4. The end labeled with 8 is of weight two while all other ends are of weight one such that the degree of C is $\Delta_4^3((4,1,0,\dots),(2,0,\dots))$, see Notation 2.2.4. The general positioned conditions C satisfies are the following: The end labeled with 1 (resp. 2) satisfies a point condition p_1 (resp. p_2). The ends labeled with $f \in [9]\backslash[2]$ satisfy codimension one tangency conditions P_f for $f \in [9]\backslash[2]$. Moreover, C satisfies the degenerated tropical cross-ratio $\lambda_1 = \{1,2,3,7\}$ at its only 4-valent vertex.

The elevator of C has weight two and is drawn dashed. Thus C has two floors C_i for $i = 1,2$, where the point p_i lies on C_i for $i = 1,2$.

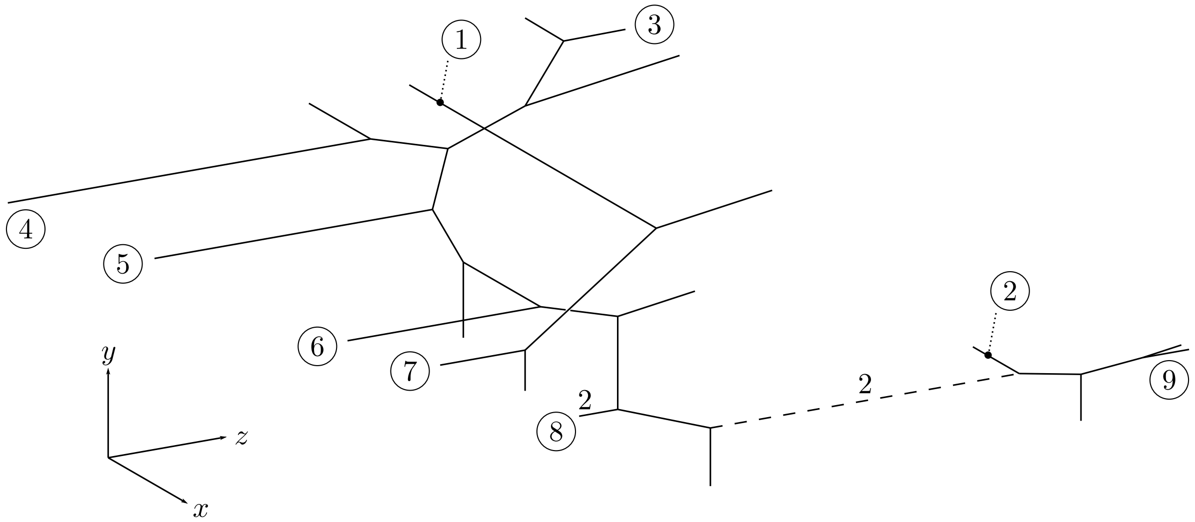


Figure 6.2: The rational tropical stable map C from Example 6.1.5 which is floor-decomposed. It has two floors C_i for $i = 1,2$, where C_1 is of size $s_1 = 3$ and C_2 is of size $s_2 = 1$. The dashed edge is the elevator of weight two of C.

Proposition 6.1.6. *Notation of Definition 3.2.4 is used. Let* $p_{[n]}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}$ *be conditions in a stretched configuration as in Definition 6.1.1. Then every rational tropical stable map that contributes to* $N_{\Delta_d^m(\alpha,\beta)}\left(p_{[n]}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}\right)$ *is floor-decomposed.*

Proof. We follow arguments used in [BM, Tor14], where an analogous statement is proved for the case without degenerated tropical cross-ratios. To incorporate degenerated tropical cross-ratios, Corollary 3.2.24 is used.

Let C be a rational tropical stable contributing to $N_{\Delta_d^m(\alpha,\beta)}\left(p_{[n]}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}\right)$. The set of all possible bounded edges' directions of C is finite because of the balancing condition and the fixed directions of ends. If ε from Definition 6.1.1 is sufficiently small compared to the distances between the points $p_{[n]}$ and all vertices of C lie inside the stripe $B_\varepsilon^{m-1} \times \mathbb{R}$, then C decomposes into parts that are connected by edges of primitive direction $\pm e_m$ (see Notation 2.2.4). So it is sufficient to show that all vertices of C lie inside $B_\varepsilon^{m-1} \times \mathbb{R}$ from Definition 6.1.1.

Assume that a vertex $v \in C$ whose x_1-coordinate is maximal lies outside of $B_\varepsilon^{m-1} \times \mathbb{R}$. Since the x_1-coordinate of v is maximal, there is an end e of direction e_0 adjacent to v. If v is not 3-valent, then there is a $j \in [l]$ such that $\lambda_j \in \lambda_v$ and the label of e appears as an entry in λ_j because of Corollary 3.2.24. Due to our assumptions on the tropical cross-ratios $\lambda_{[l]}$, the end e cannot be an entry of any of these, which is a contradiction. Hence v must be 3-valent. Denote the edges adjacent to v by e_a, e_b, e, where e is, as before, an end of direction e_0. If e_a is an end of primitive direction $\pm e_m$, then v allows a 1-dimensional movement in the direction of e_b, since e_a either satisfies no condition or satisfies a codimension two tangency condition L_k for some k with $\pi(e_b), \pi(e) \subset L_k$. This is a contradiction. If e_a is a contracted end or an end whose direction vector at v is $e_0, \dots, e_{m-1}$, then v can (since e_a cannot satisfy any condition) be moved in the direction of e_b. This is a contradiction. Thus e_a, e_b are bounded edges.

Since e is an end of direction e_0 and thus of weight 1, and v is maximal with respect to its x_1-coordinate, it follows (without loss of generality) that the x_1-coordinate of the direction vector of e_a is 0 and the x_1-coordinate of the direction vector of e_b is -1. Denote the vertex adjacent to v via e_a by v'. Notice that v' is also 3-valent, adjacent to an end e' parallel to e and a bounded edge $\tilde{e} \neq e_a$. By balancing, $e_b, e, e_a, e', \tilde{e}$ lie in the affine hyperplane $\langle e_a, e\rangle + v$ of $\mathbb{R}^m$. Thus v allows a 1-dimensional movement in the direction of e_b which is a contradiction.

Notice that similar arguments hold if v is chosen in such a way that its x_i-coordinate (for $i = 2, \dots, m-1$) is maximal or its x_j-coordinate for $j \in [m-1]$ is minimal. So in any case a 1-dimensional movement leads to a contradiction. Hence all vertices of C lie inside the stripe $B_\varepsilon^{m-1} \times \mathbb{R}$. Moreover, each floor of C satisfies exactly one point condition. Otherwise, a floor would give rise to a 1-dimensional movement parallel to $\pm e_m$. Therefore C is floor-decomposed. $\square$

6.2 Cross-ratio floor diagrams for $\mathbb{R}^2$

Cross-ratio floor diagrams for $\mathbb{R}^2$ are now introduced. Additional discrete data encodes which floor-decomposed rational tropical stable maps degenerate to a cross-ratio floor diagram.

Definition 6.2.1 (Cross-ratio floor diagrams for $\mathbb{R}^2$)**.** Notations 2.0.1 and 2.2.4 are used. Let $\left(\Delta_d^2(\alpha,\beta), l\right)$ be a degree as in Definition 2.2.3. Let $\mathcal{F}$ be a tree without ends on a totally ordered set of vertices $v_{[n]}$, then $\mathcal{F}$ is called a *cross-ratio floor diagram of degree* $\Delta_d^2(\alpha,\beta)$ if it satisfies the following:

(1) Each edge of $\mathcal{F}$ consists of two half-edges. There are two types of half-edges, *thin* and *thick* ones. A thin half-edge can only be completed to an edge with a thick half-edge and vice versa.

(2) Each vertex v is equipped with $s_v, \#\lambda_v \in \mathbb{N}$ and a set δ_v of labels that appear in $\Delta_d^2(\alpha,\beta)$, where $\#\lambda_v$ is called the *number of degenerated tropical cross-ratios* of v and s_v is called

the *size* of v such that

$$s_v = \#\{x \in \delta_v \mid l^{-1}(x) = e_1\} = \#\{x \in \delta_v \mid l^{-1}(x) = e_0\}$$

and $\emptyset = \delta_v \cap \delta_{v'}$ for all $v \neq v'$ and $\bigcup_v \delta_v$ is the set of all labels appearing in $\Delta_d^2(\alpha, \beta)$.

(3) For $\gamma = \alpha, \beta$, define

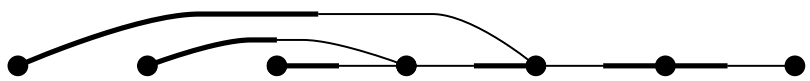
$$\delta_v^\gamma := \{x \in \delta_v \mid \text{primitive direction of } l^{-1}(x) \text{ is } \pm e_2\},$$

where $-e_2$ is considered if $\gamma = \alpha$ and $+e_2$ is considered otherwise. The total order of the vertices induces an orientation of the edges in the following way: we order the vertices on a line starting with the smallest vertex v_1 on the left and orient the edges from smaller to larger vertices. Each edge e of the graph is equipped with a *weight* $\omega(e) \in \mathbb{N}$ such that the *balancing condition*

$$0 = s_v + \sum_e \pm\omega(e) + \sum_{x \in \delta_v^\beta} \omega\left(l^{-1}(x)\right) - \sum_{x \in \delta_v^\alpha} \omega\left(l^{-1}(x)\right)$$

holds for all vertices v of $\mathcal{F}$, where $\pm$ is $+$ for outgoing edges and $-$ for incoming edges of v.

Example 6.2.2. The figure below shows a cross-ratio floor diagram of degree Δ_3^2, where all weights on the edges are 1 and where thick edges are drawn thick.



i	1	2	3	4	5	6	7
s_{v_i}	0	0	0	1	1	0	1
$\#\lambda_{v_i}$	0	0	0	0	1	0	0
δ_{v_i}	$\{1\}$	$\{2\}$	$\{3\}$	$\{4, 9\}$	$\{5, 8\}$	$\emptyset$	$\{6, 7\}$

6.2.1 Multiplicities

Recall that cross-ratio multiplicities are local and that ev-multiplicities are also local in case of $\mathbb{R}^2$ due to Corollary 3.2.35. Thus multiplicities of floor-decomposed rational tropical stable maps to $\mathbb{R}^2$ can be calculated locally at floors. Therefore we define the multiplicity of a cross-ratio floor diagram locally at its vertices such that each local multiplicity of a vertex reflects the multiplicities of floors degenerating to it.

Definition 6.2.3 (Path criterion for cross-ratio floor diagrams)**.** Let $\mathcal{F}$ be a cross-ratio floor diagram of degree $\Delta_d^2(\alpha, \beta)$ on n vertices. Let $\lambda = \{\beta_1, \ldots, \beta_4\}$ be a degenerated tropical cross-ratio with respect to $\mathcal{M}_{0,n}\left(\mathbb{R}^2, \Delta_d^2(\alpha, \beta)\right)$. Each element β_i of λ is associated to a vertex of $\mathcal{F}$ the following way:

(1) If β_i is the label of an end in $\Delta_d^2(\alpha, \beta)$, then β_i is associated to the unique vertex $v \in \mathcal{F}$ such that $\beta_i \in \delta_v$.

(2) If β_i is the label of a contracted end, i.e. $\beta_i = j \in [n]$, then β_i is associated to v_j.

Hence a pair $\{\beta_i, \beta_j\}$ induces a unique path in $\mathcal{F}$. If the paths associated to $\{\beta_{i_1}, \beta_{i_2}\}$ and $\{\beta_{i_3}, \beta_{i_4}\}$ intersect in exactly one vertex v of $\mathcal{F}$ for all pairwise different choices of $i_{[4]}$ such that $\{i_1, \ldots, i_4\}$ equal $\{1, \ldots, 4\}$, then the degenerated tropical cross-ratio λ is *satisfied* at v.

Remark 6.2.4. In Definition 6.2.3, similar to Remark 3.2.13, "all pairwise different choices of $i_{[4]}$" is equivalent to "one choice of pairwise different $i_{[4]}$". This makes it easier to check whether $\mathcal{F}$ satisfies a degenerated tropical cross-ratio.

Example 6.2.5. The cross-ratio floor diagram of Example 6.2.2 satisfies the degenerated tropical cross-ratio $\lambda = \{1, 4, 5, 6\}$ at its vertex v_5.

Definition 6.2.6 (Conditions satisfied by $\mathcal{F}$)**.** Let $\Delta_d^2(\alpha, \beta)$ be a degree. Let $p_{[n]}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}$ be in a stretched configuration with respect to $\Delta_d^2(\alpha, \beta)$. A cross-ratio floor diagram $\mathcal{F}$ *satisfies* $p_{[n]}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}$ if the following properties are fulfilled.

(1) The number of vertices of $\mathcal{F}$ equals n. The total order of the vertices $v_{[n]}$ of $\mathcal{F}$ is induced by the total order of the point conditions $p_{[n]}$ (they are ordered according to their last coordinate), where the point p_i is identified with the vertex v_i.

(2) For each degenerated tropical cross-ratio $\lambda_j \in \lambda_{[l]}$ there is a vertex of $\mathcal{F}$ that satisfies it and for each vertex v of $\mathcal{F}$ the number $\#\lambda_v$ equals the total number of degenerated tropical cross-ratios that are satisfied at v. For a vertex of a cross-ratio floor diagram that satisfies degenerated tropical cross-ratios $\lambda_{[l]}$, denote the set of all $\lambda_j \in \lambda_{[l]}$ that are satisfied at v by λ_v.

(3) The number of thick half-edges e_v^{thick} that are adjacent to a vertex v of $\mathcal{F}$ is given by

$$e_v^{\text{thick}} = \#\lambda_v + 2 - 2s_v - \#\left(\left(\delta_v^\alpha \cup \delta_v^\beta\right) \setminus \{x \in \Delta_d^2(\alpha, \beta) \mid l(x) \in \underline{\eta}^\alpha \cup \underline{\eta}^\beta\}\right), \qquad (6.1)$$

where $l(x)$ denotes the label of x in $\Delta_d^2(\alpha, \beta)$.

Definition 6.2.7 (Cutting $\mathcal{F}$ into pieces)**.** Notation of Definition 6.2.1 is used. Let $\mathcal{F}$ be a cross-ratio floor diagram of degree $\Delta_d^2(\alpha, \beta)$ such that $\mathcal{F}$ satisfies given degenerated tropical cross-ratios $\lambda_{[l]}$. Denote the ordered set of vertices of $\mathcal{F}$ by $v_{[n]}$. For $i \in [n]$, define the *i-th piece* $\left(\mathcal{F}_i, \delta_{v_i}, s_{v_i}, \#\lambda_{v_i}, \lambda_{v_i}^{\rightarrow}\right)$ of $\mathcal{F}$ the following way:

Cut all edges that connect the vertex v_i to other vertices of $\mathcal{F}$ into (thick or thin) half-edges. Denote the connected component that contains v_i by $\mathcal{F}_i$. Equip each cut edges e adjacent to v_i with the label

$$l_e := \{j \in [n] \mid e \text{ is in the shortest path connecting } v_i \text{ and } v_j\}.$$

Similar to Construction 4.1.26, we *adapt* the degenerated tropical cross-ratios λ_{v_i} to the cut edges. If $\{\beta_1, \ldots, \beta_4\} = \lambda_j \in \lambda_{[l]}$ is a degenerated tropical cross-ratio which is satisfied at v_i (i.e. $\lambda_j \in \lambda_{v_i}$), then the paths associated to λ_j in $\mathcal{F}$ (see Definition 6.2.3) might have been cut by cutting the edges connecting v_i to the rest of $\mathcal{F}$. Let $\beta_t \in \lambda_j$ be such that the path from the vertex associated to β_j to v_i is cut. If the path used to contain the edge e adjacent to v_i, then replace β_j by l_e. Notice that notation is abused here since l_e is not a natural number[1]. Denote the resulting degenerated tropical cross-ratio by $\lambda_j^{\rightarrow}$. Moreover, let $\lambda_{v_i}^{\rightarrow}$ denote the set λ_{v_i} of degenerated tropical cross-ratios that are satisfied at v_i where each of them is adapted to the cut edges. To shorten notation, $\mathcal{F}_i$ is used to refer to the i-th piece $\left(\mathcal{F}_i, \delta_{v_i}, s_{v_i}, \#\lambda_{v_i}, \lambda_{v_i}^{\rightarrow}\right)$ of $\mathcal{F}$ if the additional data $\delta_{v_i}, s_{v_i}, \#\lambda_{v_i}, \lambda_{v_i}^{\rightarrow}$ is obvious from context.

[1]However, one can encode the information l_e contains into a unique natural number N_e. If $l_e = \{x_1, \ldots, x_r\} \subset [n]$, then define $t := \min_k\{10^k \geq n\}$ and consider (in the decimal system) $N_e := 1\underbrace{0\ldots0x_1}_{\text{t digits}}\ldots\underbrace{0\ldots0x_r}_{\text{t digits}}$.

Definition 6.2.8 (Multiplicities of cross-ratio floor diagrams in $\mathbb{R}^2$)**.** Let $\Delta_d^2(\alpha,\beta)$ be a degree. Let $p_{[n]}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}$ be in a stretched configuration with respect to $\Delta_d^2(\alpha,\beta)$, see Definition 6.1.1. Let $\mathcal{F}$ be a cross-ratio floor diagram of degree $\Delta_d^2(\alpha,\beta)$ that satisfies these conditions. Let $\mathcal{F}_i$ be the i-th piece of $\mathcal{F}$ as in Definition 6.2.7.

Weights associated to $\delta_{v_i}^\alpha$ (Definition 6.2.1) and weights of incoming edges of v_i induce a partition α^i of the sum of all weights associated to $\delta_{v_i}^\alpha$ and all weights of incoming edges of v_i. Analogously, a partition β^i is induced. By abuse of notation, we can consider the degree $\Delta^2_{s_{v_i}}(\alpha^i,\beta^i)$. Define

$$\delta_{v_i}^{\text{thin}} := \{l_e \mid e \text{ is an edge of } \mathcal{F} \text{ such that its thin half-edge is adjacent to } v_i\},$$

where, for an edge e of $\mathcal{F}$, the label l_e is the one of Definition 6.2.7. Define

$$A := \left(\left(\delta_{v_i}^\alpha \cup \delta_{v_i}^\beta\right) \cap \{x \in \Delta_d^2(\alpha,\beta) \mid l(x) \in \underline{\eta}^\alpha \cup \underline{\eta}^\beta\}\right)$$

and

$$B := \left(\left(\delta_{v_i}^\alpha \cup \delta_{v_i}^\beta\right) \setminus \{x \in \Delta_d^2(\alpha,\beta) \mid l(x) \in \underline{\eta}^\alpha \cup \underline{\eta}^\beta\}\right)$$

such that $A\cup B$ is a decomposition of $\delta_{v_i}^\alpha \cup \delta_{v_i}^\beta$.

The *multiplicity* $\operatorname{mult}(\mathcal{F}_i)$ of the i-th piece $\mathcal{F}_i$ of $\mathcal{F}$ is defined as the degree of the cycle

$$Z_i := \operatorname{ev}_i^*(p_i)\cdot \prod_{f\in A} \partial \operatorname{ev}_f^*(P_f)\cdot \prod_{g\in\delta_{v_i}^{\text{thin}}} \partial \operatorname{ev}_g^*(X_g)\cdot \prod_{\lambda_j^{\to}\in\lambda_{v_i}^{\to}} \operatorname{ft}_{\lambda_j^{\to}}^*(0)\cdot \mathcal{M}_{0,1}\left(\mathbb{R}^2, \Delta^2_{s_{v_i}}\left(\alpha^i,\beta^i\right)\right),$$

where X_g are codimension one tangency conditions such that $p_i, P_A, X_{\delta_{v_i}^{\text{thin}}}, \lambda_{v_i}^{\to}$ are in general position. Notice that $\operatorname{mult}(\mathcal{F}_i)$ does not depend on the choice of the X_g by Remark 2.2.18. The *multiplicity* of $\mathcal{F}$ is defined by

$$\operatorname{mult}(\mathcal{F}) := \prod_e \omega(e)\cdot \prod_{i\in[n]} \operatorname{mult}(\mathcal{F}_i),$$

where the first product goes over all edges of $\mathcal{F}$ and $\omega(e)$ is the weight of an edge e.

Proof. We need to show that the cycle Z_i of Definition 6.2.8 is indeed zero-dimensional, i.e. we need to check whether (3.6) holds. Since $\mathcal{F}$ satisfies the given conditions (6.1) yields

$$e_{v_i}^{\text{thick}} = \#\lambda_{v_i}^{\to} + 2 - 2s_{v_i} - \#B.$$

Thus

$$2s_{v_i} + \#B + e_{v_i}^{\text{thick}} - 2 = \#\lambda_{v_i}^{\to}$$

holds and therefore

$$2s_{v_i} + \#A + \#B + \#\delta_{v_i}^{\text{thin}} + e_{v_i}^{\text{thick}} - 2 = \#\delta_{v_i}^{\text{thin}} + \#A + \#\lambda_{v_i}^{\to} \tag{6.2}$$

holds. Notice that

$$\#\Delta^2_{s_{v_i}}\left(\alpha^i,\beta^i\right) = 2s_{v_i} + \#A + \#B + \#\delta_{v_i}^{\text{thin}} + e_{v_i}^{\text{thick}},$$

which together with (6.2) yields

$$\#\Delta^2_{s_{v_i}}\left(\alpha^i,\beta^i\right) - 3 + 1 + 2 = 2\cdot 1 + 1\cdot\left(\#\delta_{v_i}^{\text{thin}} + \#A\right) + \#\lambda_{v_i}^{\to}.$$

This equation is the desired one, namely (3.6). $\square$

6.2.2 Enumeration in $\mathbb{R}^2$ using cross-ratio floor diagrams

Cross-ratio floor diagrams of degree $\Delta_d^2(\alpha,\beta)$ are now related to counts of rational tropical stable maps of degree $\Delta_d^2(\alpha,\beta)$. This is done in two steps. First, floor-decomposed rational tropical stable maps are degenerated to cross-ratio floor diagrams. Second, it is shown that the multiplicity of a cross-ratio floor diagram reflects how many floor-decomposed rational tropical stable maps degenerate to it.

Construction 6.2.9 (Floor-decomposed rational tropical stable map $\mapsto$ cross-ratio floor diagram)**.** Let $\Delta_d^2(\alpha,\beta)$ be a degree. Let $p_{[n]}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}$ be conditions in a stretched configuration with respect to $\Delta_d^2(\alpha,\beta)$ (Definition 6.1.1). Let C be a floor-decomposed rational tropical stable map of degree $\Delta_d^2(\alpha,\beta)$ that satisfies these conditions. A cross-ratio floor diagram $\mathcal{F}_C$ of degree $\Delta_d^2(\alpha,\beta)$ is associated to C the following way: Cut all elevators of C, i.e. cut all bounded edges of C whose primitive direction equals $(0,1)\in\mathbb{R}^2$. Notice that each remaining component contains exactly one contracted end which satisfies one of the point conditions $p_{[n]}$. Shrink the component C_i associated to p_i to a vertex called v_i for $i\in[n]$. The set of vertices $v_{[n]}$ inherits the order of the point conditions $p_{[n]}$ which is given by comparing x_2-coordinates. The set $v_{[n]}$ are the vertices of $\mathcal{F}_C$. Two different vertices v_i, v_j are connected by an edge if and only if the floors of C that are associated to $p_i, p_j \in [n]$ are connected by an elevator. Draw half-edges thin if they lead to a fixed component, and thick if they lead to a free component (Definition 3.2.28). Moreover, equip each vertex v_i of $\mathcal{F}_C$ with the following data: Record which non-contracted ends are adjacent to the component C_i by collecting their labels in a set δ_{v_i}. Set

$$\#\lambda_{v_i} := \sum_u \#\lambda_u,$$

where the sum goes over all vertices u of the component C_i. Finally, the balancing condition of C turns $\mathcal{F}_C$ into a cross-ratio floor diagram of degree $\Delta_d^2(\alpha,\beta)$.

Lemma 6.2.10. *The cross-ratio floor diagram $\mathcal{F}_C$ associated to C by Construction 6.2.9 satisfies the conditions $p_{[n]}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}$ if C does.*

Proof. This follows directly from Definition 6.2.6 and the calculations of the proof after Definition 6.2.8. □

Example 6.2.11. In order to illustrate Construction 6.2.9, a floor-decomposed rational tropical stable map C (see Figure 6.3) of degree Δ_3^2 is given. It satisfies the point conditions $p_{[7]}$ and the degenerated tropical cross-ratio $\lambda=\{1,4,5,6\}$. The cross-ratio floor diagram $\mathcal{F}_C$ is the one of Example 6.2.2. The floors of C are indicated by dotted lines.

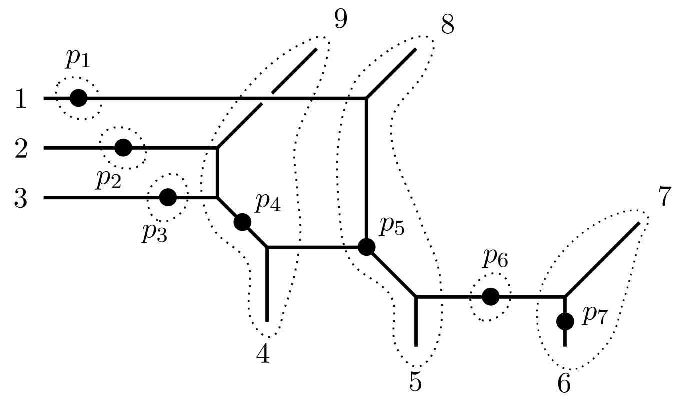


Figure 6.3: A floor-decomposed rational tropical stable map C of degree Δ_3^2.

Lemma 6.2.12. *Let G be a tree without ends such that each edge of G consists of two half-edges and there are two types of half-edges, thin and thick ones. A thin half-edge can only be completed to an edge with a thick half-edge and vice versa. Then there is a vertex of G that is only adjacent to thick half-edges.*

Proof. Induction over the number n of vertices of G is used. For $n = 2$ it is obviously true. If $n > 2$, then there is a 1-valent vertex v of G since G is a tree. There are two cases: either v is adjacent to a thick half-edge, then we are done or v is adjacent to a thin half-edge. If v is adjacent to a thin half-edge, then remove this edge and v from G. The graph G' obtained this way has one vertex less than G such that there is a vertex $v' \in G'$ that is by the induction hypothesis only adjacent to thick half-edges. Again, there are two cases: if v' is not connected to v in G, then we are done. Otherwise, the edge connecting v' to v in G is thick at v' since it is thin at v. $\square$

Theorem 6.2.13. *Notation from Notation 2.0.1, 2.2.4 and Definition 3.2.4 is used. Let $\Delta_d^2(\alpha,\beta)$ be a degree. Let $p_{[n]}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}$ be conditions in a stretched configuration with respect to $\Delta_d^2(\alpha,\beta)$, see Definition 6.1.1. Then the equality*

$$N_{\Delta_d^2(\alpha,\beta)}\left(p_{[n]}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}\right) = \sum_{\mathcal{F}} \operatorname{mult}(\mathcal{F}) \tag{6.3}$$

holds, where the sum goes over all cross-ratio floor diagrams $\mathcal{F}$ of degree $\Delta_d^2(\alpha,\beta)$ that $\mathcal{F}$ satisfy the conditions $p_{[n]}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}$.

Proof. Let $\mathcal{R}_{\Delta_d^2(\alpha,\beta)}\left(p_{[n]}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}\right)$ denote the set of rational tropical stable maps that contribute to $N_{\Delta_d^2(\alpha,\beta)}\left(p_{[n]}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}\right)$. Proposition 6.1.6 yields that all rational tropical stable maps in $\mathcal{R}_{\Delta_d^2(\alpha,\beta)}\left(p_{[n]}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}\right)$ are floor-decomposed. Let $\mathcal{F}_{\Delta_d^2(\alpha,\beta)}\left(p_{[n]}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}\right)$ denote the set of cross-ratio floor diagrams that contribute to the right-hand side of (6.3). Define the map

$$\begin{aligned} \phi : \mathcal{R}_{\Delta_d^2(\alpha,\beta)}\left(p_{[n]}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}\right) &\to \mathcal{F}_{\Delta_d^2(\alpha,\beta)}\left(p_{[n]}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}\right) \\ C &\mapsto \mathcal{F}_C \end{aligned}$$

which maps a floor-decomposed rational tropical stable map to the cross-ratio floor diagram Construction 6.2.9 associates to it. For each elevator e of C, take its x_1-coordinate $\pi(e) \in \mathbb{R}$ (Notation 2.2.9) as codimension one tangency condition X_e of Z_i in the definition of $\operatorname{mult}(\mathcal{F}_C)$ (Definition 6.2.8). Hence if $\operatorname{mult}(C)$ is nonzero, then Proposition 3.2.25 and Corollary 3.2.35 yield that $\operatorname{mult}(\mathcal{F}_{C,i})$ is also nonzero for all pieces $\mathcal{F}_{C,i}$ of $\mathcal{F}_C$. Thus $\operatorname{mult}(\mathcal{F}_C)$ is nonzero. Therefore, by Lemma 6.2.10, the map ϕ is well-defined.

We want to show that ϕ is onto by constructing preimages. Let $\mathcal{F}$ be a cross-ratio floor diagram in $\mathcal{F}_{\Delta_d^2(\alpha,\beta)}\left(p_{[n]}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}\right)$. Using Lemma 6.2.12, there is a vertex v_i of $\mathcal{F}$ such that v_i is only adjacent to thick half-edges. Let $\left(\mathcal{F}_i, \delta_{v_i}, s_{v_i}, \#\lambda_{v_i}, \lambda_{\vec{v_i}}\right)$ be the piece of $\mathcal{F}$ that contains the vertex v_i. As in Definition 6.2.8 its associated degree is $\Delta^2_{s_{v_i}}(\alpha^i, \beta^i)$. Since $\operatorname{mult}(\mathcal{F}_i)$ is nonzero there is a rational tropical stable map C_i corresponding to a point of Z_i (see Definition 6.2.8). Remove v_i and its adjacent edges from $\mathcal{F}$. The resulting graph might be disconnected. Let K be a component of this graph. Using Lemma 6.2.12, there is a vertex v_j of K such that v_j is only adjacent to thick half-edges. There are two cases:

(1) If $v_j \in \mathcal{F}$ is only adjacent to thick half-edges, then associate a rational tropical stable map C_j to v_j like we did before for v_i.

(2) There is an edge e in $\mathcal{F}$ that connects v_i and v_j such that the thick half-edge of e is adjacent to v_i. Let $\pi(e) \in \mathbb{R}$ be the x_1-coordinate of the end associated to e in C_i. We argue like above: Let $\left(\mathcal{F}_j, \delta_{v_j}, s_{v_j}, \#\lambda_{v_j}, \lambda_{\vec{v_j}}\right)$ be the piece of $\mathcal{F}$ that contains v_j. Since $\operatorname{mult}(\mathcal{F}_j)$ is nonzero there is a rational tropical stable map C_j corresponding to a point in Z_j, where for Z_j the only additional codimension one tangency condition is given by $\pi(e)$ (cf. Definition 6.2.8).

Iterating this procedure yields rational tropical stable maps C_t for each piece $\mathcal{F}_t$ of $\mathcal{F}$ such that $C_1, \dots, C_n$ can be glued together by construction. Denote the rational tropical stable map obtained from this gluing by C. The multiplicity of C is given by

$$\operatorname{mult}(C) = \prod_{t=1}^{n} \operatorname{mult}(C_t)$$

because of Proposition 3.2.25 and Corollary 3.2.35. Therefore $C \in \phi^{-1}(\mathcal{F})$.

Note that the procedure above does not depend on the choice of C_t we associated to each $\mathcal{F}_t$. Hence applying Proposition 3.2.34 yields

$$\operatorname{mult}(\mathcal{F}) = \sum_{C \in \phi^{-1}(\mathcal{F})} \operatorname{mult}(C)$$

such that in total Theorem 6.2.13 follows. □

Using Tyomkin's correspondence theorem 2.3.6 (more precisely, Corollary 3.1.20 and Proposition 3.2.7), Theorem 6.2.13 immediately yields the following corollary.

Corollary 6.2.14 (Algebro-geometric count via cross-ratio floor diagrams for $\mathbb{R}^2$)**.** *Notation 2.0.1 is used. Let $\Delta_d^2(\alpha,\beta)$ be a degree in $\mathbb{R}^2$ as in Notation 2.2.4 and let $\Sigma\left(\Delta_d^2(\alpha,\beta)\right)$ be its associated lattice polytope, see Remark 2.2.5. Let $X_{\Sigma\left(\Delta_d^2(\alpha,\beta)\right)}$ be the toric variety associated to $\Sigma\left(\Delta_d^2(\alpha,\beta)\right)$. Then the number of rational algebraic curves in $X_{\Sigma\left(\Delta_d^2(\alpha,\beta)\right)}$ over an algebraically closed field of characteristic zero that satisfy point conditions and classical cross-ratio conditions in general position can be calculated by via a weighted count of cross-ratio floor diagrams of degree $\Delta_d^2(\alpha,\beta)$ that satisfy point conditions and degenerated tropical cross-ratio conditions if each entry of each degenerated tropical cross-ratio is either the label of a contracted end or the label of and end of primitive direction $\pm e_2$ (see Notation 2.2.4).*

Remark 6.2.15. The results of this section are not restricted to rational tropical stable maps of degree $\Delta_d^2(\alpha,\beta)$, or in other words, the results are not restricted to stable maps to Hirzebruch surfaces. One can replace the degree $\Delta_d^2(\alpha,\beta)$ by another degree Δ whose associated polytope $\Sigma(\Delta)$ is *h-transverse*. That is, the results can be extended to stable maps to toric varieties of h-transverse polytopes. Cross-ratio floor diagram techniques can be extended to these degrees Δ in a straightforward way. For more about h-transverse polytopes and floor diagrams, see [AB13].

Example 6.2.16. Consider the degree Δ_3^2. Let $p_{[7]}$ be point conditions and let $\lambda = \{1,2,3,4\}$ be a degenerated tropical cross-ratio such that these conditions are in a stretched configuration. We want to determine the number $N_{\Delta_3^2}\left(p_{[7]}, \lambda\right)$ using cross-ratio floor diagrams. For that, draw all floor diagrams of degree Δ_3^2 on the 7 vertices $v_1 < \cdots < v_7$ that satisfy the point conditions $p_{[7]}$ and the degenerated tropical cross-ratio λ. Since we have seven point conditions, no floors of size 3 or 2 in floor-decomposed rational tropical stable maps that satisfy the conditions $p_{[7]}, \lambda$. Figure 6.4 shows all possible cross-ratio floor diagrams. Note that in this example we do not need all discrete data a cross-ratio floor diagram is equipped with, i.e. floors of size 1 are drawn white and floors of size 0 are drawn black (instead of specifying s_{v_i} for each floor). The number of degenerated tropical cross-ratios satisfied at each floor is obvious (we only have one degenerated tropical cross-ratio). The labels of ends adjacent to each floor are dropped here, so we need to add a factor of $(d!)^3$ to the final count. Moreover, all weights of edges are one. By considering the multiplicities of each piece $\mathcal{F}_i$ of a cross-ratio floor diagram $\mathcal{F}$ in Figure 6.4, we end up with multiplicity 1 for all cross-ratio floor diagrams shown in Figure 6.4. Hence

$$N_{\Delta_3^2}\left(p_{[7]}, \lambda\right) = 4 * (3!)^3 = 864.$$

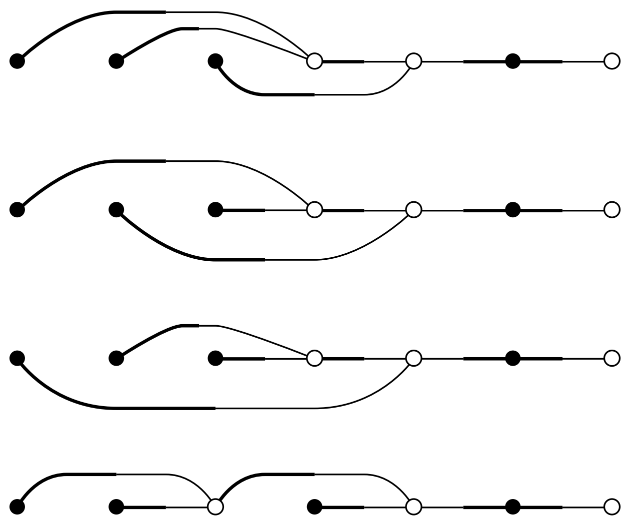


Figure 6.4: Cross-ratio floor diagrams with floors of size 0 (black) and 1 (white). The vertices $v_{[7]}$ are ordered from left to right such that the smallest vertex v_1 is left. Notice that $\#\lambda_{v_4} = 1$ for $\#\lambda_{v_4}$ of the three cross-ratio floor diagrams on the top and that $\#\lambda_{v_3} = 1$ for the cross-ratio floor diagram on the bottom.

6.3 Cross-ratio floor diagrams for $\mathbb{R}^3$

Cross-ratio floor diagrams for $\mathbb{R}^3$ are now introduced. They extend the degeneration technique of the last section to higher dimension. As a result, the algebro-geometric numbers of Tyomkin's correspondence theorem 2.3.6 can be calculated for certain 3-dimensional toric varieties.

6.3.1 Condition flows

In this subsection, *condition flows* on rational tropical stable maps are defined. They help us generalizing the "thin" and "thick" edges of cross-ratio floor diagrams of Section 6.2 which made sure that rational tropical stable maps that degenerate to a given cross-ratio floor diagram are fixed by given conditions. In other words, the motivation behind conditions flows is the following: In case of a rational tropical stable map C to $\mathbb{R}^2$ and some set S of general positioned conditions, the following implication holds (see for example [GM08]): If C satisfies all given conditions S and C has a *string*[2], then C is not fixed by the given conditions. Now that we

[2]A *string* is defined as a path between two non-contracted ends such that no contracted end that satisfies a point condition lies on it.

are in higher dimension (i.e. let C be a rational tropical stable map to $\mathbb{R}^m$), we aim for a generalization, namely the implication: If C satisfies all given conditions S and there is no *conditions flow* of type m on C, then C is not fixed by the given conditions. We remark, that condition flows are defined for rational tropical stable maps to $\mathbb{R}^m$ for arbitrary $m \in \mathbb{N}_{>0}$. A similar construction was independetly introduced by Mandel and Ruddat [MR19] to study multiplicities of tropical curves.

Definition 6.3.1 (Leaky)**.** A graph G together with a function $leak : V(G) \to \mathbb{N}$ that assigns a natural number to each vertex of the graph is called *leaky*.

Definition 6.3.2 (Flow)**.** Let G be a tree, where we allow *ends*, i.e. edges that are adjacent to a single vertex only without forming a loop. Each edge e that is adjacent to two vertices consists of two half-edges e_1, e_2. If v is a vertex of G that is adjacent to e, then we refer to the half-edge e_i ($i = 1, 2$) of e that is adjacent to v as *outgoing edge of* v and to the other half-edge of e as *incoming edge of* v. If e is an edge that is adjacent to a single vertex v, then e is considered to be an *incoming edge of* v. A *flow structure* on G is given by a map R that assigns to each half-edge and to each end of G an element of $\mathbb{N}$. We refer to the image $R(e_i)$ of an end or a half-edge e_i under the flow structure map R as *flow on* e_i. Moreover, the *flow of a vertex* v is defined by

$$\mathrm{flow}(v) := \sum_{e \text{ incoming edge of } v} R(e).$$

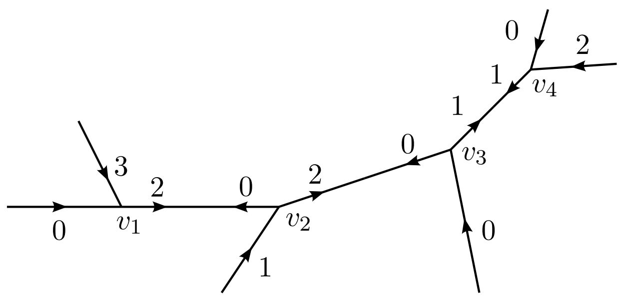


Figure 6.5: An example of a flow structure, where each half-edge and if it is considered incoming or outgoing is indicated by an arrow. The numbers on the arrows are the numbers associated to the half-edges by the flow structure. The vertices are denoted by v_i for $i \in [4]$ and $\mathrm{flow}(v_i) = 3$ for $i \in [4]$.

Definition 6.3.3 (Condition flow)**.** Let G be a leaky graph with a flow as defined in 6.3.2. The flow structure on G is called *condition flow of type* m if it satisfies the following properties:

(P1) If e is an edge of G that consists of the two half-edges e_1, e_2, then

$$R(e_1) + R(e_2) = m - 1.$$

(P2) The flow is *balanced* on each vertex, that is

$$\mathrm{flow}(v) - \mathrm{leak}(v) = 0$$

holds for all vertices v of G.

Example 6.3.4. Figure 6.5 provides an example of a condition flow of type 3, where $\mathrm{leak}(v_i) = 3$ for $i \in [4]$.

Remark 6.3.5. A condition flow of type 2 is a flow structure on a graph G such that for each edge consisting of two half-edges there is exactly one half-edge e_1 with $R(e_1) = 1$ and another half-edge e_2 with $R(e_2) = 0$. There are different ways of encoding this condition flow of type 2 into a graph G. In [GMS13] orientations on G where used to indicate half-edges e_1 with $R(e_1) = 1$, and in [CJMR21] "thick" half-edges were used. Notice that we also used thick half-edges for cross-ratio floor diagrams for $\mathbb{R}^2$. One advantage of condition flows over these ad hoc constructions is that they are applicable in higher dimensions.

Lemma 6.3.6. *A condition flow of type m on a tree is uniquely determined by its leak function and the flow on its ends.*

Proof. Assume there are two condition flows of type m with the same leak function on a tree G. First, note that the leak function determines the flows on the vertices by (P2). Assume there is at least one half-edge e_1 on which the flows differ. Since we assumed that the flows on the ends are equal, there is another half-edge e_2 adjacent to e_1. Thus the flows also differ on e_2 because of (P1). So there is an edge e of G on which the flows differ. Denote a vertex to which e is adjacent by v. If v is only adjacent to one bounded edge, namely e, then (P2) yields a contradiction. Hence there is another edge $e' \neq e$ adjacent to v on which the flows differ because of (P2). Since G is a tree, there is a vertex v' of G which is only adjacent to one bounded edge such that the flows on this edge differ, which leads to the same contradiction as above. □

Construction 6.3.7. Let G be a tree with fixed flows on its ends. We construct a flow structure on G the following way. Note that we can think of each bounded edge of G as being glued from two half-edges (by cutting it into two halves). Set all flows on all half-edges that are no ends to be zero. We use the following procedure to spread the flows of the ends to all half-edges: Choose a vertex v of G. Now spread the flows on G according to the following rule. If a vertex v has r outgoing edges $e_{\text{out},[r]}$ and $\tilde{r}$ incoming edges $e_{\text{inc},[\tilde{r}]}$ such that $e_{\text{inc},i}$ and $e_{\text{out},i}$ form an edge for $i \in [r]$, then

$$R(e_{\text{out},i}) = \begin{cases} \text{flow}(v) - R(e_{\text{inc},i}) - 1 & \text{, if flow}(v) - R(e_{\text{inc},i}) > 0 \\ 0 & \text{, if flow}(v) - R(e_{\text{inc},i}) = 0 \text{ and flow}(v) \neq 0 \end{cases} \tag{6.4}$$

for $i = 1, \ldots, r$.

Repeat with another vertex v' of G. Notice that flows on half-edges can at most increase. Stop when the flows on all edges stay the same. This construction yields a unique flow on G.

Example 6.3.8. We want to illustrate Construction 6.3.7. Figure 6.6 provides an example of a tree G on the four vertices v_i for $i \in [4]$. The flows on ends of G are indicated in Figure 6.6. Before starting with the procedure, all flows of non-end half-edges are set to be zero as in Figure 6.6.

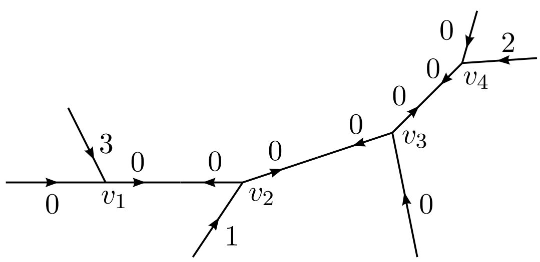


Figure 6.6: The graph G to which Construction 6.3.7 should be applied.

See Figure 6.7 for the following: In the first step, Construction 6.3.7 is applied to determine the flow on the outgoing half-edges of v_1. After that, Construction 6.3.7 is applied to determine the flows on the outgoing half-edges of v_2. If Construction 6.3.7 is then applied to v_4 and after that to v_3, then the procedure terminates. The fourth step in Figure 6.7 shows the resulting flow structure on G which is the same as the one in Figure 6.5.

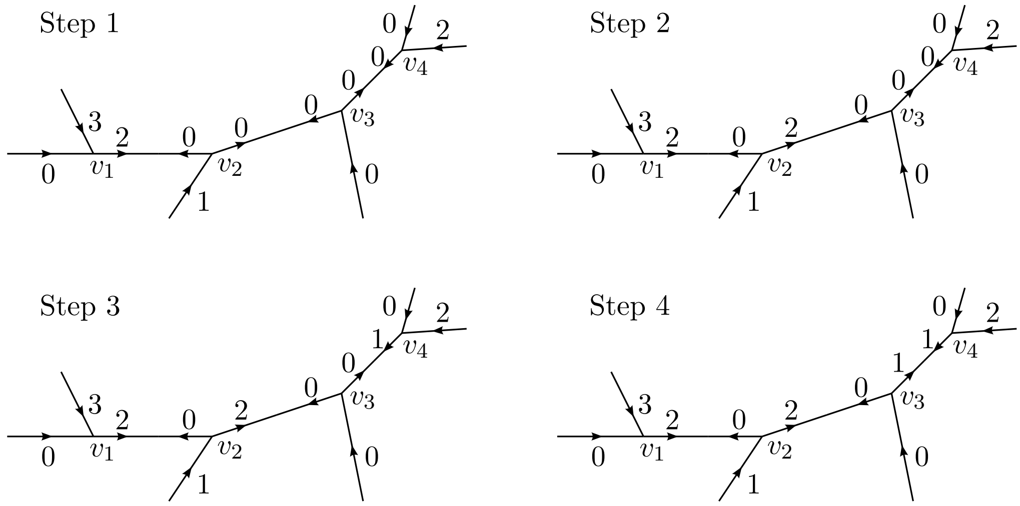


Figure 6.7: The progress of Construction 6.3.7 applied to G and the condition flow of type 3 it yields on G.

Proof that Construction 6.3.7 terminates uniquely. We use induction on the number N of vertices of G. If $N = 1$, then there is nothing to show. So let $N = 2$. Then the procedure of Construction 6.3.7 stops uniquely after at most 2 steps. For the induction step notice that G is a tree, i.e. there is a vertex v that is adjacent to exactly one edge e that is no end. The flows on the ends of G are given and thus

$$R(e_{\text{out},v}) = \begin{cases} 0 & \text{if all ends adjacent to } v \text{ are of flow zero,} \\ \text{flow}(v) - 1 & \text{else} \end{cases}$$

is uniquely determined. Let v' be the vertex adjacent to v via e. Consider the tree G' that arises from G the following way: forget e, v and all ends adjacent to v, then attach a new end e' to v' and assign the flow $R(e_{\text{out},v})$ to e'. Now run the procedure of Construction 6.3.7 on G'. By induction, this procedure terminates uniquely. Notice that the missing flow on e associated to the outgoing half-edge $e_{\text{out},v'}$ of v' is determined by (6.4). Moreover, the flow on $e_{\text{out},v'}$ does not affect the other flows on G which the procedure generated on G'. Thus Construction 6.3.7 terminates uniquely. $\square$

Definition 6.3.9 (Induced flows)**.** Consider a rational tropical stable map C that contributes to the number $N_{\Delta_d^m(\alpha,\beta)}\left(p_{[n]}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}\right)$. Associate flows $R(e)$ to ends e of C the following way: If e is a non-contracted end of C that satisfies a codimension two tangency condition L_k for some $k \in \underline{\kappa}^\gamma$, $\gamma \in \{\alpha, \beta\}$, then $R(e) := m - 2$. If e satisfies a codimension one tangency condition P_f for some $f \in \underline{\eta}^\gamma$, $\gamma \in \{\alpha, \beta\}$, then $R(e) := m - 1$. If e is a contracted end of C satisfying a point condition, then $R(e) := m$. Otherwise, set $R(e) := 0$. We refer to these flows on the ends as *induced flows* from the tangency and point conditions.

Example 6.3.10. Let C be the rational tropical stable map depicted in Figure 3.2 that contributes to the number $N_{\Delta_1^3((0,1,0,\dots),(1,0,\dots))}(p_1, L_3, P_6)$ as in Example 3.1.14. Step 1 in Figure

6.7 shows the flows conditions induce on ends of C. Example 6.3.8 shows the flow structure Construction 6.3.7 assigns so C. Notice that the resulting flow structure is a condition flow of type 3 and that the constructed flow structure does not depend on the order of the vertices from which the flows were spread.

Proposition 6.3.11. *Let C be a rational tropical stable map that contributes to the number $N_{\Delta_d^m(\alpha,\beta)}\left(p_{[n]}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}\right)$ such that flows on its ends are induced from the point and tangency conditions as in Definition 6.3.9. Then Construction 6.3.7 associates the unique condition flow of type m to C, where the leak function is given by* $\operatorname{leak}(v) = m$ *for all vertices v of C.*

Proof. Given a rational tropical stable map C as in Proposition 6.3.11, we give another interpretation of the flow constructed in 6.3.7, namely in terms of spatial restrictions the vertices of C impose on their neighbors. By restrictions we mean the following: Let Γ be the combinatorial type of C, i.e. C without its metric structure. Since C fulfills all given conditions, we are able to re-embed Γ into $\mathbb{R}^m$, i.e. we are able to reconstruct the lengths of all edges of C. To do so, we proceed in the following way: Let $v \in C$ be a vertex adjacent to a contracted end satisfying a point condition p_i for some i, then choose $\Gamma \to \mathbb{R}^m$ in such a way that $v \mapsto p_i$. Let e be a bounded edge adjacent to v and some other vertex v'. Since Γ knows the direction of e in $\mathbb{R}^m$, fixing v imposes an $(m-1)$-dimensional restriction on the position of v'. In other word, v' can only move along the direction of e. We encode this restriction from v to v' into Γ by interpreting e as two glued half-edges e_1, e_2, where the half-edge e_i adjacent to v is equipped with a number $R(e_i) = m-1$. We refer to this half-edge as outgoing edge of v or as incoming edge of v'. Iteratively, the restrictions spread along Γ, i.e. let e' be another bounded edge adjacent to v' and some other vertex $v'' \neq v$. Since we know the direction of e' in $\mathbb{R}^m$, the 1-dimensional movement of v' allows v'' to only move along two directions. More precisely, we may vary the length of e and the length of e'. Said differently, v' imposes at least an $(m-2)$-dimensional restriction on v''.

Obviously, we could also have started with a tangency condition, i.e. some other restriction incoming to a vertex via an end.

We claim that the flow structure constructed from restrictions passing from a vertex to another via half-edges fulfills the procedure equation (6.4) describes in Construction 6.3.7.

Denote the equations of (6.4) by I and II, from top to bottom.

- Let v be a vertex that is adjacent to another vertex v' via an edge e such that v gains all its spatial restrictions via the incoming half-edge e_{inc} of e. Then v does not impose a spatial restriction to v' via its outgoing half-edge e_{out} of e. Hence II holds. Said differently, a vertex cannot pass spatial directions back to an adjacent vertex from which they came.

- I holds since repeating the argument of II yields the summand $R(e_{\text{inc},i})$ of (6.4), and as we saw before, passing over a vertex lowers the number of restrictions in general by 1, where in general means that edges e, e' adjacent to the same vertex v are usually not parallel – if they are parallel, then there is (because C is fixed by the general positioned conditions) a end adjacent to v that either satisfies a point condition or some tangency condition. Notice that in both of these two special cases I holds.

Hence our flow structure on Γ defined as restrictions passing from one vertex to another is governed by the same equations as the flow structure assigned to Γ by Construction 6.3.7. Hence these two flow structures on Γ coincide. Next, we claim that the flow structure on Γ interpreted as spatial restrictions is a condition flow of type m, i.e. it satisfies (P1) and (P2) of Definition 6.3.3. Given a bounded edge e of C, cut it and stretch it to infinity. Denote the two components of C obtained that way by C_1, C_2, where e_1 is the end of C_1 that used to be e and e_2 is the analogous end of C_2. We use the following notation: Let Δ_i be the degree of C_i, let $\underline{n_i} \subset [n]$ be the point conditions satisfied by C_i, let $\underline{l_i} \subset [l]$ be the degenerated tropical cross-ratios satisfied by C_i, let $\underline{\kappa_i} \subset \underline{\kappa}^\alpha \cup \underline{\kappa}^\beta$ be the codimension two tangency conditions satisfied by C_i and let $\underline{\eta_i} \subset \underline{\eta}^\alpha \cup \underline{\eta}^\beta$ be the codimension one tangency conditions satisfied by C_i for $i = 1, 2$. Let N_i be the number of contracted ends of C_i for $i = 1, 2$. Then

$$\#\Delta_1 + \#\Delta_2 - 2 = \#\Delta_d^m(\alpha, \beta) \tag{6.5}$$

and

$$\#\Delta_i - 3 + N_i + m = m \cdot \#\underline{n_i} + \#\underline{l_i} + (m-1) \cdot \#\underline{\eta_i} + (m-2) \cdot \#\underline{\kappa_i} + R(e_i) \tag{6.6}$$

hold for $i = 1, 2$. Adding both equations from (6.6), using (6.5) and applying (3.6) yields (P1). Moreover, (P2) can be satisfied by defining the leak function this way. Then the leak function coincides with the one given in Proposition 6.3.11 since all conditions are in general position. Moreover, this condition flow is unique due to Lemma 6.3.6. □

Proposition 6.3.11 allows us to think about condition flows the way we think about strings in rational tropical stable maps to $\mathbb{R}^2$: Proposition 6.3.11 is an exclusion criterion for stable maps that are not contributing to $N_{\Delta_d^m(\alpha,\beta)}\left(p_{[n]}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}\right)$ on the level of combinatorial types. If C is the combinatorial type of a tropical stable map C' and Construction 6.3.7 does not lead to a condition flow of type m with the leak function given in Proposition 6.3.11, then C' cannot contribute to $N_{\Delta_d^m(\alpha,\beta)}\left(p_{[n]}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}\right)$.

Remark 6.3.12. Another way to think about flows is the following: each vertex v of a rational tropical stable map to $\mathbb{R}^m$ is a point in $\mathbb{R}^m$, i.e. the minimal number of affine linear equations needed to cut out v is m. The flow of v is the number of equations v needs to satisfy. These equations arise from imposing conditions to our rational tropical stable map as in the proof of Proposition 6.3.11, and these equations are affine linear since rational tropical stable maps are piecewise linear. Choosing all conditions in general conditions means to choose the minimal number of conditions needed to fix our rational tropical stable map, i.e. the matrix of affine linear equations associated to each vertex needs to have full rank, or in other words, the flow of each vertex needs to be m if each vertex should be fixed.

If there are not enough conditions to fix a rational tropical stable map, then parts of it are movable. These movable parts are encoded in the flow structure since all vertices with flow less than m are movable. The special case of one missing condition and one *movable component* for rational tropical stable maps to $\mathbb{R}^2$ was studied in Subsection 4.1.1 in order to deduce a general tropical Kontsevich's formula. We remark here, that we also could have used flows there to describe which parts of a rational tropical stable map are movable.

6.3.2 Multiplicities of floor-decomposed rational tropical stable maps to $\mathbb{R}^3$

Multiplicities of floor-decomposed rational tropical stable maps to $\mathbb{R}^3$ can not be calculated locally at vertices (see also Example 3.1.14). On the other hand, we would like to see that multiplicities of cross-ratio floor diagrams for $\mathbb{R}^3$ are local at vertices of such diagrams. Interestingly, the techniques we developed to prove the general tropical Kontsevich's formula 4.3.4 come in handy. They provide a way of expressing multiplicities of floor-decomposed rational tropical stable maps to $\mathbb{R}^3$ in terms of multiplicities of their floors.

Notation 6.3.13 (1/1 and 2/0 edges)**.** Let C be a floor-decomposed rational tropical stable map that contributes to $N_{\Delta^3_d(\alpha,\beta)}\left(p_{[n]}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}\right)$. Whenever we refer to the condition flow of C, we mean that C is equipped with the condition flow of type 3 associated to it using Construction 6.3.7. In particular, given a bounded edge e of C that consists of two half-edges e_1, e_2, we refer to e as 1/1 edge if $R(e_1) = R(e_2) = 1$, and we refer to e as 2/0 edge if either $R(e_1) = 2$ and $R(e_2) = 0$ or $R(e_1) = 0$ and $R(e_2) = 2$.

Definition 6.3.14 (Floor graph)**.** Notation 2.2.4 is used. Let $p_{[n]}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}$ be conditions in a stretched configuration with respect to the degree $\Delta^3_d(\alpha,\beta)$ as in Definition 6.1.1. Let C be a rational tropical stable map of degree $\Delta^3_d(\alpha,\beta)$ that is fixed by these conditions. Then C is floor-decomposed by Proposition 6.1.6. We associate a so-called *floor graph* Γ_C, i.e. a weighted graph on an ordered set of vertices with a flow structure, to C the following way: each vertex of Γ_C corresponds to a floor of C, an edge of Γ_C corresponds to an elevator of C and connects the vertices of Γ_C that correspond to the floors the elevator connects in C. Weights on the edges of Γ_C are induced by weights on the elevators of C. The given point conditions $p_{[n]}$ are totally ordered according to their x_3-coordinates. Thus the floors of the floor-decomposed rational tropical stable map C are also totally ordered, i.e. the vertices of Γ_C are totally ordered as well. Moreover, a flow structure on Γ_C is induced by the flows on the elevators of C (see Notation 6.3.13), i.e. if an elevator is a 1/1 (resp. 2/0) elevator, then its associated edge in Γ_C is a 1/1 (resp. 2/0) edge.

Example 6.3.15. Figure 6.8 shows the floor graph Γ_C associated to the floor-decomposed rational tropical stable map C of Example 6.1.5. Notice that the elevator of C is a 1/1 elevator.

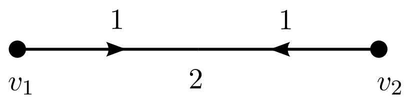


Figure 6.8: The floor graph Γ_C associated to the floor-decomposed rational tropical stable map C of Example 6.1.5. The vertex v_i of Γ_C corresponds to the floor C_i of C for $i = 1, 2$.

The following construction is similar to Definition 6.2.7. It allows us to break floor-decomposed rational tropical stable map to $\mathbb{R}^3$ into parts by cutting elevators.

Construction 6.3.16 (Cutting elevators)**.** Notation of Definition 3.2.4 is used. Let $\Delta^3_d(\alpha,\beta)$ be a degree and let $p_{[n]}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}$ be conditions in a stretched configuration with respect to $\Delta^3_d(\alpha,\beta)$ as in Definition 6.1.1. Let C be a floor-decomposed rational tropical stable map of degree $\Delta^3_d(\alpha,\beta)$ that is fixed by these conditions. If e is an elevator of C, then we construct two rational tropical stable maps C_1, C_2 from C by cutting e. The loose ends of e are stretched to infinity. These ends (with their induced weights) are denoted by $e_i \in C_i$ for $i = 1, 2$ and the vertex adjacent to e_i is denoted by v_i for $i = 1, 2$. By abuse of notation we also refer to the label of e_i by e_i.

The condition flow on C induces flow structures on C_i, where the flow on e_i is given by the flow on e that is incoming to v_i for $i = 1, 2$.

As in Definition 6.2.7 the degenerated tropical cross-ratios are *adapted* to the cutting. Denote a degenerated tropical cross-ratio λ_j for $j \in [l]$ that is adapted to e_i by $\lambda_j^{\to e_i}$.

If e is a 2/0 elevator, then the component C_i to which 2 is the incoming flow along e satisfies the codimension one tangency condition P_{e_i}, given by $\pi(v_t)$ for $i \neq t \in \{1, 2\}$, where π is the projection from Notation 2.2.9. If e is a 1/1 elevator, then each C_i satisfies a codimension two condition L_{e_i} for $i = 1, 2$, given by the projection π of the movement of the vertices v_i. Notice that ends of L_{e_i} are a priori not of standard direction. However, as we see with Corollary 6.3.23, we can assume that L_{e_i} for $i=1,2$ are — like $L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}$ — tropical multi-lines.

Denote the new sets of general positioned conditions each rational tropical stable map C_i for $i=1,2$ satisfies by $p_{\underline{n_i}}, L_{\underline{\kappa_i}^\alpha}, L_{\underline{\kappa_i}^\beta}, P_{\underline{\eta_i}^\alpha}, P_{\underline{\eta_i}^\beta}, \lambda_{\underline{l_i}}^{\to e_i}$. Moreover, denote the degree of C_i by $\Delta^3_{s_i}\left(\alpha^i,\beta^i\right)$ for $i=1,2$ as in Notation 2.2.4.

Notation 6.3.17. If Construction 6.3.16 is used to cut more than one elevator, then it can be necessary to adapt the degenerated tropical cross-ratios $\lambda_{[l]}$ to more than one cut. This is denoted by $\lambda_j^{\to}$ for $\lambda_j \in \lambda_{[l]}$.

Notation 6.3.18 (Replacing tangency conditions on 1/1 edges)**.** Let C be a floor-decomposed rational tropical stable map as in Construction 6.3.16 and let e be a 1/1 elevator. See Construction 6.3.16 for the following: cut e and obtain two new tangency conditions L_{e_1} (resp. L_{e_2}) that C_1 (resp. C_2) satisfy. Let v_i be the vertex of C_i that is adjacent to the end e_i which satisfies L_{e_i}. Let $\pi(v_i)\in\mathbb{R}^2$ denote the projection of v_i under π along the elevator direction (see also Notation 2.2.9) for $i=1,2$. Let L_{st} be a degenerated line (Definition 2.2.21) such that its vertex is translated to $\pi(v_1)$ (resp. $\pi(v_2)$). Let $C_{i,st}$ denote the rational tropical stable map that equals C_i, but where the L_{e_i} tangency condition is replaced with L_{st}, i.e. $C_{i,st}$ satisfies L_{st} instead of L_{e_i} for $i=1,2$.

Analogously to Notation 4.2.1 the multiplicities of C_i and $C_{i,st}$ may differ. In particular, the multiplicity of $C_{i,st}$ may be zero, whereas the multiplicity of C_i can be nonzero, see Example 6.3.19.

Example 6.3.19. Consider the floor C_2 of C from Example 6.1.5. The ev-multiplicity of $C_{2,10}$ equals 1 since it is the absolute value of the determinant of the ev-Matrix of Example 3.1.14. The ev-multiplicity of $C_{2,01}$ is 0 since $C_{2,01}$ is not fixed by its conditions.

Proposition 6.3.20. *Let C be a floor-decomposed rational tropical stable map that contributes to $N_{\Delta^3_d(\alpha,\beta)}\left(p_{[n]}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}\right)$. Let e be an elevator of weight $\omega(e)$ and cut C along e as in Construction 6.3.16 to obtain C_1, C_2.*

(a) If e is a 2/0 elevator, then

$$\operatorname{mult}(C)=\omega(e)\cdot\operatorname{mult}(C_1)\cdot\operatorname{mult}(C_2).$$

(b) If e is a 1/1 elevator, then

$$\operatorname{mult}(C)=\omega(e)\cdot\left|\det\left(M(C_{1,10})\right)\cdot\det\left(M(C_{2,01})\right)-\det\left(M(C_{1,01})\right)\cdot\det\left(M(C_{2,10})\right)\right|,$$

where $M(\cdot)$ denotes an ev*-matrix and tangency conditions are replaced as in Notation 6.3.18. In particular, if* $\det\left(M(C_{1,01})\right)$ *or* $\det\left(M(C_{2,10})\right)$ *vanishes, then*

$$\operatorname{mult}(C)=\omega(e)\cdot\operatorname{mult}(C_{1,10})\cdot\operatorname{mult}(C_{2,01}).$$

Proof. It is sufficient to prove (a), (b) for ev-multiplicities only since the cross-ratio multiplicities can be expressed locally at vertices, see Proposition 3.2.25. Thus contributions from vertices to cross-ratio multiplicities do not depend on cutting edges.

(a) Denote the vertices adjacent to the elevator e by v_1, v_2 such that $v_1\in C_1$ and $v_2\in C_2$ and assume without loss of generality that the half-edge e_1 of e incoming to v_1 has zero flow, i.e. $R(e_1) = 0$. Consider the ev-matrix $M(C)$ (Definition 3.1.11) of C with base point v_1, i.e.

$$M(C) = \begin{array}{rc} & \begin{array}{cccc} \text{Base } v_1 & \text{lengths in } C_1 & \text{lengths in } C_2 & l_e \end{array} \\ \begin{array}{r} \text{conditions in } C_1 \\ \\ \\ \text{conditions in } C_2 \end{array} & \left(\begin{array}{c|c|c|c} * & * & 0 & \begin{matrix}0\\ \vdots \\ 0\end{matrix} \\ \hline * & 0 & * & \begin{matrix}*\\ \vdots \\ *\end{matrix} \end{array}\right) \end{array}$$

where l_e is the length-coordinate associated to e. Let y_1 (resp. y_2) be the number of rows that belong to the conditions in C_1 (resp. C_2). Let x_1 be the number of columns that belong to the base point and the lengths in C_1. Let x_2 be the number of columns that belong to the lengths in C_2. Using notation from Definition 3.2.4 and Construction 6.3.16 with $\underline{\kappa_i} := \underline{\kappa_i}^\alpha \cup \underline{\kappa_i}^\beta$ and $\underline{\eta_i} := \underline{\eta_i}^\alpha \cup \underline{\eta_i}^\beta$ for $i = 1, 2$, we obtain

$$\begin{aligned} x_1 &= \#\Delta^3_{s_1}\left(\alpha^1, \beta^1\right) + \#\underline{n_1} - \#\underline{l_1}, \\ y_1 &= 3 \cdot \#\underline{n_1} + 2 \cdot \#\underline{\eta_1} + \#\underline{\kappa_1}. \end{aligned}$$

On the other hand, C_1 is fixed by its set of conditions since the incoming flow at v_1 along e is zero, i.e. (3.6) can be applied to obtain

$$\#\Delta^3_{s_1}\left(\alpha^1, \beta^1\right) = 2 \cdot \#\underline{n_1} + 2 \cdot \#\underline{\eta_1} + \#\underline{\kappa_1} + \#\underline{l_1}$$

which yields

$$\begin{aligned} x_1 &= 3 \cdot \#\underline{n_1} + 2 \cdot \#\underline{\eta_1} + \#\underline{\kappa_1} \\ &= y_1. \end{aligned}$$

Thus the bold red lines in $M(C)$ above divide $M(C)$ into squares, hence

$$|\det(M(C))| = \operatorname{mult}(C_1) \cdot |\det(M)|,$$

where M is the square matrix on the bottom right. Let M' denote the matrix M without its l_e-column. Define the matrix

$$M(C_{2,P}) := \begin{array}{c} \begin{array}{cc} \text{Base } P & l_e \end{array} \\ \left(\begin{array}{ccc|c|c} 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & 0 & \\ \hline & & & * & \\ & * & & \vdots & M' \\ & & & * & \end{array}\right) \end{array}$$

where the first three columns are chosen in such a way that $M(C_{2,P})$ is the ev-matrix of C_2 with respect to an additional point called P that we added to the end e_2, i.e. there is an additional 3-valent vertex of C_2 that is adjacent to e_2 and a contracted bounded end P. Notice that

$$|\det(M)| = |\det(M(C_{2,P}))|$$

holds. Since $|\det(M(C_{2,P}))|$ is independent of the choice of the base point of $C_{2,P}$ (changing the base point can be achieved performing column operations), we can choose the new base point to be v_2, i.e. we obtain the matrix $M(C_{2,v_2})$ with

$$|\det(M(C_{2,P}))| = |\det(M(C_{2,v_2}))|.$$

The l_e-column of $M(C_{2,v_2})$ has exactly one nonzero entry, namely $\omega(e)$ that is associated to evaluating the x_3-coordinate of P, where $\omega(e)$ is the weight of the cut elevator edge e. Thus, using Laplace expansion on the l_e-column, we obtain

$$|\det(M(C_{2,v_2}))| = \omega(e) \cdot |\det(M(C_2))|,$$

where C_2 satisfies a codimension one tangency condition P_{e_2} as in Construction 6.3.16. Putting everything together yields part (a) of Proposition 6.3.20.

(b) It can be assumed that the weight $\omega(L_k)$ of each multi-line L_k (Definition 2.2.19) for $k \in \underline{\kappa}^\alpha \cup \underline{\kappa}^\alpha$ equals 1 since we can pull out the factor $\omega(L_k)$ from each row of the *ev*-matrix, apply all the following arguments and multiply with $\omega(L_k)$ later.

Notation from Construction 6.3.16 is used, i.e. we denote the vertex of C_1 adjacent to the cut elevator e by v_1 and the other vertex adjacent to e by v_2. The ev-matrix $M(C)$ of C with respect to the base point v_1 is given by

$$M(C) = \begin{array}{r} \\ \text{conditions in } C_1 \\ \\ \\ \text{conditions in } C_2 \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Base } v_1 & \text{lengths in } C_1 & & \text{lengths in } C_2 & l_e \end{array} \\ \left(\begin{array}{c|c|c|c|c} & & * & & 0 \\ * & * & \vdots & 0 & \vdots \\ & & * & & 0 \\ \hline & & 0 & & * \\ * & 0 & \vdots & * & \vdots \\ & & 0 & & * \end{array}\right) \end{array}.$$

The bold red lines divide $M(C)$ into square pieces at the upper left and the lower right. This follows from similar arguments used in the proof of part (a). Let M be the matrix consisting of the lower right block of $M(C)$ whose entries (see above) are indicated by $*$ and its columns are associated to lengths in C_2. Let $A = (a_{ij})_{ij}$ be the submatrix of $M(C)$ given by the rows that belong to conditions of C_1 and by the base point's columns and the columns that are associated to lengths in C_1, i.e. A consists of all the $*$-entries above the bold red line in $M(C)$.

Consider the Laplace expansion of the rightmost column of A. Recursively, use Laplace expansion on every column that belongs to the lengths in C_1 starting with the rightmost column. Eventually, we end up with a sum in which each summand contains a factor $\det(N_i)$ for a matrix N_i, which is one of the following two matrices, namely

$$N_1 = \begin{array}{c} \begin{array}{ccc} \text{Base } v_1 & l_e & \end{array} \\ \left(\begin{array}{cc|c|c|c} * & * & 0 & 0 & \multirow{2}{*}{0} \\ * & * & 0 & 0 & \\ \hline & & * & * & \\ & & \vdots & \vdots & \\ \multicolumn{2}{c|}{*} & \vdots & \vdots & M \\ & & \vdots & \vdots & \\ & & * & * & \end{array}\right) \end{array} \quad \text{and} \quad N_2 = \begin{array}{c} \begin{array}{ccc} \text{Base } v_1 & l_e & \end{array} \\ \left(\begin{array}{cc|c|c|c} * & * & b_1 & 0 & \multirow{2}{*}{0} \\ * & * & b_2 & 0 & \\ \hline & & * & * & \\ & & \vdots & \vdots & \\ \multicolumn{2}{c|}{*} & \vdots & \vdots & M \\ & & \vdots & \vdots & \\ & & * & * & \end{array}\right) \end{array}.$$

Since the l_e column of N_1 equals $\omega(e)$ times the third column of N_1, the determinant $\det(N_1)$ is zero and thus does not occur in the Laplace expansion from above. In case of matrix N_2, at least one of the entries b_1, b_2 is 1. Moreover, if b_1 or b_2 equals 1, then this 1 is the only nonzero entry in the whole row. Thus Laplace expanding this row and dividing the l_e column by $\omega(e)$ to obtain the column $\tilde{l}_e$ (which gives the global factor of $\omega(e)$ in part (b) of Proposition 6.3.20) yields the following 3 cases.

$$M_{a_{r1}a_{r2}} := \begin{array}{c} \begin{array}{cccccc} \phantom{a_{r1}} & \phantom{a_{r2}} & \tilde{l}_e & \multicolumn{3}{c}{\text{lenghts in } C_2} \end{array} \\ \left(\begin{array}{ccc|ccc} a_{r1} & a_{r2} & 0 & 0 & \dots & 0 \\ \hline & & & & & \\ & * & & & M & \\ & & & & & \end{array}\right) \end{array},$$

where $(a_{r1}, a_{r2}) = (1, 0)$, $(a_{r1}, a_{r2}) = (0, 1)$ or $(a_{r1}, a_{r2}) = (1, -1)$ are the remaining entries of A in its r-th row after the recursive procedure. Notice that in each case the entries of the first 3 columns are of such a from that M_{st} for $st = 10, 01, 1\text{-}1$ is the ev-matrix of $C_{2,st}$ (see Notation 6.3.18) with base point v_2.

We can group the summands according to the values a_{r1}, a_{r2} and obtain in total

$$|\det(M(C))| = \omega(e) \cdot |F_{10} \cdot \det(M_{10}) + F_{01} \cdot \det(M_{01}) + F_{1\text{-}1} \cdot \det(M_{1\text{-}1})|, \tag{6.7}$$

where $F_{st} \in \mathbb{R}$ for $st = 10, 01, 1\text{-}1$ are factors occurring due to the recursive Laplace expansion. More precisely, let b' be the number of bounded edges in C_1 and define $b := b' + 1$, i.e. b is the total number of Laplace expansions we applied. Then

$$F_{st} = \sum_{r:(a_{r1}, a_{r2}) = (s,t)} \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{j=3}^{3+b} a_{\sigma(j)j}, \tag{6.8}$$

where the second sum goes over all bijections

$$\sigma : \{3, \dots, 3 + b\} \to \{1, \dots, r - 1, r + 1, \dots, b + 1\},$$

i.e. it goes over all possibilities of choosing for each column Laplace expansion was used on an entry in a row of A which is not the r-th row.

Let $A_{10}, A_{01}, A_{1\text{-}1}$ be the square matrices obtained from A by adding the new first row $(1, 0, 0, \dots, 0)$, $(0, 1, 0 \dots, 0)$ or $(1, -1, 0, \dots, 0)$ to A. Again, notice that A_{st} for $st = 10, 01, 1\text{-}1$ is the ev-matrix of $C_{1,st}$ (Definition 3.1.11, Notation 6.3.18) with base point v_1.

We claim that

$$\det(A_{10}) = F_{01} - F_{1\text{-}1} \tag{6.9}$$

holds. Let N be the number of columns and rows of A_{st}. Denote the entries of the ev-matrix $M(C)$ by $m(C)_{ij}$. Define

$$S_{st} := \{r \in [N - 1] \mid m(C)_{r1} = s,\ m(C)_{r2} = t\}$$

for $(s, t) = (1, 0), (0, 1), (1, -1)$ and notice that $\#S_{10} + \#S_{01} + \#S_{1\text{-}1} = N - 1$. Denote the entries of A_{10} by $a_{ij}^{(10)}$ and apply Leibniz' determinant formula to obtain

$$\begin{aligned}\det(A_{10}) &= \sum_{\sigma\in\mathbb{S}_N} \operatorname{sgn}(\sigma)\prod_{j=1}^{N} a_{\sigma(j)j}^{(10)}\\ &= \sum_{\substack{\sigma\in\mathbb{S}_N\\ \sigma(2)\in S_{01}}} \operatorname{sgn}(\sigma)\prod_{j=1}^{N} a_{\sigma(j)j}^{(10)} + \sum_{\substack{\sigma\in\mathbb{S}_N\\ \sigma(2)\in S_{1\text{-}1}}} \operatorname{sgn}(\sigma)\prod_{j=1}^{N} a_{\sigma(j)j}^{(10)} = F_{01} - F_{1\text{-}1},\end{aligned}$$

where the second equality holds by definition of S_{st} and the third equality holds by considering how contributions of F_{01} and $F_{1\text{-}1}$ arise as choices of entries of A, see (6.8). The minus sign comes from the factor $a_{\sigma(2),2}^{(10)} = -1$ in each product in the last sum. Thus (6.9) holds.

We can show in a similar way that

$$\det(A_{01}) = -\left(F_{10} + F_{1\text{-}1}\right) = -F_{10} - F_{1\text{-}1}, \tag{6.10}$$
$$\det(A_{1\text{-}1}) = F_{10} + F_{1\text{-}1} + F_{01} - F_{1\text{-}1} = F_{10} + F_{01} \tag{6.11}$$

hold. Solving the system of linear equations (6.9), (6.10), (6.11) for $F_{10}, F_{01}, F_{1\text{-}1}$ yields

$$\begin{pmatrix} F_{10}\\ F_{01}\\ F_{1\text{-}1}\end{pmatrix} \in \begin{pmatrix} -\det(A_{01})\\ \det(A_{10})\\ 0\end{pmatrix} + \langle \begin{pmatrix} -1\\ 1\\ 1\end{pmatrix} \rangle, \tag{6.12}$$

where the 1-dimensional part appears because of the relation

$$-\det(M_{10}) + \det(M_{01}) + \det(M_{1\text{-}1}) = 0.$$

Combining (6.7) with (6.12) proves part (b) of Proposition 6.3.20, where $A_{st} = C_{1,st}$ and $M_{st} = C_{2,st}$. In particular,

$$\begin{aligned}\operatorname{mult}(C) &= \omega(e)\cdot|\det\left(M(C_{1,10})\right)\cdot\det\left(M(C_{2,01})\right) - \det\left(M(C_{1,01})\right)\cdot\det\left(M(C_{2,10})\right)|\\ &= \omega(e)\cdot|\det\left(M(C_{1,10})\right)\cdot\det\left(M(C_{2,01})\right)|\\ &= \omega(e)\cdot|\det\left(M(C_{1,10})\right)|\cdot|\det\left(M(C_{2,01})\right)|\\ &= \omega(e)\cdot\operatorname{mult}(C_{1,10})\cdot\operatorname{mult}(C_{2,01})\end{aligned}$$

holds if $\det\left(M(C_{1,01})\right)$ or $\det\left(M(C_{2,10})\right)$ vanishes.

□

Notice that Proposition 6.3.20 and its proof is similar to Proposition 4.2.3. Moreover, notice that proof of part (b) of Proposition 6.3.20 actually gives rise to slightly different ways of expressing $\operatorname{mult}(C)$ by considering other special solutions to the system of linear equations, see (6.12). We now want to present an alternative proof of part (b) of Proposition 6.3.20 that was suggested to us by an anonymous referee. It is shorter and uses general Laplace expansion instead of a recursive procedure. However, it does not yield the different ways of expressing $\operatorname{mult}(C)$.

Alternative proof of part (b) of Proposition 6.3.20. We assume that the weights of each multi-line $\omega(L_k)$ (see Definition 2.2.19) for $k \in \underline{\kappa}^\alpha \cup \underline{\kappa}^\alpha$ equals 1 since we can pull out the factor $\omega(L_k)$ frome each row of the ev-matrix, apply all the following arguments and multiply with $\omega(L_k)$ later.

We denote the vertex of C_1 adjacent to the cut edge e by v_1 and the other vertex adjacent to e by v_2. Moreover, let M_i denote the (non-square) evaluation matrix of C_i with respect to the base point v_i for $i = 1, 2$. Similar arguments as in the proof of part (a) of Proposition 6.3.20 using (3.6) yield that

$$
M := \begin{array}{rc}
 & \begin{array}{ccccc} \text{Base } v_1 & \text{lengths in } C_1 & \text{Base } v_2 & \text{lengths in } C_2 & l_e \end{array} \\
\begin{array}{r} \\ \\ \\ \text{conditions in } C_1 \\ \\ \\ \text{conditions in } C_2 \\ \\ \end{array} &
\left(\begin{array}{c|c|c|c|c}
\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix} & 0 & \begin{matrix}-1&0&0\\0&-1&0\\0&0&-1\end{matrix} & 0 & \begin{matrix}0\\0\\\omega(e)\end{matrix} \\ \hline
\multicolumn{2}{c|}{M_1} & \multicolumn{2}{c|}{0} & \begin{matrix}0\\\vdots\\0\end{matrix} \\ \hline
\multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{M_2} & \begin{matrix}0\\\vdots\\0\end{matrix}
\end{array}\right)
\end{array}
$$

is a square matrix. Subtract the first three rows of M from the rows of M_1 in M to obtain a matrix of the form

$$
\begin{array}{rc}
 & \begin{array}{ccccc} \text{Base } v_1 & \text{lengths in } C_1 & \text{Base } v_2 & \text{lengths in } C_2 & l_e \end{array} \\
\begin{array}{r} \\ \\ \\ \text{conditions in } C_1 \\ \\ \\ \text{conditions in } C_2 \\ \\ \end{array} &
\left(\begin{array}{c|c|c|c|c}
\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix} & 0 & \begin{matrix}-1&0&0\\0&-1&0\\0&0&-1\end{matrix} & 0 & \begin{matrix}0\\0\\\omega(e)\end{matrix} \\ \hline
0 & * & * & 0 & \begin{matrix}*\\\vdots\\*\end{matrix} \\ \hline
\multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{M_2} & \begin{matrix}0\\\vdots\\0\end{matrix}
\end{array}\right)
\end{array}.
$$

Notice that there is a 3×3 block matrix in the upper left and that its associated other block in the lower right is precisely the ev-matrix of C with base point v_2. Therefore

$$\operatorname{mult}(C) = |\det(M)|. \tag{6.13}$$

On the other hand, general Laplace expansion with respect to the first three rows of M can be used to determine $|\det(M)|$: Denote the number of rows (resp. columns) of M by t and let $M(a_1, \ldots, a_r \mid b_1, \ldots b_r)$ denote the $r \times r$ minor of M that is given by the columns $a_1, \ldots, a_r$ and rows $b_1, \ldots, b_r$ of M. Then

$$|\det(M)| = |\sum_{1 \le k_1 < k_2 < k_3 \le t} \epsilon(k_1, k_2, k_3) \cdot M(k_1, k_2, k_3 \mid 1, 2, 3) \cdot M(l_1, \ldots, l_{t-3} \mid 4, \ldots, t)|, \tag{6.14}$$

where $\{l_1, \ldots, l_{t-3}\} \cup \{k_1, k_2, k_3\} = [t]$ and

$$\epsilon(k_1, k_2, k_3) := \operatorname{sgn} \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & t \\ k_1 & k_2 & k_3 & j_1 & \ldots & j_{t-3} \end{pmatrix},$$

where $j_1 < \cdots < j_{t-3}$ such that $\{k_1, k_2, k_3\} \cup \{j_1, \ldots, j_{t-3}\} = [t]$. Notice that in (6.14) a summand can only be nonzero if each k_i does not correspond to a zero-column in the first three rows of M. Moreover, $k_3 = t$ must be satisfied since otherwise $M(l_1, \ldots, l_{t-3} \mid 4, \ldots, t)$ is zero due to a zero-column (the l_e-column). Hence

$$|\det(M)| = |\sum_{1 \le k_1 < k_2 < t} \epsilon(k_1, k_2, t) \cdot M(k_1, k_2, t \mid 1, 2, 3) \cdot M(l_1, \ldots, l_{t-3} \mid 4, \ldots, t)|, \tag{6.15}$$

where k_1, k_2 are columns associated to the base point v_1 or v_2. Therefore (6.15) breaks down to four summands. Let $M_{i,\hat{j}}$ denote the matrix M_i without its j-th column. The four relevant summands of (6.15) are

$$\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega(e) \end{pmatrix} \cdot \det\left(\begin{array}{c|c|c} M_{1,\hat{1},\hat{2}} & 0 & 0 \\ \hline 0 & * & M_{2,\hat{1}} \end{array}\right),$$
$$\det\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \omega(e) \end{pmatrix} \cdot \det\left(\begin{array}{c|c|c} M_1 & * & 0 \\ \hline 0 & 0 & M_{2,\hat{1},\hat{2}} \end{array}\right),$$
$$\det\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \omega(e) \end{pmatrix} \cdot \det\left(\begin{array}{c|c} M_{1,\hat{2}} & 0 \\ \hline 0 & M_{2,\hat{1}} \end{array}\right),$$
$$\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \omega(e) \end{pmatrix} \cdot \det\left(\begin{array}{c|c} M_{1,\hat{1}} & 0 \\ \hline 0 & M_{2,\hat{2}} \end{array}\right).$$

The bold red lines indicate square boxes in the matrices. Notice that the first two listed summands vanish. Using Notation 6.3.18 yields that

$$\det\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \omega(e) \end{pmatrix} \cdot \det\left(\begin{array}{c|c} M_{1,\hat{2}} & 0 \\ \hline 0 & M_{2,\hat{1}} \end{array}\right) = -\omega(e) \cdot \det\left(M(C_{1,01})\right) \cdot \det\left(M(C_{2,10})\right)$$

and

$$\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \omega(e) \end{pmatrix} \cdot \det\left(\begin{array}{c|c} M_{1,\hat{1}} & 0 \\ \hline 0 & M_{2,\hat{2}} \end{array}\right) = \omega(e) \cdot \det\left(M(C_{1,10})\right) \cdot \det\left(M(C_{2,01})\right).$$

The signs $\epsilon(k_1, k_2, k_3)$ appearing in (6.15) are all the same as one can see from a case-by-case analysis distinguishing if the numbers of columns of M_1 (resp. M_2) are even or odd. Thus (6.15) yields

$$|\det(M)| = \omega(e) \cdot |\det\left(M(C_{1,10})\right) \cdot \det\left(M(C_{2,01})\right) - \det\left(M(C_{1,01})\right) \cdot \det\left(M(C_{2,10})\right)|$$

The desired equation now follows from (6.13). □

Future research 6.3.21. Proposition 4.2.3 states how the multiplicity of a rational tropical stable map to $\mathbb{R}^2$ can be expressed when a contracted bounded edge is cut. Notice that Proposition 4.2.3 and its proof is similar to Proposition 6.3.20. Thus, in terms of multiplicities, a contracted bounded edge of a rational tropical stable map to $\mathbb{R}^2$ behaves like an elevator of a floor-decomposed rational tropical stable map to $\mathbb{R}^3$. This may be a hint on how to express multiplicities of floor-decomposed rational tropical stable maps to $\mathbb{R}^m$ for $m > 3$ in terms of multiplicities of their floors.

6.3.3 Pushing forward conditions along elevators

The aim of this subsection is to prove the following proposition, which determines how the 1-dimensional conditions a floor-decomposed rational tropical stable map exchanges via its 1/1 elevators look like. More precisely, cutting a 1/1 elevator q adjacent to the floors C_i, C_j leads to loose edges that can move in a 1-dimensional way, i.e. the floor C_i adjacent to q gives rise to a 1-dimensional cycle $Y_{i,q}$ that can be pushed forward to $\mathbb{R}^2$ using $\partial\operatorname{ev}_q$. The cycle $\partial\operatorname{ev}_{q,*}(Y_{i,q})$ is the 1-dimensional restriction C_i imposes on C_j via the elevator q.

Proposition 6.3.22. *Notation from Theorem 2.2.20, Definition 3.2.4, notations 2.0.1, 6.3.17 and Construction 6.3.16 is used. Let C_i be a floor of a floor-decomposed rational tropical stable map $C \in \mathcal{M}_{0,n}\left(\mathbb{R}^3, \Delta_d^3(\alpha,\beta)\right)$ which satisfies general positioned conditions, namely $p_{[n]}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}$. Let $\Delta^3_{s_i}\left(\alpha^i,\beta^i\right)$ be the degree of C_i and let $q \in \Delta^3_{s_i}\left(\alpha^i,\beta^i\right)$ be the label of an end whose primitive direction is $(0,0,\pm 1)$. The cycle*

$$Y_{i,q} := \prod_{k\in\underline{\kappa_i}^\alpha\cup\underline{\kappa_i}^\beta} \partial\operatorname{ev}_k^*(L_k)\cdot \prod_{f\in\underline{\eta_i}^\alpha\cup\underline{\eta_i}^\beta} \partial\operatorname{ev}_f^*(P_f)\cdot \prod_{j\in\underline{l_i}} \operatorname{ft}^*_{\lambda_j^{\rightarrow}}(0)\cdot \operatorname{ev}_i^*(p_i)\cdot \mathcal{M}_{0,1}\left(\mathbb{R}^3, \Delta^3_{s_i}\left(\alpha^i,\beta^i\right)\right)$$

has the following properties.

(1) The recession fan of the push-forward $\partial\operatorname{ev}_{q,*}(Y_{i,q})$ does only contain ends of standard directions.

(2) Each unbounded cell $\sigma \in Y_{i,q}$ that is mapped to an end of the recession fan of $\partial\operatorname{ev}_{q,*}(Y_{i,q})$ under the push-forward $\partial\operatorname{ev}_{q,*}$ satisfies the following: If $C_\sigma \in \sigma$ is a rational tropical stable map in the interior of σ, then q is adjacent to a 3-valent vertex, which is adjacent to another end $E \neq q$ such that $\pi(E) \subset \mathbb{R}^2$ is an end of standard direction.

An immediate consequence of Proposition 6.3.22 is the following corollary, which yields that all restrictions exchanged via a 1/1 elevator are in fact rational tropical stable maps to $\mathbb{R}^2$ with ends of standard direction.

Corollary 6.3.23. *Let C be a floor-decomposed rational tropical stable map that contributes to $N_{\Delta_d^3(\alpha,\beta)}\left(p_{[n]}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}\right)$. Then the codimension two tangency conditions each 1/1 elevator passes on to its neighbors have ends of standard direction only. In particular, we can assume that if we cut all elevators as in Construction 6.3.16, then the appearing codimension two tangency conditions have ends of standard directions.*

Proof. Apply part (1) of Proposition 6.3.22 inductively by cutting one 1/1 elevator after another. □

Remark 6.3.24. Notice that Proposition 6.3.22 can also be shown the way Corollary 4.1.31 was shown, i.e. by using Proposition 4.1.1. Since Proposition 4.1.1 is actually a stronger statement than Proposition 6.3.22 there is is no need to evoke the machinery developed in Chapter 4.

Proof of Proposition 6.3.22. Let L_{10} be a degenerated tropical line in $\mathbb{R}^2$ which is parallel to the y-axis as in Definition 2.2.21. The projection formula (Proposition 2.1.13 in case of abstract cycles) yields

$$L_{10}\cdot\partial\operatorname{ev}_{q,*}(Y_{i,q}) = \partial\operatorname{ev}_{q,*}\left(\partial\operatorname{ev}_q^*(L_{10})\cdot Y_{i,q}\right). \tag{6.16}$$

Assume that the degenerated line L_{10} is shifted in the direction $(-1,0)\in\mathbb{R}^2$ such that L_{10} intersects $\partial\operatorname{ev}_{q,*}(Y_{i,q}) \subset \mathbb{R}^2$ only in 1-dimensional ends of $\partial\operatorname{ev}_{q,*}(Y_{i,q})$.

Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection that forgets the z-coordinate and let

$$\tilde{\pi} : \mathcal{M}_{0,1}\left(\mathbb{R}^3, \Delta^3_{s_i}\left(\alpha^i, \beta^i\right)\right) \to \mathcal{M}_{0,1+|\alpha^i|+|\beta^i|}\left(\mathbb{R}^2, \pi\left(\Delta^3_{s_i}\left(\alpha^i, \beta^i\right)\right)\right)$$

be its induced map π on the moduli spaces as in Notation 2.2.9.

Each tropical stable map corresponding to a point of $\tilde{\pi}_*(Y_{i,q})$ can be lifted uniquely to a tropical stable map corresponding to a point in $Y_{i,q}$ as in the proof of Proposition 6.3.31. Thus for

$$\begin{aligned} Y_{\pi,i,q} := & \prod_{k \in \underline{\kappa_i}^{\alpha} \cup \underline{\kappa_i}^{\beta}} \operatorname{ev}_k^*(L_k) \\ & \cdot \prod_{f \in \underline{\eta_i}^{\alpha} \cup \underline{\eta_i}^{\beta}} \operatorname{ev}_f^*(P_f) \cdot \prod_{j \in \underline{l_i}} \operatorname{ft}^*_{\lambda_j^{\to q}}(0) \cdot \operatorname{ev}_i^*\left(\pi(p_i)\right) \cdot \mathcal{M}_{0,1+|\alpha^i|+|\beta^i|}\left(\mathbb{R}^2, \pi\left(\Delta^3_{s_i}\left(\alpha^i, \beta^i\right)\right)\right) \end{aligned}$$

the equality

$$\tilde{\pi}_*(Y_{i,q}) = Y_{\pi,i,q} \tag{6.17}$$

holds on the level of sets. To see that (6.17) also holds on the level of cycles, multiplicities are compared. Notice that each multiplicity of a top-dimensional cell of $\tilde{\pi}_*(Y_{i,q})$ (resp. $Y_{\pi,i,q}$) arises as a product of a cross-ratio multiplicity and an index of an ev-matrix, see Definition 3.1.11. The lifting of the proof of Proposition 6.3.31 guarantees that the cross-ratio multiplicity part coincides. Let σ be a top-dimension cell of $Y_{i,q}$ the ev-multiplicity part of σ is given by the absolute value of the index of the ev-Matrix $M(\sigma)$ associated to σ, see Definition 3.1.11. We choose p_i as base point for the local coordinates used for $M(\sigma)$. Then

$$M(\sigma) = \begin{array}{c} \text{Base } p_i \quad\quad\quad\quad \\ \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \hline & & * & & & \\ & * & \vdots & & * & \\ & & * & & & \end{array}\right) \end{array}.$$

The ev-matrix $M(\pi(\sigma))$ is obtained from $M(\sigma)$ by erasing the third column and row which intersect in the z-coordinate of the base point, which is 1. Recall that $\partial\operatorname{ev}$ is by definition $\operatorname{ev} \circ \pi$, i.e. the indices of $M(\sigma)$ and $M(\pi(\sigma))$ are equal. Therefore (6.17) holds on the level of cycles.

By definition of $\partial\operatorname{ev}_q$ and the arguments from before, the right-hand side of (6.16) is equal to $\operatorname{ev}_{q,*}\left(\operatorname{ev}_q^*(L_{10}) \cdot Y_{\pi,i,q}\right)$, which is, by the projection formula, equal to $L_{10} \cdot \operatorname{ev}_{q,*}\left(Y_{\pi,i,q}\right)$. By shifting L_{10} to the left as before, we can assume that L_{10} intersects $\operatorname{ev}_{q,*}\left(Y_{\pi,i,q}\right)$ only in ends of $\operatorname{ev}_{q,*}\left(Y_{\pi,i,q}\right)$.

We claim that each tropical stable map C that contributes to the 0-dimensional cycle $\operatorname{ev}_q^*(L_{10}) \cdot Y_{\pi,i,q}$ has an end of direction $(-1,0) \in \mathbb{R}^2$ that is adjacent to a 3-valent vertex v which in turn is adjacent to the contracted end q. To prove the claim, it is sufficient to show that C has no vertex that is not adjacent to q whose x-coordinate is smaller or equal to the one of L_{10}. Assume that there is a vertex v of C that is not adjacent to q and that the x-coordinate of v is minimal. Assume also that v is adjacent to an end e of direction $(-1,0) \in \mathbb{R}^2$. Since each entry of a given tropical cross-ratio is a contracted end, Corollary 3.2.24 yields that v is 3-valent. If v is adjacent to a contracted end e, then this end needs to satisfy a condition, otherwise v allows a 1-dimensional movement which is a contradiction. Since L_{10} was moved sufficiently into the direction of $(-1,0)$ in $\mathbb{R}^2$, we know that the condition e satisfies can only be a multi-line condition that locally around v is parallel to the x-axis of $\mathbb{R}^2$. Hence v allows again a 1-dimensional movement which is a contradiction. In total, v is 3-valent, adjacent to an end of direction $(-1,0)$ and is not adjacent to a contracted end.

Assume additionally that the y-coordinate of v is minimal among the vertices with minimal x-coordinate that are not adjacent to q and that are adjacent to an end of direction $(-1,0)$. We distinguish two cases:

- In the fist case, the x-coordinate of v is strictly smaller than the one of L_{10}. Hence v is by balancing (all ends have weight 1) adjacent to an edge of direction $(0,-1)$. If this edge is an end, then v gives rise to a 1-dimensional movement which is a contradiction. If this edge leads to a vertex v' that is adjacent to a contracted end t, then t can only satisfy a multi-line condition that locally around t is parallel to the x-axis of $\mathbb{R}^2$. By our assumption, there is no vertex with the same x-coordinate as v' below v' that is adjacent to an end of direction $(0,-1)$. Hence v allows a 1-dimensional movement which is a contradiction.

- In the second case, the x-coordinate of v equals the x-coordinate of L_{10} and v is adjacent to a vertex v' that is in turn adjacent to the contracted end q such that the y-coordinate of v' is smaller than the one of v (if there is no such vertex v', then we end up with the same contradiction as in case one). By Corollary 3.2.24, v' cannot be adjacent to an end of direction $(-1,0)$ and by minimality of the y-coordinate of v, there is an end e' of direction $(0,-1)\in\mathbb{R}^2$ adjacent to v' which is parallel to L_{10}. Thus v' allows a 1-dimensional movement which is a contradiction.

Thus the claim is true.

Since the x-coordinate of L_{10} is so small that L_{10} intersects $\operatorname{ev}_{q,*}(Y_{\pi,i,q})$ in ends only, we can use the claim from above to determine the directions of those ends: Consider a point C in $\operatorname{ev}_q^*(L_{10})\cdot Y_{\pi,i,q}$. Then C is also a point of $Y_{\pi,i,q}$. Our claim tells us that the contracted end q of C is adjacent to a 3-valent vertex which in turn is adjacent to an end of direction $(-1,0)\in\mathbb{R}^2$. Thus C gives rise to a ray of $Y_{\pi,i,q}$ whose direction under $\operatorname{ev}_{q,*}$ is $(-1,0)\in\mathbb{R}^2$, which is a standard direction. Since we shifted L_{10} sufficiently to the left, all rays of $\operatorname{ev}_{q,*}(Y_{\pi,i,q})$ whose direction vector's x-coordinates are negative occur this way. Notice that by definition of $Y_{\pi,i,q}$ this implies that all ends of $\partial\operatorname{ev}_{q,*}(Y_{i,q})$ whose direction vector's x-coordinate is negative (i.e. such ends that intersect a shifted line parallel to the y-axis) are actually of the standard direction $(-1,0)\in\mathbb{R}^2$.

We can use similar arguments for L_{10} if the x-coordinate of L_{10} is so large that it intersects $\partial\operatorname{ev}_{q,*}(Y_{i,q})$ in ends only, and we can use similar arguments for L_{01} with small (resp. large) y-coordinate. In total, it follows that ends of $\partial\operatorname{ev}_{q,*}(Y_{i,q})$ are of standard direction and that their weights are given as in Proposition 6.3.22. □

6.3.4 Cross-ratio floor diagrams for $\mathbb{R}^3$ and their multiplicities

Cross-ratio floor diagrams for $\mathbb{R}^3$ are now introduced. Similar to the $\mathbb{R}^2$-case additional discrete data is used to encode which floor-decomposed rational tropical stable maps degenerate to a cross-ratio floor diagram. Multiplicities of cross-ratio floor diagrams are then defined. We want to point out that these multiplicities can be obtained from enumerating rational tropical stable maps to $\mathbb{R}^2$ such that results of previous chapters can be used.

Definition 6.3.25 (Cross-ratio floor diagrams for $\mathbb{R}^3$)**.** Notation of Definition 3.2.4 is used. Let $\Delta_d^3(\alpha,\beta)$ be a degree. Let $\mathcal{F}$ be a tree without ends on a totally ordered set of vertices $v_{[n]}$, then $\mathcal{F}$ is called *cross-ratio floor diagram of degree* $\Delta_d^3(\alpha,\beta)$ if it satisfies the following:

(1) The graph $\mathcal{F}$ is equipped with a flow structure that is a condition flow of type 3.

(2) Each vertex v_i is equipped with a (possibly empty) set of labels that appear in $\Delta_d^3(\alpha,\beta)$ such that $\delta_{v_i}\cap\delta_{v_j}=\emptyset$ for all $i\neq j$ and $\bigcup_{i=1}^n\delta_{v_i}$ is the set of all labels appearing in $\Delta_d^3(\alpha,\beta)$. Moreover, each vertex v_i is equipped with an integer $\#\lambda_{v_i}\in\mathbb{N}$.

(3) Each edge e of $\mathcal{F}$ (consisting of two half-edges) is equipped with a *weight* $\omega(e)\in\mathbb{N}_{>0}$ such that vertices v_i of $\mathcal{F}$ are *balanced* with respect to these weights, i.e.

$$\#\delta_{v_i}^{(1,1,1)}+\sum_{e'\in\delta_{v_i}^{\beta}}\omega(e')+\sum_{\substack{e\text{ an edge}\\ \text{between } v_i<v_j}}\omega(e)-\sum_{e'\in\delta_{v_i}^{\alpha}}\omega(e')-\sum_{\substack{e\text{ an edge}\\ \text{between } v_j<v_i}}\omega(e)=0$$

holds for all $i\in[n]$, where $\delta_{v_i}^{(1,1,1)}$ is the subset of δ_{v_i} that contains all labels of ends of primitive direction $(1,1,1)$, $\delta_{v_i}^{\beta}$ is the subset of δ_{v_i} that contains all labels of ends of primitive direction $(0,0,1)$ and $\delta_{v_i}^{\alpha}$ is the subset of δ_{v_i} that contains all labels of ends of direction $(0,0,-1)$.

Remark 6.3.26. The path criterion for cross-ratio floor diagrams (Definition 6.2.3) can be defined analogously for cross-ratio floor diagrams in case of $\mathbb{R}^3$ by replacing $\Delta_d^2(\alpha,\beta)$ with $\Delta_d^3(\alpha,\beta)$. This is used in the following definition.

Definition 6.3.27 (Conditions satisfied by $\mathcal{F}$)**.** Let $\Delta_d^3(\alpha,\beta)$ be a degree as in Notation 2.2.4. Let $p_{[n]},L_{\underline{\kappa}^\alpha},L_{\underline{\kappa}^\beta},P_{\underline{\eta}^\alpha},P_{\underline{\eta}^\beta},\lambda_{[l]}$ be in a stretched configuration with respect to $\Delta_d^3(\alpha,\beta)$. A cross-ratio floor diagram $\mathcal{F}$ *satisfies* $p_{[n]},L_{\underline{\kappa}^\alpha},L_{\underline{\kappa}^\beta},P_{\underline{\eta}^\alpha},P_{\underline{\eta}^\beta},\lambda_{[l]}$ if the following properties are fulfilled.

(1) The number of vertices of $\mathcal{F}$ equals n. The total order of the vertices $v_{[n]}$ of $\mathcal{F}$ is induced by the total order of the point conditions $p_{[n]}$ (they are ordered according to their last coordinate), where the point p_i is identified with the vertex v_i.

(2) For each degenerated tropical cross-ratio $\lambda_j\in\lambda_{[l]}$ there is a vertex of $\mathcal{F}$ that satisfies it (Remark 6.3.26) and for each vertex v of $\mathcal{F}$ the number $\#\lambda_v$ equals the total number of degenerated tropical cross-ratios that are satisfied at v. For a vertex of a cross-ratio floor diagram that satisfies degenerated tropical cross-ratios $\lambda_{[l]}$, denote the set of all $\lambda_j\in\lambda_{[l]}$ that are satisfied at v by λ_v.

(3) Define

$$A(v_i):=$$
$$3\#\delta_{v_i}^{(1,1,1)}+\#\delta_{v_i}^{\alpha}+\#\delta_{v_i}^{\beta}+\operatorname{val}(v_i)-2-\#\lambda_{v_i}-2(\#\delta_{v_i}^{\alpha,P}+\#\delta_{v_i}^{\beta,P})-\#\delta_{v_i}^{\alpha,L}-\#\delta_{v_i}^{\beta,L}$$

for every vertex v_i, where $\operatorname{val}(v_i)$ is the valence of v_i in $\mathcal{F}$ and $\delta_{v_i}^{\alpha},\delta_{v_i}^{\beta},\delta_{v_i}^{\alpha,P},\delta_{v_i}^{\beta,P},\delta_{v_i}^{\alpha,L},\delta_{v_i}^{\beta,L}$ are subsets of δ_{v_i} such that

- $\delta_{v_i}^{\alpha}$ (resp. $\delta_{v_i}^{\beta}$) are the ends that are associated to α (resp. β),
- $\delta_{v_i}^{\alpha,P}\subset\delta_{v_i}^{\alpha}$ (resp. $\delta_{v_i}^{\beta,P}\subset\delta_{v_i}^{\beta}$) are the ends that satisfy some codimension one tangency conditions (see Notation 3.1.5),
- $\delta_{v_i}^{\alpha,L}\subset\delta_{v_i}^{\alpha}$ (resp. $\delta_{v_i}^{\beta,L}\subset\delta_{v_i}^{\beta}$) are the ends that satisfy some codimension two tangency conditions (see Notation 3.1.5).

The leak function of $\mathcal{F}$ is given by

$$\operatorname{leak}(v_i) = \begin{cases} 0 & \text{, if } \delta_{v_i}^{(1,1,1)} = \emptyset \\ A(v_i) & \text{, else} \end{cases}$$

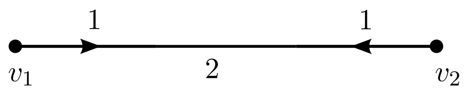
Notice that the leak function determines the condition flow of type 3 on $\mathcal{F}$ uniquely by Lemma 6.3.6.

Example 6.3.28. Let $\Delta_4^3\left((4,1,0,\dots),(2,0,\dots)\right)$ denote a degree as in Notation 2.2.4 whose labeling is:

Ends of primitive direction...	e_3	$-e_3$	$-e_1$	$-e_2$	e_0
its associated labels	$3,9$	$4,5,6,7,8$	$10,11,12,19$	$13,14,15,20$	$16,17,18,21$

,

such that the end of weight two is labeled by 8. Let $p_{[2]}, P_{[9]\setminus[2]}, \lambda_1 = \{1,2,3,7\}$ be conditions in a stretched configuration with notation from Definition 3.2.4. Recall the floor graph from Example 6.3.15. Equipping it with discrete data as below turns it into a cross-ratio floor diagram $\mathcal{F}$ of degree $\Delta_4^3\left((4,1,0,\dots),(2,0,\dots)\right)$ that satisfies $p_{[2]}, P_{[9]\setminus[2]}, \lambda_1$.



i	1	2
δ_{v_i}	$[18]\setminus\{2,9\}$	$\{2,9,19,20,21\}$
$\delta_{v_i}^{(1,1,1)}$	$\{16,17,18\}$	$\{21\}$
$\delta_{v_i}^{\alpha}$	$\{4,5,6,7,8\}$	$\emptyset$
$\delta_{v_i}^{\beta}$	$\{3\}$	$\{9\}$
λ_{v_i}	$\{\lambda_1\}$	$\emptyset$
$\delta_{v_i}^{\alpha,P}$	$\{4,5,6,7,8\}$	$\emptyset$
$\delta_{v_i}^{\beta,P}$	$\{3\}$	$\{9\}$
$\delta_{v_i}^{\alpha,L}$	$\emptyset$	$\emptyset$
$\delta_{v_i}^{\beta,L}$	$\emptyset$	$\emptyset$

Definition 6.3.29 (Multiplicity of a cross-ratio floor diagrams in $\mathbb{R}^2$)**.** Let $\Delta_d^3(\alpha,\beta)$ be a degree as in Notation 2.2.4. Let $p_{[n]}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}$ be in a stretched configuration with respect to $\Delta_d^3(\alpha,\beta)$. Let $\mathcal{F}$ be a cross-ratio floor diagram of degree $\Delta_d^3(\alpha,\beta)$ that satisfies these conditions. Let $v_{[n]}$ denote the totally ordered set of vertices of $\mathcal{F}$. For $\gamma=\alpha,\beta$, define the following: Let $\underline{2/0_i}^\gamma$ be the 2/0 edges adjacent to v_i and v_j (with $j\neq i$) in $\mathcal{F}$ such that $v_j<v_i$ if $\gamma=\alpha$ and $v_i<v_j$ if $\gamma=\beta$. Let $\underline{1/1_i}^\gamma$ be the 1/1 edges adjacent to v_i, where $\gamma\in\{\alpha,\beta\}$ is defined analogously. For a vertex v_i of the cross-ratio floor diagram $\mathcal{F}$, its *multiplicity* $\operatorname{mult}(v_i)$

is defined as

$$\operatorname{mult}(v_i) := N_{\Delta^3_{\#\delta_{v_i}^{(1,1,1)}}(\alpha^i,\beta^i)}\left(p_i, L_{\delta_{v_i}^{\alpha,L}\cup\underline{1/1_i}^{\alpha}}, L_{\delta_{v_i}^{\beta,L}\cup\underline{1/1_i}^{\beta}}, P_{\delta_{v_i}^{\alpha,P}\cup\underline{2/0_i}^{\alpha}}, P_{\delta_{v_i}^{\beta,P}\cup\underline{2/0_i}^{\beta}}, \lambda_{v_i}^{\rightarrow}\right),$$

with the notation from Definition 6.3.25 and Notation 2.0.1, where α^i (resp. β^i) of the degree $\Delta^3_{\#\delta_{v_i}^{(1,1,1)}}(\alpha^i,\beta^i)$ arises from $\delta_{v_i}^{\alpha}$ (resp. $\delta_{v_i}^{\beta}$) and edges contributing to $\operatorname{val}(v_i)$ in $\mathcal{F}$, and where $L_{\underline{1/1_i}^{\alpha}}$ (resp. $L_{\underline{1/1_i}^{\beta}}$) are collections of tropical multi-line conditions with ends of weight 1. Moreover, the cross-ratios λ_{v_i} are adapted to cutting the edges adjacent to v_i similar to Construction 6.3.16 and Notation 6.3.17. The *multiplicity* of the entire cross-ratio floor diagram $\mathcal{F}$ is defined to be the product of the vertices' multiplicities times the edges' weights, i.e.

$$\operatorname{mult}(\mathcal{F}) := \prod_{\substack{e \text{ edge} \\ \text{of } \mathcal{F}}} \omega(e) \cdot \prod_{i=1}^{n} \operatorname{mult}(v_i).$$

Example 6.3.30. Let $\mathcal{F}$ be the cross-ratio floor diagram of Example 6.3.28. Label its edge of weight two by 22. The multiplicities of the vertices v_1, v_2 of $\mathcal{F}$ are

$$\begin{aligned}
\operatorname{mult}(v_1) &= N_{\Delta^3_3((4,1,0,\dots),(1,1,0,\dots))}\left(p_1, L_{22}, P_{[8]\setminus[2]}, \lambda_1^{\rightarrow 22}\right),\\
\operatorname{mult}(v_2) &= N_{\Delta^3_1((0,1,0,\dots),(1,0,\dots))}\left(p_2, L_{22}, P_9\right),
\end{aligned}$$

where $\lambda_1^{\rightarrow 22} = \{1, 22, 3, 7\}$ and L_{22} is a tropical multi-line with ends of weight 1 (Notation 3.1.5 is used).

The following Proposition reduces the calculation of the multiplicity of a cross-ratio floor diagram in $\mathbb{R}^3$ to the enumeration of rational tropical stable maps to $\mathbb{R}^2$ satisfying point, multi-line and degenerated tropical cross-ratio conditions.

Proposition 6.3.31. *For notation, see Notation 2.0.1, 2.2.9 and Definition 6.3.29. Let $\mathcal{F}$ be a cross-ratio floor diagram of degree $\Delta^3_d(\alpha,\beta)$ that satisfies conditions $p_{[n]}, L_{\underline{\kappa}^{\alpha}}, L_{\underline{\kappa}^{\beta}}, P_{\underline{\eta}^{\alpha}}, P_{\underline{\eta}^{\beta}}, \lambda_{[l]}$. The multiplicity* $\operatorname{mult}(v_i)$ *of a vertex v_i of $\mathcal{F}$ equals the degree of the cycle*

$$\prod_{k\in\underline{\kappa_i}^{\alpha}\cup\underline{\kappa_i}^{\beta}} \operatorname{ev}_k^*(L_k)\cdot \prod_{f\in\underline{\eta_i}^{\alpha}\cup\underline{\eta_i}^{\beta}} \operatorname{ev}_f^*(P_f)\cdot \prod_{\lambda_j\in\lambda_{v_i}} \operatorname{ft}^*_{\lambda_j^{\rightarrow}}(0)\cdot \operatorname{ev}_i^*(\pi(p_i))$$
$$\cdot\, \mathcal{M}_{0,1+|\alpha^i|+|\beta^i|}\left(\mathbb{R}^2, \pi\left(\Delta^3_{\#\delta_{v_i}^{(1,1,1)}}\left(\alpha^i,\beta^i\right)\right)\right), \tag{6.18}$$

where $\underline{\kappa_i}^{\gamma} := \delta_{v_i}^{\gamma,L}\cup\underline{1/1_i}^{\gamma}$ and $\underline{\eta_i}^{\gamma} := \delta_{v_i}^{\gamma,P}\cup\underline{2/0_i}^{\gamma}$ for $\gamma=\alpha,\beta$.

Proof. Notice that since the given conditions $p_i, L_{\underline{\kappa_i}^{\alpha}}, L_{\underline{\kappa_i}^{\beta}}, P_{\underline{\eta_i}^{\alpha}}, P_{\underline{\eta_i}^{\beta}}, \lambda_{v_i}^{\rightarrow}$ are in general position with respect to $\Delta^3_{\#\delta_{v_i}^{(1,1,1)}}\left(\alpha^i,\beta^i\right)$ and there is only one point condition p_i, we can assume that the conditions $\pi(p_i), L_{\underline{\kappa_i}^{\alpha}}, L_{\underline{\kappa_i}^{\beta}}, P_{\underline{\eta_i}^{\alpha}}, P_{\underline{\eta_i}^{\beta}}, \lambda_{v_i}^{\rightarrow}$ are also in general position with respect to the degree $\pi\left(\Delta^3_{\#\delta_{v_i}^{(1,1,1)}}\left(\alpha^i,\beta^i\right)\right)$. Using (3.6), we see that the cycle (6.18) is indeed 0-dimensional. Thus considering its degree makes sense.

Let C be a rational tropical stable map that contributes to $\operatorname{mult}(v_i)$. Applying the map $\tilde{\pi}$ from Notation 2.2.9 induced by the projection $\pi:\mathbb{R}^3\to\mathbb{R}^2$ that forgets the x_3-coordinate leads to a rational tropical stable map $\tilde{\pi}(C)$ that contributes to (6.18) by Corollary 3.2.10. The other way round, a rational tropical stable map C' that contributes to (6.18) can be lifted uniquely to a rational tropical stable map C that contributes to $\mathrm{mult}(v_i)$, because the x_3-coordinates of the directions of the edges can be recovered from the balancing condition and the overall x_3-position of C is fixed by the x_3-coordinate of p_i. Hence π induces a bijection between rational tropical stable maps C that contribute to $\mathrm{mult}(v_i)$ and rational tropical stable maps $\tilde{\pi}(C)$ that contribute to (6.18).

It remains to show that the multiplicities of C and $\tilde{\pi}(C)$ coincide. For that, notice that the cross-ratio multiplicities of every vertex $v \in C$ and its image $\pi(v_i)$ in $\tilde{\pi}(C)$ coincide. Thus is remains to show that the ev-multiplicities coincide as well. The ev-multiplicity of C (resp. $\tilde{\pi}(C)$) is given by the absolute value of the determinant of the ev-Matrix $M(C)$ (resp. the ev-matrix $M(\tilde{\pi}(C))$) associated to C (resp. $\tilde{\pi}(C)$), see Definition 3.1.11. We choose p_i as base point for the local coordinates used for $M(C)$ (resp. $\pi(p_i)$ as base point for $M(\tilde{\pi}(C))$) which are the lengths of the edges of C (resp. $\tilde{\pi}(C)$). The matrix $M(\tilde{\pi}(C))$ is obtained from $M(C)$ by erasing the third column and row which intersect in the x_3-coordinate of the base point, which is 1 (see below). The matrices $M(C)$ and $M(\tilde{\pi}(C))$ look like follows

$$M(C) = \begin{array}{c} \text{Base } p_i \\ \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \hline & & * & & & \\ & B & \vdots & & M & \\ & & * & & & \end{array}\right) \end{array} \quad \text{and} \quad M(\tilde{\pi}(C)) = \begin{array}{c} \text{Base } \pi(p_i) \\ \left(\begin{array}{cc|ccc} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \hline & & & & \\ & B & & M & \\ & & & & \end{array}\right) \end{array}.$$

Recall that $\partial\,\mathrm{ev}$ equals $\mathrm{ev} \circ \pi$, i.e. the submatrices B, M marked above are equal. Therefore

$$|\det(M(C))| = |\det(M(\tilde{\pi}(C))|$$

follows from using Laplace expansion on the third row of $M(C)$. □

The general tropical Kontsevich's formula 4.3.4 recursively calculates the weighted number of rational tropical stable maps to $\mathbb{R}^2$ that satisfy point, multi-line and degenerated tropical cross-ratio conditions. As a consequence, the multiplicity of a cross-ratio floor diagram in $\mathbb{R}^3$ can be determined recursively.

Corollary 6.3.32. *The multiplicity* $\mathrm{mult}(v_i)$ *of a vertex* v_i *of a cross-ratio floor diagram of degree* $\Delta^3_d(\alpha, \beta)$ *can be calculated recursively using the general tropical Kontsevich's formula 4.3.4.*

Example 6.3.33. Let $\mathcal{F}$ be the cross-ratio floor diagram of Example 6.3.28. Using Proposition 6.3.31 to express $\mathrm{mult}(v_1), \mathrm{mult}(v_2)$ of Example 6.3.30 yields

$$\begin{aligned} \mathrm{mult}(v_1) &= N_{\Delta^2_3}\left(\pi(p_1), \pi\left(P_{[8]\setminus[2]}\right), L_{22}, \lambda_1^{\to 22}\right), \\ \mathrm{mult}(v_2) &= N_{\Delta^2_1}\left(\pi(p_2), \pi\left(P_9\right), L_{22}\right). \end{aligned}$$

This allows us to use Corollary 6.3.32, resp. general tropical Kontsevich's formula 4.3.4. Hence

$$\begin{aligned} \mathrm{mult}(v_1) &= 5, \\ \mathrm{mult}(v_2) &= 1. \end{aligned}$$

Therefore $\mathrm{mult}(\mathcal{F}) = 10$.

6.3.5 Enumeration in $\mathbb{R}^3$ using cross-ratio floor diagrams

Cross-ratio floor diagrams of degree $\Delta_d^3(\alpha,\beta)$ are now related to counts of rational tropical stable maps of degree $\Delta_d^3(\alpha,\beta)$. This is done in two steps. First, floor-decomposed rational tropical stable maps are degenerated to cross-ratio floor diagrams. Second, it is shown that the multiplicity of a cross-ratio floor diagram reflects how many floor-decomposed rational tropical stable maps degenerate to it.

Theorem 6.3.34. *Notation of Definition 3.2.5, 6.1.1, 6.3.25 and Notation 2.2.4 is used. Let $\Delta_d^3(\alpha,\beta)$ be a degree. Let $p_{[n]}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}$ be in a stretched configuration with respect to $\Delta_d^3(\alpha,\beta)$. Then*

$$N_{\Delta_d^3(\alpha,\beta)}\left(p_{[n]}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}\right) = \sum_{\mathcal{F}} \operatorname{mult}(\mathcal{F}) \tag{6.19}$$

holds, where the sum goes over all cross-ratio floor diagrams of degree $\Delta_d^3(\alpha,\beta)$ that satisfy the given conditions.

Construction 6.3.35 (Floor-decomposed rational tropical stable map $\mapsto$ cross-ratio floor diagram)**.** Let C be a floor-decomposed rational tropical stable map that contributes to the number $N_{\Delta_d^3(\alpha,\beta)}\left(p_{[n]}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}\right)$. We want to construct a cross-ratio floor diagram $\mathcal{F}$ that satisfies the given conditions $p_{[n]}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}$.

Let $\mathcal{F}$ denote the floor graph associated to C, see Definition 6.3.14. To turn $\mathcal{F}$ into a cross-ratio floor diagram of degree $\Delta_d^3(\alpha,\beta)$, define δ_{v_i} (notation of Definition 6.3.25 is used) as the set of labels of ends adjacent to the floor C_i of C which satisfies the point condition p_i. Moreover, $\mathcal{F}$ is balanced in the sense of Definition 6.3.25 since C is balanced. Thus $\mathcal{F}$ is indeed a cross-ratio floor diagram of degree $\Delta_d^3(\alpha,\beta)$ if its flow structure is a conditions flow of type 3. The flow structure of $\mathbb{F}$ is discussed below.

We now check whether $\mathcal{F}$ satisfies the given conditions, i.e. we need to check if properties (1), (2) and (3) of Definition 6.2.6 are satisfied. Property (1) is satisfied by definition of the floor graph from which $\mathcal{F}$ was constructed. For (2), define λ_{v_i} as the union over all λ_{u_j} (the set of cross-ratios satisfied at u_j), where u_j is a vertex of the floor C_i, and use the path criterion (Corollary 3.2.12) to verify that $\mathcal{F}$ satisfies the degenerated tropical cross-ratios $\lambda_{[l]}$ if C does. Property (3) is more technical: If the floor C_i does not contain an end of direction $(1,1,1)$ (i.e. if $\delta_{v_i}^{(1,1,1)} = \emptyset$ with the notation from Definition 6.3.27), then $\operatorname{flow}(v_i) = 0$ since C_i consists of a single vertex satisfying the point condition p_i that gains all its flow in C_i via a contracted end which is not contained in $\mathcal{F}$. Let now $\delta_{v_i}^{(1,1,1)} \neq \emptyset$ and let the *elevator flow* $\operatorname{flow}_{\text{elevator}}(C_i)$ of C_i be the total flow incoming to vertices of C_i via elevators. Let the *end flow* $\operatorname{flow}_{\text{end}}(C_i)$ of C_i be the total flow incoming to vertices of C_i via non-contracted ends (notice that notation is abused here as indicated in Definition 6.1.3). Since C_i, that is of degree $\Delta_{s_i}^3\left(\alpha^i,\beta^i\right)$ (notation of Construction 6.3.16), is fixed by all restrictions imposed to it via its ends and edges, we can use Equation (3.6) and the notation of Definition 6.3.27 to obtain

$$\#\Delta_{s_i}^3\left(\alpha^i,\beta^i\right) - 2 - \#\lambda_{v_i} - \operatorname{flow}_{\text{end}}(C_i) = \operatorname{flow}_{\text{elevator}}(C_i),$$

where -2 comes from the point condition p_i that is satisfied by C_i. Using

$$\#\Delta_{s_i}^3\left(\alpha^i,\beta^i\right) = 3\cdot\#\delta_{v_i}^{(1,1,1)} + \#\delta_{v_i}^{\alpha} + \#\delta_{v_i}^{\beta} + \operatorname{val}(v_i)$$

and

$$\operatorname{flow}_{\text{end}}(C_i) = 2\cdot\#\delta_{v_i}^{\alpha,P} + 2\cdot\#\delta_{v_i}^{\beta,P} + \#\delta_{v_i}^{\alpha,L} + \#\delta_{v_i}^{\beta,L}$$

turns the induced flow on $\mathcal{F}$ into a condition flow of type 3 if the leak function is defined as $\text{leak}(v_i) := \text{flow}_{\text{elevator}}(C_i)$ for all $i \in [n]$. Notice that this leak function coincides with the one of Definition 6.3.27. In this case, the condition flow is uniquely determined by its leak function (see Lemma 6.3.6). Hence $\mathcal{F}$ is a cross-ratio floor diagram that satisfies the given conditions. We say that C *degenerates* to $\mathcal{F}$ and denote it by $C \to \mathcal{F}$.

Example 6.3.36. Let $\mathcal{F}$ be the cross-ratio floor diagram of Example 6.3.28. Observe that the floor-decomposed rational tropical stable map C of Example 6.1.5 degenerates to $\mathcal{F}$ if the labels of ends that are not shown in Figure 6.2 are chosen appropriately, i.e. to fit Example 6.3.28.

Figure 6.9 shows another floor-decomposed rational tropical stable map D. It satisfies the degenerated tropical cross-ratio $\lambda_1 = \{1, 2, 3, 7\}$. Moreover, we claim that it is possible to assign lengths to the bounded edges of D in its schematic representation in Figure 6.9 in such a way that D satisfies the same conditions as C. The conditions in question are point conditions p_1, p_2 and the codimension one tangency conditions $P_{[9]\setminus[2]}$. Cut the elevators of C and D, project the floors to $\mathbb{R}^2$ using π as in the proof of Proposition 6.3.31. Notice that it is sufficient to check whether the projections of the floors C_1 and D_1 satisfy the same conditions. For that, use the cross-ratio lattice path algorithm of Chapter 5 with the degenerated tropical cross-ratio λ_1 to obtain the projections $\pi(C_1)$ and $\pi(D_1)$ that then satisfy $\pi\left(p_{[2]}\right)$ and $\pi\left(P_{[9]\setminus[2]}\right)$. Lifting $\pi(C_1)$ and $\pi(D_1)$ yields the desired lengths. The lattice path calculation can be found in Example 5.1.17 and Figure 5.5. More precisely, $\pi(C_1)$ corresponds to the entry (read as a matrix) $(6, 2)$, and $\pi(D_1)$ corresponds to the entry $(4, 2)$ of Figure 5.5 (see also Example 5.2.5).

Thus D degenerates to $\mathcal{F}$ as well if the missing labels in Figure 6.9 are chosen appropriately.

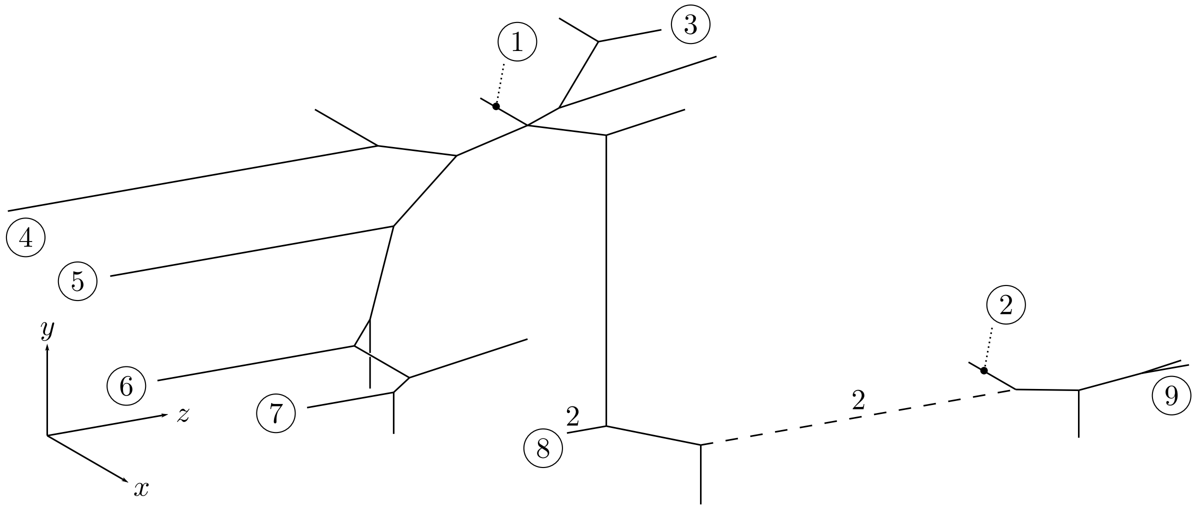


Figure 6.9: The floor-decomposed rational tropical stable map D of Example 6.3.36. It has two floors D_i for $i = 1, 2$. The dashed edge is the elevator of weight two of D.

Proof of Theorem 6.3.34. By Proposition 6.1.6 every rational tropical stable map that contributes to the left-hand side of (6.19) is floor-decomposed. Hence Construction 6.3.35 associates a cross-ratio floor diagram that contributes to the right-hand side of (6.19) to each such rational tropical stable map. Therefore it is sufficient to show for a fixed cross-ratio floor diagram $\mathcal{F}$ of degree $\Delta^3_d(\alpha, \beta)$ which satisfies the given conditions that

$$\text{mult}(\mathcal{F}) = \sum_{C \to \mathcal{F}} \text{mult}(C) \tag{6.20}$$

holds, where the sum goes over all C degenerating to $\mathcal{F}$. In other words, we need to show that the multiplicity with which a cross-ratio floor diagram $\mathcal{F}$ is counted equals the sum of the multiplicities of all rational tropical stable maps C that contribute to the left-hand side of (6.19) such that C degenerates to $\mathcal{F}$. So fix a cross-ratio floor diagram $\mathcal{F}$ that contributes to the right-hand side of (6.19).

To shorten notation, let $B := \{p_{[n]}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}\}$ be the set of conditions that $\mathcal{F}$ satisfies, and let $N(B)$ (resp. $N^{\text{floor}}(B)$) denote the number on the left-hand side (resp. the right-hand side) of (6.19). Assume that $\mathcal{F}$ has more than 1 vertex, because otherwise there is nothing to show. Since $\mathcal{F}$ is a tree, there is a 1-valent vertex v_i of $\mathcal{F}$ adjacent to a vertex v_j via an edge q such that $v_i < v_j$. There are two cases: q is either a 1/1 edge or a 2/0 edge. First, assume that q is a 1/1 edge. Let $L^{(j)}$ (resp. $L^{(i)}$) be the codimension two tangency condition of Corollary 6.3.23 which v_i passes to v_j via q (resp. v_j passes to v_i). Notice that the conditions $L^{(j)}$ (resp. $L^{(i)}$) are cycles in $\mathbb{R}^2$ as in Proposition 6.3.22 that do not depend on the tropical stable maps that degenerate to $\mathcal{F}$.

Similar to the proof of Lemma 4.3.3, we follow the idea of recursively moving conditions in such a way that $\operatorname{mult}(C)$ can be calculated as a product in which each factor is associated to a floor. Cut q to obtain two new cross-ratio floor diagrams $\mathcal{F}_i$ and $\mathcal{F}_j$, where $\mathcal{F}_i$ consists of a single vertex v_i and $\mathcal{F}_j$ is given by $\mathcal{F}$ without v_i. Decomposing $\mathcal{F}$ into $\mathcal{F}_i$ and $\mathcal{F}_j$ decomposes B into B_i and B_j as well, more precisely, let $B_i \subset B$ (resp. $B_j \subset B$) be the subset of conditions $\mathcal{F}_i \subset \mathcal{F}$ (resp. $\mathcal{F}_j$) satisfies. Notice that the set of all conditions $\mathcal{F}_i$ (resp. $\mathcal{F}_j$) satisfies is $B_i \cup L^{(i)}$ (resp. $B_j \cup L^{(j)}$).

If we change the x_1- and x_2-coordinates of the conditions in B_i and B_j in such a way that all conditions are still in a stretched configuration, then any rational tropical stable map C with $C \to \mathcal{F}$ is still floor-decomposed and still degenerates to $\mathcal{F}$. Notice that moving conditions as above moves $L^{(i)}$ and $L^{(j)}$ accordingly. More precisely, we are allowed to move the x_1- and x_2-coordinates of the conditions in B_i and B_j inside a box, which is small compared to the stretching of the conditions in x_3-direction. Notice that moving conditions as above moves $L^{(i)}$ and $L^{(j)}$ accordingly. Hence we can achieve (by bringing the x_1, x_2-positions of conditions in B_j (resp. B_i) arbitrarily close together) that $L^{(i)}$ and $L^{(j)}$ intersect in the following way: all points of the intersection of $L^{(i)}$ and $L^{(j)}$ are on the ends of $L^{(i)}$ that are of primitive direction $(0,-1) \in \mathbb{R}^2$, and on the ends of $L^{(j)}$ that are of primitive direction $(-1,0) \in \mathbb{R}^2$, see Figure 6.10. Thus, using part (b) of Proposition 6.3.20 and Notation 6.3.18, we have

$$\sum_{C \to \mathcal{F}} \operatorname{mult}(C) = \omega(q) \sum_{C \to \mathcal{F}} \operatorname{mult}(C_{i,10}) \operatorname{mult}(C_{j,01}),$$

where $\omega(q)$ is the weight of the cut edge q and C_i, C_j are the pieces obtained from C by cutting the elevator that corresponds to q.

We claim that

$$\sum_{C \to \mathcal{F}} \operatorname{mult}(C_{i,10}) \operatorname{mult}(C_{j,01}) = N(B_i \cup \{E^{(i)}\}) \sum_{C_j \to \mathcal{F}_j} \operatorname{mult}(C_j), \tag{6.21}$$

where $\mathcal{F}_j$ is understood as a cross-ratio floor diagram that satisfies the conditions $B_j \cup \{E^{(j)}\}$, and where $E^{(i)}$ and $E^{(j)}$ are codimension two tangency conditions that are tropical multi-lines in $\mathbb{R}^2$ with ends of weight 1. To see this, let $E^{(i)}$ and $E^{(j)}$ be two tropical lines with ends of weight 1 whose positions are chosen according to Figure 6.10, i.e. choose $E^{(i)}$ (resp. $E^{(j)}$) in such a way that it intersects $L^{(j)}$ (resp. $L^{(i)}$) only in its ends of primitive direction $(-1,0)$ (resp. $(0,-1)$) and such that each point of intersection locally looks like the x_1 and x_2 axes' intersection.

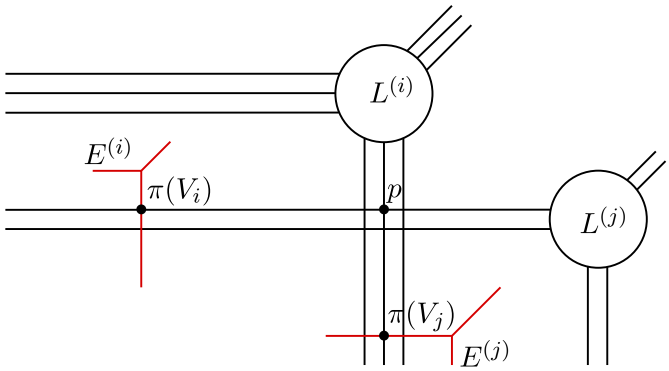


Figure 6.10: Codimension two tangency conditions $L^{(i)}$ and $L^{(j)}$ after movement, together with the codimension two tangency conditions $E^{(i)}$ and $E^{(j)}$, where $p \in L^{(i)} \cap L^{(j)}$ is the point associated to $\pi(V_i)$ and $\pi(V_j)$.

Each rational tropical stable map C_i that contributes to $N(B_i \cup \{E^{(i)}\})$ has an end q parallel to the x_3-axis whose adjacent vertex V_i is 3-valent and satisfies $\pi(V_i) \in L^{(j)} \cap E^{(i)}$, where π is the natural projection that forgets the x_3-coordinate. This is true due to Proposition 6.3.22 and since C_i satisfies $L^{(j)}$ by definition of $L^{(j)}$. Analogously, by definition of $L^{(i)}$ and Proposition 6.3.22, each rational tropical stable map C_j from the right-hand side of (6.21) has an end q parallel to the x_3-axis whose adjacent vertex V_j is 3-valent and satisfies $\pi(V_j) \in L^{(i)} \cap E^{(j)}$. Since V_i (resp. V_j) is by Proposition 6.3.22 adjacent to an end of C_i (resp. C_j), we can move V_i and V_j as in Figure 6.10 to the corresponding point of intersection of $L^{(j)} \cap L^{(i)}$ such that the combinatorial types of C_i and C_j do not change and such that the multiplicities of C_i and C_j understood as rational tropical stable maps that contribute to the right-hand side of (6.21) do not change. Since we moved V_i and V_j to one point, C_i and C_j can be glued to obtain a rational tropical stable map C such that $C \to \mathcal{F}$ and the multiplicities of C_i and C_j (understood as rational tropical stable maps that contribute to the right-hand side of (6.21)) are equal to $\operatorname{mult}(C_{i,10})$ and $\operatorname{mult}(C_{j,01})$ by our special choice of the positions of $L^{(i)}$ and $L^{(j)}$. Reversing the process of gluing yields a bijection between factors of the left and factors of the right-hand side of (6.21).

The multiplicity of the vertex v_i of $\mathcal{F}$ equals $N(B_i \cup \{E^{(i)}\})$ by Definition 6.3.29. Moreover, if q is a 2/0 edge instead, then part (a) of Proposition 6.3.20 guarantees that multiplicities split nicely if edges are cut. So in total (6.21) gives rise to a recursion that eventually yields

$$\sum_{C \to \mathcal{F}} \operatorname{mult}(C_{i,10}) \operatorname{mult}(C_{j,01}) = \prod_{i=1}^{n} \operatorname{mult}(v_i)$$

since $\mathcal{F}$ is a tree. Hence (6.20) holds.

Notice that $L^{(i)}$ and $L^{(j)}$ depend on the choice of the floor diagram $\mathcal{F}$. So we should use the notation $L^{(i)}_{\mathcal{F}}$ and $L^{(j)}_{\mathcal{F}}$ instead. It remains to show that we can bring $L^{(i)}_{\mathcal{F}}$ and $L^{(j)}_{\mathcal{F}}$ in a position as above for each choice of floor diagram $\mathcal{F}$ without effecting the overall weighted count of cross-ratio floor diagrams of degree $\Delta^3_d(\alpha, \beta)$ that satisfy the given conditions $p_{[n]}, L_{\underline{\kappa}^\alpha}, L_{\underline{\kappa}^\beta}, P_{\underline{\eta}^\alpha}, P_{\underline{\eta}^\beta}, \lambda_{[l]}$. Moving conditions as above does not lead to a rational tropical stable map that degenerates to another cross-ratio floor diagram then it initially did, and the cycle obtained by moving conditions (i.e. by relaxing some of the initially given conditions) as above is balanced. Therefore we can assume that $L^{(i)}_{\mathcal{F}}$ and $L^{(j)}_{\mathcal{F}}$ are always in a position as shown in Figure 6.10. $\square$

Using Tyomkin's correspondence theorem 2.3.6 (more precisely, Corollary 3.1.20 and Proposition 3.2.7), Theorem 6.3.34 immediately yields the following corollary.

Corollary 6.3.37 (Algebro-geometric count via cross-ratio floor diagrams for $\mathbb{R}^3$)**.** *Notation 2.0.1 is used. Let $\Delta_d^3(\alpha,\beta)$ be a degree in $\mathbb{R}^3$ as in Notation 2.2.4 and let $\Sigma\left(\Delta_d^3(\alpha,\beta)\right)$ be its associated lattice polytope, see Remark 2.2.5. Let $X_{\Sigma\left(\Delta_d^3(\alpha,\beta)\right)}$ be the toric variety associated to $\Sigma\left(\Delta_d^3(\alpha,\beta)\right)$. Then the number of rational algebraic curves in $X_{\Sigma\left(\Delta_d^3(\alpha,\beta)\right)}$ over an algebraically closed field of characteristic zero that satisfy point conditions and classical cross-ratio conditions in general position can be calculated via a weighted count of cross-ratio floor diagrams of degree $\Delta_d^3(\alpha,\beta)$ that satisfy point conditions and degenerated tropical cross-ratio conditions if each entry of each degenerated tropical cross-ratio is either the label of a contracted end or the label of and end of primitive direction $\pm e_3$ (see Notation 2.2.4).*

Notice that Corollary 6.3.37 indeed provides a way of explicitly calculating the numbers in question since there are only finitely many cross-ratio floor diagrams of degree $\Delta_d^3(\alpha,\beta)$ which can be found when going through all possible cases.

Remark 6.3.38. As in Section 6.2 the results of the current section are not restricted to rational tropical stable maps of degree $\Delta_d^3(\alpha,\beta)$. They can be extended to a slightly larger class of degrees similar to the ones of Remark 6.2.15.

Part II

Enumeration in tropical mirror symmetry

Chapter 7

Preliminaries

In this preliminary section, we give a brief overview of the necessary Gromov-Witten invariants and Feynman integrals. The part about Gromov-Witten invariants is split into two parts, namely Gromov-Witten invariants of an elliptic curve E and Gromov-Witten invariants of $E \times \mathbb{P}^1$. The part about Gromov-Witten invariants of E provides preliminaries necessary for Chapter 8. The part about Gromov-Witten invariants of $E \times \mathbb{P}^1$ is for Chapter 9. Feynman integrals are important for both, Chapter 8 and Chapter 9.

7.1 Covers of an elliptic curve E

Descendant Gromov-Witten invariants of E (i.e. Gromov-Witten invariants which involve psi-conditions) are recalled. Moreover, relative Gromov-Witten invariants of $\mathbb{P}^1$ are recalled as well since they turn out to be useful later. Then, the tropical counterparts to such invariants are defined and correspondence theorems are recalled.

7.1.1 Descendant Gromov-Witten invariants

Gromov-Witten invariants are virtual enumerative intersection numbers on moduli spaces of stable maps. Let E be an elliptic curve. Gromov-Witten invariants of E do not depend on its complex structure. A *stable map* of degree d from a curve of genus g to E with n markings is a map $f : C \to E$, where C is a connected projective curve with at worst nodal singularities, and with n distinct nonsingular marked points $x_1, \ldots, x_n \in C$, such that $f_*([C]) = d[E]$ and f has a finite group of automorphism. The moduli space of stable maps, denoted $\overline{\mathcal{M}}_{g,n}(E, d)$, is a proper Deligne-Mumford stack of virtual dimension $2g-2+n$ [Beh97, BF97]. The ith evaluation morphism is the map $\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(E, d) \to E$ that sends a point $[C, x_1, \ldots, x_n, f]$ to $f(x_i) \in E$. The ith cotangent line bundle $\mathbb{L}_i \to \overline{\mathcal{M}}_{g,n}(E, d)$ is defined by a canonical identification of its fiber over a moduli point $(C, x_1, \ldots, x_n, f)$ with the cotangent space $T^*_{x_i}(C)$. The first Chern class of the cotangent line bundle is called a *psi class* ($\psi_i = c_1(\mathbb{L}_i)$).

Definition 7.1.1 (Stationary descendant Gromov-Witten invariants of E)**.** Let E be an elliptic curve. Fix g, n, d and let $k_1, \ldots, k_n$ be non-negative integers with

$$k_1 + \ldots + k_n = 2g - 2.$$

The *stationary descendant Gromov-Witten invariant* $\langle \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) \rangle^{E,d}_{g,n}$ is defined by:

$$\langle \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) \rangle^{E,d}_{g,n} = \int_{[\overline{\mathcal{M}}_{g,n}(E,d)]^{vir}} \prod_{i=1}^{n} ev_i^*(pt)\psi_i^{k_i}, \tag{7.1}$$

where pt denotes the class of a point in E.

Remark 7.1.2. It follows from the Gromov-Witten/Hurwitz correspondence in [OP06] that a stationary descendant Gromov-Witten invariant with $k_i = 1$ for all i is a Hurwitz number which counts covers of the given degree and genus and with n fixed simple branch points.

Similarly to Definition 7.1.1, *relative Gromov-Witten inviariants* of $\mathbb{P}^1$ can be defined. For that, let $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, \mu, \nu, d)$ denote the moduli spaces of *relative stable maps*, where part of the data specified are the ramification profiles μ and ν which we fix over 0 resp. $\infty \in \mathbb{P}^1$. The preimages of 0 and ∞ are marked. A detailed discussion of spaces of relative stable maps to $\mathbb{P}^1$ and their boundary is not necessary for our purpose, but can be found in [Vak08]. Relative Gromov-Witten invariants of $\mathbb{P}^1$ are useful to study descendant Gromov-Witten invariants of E as we see later.

Definition 7.1.3 (Relative descendant Gromov-Witten inviariants of $\mathbb{P}^1$)**.** Fix g, n, d and let $k_1, \ldots, k_n$ be non-negative integers with

$$k_1 + \ldots + k_n = 2g - 2.$$

The *relative descendant Gromov-Witten invariant* $\langle \mu | \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) | \nu \rangle_{g,n}^{\mathbb{P}^1, d}$ is defined by:

$$\langle \mu | \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) | \nu \rangle_{g,n}^{\mathbb{P}^1, d} = \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, \mu, \nu, d)]^{\mathrm{vir}}} \prod_{i=1}^{n} \mathrm{ev}_i^*(pt) \psi_i^{k_i},$$

where pt denotes the class of a point in $\mathbb{P}^1$.

Notation 7.1.4. One can allow source curves to be disconnected, and introduce *disconnected Gromov-Witten invariants.* We add the superscript $\bullet$ anytime we wish to refer to the disconnected theory.

Definition 7.1.5 ($\mathcal{S}$-function)**.** Let sinh denote the sinus hyperbolicus. Define the formal series

$$\mathcal{S}(z) := \frac{\sinh(z/2)}{z/2}$$

in the variable z and call it *$\mathcal{S}$-function.* Note that the $\mathcal{S}$-function is an even function (i.e. $\mathcal{S}(-z) = \mathcal{S}(z)$), since sinh is an odd function and quotients of odd functions are even.

The following theorem provides a nice form for the generating series of relative one-point descendant Gromov-Witten invariants.

Theorem 7.1.6 (Okounkov-Pandharipande's one-point series, Theorem 2 of [OP06])**.** *The generating series of relative one-point descendant Gromov-Witten invariants with respect to g can be expressed in terms of the $\mathcal{S}$-function. More precisely, the equation*

$$\sum_{g \geq 0} \langle \mu | \tau_{2g-2+\ell(\mu)+\ell(\nu)}(pt) | \nu \rangle_{g,1}^{\mathbb{P}^1, d} \cdot z^{2g} = \frac{\prod \mathcal{S}(\mu_i z) \cdot \prod \mathcal{S}(\nu_i z)}{\mathcal{S}(z)} \tag{7.2}$$

holds, where the product goes over all entries μ_i (resp. ν_i) of the fixed partition μ (resp. ν).

7.1.2 Tropical descendant Gromov-Witten invariants

The following definition generalizes Definition 2.2.1 of abstract rational tropical curves to abstract tropical curves of any genus.

Definition 7.1.7 (Abstract tropical curves)**.** An *abstract tropical curve* is a connected metric graph Γ together with a genus function $g : \Gamma \to \mathbb{Z}_{\geq 0}$, such that Γ can have unbounded non-loop edges adjacent to only one vertex called *ends* which have infinite length, and the genus function g has finite support. Locally around a point p, the metric graph Γ is homeomorphic to a star with r halfrays. The number r is called the *valence* of the point p and is denoted by $\mathrm{val}(p)$. The vertex set of Γ is identified with the points where the genus function is nonzero, together with points of valence different from 2. Besides *edges*, the notion of *flags* of Γ is introduced. A flag is a pair (V, e) of a vertex V and an edge e incident to it ($V \in \partial e$). Edges that are not ends are required to have finite length and are referred to as *bounded* edges.

A *marked tropical curve* is an abstract tropical curve whose leaves are labeled. An isomorphism of abstract tropical curves is an isometry that respects the leaf markings and the genus function. The *genus* of an abstract tropical curve Γ is defined as

$$g(\Gamma) := b_1(\Gamma) + \sum_{p \in \Gamma} g(p),$$

where $b_1(\Gamma)$ denotes the first Betti number of Γ. An abstract tropical curve of genus zero is called *rational* and an abstract tropical curve that satisfies $g(v) = 0$ for all v is called *explicit*. The *combinatorial type* of an abstract tropical curve is the equivalence class of abstract tropical curves obtained by identifying any two abstract tropical curves which differ only by edge lengths.

Example 7.1.8. We denote the tropical numbers $\mathbb{R} \cup \{-\infty\}$ by $\mathbb{T}$. The tropical projective line, $\mathbb{P}^1_{\mathbb{T}}$, equals $\mathbb{R} \cup \{\pm\infty\}$. As in algebraic geometry, it is glued from two copies of the affine line $\mathbb{T}$ using the tropicalization of the identification map, i.e. using $x \mapsto -x$ on $\mathbb{R}$.

A (nondegenerate) tropical elliptic curve $E_{\mathbb{T}}$ is a circle with a fixed length.

Definition 7.1.9 (Tropical covers)**.** A *tropical cover* $\pi : \Gamma_1 \to \Gamma_2$ is a surjective balanced map of abstract tropical curves. The map π is piecewise integer affine linear on each edge, the slope of π on a flag or edge e is a non-negative integer called the *expansion factor* $\omega(e) \in \mathbb{N}$.

The expansion factor of e can be zero only if e is an end. We fix the convention that the ends marked $1, \ldots, n$ are the ones with expansion factor 0. These ends are called *contracted ends.*

For a point $v \in \Gamma_1$, the *local degree d_v of π at v* is defined as follows. Choose a flag f' adjacent to $\pi(v)$, and add the expansion factors of all flags f adjacent to v that map to f', i.e.

$$d_v := \sum_{f \mapsto f'} \omega(f). \tag{7.3}$$

We define the *balancing condition* (or *harmonicity condition*, see [ABBR15]) to be the fact that for each point $v \in \Gamma_1$, the local degree at v is well defined (i.e. independent of the choice of f'). The map π is called *balanced* if it satisfies the balancing condition.

The *degree* d of a tropical cover is the sum over all local degrees of preimages of a point a, i.e.

$$d := \sum_{p \mapsto a} d_p.$$

By the balancing condition, this definition does not depend on the choice of $a \in \Gamma_2$. For a flag f of Γ_2, let μ_f be the partition of expansion factors of the flags of Γ_1 mapping onto f. We call μ_f the *ramification profile* above f.

Definition 7.1.10 (Psi- and point conditions)**.** We say that a tropical cover $\pi : \Gamma_1 \to \Gamma_2$ with a marked end i *satisfies a psi-condition* with power k at i, if the vertex V to which the marked end i is adjacent has valence $k + 3 - 2g(V)$. We say $\pi : \Gamma_1 \to \Gamma_2$ *satisfies the point conditions* $p_1, \ldots, p_n \in \Gamma_2$ if

$$\{\pi(1), \ldots, \pi(n)\} = \{p_1, \ldots, p_n\},$$

where $1, \ldots, n$ are the labels of the contracted ends of Γ_1.

In our case, the abstract tropical curve Γ_2 is either a tropical elliptic curve $E_{\mathbb{T}}$ or a tropical projective line $\mathbb{P}^1_{\mathbb{T}}$ as in Example 7.1.8. If Γ_2 is a tropical elliptic curve such that all ends of a tropical cover of $E_{\mathbb{T}}$ are contracted ends with image points the points p_i we fix as conditions in $E_{\mathbb{T}}$, then it is easy to count the vertices of the tropical source curve:

Lemma 7.1.11. *Fix $g, n, d \in \mathbb{N}_{>0}$ and let $k_1, \ldots, k_n$ be non-negative integers with*

$$k_1 + \ldots + k_n = 2g - 2.$$

Let $\pi : \Gamma \to E_{\mathbb{T}}$ be a tropical cover of degree d such that Γ is of genus g and has n marked ends. Fix n distinct points $p_1, \ldots, p_n \in E_{\mathbb{T}}$. If at the marked end i, a psi-condition with power k_i is satisfied, and the point conditions are satisfied, then Γ has exactly n vertices, each adjacent to one marked end.

Proof. Let V^i_Γ be the set of vertices of Γ such that each vertex in V^i_Γ is adjacent to one marked end. Notice that the marked ends must be adjacent to different vertices, since they satisfy different point conditions. Thus $|V^i_\Gamma| = n$. Let V_Γ be the set of vertices of Γ and let V'_Γ be the complement of V^i_Γ in V_Γ. Denote $a := |V'_\Gamma|$. The Euler characteristic $\chi(\Gamma)$ of the graph Γ (as a simplicial complex) is by definition

$$\chi(\Gamma) = -|V_\Gamma| + |E_\Gamma|,$$

where E_Γ is the set of edges of Γ. There can not be a non-contracted end in Γ since the degree d is finite. Therefore

$$\chi(\Gamma) = -(n+a) + \frac{1}{2}\left(n + \left(\sum_{v_i \in V^i_\Gamma} (k_i + 3 - 2g(v_i))\right) + \left(\sum_{v' \in V'_\Gamma} \big(\mathrm{val}(v') + 2g(v') - 2g(v')\big)\right)\right)$$

holds by the handshaking lemma. Using $k_1 + \ldots + k_n = 2g - 2$ and the definition of $g(\Gamma)$, we obtain

$$\begin{aligned}\chi(\Gamma) &= -(n+a) + 2n - g + h_1(\Gamma) + g - 1 + \frac{1}{2}\left(\sum_{v' \in V'_\Gamma} (\mathrm{val}(v') + 2g(v'))\right) \\ &= -n - a + 2n + h_1(\Gamma) - 1 + \frac{1}{2}\left(\sum_{v' \in V'_\Gamma} (\mathrm{val}(v') + 2g(v'))\right).\end{aligned} \tag{7.4}$$

On the other hand, removing all n ends of Γ and $b_1(\Gamma)$ additional edges yields a tree which has $|V_\Gamma| - 1$ edges. Thus

$$\chi(\Gamma) = -1 + n + b_1(\Gamma). \tag{7.5}$$

Combining (7.4) and (7.5) yields

$$|V'_\Gamma| = a = \frac{1}{2}\left(\sum_{v' \in V'_\Gamma} \big(\mathrm{val}(v') + 2g(v')\big)\right).$$

By definition of vertices of Γ it holds that $2g(v') > 1$ or $\mathrm{val}(v') > 2$. This yields a contradiction unless $V'_\Gamma = \emptyset$. □

Definition 7.1.12 (Local vertex multiplicities)**.** Let $\pi : \Gamma \to \Gamma'$ be a tropical cover of degree d, where $\Gamma' = E_{\mathbb{T}}$ or $\Gamma' = \mathbb{P}^1_{\mathbb{T}}$ (see Example 7.1.8) such that Γ is of genus g and satisfies given point conditions and psi-conditions as in Definition 7.1.10. Locally at the marked end i, the cover sends the vertex to an interval consisting of two flags f and f'. Define the *local vertex multiplicity* $\mathrm{mult}_i(\pi)$ to be a one-point relative descendant Gromov-Witten invariant, i.e.

$$\mathrm{mult}_i(\pi) := \langle \mu_f | \tau_{k_i}(pt) | \mu_{f'} \rangle^{\mathbb{P}^1, d_i}_{g_i, 1}, \tag{7.6}$$

where g_i denotes the genus of the vertex adjacent to the marked end i, d_i its local degree, and μ_f resp. $\mu_{f'}$ the ramification profiles above the two flags of the image interval.

Define the multiplicity of the cover π to be

$$\frac{1}{|\operatorname{Aut}(\pi)|} \cdot \prod_{i=1}^{n} \mathrm{mult}_i(\pi) \cdot \prod_e \omega(e), \tag{7.7}$$

where the last product goes over the bounded edges e of Γ.

Definition 7.1.13 (Tropical stationary descendant Gromov-Witten invariant of $E_{\mathbb{T}}$)**.** Fix $g, n, d \in \mathbb{N}_{>0}$ and let $k_1, \ldots, k_n$ be non-negative integers with

$$k_1 + \ldots + k_n = 2g - 2.$$

Define the *tropical stationary descendant Gromov-Witten invariant*

$$\langle \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) \rangle^{E,d,\mathrm{trop}}_{g,n}$$

to be the weighted count of tropical genus g degree d covers of $E_{\mathbb{T}}$ with n distinct marked points such that each such tropical cover satisfies point and psi-conditions as above and is counted with its multiplicity as defined in (7.7).

Remark 7.1.14. The metric structure of the source curves of covers contributing to a tropical descendant Gromov-Witten invariant is implicit in the metric data of $E_{\mathbb{T}}$ and the chosen point conditions. We can thus neglect length data in the source curve.

Example 7.1.15. Fix three different points p_1, p_2, p_3 on $E_{\mathbb{T}}$ and let $d = 3$, $g = 2$, $k_1 = 2$, $k_2 = 0$, $k_3 = 0$. Notice that $\sum_i k_i = 2g - 2$ is satisfied. We list all covers that contribute to $\langle \tau_2(pt)\tau_0(pt)\tau_0(pt) \rangle^{E,3,\mathrm{trop}}_{2,3}$ in Figure 7.1 below. Figure 7.1 shows schematic representations of the source curves of all covers contributing, where we assume that the top vertex of each such representation is mapped to p_1, the right vertex is mapped to p_2 and the left one is mapped to p_3. This convention gives us one choice out of 3! choices of an order of labeled vertices of the source curve mapping to p_1, p_2, p_3 on $E_{\mathbb{T}}$. A circled number indicates that there is a nonzero genus g_i at a vertex i. The other numbers are the weights of the edges that are greater than 1. Notice that the valence of a vertex i is given by $k_i + 3 - 2g_i$ when taking the contracted ends into account. When neglecting marked ends, the underlying graph is either a figure 8 or a loop (see Example 7.3.4). In each case, every loop is mapped to $E_{\mathbb{T}}$. When drawing a curl in an edge, it means that the edge is mapped once around $E_{\mathbb{T}}$.

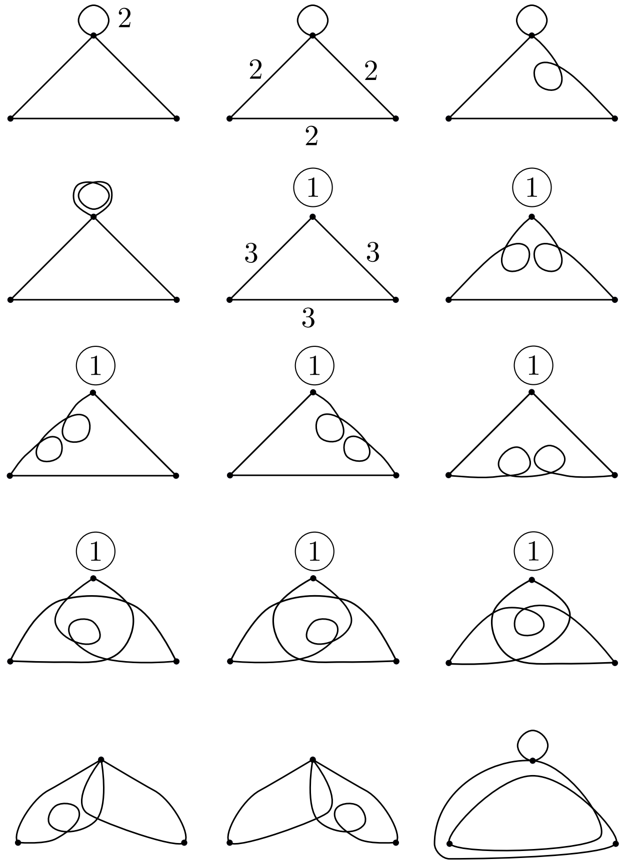


Figure 7.1: Schematic representations of source curves.

The multiplicity with which each curve contributes is given by (7.7). The local multiplicities $\operatorname{mult}_i(\pi)$, which are one-point relative descendant Gromov-Witten invariants, can be calculated explicitly using the one-point series (7.2). Each entry of the tabular below corresponds to one source curve of Figure 7.1 in the obvious way. An entry is the multiplicity of the corresponding cover π, where the first factor equals $|\operatorname{Aut}(\pi)|^{-1}$, the second factor equals $\prod_i \operatorname{mult}_i(\pi)$ and the third factor equals $\prod_e \omega(e)$.

$1 \cdot 1 \cdot 2$	$1 \cdot 1 \cdot 8$	$1 \cdot 1 \cdot 1$	$1 \cdot 1 \cdot 1$	$1 \cdot 1 \cdot 1$
$1 \cdot 1 \cdot 1$	$1 \cdot \frac{17}{24} \cdot 27$	$1 \cdot \frac{1}{24} \cdot 1$	$1 \cdot \frac{1}{24} \cdot 1$	$1 \cdot \frac{1}{24} \cdot 1$
$1 \cdot \frac{1}{24} \cdot 1$	$1 \cdot \frac{1}{24} \cdot 1$	$1 \cdot \frac{1}{24} \cdot 1$	$1 \cdot 1 \cdot 4$	$1 \cdot 1 \cdot 4$
$1 \cdot \frac{1}{24} \cdot 1$	$1 \cdot \frac{1}{24} \cdot 1$	$1 \cdot \frac{1}{24} \cdot 1$	$1 \cdot 1 \cdot 1$	$1 \cdot 1 \cdot 1$
$1 \cdot 1 \cdot 1$	$1 \cdot 1 \cdot 1$	$1 \cdot 1 \cdot 1$		

Summing over all entries and considering the factor 3! yields

$$\begin{aligned} \langle \tau_2(pt)\tau_0(pt)\tau_0(pt)\rangle_{2,3}^{E,3,\text{trop}} &= 3! \cdot \frac{93}{2} \\ &= 279. \end{aligned}$$

Notation 7.1.16. For a partition μ, let $\ell(\mu)$ denote its length, i.e. its number of entries.

Definition 7.1.17 (Tropical relative stationary descendant Gromov-Witten invariant of $\mathbb{P}^1_{\mathbb{T}}$)**.** Let $g, d \in \mathbb{N}$. Let μ, ν be two partitions of the degree d. Let $k_1, \ldots, k_n$ be non-negative integers with

$$k_1 + \ldots + k_n = 2g + \ell(\mu) + \ell(\nu) - 2,$$

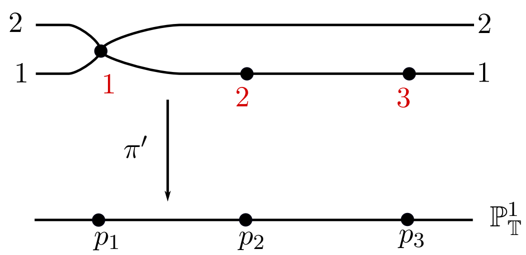
where Notation 7.1.16 is used. Consider tropical covers of $\mathbb{P}^1_{\mathbb{T}}$ such that the ramification profile over $-\infty$ equals μ and the ramification profile over ∞ equals ν. That is, in addition to the contracted ends that we use to impose point conditions, the source curve Γ has $\ell(\mu) + \ell(\nu)$ marked ends which map to $\pm\infty$ with expansion factors imposed by μ and ν. Define the *tropical relative stationary descendant Gromov-Witten invariant*

$$\langle \mu | \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) | \nu \rangle_{g,n}^{\mathbb{P}^1,d,\mathrm{trop}}$$

to be the weighted count of tropical genus g degree d covers of $\mathbb{P}^1_{\mathbb{T}}$ with n marked points such that each of these tropical covers satisfies point and psi-conditions (Definition 7.1.10), and such that the expansion factors of the unmarked ends are imposed by μ and ν. The multiplicity with which each such tropical cover is counted is given by (7.7).

For more details about tropical relative stationary descendant Gromov-Witten invariant of $\mathbb{P}^1_{\mathbb{T}}$, see Definition 3.1.1 of [CJMR18].

Example 7.1.18. Choose three different points p_1, p_2, p_3 on $E_{\mathbb{T}}$ and let $d = 3$, $g = 2$, $k_1 = 2$, $k_2 = 0$, $k_3 = 0$ be as in Example 7.1.15. Let p_0 be a base point on $E_{\mathbb{T}}$ such that p_0, p_1, p_2, p_3 are pairwise different and ordered this way on $E_{\mathbb{T}}$. Consider the source curve of a cover π of $E_{\mathbb{T}}$ depicted in the upper left corner of Figure 7.1 and cut it along $\pi^{-1}(p_0)$. Stretching the cut edges to infinity yields the cover shown below (we let i be mapped to p_i). Note that this is a cover π' of $\mathbb{P}^1_{\mathbb{T}}$ that contributes to $\langle (2,1) | \tau_2(pt)\tau_0(pt)\tau_0(pt) | (2,1) \rangle_{0,3}^{\mathbb{P}^1,3,\mathrm{trop}}$.



7.1.3 Correspondence theorems for descendant Gromov-Witten invariants

The following correspondence theorems relate descendant Gromov-Witten invariants of E and relative descendant Gromov-Witten invariants of $\mathbb{P}^1$ to their tropical counterparts, and thus establish tropical geometry as a tool to study such Gromov-Witten invariants.

Theorem 7.1.19 (Stationary correspondence theorem, Theorem 3.2.1 of [CJMR18])**.** *A stationary descendant Gromov-Witten invariant of E from Definition 7.1.1 coincides with its tropical counterpart from Definition 7.1.13, i.e.*

$$\langle \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) \rangle_{g,n}^{E,d} = \langle \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) \rangle_{g,n}^{E,d,\mathrm{trop}}.$$

Theorem 7.1.20 (Relative correspondence theorem, Theorem 3.1.2 of [CJMR18])**.** *A relative stationary descendant Gromov-Witten invariant of $\mathbb{P}^1$ from Definition 7.1.3 coincides with its tropical counterpart from Definition 7.1.17, i.e.*

$$\langle \mu | \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) | \nu \rangle_{g,n}^{\mathbb{P}^1,d} = \langle \mu | \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) | \nu \rangle_{g,n}^{\mathbb{P}^1,d,\mathrm{trop}}.$$

Notation 7.1.21. As in Notation 7.1.4, a superscript $\bullet$ is added to the notation to refer to the disconnected theory, where it is allowed for tropical source curves to be disconnected.

Remark 7.1.22 (Leaking)**.** We can tweak the definition of tropical covers of $E_{\mathbb{T}}$ (resp. $\mathbb{P}^1_{\mathbb{T}}$) that satisfy point and psi-conditions as follows: fix a direction for the target curve and specify for each end i of the source curve an integer l_i. Change the balancing condition in such a way that for the two flags f_1 and f_2 adjacent to $\pi(i) \in \{p_1, \dots, p_n\}$ (where we chose the notation to match the direction), the local degrees are not equal but differ by l_i:

$$\sum_{f' \mapsto f_1} \omega(f') = \sum_{f'' \mapsto f_2} \omega(f'') - l_i.$$

We call such covers *leaky tropical covers.* Leaky tropical covers show up as floor diagrams representing counts of tropical curves in toric surfaces (see e.g. [BM08, FM10, AB17]). They are of interest here, since they can be treated in terms of Feynman integrals analogously to their balanced versions.

7.2 Curves in $E \times \mathbb{P}^1$

Let E be an elliptic curve. We recall Gromov-Witten invariants of $E \times \mathbb{P}^1$. A *stable map* of bidegree (d_1, d_2) from a curve of genus g to $E \times \mathbb{P}^1$ with n markings is a map $f : C \to E$, where C is a connected projective curve with at worst nodal singularities, and with n distinct nonsingular marked points $x_1, \dots, x_n \in C$, such that $f_*([C])$ is of class (d_1, d_2) and f has a finite group of automorphism. The moduli space of stable maps, denoted $\overline{\mathcal{M}}_{g,n}(E \times \mathbb{P}^1, (d_1, d_2))$, is a proper Deligne-Mumford stack of virtual dimension $2d_2 + g - 1 + n$ [Beh97, BF97]. The ith evaluation morphism is the map $\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(E \times \mathbb{P}^1, (d_1, d_2)) \to E \times \mathbb{P}^1$ that sends a point $[C, x_1, \dots, x_n, f]$ to $f(x_i) \in E \times \mathbb{P}^1$.

Definition 7.2.1 (Gromov-Witten invariants of $E\times\mathbb{P}^1$)**.** Let $g, n \in \mathbb{N}_{>0}$ and let $(d_1, d_2) \in \mathbb{N}\times\mathbb{N}$ such that $n = 2d_2 + g - 1$ holds. Define the *Gromov-Witten invariant*

$$\langle \tau_0(pt)^n \rangle_{g,n}^{E\times\mathbb{P}^1,(d_1,d_2)} := \int_{[\overline{\mathcal{M}}_{g,n}(E\times\mathbb{P}^1,(d_1,d_2))]^{vir}} \prod_{i=1}^{n} \mathrm{ev}_i^*(pt),$$

where pt denotes the class of a point in $E \times \mathbb{P}^1$. In order to shorten notation, we also write

$$N_{(d_1,d_2,g)} := \langle \tau_0(pt)^n \rangle_{g,n}^{E\times\mathbb{P}^1,(d_1,d_2)}.$$

In order to relate Gromov-Witten invariants of $E \times \mathbb{P}^1$ to their tropical counterparts (see Theorem 9.1.16), *relative Gromov-Witten invariants* of $\mathbb{P}^1 \times \mathbb{P}^1$ are required. For that, let μ^+, ϕ^+, μ^- and ϕ^- be partitions such that the sum d_1 of the parts in μ^+ and ϕ^+ equals the sum of the parts in μ^- and ϕ^-. Let

$$n_1 := \ell(\phi^+) + \ell(\phi^-) \quad \text{and} \quad n_2 := \ell(\mu^+) + \ell(\mu^-),$$

where Notation 7.1.16 is used. Consider the moduli space of *relative stable maps to* $\mathbb{P}^1 \times \mathbb{P}^1$

$$\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1 \times \mathbb{P}^1, (\mu^+, \phi^+), (\mu^-, \phi^-), (d_1, d_2)),$$

where part of the data specified are the partitions of contact orders (μ^+, ϕ^+) resp. (μ^-, ϕ^-) which we fix over the 0- resp. ∞-section in $\mathbb{P}^1 \times \mathbb{P}^1$. The points of contact with the 0- and ∞-section are marked. We want to fix the points with contact orders given by ϕ^+ and ϕ^-, the ones with contact orders given by μ^+ and μ^- are allowed to move. A detailed discussion of spaces of relative stable maps and their boundary can be found e.g. in [Vak08]. This moduli space is a Deligne-Mumford stack of virtual dimension $(g-1)+2d_2+n+n_1+n_2$. For $i=1,\dots,n$, the ith evaluation morphism is the map

$$\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1\times\mathbb{P}^1,(\mu^+,\phi^+),(\mu^-,\phi^-),(d_1,d_2)) \to \mathbb{P}^1\times\mathbb{P}^1$$

that sends a point $[C,x_1,\dots,x_n,f]$ to $f(x_i)\in\mathbb{P}^1\times\mathbb{P}^1$. The points marking the contact points with the 0- and ∞-section give rise to evaluation morphisms

$$\widehat{\mathrm{ev}}_i : \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1\times\mathbb{P}^1,(\mu^+,\phi^+),(\mu^-,\phi^-),(d_1,d_2)) \to \mathbb{P}^1,$$

where the target $\mathbb{P}^1$ is the 0-section for ϕ^- and μ^-, and the ∞-section for ϕ^+ and μ^+.

Definition 7.2.2 (Relative Gromov-Witten invariants of $\mathbb{P}^1\times\mathbb{P}^1$)**.** A *relative Gromov–Witten* invariant is defined as the following intersection number on the moduli space of relative stable maps $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1\times\mathbb{P}^1,(\mu^+,\phi^+),(\mu^-,\phi^-),(d_1,d_2))$:

$$\langle(\phi^-,\mu^-)|\tau_0(pt)^n|(\phi^+,\mu^+)\rangle_{g,n}^{\mathbb{P}^1\times\mathbb{P}^1,(d_1,d_2)} = \int\prod_{j=1}^{n}\mathrm{ev}_j^*(pt)\prod_{i=n+1}^{n+n_1}\widehat{\mathrm{ev}}_i^*(pt). \tag{7.8}$$

Notation 7.2.3. Again, one can allow curves to be disconnected. Whenever we refer to the *disconnected Gromov-Witten invariants*, a superscript $\bullet$ is added.

The following statement is a consequence of the degeneration formula [Li01, Li02], see also Theorem 2.3.2 in [CJMR18]:

Proposition 7.2.4. *A Gromov-Witten invariant of $E\times\mathbb{P}^1$ equals a weighted sum of relative Gromov-Witten invariants of $\mathbb{P}^1\times\mathbb{P}^1$:*

$$N^{\bullet}_{(d_1,d_2,g)} = \sum_{(\mu,\phi)\,\vdash d_1}\frac{\prod_i\mu_i\prod_j\phi_j}{|\operatorname{Aut}(\mu)||\operatorname{Aut}(\phi)|}\langle(\mu,\phi)|\tau_0(pt)^n|(\phi,\mu)\rangle^{\mathbb{P}^1\times\mathbb{P}^1,(d_1,d_2),\bullet}_{g-\ell(\mu)-\ell(\phi),n}.$$

Here, the sum goes over all tuples of partitions which together form a partition (μ,ϕ) of d_1.

7.3 Feynman integrals

Feynman integrals are now defined as coefficients of a formal series as in [BBBM17]. Notice that such coefficients can be computed using a computer algebra system. They are called "integrals" because in a special case there is an interpretation of Feynman integrals in terms of path integrals using complex analysis.

Definition 7.3.1 (Propagator)**.** Define the *propagator* as a (formal) series in x and q:

$$P(x,q) := \sum_{w=1}^{\infty} w\cdot x^w + \sum_{a=1}^{\infty}\left(\sum_{w|a} w\left(x^w+x^{-w}\right)\right)q^a.$$

Definition 7.3.2 (Feynman graphs)**.** Fix $n>1$. A *Feynman graph* is a (non-metrized) graph Γ without ends with n vertices which are labeled $x_1,\dots,x_n$ and with labeled edges $q_1,\dots,q_r$.

Notation 7.3.3. We do not fix the number of edges for a Feynman graph, the index r can vary from graph to graph. The letter r is always used for the number of edges in a fixed Feynman graph Γ. For $k>s$, denote the vertices adjacent to the (non-loop) edge q_k by x_{k^1} and x_{k^2}, where we assume $x_{k^1}<x_{k^2}$ in Ω.

Example 7.3.4. Recall Example 7.1.15, where we provided all covers contributing to the number $\langle \tau_2(pt)\tau_0(pt)\tau_0(pt)\rangle_{2,3}^{E,3,\mathrm{trop}}$. We can label their source curves, turning them into Feynman graphs, see Figure 7.2.

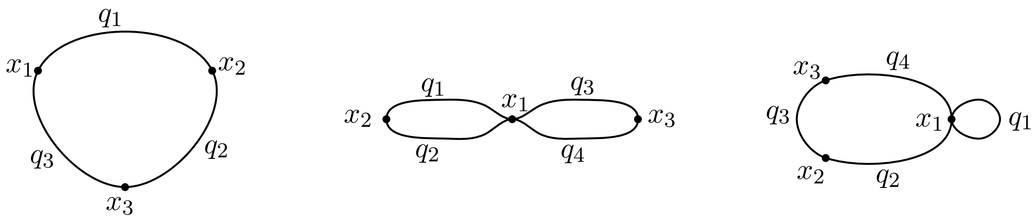


Figure 7.2: Examples of Feynman graphs with three vertices.

Definition 7.3.5 (Feynman integrals)**.** Let Γ be a Feynman graph without loops. Let Ω be an order of the n vertices of Γ. Denote the vertices adjacent to the edge q_k by x_{k^1} and x_{k^2} as in Notation 7.3.3

For integers $l_1, \ldots, l_n$, define the *Feynman integral* for Γ and Ω to be

$$I_{\Gamma,\Omega}^{l_1,\ldots,l_n}(q) := \mathrm{Coef}_{[x_1^{l_1}\ldots x_n^{l_n}]} \prod_{k=1}^{r} P\Big(\frac{x_{k^1}}{x_{k^2}}, q\Big)$$

and the *refined Feynman integral* to be

$$I_{\Gamma,\Omega}^{l_1,\ldots,l_n}(q_1,\ldots,q_r) := \mathrm{Coef}_{[x_1^{l_1}\ldots x_n^{l_n}]} \prod_{k=1}^{r} P\Big(\frac{x_{k^1}}{x_{k^2}}, q_k\Big).$$

Finally, we set

$$I_{\Gamma}^{l_1,\ldots,l_n}(q) := \sum_{\Omega} I_{\Gamma,\Omega}^{l_1,\ldots,l_n}(q),$$

where the sum goes over all $n!$ orders of the vertices of Γ, and

$$I_{\Gamma}^{l_1,\ldots,l_n}(q_1,\ldots,q_r) := \sum_{\Omega} I_{\Gamma,\Omega}^{l_1,\ldots,l_n}(q_1,\ldots,q_r).$$

If the superscript $l_1, \ldots, l_n$ in the notations above is dropped, then this stands for $l_i = 0$ for all i.

Complex analysis and Feynman integrals Often, Feynman integrals are defined as path integrals using complex analysis. The advantage of our definition of Feynman integrals is that it is more general. That is, Feynman integrals can be viewed as path integrals in case of $l_i = 0$ for all i. Our general definition allows us to also relate counts of leaky tropical covers (Remark 7.1.22) to Feynman integrals.

To recall how Feynman integrals and path integrals are related, define

$$p(z,q) := -\frac{1}{4\pi^2}\wp(z,q) - \frac{1}{12}E_2(q^2)$$

in terms of the Weierstraß-P-function $\wp$ and the Eisenstein series

$$E_2(q) := 1 - 24\sum_{a=1}^{\infty}\sigma(a)q^a,$$

where σ denotes the sum-of-divisors function $\sigma(a) = \sum_{w|a} w$. A coordinate change $x = e^{i\pi z}$, yields that p has the following nice form (see Theorem 2.22 [BBBM17]):

$$P(x,q) = \sum_{w=1}^{\infty} w \cdot x^w + \sum_{a=1}^{\infty} \left(\sum_{w|a} w \left(x^w + x^{-w} \right) \right) q^a, \tag{7.9}$$

where $|x| < 1$ is assumed in order to use a geometric series expansion (see Lemma 2.23 [BBBM17]). Notice that (7.9) is precisely the propagator from Definition 7.3.1.

Definition 7.3.6 (Feynman integrals in complex analysis)**.** Let Γ be a Feynman graph without loops and let Ω be an order as in Definition 7.3.5. Pick starting points of the form $iy_1, \ldots, iy_n$ in the complex plane, where the y_j are pairwise different small real numbers. Define integration paths $\gamma_1, \ldots, \gamma_n$ by

$$\gamma_j : [0,1] \to \mathbb{C} : t \mapsto iy_j + t,$$

such that the order of the real coordinates y_j of the starting points of the paths equals Ω. We then define the integral

$$I'_{\Gamma,\Omega}(q) := \int_{z_j \in \gamma_j} \prod_{k=1}^{r} \left(p(z_{k^1} - z_{k^2}, q) \right), \tag{7.10}$$

and the refined version

$$I'_{\Gamma,\Omega}(q_1, \ldots, q_r) := \int_{z_j \in \gamma_j} \prod_{k=1}^{r} \left(p(z_{k^1} - z_{k^2}, q_k) \right).$$

Here, as in Definition 7.3.5, x_{k^1} and x_{k^2} are the two vertices adjacent to an edge q_k. Finally, we set

$$I'_{\Gamma}(q) := \sum_{\Omega} I'_{\Gamma,\Omega}(q) \quad \text{and} \quad I'_{\Gamma}(q_1, \ldots, q_r) := \sum_{\Omega} I'_{\Gamma,\Omega}(q_1, \ldots, q_r).$$

Since p is an even function in z (because $\wp(z,q)$ is by definition an even function in z), it is not important here in which way the vertices x_{k^1} and x_{k^2} of q_k are ordered. The order Ω is only important for the arrangement of the integration paths.

The following theorem is a consequence of Lemma 2.24 and Lemma 2.25 of [BBBM17].

Theorem 7.3.7. *The complex analysis version of Feynman integrals agrees with the version from Definition 7.3.5 that used constant coefficients of formal series. More precisely:*

$$I_{\Gamma,\Omega}(q) = I'_{\Gamma,\Omega}(q) \quad \textit{and} \quad I_{\Gamma,\Omega}(q_1, \ldots, q_r) = I'_{\Gamma,\Omega}(q_1, \ldots, q_r).$$

Chapter 8

(Tropical) mirror symmetry for elliptic curves

In this chapter, which is based on joint work with Janko Böhm and Hannah Markwig [BGM18], the tropical mirror symmetry relation for tropical elliptic curves is extended to tropical descendant Gromov-Witten invariants (Theorem 8.1.9). For that, a bijection as in [BBBM17] is used. Thus leading question (Q6), which asks whether the tropical mirror symmetry relation of Hurwitz numbers of an elliptic curve can be extended, is answered. As a consequence, statements about quasimodularity of generating functions of certain tropical covers of a tropical elliptic curve are obtained, see Corollary 8.1.20.

Moreover, another approach to mirror symmetry of an elliptic curve is studied. For that, tropical Hurwitz numbers are directly linked to matrix elements on the bosonic Fock space. To do so, a version of Wick's Theorem (similar to [CJMR18]) is used, which encodes matrix elements in a bosonic Fock space as weighted sums of graphs, which can then directly be related to tropical Hurwitz covers. Notice that this partially answers leading question (Q7). To answer Question (Q7) completely (i.e. to relate tropical descendant Gromov-Witten invariants to the Fock space), only the amount of notation has to be increased, which we omit here.

8.1 Generating series and Feynman integrals

The aim of this section is to prove Theorem 8.1.9, which is a tropical mirror symmetry theorem for tropical elliptic curves that involves tropical descendant Gromov-Witten invariants. Theorem 8.1.9 then yields a classical analogue, namely Theorem 8.1.4 for elliptic curves which generalizes Dijkgraaf's mirror symmetry theorem.

8.1.1 Generalizing Feynman integrals

Feynman integrals are now generalized to take loop-edges and genus at vertices of the underlying Feynman graphs into account.

Notation 8.1.1. Let Γ be a Feynman graph with labeled edges $q_1,\dots,q_r$ as in Definition 7.3.2. By convention, we assume that $q_1,\dots,q_s$ are loop edges and $q_{s+1},\dots,q_r$ are non-loop edges.

Definition 8.1.2 (Loop Propagator and refined Feynman integrals)**.** In addition to the propagator $P(x,q)$ from Definition 7.3.1, we introduce another formal series in q, namely

$$P^{\text{loop}}(q) := \sum_{a=1}^{\infty}\left(\sum_{w|a} w\right) q^a.$$

$P^{\text{loop}}(q)$ is called *loop-propagator* and it should be viewed as the propagator for loop-edges of a Feynman graph, i.e. use Notation 8.1.1 and define the *refined Feynman integral* for a Feynman graph Γ with loop-edges as

$$I_{\Gamma,\Omega}^{l_1,\ldots,l_n}(q_1,\ldots,q_r) = \operatorname{Coef}_{[x_1^{l_1}\ldots x_n^{l_n}]} \prod_{k=1}^{s} P^{\text{loop}}(q_k) \prod_{k=s+1}^{r} P\Big(\frac{x_{k^1}}{x_{k^2}}, q_k\Big).$$

Analogously to Definition 7.3.5, we also define $I_{\Gamma}^{l_1,\ldots,l_n}(q_1,\ldots,q_r)$ for Feynman graphs with loop-edges.

Definition 8.1.3 (Feynman integrals with vertex contributions)**.** Let Γ be a Feynman graph, and equip it with an additional genus function $\underline{g}$ that associates a nonnegative integer g_i to every vertex x_i of Γ. Let Ω be an order of the n vertices of Γ. We adapt our notion of propagators from definitions 7.3.1, 8.1.2 and 7.3.5 to include vertex contributions. For non-loop-edges, we set

$$\begin{aligned}\tilde{P}(\frac{x_{k^1}}{x_{k^2}}, q) :=& \sum_{w=1}^{\infty} \mathcal{S}(wz_{k^1})\mathcal{S}(wz_{k^2}) \cdot w \cdot \left(\frac{x_{k^1}}{x_{k^2}}\right)^w \\ &+ \sum_{a=1}^{\infty} \left(\sum_{w|a} \mathcal{S}(wz_{k^1})\mathcal{S}(wz_{k^2}) \cdot w \cdot \left(\left(\frac{x_{k^1}}{x_{k^2}}\right)^w + \left(\frac{x_{k^2}}{x_{k^1}}\right)^w \right) \right) \cdot q^a,\end{aligned}$$

where Notation 7.3.3 is used and $\mathcal{S}$ denotes the $\mathcal{S}$-function (Definition 7.1.5). For loop-edges connecting the vertex x_{k^1} to itself, we set

$$\tilde{P}^{\text{loop}}(q) := \sum_{a=1}^{\infty} \left(\sum_{w|a} \mathcal{S}(wz_{k^1})^2 \cdot w \right) q^a.$$

We define the *Feynman integral with vertex contributions* for Γ, $\underline{g}$ and Ω to be

$$I_{\Gamma,\underline{g},\Omega}^{l_1,\ldots,l_n}(q) := \operatorname{Coef}_{[z_1^{2g_1}\ldots z_n^{2g_n}]} \operatorname{Coef}_{[x_1^{l_1}\ldots x_n^{l_n}]} \prod_{i=1}^{n} \frac{1}{\mathcal{S}(z_i)} \prod_{k=1}^{s} \tilde{P}^{\text{loop}}(q) \prod_{k=s+1}^{r} \tilde{P}(\frac{x_{k^1}}{x_{k^2}}, q)$$

and the *refined Feynman integral with vertex contributions*

$$I_{\Gamma,\underline{g},\Omega}^{l_1,\ldots,l_n}(q_1,\ldots,q_r) := \operatorname{Coef}_{[z_1^{2g_1}\ldots z_n^{2g_n}]} \operatorname{Coef}_{[x_1^{l_1}\ldots x_n^{l_n}]} \prod_{i=1}^{n} \frac{1}{\mathcal{S}(z_i)} \prod_{k=1}^{s} \tilde{P}^{\text{loop}}(q_k) \prod_{k=s+1}^{r} \tilde{P}(\frac{x_{k^1}}{x_{k^2}}, q_k).$$

Moreover, we set

$$I_{\Gamma,\underline{g}}^{l_1,\ldots,l_n}(q) := \sum_{\Omega} I_{\Gamma,\underline{g},\Omega}^{l_1,\ldots,l_n}(q),$$

where the sum goes over all $n!$ orders of the vertices, and

$$I_{\Gamma,\underline{g}}^{l_1,\ldots,l_n}(q_1,\ldots,q_r) := \sum_{\Omega} I_{\Gamma,\underline{g},\Omega}^{l_1,\ldots,l_n}(q_1,\ldots,q_r).$$

The superscript $l_1,\ldots,l_n$ is dropped in case of $l_i = 0$ for all i.

8.1.2 Mirror symmetry for (tropical) elliptic curves

We now state theorems 8.1.4 and 8.1.9. The latter is a a tropical mirror symmetry theorem for $E_{\mathbb{T}}$ and is one of the main results of Chapter 8. It expresses generating series of so-called labeled tropical descendant Gromov-Witten invariants in terms of Feynman integrals. Moreover, we shown that it implies the classical mirror symmetry theorem 8.1.4, which is a generalization of Dijkgraaf's mirror symmetry theorem for elliptic curves.

Theorem 8.1.4 (Mirror symmetry for E)**.** *Fix $g \geq 2$, $n \geq 1$ and $k_1, \ldots, k_n \geq 1$ such that*

$$k_1 + \ldots + k_n = 2g - 2$$

holds. The generating series of descendant Gromov-Witten invariants of E can be expressed in terms of Feynman integrals:

$$\sum_{d \geq 1} \langle \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) \rangle_{g,n}^{E,d} q^d = \sum_{(\mathrm{ft}(\Gamma), \underline{g})} \frac{1}{|\operatorname{Aut}(\mathrm{ft}(\Gamma), \underline{g})|} I_{\Gamma, \underline{g}}(q), \tag{8.1}$$

where Γ is a Feynman graph (see Definition 7.3.2) with a genus function $\underline{g}$, such that the vertex x_i has genus g_i and valence $k_i + 2 - 2g_i$, and such that

$$b_1(\Gamma) + \sum g_i = g,$$

where $b_1(\Gamma)$ denotes the first Betti number of Γ. Moreover, in (8.1) *automorphisms of unlabeled graphs are considered (*ft *is the forgetful map that forgets all labels of a Feynman graph Γ, see Definition 8.1.7) that are required to respect the genus function.*

A version of Theorem 8.1.4 is proved in [Li11b], Proposition 6.7 (resp. [Li11a], Proposition 3.4) using the Fock space approach common in mathematical physics to which we relate the tropical approach in Section 8.2. In our approach, Theorem 8.1.4 becomes an easy corollary obtained by combining our tropical mirror symmetry Theorem 8.1.9 with the Correspondence Theorem 7.1.19.

Example 8.1.5 (Automorphisms)**.** Consider the middle Feynman graph of Example 7.3.4, denote it by Γ and let its genus function be $\underline{g} = 0$, i.e. there is no genus at the vertices. The automorphisms appearing in Theorem 8.1.4 are automorphisms respecting the underlying graph structure and the genus function of $(\Gamma, \underline{g})$. In other words, we forget the labels of Γ before determining its automorphisms. In case of Γ as above, the automorphism group is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, because we can exchange the edges q_1 and q_2 (see Example 7.3.4) which gives a factor of $\mathbb{Z}_2$, we can exchange the edges q_3 and q_4 and we can exchange the vertices x_2 and x_3 in such a way that the edge q_1 maps to q_3 and the edge q_2 maps to q_4, see also the left side of Figure 8.1.

In Section 8.1.4, we deal with unlabeled tropical covers, but with fixed *order*. That is, we fix which end i maps to which point p_j on the elliptic curve. In such a case, on the Feynman integral side, we deal with automorphisms of the underlying Feynman graph with vertex labels (see Corollary 8.1.18). If we choose $(\Gamma, \underline{g})$ as above, then the automorphism group of the graph with vertex labels is $\mathbb{Z}_2 \times \mathbb{Z}_2$ since we cannot exchange the vertices x_2 and x_3 anymore, they are now distinguishable (see also the right side of Figure 8.1).

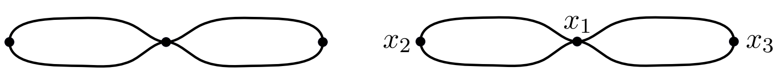


Figure 8.1: A non-labeled and a partially labeled graph.

Remark 8.1.6. If $k_i = 1$ for all i, then the condition for the valences implies that the genus at each vertex is 0 and the vertices are 3-valent. When forming the integral, the in the z_i constant coefficient is just 1, so we can neglect the z_i and obtain Feynman integrals without vertex contributions in this case.

By Remarks 7.1.2 and 8.1.6, the equality in Theorem 8.1.4 specializes to the well-known relation involving the generating series of Hurwitz numbers and Feynman graphs, see e.g. the mirror symmetry theorem for elliptic curves of Dijkgraaf [Dij95] and Theorem 2.6 of [BBBM17].

Using the Correspondence Theorem 7.1.19, we can formulate a version of the mirror symmetry relation in Theorem 8.1.4, where instead of the generating function of descendant Gromov-Witten invariants the generating function of tropical descendant Gromov-Witten invariants are used. It turns out however that a finer version of a mirror symmetry relation naturally holds in the tropical world, which uses labeled tropical covers, multidegrees and refined Feynman integrals:

Definition 8.1.7 (Labeled tropical cover)**.** Let π be a tropical cover that satisfies given psi-conditions with powers $k_1, \ldots, k_n$ and denote the genus of a vertex of the source curve which is adjacent to end i by g_i, where g_i is given by k_i via the psi-conditions (see Definition 7.1.10). We can shrink the ends of the source curve and label the vertex that used to be adjacent to end i with x_i. The cover π is called *labeled tropical cover* if there is an isomorphism of multigraphs sending a Feynman graph $(\Gamma, \underline{g}')$ with a genus function (see Definition 7.3.2) to the combinatorial type of the source curve of π, where the ends of the source curve are shrunk, such that $g'_i = g_i$ for all vertices. We say that π is of *type* Γ.

Shortly, a labeled tropical cover is a tropical cover for which we label the vertices and edges of the source (vertices of different genus are distinguishable) according to a Feynman graph.

Definition 8.1.8 (Multidegree and labeled descendant invariants)**.** Fix a point $p_0 \in E_{\mathbb{T}}$ on a tropical elliptic curve. For a labeled tropical cover of $E_{\mathbb{T}}$ of type Γ, we define its *multidegree* as the vector $\underline{a}$ in $\mathbb{N}^r$ with k-th entry

$$a_k = |\pi^{-1}(p_0) \cap q_k| \cdot \omega(q_k),$$

where $\omega(q_k)$ denotes the expansion factor of the edge q_k. We define a *labeled tropical descendant Gromov-Witten invariant*

$$\langle \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) \rangle_{\Gamma,n}^{E,\underline{a},\mathrm{trop}}$$

as a count of labeled tropical covers of type Γ and with multidegree $\underline{a}$ satisfying the prescribed point- and psi-conditions, again counted with multiplicity as in Equation (7.7). (Note that there are no nontrivial automorphism for a labeled tropical cover since all edges and vertices are distinguishable by their labeling.)

The following Theorem is a main result of this chapter. Using Correspondence Theorem 7.1.19, we show below that it indeed implies the classical mirror symmetry theorem 8.1.4.

Theorem 8.1.9 (Tropical mirror symmetry for $E_{\mathbb{T}}$)**.** *Fix $g \geq 2$, $n \geq 1$ and $k_1, \ldots, k_n \geq 1$ such that*

$$k_1 + \ldots + k_n = 2g - 2$$

holds. Let Γ be a Feynman graph (Definition 7.3.2) such that vertex x_i has valence $k_i + 2 - 2g_i$, and record the numbers g_i in a genus vector $\underline{g}$. Then generating series of labeled descendant

Gromov-Witten invariants of $E_{\mathbb{T}}$ of type Γ (Definition 8.1.8) can be expressed in terms of a Feynman integral with vertex contributions (Definition 8.1.3):

$$\sum_{\underline{a}\in\mathbb{N}^r} \langle \tau_{k_1}(pt)\dots\tau_{k_n}(pt)\rangle_{\Gamma,n}^{E,\underline{a},\mathrm{trop}} q_1^{a_1}\cdot\dots\cdot q_r^{a_r} = I_{\Gamma,\underline{g}}(q_1,\dots,q_r).$$

Theorem 8.1.9 is proved in Section 8.1.3 using Theorem 8.1.14, which establishes a bijection between labeled tropical covers contributing to a labeled tropical descendant Gromov-Witten invariant and monomials contributing to a term of the series used for the Feynman integrals.

Proof of Theorem 8.1.4 using Theorem 8.1.9. In order to deduce Theorem 8.1.4 from Theorem 8.1.9, we follow the "Proof of Theorem 2.14 using Theorem 2.20" in [BBBM17].

Fix a Feynman graph $(\Gamma,\underline{g})$ with a genus function $\underline{g}$. Let $\langle \tau_{k_1}(pt)\dots\tau_{k_n}(pt)\rangle_{\Gamma,n}^{E,d,\mathrm{trop}}$ be the number of (unlabeled) tropical covers of degree d (satisfying the conditions), where the combinatorial type of the source curve C is Γ and the genus function of C is given by $\underline{g}$. Each cover $\pi: C\to E_{\mathbb{T}}$ is counted with multiplicity $\frac{1}{|\operatorname{Aut}(\pi)|}\cdot\prod_{i=1}^n \mathrm{mult}_i(\pi)\cdot\prod_e \omega(e)$ (see (7.7)).

There is a forgetful map ft from the set of labeled tropical covers satisfying the conditions to the set of unlabeled covers by forgetting labels of edges and vertices. Let $\pi: C\to E_{\mathbb{T}}$ be an unlabeled cover as above. The automorphism group $\operatorname{Aut}(\mathrm{ft}(\Gamma))$ (by abuse of notations $\mathrm{ft}(\Gamma)$ is the forgetful map that forgets all labels of the Feynman graph Γ) acts transitively on $\mathrm{ft}^{-1}(\pi)$ by relabeling edges and vertices while respecting the genus function. Since the stabilizer of this action is $\operatorname{Aut}(\pi)$, the size of the orbit $\mathrm{ft}^{-1}(\pi)$ is

$$|\,\mathrm{ft}^{-1}(\pi)| = \frac{|\operatorname{Aut}(\mathrm{ft}(\Gamma))|}{|\operatorname{Aut}(\pi)|}.$$

Each labeled cover $\tilde{\pi}$ in $\mathrm{ft}^{-1}(\pi)$ is counted with multiplicity $\frac{1}{|\operatorname{Aut}(\tilde{\pi})|}\cdot\prod_i \mathrm{mult}_i(\pi)\cdot\prod_e\omega(e)$, where $|\operatorname{Aut}(\tilde{\pi})|=1$ since $\tilde{\pi}$ is labeled. Hence

$$\begin{aligned}\sum_{\substack{\underline{a}:\\ \sum a_i=d}} \langle \tau_{k_1}(pt)\dots\tau_{k_n}(pt)\rangle_{\Gamma,n}^{E,\underline{a},\mathrm{trop}} &= \sum_{\tilde{\pi}:C\to E_{\mathbb{T}}}\prod_e \omega(e)\prod_{i=1}^n \mathrm{mult}_i(\pi)\\ &= \sum_{\pi:C\to E_{\mathbb{T}}}\sum_{\substack{\tilde{\pi}:C\to E_{\mathbb{T}}:\\ \mathrm{ft}(\tilde{\pi})=\pi}}\prod_e \omega(e)\prod_{i=1}^n \mathrm{mult}_i(\pi)\\ &= \sum_{\pi:C\to E_{\mathbb{T}}}\frac{|\operatorname{Aut}(\mathrm{ft}(\Gamma))|}{|\operatorname{Aut}(\pi)|}\prod_e \omega(e)\prod_{i=1}^n \mathrm{mult}_i(\pi)\\ &= |\operatorname{Aut}(\mathrm{ft}(\Gamma))|\cdot\langle \tau_{k_1}(pt)\dots\tau_{k_n}(pt)\rangle_{g,n}^{E,d},\end{aligned}$$

where the second sum goes over all labeled covers with fixed multidegree $\underline{a}$, genus function $\underline{g}$ and combinatorial type Γ. The third sum goes over all unlabeled covers that satisfy the given conditions.

Summing over all degrees d and using Theorem 8.1.9 with $q_1=\dots=q_r=q$ gives us

$$\sum_{d\geq 1} |\operatorname{Aut}(\mathrm{ft}(\Gamma))|\cdot\langle \tau_{k_1}(pt)\dots\tau_{k_n}(pt)\rangle_{\Gamma,n}^{E,d,\mathrm{trop}}\cdot q^d = I_{\Gamma,\underline{g}}(q)$$

for a fixed Feynman graph Γ with genus function $\underline{g}$. If we sum over all Feynman graphs Γ with a genus function $\underline{g}$ and use Theorem 7.1.19, then Theorem 8.1.4 follows. □

Example 8.1.10. Fix $g = 2$. We want to use Theorem 8.1.9 to calculate contributions to $\langle\tau_2(pt)\tau_0(pt)\tau_0(pt)\rangle_{\Gamma,3}^{E,\underline{a},\mathrm{trop}}$ for two cases, where in the first case the covers contributing have a source curve with a nonzero genus function and in the second case the source curves have a loop.

First case: we choose $a = (0,0,3)$ and Γ as the left Feynman graph of Example 7.3.4. So Theorem 8.1.9 tells us that we need to calculate the $q_1^0q_2^0q_3^3$-coefficient of $I_{\Gamma,\underline{g}}(q_1,q_2,q_3)$ with $\underline{g} = (1,0,0)$. We fix an order Ω, namely the identity as we did in Example 7.1.15. That is, we require that end i is mapped to the point p_i on $E_{\mathbb{T}}$. Notice that the covers contributing to $\langle\tau_2(pt)\tau_0(pt)\tau_0(pt)\rangle_{\Gamma,3}^{E,(3,0,0),\mathrm{trop}}$ for Ω as above are the ones corresponding to the entries[1] $(2,2)$ and $(3,1)$ in the table given in Example 7.1.15. So we expect

$$\frac{1}{24} + \frac{17\cdot 27}{24} = \frac{115}{6} \tag{8.2}$$

as the contribution. We start by calculating terms of the propagators that contribute to the $q_1^0q_2^0q_3^3$-coefficient (we first let $w = 1$ for a_3) in the product of the propagators such that their product is constant in the x_i, i.e. $l_1 = l_2 = l_3 = 0$,

$$\begin{aligned}
\tilde{P}\Big(\frac{x_1}{x_3}, q_3\Big) &= \frac{4\sinh\left(\frac{z_1}{2}\right)\sinh\left(\frac{z_3}{2}\right)\left(\frac{x_1}{x_3}+\frac{x_3}{x_1}\right){q_3}^3}{z_1z_3} + \ldots,\\
\tilde{P}\Big(\frac{x_2}{x_3}, q_2\Big) &= \frac{4\sinh\left(\frac{z_2}{2}\right)\sinh\left(\frac{z_3}{2}\right)x_2}{z_2z_3x_3} + \ldots,\\
\tilde{P}\Big(\frac{x_1}{x_2}, q_1\Big) &= \frac{4\sinh\left(\frac{z_1}{2}\right)\sinh\left(\frac{z_2}{2}\right)x_1}{z_1z_2x_2} + \ldots.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{Coef}_{[q_1^0q_2^0q_3^3]}\,\frac{\tilde{P}(\frac{x_1}{x_3},q_3)\tilde{P}(\frac{x_2}{x_3},q_2)\tilde{P}(\frac{x_1}{x_2},q_1)}{\mathcal{S}(z_3)\mathcal{S}(z_2)\mathcal{S}(z_1)} &= \frac{8\sinh\left(\frac{z_1}{2}\right)\sinh\left(\frac{z_2}{2}\right)\sinh\left(\frac{z_3}{2}\right)}{z_1z_2z_3}\\
= \cdots + \frac{1}{1920}{z_1}^4 + \frac{1}{576}{z_1}^2{z_2}^2 + \frac{1}{576}{z_1}^2{z_3}^2 &+ \frac{1}{1920}{z_2}^4 + \frac{1}{576}{z_2}^2{z_3}^2\\
&+\frac{1}{1920}{z_3}^4 + \frac{1}{24}{z_1}^2 + \frac{1}{24}{z_2}^2 + \frac{1}{24}{z_3}^2 + 1
\end{aligned}$$

and, hence, the $z_1^2z_2^0z_3^0$-coefficient is $\frac{1}{24}$, which is precisely the first summand of (8.2). The second summand is obtained by letting $w = 3$ for a_3 such that

$$\begin{aligned}
\tilde{P}\Big(\frac{x_1}{x_3}, q_3\Big) &= \frac{4\sinh\left(\frac{3z_1}{2}\right)\sinh\left(\frac{3z_3}{2}\right)\left(\frac{x_1^3}{x_3^3}+\frac{x_3^3}{x_1^3}\right){q_3}^3}{3z_1z_3} + \ldots,\\
\tilde{P}\Big(\frac{x_2}{x_3}, q_2\Big) &= \frac{4\sinh\left(\frac{3z_2}{2}\right)\sinh\left(\frac{3z_3}{2}\right)x_2^3}{3z_2z_3x_3^3} + \ldots,\\
\tilde{P}\Big(\frac{x_1}{x_2}, q_1\Big) &= \frac{4\sinh\left(\frac{3z_1}{2}\right)\sinh\left(\frac{3z_2}{2}\right)x_1^3}{3z_1z_2x_2^3} + \ldots
\end{aligned}$$

and therefore

$$\begin{aligned}
\mathrm{Coef}_{[q_1^0q_2^0q_3^3]}\,\frac{\tilde{P}(\frac{x_1}{x_3},q_3)\tilde{P}(\frac{x_2}{x_3},q_2)\tilde{P}(\frac{x_1}{x_2},q_1)}{\mathcal{S}(z_3)\mathcal{S}(z_2)\mathcal{S}(z_1)} &= \frac{8\left(\sinh\left(\frac{3z_1}{2}\right)\right)^2\left(\sinh\left(\frac{3z_2}{2}\right)\right)^2\left(\sinh\left(\frac{3z_3}{2}\right)\right)^2}{27\sinh\left(\frac{z_1}{2}\right)\sinh\left(\frac{z_2}{2}\right)\sinh\left(\frac{z_3}{2}\right)z_1z_2z_3}\\
= \ldots \frac{3369}{640}{z_1}^4 + \frac{867}{64}{z_1}^2{z_2}^2 &+ \frac{867}{64}{z_1}^2{z_3}^2 + \frac{3369}{640}{z_2}^4 + \frac{867}{64}{z_2}^2{z_3}^2\\
&+\frac{3369}{640}{z_3}^4 + \frac{153}{8}{z_1}^2 + \frac{153}{8}{z_2}^2 + \frac{153}{8}{z_3}^2 + 27,
\end{aligned}$$

[1]Given a table, matrix notation is used to refer to its entries.

where the $z_1^2 z_2^0 z_3^0$-coefficient is $\frac{153}{8}$ which equals the second summand of (8.2).

Second case: we choose $a = (2,0,0,1)$ and Γ as the right Feynman graph of Example 7.3.4. By Theorem 8.1.9, we need to calculate the $q_1^2 q_2^0 q_3^0 q_4^1$-coefficient of $I_{\Gamma,\underline{g}}(q_1,q_2,q_3,q_4)$ with $\underline{g} = 0$. Again, we pick Ω as the order given by the identity. As before, we calculate the terms of the propagators that contribute to the $q_1^2 q_2^0 q_3^0 q_4^1$-coefficient in the product of the propagators such that their product is constant in the x_i, i.e. $l_1 = l_2 = l_3 = l_4 = 0$, and let $w = 2$ for a_1, then

$$
\begin{aligned}
\tilde{P}^{\text{loop}}(q_1) &= \frac{2\left(\sinh\left(z_1\right)\right)^2 {q_1}^2}{{z_1}^2},\\
\tilde{P}\Big(\frac{x_1}{x_2}, q_2\Big) &= \frac{4\sinh\left(\frac{z_1}{2}\right)\sinh\left(\frac{z_2}{2}\right)x_1}{z_1 z_2 x_2} + \dots,\\
\tilde{P}\Big(\frac{x_2}{x_3}, q_3\Big) &= \frac{4\sinh\left(\frac{z_2}{2}\right)\sinh\left(\frac{z_3}{2}\right)x_2}{z_2 z_3 x_3} + \dots,\\
\tilde{P}\Big(\frac{x_1}{x_3}, q_4\Big) &= \frac{4\sinh\left(\frac{z_1}{2}\right)\sinh\left(\frac{z_3}{2}\right)\left(\frac{x_1}{x_3}+\frac{x_3}{x_1}\right)q_4}{z_1 z_3} + \dots
\end{aligned}
$$

and

$$
\begin{aligned}
\operatorname{Coef}_{[q_1^2 q_2^0 q_3^0 q_4^1]} & \frac{\tilde{P}^{\text{loop}}(q_1)\tilde{P}(\frac{x_1}{x_3}, q_4)\tilde{P}(\frac{x_2}{x_3}, q_3)\tilde{P}(\frac{x_1}{x_2}, q_2)}{\mathcal{S}(z_3)\mathcal{S}(z_2)\mathcal{S}(z_1)}\\
&= \frac{16\left(\sinh\left(z_1\right)\right)^2\sinh\left(\frac{z_1}{2}\right)\sinh\left(\frac{z_2}{2}\right)\sinh\left(\frac{z_3}{2}\right)}{{z_1}^3 z_2 z_3}\\
&= 2 + \frac{3}{4}{z_1}^2 + \frac{1}{12}{z_2}^2 + \frac{1}{12}{z_3}^2 + \frac{113}{960}{z_1}^4 + \frac{1}{32}{z_1}^2{z_2}^2\\
&\quad + \frac{1}{32}{z_3}^2{z_1}^2 + \frac{1}{960}{z_2}^4 + \frac{1}{288}{z_3}^2{z_2}^2 + \frac{1}{960}{z_3}^4 + \dots,
\end{aligned}
$$

where the constant coefficient in the z_i is 2. If we let $w = 1$ for a_1, we get 1. This makes 3 in total, which is the number we expect when using the table from Example 7.1.15 again (entries $(1,1)$ and $(2,1)$).

8.1.3 The bijection

In this section, the tropical mirror symmetry theorem 8.1.9 is proved. The main ingredient is a bijection of graph covers and monomials that contribute to a Feynman integral. It generalizes the bijective approach used in [BBBM17] to graph covers with loop-edges. For an overview of the bijective method, see also Figure 8.3.

Definition 8.1.11 (Graph covers of fixed order)**.** Let Γ be a Feynman graph (see Definition 7.3.2). Fix a multidegree $\underline{a} \in \mathbb{N}^r$ and an order Ω. The order Ω can be viewed as an element in the symmetric group on n elements, associating to i the place $\Omega(i)$ that the vertex x_i of Γ takes in the order Ω. Moreover, fix an orientation of $E_{\mathbb{T}}$ and points $p_0, p_1, \dots, p_n$ ordered in this way when going around $E_{\mathbb{T}}$ in the fixed orientation starting at p_0.

A *graph cover* of type Γ, order Ω and multidegree $\underline{a}$ is a (possibly leaky w.r.t. $(l_1,\dots,l_n)$, see Remark 7.1.22) tropical cover $\pi : \Gamma' \to E_{\mathbb{T}}$, where Γ' is a metrization of Γ, such that the multidegree of π at p_0 is $\underline{a}$ and such that $\pi^{-1}(p_{\Omega(i)})$ contains x_i. Notice that since there are n point conditions and n vertices, it follows from Lemma 7.1.11 that there is precisely one vertex of Γ in each preimage $\pi^{-1}(p_j)$.

Define $N_{\Gamma,\underline{a},\Omega}^{l_1,\dots,l_n}$ to be the weighted count of $(l_1,\dots,l_n)$-leaky graph covers of type Γ, order Ω and multidegree $\underline{a}$, where each such $(l_1,\dots,l_n)$-leaky graph cover is counted with the product of the expansion factors of its edges. If there is no mentioning of $l_1, \ldots, l_n$, then we refer to the case of no leaking as usual.

Lemma 8.1.12 (Graph covers and labeled tropical covers)**.** *Let $g \geq 2$, $n \geq 1$ and $k_1, \ldots, k_n \geq 1$ be natural numbers such that*

$$k_1 + \ldots + k_n = 2g - 2$$

is satisfied. Let Γ be a Feynman graph such that for each vertex x_i of Γ, the number $k_i + \mathrm{val}(x_i)$ is even. Fix a multidegree $\underline{a}$ and an order Ω.

Then there is a bijection between graph covers of type Γ, order Ω and multidegree $\underline{a}$ (Definition 8.1.11) and labeled tropical covers $\pi : \Gamma' \to E_{\mathbb{T}}$ (definitions 8.1.7, 8.1.8) that contribute (possibly with weight zero) to

$$\langle \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) \rangle_{\Gamma,n}^{E,\underline{a},\mathrm{trop}}$$

and satisfy $\pi(i) = p_{\Omega(i)}$.

Proof. Let $\pi : \Gamma' \to E_{\mathbb{T}}$ be such a labeled tropical cover. We can describe the bijection as the map sending π to a graph cover $\tilde{\pi}$ by shrinking marked ends of Γ', labeling the vertex that used to be adjacent to end i by x_i, and forgetting the genus at vertices. By definition of $\langle \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) \rangle_{\Gamma,n}^{E,\underline{a},\mathrm{trop}}$, the graph cover is of type Γ. The multidegree is the same for the tropical cover and the graph cover. The set $\pi^{-1}(p_{\Omega(i)})$ contains x_i, since the marked end i is mapped to $p_{\Omega(i)}$ by π. The inverse map associates the genus $\frac{k_i+2-\mathrm{val}(x_i)}{2}$ to the vertex x_i (which is an integer by our assumption), and attaches the end marked with i. Then the valence is $k_i + 3 - 2g_i$ and the psi-condition is satisfied according to Definition 7.1.10. □

Remark 8.1.13. Let $\langle \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) \rangle_{\Gamma,n,\Omega}^{E,\underline{a},\mathrm{trop}}$ denote the weighted count of tropical covers that contribute to $\langle \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) \rangle_{\Gamma,n}^{E,\underline{a},\mathrm{trop}}$ such that $\pi(i) = p_{\Omega(i)}$ for a fixed order Ω. Then Lemma 8.1.12 establishes a one-to-one correspondence between contributions (possibly with weight zero) to $\langle \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) \rangle_{\Gamma,n,\Omega}^{E,\underline{a},\mathrm{trop}}$ and graph covers that possibly contribute to $N_{\Gamma,\underline{a},\Omega}$ from Definition 8.1.11. Notice that this does not imply that $\langle \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) \rangle_{\Gamma,n,\Omega}^{E,\underline{a},\mathrm{trop}}$ and $N_{\Gamma,\underline{a},\Omega}$ are the same number since the weight of an element contributing to $N_{\Gamma,\underline{a},\Omega}$ is a product of expansion factors, whereas the weight of an element contributing to $\langle \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) \rangle_{\Gamma,n,\Omega}^{E,\underline{a},\mathrm{trop}}$ is given by (7.7).

Theorem 8.1.14 (Bijection of graph covers and tuples in Feynman integrals)**.** *Notations 7.3.3, 8.1.1 are used. Let Γ be a Feynman graph as in Definition 7.3.2. Fix a multidegree $\underline{a} \in \mathbb{N}^r$ with $a_k > 0$ for all $k \leq s$, an order Ω, and integers $l_1, \ldots, l_n$.*

There is a bijection between $(l_1, \ldots, l_n)$-leaky graph covers of type Γ, order Ω and multidegree $\underline{a}$ (Definition 8.1.11), and tuples

$$\left(\Big(w_k\Big)_{k=1,\ldots,s}, \Big(\big(a_k, w_k \cdot (\frac{x_{k^i}}{x_{k^j}})^{w_k}\big) \Big)_{k=s+1,\ldots,r} \right), \tag{8.3}$$

where $i = 1$ and $j = 2$ if $a_k = 0$, and $\{i, j\} = \{1, 2\}$ otherwise, where w_k divides a_k if $a_k \neq 0$, and where the product of the fractions has exponent l_i in x_i.

Moreover, the weighted count of graph covers equals the $q_1^{a_1} \ldots q_n^{a_n}$-coefficient of the refined Feynman integral (Definition 7.3.5):

$$N_{\Gamma,\underline{a},\Omega}^{l_1,\ldots,l_n} = \mathrm{Coef}_{[q_1^{a_1} \ldots q_r^{a_r}]} \, I_{\Gamma,\Omega}^{l_1,\ldots,l_n}(q_1, \ldots, q_r). \tag{8.4}$$

Remark 8.1.15 (Tuples and Feynman integrals)**.** Note that the products of second entries for $k > s$ of a tuple as in (8.3) are precisely the contributions showing up in the series

$$\prod_{k=s+1}^{r} P\Big(\frac{x_{k^1}}{x_{k^2}}, q_k\Big) = \prod_{k=s+1}^{r}\left(\sum_{w=1}^{\infty} w\cdot\Big(\frac{x_{k^1}}{x_{k^2}}\Big)^{w} + \sum_{a=1}^{\infty}\left(\sum_{w|a} w\left(\Big(\frac{x_{k^1}}{x_{k^2}}\Big)^{w} + \Big(\frac{x_{k^1}}{x_{k^2}}\Big)^{-w}\right)\right) q_k^a\right)$$

with the exponents of the x_i given by the l_i, and the exponents of the q_i given by the a_i. By definition of the refined Feynman integral (see Definition 8.1.2), adding a choice of summand w_k for each loop-edge q_k, $k \leq s$, each tuple contributes exactly $w_1 \cdot \ldots \cdot w_r$ to the $q_1^{a_1}\ldots q_n^{a_n}$-coefficient of the refined Feynman integral $I_{\Gamma,\Omega}^{l_1,\ldots,l_n}(q_1,\ldots,q_r)$.

In particular, if $a_k = 0$ for some $k \leq s$, the $q_1^{a_1}\ldots q_n^{a_n}$-coefficient of the refined Feynman integral $I_{\Gamma,\Omega}^{l_1,\ldots,l_n}(q_1,\ldots,q_r)$ is zero, and there are no tuples.

Proof of Theorem 8.1.14. Given a tuple as in (8.3), we construct a graph cover as follows. We keep track of the cover by drawing the vertices and edges projecting onto their images. To ease the drawing, we think of $E_{\mathbb{T}}$ as being cut off at p_0 (see Example 7.1.18).

We start by drawing vertices x_i above the points $p_{\Omega(i)}$ in $E_{\mathbb{T}}$.

For $k > s$ and for an entry $w_k\cdot\big(\frac{x_{k^i}}{x_{k^j}}\big)^{w_k}$, we draw an edge with expansion factor w_k connecting vertex x_{k^i} to vertex x_{k^j}. If $a_k = 0$, we draw this edge in our cut picture direct, without passing over p_0. If $a_k > 0$, we "curl it", passing over p_0 exactly $\frac{a_k}{w_k}$ times before it reaches its end vertex. We assume in our tuple that $i = 1$ and $j = 2$ if $a_k = 0$. By Notation 7.3.3, $x_{k^1} < x_{k^2}$ in Ω, which implies that in our picture, the vertex x_{k^1} is drawn before x_{k^2} (in the orientation of $E_{\mathbb{T}}$), which makes it possible to draw the edge q_k directly without passing p_0. Since w_k divides a_k, it is possible to "curl" the edges q_k with $a_k > 0$ as required.

For $k \leq s$ and an entry w_k, we draw a loop-edge of weight w_k connecting the vertex of q_k to itself, "curled" over p_0 exactly $\frac{a_k}{w_k}$ times.

In the drawing we created for the tuple (8.3), we have obviously drawn a graph cover with source curve of type Γ, since we connected the vertices x_{k^1} and x_{k^2} with the edge q_k. Furthermore, the multidegree is $\underline{a}$ because of our curling requirement. The order Ω is respected by the way we have drawn the vertices. It remains to show that the graph cover is $(l_1,\ldots,l_n)$-leaky. To see this, notice that the edges adjacent to vertex x_i correspond to tuples whose fraction contains a power of x_i, and that the exponent equals $\pm$ the expansion factor of the edge, where the sign is positive if the edge leaves x_i and negative if it enters x_i (w.r.t. the orientation of $E_{\mathbb{T}}$). Since we require the total power in x_i to be l_i, the cover leaks l_i at vertex x_i.

Clearly, the process can be reversed associating a tuple to a graph cover, and using the same arguments as before, the tuple satisfies the requirements from above. In particular, the entry a_k of the multidegree of a cover with a loop-edge q_k is nonzero. Thus, we have a bijection between graph covers and tuples.

Equality 8.4 follows from Remark 8.1.15, taking into account that a graph cover is counted with multiplicity the product of its expansion factors (which are the w_i) in $N_{\Gamma,\underline{a},\Omega}^{l_1,\ldots,l_n}$. □

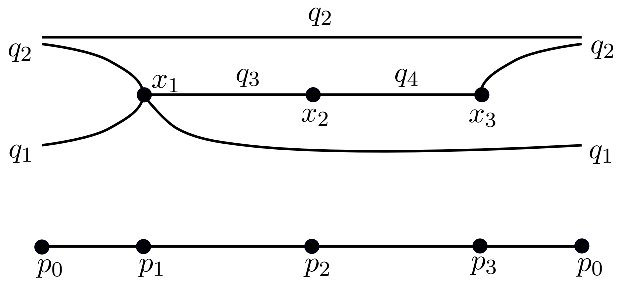


Figure 8.2: The graph cover constructed from (8.5). Notice that this graph cover arises from cutting (and labeling) the upper right source curve in Figure 7.1 along the preimages of a point p_0 (see also Example 7.1.18).

Example 8.1.16. We illustrate the proof of Theorem 8.1.14 by constructing a graph cover from a given tuple as in (8.3). We let Γ be the right graph of Example 7.3.4, Ω the identity, and $l_i = 0$ for all i. Consider the tuple

$$\left(1, \left(2, 1 \cdot \left(\frac{x_1}{x_3}\right)^{-1}\right), \left(0, 1 \cdot \left(\frac{x_1}{x_2}\right)^{1}\right), \left(0, 1 \cdot \left(\frac{x_2}{x_3}\right)^{1}\right)\right). \tag{8.5}$$

Notice that it is not leaky. See Figure 8.2 for the following: We start by drawing the vertices x_1, x_2, x_3 above p_1, p_2, p_3. Then we add the non-curled edges q_3, q_4 which are given by the third and fourth entry of our tuple above. The edge q_2 is obtained by starting at x_1 and going left (we have a negative exponent in the second entry of our tuple), curling once (we want to pass p_0 twice with q_2) and ending at x_3. There is also one loop edge (the first entry of the tuple) adjacent to x_1 which does not curl. Since all weights of edges are 1, we can also see from the graph that it is not leaky as we expected. The upper graph in Figure 8.2 inherits a metrization from downstairs. Thus a graph cover is obtained.

Figure 8.3 illustrates the structure of the proof of Theorem 8.1.9. In particular, it sums up the idea behind the bijective method, namely to establish bijections (the double sided arrows in Figure 8.3) and then, in a second step, to compare the weights of the elements that are identified under the concatenation of the bijections. Lemma 8.1.12 states that graph covers are closely related to tropical covers that show up in a tropical descendant Gromov-Witten invariant. However, the multiplicity of a tropical cover contains, besides the expansion factors for edges which already appear in the bijection in Theorem 8.1.14, also factors for each vertex which can be viewed as local descendant Gromov-Witten invariants (see Equation (7.6)). By Theorem 7.1.6, such contributions of local descendant Gromov-Witten invariants can be computed using the $\mathcal{S}$-function. Thus we need to consider Feynman integrals with vertex contributions (see Definition 8.1.3) instead of refined Feynman integrals (Definition 7.3.5) as we did in Theorem 8.1.14

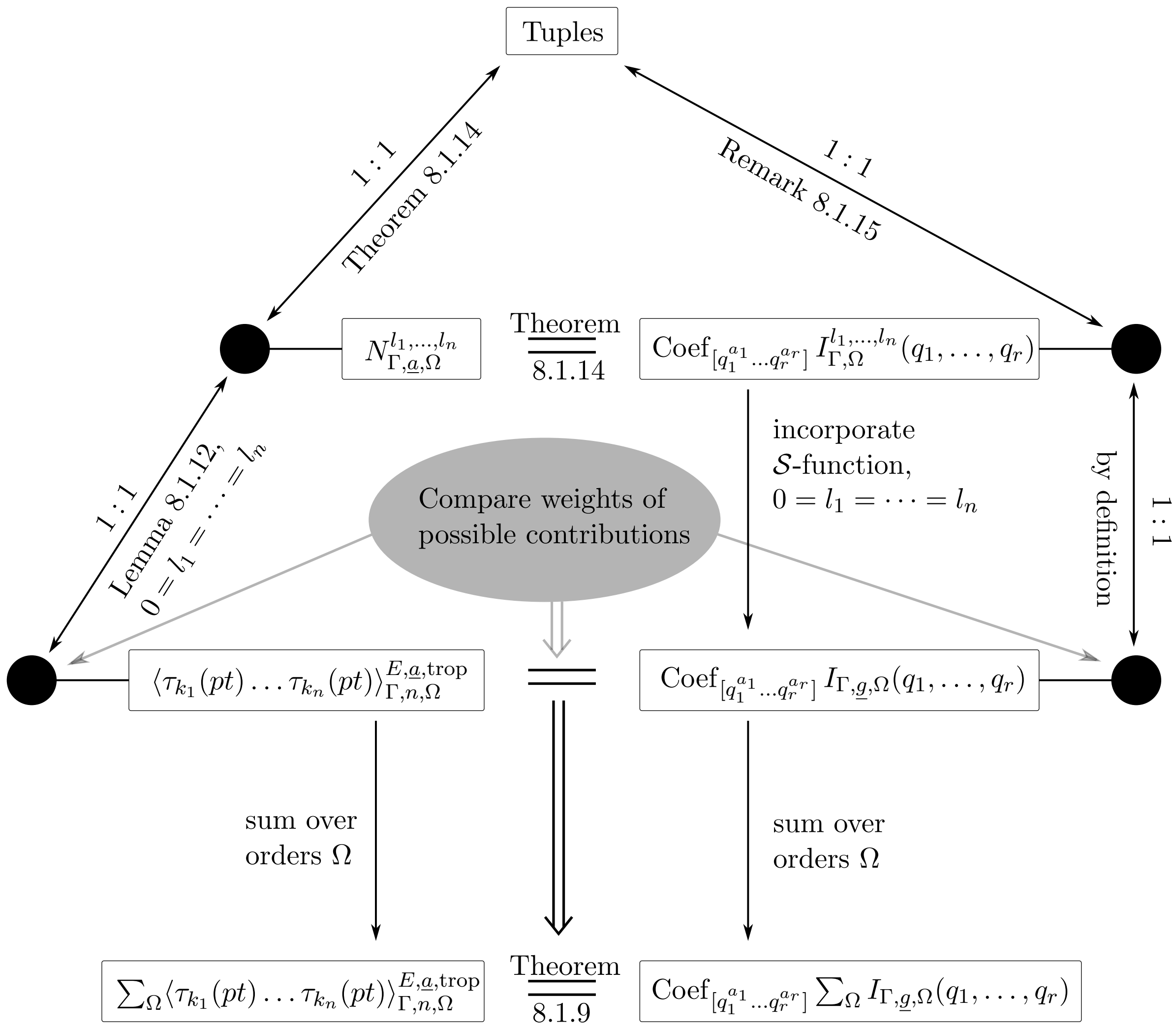


Figure 8.3: Overview of the structure of the proof of Theorem 8.1.9. The black dots indicate contributions (possibly with weight zero) to the box they are connected to. For example, the arrow in the upper right indicates that Remark 8.1.15 establishes a one-to-one correspondence between certain tuples and elements that contribute to $\mathrm{Coef}_{[q_1^{a_1}\dots q_r^{a_r}]}\, I^{l_1,\dots,l_n}_{\Gamma,\Omega}(q_1,\dots,q_r)$.

Proof of Theorem 8.1.9. We prove the equality by restricting to the $q_1^{a_1}\dots q_r^{a_r}$-Coefficient on each side. It follows from Lemma 8.1.12 that we can expand the left side as a sum over orders Ω, which we can do by definition of Feynman integral also on the right. We thus have to show that the weighted count of labeled tropical covers contributing to $\langle \tau_{k_1}(pt)\dots\tau_{k_n}(pt)\rangle^{E,\underline{a},\mathrm{trop}}_{\Gamma,n}$ and satisfying $\pi(i) = p_{\Omega(i)}$ equals $\mathrm{Coef}_{[q_1^{a_1}\dots q_r^{a_r}]}\, I_{\Gamma,\underline{g},\Omega}(q_1,\dots,q_r)$.

To see this, note that by Remark 8.1.15 we deal with tuples as in Theorem 8.1.14 when computing $\mathrm{Coef}_{[q_1^{a_1}\dots q_r^{a_r}]}\, I_{\Gamma,\Omega}(q_1,\dots,q_r)$, however since we compute a Feynman integral with vertex contributions (see Definition 8.1.3) now each second entry $w_k\cdot\left(\frac{x_{k_i}}{x_{k_j}}\right)^{w_k}$ showing up in a tuple meets $\mathcal{S}(w_k z_{k^1})\mathcal{S}(w_k z_{k^2})$ first. By Theorem 8.1.14, the tuples are in bijection with graph covers. For a fixed graph cover corresponding to a fixed tuple, the vertex contributions in the Feynman integral thus produce factors of $\mathcal{S}(w_k z_{k^1})\mathcal{S}(w_k z_{k^2})$ for an edge of expansion factor w_k connecting the vertices x_{k^1} and x_{k^2}. Collecting those factors, sorting by z_i, and adding in the factor $\frac{1}{\mathcal{S}(z_i)}$ we have in the definition of Feynman integral with vertex contributions (see

Definition 8.1.3), we obtain for each vertex x_i a contribution of

$$\frac{\prod \mathcal{S}(\mu_j z_i) \cdot \prod \mathcal{S}(\nu_j z_i)}{\mathcal{S}(z_i)}.$$

Here, the notation is set up as follows: we collect the expansion factors of all incoming edges adjacent to x_i in the partition μ and those of all outgoing edges in the partition ν. Taking the $z_i^{2g_i}$-coefficient, we obtain a local vertex contribution of

$$\langle \mu | \tau_{k_i}(pt) | \nu \rangle_{g_i,1}^{\mathbb{P}^1,|\mu|}$$

by the one-point series from Equation (7.2). By Equation (7.6), this is exactly the local vertex multiplicity we need to take into account for the labeled tropical cover. □

Remark 8.1.17. Let us compare the Tropical mirror symmetry Theorem 8.1.9 for tropical descendant invariants with the version for Hurwitz numbers (Theorem 2.20 [BBBM17]). As we saw in remarks 7.1.2 and 8.1.6, in the version for Hurwitz numbers, we only have to take 3-valent graphs into account such that all vertices have genus zero. Adding in descendants requires us to generalize in two ways: we need to include graphs whose vertices have other valencies, and whose vertices have genus. The main ingredient in our proof of tropical mirror symmetry is the bijection between graph covers and monomials contributing to a Feynman integral, see Theorem 8.1.14 or the upper triangle in Figure 8.3. Graphs with vertices of valence bigger 3 fit into this context. The genus at vertices requires us to use local vertex multiplicities for the tropical covers, which are hard to translate to the Feynman integral world. The fact that the one-point series (7.2) can be expressed in a way separating contributions for the edges adjacent to a vertex makes it possible to incorporate these multiplicities in a Feynman integral with vertex contributions as in Definition 8.1.3.

8.1.4 Quasimodularity

Quasimodularity of a generating function is desirable because it controls its asymptotic. A series in q is *quasimodular* if and only if it is in the polynomial ring $\mathbb{C}[E_2, E_4, E_6]$ generated by the three Eisenstein series E_2, E_4 and E_6 [KZ95]. The *weight* of a quasimodular form refers to its degree when viewed as a polynomial in the Eisenstein series. A series is called a quasimodular form of weight w if it is a homogeneous polynomial of degree w in the Eisenstein series, and it is called a quasimodular form of mixed weight if it is a non-homogeneous polynomial in the Eisenstein series.

In case that all $k_i = 1$, the mirror symmetry theorem 8.1.4 specializes to the well-known relation involving the generating series of Hurwitz numbers and Feynman integrals for 3-valent graphs without vertex contributions (see Remark 8.1.6). This special case of the mirror symmetry relation was used in [Dij95, KZ95] to prove that the generating function of Hurwitz numbers for $g \geq 2$ is a quasimodular form of weight $6g - 6$. Note that quasimodularity of generating functions of covers is a phenomenon that was studied beyond the case considered here, see e.g. [EO06, EOP08].

From the tropical mirror symmetry theorem 8.1.9, the generating function of descendant Gromov-Witten invariants of an elliptic curve obtains a natural stratification as sum over Feynman graphs, and, even finer, as sum over orders Ω for each Feynman graph (see Corollary 8.1.18). If we fix a Feynman graph Γ and a suitable genus function $\underline{g}$ (if $k_i = 1$ for all i, this means that we fix a 3-valent graph with genus 0 at each vertex), then we can study quasimodularity of individual summands. We can consider summands $I_{\Gamma,\underline{g}}(q)$, or we can even break the sum into finer contributions by considering $I_{\Gamma,\underline{g},\Omega}(q)$ for a fixed order Ω.

For the case that $k_i = 1$ for all i, this study was initiated in [BBBM17], where it is conjectured that $I_{\Gamma,\underline{g}}(q)$ is quasimodular of weight $6g-6$. In [GM20], Goujard and Möller provide tools to study quasimodularity of generating series depending on Feynman graphs, and they prove that if $k_i = 1$ for all i, each summand $I_{\Gamma,\underline{g},\Omega}(q)$ is a quasimodular form of mixed weight, where the highest appearing weight is $6g-6$. They also compute examples where lower weights appear. Since the whole sum (over all graphs, and over all orders) is quasimodular of weight $6g-6$, the contributions of lower weights must cancel in the sum. It remains an open question whether they already cancel in a summand $I_{\Gamma,\underline{g}}(q)$, i.e. when we sum over all orders Ω, but for a fixed graph Γ.

In case of arbitrary k_i, quasimodularity (of mixed weight) of the whole generating series (the sum over all graphs, and all orders) was studied before [Li16].

We now deduce from [GM20] that $I_{\Gamma,\underline{g},\Omega}(q)$ is a quasimodular form of mixed weight even in case of arbitrary k_i. For that, we start with interpreting $I_{\Gamma,\underline{g},\Omega}(q)$ as a generating function of tropical covers:

Corollary 8.1.18. *Let $g \geq 2$, $n \geq 1$ and $k_1, \ldots, k_n \geq 1$ be natural numbers such that*

$$k_1 + \ldots + k_n = 2g - 2$$

is satisfied. Fix a Feynman graph Γ such that the vertex x_i has valence $k_i + 2 - 2g_i$ with $g_i \in \mathbb{N}$. Record the numbers g_i in the genus function $\underline{g}$. Fix an order Ω. For $d \in \mathbb{N}$, let

$$\langle \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) \rangle_{\Gamma,n,\Omega}^{E,d,\mathrm{trop}}$$

denote the number of (unlabeled) tropical covers (counted with multiplicity) that contribute to $\langle \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) \rangle_{g,n}^{E,d,\mathrm{trop}}$, for which the source curve has combinatorial type Γ after shrinking the ends and satisfying $\pi(i) = p_{\Omega(i)}$.

Then we can express the generating function of these invariants in terms of the Feynman integral

$$\sum_{d \in \mathbb{N}} \langle \tau_{k_1}(pt) \ldots \tau_{k_n}(pt) \rangle_{\Gamma,n,\Omega}^{E,d,\mathrm{trop}} q^d = \frac{1}{|\operatorname{Aut}(\mathrm{ft}_{\mathrm{edge}}(\Gamma), \underline{g})|} I_{\Gamma,\underline{g},\Omega}(q),$$

where $\mathrm{ft}_{\mathrm{edge}}$ *is the map that forgets the edge labels of a Feynman graph, and automorphisms respect the remaining vertex labels and the genus function (see Example 8.1.5).*

Proof. Consider Theorem 8.1.9 and let $q_1 = \cdots = q_r = q$. With a similar argument as we use to deduce Theorem 8.1.4 from Theorem 8.1.9, we also obtain an automorphism factor here. Fixing the order leads to labels on the vertices of the source curves, i.e. we need to consider automorphisms which respect partially labeled graphs as in Example 8.1.5. □

Example 8.1.19. We want to express $I_{\Gamma,\underline{g},\Omega}(q)$ as polynomial in the Eisenstein series, where Ω is the identity, $\underline{g} = (0,0,0)$ or $\underline{g} = (1,0,0)$ and Γ is any Feynman graph as shown in Example 7.3.4. So this example is a continuation of Examples 7.1.15 and 8.1.10.

First, let Γ_1 be the left Feynman graph of Example 7.3.4 and let $\underline{g}_1 = (1,0,0)$. We calculate that

$$\begin{aligned} I_{\Gamma_1,\underline{g}_1,\Omega}(q) &= \frac{1}{20736} E_6(q) - \frac{1}{13824} E_2(q) E_4(q) + \frac{1}{41472} E_2^3(q) + \frac{1}{20736} E_4^2(q) \\ &\quad - \frac{1}{10368} E_2(q) E_6(q) + \frac{1}{20736} E_2^2(q) E_4(q) \\ &= \frac{1}{24} q + \frac{5}{2} q^2 + \frac{39}{2} q^3 + \frac{278}{3} q^4 + \frac{1025}{4} q^5 + 738 q^6 + \frac{4165}{3} q^7 + 3080 q^8 + \ldots \end{aligned}$$

Notice that $I_{\Gamma_1,\underline{g}_1,\Omega}$ is of of mixed weight since E_6 and E_2E_6 are of different weights. Recall that we calculated $\frac{115}{6}$ as contribution to the q^3-coefficient in Example 8.1.10. The other covers contributing are shown in Figure 7.1 of Example 7.1.15 and are the ones corresponding to the following entries of the table of Example 7.1.15 (read as a matrix): $(2,3)$, $(2,4)$, $(2,5)$, $(3,2)$, $(3,3)$, $(4,1)$, $(4,2)$, $(4,3)$. Each of these covers contributes with $\frac{1}{24}$ such that in total we expect (see Corollary 8.1.18)

$$\mathrm{Coef}_{[q^3]}\, I_{\Gamma_1,\underline{g}_1,\Omega}(q) = \frac{115}{6} + \frac{8}{24} = \frac{39}{2},$$

which matches our calculation.

Second, we choose Γ_2 to be the right Feynman graph of Example 7.3.4 and let $\underline{g}_2 = 0$. We calculate that

$$\begin{aligned} I_{\Gamma_2,\underline{g}_2,\Omega}(q) = &-\frac{1}{20736}E_6(q) + \frac{1}{13824}E_2(q)E_4(q) - \frac{1}{41472}E_2^3(q) + \frac{1}{20736}E_2(q)E_6(q) \\ &- \frac{1}{13824}E_2^2(q)E_4(q) + \frac{1}{41472}E_2^4(q) \\ = &\; q^2 + 15q^3 + 76q^4 + 275q^5 + 720q^6 + 1666q^7 + 3440q^8 + 6129q^9 + \ldots \end{aligned}$$

Notice that, again, $I_{\Gamma_2,\underline{g}_2,\Omega}$ is of mixed weight, but $I_{\Gamma_1,\underline{g}_1,\Omega} + I_{\Gamma_2,\underline{g}_2,\Omega}$ is homogeneous. As above, we can verify the q^3-coefficient using Example 7.1.15.

Third, we choose Γ_3 to be the middle Feynman graph of Example 7.3.4 and let $\underline{g}_3 = 0$. In this case, we obtain the homogeneous expression

$$\begin{aligned} I_{\Gamma_3,\underline{g}_3,\Omega}(q) &= \frac{1}{20736}E_4^2(q) - \frac{1}{10368}E_2^2(q)E_4(q) + \frac{1}{20736}E_2^4(q) \\ &= 4q^2 + 48q^3 + 240q^4 + 800q^5 + 2160q^6 + 4704q^7 + 9920q^8 + 17280q^9 + \ldots \\ &= 4\cdot(q^2 + 12q^3 + 60q^4 + 200q^5 + 540q^6 + 1176q^7 + 2480q^8 + 4320q^9 + \ldots), \end{aligned}$$

where the factor 4 in the last row is due to the automorphisms of the underlying Feynman graph, see Corollary 8.1.18. Again, we can verify the q^3-coefficient using Example 7.1.15.

Corollary 8.1.20. *Let $g \geq 2$, $n \geq 1$ and $k_1, \ldots, k_n \geq 1$ be natural numbers such that*

$$k_1 + \ldots + k_n = 2g - 2$$

is satisfied. Let Γ be a Feynman graph with r edges (see Definition 7.3.2) and let $\underline{g}$ be a genus function that satisfies

$$b_1(\Gamma) + \sum_{i=1}^{n} g_i = g,$$

where $b_1(\Gamma)$ denotes the first Betti number of Γ. Fix an order Ω.

Then the Feynman integral $I_{\Gamma,\underline{g},\Omega}(q)$ — i.e. the generating function counting tropical covers for the tropical descendant Gromov-Witten invariant $\langle \tau_{k_1}(pt)\ldots\tau_{k_n}(pt)\rangle_{\Gamma,n}^{E,d,\mathrm{trop}}$ of type Γ and order Ω, see Corollary 8.1.18 — is a quasimodular form of mixed weight, with highest occurring weight $2\cdot(r + \sum_{i=1}^{n} g_i)$.

Proof. This follows from Theorem 6.1 of [GM20], since the local vertex contributions we have to take into account for a vertex x_i are polynomial of even degree $2g_i$ in the expansion factors of the adjacent edges by Theorem 4.1 in [GM20] (see [OP06], [SSZ12]). $\square$

This statement is essentially a byproduct of Corollary 6.2 of [GM20] which states that the generating series of tropical covers with fixed ramification profiles (see [CJMR21], Definition 2.1.3) and with fixed underlying graph Γ and order Ω is a quasimodular form of mixed weight. The proof in [GM20] detours by deducing the quasimodularity of the function above from the quasimodularity of our $I_{\Gamma,\underline{g},\Omega}(q)$ (without explicitely stating this). The descendant Gromov-Witten invariants we focus on here are called Hurwitz numbers with completed cycles in [GM20], which is explained by the Okounkov-Pandharipande GW/H correspondence in [OP06], see also [SSZ12].

8.2 Tropical mirror symmetry and Boson-Fermion correspondence

The purpose of this section is to reveal the close relation between the proof of Theorem 8.1.4 in mathematical physics, using Fock spaces, and our tropical approach. Since the tropical setting requires a labeling of the underlying Feynman graphs and the use of the variables $q_1, \ldots, q_r$ to distinguish degree contributions from the different edges, we enrich the Fock space approach by incorporating adequate labelings. This enlarges the set of operators, but makes it easier to distinguish contributions for a fixed Feynman graph to a matrix element. In this way, we extend the Fock space approach so that it gives an alternative proof of the tropical mirror symmetry Theorem 8.1.9, which holds on a finer level. Our main ingredient is Theorem 8.2.10, proving the equality of the number of labeled tropical covers with fixed underlying source graph, fixed multidegree and order and a sum of matrix elements in a bosonic Fock space.

For the sake of explicitness, we limit our considerations to the case of Hurwitz numbers, i.e. $k_i = 1$ for all i, and we do not have vertex contributions for Feynman integrals (see remarks 7.1.2 and 8.1.6). In particular, all our graphs are 3-valent, have no loops and genus 0 at vertices. Higher descendants resp. vertex contributions can be incorporated into our discussion also, but would increase the amount of notation largely — we would have to consider more summands for a bosonic vertex operator, and the tropical local vertex multiplicities would have to show up as coefficients of the bosonic vertex operator [CJMR18].

As shown in Figure 1.3, tropical geometry hands us a short-cut in the Fock space setting: we can relate the generating series of Hurwitz numbers directly to operators on the bosonic Fock space and do not need to evoke the fermionic Fock space and the Boson-Fermion Correspondence, which is often viewed as the essence of mirror symmetry for elliptic curves.

8.2.1 Hurwitz numbers as matrix elements

We shortly review the bosonic Fock space approach for generating series of Hurwitz numbers: The bosonic Heisenberg algebra $\mathcal{H}$ is the Lie algebra with basis α_n for $n \in \mathbb{Z}$ such that for $n \neq 0$ the following commutator relations are satisfied:

$$[\alpha_n, \alpha_m] = (|n| \cdot \delta_{n,-m})\, \alpha_0,$$

where $\delta_{n,-m}$ is the Kronecker symbol and

$$[\alpha_n, \alpha_m] := \alpha_n \alpha_m - \alpha_m \alpha_n$$

The bosonic Fock space F is a representation of $\mathcal{H}$. It is generated by a single "vacuum vector" $v_\emptyset$. The positive generators annihilate $v_\emptyset$: $\alpha_n \cdot v_\emptyset = 0$ for $n > 0$, α_0 acts as the identity and the negative operators act freely. That is, F has a basis b_μ indexed by partitions, where

$$b_\mu = \alpha_{-\mu_1} \dots \alpha_{-\mu_m} \cdot v_\emptyset.$$

We define an inner product on F by declaring $\langle v_\emptyset | v_\emptyset \rangle = 1$ and α_n to be the adjoint of α_{-n}.

We write $\langle v|A|w\rangle$ for $\langle v|Aw\rangle$, where $v, w \in F$ and the operator A is a product of elements in $\mathcal{H}$, and we write $\langle A \rangle$ for $\langle v_\emptyset|A|v_\emptyset\rangle$. The first is called a *matrix element*, the second a *vacuum expectation*.

Example 8.2.1. To get used to the inner product on F, two easy examples are calculated. First,

$$\langle \alpha_{-3} \cdot v_\emptyset | \alpha_{-5} \cdot v_\emptyset \rangle = \langle v_\emptyset | \alpha_3 \alpha_{-5} \cdot v_\emptyset \rangle = 0$$

since α_3 and α_{-5} commute such that α_3 meets $v_\emptyset$ and therefore vanishes. We observe from this example that if we want to obtain not zero from our inner product, we need to have for each operator on the left of the inner product one on the right with the same index. For example:

$$\begin{aligned} \langle \alpha_{-3}\alpha_{-5} \cdot v_\emptyset | \alpha_{-5}\alpha_{-3} \cdot v_\emptyset \rangle &= \langle v_\emptyset | \alpha_5 \alpha_3 \alpha_{-5} \alpha_{-3} \cdot v_\emptyset \rangle \\ &= \langle v_\emptyset | \alpha_5 \alpha_{-5} \alpha_3 \alpha_{-3} \cdot v_\emptyset \rangle \\ &= \langle v_\emptyset | 5 \cdot 3 \cdot v_\emptyset \rangle \\ &= 15, \end{aligned}$$

where the commutator relation was used once for the second equation and it was used twice for the third equation.

Definition 8.2.2. The *cut-join* operator is defined by:

$$M := \frac{1}{2} \sum_{k>0} \sum_{\substack{0<i,j \\ i+j=k}} \alpha_{-j}\alpha_{-i}\alpha_k + \alpha_{-k}\alpha_i\alpha_j. \tag{8.6}$$

The relative invariants of $\mathbb{P}^1$ can be interpreted as a matrix element involving M (notice that the invariants in questions are equal to double Hurwitz numbers by Okounkov-Pandharipande's GW/H correspondence [OP06]):

Proposition 8.2.3. *Notation 7.1.4 is used. A relative Gromov-Witten invariant of $\mathbb{P}^1$, resp. a double Hurwitz number, equals a matrix element on the bosonic Fock space:*

$$\langle \mu | \tau_1(pt)^n | \nu \rangle_{g,n}^{\mathbb{P}^1,d,\bullet} = \frac{n!}{\prod_i \mu_i \cdot \prod_j \nu_j} \langle b_\mu | M^n | b_\nu \rangle.$$

This statement follows by combining Wick's Theorem with the Correspondence Theorem 7.1.20: Wick's Theorem ([Wic50], Proposition 5.2 [BG16], Theorem 5.4.3 [CJMR18]) expresses a matrix element as a weighted count of graphs that are obtained by completing local pictures. It turns out that the graphs in question are exactly the tropical covers we enumerate to obtain $\langle \mu | \tau_1(pt)^n | \nu \rangle_{g,n}^{\mathbb{P}^1,d,\mathrm{trop}}$, where $n!$ arises from fixing a set of points to which labeled ends are mapped to (rather than prescribing a point a labeled end should map to, see Definition 7.1.10).

Notice that we have to use the disconnected theory here ($\bullet$), since the matrix element encodes *all* graphs completing the local pictures and cannot distinguish connected and disconnected graphs.

The local pictures are built as follows: we draw one vertex for each cut-join operator. For an α_n with $n > 0$, we draw an edge germ of weight n pointing to the right. If $n < 0$, we draw an edge germ of weight n pointing to the left. For the two Fock space elements b_μ and b_ν, we draw germs of ends: of weights μ_i on the left pointing to the right, of weights ν_i on the right pointing to the left. Wick's Theorem states that the matrix element $\langle b_\mu | M^n | b_\nu \rangle$ equals a sum of graphs completing all possible local pictures, where each graph contributes the product of the weights of all its edges (including the ends). A completion of the local pictures can be interpreted as a tropical cover of $\mathbb{P}^1_{\mathbb{T}}$ (with suitable metrization).

The cut-join operator sums over all the possibilities of the local pictures for the graphs, i.e. it sums over all possibilities how a vertex of a tropical cover can look like (see Figure 8.4).

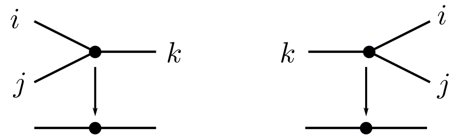


Figure 8.4: Local pictures of graphs with weights on the edges.

Example 8.2.4. Consider the local pieces shown below. There are three ways of completing them to a graph with local pictures like in Figure 8.4.

The completed graphs are shown in Figure 8.5. The product of the upper graph's edge weights (including the ends) is 12, 4 for the middle graph and 4 for the lower graph. Hence Wick's Theorem and Proposition 8.2.3 yield

$$\langle (2,1) | \tau_1(pt)^2 | (2,1) \rangle_{2,2}^{\mathbb{P}^1,3,\bullet} = 2! \cdot \left(3 + 1 + \frac{1}{2} \right) = 9,$$

where we have to divide the last summand by two because there is an automorphism exchanging the two edges that connect the vertices in the lower graph of Figure 8.5.

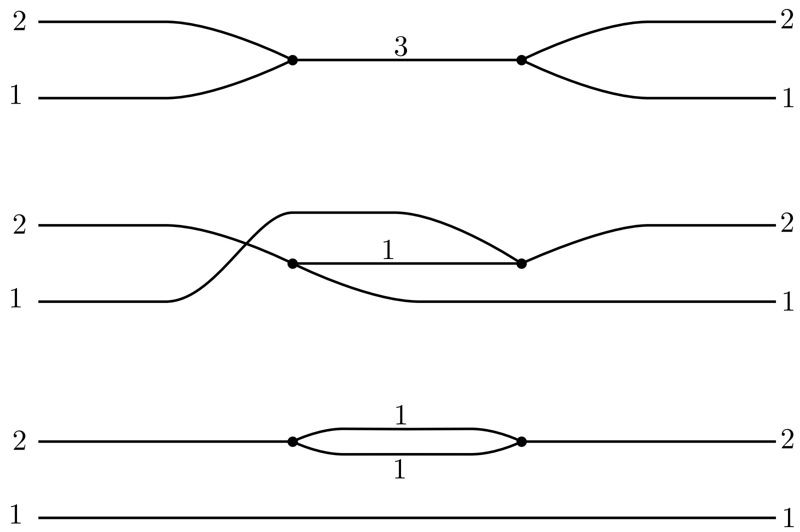


Figure 8.5: Completions of the local pieces above. Note that there is an automorphism exchanging the two bounded edges in the lower graph.

Combining Proposition 8.2.3 with a degeneration argument, we can express Gromov-Witten invariants, resp. Hurwitz numbers of the elliptic curve in terms of matrix elements:

Proposition 8.2.5. *A Hurwitz number of the elliptic curve equals a weighted sum of double Hurwitz numbers of* $\mathbb{P}^1$*:*

$$\langle \tau_1(pt)^n \rangle_{g,n}^{E,d,\bullet} = \sum_{\mu \vdash d} \frac{\prod_i \mu_i}{|\operatorname{Aut}(\mu)|} \langle \mu | \tau_1(pt)^n | \mu \rangle_{g-\ell(\mu),n}^{\mathbb{P}^1,d,\bullet}.$$

Here, the sum goes over all partitions μ *of* d*,* μ_i *denotes their entries, and* $\ell(\mu)$ *the length.*

Proposition 8.2.3 which relates relative Gromov-Witten invariants to matrix elements yields the following corollary.

Corollary 8.2.6. *A Hurwitz number of the elliptic curve* E *equals a sum of matrix elements on the bosonic Fock space:*

$$\langle \tau_1(pt)^n \rangle_{g,n}^{E,d,\bullet} = \sum_{\mu \vdash d} \frac{n!}{|\operatorname{Aut}(\mu)| \prod_i \mu_i} \langle b_\mu | M^n | b_\mu \rangle.$$

Proposition 8.2.5 is a corollary from the two Correspondence Theorems 7.1.19 and 7.1.20: given a tropical cover of $E_{\mathbb{T}}$, let μ be the partition encoding the weights of the edges mapping to the base point p_0. We mark the preimages of p_0, for which we have $|\operatorname{Aut}(\mu)|$ choices. For each choice, we cut off $E_{\mathbb{T}}$ at p_0 and the covering curve at the preimages of p_0, obtaining a cover of $\mathbb{P}^1_{\mathbb{T}}$ with ramification profiles μ and μ above $\pm\infty$. The cut off tropical cover contributes to $\langle \mu | \tau_1(pt)^n | \mu \rangle_{g,n}^{\mathbb{P}^1,d,\bullet,\mathrm{trop}}$, but its multiplicity differs from the multiplicity of the cover of $E_{\mathbb{T}}$ by a factor of $\prod \mu_i$, since the edges we cut off are no longer bounded.

Example 8.2.7. We want to calculate $\langle \tau_1(pt)^2 \rangle_{2,2}^{E,3,\bullet}$ using Corollary 8.2.6 and Wick's Theorem. The partitions of 3 are $(1,1,1)$, $(2,1)$ and (3). The summand of $(2,1)$ follows from Example 8.2.4, namely $9 \cdot 2 = 18$. The figure below shows how to complete the local pieces given by the partitions $(1,1,1)$ and (3).

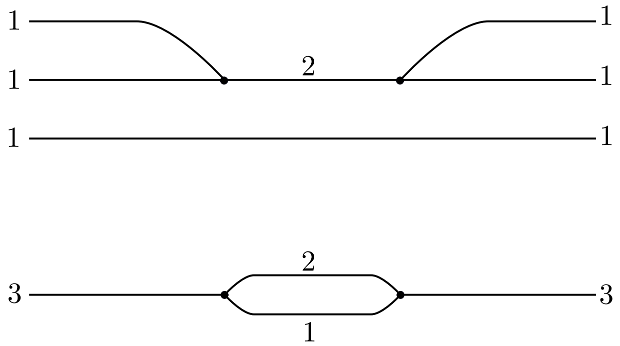


Figure 8.6: More completions of local pieces. Note that there are automorphisms of the upper graph that exchange the edges of weight one adjacent to a 3-valent vertex.

Note that there are in fact 9 choices of how to complete the local pieces of $(1,1,1)$ since we can choose which ends (in the upper graph) the straight line should connect. Thus the upper graphs contribute (9 of them) $2! \cdot \frac{9 \cdot 2}{4} \cdot \frac{1}{3!} = \frac{3}{2}$ and the lower graph contributes $2! \cdot 2 \cdot 3 = 12$. Therefore $\langle \tau_1(pt)^2 \rangle_{2,2}^{E,3,\bullet} = \frac{63}{2}$.

8.2.2 Labeled matrix elements for labeled tropical covers

Now we would like to link this Fock space language for Gromov-Witten invariants resp. Hurwitz numbers with tropical mirror symmetry. Recall that tropical mirror symmetry holds naturally on a fine level, giving an equality of the $q_1^{a_1} \dots q_r^{a_r}$-coefficient of a Feynman integral $I_{\Gamma,\Omega}(q_1, \dots, q_r)$ and the number $N_{\Gamma,\underline{a},\Omega}$, which counts labeled tropical covers of type Γ, with multidegree $\underline{a}$ and such that the order Ω is satisfied, i.e. (in case of Hurwitz numbers) the contribution to $\langle \tau_1(pt)^n \rangle_{\Gamma,n}^{E,\underline{a},\text{trop}}$ of covers π satisfying $\pi(i) = p_{\Omega(i)}$ (see Lemma 8.1.12, Theorem 8.1.14 and Theorem 8.1.9).

Fix a Feynman graph Γ, a multidegree $\underline{a}$ and an order Ω. Remember that Γ is a 3-valent connected graph with first Betti number g, because of our restriction that $k_i = 1$ for all i. In particular, Γ has no loops.

Notation 8.2.8. Our expression for $N_{\Gamma,\underline{a},\Omega}$ in terms of matrix elements (see Theorem 8.2.10 below) involves a sum over all tuples $(w_k)_{k:a_k>0}$ with $w_k | a_k$ for all k with $a_k > 0$, since we incorporate the degeneration idea from Proposition 8.2.5.

For a fixed choice of $(w_k)_k$, let Γ' be the graph obtained from Γ by cutting the edge q_k exactly $\frac{a_k}{w_k}$ times. We introduce the following labels for the (cut) edges of Γ': we denote the pieces by $q_{k,1}, \dots, q_{k,\frac{a_k}{w_k}+1}$. There are at most $a_k + 1$ pieces, depending on w_k. For an edge which is not cut, i.e. $a_k = 0$, we call it $q_{k,1}$ to consistently have two indices for the edge labels in Γ'.

We enlarge our set of operators in a way that allows to distinguish the edges of the cut graph Γ': Let the $\alpha_n^{k,j}$, for each $k = 1, \dots, r$, $j = 1, \dots, a_k + 1$, and $n \in \mathbb{Z} \setminus \{0\}$, satisfy the commutator relations

$$[\alpha_n^{k,j}, \alpha_m^{l,i}] := (|n| \cdot \delta_{k,l} \cdot \delta_{j,i} \cdot \delta_{n,-m})\, \alpha_0.$$

As before, we let the bosonic Fock space F be generated by $v_\emptyset$, following the rules from before: $\alpha_n^{k,j} \cdot v_\emptyset = 0$ for $n > 0$, α_0 acts as identity, and the operators with negative subscript act freely. We let $\langle v_\emptyset | v_\emptyset \rangle = 1$ and let $\alpha_n^{k,j}$ be the adjoint of $\alpha_{-n}^{k,j}$.

Definition 8.2.9. Let Γ, $\underline{a}$ and $(w_k)_k$ be as in Notation 8.2.8. Let x_i be a vertex of Γ. We denote its three adjacent edges by q_{i_1}, q_{i_2} and q_{i_3}. For $l = 1, 2, 3$ set $c_l = \frac{a_{i_l}}{w_{i_l}} + 1$ if $a_{i_l} > 0$ and $c_l = 1$ else. We also set $d_{m_l} = c_l$ if $m_l > 0$ and $d_{m_l} = 1$ otherwise. Define the *labeled cut-join operator* for the vertex x_i as

$$M_i := \sum_{\substack{m_1, m_2, m_3 \in \mathbb{Z} \setminus \{0\} \\ m_1 + m_2 + m_3 = 0}} \alpha_{m_1}^{i_1, d_{m_1}} \alpha_{m_2}^{i_2, d_{m_2}} \alpha_{m_3}^{i_3, d_{m_3}}.$$

Since the first superscript differs for the α-operators in a summand, the commutator relations imply that these factors can be permuted within a summand without changing the cut-join operator.

This operator sums over all possibilities of how, locally, a vertex with its adjacent edge germs can be arranged, as shown in Figure 8.7.

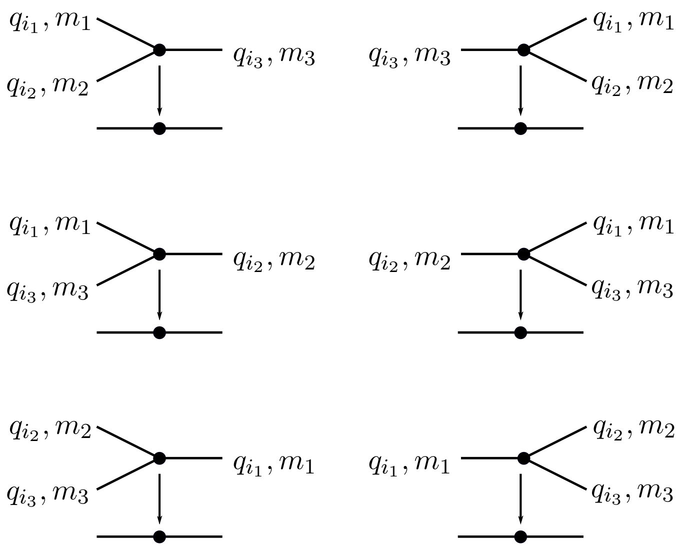


Figure 8.7: Local pictures of graphs with weights m_1, m_2, m_3 on the labeled edges $q_{i_1}, q_{i_2}, q_{i_3}$.

Notice that, compared to the (unlabeled) cut-join operator, we do not need a factor of $\frac{1}{2}$ which had to be there to take automorphisms into account resp. to undo overcounting by distinguishing edges which are not distinguishable. Here, all edges are labeled and thus distinguishable.

Theorem 8.2.10. *For a fixed* 3*-valent Feynman graph* Γ *without loops, multidegree* $\underline{a}$ *and an order* Ω*, the count of labeled tropical covers of* $E_{\mathbb{T}}$ *of type* Γ *and of the right multidegree and order (see Lemma 8.1.12, Theorem 8.1.14 and Theorem 8.1.9) equals a sum of matrix elements:*

$$N_{\Gamma,\underline{a},\Omega} = \sum_{\substack{(w_k)_k \\ w_k | a_k}} \prod_{k=1}^{r} \left(\frac{1}{w_k}\right)^{\frac{a_k}{w_k}} \cdot \left\langle \prod_{k=1}^{r} \prod_{l=1}^{\frac{a_k}{w_k}} \alpha^{k,l}_{-w_k} v_\emptyset \middle| \prod_{i=1}^{n} M_{\Omega^{-1}(i)} \middle| \prod_{k=1}^{r} \prod_{l=2}^{\frac{a_k}{w_k}+1} \alpha^{k,l}_{-w_k} v_\emptyset \right\rangle. \tag{8.7}$$

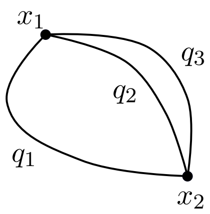
Proof. We use Wick's Theorem: the right hand side is a sum over all possible ways to combine the local pictures given by the cut-join operators to a graph Γ' that covers $\mathbb{P}^1_{\mathbb{T}}$. Our local pictures are now vertices with labels $x_{\Omega^{-1}(i)}$. For each $\alpha^{k,j}_n$, we have an adjacent edge germ with label $q_{k,j}$ of weight $|n|$, pointing to the right if n is positive and to the left otherwise. Fix a graph Γ' which is a completion of such local pictures. The preimages of $-\infty$ are leaf vertices of Γ' whose adjacent edges are labeled $q_{k,1}, \ldots, q_{k,\frac{a_k}{w_k}}$ and are of weight w_k (for all k). The preimages of ∞ are leaf vertices whose adjacent edges have labels $q_{k,2}, \ldots, q_{k,\frac{a_k}{w_k}+1}$, also of weight w_k. Since the α-operators in the cut-join operator only have the values 1 or $\frac{a_k}{w_k}+1$ as their second superscript, the commutator relation guarantees that the leaves of $q_{k,2} \ldots, q_{k,\frac{a_k}{w_k}}$ over $-\infty$ have to be connected to the leaves with the corresponding label over ∞. The leaf adjacent to $q_{k,1}$ over $-\infty$ is merged with an interior vertex adjacent to q_k, by definition of the labeled cut-join operator which depends on Γ. The same holds for the leaf adjacent to $q_{k,\frac{a_k}{w_k}+1}$.

To produce a tropical cover of $E_{\mathbb{T}}$, we glue Γ' as follows: for all k and for $i = 1, \ldots, \frac{a_k}{w_k}$, the leaf of $q_{k,i}$ over $-\infty$ is attached to the leaf of $q_{k,i+1}$ over ∞. Identifying the edges $q_{k_1}, \ldots, q_{k,\frac{a_k}{w_k}+1}$ (which are subdivided by 2-valent vertices obtained from gluing) to one edge q_k, we obtain a graph cover of $E_{\mathbb{T}}$ of type Γ which is of the right order and multidegree: the order is imposed by the order in which we multiply the cut-join operators, the multidegree is given by the "curls" of the edge q_k, which has weight w_k and which is curled $\frac{a_k}{w_k}$ times by our way of gluing.

Obviously, each tropical cover of type Γ and multidegree $\underline{a}$ with order Ω can be obtained by gluing a graph Γ' that arises with Wick's Theorem from the right hand side.

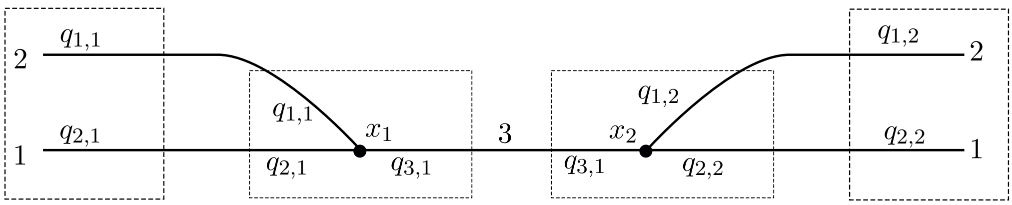
On the right hand side, a graph Γ' that we produce with Wick's Theorem contributes with the product of the weights of all edges which are connected, including the ends. For an edge q_k (which is cut into $w_k^{\frac{a_k}{w_k}+1}$ pieces in Γ') with $a_k > 0$, we thus obtain a factor of $w_k^{\frac{a_k}{w_k}+1}$, where we actually only want w_k for the tropical multiplicity. This is taken care of by the pre-factor before the summands on the right. $\square$

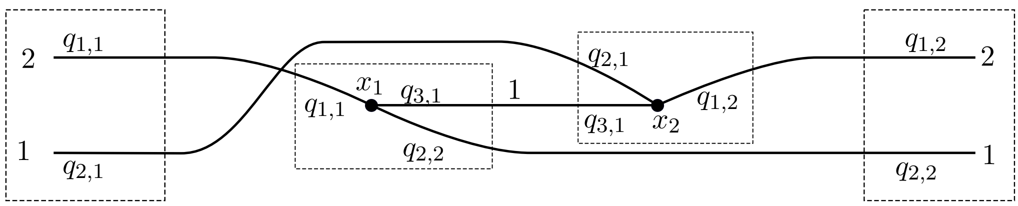
Example 8.2.11. Fix the multidegree $\underline{a} = (2, 1, 0)$, an order Ω on the vertices x_1, x_2 such that $x_1 < x_2$ and the following Feynman graph:



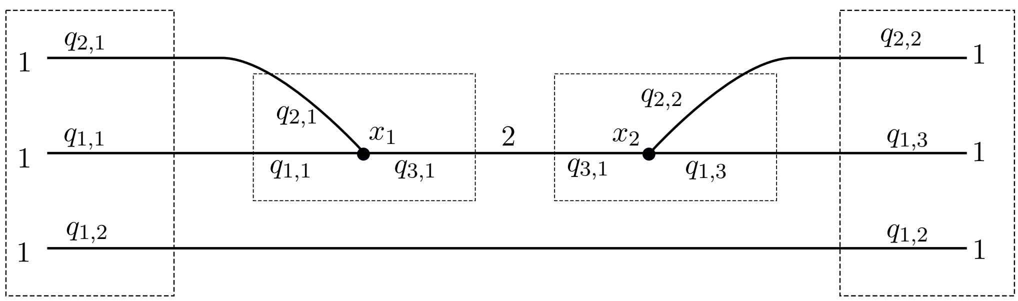
Fix points p_1, p_2 on $E_{\mathbb{T}}$. We obtain a labeled tropical cover of $\mathbb{P}^1_{\mathbb{T}}$ that can be glued to a cover of $E_{\mathbb{T}}$ of type Γ by choosing local pieces (see Figure 8.7).

Notice that there are two choices of the expansion factor w_1, namely $w_1 = 1$ or $w_1 = 2$. We start with $w_1 = 2$ and obtain the following source curve of a tropical cover to $\mathbb{P}^1_{\mathbb{T}}$, where the local pieces are indicated by boxes.





In case of $w_1 = 2$, there are no other choices of local pieces that fit Γ than the ones shown above. If we choose $w_1 = 1$, then another valid choice of local pieces is shown below.



Note that these two graphs are labeled version of the middle and upper graph of Figure 8.5 and the upper graph of Figure 8.6. However, we do not get all graphs of examples 8.2.4 and 8.2.7 since we fixed Γ and $\underline{a}$.

8.2.3 From matrix elements to Feynman integrals

Finally, we link the matrix elements on the right of (8.7) in Theorem 8.2.10 with Feynman integrals. To do so, we introduce formal variables for vertices in the labeled cut-join operators:

Definition 8.2.12. Let Γ, $\underline{a}$ and $(w_k)_k$ be as in Notation 8.2.8. For $l = 1, 2, 3$ set $c_l = \frac{a_{i_l}}{w_{i_l}} + 1$ if $a_{i_l} > 0$ and $c_l = 1$ otherwise. We also set $d_{m_l} = c_l$ if $m_l > 0$ and $d_{m_l} = 1$ otherwise. Define the labeled cut-join operator for the vertex x_i as

$$M(x_i) := \sum_{m_1,m_2,m_3 \in \mathbb{Z}\setminus\{0\}} \alpha_{m_1}^{i_1,d_{m_1}} x_i^{m_1} \alpha_{m_2}^{i_2,d_{m_2}} x_i^{m_2} \alpha_{m_3}^{i_3,d_{m_3}} x_i^{m_3}.$$

Here, the labeled cut-join operators are treated as formal series in $x_1, \ldots, x_n$. With this, we can rewrite equation (8.7) of Theorem 8.2.10, namely

$$N_{\Gamma,\underline{a},\Omega} = \mathrm{Coef}_{[x_1^0 \ldots x_n^0]} \sum_{\substack{(w_k)_k \\ w_k | a_k}} \prod_{k=1}^{r} \left(\frac{1}{w_k}\right)^{\frac{a_k}{w_k}} \cdot \left\langle \prod_{k=1}^{r} \prod_{l=1}^{\frac{a_k}{w_k}} \alpha_{-w_k}^{k,l} v_\emptyset \middle| \prod_{i=1}^{n} M(x_{\Omega^{-1}(i)}) \middle| \prod_{k=1}^{r} \prod_{l=2}^{\frac{a_k}{w_k}+1} \alpha_{-w_k}^{k,l} v_\emptyset \right\rangle. \tag{8.8}$$

Each matrix element on the right-hand side of (8.8) is now a series in $x_1, \ldots, x_n$ when evaluated.

Lemma 8.2.13. *Fix Γ, $\underline{a}$ and Ω as in Notation 8.2.8. The right-hand side of Equation* (8.8), *satisfies the following:*

$$\begin{aligned} &\mathrm{Coef}_{[x_1^0 \ldots x_n^0]} \sum_{\substack{(w_k)_k \\ w_k | a_k}} \prod_{k=1}^{r} \left(\frac{1}{w_k}\right)^{\frac{a_k}{w_k}} \cdot \left\langle \prod_{k=1}^{r} \prod_{l=1}^{\frac{a_k}{w_k}} \alpha_{-w_k}^{k,l} v_\emptyset \middle| \prod_{i=1}^{n} M(x_{\Omega^{-1}(i)}) \middle| \prod_{k=1}^{r} \prod_{l=2}^{\frac{a_k}{w_k}+1} \alpha_{-w_k}^{k,l} v_\emptyset \right\rangle \\ &= \mathrm{Coef}_{[x_1^0 \ldots x_n^0]} \prod_{k: a_k > 0} w_k \cdot \left(\left(\frac{x_{k^1}}{x_{k^2}}\right)^{w_k} + \left(\frac{x_{k^2}}{x_{k^1}}\right)^{w_k} \right) \cdot \prod_{k: a_k = 0} \left(\sum_{w_k > 0} w_k \cdot \left(\frac{x_{k^1}}{x_{k^2}}\right)^{w_k} \right). \end{aligned}$$

Here, x_{k^1} and x_{k^2} denote the vertices adjacent to the edge q_k, where in the order Ω we have $x_{k^1} < x_{k^2}$ as in Notation 7.3.3.

Proof. Let q_k be an edge with $a_k = 0$. Since $a_k = 0$, an α with first superscript k does not show up in the vectors of the matrix element, only in the labeled cut-join operators. Also, the second superscript must be 1, and it appears for exactly two cut-join operators, namely the one for x_{k^1} and the one for x_{k^2}. Thus, we draw an edge germ labeled $q_{k,1}$ at x_{k^1} and an edge germ labeled $q_{k,1}$ at x_{k^2}. These are the only edge germs with this label. To obtain a nonzero contribution to the matrix element, the edge germ at x_{k^1} must point to the right and the one at x_{k^2} must point to the left. Furthermore, they must have the same weight w_k. There is no restriction on the weight w_k. (The balancing condition is imposed only after we take the $x_1^0 \ldots x_n^0$-coefficient.) So, for any $w_k > 0$, we have nonzero contributions to the matrix elements above with an $\alpha_{w_k}^{k,1}$ in the cut-join operator $M(x_{k^1})$ and an $\alpha_{-w_k}^{k,1}$ in the cut-join operator $M(x_{k^2})$. Combining those α-operators with the respective power of the variable, we obtain $\alpha_{w_k}^{k,1} \cdot x_{k^1}^{w_k} \cdot \alpha_{-w_k}^{k,1} \cdot x_{k^2}^{-w_k}$, which, after applying the commutator relation and simplifying becomes $w_k \cdot \left(\frac{x_{k^1}}{x_{k^2}}\right)^{w_k}$.

We have treated the sum of matrix elements as a weighted sum of graphs. Any nonzero summand must have an edge connecting the edge germs above, and it can be of any weight. More precisely, if we have a graph with such an edge of a certain weight, we also have all summands that correspond to the same graph, but with the weight of the edge varying. Thus, we can pull out a factor

$$\sum_{w_k>0} w_k \cdot \left(\frac{x_{k^1}}{x_{k^2}}\right)^{w_k}$$

for the edge q_k.

Let us now consider an edge q_k with $a_k > 0$. The matrix elements on the left are summed over all $w_k|a_k$. For the local pictures, we draw end germs of weight w_k on the left pointing to the right, with labels $q_{k,1}, \ldots, q_{k,\frac{a_k}{w_k}}$, and on the right, pointing to the left, with labels $q_{k,2}, \ldots, q_{k,\frac{a_k}{w_k}+1}$. We use the commutator relations for the α in charge of connecting the "curls" $q_{k,2}, \ldots, q_{k,\frac{a_k}{w_k}}$, they produce a factor of w_k which is cancelled by the pre-factor. The end germ $q_{k,1}$ must be connected to an edge germ appearing in a cut-join operator, that can be either $M(x_{k^1})$ or $M(x_{k^2})$. The end germ $q_{k,\frac{a_k}{w_k}+1}$ must also be connected to an edge germ of a cut-join operator, necessarily the other on in the choice of $M(x_{k^1})$ or $M(x_{k^2})$. Thus, we either have

$$\alpha^{k,1}_{w_k} \cdot \alpha^{k,1}_{-w_k} \cdot x_{k^1}^{-w_k} \cdot \alpha^{k,\frac{a_k}{w_k}+1}_{w_k} \cdot x_{k^2}^{w_k} \cdot \alpha^{k,\frac{a_k}{w_k}+1}_{-w_k} \quad \text{or}$$
$$\alpha^{k,1}_{w_k} \cdot \alpha^{k,1}_{-w_k} \cdot x_{k^2}^{-w_k} \cdot \alpha^{k,\frac{a_k}{w_k}+1}_{w_k} \cdot x_{k^1}^{w_k} \cdot \alpha^{k,\frac{a_k}{w_k}+1}_{w_k}$$

(notice the subscript changes sign when we let factors jump in the scalar product, by convention of the adjoints). Taking the commutator relations into account, and realizing that one factor of w_k is again cancelled by the pre-factor, we obtain either $w_k \cdot (\frac{x_{k^1}}{x_{k^2}})^{w_k}$ or $w_k \cdot (\frac{x_{k^2}}{x_{k^1}})^{w_k}$. Also, w_k was imposed by the summand we picked on the left hand side. But for a given graph with an edge of weight w_k, we also have the analogous graph (with fewer or more curls) where the edge has another weight which divides a_k. Thus, for the edge q_k we obtain a total factor of

$$\sum_{w_k|a_k} w_k \cdot \left(\left(\frac{x_{k^1}}{x_{k^2}}\right)^{w_k} + \left(\frac{x_{k^2}}{x_{k^1}}\right)^{w_k}\right).$$

Taking the $x_1^0 \ldots x_n^0$-coefficient on the right-hand side of our desired equation ensures that there are no vertices (in the graph associated to a factor of the right-hand side) whose three adjacent edge germs are pointing in the same direction. Thus Lemma 8.2.13 follows. □

Having this, we can give an alternative proof of Theorem 8.1.9 (in the case $k_i = 1$ for all i), which follows the more traditional Fock space approach, now with a larger set of operators in charge of the labels. Thus using tropical geometry, we can take a shortcut that avoids the fermionic Fock space and relies on Wick's Theorem instead, see Figure 1.3.

Proof of Theorem 8.1.9 in the case $k_i = 1$ for all i. We prove the equality by restricting to the $q_1^{a_1} \ldots q_r^{a_r}$-Coefficient on each side. It follows from Lemma 8.1.12 that we can expand the left side as a sum over orders Ω, which we can do by definition of Feynman integral also on the right. We thus have to show that the weighted count $N_{\Gamma,\underline{a},\Omega}$ of labeled tropical covers contributing to $\langle \tau_1(pt)^n \rangle^{E,\underline{a},\text{trop}}_{\Gamma,n}$ and satisfying $\pi(i) = p_{\Omega(i)}$ equals $\mathrm{Coef}_{[q_1^{a_1} \ldots q_r^{a_r}]}\, I_{\Gamma,\underline{g},\Omega}(q_1, \ldots, q_r)$. By Definition 7.3.5 of a Feynman integral, the $x_1^0 \ldots, x_n^0$-Coefficient on the right-hand side of Lemma 8.2.13 equals the $q_1^{a_1} \ldots q_r^{a_r}$-Coefficient of $I_{\Gamma,\Omega}(q_1, \ldots, q_r)$. Using Lemma 8.2.13 and Equation (8.8) (which follows from Theorem 8.2.10 relying on Wick's Theorem), it follows that it also equals $N_{\Gamma,\underline{a},\Omega}$. Thus the statement is proved. □

Chapter 9

Tropical mirror symmetry for $E \times \mathbb{P}^1$

In this chapter, which is based on joint work with Janko Böhm and Hannah Markwig [BGM20], a mirror symmetry relation for $E \times \mathbb{P}^1$ involving counts of curves and Feynman integrals is established, see Corollary 9.3.5. To do so, a correspondence theorem 9.1.16 is used to count tropical stable map to tropical $E\times\mathbb{P}^1$ instead. The floor decomposition techniques (see Chapter 6) of tropical stable maps allow us to pass from tropical stable maps to tropical $E\times\mathbb{P}^1$ to certain tropical covers of a tropical elliptic curve called *curled pearl chains*. Hence results of Chapter 8 can be applied, which yield a tropical mirror symmetry relation that relates generating series of curled pearl chains to sums of Feynman integrals, see Theorem 9.3.1.

As in Chapter 8, an even finer version of the tropical mirror symmetry relation holds naturally (Theorem 9.3.11). Moreover, our tropical approach again allows us to deduce quasi-modularity results for certain generating series of tropical stable maps to tropical $E \times \mathbb{P}^1$, see theorems 9.4.2, 9.4.5.

9.1 Tropical stable maps to tropical $E \times \mathbb{P}^1$

For an elliptic curve E, denote the tropical analogue of the surface $E \times \mathbb{P}^1$ by $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$. We define it as an infinite cylinder surface arising from $E_{\mathbb{T}}$ and $\mathbb{P}^1_{\mathbb{T}}$, see 9.1. It is a tropical surface in the sense of Definition 3.1 [Sha15]. Correspondence theorem 9.1.16 shows that $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ should indeed be viewed as the tropical analogue of $E \times \mathbb{P}^1$.

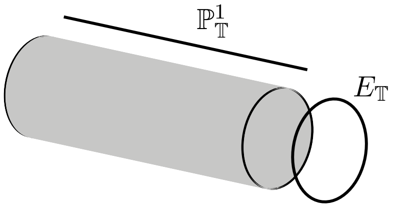


Figure 9.1: An illustration of $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$.

9.1.1 Tropical stable maps to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ via cut open ones

To define tropical stable maps to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$, we consider tropical stable maps to $\mathbb{R}^2$ of a degree that allows us to glue some of its ends together such that the resulting tropical curve winds around an infinite cylinder. For that, notice that a nice feature of tropical geometry is that often, we do not have to use compactifications to get sufficient geometric information. This holds true for our counts of curves. Therefore, we can consider tropical versions of stable maps as maps to $\mathbb{R}^2$. The compactification is implicit in the choice of directions of the ends, resp. in the way they glue.

In Definition 9.1.1, we recall the basic concept of a tropical stable map to $\mathbb{R}^2$ (cf. Definition 2.2.7 for rational tropical stable maps). Next, in 9.1.2, this concept is specified in such a way that the tropical stable maps in question can be interpreted as (building blocks of) maps to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ by gluing opposite ends appropriately. In this definition, we introduce the concept of a leaky degree first. We do this with a view towards further generalizations. Most natural, and most important for our main applications here, is the case where the leaky degree is zero. The general case of leaky degrees is included for the sake of completeness.

Definition 9.1.1. A *tropical stable map* to $\mathbb{R}^2$ is a tuple (Γ, f) where Γ is an explicit (i.e. there is no genus function) and not necessarily connected abstract tropical curve (see Definition 7.1.7) with n marked ends denoted $x_1, \dots, x_n$ and $f : \Gamma \to \mathbb{R}^2$ is a map satisfying:

1. *Integer affine on each edge:* On each edge e of Γ, the map f is of the form
$$t \mapsto a + t \cdot v \text{ with } v \in \mathbb{Z}^2,$$
where we parametrize e as an interval of size the length $l(e)$ of e. The vector v, called the *direction*, arising in this equation is defined up to sign, depending on the starting vertex of the parametrization of the edge. We sometimes speak of the direction of a flag $v(V, e)$ at a vertex V. If e is an end we use the notation $v(e)$ for the direction of its unique flag.

2. *Balancing condition:* At every vertex, we have
$$\sum_{V \in \partial e} v(V, e) = 0.$$

For an edge with direction $v = (v_1, v_2) \in \mathbb{Z}^2 \backslash \{0\}$, we call $w := \gcd(v_1, v_2)$ the *expansion factor* and $\frac{1}{w} \cdot v$ the *primitive direction* of e. If $v = 0$, then we set $w := 0$ as expansion factor.

An isomorphism of tropical stable maps is an isomorphism of the underlying tropical curves respecting the map. The *combinatorial type* of a tropical stable map is the data obtained when dropping the metric of the underlying graph. More explicitly, it consists of the data of a finite graph Γ, and for each edge e of Γ, the direction of e.

Definition 9.1.2. A (cut, open) *tropical stable map to* $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ of *leaky degree*
$$\Delta = \{L_1, \dots, L_{d_2}\}$$
is a tropical stable map to $\mathbb{R}^2$, with N marked ends satisfying:

1. Δ is a multiset containing elements of $\mathbb{Z}$ such that $\sum_{i=1}^{d_2} L_i = 0$.

2. *End directions:* The directions of the ends are given as follows:
   - The marked ends x_i, $i = 1, \dots, n < N$, are contracted: $v(x_i) = 0$.
   - There are d_2 ends of direction $(0, -1)$. All these ends are unmarked.
   - There are d_2 (unmarked) ends of direction $(L_i, 1)$.
   - The remaining ends are marked and of primitive direction $(\pm 1, 0)$.

3. *Gluing:* The ends of primitive direction $(\pm 1, 0)$ come in pairs, one of direction $(1, 0)$ and one of direction $(-1, 0)$, with the same expansion factor and the same y-coordinate.

When $L_i = 0$ for all $i = 1, \ldots, d_2$, we refer to these tropical stable maps as maps without *leaking* or of *leaky degree zero*.

Construction 9.1.3. In two steps, we can produce a tropical stable map to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ from a cut, open tropical stable map:

1. We glue the pairs of ends in (2) of Definition 9.1.2 and partially compactify by adding 1-valent vertices at infinity for the ends of direction $(0, -1)$. The markings for the ends which we glued are forgotten. The graph obtained this way is denoted by Γ'. The map f extends to Γ' and the image $f(\Gamma')$ is a non-compact tropical stable map to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$.

2. To produce a tropical stable map to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$, we need to mark the ends of directions $(L_i, 1)$ and fix the following additional data:

   (a) A partition of Δ into subsets $\Delta_i = \{L_{i_1}, \ldots, L_{i_{k_i}}\}$ satisfying $\sum_{j=1}^{k_i} L_{i_j} = 0$ for all i.

   (b) For each subset Δ_i a 3-valent tree T_i that satisfies the following.

      i. The tree T_i has $k_i + 1$ leaves, where each L_{i_j} for $j = 1, \ldots, k_i$ appears as a label of a leaf and exactly one leaf, called the *root vertex*, is unlabeled.

      ii. The tree T_i is balanced in the following sense: Equip T_i with the orientation that is induced by the root vertex such that the edge adjacent to the root vertex points towards the root vertex (notice that every non-leaf vertex of T_i has precisely 2 incoming edges and 1 outgoing edge). Equip each edge adjacent to a leaf labeled by L_{i_j} with the weight $L_{i_j} \in \mathbb{Z}$. For every non-leaf vertex V of T_i, define the adjacent edges' weights by *balancing*, i.e. the outgoing edge's weight of V is the sum of the two incoming edges' weights. We do not allow a vertex V of T_i to have two incoming edges of the same weight .

   (c) Each non-leaf vertex V of T_i is decorated with a number $n_V \in \mathbb{N}_{>0}$.

   With this additional data, we can produce a (compact) tropical curve in $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ from the non-compact one: If two leaves L_1 and L_2 are adjacent to a vertex V in T_i (in particular, $L_1 \neq L_2$ by condition (2b)), the two open ends of direction $(L_1, 1)$ and $(L_2, 1)$ meet in $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$. At their n_V-th meeting point, we let them merge to a 3-valent vertex and start a new open end at that vertex whose direction is determined by the balancing condition. We cut the cherry corresponding to L_1 and L_2 from the tree T_i and continue recursively with the new tree. Finally, we end up with the edge adjacent to the root vertex that produces an end of vertical direction $(0, k_i)$, which we compactify by adding a vertex at infinity.

   In this way, we produce a tropical stable map to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ whose upper vertical ends may have non-trivial expansion factors.

Here, we focus on the non-compact tropical stable maps we obtain using step (1) above. In particular, we provide methods to count such non-compact tropical curves and study their generating functions. We believe that our methods can be used in future research focusing on counts of curves in $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ with ends of non-trivial expansion factors, i.e. tropicalizations of curves in $E \times \mathbb{P}^1$ satisfying tangency conditions with the ∞-section.

Note that for tropical stable maps without leaking, step (2) of Construction 9.1.3 is trivial: we only compactify by adding vertices at infinity for the vertical ends of direction $(0, 1)$. We also neglect markings for these upper vertical ends. These curves, providing counts of curves satisfying point conditions in $E \times \mathbb{P}^1$, play the main role in this chapter.

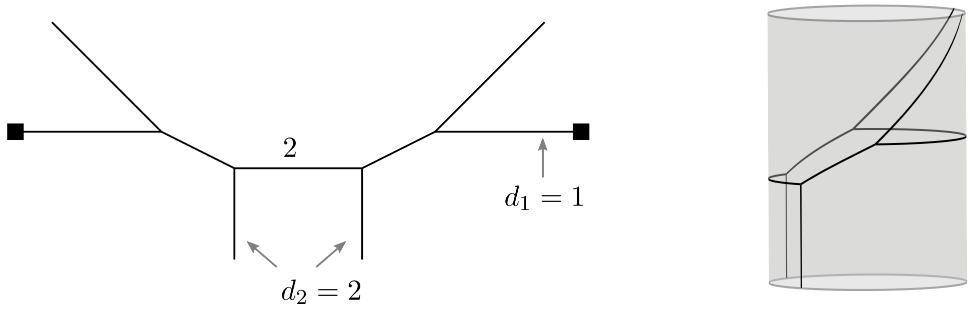


Figure 9.2: Left: A cut tropical stable map (Γ, f) to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ of bidegree $(d_1, d_2) = (1, 2)$ and genus 1. Right: A non-cut picture, where step 1 of Construction 9.1.3 is used. The leaky degree is $\{-1, 1\}$.

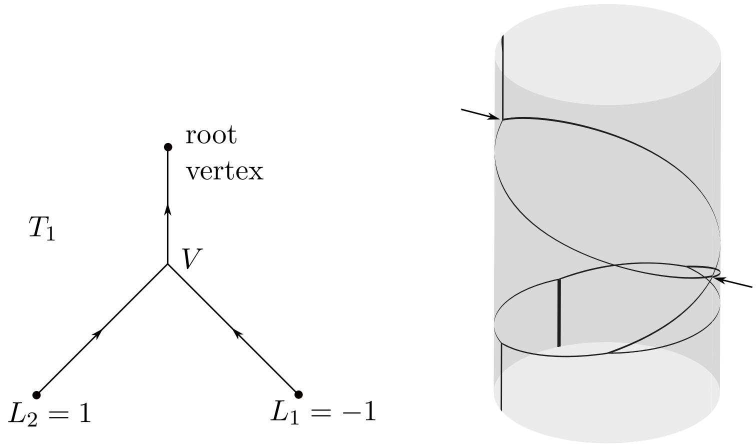


Figure 9.3: Left: An example of a tree T_1 as in step 2 of Construction 9.1.3 with its orientation and labels. Right: A glued picture that is produced by Construction 9.1.3 by taking the cut tropical stable map on the left of Figure 9.2, the tree T_1 and $n_V = 2$. The two arrows mark the intersections which we need to take into account for $n_V = 2$.

Definition 9.1.4. Two tropical stable maps are *equivalent*, if they differ only in the markings for the ends of primitive direction $(\pm 1, 0)$, i.e. the glued graph and the map to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ from Construction 9.1.3(1) coincide. By abuse of notation, we consider tropical stable maps only up to equivalence.

Depending on the image of their end vertices, an end of direction $(L_1, 1)$ and an end of direction $(L_2, 1)$ of a cut tropical stable map can intersect. In the dual subdivision for $f(\Gamma)$, such an intersection corresponds to a parallelogram. Consider the dual subdivision without these parallelograms, and let d_1 be the minimal distance of its vertices to its base line (see Chapter 5 for more about dual subdivisions). Let d_2 be the number of ends of direction $(0, -1)$ which equals the number of ends of direction $(L_i, 1)$ for all i. We call (d_1, d_2) the *bidegree* (see Figure 9.2) of the tropical stable map (Γ, f) for leaky degree Δ to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$.

We define the *genus* of a tropical stable map (Γ, f) to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ to be the genus of the graph Γ' obtained by gluing pairs of ends of Γ as in Construction 9.1.3(1). We say that a tropical stable map to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ is *connected* if Γ' is connected. If $\nu \vdash d_1$ is the partition of expansion factors of the ends of Γ of direction $(1, 0)$ (equivalently, $(-1, 0)$) and the genus of (Γ, f) is g, then Γ has genus $g - \ell(\nu)$, where $\ell(\nu)$ denotes the length of the partition ν.

Definition 9.1.5 (Point conditions)**.** For points $p_1, \ldots, p_n \in \mathbb{R}^2$, a tropical stable map (Γ, f) to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ *satisfies the point conditions* $(p_1, \ldots, p_n)$ if the point $f(x_i)$ to which the end marked x_i is contracted equals p_i.

Remark 9.1.6. Notice that a tropical stable map to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ satisfying point conditions can be viewed as a tropical stable map to the $\mathbb{R}^2$ that satisfies end and point conditions, where the end conditions are imposed by gluing (see Definition 9.1.11). In particular, the theory of counts of plane tropical curves applies, see [Mik05, GM07a, GM07b, GM08]. The end conditions that are imposed by gluing are not necessarily in general position, for example there can be an edge of the glued graph Γ' in Construction 9.1.3(1) that is "curled" several times. In Γ, this edge is cut into several connected components which are all mapped to the same horizontal line. If we shift the end conditions slightly, we have a tropical stable map to $\mathbb{R}^2$ satisfying general conditions, and thus it is 3-valent and, away from the contracted marked ends, locally an embedding.

Example 9.1.7. Let $p_1, \ldots, p_5 \in \mathbb{R}^2$ be general positioned points. Figure 9.4 shows a tropical stable map to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ of leaky degree $\Delta = \{-1, 1\}$ with a curled edge such that the point conditions are satisfied. Note that we need to fix pairs of ends with the same y-coordinate to make the gluing unique.

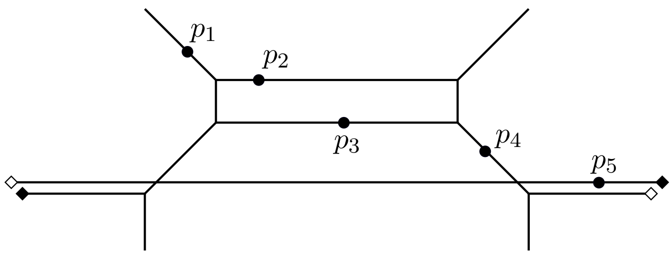


Figure 9.4: A (cut) tropical stable map satisfying point conditions. Note that one edge is curled once. The different pairs of end markings tell us how to glue, but note that we shifted the y-coordinates of glued ends a bit in order to get a better picture (in fact their y-coordinates are the same).

For $n := 2d_2 + g - 1$ points $p_1, \ldots, p_n \in \mathbb{R}^2$ in (tropical) general position, there are finitely many tropical stable maps (Γ, f) to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ of bidegree (d_1, d_2) and genus g satisfying the point conditions. Furthermore, each Γ is 3-valent, and f is an embedding locally around the vertices which are not adjacent to contracted ends. In particular, the multiplicity of a tropical stable map can be defined as in the original count of plane tropical curves [Mik05]:

Definition 9.1.8. Let (Γ, f) be a tropical stable map to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ such that Γ is 3-valent, and f is an embedding locally around the vertices which are not adjacent to marked ends. For a vertex V of Γ which is not adjacent to a marked contracted end, we define its *vertex multiplicity* $\operatorname{mult}_V(\Gamma, f)$ to be $|\det(v_1, v_2)|$, where v_1 and v_2 denote the direction vectors of two of its adjacent edges.

We define the *multiplicity* of (Γ, f) to be the product of its vertex multiplicities, i.e.

$$\operatorname{mult}(\Gamma, f) = \prod_V \operatorname{mult}_V(\Gamma, f),$$

where the product goes over all vertices not adjacent to marked contracted ends.

Definition 9.1.9. Fix positive integers d_1, d_2 and g. Fix $n = 2d_2+g-1$ points $p_1, \dots, p_n \in \mathbb{R}^2$ in (tropical) general position. Let $N^{\text{trop}}_{(\Delta,d_1,d_2,g)}$ be the *number of connected tropical stable maps to* $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ of leaky degree Δ, bidegree (d_1, d_2) and genus g satisfying the point conditions $p_1, \dots, p_n$, counted with multiplicity as in Definition 9.1.8. If we only list 3 subscripts, then this refers to the case without leaking, i.e.

$$N^{\text{trop}}_{(d_1,d_2,g)} := N^{\text{trop}}_{(\{0,\dots,0\},d_1,d_2,g)}.$$

As usual, the analogous count of not necessarily connected tropical stable maps is denoted by $N^{\text{trop},\bullet}_{(\Delta,d_1,d_2,g)}$, resp. $N^{\text{trop},\bullet}_{(d_1,d_2,g)}$.

Remark 9.1.10. The numbers of Definition 9.1.9 are independent of the exact positions of the point conditions by Remark 9.1.6.

9.1.2 Relative tropical stable maps

The correspondence theorem we prove in Subsection 9.1.3 relies on cutting open the infinite cylinder and obtaining from the tropical stable maps within tropical stable maps to $\mathbb{R}^2$ with tangency conditions on the cut line. The latter are known to produce relative Gromov-Witten invariants. We introduce these tropical stable maps in Definition 9.1.11, and relate them to tropical stable maps to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ in Proposition 9.1.14.

Definition 9.1.11 (Relative tropical stable map)**.** Fix partitions μ^+, ϕ^+, μ^- and ϕ^- such that the sum d_1 of the parts in μ^+ and ϕ^+ equals the sum of the parts in μ^- and ϕ^-. Let $n_1 := \ell(\phi^+)+\ell(\phi^-)$ and $n_2 := \ell(\mu^+)+\ell(\mu^-)$. Fix $n \in \mathbb{N}_{>0}$ and a leaky degree $\Delta = \{L_1, \dots, L_{d_2}\}$ which is a multiset containing elements of $\mathbb{Z}$ such that $\sum_{i=1}^{d_2} L_i = 0$.

A *relative tropical stable map* matching the discrete data above is a (not necessarily connected) tropical stable map to $\mathbb{R}^2$ of leaky degree Δ, with $n + n_1 + n_2$ marked ends satisfying:

1. The direction of the ends are imposed as follows:
   - The marked ends x_i for $i = 1, \dots, n$ are contracted, i.e. $v(x_i) = 0$.
   - The other marked ends are of primitive direction $(\pm 1, 0)$.
   - There are d_2 ends of direction $(0, -1)$.
   - There is an end of direction $(L_i, 1)$ for each $i = 1, \dots, d_2$.
   - The partition of expansion factors of the marked ends with primitive direction $(1, 0)$ is (μ^+, ϕ^+).
   - The partition of expansion factors of the marked ends with primitive direction $(-1, 0)$ is (μ^-, ϕ^-).

The *genus* of a relative tropical stable map (Γ, f) is defined to be the genus of Γ. We say (Γ, f) is *connected* if Γ is.

For relative tropical stable maps, *end conditions* can be imposed by requiring that the y-coordinate of the horizontal line to which $f(x_i)$ is mapped equals a fixed value y_i for the end marked i. By convention, we fix y-coordinates for the ends corresponding to ϕ^+ and ϕ^-. For points and end conditions in general position, every relative tropical stable map satisfying the conditions has a 3-valent source graph and is locally an embedding. We can define its multiplicity similar to Definition 9.1.8:

Definition 9.1.12. Let μ^+, ϕ^+, μ^- and ϕ^-, Δ, d_1, d_2, n, n_1 and n_2 be as in Definition 9.1.11. Fix n points in general position and n_1 y-coordinates for ends. For a relative tropical stable map (Γ, f) satisfying the point and end conditions, we define its multiplicity as

$$\mathrm{mult}(\Gamma, f) := \frac{1}{\prod_i \phi_i^+ \prod_j \phi_j^-} \prod_V \mathrm{mult}_V(\Gamma, f),$$

where the last product goes over all vertices V not adjacent to contracted ends and $\mathrm{mult}_V(\Gamma, f)$ is defined in 9.1.8.

Definition 9.1.13. Fix μ^+, ϕ^+, μ^- and ϕ^-, Δ, d_1, d_2, n, n_1 and n_2 as in Definition 9.1.11, and a genus g. Fix $n := n_2 + 2d_2 + g - 1$ points in general position and n_1 y-coordinates for ends. For the invariant

$$\langle (\phi^-, \mu^-) | \tau_0(pt)^n | (\phi^+, \mu^+) \rangle_{g,n}^{\mathrm{trop}, \Delta, (d_1, d_2)},$$

we count connected *relative tropical stable maps of genus g, matching the data and satisfying the conditions* with their multiplicity as defined in 9.1.12. The invariant

$$\langle (\phi^-, \mu^-) | \tau_0(pt)^n | (\phi^+, \mu^+) \rangle_{g,n}^{\mathrm{trop}, \Delta, (d_1, d_2), \bullet}$$

denotes the analogous count of not necessarily connected relative tropical stable maps. For the case $\Delta = \{0, \ldots, 0\}$, we drop the superscript Δ from the notation above.

Proposition 9.1.14. *The number $N^{\mathrm{trop},\bullet}_{(\Delta, d_1, d_2, g)}$ of tropical stable maps to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ of leaky degree Δ, bidegree (d_1, d_2) and genus g satisfying general point conditions equals a sum of counts of relative tropical stable maps:*

$$N^{\mathrm{trop},\bullet}_{(\Delta, d_1, d_2, g)} = \sum_{(\mu, \phi) \,\vdash d_1} \frac{\prod_i \mu_i \prod_j \phi_j}{|\mathrm{Aut}(\mu)| |\mathrm{Aut}(\phi)|} \langle (\mu, \phi) | \tau_0(pt)^n | (\phi, \mu) \rangle_{g - \ell(\mu) - \ell(\phi), n}^{\mathrm{trop}, \Delta, (d_1, d_2), \bullet}.$$

Here, the sum goes over all pairs of partitions (μ, ϕ) of d_1.

In particular, the corresponding equality holds for the case $\Delta = \{0, \ldots, 0\}$, for which the superscript Δ is dropped from the notation above.

Proof. Fix a tropical stable map (Γ, f) to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ contributing to $N^{\mathrm{trop},\bullet}_{(\Delta, d_1, d_2, g)}$. Consider Γ minus the closures of the contracted ends. Since the point conditions are in general position, each connected component contains at least one end. There can be connected components of Γ which just consist of a single unbounded edge of primitive direction $(\pm 1, 0)$, we consider the left and the right part as an end. Such connected components arise from edges of the glued graph Γ' from Construction 9.1.3(1) which are curled multiple times. Let us first consider ends which are not part of such connected components.

An end is fixed by the point conditions if it is the unique end in its connected component. The other ends are moving: we can form a 1-parameter family of tropical stable maps of the same combinatorial type that still meet the point conditions by shifting one of the moving ends slightly and letting the other edges follow.

We treat one end of a component consisting of a single unbounded edge as a fixed end, and the other as a moving end, opposite to the assignment of the end they glue to.

Let ϕ be the partition of expansion factors of ends of primitive direction $(-1, 0)$ which are fixed by the point conditions, and μ the partition of expansion factors of ends of primitive direction $(-1, 0)$ which are moving. Then (μ, ϕ) is a partition of d_1. Furthermore, the gluing condition implies that the expansion factors of the ends of primitive direction $(1, 0)$ are also given by (μ, ϕ), however, the ones corresponding to ϕ must be moving, since the gluing already imposes a condition on them. Vice versa, the ones corresponding to μ must be fixed, since they impose conditions by gluing.

In this way, we can interpret each tropical stable map to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ as a relative tropical stable map.

The factors of $\prod_i \mu_i \prod_j \phi_j$ show up because the multiplicity of the relative tropical stable maps differs from the multiplicity of the tropical stable map to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ by factors of $\frac{1}{w}$ for each expansion factor w of fixed ends, and the ends with expansion factors ϕ_j are fixed on the left while the ones with expansion factors μ_i are fixed on the right.

The factors of $\frac{1}{|\operatorname{Aut}(\mu)||\operatorname{Aut}(\phi)|}$ show up because the relative tropical stable maps have marked ends of primitive direction $(\pm 1, 0)$, and we forget such markings for tropical stable maps to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$. □

9.1.3 Correspondence theorems

Correspondence Theorems for relative Gromov-Witten invariants of Hirzebruch surfaces have been studied before [GM07a, CJMR18]:

Theorem 9.1.15. *Relative Gromov-Witten invariants of $\mathbb{P}^1 \times \mathbb{P}^1$ are equal to their tropical counterparts. This holds both for the connected and the disconnected theory:*

$$\langle (\mu, \phi) | \tau_0(pt)^n | (\phi, \mu) \rangle_{g,n}^{\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2)} = \langle (\mu, \phi) | \tau_0(pt)^n | (\phi, \mu) \rangle_{g,n}^{\text{trop}, \{0, \dots, 0\}, (d_1, d_2)}$$
$$\langle (\mu, \phi) | \tau_0(pt)^n | (\phi, \mu) \rangle_{g,n}^{\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2), \bullet} = \langle (\mu, \phi) | \tau_0(pt)^n | (\phi, \mu) \rangle_{g,n}^{\text{trop}, \{0, \dots, 0\}, (d_1, d_2), \bullet}.$$

Using the degeneration formula in Proposition 7.2.4 together with the tropical relation of counts of stable maps to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ and relative tropical stable maps, Proposition 9.1.14, we can deduce:

Theorem 9.1.16 (Correspondence theorem for $E \times \mathbb{P}^1$)**.** *Notation of definitions 7.2.1, 9.1.9 is used. Gromov-Witten invariants of $E \times \mathbb{P}^1$ agree with the corresponding tropical counts of stable maps to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ without leaking:*

$$N^{\bullet}_{(d_1, d_2, g)} = N^{\text{trop}, \bullet}_{(d_1, d_2, g)}.$$

Since connectedness can be read off the dual graph of a degeneration, we can also deduce the version for connected numbers:

$$N_{(d_1, d_2, g)} = N^{\text{trop}}_{(d_1, d_2, g)}.$$

9.2 Pearl chains

Using a floor diagram technique as in Chapter 6, we introduce a finite method to list all tropical stable maps of genus g, leaky degree Δ and bidegree (d_1, d_2) to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ — we count *curled pearl chains.*

A floor diagram technique relies on picking particular point conditions for which a further degeneration for the tropical stable maps is achieved: as in Figure 9.7, each tropical stable map satisfying the particular point conditions can be split into its horizontal edges (of which each meets a point) and its so-called floors. These remaining parts, the floors, each contain one downward- and one upward-pointing end and meet precisely one point condition also. By shrinking the floors to white vertices, we obtain a bipartite graph, which, due to its embedding in $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$, comes with a natural projection to $E_{\mathbb{T}}$. This projection is what we call a curled pearl chain, the source is what we call a pearl chain. The process how to obtain a curled pearl chain from a tropical stable map to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ is described in Construction 9.2.6.

9.2.1 Pearl chains and curled pearl chains

Definition 9.2.1 (Pearl chain)**.** Let d_2 and g be positive integers. A *pearl chain* of type (d_2, g) is a (non-metric) connected graph $\mathcal{P}$ of genus g. It has d_2 white and $d_2 + g - 1$ black vertices. Edges can only connect a white with a black vertex, but not vertices of the same color. Black vertices must be 2-valent, white vertices can have any valence. There are no cycles of length two.

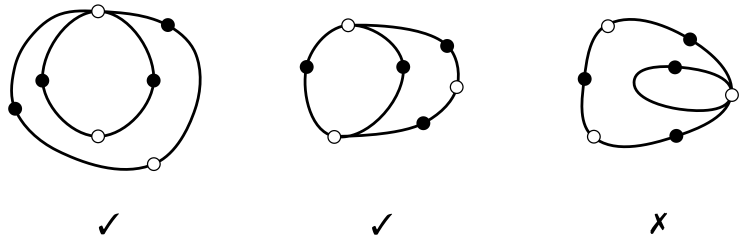


Figure 9.5: Some examples of pearl chains of type $(3, 2)$ and a non-example (the right picture contains a cycle of length 2).

To define curled pearl chains, leaky tropical covers are recalled and their degree is defined:

Definition 9.2.2 (Leaky tropical covers)**.** Consider a tropical cover $\pi : \Gamma \to E_{\mathbb{T}}$ of a tropical elliptic curve as in Definition 7.1.9, and assume that the images of the vertices of Γ are distinct. Let $p \in E_{\mathbb{T}}$ be the image of the vertex v. Fix an orientation of $E_{\mathbb{T}}$. Let f_1 and f_2 be the two flags of $E_{\mathbb{T}}$ adjacent to p, ordered such that the orientation is respected. We say that there is a *leaking* of $L \in \mathbb{Z}$ at p (resp. at v) if

$$d_{v,f_1} - d_{v,f_2} = L$$

holds for the local degrees of π at v with respect to f_i for $i = 1, 2$. If not all vertices of a tropical cover $\pi : \Gamma \to E_{\mathbb{T}}$ satisfy balancing, but some have leaking, we say that it is a *leaky tropical cover.*

For a leaky tropical cover $\pi : \Gamma \to E_{\mathbb{T}}$, define the *degree* d to be the minimum of all sums over all local degrees of preimages of a point a with respect to an adjacent flag f', i.e.

$$d := \sum_{p \mapsto a} d_{p,f'}.$$

Definition 9.2.3 (Curled pearl chains)**.** Fix $d_2, g \in \mathbb{N}_{>0}$. Fix a multiset $\Delta = \{L_1, \dots, L_{d_2}\}$ that contains elements of $\mathbb{Z}$ such that their sum vanishes, i.e. $\sum_{i=1}^{d_2} L_i = 0$. Let $n := 2d_2 + g - 1$, and let $E_{\mathbb{T}}$ be a tropical elliptic curve on which we fix $n+1$ points $p_0, \dots, p_n$ ordered this way. Notice that this choice fixes an orientation of $E_{\mathbb{T}}$. Let $\mathcal{P}$ be a pearl chain of type (d_2, g). A *curled pearl chain* of type (Δ, d_2, g) is a leaky tropical cover $\pi : \mathcal{P}' \to E_{\mathbb{T}}$, where $\mathcal{P}'$ is a metrization of $\mathcal{P}$, such that each $\pi^{-1}(p_i)$ contains one vertex for $i = 1, \dots, n$, and such that

(1) each element L of Δ corresponds to a white vertex with leaking L (in particular, there are d_2 white vertices), and

(2) the black vertices are balanced.

Definition 9.2.4. Let $N^{\text{pearl}}_{(\Delta,d_1,d_2,g)}$ be the weighted *count of curled pearl chains* of type (Δ, d_2, g) and degree d_1. Each curled pearl chain is counted with multiplicity $\prod_e w_e$, the product goes over all expansion factors of edges. If we write $N^{\text{pearl}}_{(d_1,d_2,g)}$, this refers to the case without leaking $\Delta = \{0,\dots,0\}$, i.e.

$$N^{\text{pearl}}_{(d_1,d_2,g)} := N^{\text{pearl}}_{\{0,\dots,0\}(d_1,d_2,g)}.$$

Example 9.2.5. We want to determine $N^{\text{pearl}}_{(2,2,1)}$. We list curled pearl chains of type $(2,1)$ and degree 2 below, where we suppress $E_{\mathbb{T}}$ and the map π and fix the upper white vertex as preimage of p_1 instead (the numbers i of the vertices in Figure 9.6 indicate to which point p_i on $E_{\mathbb{T}}$ the vertices are mapped to). One curled pearl chain has multiplicity $2^4 = 16$ and the rest has multiplicity 1. So Figure 9.6 yields $16 + 14 = 30$ curled pearl chains counted with multiplicity. Since all vertices are 2-valent, we can exchange the colors (i.e. fix a black vertex as preimage of p_1) and obtain a factor 2. Therefore

$$N^{\text{pearl}}_{(2,2,1)} = 60.$$

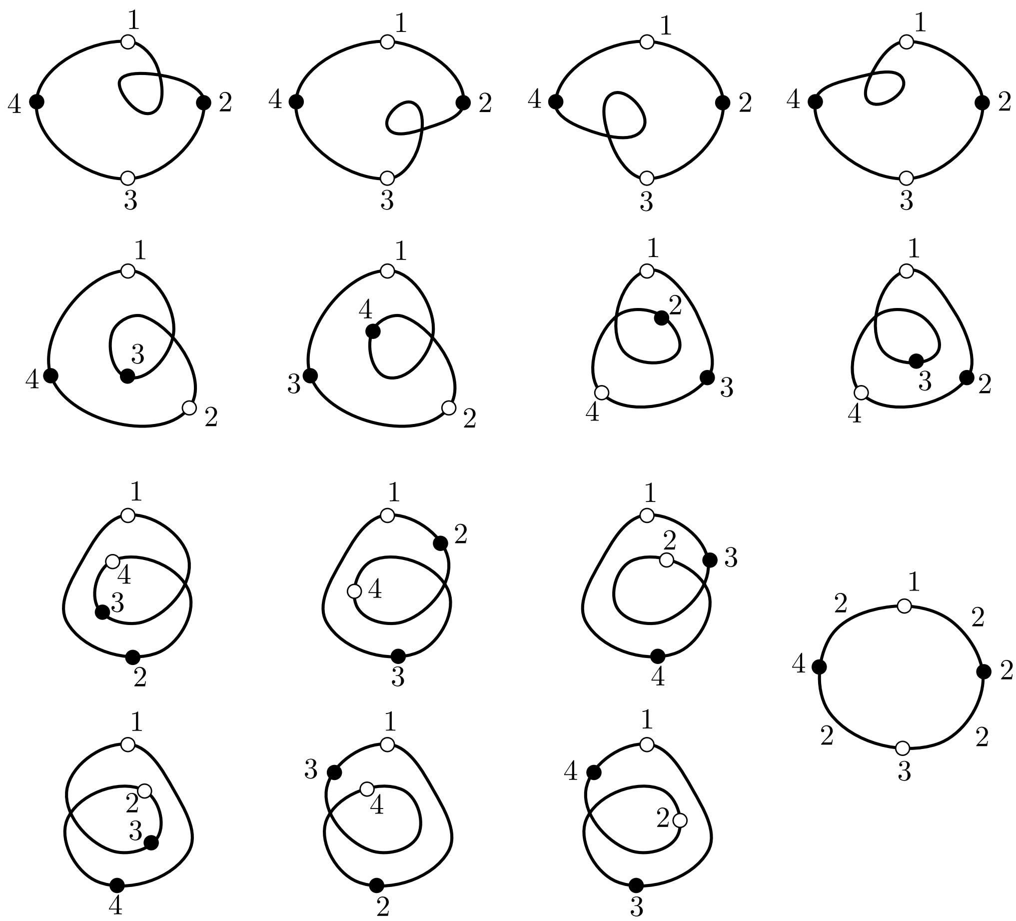


Figure 9.6: Curled pearl chains contributing to $N^{\text{pearl}}_{(2,2,1)}$.

9.2.2 Floor-decomposed tropical stable maps to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$

Since the images of tropical stable maps to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ can be viewed as tropical plane curves, we can also make use of the floor diagram technique [BM08, FM10] as in Chapter 6 by picking a horizontally stretched set of point conditions (similar to Definition 6.1.1).

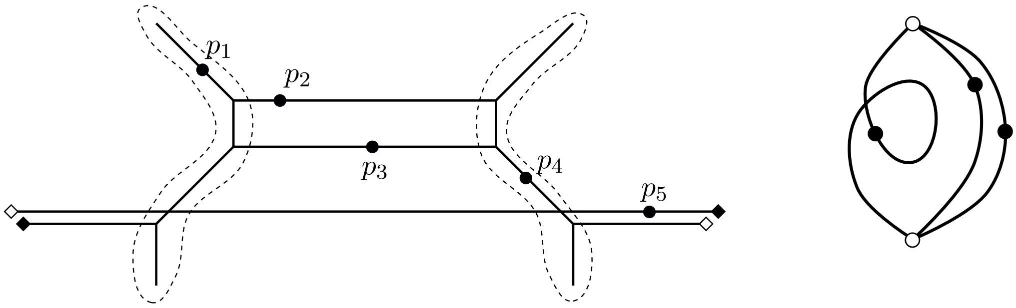


Figure 9.7: Left: The tropical stable map to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ of leaky degree $\{-1, 1\}$ with a curled edge from Example 9.1.7, where we indicated the floors. Right: The curled pearl chain Construction 9.2.6 associates to this tropical stable map (we suppressed the map π).

Every tropical stable map (Γ, f) of leaky degree $\Delta = \{L_1, \ldots, L_{d_2}\}$ satisfying these conditions is *floor-decomposed*, i.e. every connected component of Γ minus the edges of primitive direction $(\pm 1, 0)$ (called a *floor*) contains exactly one marked end which satisfies a point condition, one end of direction $(0, -1)$ and one end of some direction $(L_i, 1)$. Furthermore, each edge of primitive direction $(\pm 1, 0)$ (called an *elevator*), except for n_1 ends whose y-coordinates are imposed by the gluing condition, is adjacent to precisely one marked end that satisfies a point condition.

Construction 9.2.6 (Tropical stable map $\mapsto$ curled pearl chain)**.** Given a floor-decomposed tropical stable map (Γ, f) to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ of bidegree (d_1, d_2) and genus g satisfying $n = 2d_2 + g - 1$ horizontally stretched point conditions, a curled pearl chain $\pi : \mathcal{P}' \to E_{\mathbb{T}}$ can can be associated to it as follows:

As in Construction 9.1.3(1), let Γ' be the graph obtained from Γ by gluing the pairs of ends of primitive direction $(\pm 1, 0)$. Shrink each floor in Γ' to a white vertex. Shrink the marked points on elevators to black vertices. The graph obtained this way is $\mathcal{P}'$. Its edges correspond to elevator edges of Γ', see Figure 9.7.

Let pr denote the vertical projection of $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ to a tropical elliptic curve and let $p_1, \ldots, p_n$ be the images of the horizontally stretched point conditions. The map π can be viewed as the composition of (the extension of) f with pr. It maps the white vertex arising from the floor containing x_i to p_i in E, and the black vertex arising from x_j to p_j. The edge of $\mathcal{P}$ arising from an elevator edge of Γ' is mapped as the projection of the elevator edge. The expansion factors w_e of the edges of $\mathcal{P}$ are given as the expansion factors of the corresponding elevator edges of Γ'. Note that the leaky degree of the tropical stable map and the leaking of the associated curled pearl chain coincide.

Lemma 9.2.7. *Construction 9.2.6 associates a curled pearl chain $\pi : \mathcal{P}' \to E_{\mathbb{T}}$ of type (Δ, d_2, g) and degree d_1 to a floor-decomposed connected tropical stable map (Γ, f) to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ of leaky degree Δ, bidegree (d_1, d_2) and genus g.*

Proof. Denote $\Delta = \{L_1, \ldots, L_{d_2}\}$. The tropical stable map has d_2 ends of direction $(0, -1)$ and exactly one end of directions $(L_i, 1)$ for each $i = 1, \ldots, d_2$. Each floor of its floors contains one end of direction $(0, -1)$ and one of direction $(L_i, 1)$ for some i. The white vertices of $\mathcal{P}'$ come from the d_2 floors. The black vertices come from the remaining $d_2 + g - 1$ points. Every elevator edge of Γ' must be fixed by a point, so the corresponding edge in $\mathcal{P}$ must be adjacent to a black vertex. An edge cannot connect two black vertices since this would correspond to an elevator edge adjacent to two contracted ends, which would then be mapped to the same horizontal line. Since the point conditions are general, two contracted ends cannot be mapped to the same horizontal line. It follows that each edge of $\mathcal{P}'$ is adjacent to one black and one white vertex. A black vertex comes from a contracted end and its two adjacent elevator edges, thus it must be 2-valent. The graph $\mathcal{P}'$ arises from the glued graph Γ' of Construction 9.1.3(1) by shrinking floors resp. ends, hence it has the same genus. Assume $\mathcal{P}'$ had a cycle of length 2, which necessarily connects a white with a black vertex. This would correspond to an elevator edge of Γ', starting at the floor corresponding to the white vertex and returning to the same floor with the same y-coordinate. The balancing condition implies that the floor cannot have a 3-valent vertex adjacent to those two edges, the elevator loop would form a separate connected component, which we excluded. Thus $\mathcal{P}'$ has no cycles of length 2. It follows that $\mathcal{P}$, which equals $\mathcal{P}'$ after forgetting metric data, is a pearl chain of type (d_2, g). From the construction of the map π, it is clear that the preimage of each p_i contains exactly one vertex. Also, the two flags adjacent to a black vertex are mapped to the two flags in E adjacent to the image vertex, and their expansion factors agree: they correspond to the two elevator edges adjacent to a contracted end, which by the balancing condition must have direction $(w_e, 0)$ resp. $(-w_e, 0)$. For a white vertex, which represents a floor of (Γ, f), note that it has two non-contracted ends of direction $(0, -1)$ and $(L_i, 1)$ for some i. The other edges leaving the floor are the elevators, they are of primitive direction $(\pm 1, 0)$. It follows from the balancing condition that there is leaking of Δ at the white vertices. Thus, $\pi : \mathcal{P}' \to E_{\mathbb{T}}$ is a curled pearl chain of type (Δ, d_2, g). For a floor-decomposed tropical stable map, the first entry of the bidegree (d_1, d_2) equals the minimum of the sums of expansion factors of all elevators passing a fixed vertical line. This equals the degree of the curled pearl chain constructed from (Γ, f). □

9.2.3 Counts of leaky tropical stable maps to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ and pearl chains

Theorem 9.2.8. *The count of curled pearl chains of type (Δ, d_2, g) and degree d_1 from Definition 9.2.4 equals the number of connected tropical stable maps to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ of leaky degree Δ, genus g and bidegree (d_1, d_2) (see Definition 9.1.2):*

$$N^{\text{pearl}}_{(\Delta, d_1, d_2, g)} = N^{\text{trop}}_{(\Delta, d_1, d_2, g)}.$$

In particular, for the case without leaking, we have

$$N^{\text{pearl}}_{(d_1, d_2, g)} = N^{\text{trop}}_{(d_1, d_2, g)}.$$

Proof. Given a tropical stable map contributing to $N^{\text{trop}}_{(\Delta, d_1, d_2, g)}$, we know from Construction 9.2.6 and Lemma 9.2.7 how to construct a curled pearl chain contributing to $N^{\text{pearl}}_{(\Delta, d_1, d_2, g)}$. Here, we want to show that Construction 9.2.6 yields a bijection between the set of tropical stable maps contributing to $N^{\text{trop}}_{(\Delta, d_1, d_2, g)}$ and the set of curled pearl chains contributing to $N^{\text{pearl}}_{(\Delta, d_1, d_2, g)}$, and that each tropical stable map is counted with the same weight as its associated curled pearl chain.

Pick general positioned horizontally stretched points $p'_1, \ldots, p'_n$ in $\mathbb{R}^2$. Given a curled pearl chain $\pi : \mathcal{P}' \to E_{\mathbb{T}}$, each vertex of $\mathcal{P}$ corresponds to a point $p_1, \ldots, p_n \in E_{\mathbb{T}}$ via its image under the map π. We construct a tropical stable map (Γ, f) satisfying the point conditions $p'_1, \ldots, p'_n$ starting from local pieces of the image $f(\Gamma) \subset \mathbb{R}^2$.

For a black vertex mapping to $p_i \in E_{\mathbb{T}}$, we draw germs of elevator edges adjacent to p'_i. Each elevator edge of the glued graph Γ' must be fixed by a unique point. We draw end germs for elevator edges of Γ which are moving ends, but whose y-coordinate is imposed by the gluing conditions. If an edge of $\mathcal{P}$ curls multiple times, this corresponds to multiple connected components of Γ consisting of a single edge which is mapped to a horizontal line.

Consider a white vertex mapping to $p_j \in E_{\mathbb{T}}$. It corresponds to a floor which satisfies the point condition p'_j. In Γ, a floor can be viewed as a path connecting an end of direction

$(0,-1)$ with an end of direction $(L_i, 1)$ — the choice of i here depends on the leaking of the corresponding white vertex. The edges adjacent to this path in Γ are elevator edges, and they correspond to the edges of $\mathcal{P}$ adjacent to the white vertex. There is a unique way to connect the corresponding elevator edges (for which the horizontal line to which they map is already fixed) via a path, and to map this floor with f, such that it meets the point p'_j. In this way, we have constructed a floor-decomposed stable map (Γ, f). This construction is obviously inverse to Construction 9.2.6, and so it follows in particular that the discrete data matches.

The multiplicity of (Γ, f) is given by the product of its local vertex multiplicities (see Definition 9.1.8). Each 3-valent vertex V which is not adjacent to a contracted end is contained in a floor. Thus each such vertex V is adjacent to an elevator e. Hence the multiplicity $\operatorname{mult}_V(\Gamma, f)$ of V is given by

$$\operatorname{mult}_V(\Gamma, f) = |\det \begin{pmatrix} w_e & * \\ 0 & \pm 1 \end{pmatrix}| = w_e,$$

where the first column is the direction of the elevator e and the second column is the direction of another edge adjacent to V. From the directions of the non-contracted ends that belong to the floor — $(0,-1)$ and $(L_i, 1)$ — and the balancing condition, it follows that the y-coordinate of this direction vector must be ± 1. Thus every such vertex V contributes a factor of the expansion factor of its adjacent elevator edge. Vice versa, every elevator edge is adjacent to one contracted end and one such vertex V. It follows that $\operatorname{mult}(\Gamma, f)$ equals the product of the expansion factors of its elevators, which equals the weight with which the associated curled pearl chain contributes to $N^{\text{pearl}}_{(\Delta, d_1, d_2, g)}$ by Construction 9.2.6 and Definition 9.2.4. □

9.3 Generating series and Feynman integrals

A pearl chain $\mathcal{P}$ of type (d_2, g) can be turned into a Feynman graph by labeling its vertices by $x_1, \ldots, x_n$ and by labeling its edges by $q_1, \ldots, q_r$, see Definition 7.3.2. Therefore we can form a Feynman integral $I^{l_1,\ldots,l_n}_{\mathcal{P},\Omega}(q)$ for a total order Ω of the n vertices of $\mathcal{P}$, see Notation 7.3.3 and Definition 7.3.5. Thus results of Chapter 8 can be applied to study generating series of the numbers $N^{\text{trop}}_{(\Delta, d_1, d_2, g)}$ in terms of Feynman integrals.

9.3.1 The tropical mirror symmetry theorem for $E \times \mathbb{P}^1$

We now state the main mirror symmetry theorem of this chapter and its corollaries involving generating series of counts of curled pearl chains, tropical stable maps of leaky degree, and Gromov-Witten invariants.

Theorem 9.3.1. *Fix positive integers d_2, g and a multiset $\Delta = \{L_1, \ldots, L_{d_2}\}$ containing elements of $\mathbb{Z}$ such that $\sum_{i=1}^{d_2} L_i = 0$. The generating series of counts of curled pearl chains of type (Δ, d_2, g) (see Definition 9.2.4) equals a sum of Feynman integrals:*

$$\sum_{d_1} N^{\text{pearl}}_{(\Delta, d_1, d_2, g)} q^{d_1} = \sum_{\mathcal{P}} \frac{1}{|\operatorname{Aut}(\mathcal{P})|} \sum_{v} \sum_{\Omega} I^{v}_{\mathcal{P},\Omega}(q).$$

The first sum on the right hand side goes over all pearl chains $\mathcal{P}$ of type (d_2, g) (see Definition 9.2.1), the second sum goes over all vectors v which associate the leakings $L_1, \ldots, L_{d_2}$ to the white vertices of $\mathcal{P}$, and 0 to the black vertices.

Definition 9.3.2. We call the vectors v over which we sum in Theorem 9.3.1 the suitable *leaking vectors.*

Remark 9.3.3. Notice that the generating series on the left of the equality in Theorem 9.3.1 can be stratified into summands for pearl chains also — counting only those curled pearl chains $\pi : \mathcal{P}' \to E_{\mathbb{T}}$ for a fixed pearl chain $\mathcal{P}$. The equality holds for each summand indexed by a pearl chain $\mathcal{P}$.

Using Theorem 9.2.8 and, for $\Delta = \{0, \dots, 0\}$, the Correspondence Theorem 7.1.19, we can interpret the generating series on the left as generating series of counts of tropical stable maps to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ of some leaky degree Δ, and, for $\Delta = \{0, \dots, 0\}$, also as the generating series of Gromov-Witten invariants of $E \times \mathbb{P}^1$.

Corollary 9.3.4 (Tropical mirror symmetry for $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$)**.** *Fix positive integers d_2, g and a leaky degree Δ. The generating series of counts of connected tropical stable maps to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ of leaky degree, genus g and bidegree (d_1, d_2) (see Definition 9.1.9) equals a sum of Feynman integrals:*

$$\sum_{d_1} N^{\mathrm{trop}}_{(\Delta, d_1, d_2, g)} q^{d_1} = \sum_{\mathcal{P}} \frac{1}{|\operatorname{Aut}(\mathcal{P})|} \sum_{v} \sum_{\Omega} I^{v}_{\mathcal{P},\Omega}(q).$$

The first sum on the right hand side goes over all pearl chains of type (d_2, g) (see Definition 9.2.1), the second over all suitable leaking vectors.

Proof. This follows from Theorem 9.2.8 and Theorem 9.3.1 since the generating series on the left are equal. □

Also here, the left hand side can be stratified into sums corresponding to a fixed pearl chain, where we sum only over those tropical stable maps (Γ, f) whose glued graph Γ' (see Construction 9.1.3(1)) equals $\mathcal{P}$ after shrinking floors and contracted ends and forgetting the metric. We denote those counts by $N^{\mathrm{trop},\mathcal{P}}_{(\Delta, d_1, d_2, g)}$. For each pearl chain $\mathcal{P}$, we have

$$\sum_{d_1} N^{\mathrm{trop},\mathcal{P}}_{(\Delta, d_1, d_2, g)} q^{d_1} = \frac{1}{|\operatorname{Aut}(\mathcal{P})|} \sum_{v} \sum_{\Omega} I^{v}_{\mathcal{P},\Omega}(q),$$

where the sum goes over all suitable leaking vectors.

Corollary 9.3.5 (Mirror symmetry for $E \times \mathbb{P}^1$)**.** *Fix positive integers d_2 and g. The generating series of Gromov-Witten invariants of $E \times \mathbb{P}^1$ of genus g and bidegree (d_1, d_2) (see Definition 7.2.1) equals a sum of Feynman integrals:*

$$\sum_{d_1} N_{(d_1, d_2, g)} q^{d_1} = \sum_{\mathcal{P}} \frac{1}{|\operatorname{Aut}(\mathcal{P})|} \sum_{\Omega} I_{\mathcal{P},\Omega}(q).$$

The first sum on the right-hand side goes over all pearl chains $\mathcal{P}$ of type (d_2, g) (see Definition 9.2.1).

Proof. This follows from Corollary 9.3.4 using the correspondence theorem 9.1.16 and taking into account that the sum over all suitable leaking vectors v in Corollary 9.3.4 becomes trivial for $\Delta = \{0, \dots, 0\}$. □

Using tropical geometry, we can stratify the generating series on the left again into summands corresponding to a pearl chain: close to the tropical limit, we can take those stable maps which degenerate to a tropical stable map with a fixed underlying pearl chain. If we denote these numbers by $N^{\mathcal{P}}_{(d_1, d_2, g)}$, then we have

$$\sum_{d_1} N^{\mathcal{P}}_{(d_1, d_2, g)} q^{d_1} = \frac{1}{|\operatorname{Aut}(\mathcal{P})|} \sum_{\Omega} I_{\mathcal{P},\Omega}(q). \tag{9.1}$$

Remark 9.3.6. Notice that the Feynman integrals that appear on the right-hand side of the equation of Corollary 9.3.5 and in Equation (9.1) are equal to the complex analysis version of Feynman integrals, see Theorem 7.3.7.

9.3.2 Labeled curled pearl chains and the proof of Theorem 9.3.1

The proof of Theorem 9.3.1 relies on methods developed in [BBBM17] and Chapter 8. The main ingredient is a bijection between certain covers of graphs and monomials contributing to a Feynman integral, see Theorem 8.1.14. We can apply this here to our case of curled pearl chains, but to do this we need to introduce labeled curled pearl chains whose underlying tropical cover is labeled.

Definition 9.3.7 (Labeled curled pearl chain)**.** A *labeled curled pearl chain* $\pi : \mathcal{P}' \to E_{\mathbb{T}}$ is a curled pearl chain (Definition 9.2.3) for which we fix labels $x_1, \ldots, x_n$ for the vertices of $\mathcal{P}$ and $q_1, \ldots, q_r$ for the edges.

Remark 9.3.8. By definition, a labeled curled pearl chain is a graph cover with a particular source graph, namely a pearl chain, for which only suitable leaking vectors v are allowed, see definitions 9.2.3, 9.3.2.

Definition 9.3.9. See definitions 8.1.8, 8.1.11, 9.2.1, 9.3.2 for the following. Let $\mathcal{P}$ be a pearl chain of type (d_2, g), Ω and order and $\underline{a}$ a multidegree. Let v be a suitable leaking vector for Δ.

We define $N^v_{\mathcal{P},\underline{a},\Omega}$ to be the *number of labeled curled pearl chains* whose source curve is a metrization of $\mathcal{P}$, of order Ω and multidegree $\underline{a}$, where the leaking is imposed by v. Each labeled curled pearl chain is counted with the product of the expansion factors w_e of the edges e of $\mathcal{P}$.

Remark 9.3.10. Notice that $N^v_{\mathcal{P},\underline{a},\Omega}$ is a count of graph covers with fixed order and multidegree as in Definition 8.1.11, for which we chose a particular source graph (namely a pearl chain) and a suitable leaking vector.

Theorem 9.3.11. *Fix positive integers d_2, g and a leaky degree Δ. Fix a labeled pearl chain $\mathcal{P}$, an order Ω and a multidegree $\underline{a}$. Let v be a suitable leaking vector.*

The numbers $N^v_{\mathcal{P},\underline{a},\Omega}$ of labeled curled pearl chains of Definition 9.3.7 are coefficients of a refined Feynman integral:

$$N^v_{\mathcal{P},\underline{a},\Omega} = \mathrm{Coef}_{[q_1^{a_1}\ldots q_r^{a_r}]}\, I^v_{\mathcal{P},\Omega}(q_1, \ldots, q_r).$$

Proof. This follows from Theorem 8.1.14 using Remark 9.3.10. □

The following is an immediate corollary of Theorem 9.3.11.

Corollary 9.3.12. *Fix positive integers d_2, g and a leaky degree Δ. Fix a labeled pearl chain $\mathcal{P}$, an order Ω and a multidegree $\underline{a}$. Let v be a suitable leaking vector.*

The generating series of the numbers of labeled curled pearl chains (see Definition 9.3.7) equals a refined Feynman integral:

$$\sum_{\underline{a}\in\mathbb{N}^r} N^v_{\mathcal{P},\underline{a},\Omega} q_1^{a_1} \cdot \ldots \cdot q_r^{a_r} = I^v_{\mathcal{P},\Omega}(q_1, \ldots, q_r).$$

Proof of Theorem 9.3.1. The equality of the generating series and the sum of Feynman integrals now follows from Corollary 9.3.12 by summing over all pearl chains $\mathcal{P}$ of type (d_2, g) and leaking

Δ, and over all suitable leaking vectors v, setting the q_k equal to q again for all k and keeping track of automorphisms as in the proof of Theorem 8.1.4 using Theorem 8.1.9:

Fix a pearl chain, a leaking vector v, an order Ω and a degree d_1. Let $N^v_{\mathcal{P},\Omega,d_1}$ be the number of curled pearl chains of degree d_1, order Ω and with underlying pearl chain $\mathcal{P}$. Each curled pearl chain $\pi : \mathcal{P}' \to E_{\mathbb{T}}$ is counted with multiplicity $\prod_e \omega(e)$. There is a forgetful map ft from the set of labeled curled pearl chains (with $\mathcal{P}, d_1, v, \Omega$) to the set of curled pearl chains by forgetting labels of edges and vertices.

Let $\pi : \mathcal{P}' \to E_{\mathbb{T}}$ be a curled pearl chain as above. The automorphism group $\mathrm{Aut}(\mathcal{P})$ acts transitively on $\mathrm{ft}^{-1}(\pi)$ by relabeling edges and vertices. Since the stabilizer of this action is $\mathrm{Aut}(\pi)$, the size of the orbit $\mathrm{ft}^{-1}(\pi)$ is

$$|\,\mathrm{ft}^{-1}(\pi)| = \frac{|\,\mathrm{Aut}(\mathcal{P})|}{|\,\mathrm{Aut}(\pi)|}.$$

Note that $|\,\mathrm{Aut}(\pi)| = 1$ since Ω is fixed and there are no multiple edges. Each labeled curled pearl chain $\tilde{\pi}$ in $\mathrm{ft}^{-1}(\pi)$ is counted with multiplicity $\prod_e \omega(e)$. Hence

$$\begin{aligned}
\sum_{\underline{a}:\sum a_i = d_1} N^v_{\mathcal{P},\underline{a},\Omega} &= \sum_{\tilde{\pi}:\mathcal{P}'\to E_{\mathbb{T}}} \prod_e \omega(e) \\
&= \sum_{\pi:\mathcal{P}'\to E_{\mathbb{T}}} \sum_{\substack{\tilde{\pi}:\mathcal{P}'\to E_{\mathbb{T}}: \\ \mathrm{ft}(\tilde{\pi})=\pi}} \prod_e \omega(e) \\
&= \sum_{\pi:\mathcal{P}'\to E_{\mathbb{T}}} |\,\mathrm{Aut}(\mathcal{P})| \prod_e \omega(e) \\
&= |\,\mathrm{Aut}(\mathcal{P})| \cdot N^v_{\mathcal{P},\Omega,d_1},
\end{aligned}$$

where the second sum goes over all labeled curled pearl chains with fixed multidegree $\underline{a}$ and underlying pearl chain $\mathcal{P}$.

Summing over all degrees d_1 and using Corollary 9.3.12 with $q_1 = \cdots = q_r = q$ gives us

$$\sum_{d_1 \geq 1} |\,\mathrm{Aut}(\mathcal{P})| \cdot N^v_{\mathcal{P},\underline{a},\Omega} q^{d_1} = I^v_{\mathcal{P},\Omega}(q)$$

for a fixed pearl chain $\mathcal{P}$ of type (Δ, d_2, g) such that v is a suitable leaking vector. If we sum over all $\mathcal{P}$, all Ω and all v of type Δ, then Theorem 9.3.1 follows. □

Example 9.3.13. Let $\underline{a} = (1,0,0,1)$ be a multidegree and let v be zero, i.e. there is no leaking. Fix points $p_0, \ldots, p_4$ on $E_{\mathbb{T}}$ and let Ω be the order associated to the identity in the permutation group of 4 elements. We want to determine $N^v_{\mathcal{P},\underline{a},\Omega}$, where $\mathcal{P}$ and its labels are depicted on the left in Figure 9.8. Using Figure 9.6 from Example 9.2.5, we can see that the only labeled curled pearl chain of multidegree $(1,0,0,1)$ and order Ω is the one shown on the right below. The gray dots indicate the preimages of $p_0 \in E_{\mathbb{T}}$. Since this labeled curled pearl chain has multiplicity 1, we have $N_{\mathcal{P},\underline{a},\Omega} = 1$. This count can be verified using Theorem 9.3.11.

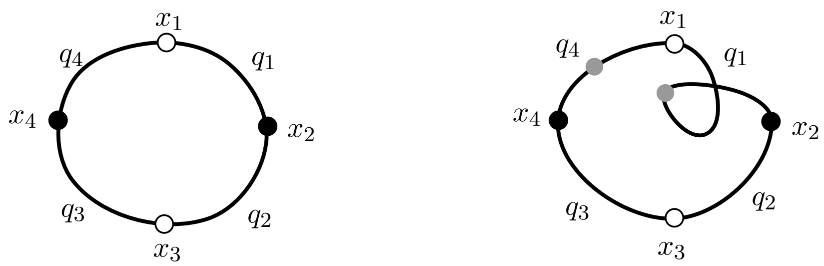


Figure 9.8: Left: A pearl chain $\mathcal{P}$ with labels. Right: A labeled curled pearl chain.

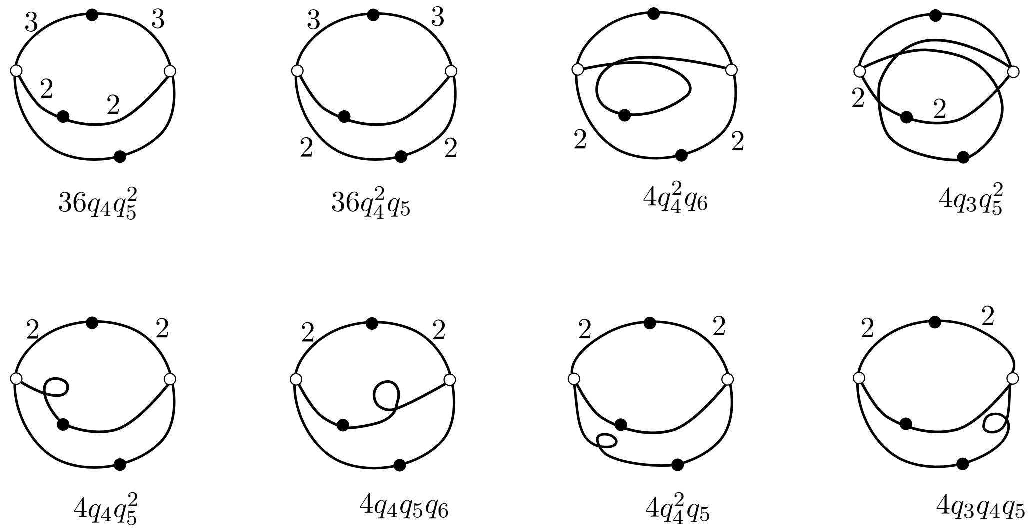


Figure 9.10: Curled pearl chains and their contributions to the refined Feynman integral.

Example 9.3.14. We want to provide a larger example illustrating Theorem 9.3.11. First of all, fix the following data: we consider no leaking, the pearl chain $\mathcal{P}$ (which is of genus 2) is shown in Figure 9.9, the order Ω corresponds to the identical permutation and the total degree d should be 3. First, we want to determine the left hand side of Theorem 9.3.11 for the given input data, i.e. $\sum_{\underline{a}} N_{\mathcal{P},\underline{a},\Omega}$, where the sum goes over all $\underline{a}$ contributing to a total degree of 3. After that, we calculate the refined Feynman integral on the right-hand side and compare its coefficients to the combinatorial count of labeled pearl chains from before.

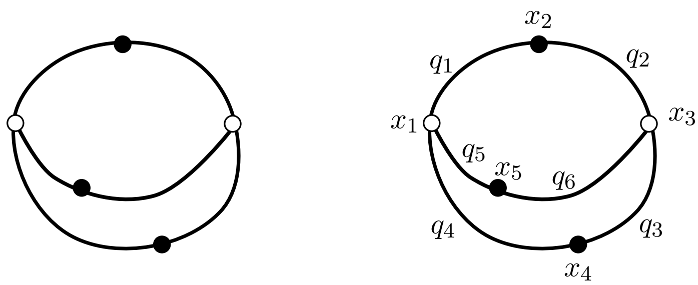


Figure 9.9: Left: A pearl chain $\mathcal{P}$. Right: $\mathcal{P}$ with labels.

Since we fixed an order Ω, we should label the pearl chain's vertices and edges (see Figure 9.9) before curling them. Using Figure 9.10, we determine $\sum_{\underline{a}} N_{\mathcal{P},\underline{a},\Omega}$. Figure 9.10 shows schematic representations of labeled curled pearl chains: we suppress the elliptic curve $E_{\mathbb{T}}$ and only show the source curve of each cover. The labels of the source curve are also suppressed, but can easily be seen from comparing Figure 9.10 to Figure 9.9. Each curled pearl chain contributes with the product of its source curve's edge expansion factors (the edge expansion factors not equal to 1 are shown in Figure 9.10), thus

$$\sum_{\underline{a}} N_{\mathcal{P},\Omega,\underline{a}} = 96.$$

The sum of the refined Feynman integral's terms that are of degree 3 is

$$40q_4^2q_5 + 40q_4q_5^2 + 4q_4q_5q_6 + 4q_3q_4q_5 + 4q_3q_5^2 + 4q_4^2q_6,$$

which makes $96q^3$ if we set $q_1 = \cdots = q_4 = q$, i.e. we obtained exactly the coefficient which we expected from our combinatorial count. The contributions of each term can directly be seen from our curled pearl chains. These contributions can be found under each curled pearl chain in Figure 9.10.

9.4 Quasimodularity

Following the study of quasimodularity for generating series of Gromov-Witten invariants of an elliptic curve, Oberdieck and Pixton conjectured a quasimodularity statement for cycle-valued generating series of Gromov-Witten invariants of elliptic fibrations (Conjecture A [OP18]), which they prove for the case of certain products involving an elliptic curve (Corollary 2).

As in Section 8.1.4 of Chapter 8, our tropical mirror symmetry results (see Theorem 9.3.1, Corollary 9.3.4 and Corollary 9.3.5) hand us a new way to study quasimodularity by making use of Feynman integrals. The quasimodularity (of mixed weight) of a summand $I_{\mathcal{P}}(q)$ for a fixed pearl chain $\mathcal{P}$ can be deduced from the recent study of quasimodularity for graph sums in [GM20].

When passing to the tropical limit, the summands $I_{\mathcal{P}}(q)$ obtain a meaning as summands of the generating series corresponding to a fixed pearl chain:

Remark 9.4.1. Fix positive integers d_2, g, and a pearl chain $\mathcal{P}$ of type (d_2, g).

Then the following generating series coincide (see also Figure 1.4):

(1) the generating series of Gromov-Witten invariants of $E \times \mathbb{P}^1$ of bidegree (d_1, d_2) and genus g which correspond to a fixed pearl chain $\mathcal{P}$ close to the tropical limit,

(2) the generating series of tropical stable maps to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ of bidegree (d_1, d_2) and genus g with fixed underlying pearl chain $\mathcal{P}$,

(3) the generating series of curled pearl chains of type (d_2, g) without leaking with fixed underlying pearl chain $\mathcal{P}$.

In other words,

$$\underbrace{\sum_{d_1} N^{\mathcal{P}}_{(d_1,d_2,g)} q^{d_1}}_{(1)} = \underbrace{\sum_{d_1} N^{\mathrm{trop},\mathcal{P}}_{(d_1,d_2,g)} q^{d_1}}_{(2)} = \underbrace{\sum_{d_1} N^{\mathrm{pearl},\mathcal{P}}_{(d_1,d_2,g)} q^{d_1}}_{(3)} . \tag{9.2}$$

By Theorem 9.3.1, the series in Equation (9.2) equal a sum of Feynman integrals, namely

$$\frac{1}{|\operatorname{Aut}(\mathcal{P})|} \sum_{\Omega} I_{\mathcal{P},\Omega}(q).$$

Theorem 9.4.2. *Fix positive integers d_2, g and a pearl chain $\mathcal{P}$ of type (d_2, g). Each summand $I_{\mathcal{P},\Omega}(q)$ corresponding to an order Ω in the generating series (9.2) is a quasimodular forms of mixed weight at most $4(d_2 + g - 1)$.*

This follows from Theorem 1.1 resp. Theorem 6.1 in [GM20]. These results imply that each summand $I_{\mathcal{P},\Omega}(q)$ corresponding to an order Ω is a quasimodular form of mixed weight at most $2r$, where r is the number of edges of $\mathcal{P}$. Note that r equals two times the number of black vertices of $\mathcal{P}$ (which is by definition $d_2 + g - 1$) because black vertices are 2-valent and no black vertex is adjacent to a black vertex.

As a consequence, the generating series which is the sum of the ones in Equation (9.2), summing over all pearl chains $\mathcal{P}$, is a quasimodular form:

Corollary 9.4.3. *Let d_2, g be positive integers. Then the generating series of Gromov-Witten invariants of $E \times \mathbb{P}^1$ (resp. of tropical stable maps to $E_{\mathbb{T}} \times \mathbb{P}^1_{\mathbb{T}}$ of bidegree (d_1, d_2) and genus g, resp. of curled pearl chains of type (d_2, g))*

$$\sum_{d_1} N_{(d_1,d_2,g)} q^{d_1} = \sum_{d_1} N^{\text{trop}}_{(d_1,d_2,g)} q^{d_1} = \sum_{d_1} N^{\text{pearl}}_{(d_1,d_2,g)} q^{d_1}$$

is a quasimodular form of mixed weight at most $4(d_2 + g - 1)$.

Proof. This follows from the quasimodularity of each summand corresponding to a pearl chain $\mathcal{P}$ and an order Ω as considered in Theorem 9.4.2. The maximal weight arises because a pearl chain of type (d_2, g) has $2(d_2+g-1)$ edges, where d_2+g-1 is the number of black vertices. □

In the following, we give an example illustrating Corollary 9.4.3 which states that the series $\sum_{d_1} N^{\text{pearl}}_{(d_1,d_2,g)} q^{d_1}$ is a quasimodular form. Recall that $\sum_{d_1} N^{\text{pearl}}_{(d_1,d_2,g)} q^{d_1}$ is stratified as a sum over orders Ω. We observe that for a choice of input data d_2, g each of these strata contributes with a summand that is a quasimodular form of mixed weight, but the sum over all orders yields a homogeneous quasimodular form.

This is in accordance with the situation of generating series of Hurwitz numbers, where each summand is quasimodular of mixed weight, but the whole sum is homogeneous [GM20].

Example 9.4.4. Let $d_2 = 2$ and $g = 1$. Hence there is only one pearl chain $\mathcal{P}$ contributing to $\sum_{d_1} N^{\text{pearl}}_{(d_1,d_2,g)} q^{d_1}$, namely the one shown in Example 9.3.13. Using Corollary 9.3.12, we can calculate the generating series for our pearl chain $\mathcal{P}$ and any order Ω in terms of refined Feynman integrals. We observe that there are 8 orders Ω that are the elements of the symmetry group

$$D_4 = \{(), (1,2,3,4), (1,3)(2,4), (1,4,3,2), (2,4), (1,3), (1,2)(3,4), (1,4)(2,3)\}$$

of the square, which all lead to the same generating series

$$\begin{aligned} g_1 := \ldots 15092q^{11} + 13560q^{10} + 7701q^9 + 5680q^8 + 2520q^7 + 1872q^6 + 670q^5 \\ + 344q^4 + 92q^3 + 20q^2 + q. \end{aligned}$$

The series g_1 is a quasimodular form of mixed weight since it can be expressed in Eisenstein series as

$$\begin{aligned} g_1 = &-\frac{1}{1080}E_6(q) + \frac{1}{1080}E_2(q)E_4(q) + \frac{1}{6912}E_4^2(q) - \frac{1}{2592}E_2(q)E_6(q) \\ &+ \frac{1}{3456}E_2^2(q)E_4(q) - \frac{1}{20736}E_2^4(q). \end{aligned}$$

Moreover, we observe that the remaining 16 orders all lead to the same generating series

$$\begin{aligned} g_2 := \ldots 440q^{11} + 2220q^{10} + 888q^9 + 1000q^8 + 112q^7 + 360q^6 + 40q^5 + 52q^4 \\ + 8q^3 + 2q^2, \end{aligned}$$

which is also a quasimodular form of mixed weight since it can be expressed in Eisenstein series as

$$\begin{aligned} g_2 = &\frac{1}{2160}E_6(q) - \frac{1}{2160}E_2(q)E_4(q) + \frac{1}{6912}E_4^2(q) - \frac{1}{2592}E_2(q)E_6(q) \\ &+ \frac{1}{3456}E_2^2(q)E_4(q) - \frac{1}{20736}E_2^4(q). \end{aligned}$$

Taking the sum, we obtain that the generating series

$$\begin{aligned}\sum_{d_1} N^{\text{pearl}}_{(d_1,2,1)} q^{d_1} &= 8g_1 + 16g_2 \\ &= \frac{1}{288}E_4^2(q) - \frac{1}{108}E_2(q)E_6(q) + \frac{1}{144}E_2^2(q)E_4(q) - \frac{1}{864}E_2^4(q)\end{aligned}$$

is a homogeneous quasimodular form of weight 8.

The observations from Example 9.4.4 are true in general, which follows from the study of Oberdieck and Pixton (Theorem 7, Appendix A, [OP18]):

Theorem 9.4.5. *Fix positive integers d_2, g and a pearl chain $\mathcal{P}$ of type (d_2, g). Then the generating series $\sum_{d_1} N^{\text{trop},\mathcal{P}}_{(d_1,d_2,g)} q^{d_1}$ obtained for a fixed pearl chain as the sum of the contributions over all orders is a homogeneous quasimodular form of weight $4(d_2 + g - 1)$.*

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